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Dynamic Pricing and Inventory Management: Theory and Applications

Renyu Zhang
Washington University in St. Louis

WASHINGTON UNIVERSITY IN ST. LOUIS

Olin Business School

Dissertation Examination Committee:

Nan Yang, Chair

Fuqiang Zhang, Co-Chair

Amr Farahat

Jake Feldman

John Nachbar

Dynamic Pricing and Inventory Management: Theory and Applications

by

Renyu Zhang

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ABSTRACT OF THE DISSERTATION

Dynamic Pricing and Inventory Management: Theory and Applications

by

Renyu Zhang

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Professor Nan Yang, Chair

Professor Fuqiang Zhang, Co-Chair

We develop the models and methods to study the impact of some emerging trends in technology, marketplace, and society upon the pricing and inventory policy of a firm. We focus on the situation where the firm is in a dynamic, uncertain, and (possibly) competitive market environment. The market trends of particular interest to us are: (a) social networks, (b) sustainability concerns, and (c) customer behaviors. The two main running questions this dissertation aims to address are: (a) How these emerging market trends would influence the operations decisions and profitability of a firm; and (b) What pricing and inventory strategies a firm could use to leverage these trends. We also develop an effective comparative statics analysis method to address these two questions under different market trends.

Overall, our results suggest that the current market trends of social networks, sustainability concerns, and customer behaviors have significant and interesting impact upon the operations policy of a firm, and that the firm could adopt some innovative pricing and inventory strategies to exploit these trends and substantially improve its profit. Our main findings are:

(a) *Network externalities (the monopoly setting).* We find that network externalities prompt a firm to face the tradeoff between generating current profits and inducing future demands when making the price and inventory decisions, so that it should increase the base-stock level, and to decrease [increase] the sales price when the

network size is small [large]. Our extensive numerical experiments also demonstrate the effectiveness of the heuristic policies that leverage network externalities by balancing generating current profits and inducing demands in the near future. (Chapter 2.)

(b) *Network externalities (the dynamic competition setting).* In a competitive market with network externalities, the competing firms face the tradeoff between generating current profits and winning future market shares (i.e., the exploitation-induction tradeoff). We characterize the pure strategy Markov perfect equilibrium in both the simultaneous competition and the promotion-first competition. We show that, to balance the exploitation-induction tradeoff, the competing firms should increase promotional efforts, offer price discounts, and improve service levels. The exploitation-induction tradeoff could be a new driving force for the fat-cat effect (i.e., the equilibrium promotional efforts are higher under the promotion-first competition than those under the simultaneous competition). (Chapter 3.)

(c) *Trade-in remanufacturing.* We show that, with the adoption of the very commonly used trade-in remanufacturing program, the firm may enjoy a higher profit with strategic customers than with myopic customers. Moreover, trade-in remanufacturing creates a tension between firm profitability and environmental sustainability with strategic customers, but benefits both the firm and the environment with myopic customers. We also find that, with either strategic or myopic customers, the socially optimal outcome can be achieved by using a simple linear subsidy and tax scheme. The commonly used government policy to subsidize for remanufacturing alone, however, does not induce the social optimum in general. (Chapter 4.)

(d) *Scarcity effect of inventory.* We show that the scarcity effect drives both optimal prices and order-up-to levels down, whereas increased operational flexibilities (e.g., the inventory disposal and inventory withholding opportunities) mitigate the demand loss caused by high excess inventory and increase the optimal order-up-to levels and sales prices. Our extensive numerical studies also demonstrate that dynamic pricing leads to a much more significant profit improvement with the scarcity effect of inventory than without. (Chapter 5.)

(e) *Comparative statics analysis method.* We develop a comparative statics method to study a general joint pricing and inventory management model with multiple demand segments, multiple suppliers, and stochastically evolving market conditions. Our new method makes componentwise comparisons between the focal decision variables under different parameter values, so it is capable of performing comparative statics analysis in a model where part of the decision variables are non-monotone, and it is well scalable. Hence, our new method is promising for comparative statics analysis in other operations management models. (Chapter 6.)

1. Introduction

1.1 Motivation

Price and inventory are definitely two key operations decisions of any firm that delivers (physical) products to customers. The development of advanced information technologies facilitates the sellers to plan, implement and take advantage of the dynamic pricing strategies. Thanks to the IT decision support applications, sellers are now able to optimize sales prices and inventory control policies based on complex analytics and optimization methods. Therefore, the joint dynamic pricing and inventory management strategies have been extensively studied in the literature, and widely used in practice. For example, Amazon not only dynamically adjusts the sales prices of thousands of its items everyday, but also adopts a complex procurement and delivery system to manage its inventories.

The emerging trends in technology, marketplace, and society have led to unprecedented challenges to optimize their pricing and inventory control policy. The primary goal of this dissertation is to develop the models and methods to understand the impact of some emerging market trends upon a firm's pricing and inventory policy. Specifically, we consider three types of current market trends: (a) social networks, (b) sustainability concerns, and (c) customer behaviors.

- *Social networks.* The recent fast development of online social media has significantly intensified the interactions between customers. The social networks make customers easily know and follow their friends' purchasing decisions, thus giving rise to (positive) *network externalities* for almost all products. That is, customers are more likely to purchase a product if there are more other customers who purchase the same product. Network externalities enable firms to use current customers to attract future customers and, thus, may have interesting implications on the pricing and inventory policy of a firm.

- *Sustainability concerns.* In the recent years, the society embraces an increasing trend of sustainability/environmental concerns. Remanufacturing, and the associated trade-in program to collect used products for remanufacturing, have been increasingly used for the sake of its environmental benefit. We are especially interested in characterizing how trade-in remanufacturing would influence the pricing and production policy of a firm, and the economic and environmental values of this business practice. From the government's perspective, it is also interesting to study the public policy that could improve the social welfare when taking into account firm profit, customer surplus, and environmental impact.

- *Customer behaviors.* We study two customer behaviors in this dissertation. The first is the strategic waiting behavior of customers. With this behavior, customers will strategically seek for future discount and trade-in opportunities. We are curious about the impact of strategic customer behavior upon the economic and environmental values of trade-in remanufacturing. The second customer behavior studied in this dissertation is the scarcity effect of inventory, which refers to the phenomenon that customers are discouraged by high inventory and encouraged by low inventory available to them. The operational implications of the scarcity effect of inventory have also been analyzed in this dissertation.

1.2 Contribution

In this dissertation, we establish dynamic programming and game theoretic models to study the dynamic pricing and inventory control issues under the presence of these new market trends. Our focus is to address two main questions: (a) How these emerging market trends would influence the operations decisions and profitability of a firm; and (b) What pricing and inventory strategies a firm could use to leverage these trends. Our analysis reveals that the current market trends of social networks, sustainability concerns, and customer behaviors give rise to some new tradeoffs the firm has to balance and, thus, have significant and interesting impact upon the operations policy of a firm. On the other hand, the firm could adopt some innovative pricing and inventory strategies to exploit these trends and substantially improve its profit. To facilitate the analysis of the two main questions, we also develop an effective comparative statics analysis method for a general class of joint pricing and inventory management models.

Network externalities (the monopoly setting, Chapter 2). We study the impact of network externalities upon a firms pricing and inventory policy under demand uncertainty. The firm sells a product associated with an online service or communication network, which is formed by (part of) the customers who have purchased the product. The product exhibits *network externalities*, i.e., a customer's willingness-to-pay and, thus, the potential demand are increasing in the size of the associated network. We show that a network-size-dependent base-stock/list-price policy is optimal. Moreover, the inventory dynamics of the firm do not influence the optimal policy as long as the initial inventory is below the initial base-stock level. Hence, we can reduce the dynamic program to characterize the optimal policy to one with a single-dimensional state-space (the network size). Network externalities give rise to the tradeoff between generating current profits and inducing future demands, thus having several important implications upon the firm's operations decisions. Compared with the benchmark case without network externalities, the firm under network externalities sets a higher base-stock level, and charges a lower [higher] sales price when the network size is small [large]. When the market is stationary, the firm adopts the introductory price strategy, i.e., it charges a lower price at the beginning of the sales season to induce higher future demands. The price discrimination and network expanding promotion strategies can effectively leverage network externalities and improve the firm's profit. Both strategies facilitate the firm to (partially) separate generating current profits and inducing future demands through network externalities. Finally, we perform extensive numerical studies to demonstrate the significant profit loss of ignoring network externalities. We also propose near-optimal heuristic policies that leverage network externalities by balancing generating current profits and inducing demands in the *near future*.

Network externalities (the dynamic competition setting, Chapter 3). We study a dynamic competition model, in which retail firms periodically compete on promotional effort, sales price, and service level over a finite planning horizon. The key feature of our model is that the current decisions influence the future market sizes through the service effect and the network effect, i.e., the firm with a higher current service level and a higher current demand is more likely to have larger future market sizes and vice versa. Hence, the competing firms face the tradeoff between generating current profits and inducing future demands (i.e., the *exploitation-induction tradeoff*). Using the linear separability approach, we characterize the pure strategy Markov perfect equilibrium in both the simultaneous competition and the promotion-first competition. The exploitation-induction tradeoff has several important managerial implications under both competitions. First, to balance the exploitation-induction tradeoff, the competing firms should increase promotional efforts, offer price discounts, and improve service levels under the service effect and the network effect. Second, the exploitation-induction tradeoff is more intensive at an earlier stage of the sales season than at later stages, so the equilibrium sales prices are increasing, whereas the equilibrium promotional efforts and service levels are decreasing, over the planning horizon. Third, the competing firms need to balance the exploitation-induction tradeoff inter-temporally under the simultaneous competition, whereas they need to balance this tradeoff both inter-temporally and intra-temporally under the promotion-first competition. Finally, we show that, in the dynamic game with market size dynamics, the exploitation-induction tradeoff could be a new driving force for the "fat-cat" effect (i.e., the equilibrium promotional efforts are higher under the promotion-first competition than those under the simultaneous competition).

Trade-in remanufacturing (Chapter 4). We investigate the impact of strategic customer behavior on the economic and environmental values of the trade-in remanufacturing practice. There are several major findings. First, under trade-in remanufacturing, a firm may earn a higher profit with strategic customers than with myopic customers, which differs from the common belief that firms dislike forward-looking customer behavior due to its detrimental effect on profit. This is because strategic customers can anticipate the future price discount brought by the trade-in option, so when the revenue-generating effect of remanufacturing is strong enough, they might be willing to pay a higher first-period price than the myopic customers. Second, we show that strategic customer behavior may create a tension between profitability and sustainability: On one hand, by exploiting the forward-looking customer behavior, trade-in remanufacturing is more valuable to the firm with strategic customers than with myopic customers; on the other hand, with strategic customers, trade-in remanufacturing may have a negative impact on the environment and also on social welfare, since it may give rise to a significantly higher production quantity without improving customer surplus. Therefore, our research demonstrates that it is important to understand the interaction between trade-in remanufacturing and strategic customer behavior. Finally, to resolve the above tension, we study how a social planner (e.g., the government) should design a public policy to maximize social welfare. It has been shown that subsidizing remanufactured products alone may lead to undesired outcomes; however, the social optimum can be achieved by using a simple linear subsidy and tax scheme for all product versions.

Scarcity effect of inventory (Chapter 5). We analyze a finite horizon periodic review joint pricing and inventory management model for a firm that replenishes and sells a product under the scarcity effect of inventory. The demand distribution in each period depends negatively on the sales price and customer-accessible inventory level at the beginning of the period. The firm can withhold or dispose of its on-hand inventory to deal with the scarcity effect. We show that a customer-accessible-inventory-dependent order-upto/dispose-down-to/display-up-to list-price policy is optimal. Moreover, the optimal order-up-to/display-up-to and list-price levels are decreasing in the customer-accessible inventory level. When the scarcity effect of inventory is sufficiently strong, the firm should display no positive inventory and deliberately make every customer wait. The analysis of two important special cases wherein the firm cannot withhold (or dispose of) inventory delivers sharper insights showing that the inventory-dependent demand drives both optimal prices and order-up-to levels down. In addition, we demonstrate that an increase in the operational flexibility (e.g., a higher salvage value or the inventory withholding opportunity) mitigates the demand loss caused by high excess inventory and increases the optimal order-up-to levels and sales prices. We also generalize our model by incorporating responsive inventory reallocation after demand realizes. Finally, we perform extensive numerical studies to demonstrate that both the profit loss of ignoring the scarcity effect and the value of dynamic pricing under the scarcity effect are significant

Comparative statics analysis method (Chapter 6). We consider a general joint pricing and inventory management model, in which a firm sources from multiple supply channels to serve a market with multiple demand segments. Moreover, both the market size of each demand segment and the reference procurement cost of each supply channel are fluctuating over the planning horizon according to an exogenous Markov process. Comparative statics analysis is essential in this model, but the commonly used implicit function theorem (IFT) approach and monotone comparative statics (MCS) approach are not amenable. Hence, we develop a new comparative statics method for this model. We utilize the method to characterize the structure of the optimal policy and the impact of market fluctuation, demand segmentation, and supply diversification upon the optimal policy in each period. The new method establishes the desired comparative statics results by iteratively linking the comparisons between optimizers and those between the partial derivatives of the objective functions. The method makes componentwise comparisons between the optimizers with different parameter values, so it applies to the models where *not all* of the optimal decision variables are monotone in the parameter, and it is well scalable. The method does not require the objective function to be twice continuously differentiable or jointly supermodular. We also employ this comparative statics method to study a joint price and effort competition model.

1.3 Organization of the Dissertation

The remainder of this dissertation is organized as follows. Chapters 2 and 3 examine the impact of network externalities upon the pricing and inventory management policy in the monopoly and dynamic competition settings, respectively. In Chapter 4, we study how strategic customer behavior would influence the economic and environmental values of trade-in remanufacturing. Chapter 5 presents the analysis of the combined pricing and inventory control issue under the scarcity effect of inventory. Chapter 6 is devoted to the development of a new comparative statics analysis method for a general class of joint pricing and inventory management models. We conclude the dissertation in Chapter 7, where we also discuss potential directions for future research. All proofs are relegated to the Appendices. For Chapters 2 to 6, the notations within each chapter are self-contained, so the same notation may have different meanings in different chapters.

2. Operations Impact of Network Externalities: the Monopoly Setting

2.1 Introduction

[1]Network externalities refer to the general phenomenon that a customer's utility of purchasing a product is increasing in the number of other customers who buy the same product. See, e.g., [66]. With the fast development of information technology, network externalities have become a key driver of profitability for a high-tech firm. Take Apple as an example. Around year 2000, Apple computers were better, by all accounts, than the PCs with the Windows system. However, the vast majority of desktop and laptop computers ran Windows as their operating systems because of network externalities (see, e.g., [107]). Due to Windows' dominating role in the operating system market, software developers made only one sixth as many applications for Macintosh as they did for Windows by the time of Microsoft's antitrust trial. This, in turn, made Apple computers unattractive to new consumers, despite its functional advantages (see [65]). At the era of smartphones, however, Apple becomes the winning side of the network externalities game. Since the launch of App Store in 2008, there have been more than 1.4 million mobile apps with more than 75 billion downloads on this digital distribution platform. The App Store not only generates huge revenues (Apple takes 30% of all revenues generated through apps), but also creates large availability of apps for iPhones, thus enabling Apple to exploit network externalities to a large extent. As a consequence, iPhones have a market share of 47.4% among all smartphones in November 2014 (see, e.g., [101]).

The example of Apple clearly demonstrates the importance of network externalities upon a firm's success in the market. In particular, the online mobile software distributing platform App Store plays an important role in strengthening the network externalities of Apple products, and in boosting the sales of iPhones. As an analogous example, Xbox Live, the online multiplayer gaming network for Xbox game consoles, significantly intensifies the network externalities of Xbox consoles. This is because the value of an

[1]This chapter is based on the author's earlier work [190]

Xbox to an user increases if she has more opportunities to play games with her friends on Xbox Live (see, also, [127]). Thus, the size of the online gaming network Xbox Live is crucial to Microsoft's game console business, and the firm should manage the size of this network carefully. Being aware of this, Microsoft offered a discount of $50 for Xbox One customers who guaranteed to sign up for Xbox Live Gold membership for at least one year ([85]). This strategy helps Microsoft price discriminate in favor of the customers who would join Xbox Live. In another promotion, the 12-month Xbox Live Gold membership was discounted by 33% in February 2015 to attract Xbox customers into the online gaming network ([153]).

Firms like Apple and Microsoft naturally face the question of how to optimally coordinate the price and inventory policy of their products (iPhone and Xbox One). To address this question, we study a periodic-review single-item dynamic pricing and inventory management model under network externalities. The firm may launch an online service network associated with the product (e.g., App Store and Xbox Live). With the recent trends of online social media, the associated network can also be in the form of a social communication network (e.g., Facebook), where customers share their purchasing and consumption experiences of the product. To model network externalities, we assume that a customer's willingness-to-pay is increasing in the size of the associated network. Moreover, in each period, a fraction of the customers who make a purchase would join the network, whereas the rest directly leave the market. We call the former customers the social customers, and the latter ones the individual customers. The firm may generate revenues from the network via, e.g., service fees. This model enables us to characterize the optimal pricing and inventory policy of a profit-maximizing firm under network externalities. Our analysis highlights the impact of network externalities upon the firm's optimal price and inventory policy, and identifies effective strategies to exploit network externalities.

To the best of our knowledge, we are the first in the literature to study the dynamic pricing and inventory management problem under network externalities. We show that a network-size-dependent base-stock/list-price policy is optimal. Moreover, we make an interesting technical contribution in this chapter: The inventory dynamics of the firm would not affect its optimal policy. As a consequence, the optimal policy can be characterized by a dynamic program with a single-dimensional state space (the network size). We perform a sample path analysis of the inventory system and show that, if the firm adopts the optimal policy and the initial inventory is below the initial base-stock level, the inventory level of the firm will stay below the optimal base-stock level in each period throughout the planning horizon with probability 1. Under the base-stock/list-price policy, inventory will not affect the optimal policy if it is below the base-stock level. Therefore, although the firm carries inventory, the optimal policy does not depend on the inventory dynamics once it falls below the base-stock level of any decision period. With a simple transformation to normalize the value of current inventory, we can reduce the dynamic program that characterizes the optimal policy to one with a single-dimensional state space (the network size). This dimensionality reduction result significantly simplifies the analysis, and enables us to deliver sharper insights on the managerial implications of network externalities.

Our analysis reveals that network externalities drive the firm to balance the tradeoff between generating current profits and inducing future demands. Under network externalities, since customers have a higher willingness-to-pay with a larger network size, the optimal list-price are increasing in the current network size. The optimal expected demand and base-stock level, however, may be either increasing or decreasing in the current network size. Moreover, network externalities give rise to higher potential demand, thus driving the firm to increase the base-stock level in each period. The optimal sales price, however, may be higher or lower under network externalities, because the firm should decrease the sales price to induce higher future demands when the network size is small, and increase the sales price to exploit the better market condition when the network size is big. From the intertemporal perspective, the firm should put more weight on inducing future demands at the early stage of a sales season. Thus, when the market is stationary, the firm charges lower prices at the beginning of the planning horizon. Hence, the widely-adopted introductory price strategy (offering price discounts when starting the sales season of a product) may stem from network externalities.

We demonstrate the effectiveness of two commonly adopted strategies in the presence of network externalities: (a) price discrimination and (b) network expanding promotion. The key uniform idea of both strategies is that, the firm employs an additional leverage (price or promotion) to (partially) separate generating current profits and inducing future demands through network externalities. Under the price discrimination strategy, the firm tailors (potentially) different prices to different customer segments based on their social influences. The prices for both the social and individual customers help generate current profits, but the price for the social customers has the additional role of inducing future demands via network externalities. Therefore, it is optimal for the firm to offer discounts to social customers to induce future demands, and compensate for the reduced margin in the social segment with an increased margin in the individual segment.

Our model validates the use of (costly) network expanding promotions (e.g., offering discounts for the service fee of the associated network or investing in social media marketing strategies). When network externalities are sufficiently strong or the marginal profit of the associated network is sufficiently high, it is optimal for the firm to offer network expanding promotion, regardless of its inventory level. The optimal sales price in each period is higher with network expanding promotion than without. In other words, the firm employs network expanding promotions to induce future demands via network externalities, while charging a premium product price to generate higher current profits from selling the product.

We perform extensive numerical studies to demonstrate that (a) the profit loss of ignoring network externalities is significant, and (b) some easy-to-implement heuristic policies can effectively exploit network externalities and achieve low optimality gaps. Our numerical results show that ignoring the demand-induction effect of network externalities leads to a significant profit loss, especially when the network externalities intensity, the social customer proportion, or the network size carry-through rate is high. In this case, the firm faces a strong tradeoff between generating current profits and inducing future demands, so ignoring network externalities yields a misleading myopic policy. On the other hand, the heuristic policies that dynamically maximize the profit in a moving time window of no more than 5 periods enable the firm to leverage network externalities to a large extent, and achieve low profit losses relative to the optimal policy. Although completely ignoring network externalities gives rise to significant profit losses, the firm can effectively exploit network externalities by balancing the current profits and the *near future* demands.

The rest of this chapter is organized as follows. In Section 2.2, we position this chapter in the related literature. Section 2.3 presents the basic formulation, notations and assumptions of our model. Section 2.4 analyzes the base model. We discuss how price discrimination and network expanding promotion strategies help exploit network externalities in Section 2.5. The numerical studies are reported in Section 2.6. In Section 2.7, we conclude this chapter by summarizing our main findings. All proofs are relegated to Appendix A.1.

2.2 Related Research

This chapter is built upon two streams of research in the literature: (a) network externalities and (b) joint pricing and inventory management.

Network externalities have been extensively studied in the economics literature. In their seminal papers, [102, 103] characterize the impact of network externalities upon market competition, product compatibility, and technology adoption. [62, 67] study the network externality in financial markets. Several papers also study dynamic pricing under network externalities. For example, [61] characterize the optimal nonlinear pricing strategy for a network product with heterogenous customers. [19] consider the optimal dynamic monopoly pricing under network externalities and show that the equilibrium prices increase as time passes. [38] show that, for a monopolist, the introductory price strategy is optimal under demand information incompletion or asymmetry. [36] study the optimal pricing strategy in a network with given network structure, and characterize the relationship between optimal prices and consumers' centrality. Recently, the operations management (OM) literature starts to take into account the impact of network externalities upon a firm's operations strategy. For example, [185] propose and analyze the consumer choice models that endogenize network externalities.

The literature on the joint pricing and inventory management problem under stochastic demand is rich. [137] give a comprehensive review on the single period joint pricing and inventory control problem, and extend the results in the newsvendor problem with pricing. [70] show that a list-price/order-up-to policy is optimal for a general periodic-review joint pricing and inventory management model. When the demand distribution is unknown, [138] address the joint pricing and inventory management problem under demand learning. [47, 48, 49] analyze the joint pricing and inventory control problem with fixed ordering cost. They show that (s, S, p) policy is optimal for finite horizon, infinite horizon and continuous review models. [52] and [96], among others, study the joint pricing and inventory control problem under lost sales. In the case of a single unreliable supplier with random yield, [112] show that supply uncertainty drives the firm to charge a higher price. [88] and [43] characterize the joint dynamic pricing and dual-sourcing policy of an inventory system facing the random yield risk and the disruption risk, respectively. When the replenishment leadtime is positive, the joint pricing and inventory control problem under periodic review is extremely difficult. For this problem, [136] partially characterize the structure of the optimal policy, whereas [26] develop a simple heuristic that resolves the computational complexity. [46] characterize the optimal joint pricing and inventory control policy with positive procurement leadtime and perishable inventory. When the firm adopts supply diversification to complement its pricing strategy, [195] characterize the optimal dynamic pricing/dual-sourcing strategy, whereas [173] demonstrate how a firm should coordinate its pricing and sourcing strategies to address procurement cost fluctuation. We refer interested readers to [50] for a comprehensive survey on joint pricing and inventory control models.

This chapter contributes to the above two streams of research by incorporating network externalities into the standard joint pricing and inventory management model, studying the impact of network externalities upon a firm's pricing and inventory policy, and identifying effective strategies and heuristics to exploit network externalities.

Finally, from the modeling perspective, this chapter is related to the literature on inventory systems with positive intertemporal demand correlations (see, e.g., [100, 89, 16]). The key difference between our work and this line of research is that we endogenize the pricing decision in out model and, thus, the firm can partially control the demand process via network externalities. As a consequence, our focus is on the tradeoff between generating current profits and inducing future demands, whereas that literature focuses on the demand learning and inventory control issues with intertemporally correlated demands. The new perspective and focus of our work enable us to deliver new insights on the managerial implications of network externalities to the literature on inventory management with intertemporal demand correlations.

2.3 Model Formulation

Consider a periodic-review backlog joint pricing and inventory management model of a firm who sells a network product (e.g., a smartphone or a video game console) over a T-period planning horizon, labeled backwards as $\{T, T-1, \cdots, 1\}$. We assume that there is an online service network associated with the product (e.g., the App Store or the Xbox Live) or an online social communication network (e.g. Facebook), so that (part of) the customers who purchase the product can join the network and exhibit network externalities onto potential customers in the future. More specifically, in each period t, a continuum of infinitesimal customers arrive at the market. Each customer requests at most one product. Following [102], we assume that the willingness-to-pay of a new customer in period t is given by $V + \gamma(N_t)$, where V is the customer type uniformly distributed on the interval $(-\infty, \bar{V}_t]$ with density 1, and $\gamma(\cdot)$ is a nonnegative, concavely increasing, and twice continuously differentiable function of the network size at the beginning of period t, N_t. Hence, V is the type-V customer's intrinsic valuation of the product that is independent of network externalities, whereas $\gamma(\cdot)$ captures the network externalities of the product, i.e., the larger the associated network, the greater utilities customers gain to purchase the product. We call $\gamma(\cdot)$ the network externalities function hereafter. For technical tractability, we assume that the customers are bounded rational so that they base their purchasing decisions on the current sales price and network size, instead of rational expectations on future prices and network sizes. Therefore, a type-V customer would make a purchase in period t if and only if $V + \gamma(N_t) \geq p_t$, where $p_t \in [\underline{p}, \bar{p}]$ is the product price the firm charges in period t. In each period t, there exists a random additive demand shock, ξ_t, which captures other uncertainties not explicitly modeled (e.g., the macro-economic condition of period t). Hence, the actual demand in period t is given by:

$$D_t(p_t, N_t) := \bar{V}_t + \gamma(N_t) - p_t + \xi_t,$$

where ξ_t is independent of the price p_t and the network size N_t with $\mathbb{E}[\xi_t] = 0$. Moreover, $\{\xi_t : t = T, T-1, \cdots, 1\}$ are *i.i.d.* continuously distributed random variables. Without loss of generality, we assume that $D_t(p_t, N_t) \geq 0$ with probability 1, for all $p_t \in [\underline{p}, \bar{p}]$ and $N_t \geq 0$.

We now introduce the dynamics of the network sizes $\{N_t : t = T, T-1, \cdots 1\}$. Given the current network size N_t, the network size of the next period, N_{t-1}, is determined by two effects. First, some customers may leave the network. For example, a game player may lose its enthusiasm in online gaming three years after purchasing the Xbox console. Analogously, an iPhone user may switch to Samsung for her next smartphone. Thus, given N_t, let ηN_t be the remaining number of customers staying in the network in period

$t-1$, where $\eta \in [0,1]$ is the carry-through rate of the network size. Second, a fraction of new customers who purchase the product in period t would join the network. Not all new customers will join the network and exhibit positive externalities onto potential customers in the future, because, e.g., some Xbox players only play the games off-line and, thus, are not part of the Xbox Live network. Clearly, these players exert few, if any, network externalities onto other customers. For any given (p_t, N_t), let $\theta D_t(p_t, N_t)$ be the number of new customers who opt to join the network associated with the product, which we call the *social customers* hereafter, where $\theta \in (0,1]$ is the proportion of such customers in the market. The other $(1-\theta)D_t(p_t, N_t)$ customers who exert no network externalities are called *individual customers* hereafter. Although we implicitly assume that the utility functions of the social and individual customers are identical, most of the results in this chapter (except Theorem 2.5.1) continue to hold if $\bar{V}_t$ and $\gamma(\cdot)$ are different for the social and individual customers. To capture the market size dynamics, we notice that, due to demand uncertainty and limited inventory availability, not all customers request a product can get one in the current period. We assume that the social customers who purchase but not get the product still join the network. It is commonly observed in practice that customers exert network externalities upon future potential buyers before receiving the product. For example, before obtaining the pre-ordered product, a customer may comment on her excitement in waiting for and expecting the product on Facebook, thus exerting network externalities upon potential buyers. Moreover, by Theorem 2.4.1(c) below, if the firm adopts the optimal pricing and inventory policy, all backlogged demand will be fulfilled in the next period, so the backlogged social customers will get the product and join the network shortly. For simplicity, we ignore the differences in the timing of joining the network between the customers who get the product upon request and those who are backlogged to the next period. Therefore, given N_t, the network size at the beginning of period $t-1$ is given by:

$$N_{t-1} = \eta N_t + \theta D_t(p_t, N_t) + \epsilon_t, \tag{2.1}$$

where ϵ_t is the additive random shock in the network size dynamics not explicitly captured in our model. We assume that ϵ_t is independent of the price p_t and the network size N_t with $\mathbb{E}[\epsilon_t] = 0$. Moreover, $\{\epsilon_t : t = T, T-1, \cdots, 1\}$ are *i.i.d.* continuously distributed random variables.

If the associated network is a service network, the firm can generate profits via this network by charging service/subscription fees. For example, Microsoft charges an annual subscription fee of $59.99 for the Xbox Live Gold membership, whereas Apple takes 30% of all revenues generated through apps in the App Store. For any network size $N \geq 0$, let $r_n(N) \geq 0$ denote the per-period profit the firm earns from the network. Without loss of generality, we assume that $r_n(\cdot)$ is a concavely increasing and continuously differentiable function with $r_n(0) = 0$. To focus on the firm's pricing and inventory policy of its *product*, we do not explicitly model the firm's price decision of its network service. Hence, the per-period profit function of the network, $r_n(\cdot)$, is assumed to be exogenously given. Without loss of generality, we assume that the service fees are paid at the end of each period. Hence, the total expected profit the firm obtains from the associated network in period t is given by: $\mathbb{E}[r_n(\eta N_t + \theta D_t(p_t, N_t) + \epsilon_t)]$. If the associated network is a social communication network where the social customers share their purchasing and consumption experiences, the firm obtains no profit from this network, i.e., $r_n(\cdot) \equiv 0$.

The state of the inventory system is given by $(I_t, N_t) \in \mathbb{R} \times \mathbb{R}_+$, where

$$I_t = \text{the starting inventory level before replenishment in period } t,\ t = T, T-1, \cdots, 1;$$
$$N_t = \text{the starting network size of the product in period } t,\ t = T, T-1, \cdots, 1.$$

The decisions of the firm is given by $(x_t, p_t) \in \mathcal{F}(I_t) := [I_t, +\infty) \times [\underline{p}, \bar{p}]$, where

$$x_t = \text{the inventory level after replenishment in period } t,\ t = T, T-1, \cdots, 1;$$
$$p_t = \text{the sales price charged in period } t,\ t = T, T-1, \cdots, 1.$$

In each period, the sequence of events unfolds as follows: At the beginning of period t, after observing the inventory level I_t and the network size N_t, the firm simultaneously chooses the inventory stocking level $x_t \geq I_t$ and the sales price p_t, and pays the ordering cost $c(x_t - I_t)$. The inventory procurement leadtime is assumed to be zero, so that the replenished inventory is received immediately. The demand $D_t(p_t, N_t)$ then realizes. The revenue from selling the product, $p_t\mathbb{E}[D_t(p_t, N_t)]$, and the profit from the associated network, $\mathbb{E}[r_n(\eta N_t + \theta D_t(p_t, N_t) + \epsilon_t)]$, are collected. Unmet demand is fully backlogged. At the end of period t, the holding and backlogging costs are paid, the net inventory is carried over to the next period, and the network size is updated according to the network size dynamics (2.1).

We introduce the following model primitives:

$$\alpha = \text{discount factor of revenues and costs in future periods, } 0 < \alpha \leq 1;$$
$$c = \text{inventory purchasing cost per unit ordered;}$$
$$b = \text{backlogging cost per unit backlogged at the end of a period;}$$
$$h = \text{holding cost per unit stocked at the end of a period.}$$

Without loss of generality, we make the following assumptions on the model primitives:

$b > (1 - \alpha)c$: the backlogging penalty is higher than the saving from delaying an order to the next period, so that the firm will not backlog all of its demand;

$\underline{p} > b + \alpha c$: positive margin for backlogged demand.

The above assumptions are common in the joint pricing and inventory management literature (see, e.g., [189]).

For technical tractability, we make the following assumption throughout our analysis.

Assumption 2.3.1 *For each period t, $R_t(\cdot, \cdot)$ is jointly concave in $(p_t, N_t) \in [\underline{p}, \bar{p}] \times [0, +\infty)$, where*

$$R_t(p_t, N_t) := (p_t - b - \alpha c)(\bar{V}_t - p_t + \gamma(N_t)). \tag{2.2}$$

Given the sales price, p_t, and the network size, N_t, of period t, $R_t(p_t, N_t)$ is the expected difference between the revenue and the total cost, which consists of ordering and backlogging costs, to satisfy the current demand in the next period. Hence, the joint concavity of $R_t(\cdot, \cdot)$ implies that such difference has decreasing marginal values with respect to the current sales price and network size. We remark that $R_t(\cdot, N_t)$ is strictly concave in p_t for any given N_t. Moreover, the monotonicity of $\gamma(\cdot)$ suggests that $R_t(\cdot, \cdot)$ is supermodular in (p_t, N_t). The following lemma gives the necessary and sufficient condition for Assumption 2.3.1.

Lemma 1 *Assumption 2.3.1 holds for period t, if and only if, for all $N_t \geq 0$,*

$$-2(\underline{p} - \alpha c - b)\gamma''(N_t) \geq (\gamma'(N_t))^2. \tag{2.3}$$

Based on Lemma 1, we give more specific conditions on the network externalities function $\gamma(\cdot)$ for Assumption 2.3.1 to hold in Appendix A.2. In a nutshell, Assumption 2.3.1 holds when (a) the curvature of the network externalities function $\gamma(\cdot)$ is not too small in the region network externalities exist (i.e., $\gamma'(\cdot) > 0$), and (b) the price elasticity of demand (i.e., $|(\mathrm{d}\mathbb{E}[D_t(p_t, N_t)]/\mathbb{E}[D_t(p_t, N_t)])/(\mathrm{d}p_t/p_t)|$) is sufficiently big relative to the network size elasticity of demand (i.e., $|(\mathrm{d}\mathbb{E}[D_t(p_t, N_t)]/\mathbb{E}[D_t(p_t, N_t)])/(\mathrm{d}N_t/N_t)|$).

2.4 Analysis of the Base Model

In this section, we analyze the base model suitable for the usual sales season of the network product, when the firm charges a single regular price for all customers without any promotional campaigns. In Section 2.5, we introduce price discrimination and network expanding promotion strategies, and analyze their effectiveness in leveraging network externalities.

We first characterize the structure of the optimal pricing and inventory policy in our model. Then, we show that the state space dimension of the dynamic program for the joint pricing and inventory replenishment problem can be reduced to 1. Finally, we study the managerial implications of network externalities.

2.4.1 Optimal Policy

We now formulate the planning problem as a dynamic program. Define

$v_t(I_t, N_t) :=$ the maximum expected discounted profits in periods $t, t-1, \cdots, 1$, when starting period t with an inventory level I_t and network size N_t.

Without loss of generality, we assume that, in the last period (period 1), the excess inventory is salvaged with unit value c, and the backlogged demand is filled with ordering cost c, i.e., $v_0(I_0, N_0) = cI_0$ for any (I_0, N_0). The optimal value function $v_t(I_t, N_t)$ satisfies the following recursive scheme:

$$v_t(I_t, N_t) = cI_t + \max_{(x_t, p_t) \in \mathcal{F}(I_t)} J_t(x_t, p_t, N_t), \tag{2.4}$$

where $\mathcal{F}(I_t) := [I_t, +\infty) \times [\underline{p}, \bar{p}]$ denotes the set of feasible decisions and,

$$
\begin{aligned}
J_t(x_t, p_t, N_t) &= -cI_t + \mathbb{E}\{p_t D_t(p_t, N_t) - c(x_t - I_t) - h(x_t - D_t(p_t, N_t))^+ \\
&\quad -b(x_t - D_t(p_t, N_t))^- + r_n(\theta D_t(p_t, N_t) + \eta N_t + \epsilon_t) \\
&\quad +\alpha v_{t-1}(x_t - D_t(p_t, N_t), \theta D_t(p_t, N_t) + \eta N_t + \epsilon_t)|N_t\}, \\
&= (p_t - \alpha c - b)(\bar{V}_t - p_t + \gamma(N_t)) + (b - (1-\alpha)c)x_t \\
&\quad +\mathbb{E}\{r_n(\theta(\bar{V}_t - p_t + \gamma(N_t) + \xi_t) + \eta N_t + \epsilon_t) \\
&\quad -(h+b)(x_t - \bar{V}_t + p_t - \gamma(N_t) - \xi_t)^+ \\
&\quad +\alpha[v_{t-1}(x_t - \bar{V}_t + p_t - \gamma(N_t) - \xi_t, \theta(\bar{V}_t - p_t + \gamma(N_t) + \xi_t) + \eta N_t + \epsilon_t) \\
&\quad -c(x_t - \bar{V}_t + p_t - \gamma(N_t) - \xi_t)]|N_t\} \\
&= R_t(p_t, N_t) + \beta x_t + \Lambda(x_t - \bar{V}_t + p_t - \gamma(N_t)) \\
&\quad +\Psi_t(x_t - \bar{V}_t + p_t - \gamma(N_t), \theta(\bar{V}_t - p_t + \gamma(N_t)) + \eta N_t), \qquad (2.5)
\end{aligned}
$$

$$
\begin{aligned}
\text{with } \Psi_t(x, y) &:= \mathbb{E}\{r_n(y + \theta\xi_t + \epsilon_t) + \alpha[v_{t-1}(x - \xi_t, y + \theta\xi_t + \epsilon_t) - cx]\}, \\
\Lambda(x) &:= \mathbb{E}\{-(b+h)(x - \xi_t)^+\}, \\
\beta &:= b - (1-\alpha)c = \text{the monetary benefit of ordering one unit of inventory.}
\end{aligned}
$$

Hence, for each period t, the firm selects

$$
(x_t^*(I_t, N_t), p_t^*(I_t, N_t)) := \text{argmax}_{(x_t, p_t) \in \mathcal{F}(I_t)} J_t(x_t, p_t, N_t) \qquad (2.6)
$$

as the optimal price and inventory policy contingent on the state variable (I_t, N_t).

We begin our analysis by characterizing the preliminary concavity and differentiability properties of the value and objective functions in the following lemma.

Lemma 2 *For each $t = T, T-1, \cdots, 1$, the following statements hold:*

(a) $\Psi_t(\cdot, \cdot)$ is jointly concave and continuously differentiable in (x, y). Moreover, $\Psi_t(x, y)$ is decreasing in x and increasing in y.

(b) $J_t(\cdot, \cdot, \cdot)$ is jointly concave and continuously differentiable in (x_t, p_t, N_t).

(c) $v_t(\cdot, \cdot)$ is jointly concave and continuously differentiable in (I_t, N_t). Moreover, $v_t(I_t, N_t)$ is increasing in N_t, and $v_t(I_t, N_t) - cI_t$ is decreasing in I_t.

Lemma 2 proves that, in each period t, the objective function is concave and continuously differentiable, and the value function is jointly concave and continuously differentiable. Moreover, the normalized value function $v_t(I_t, N_t) - cI_t$ is decreasing in the inventory level I_t and increasing in the network size N_t. Lemma 2 is a standard result in the joint pricing and inventory management literature (see, e.g., Theorem 1 in [70]). The concavity and continuous differentiability of $J_t(\cdot,\cdot,\cdot)$ ensure that, the optimal price and inventory policy, $(x_t^*(I_t, N_t), p_t^*(I_t, N_t))$, is well-defined and can be obtained via first-order conditions. Moreover, we can define the inventory-independent optimizer $(x_t(N_t), p_t(N_t))$ as follows:

$$(x_t(N_t), p_t(N_t)) := \text{argmax}_{x_t\in\mathbb{R}, p_t\in[\underline{p},\bar{p}]} J_t(x_t, p_t, N_t). \tag{2.7}$$

In case of multiple optimizers, we select the lexicographically smallest one. We define $y_t(N_t) := \bar{V}_t - p_t(N_t) + \gamma(N_t)$ as the optimal inventory-independent expected demand of period t. With Lemma 2, we characterize the optimal pricing and inventory policy in the following theorem.

Theorem 2.4.1 *For any t, the following statements hold:*

(a) If $I_t \leq x_t(N_t)$, $(x_t^*(I_t, N_t), p_t^*(I_t, N_t)) = (x_t(N_t), p_t(N_t))$.

(b) If $I_t > x_t(N_t)$, $x_t^*(I_t, N_t) = I_t$ and $p_t^*(I_t, N_t) = \text{argmax}_{p_t\in[\underline{p},\bar{p}]} J_t(I_t, p_t, N_t)$.

(c) For any $I_t \in \mathbb{R}$ and $N_t \geq 0$, $x_t^*(I_t, N_t) > 0$.

Theorem 2.4.1 shows that the optimal policy in the base model is a network-size-dependent base-stock/list-price policy. If the starting inventory level I_t is below the network-size-dependent base-stock level $x_t(N_t)$, it is optimal to order up to this base-stock level, and charge the network-size-dependent list-price $p_t(N_t)$. If the starting inventory level is above the network-size-dependent base-stock level, it is optimal not to order anything, and charge an inventory-dependent sales price $p_t^*(I_t, N_t)$. Moreover, as shown in Theorem 2.4.1(c), the optimal order-up-to level $x_t^*(I_t, N_t)$ is always positive for any inventory level I_t and network size N_t. This implies that, under the optimal policy, all backlogged demand in any period t will be satisfied in the next period (i.e., period $t-1$).

2.4.2 State Space Dimension Reduction

The original dynamic program to characterize the optimal pricing and inventory policy (2.4) has a state space of two dimensions (inventory level I_t and network size N_t). Hence, it is difficult to work with (2.4) directly. In this subsection, we demonstrate that the dynamic program (2.4) can be reduced to a much simpler one with a single-dimensional state space (network size N_t). Moreover, with probability 1, the optimal policy in each period t, $(x_t^*(I_t, N_t), p_t^*(I_t, N_t))$, is independent of the dynamics of inventory level $\{I_s : s = T, T-1, \cdots, t\}$, as long as the initial inventory level I_T is below the optimal period-T base-stock level $x_T(N_T)$. The state space dimension reduction, as we will show in Section 2.4.3 and Section 2.5, enables us to deliver sharper insights on the managerial implications of network externalities and the effective strategies to exploit network externalities.

To begin with, we employ the sample path analysis approach to characterize the behavior of the inventory level dynamics under the optimal pricing and inventory policy.

Lemma 3 *For each period t, the following statements hold:*

(a) For all network sizes N_t and N_{t-1}, we have

$$\mathbb{P}[x_t(N_t) - D_t(p_t(N_t), N_t) \le x_{t-1}(N_{t-1})] = 1.$$

(b) For all $N_t \ge 0$, $x_t(N_t) = \Delta^* + y_t(N_t)$, where $\Delta^* := \text{argmax}_\Delta\{\beta\Delta + \Lambda(\Delta)\}$.

Lemma 3(a) shows that, if the firm adopts the optimal policy and the starting inventory level in period t, I_t, is below the period-t base-stock level $x_t(N_t)$, the starting inventory level in period $t-1$, $I_{t-1} = x_t(N_t) - D_t(p_t(N_t), N_t)$, is below the period-$(t-1)$ base-stock level, $x_{t-1}(N_{t-1})$, with probability 1. Lemma 3(a) also implies that once the starting inventory level falls below the optimal base-stock level of one period, the firm should replenish in each period thereafter throughout the planning horizon with probability 1. Since our model best fits the network product that is either a new product (e.g., the first-generation iPhone) or a new generation of an existing product (e.g., Xbox One), zero inventory is stocked at the beginning of the sales season, i.e., $I_T = 0$. Therefore, Theorem 2.4.1(c) and Lemma 3(a) imply that $I_t \le x_t(N_t)$ with probability 1 for each period t. As a corollary of Lemma 3(a), Lemma 3(b) shows that, if the starting inventory is below the optimal base-stock level (i.e., $I_t \le x_t(N_t)$), the optimal safety-stock Δ^* is invariant with respect to the period t and the network size N_t, and can be obtained by solving a one-dimensional convex optimization.

Based on Lemma 3, we now show that the bivariate value functions of the dynamic program (2.4), $\{v_t(\cdot,\cdot) : t = T, T-1, \cdots, 1\}$, can be transformed into a univariate function $\pi_t(\cdot)$ of the current network size N_t by normalizing the value of the starting inventory cI_t.

Moreover, the normalized value function $\pi_t(\cdot)$ is concavely increasing and continuously differentiable in N_t.

Lemma 4 *There exists a sequence of functions $\{\pi_t(\cdot) : 1 \leq t \leq T\}$, such that, (i) $\pi_t(N_t) = \max\{J_t(x_t, p_t, N_t) : x_t \geq 0, , p_t \in [\underline{p}, \bar{p}]\}$ for all $N_t \geq 0$; (ii) for each t, $\pi_t(\cdot)$ is concavely increasing and continuously differentiable in N_t; (iii) $v_t(I_t, N_t) = cI_t + \pi_t(N_t)$ for all $N_t \geq 0$ and $I_t \leq x_t(N_t)$; (iv) for all $N_t \geq 0$,*

$$J_t(x_t, p_t, N_t) = R_t(p_t, N_t) + \beta x_t + \Lambda(x_t - \bar{V}_t + p_t - \gamma(N_t)) + G_t(\theta(\bar{V}_t - p_t + \gamma(N_t)) + \eta N_t), \quad (2.8)$$

where $G_t(y) := \mathbb{E}[r_n(y + \theta\xi_t + \epsilon_t) + \alpha\pi_{t-1}(y + \theta\xi_t + \epsilon_t)]$ and $x_t - \bar{V}_t + p_t - \gamma(N_t) \leq \Delta^*$; and (v) $(x_t(N_t), p_t(N_t))$ maximizes the right-hand side of equation (2.8).

Lemma 4 enables us to reduce the original dynamic program (2.4), which has a two-dimension state-space, to one with a single-dimension state space. More specifically, Lemma 4 implies that the optimal network-size-dependent base-stock level and list-price in each period t, $(x_t(N_t), p_t(N_t))$, can be recursively determined by solving the following dynamic program with a single dimensional state-space of network size N_t:

$$\begin{aligned}
\pi_t(N_t) &= \max_{x_t \geq 0, p_t \in [\underline{p}, \bar{p}]} J_t(x_t, p_t, N_t), \qquad (2.9)\\
\text{where } J_t(x_t, p_t, N_t) &= R_t(p_t, N_t) + \beta x_t + \Lambda(x_t - \bar{V}_t + p_t - \gamma(N_t))\\
&\quad + G_t(\theta(\bar{V}_t - p_t + \gamma(N_t)) + \eta N_t),\\
\text{with } G_t(y) &:= \mathbb{E}\{r_n(y + \theta\xi_t + \epsilon_t) + \alpha\pi_{t-1}(y + \theta\xi_t + \epsilon_t)\}, \text{ and } \pi_0(\cdot) \equiv 0,
\end{aligned}$$

Summarizing Theorem 2.4.1, Lemma 3, and Lemma 4, we have the following sharper characterization of the optimal policy in each period.

Theorem 2.4.2 *Assume that $I_T \leq x_T(N_T)$. In each period t and for each I_t and N_t, $(x_t^*(I_t, N_t), p_t^*(I_t, N_t)) = (x_t(N_t), p_t(N_t))$ with probability 1. Moreover, $\{(x_t(N_t), p_t(N_t)) : t = T, T-1, \cdots, 1\}$ is the solution to the Bellman equation* (2.9).

Theorem 2.4.2 shows that, as long as the planning horizon starts with an inventory level below the optimal period-T base-stock level (i.e., $I_T \leq x_T(N_T)$), the optimal pricing and inventory policy in each period t, $(x_t^*(I_t, N_t), p_t^*(I_t, N_t))$, is identical to the optimal base-stock level and list-price, $(x_t(N_t), p_t(N_t))$, with probability 1. Although the firm holds inventory throughout the sales horizon, the optimal policy is independent of the inventory dynamics if the initial inventory level I_T is sufficiently low. As discussed above, in most applications, the firm holds zero initial inventory at the beginning of the sales season, i.e., $I_T = 0$. Hence, $(x_t^*(I_t, N_t), p_t^*(I_t, N_t)) = (x_t(N_t), p_t(N_t))$ for all (I_t, N_t) with probability 1. Therefore, we will focus on analyzing the properties of the optimal inventory-independent base-stock level and list-price $(x_t(N_t), p_t(N_t))$ for the rest of this section.

2.4.3 Managerial Implications of Network Externalities

This subsection studies the impact of network externalities upon the firm's optimal price and inventory decisions in each period. Specifically, we strive to answer the following questions: (a) How should the firm adjust its price and inventory policy in response to the network size evolution? (b) How do network externalities directly impact the optimal policy of the firm? (c) How should the firm adjust its price and inventory policy intertemporally throughout the sales season? And (d) how to balance earning profits directly from selling the product and from the service fees of the network? The answers to these questions shed lights on the managerial implications of network externalities.

To begin with, we characterize the impact of network size upon the firm's optimal pricing and inventory policy in the following theorem.

Theorem 2.4.3 *For each period t, assume that $\hat{N}_t > N_t$. We have: (a) $p_t(\hat{N}_t) \geq p_t(N_t)$; (b) if $I_t \leq x_t(N_t)$, $\mathbb{E}[N_{t-1}|\hat{N}_t] \geq \mathbb{E}[N_{t-1}|N_t]$; (c) if $\gamma(\hat{N}_t) = \gamma(N_t)$, then $y_t(\hat{N}_t) \leq y_t(N_t)$ and $x_t(\hat{N}_t) \leq x_t(N_t)$; and (d) if $\eta = 0$, then $y_t(\hat{N}_t) \geq y_t(N_t)$ and $x_t(\hat{N}_t) \geq x_t(N_t)$.*

Theorem 2.4.3 characterizes how the current network size influences the optimal joint pricing and inventory policy, the optimal expected current-period demand, and the optimal expected next-period network size. More specifically, we show that the optimal list-price, $p_t(N_t)$, and the optimal expected network size in the next period, $\mathbb{E}[N_{t-1}|N_t] = \theta y_t(N_t) + \eta N_t$, are increasing in the current network size N_t. The optimal expected demand $y_t(N_t)$, and the optimal base-stock level $x_t(N_t)$, however, may not necessarily be increasing or decreasing in N_t (see Theorem 2.4.3(c,d)). Under network externalities, a larger current network size N_t gives rise to a higher potential demand, so the firm charges a higher price to exploit the better market condition. Hence, with a larger current network size, the combination of a better market condition and an increased sales price may drive the resulting optimal expected demand and the optimal base-stock level either higher or lower.

In the joint pricing and inventory management model without network externalities (e.g., [70]), the optimal policy in each period is independent of either past demands or past decisions. Since the current network size N_t is positively correlated with past demands, Theorem 2.4.3 implies that network externalities create intertemporal correlations between demands and optimal decisions throughout the planning horizon. Hence, the firm can employ the current price and inventory decisions to control future demands. Therefore, the firm needs to dynamically balance the tradeoff between generating current profits and inducing future demands through network externalities.

Theorems 2.4.1-2.4.3 are silent on the properties of the optimal policy when the starting inventory exceeds the optimal base-stock level (i.e., $I_t > x_t(N_t)$). Though this scenario occurs with probability 0 as long as $I_T \leq x_T(N_T)$ (see Theorem 2.4.2), we give the following theorem that characterizes the structure of the optimal policy therein.

Theorem 2.4.4 *Assume that $\eta = 0$. For each period t, the following statements hold,*

(a) $v_t(I_t, N_t)$ is supermodular in (I_t, N_t).

(b) $x_t^*(I_t, N_t)$ is continuously increasing in I_t and N_t.

(c) $p_t^*(I_t, N_t)$ is continuously decreasing in I_t, and continuously increasing in N_t.

(d) The optimal expected demand $y_t^*(I_t, N_t) := \bar{V}_t - p_t^*(I_t, N_t) + \gamma(N_t)$ is continuously increasing in I_t and N_t. Hence, $\mathbb{E}[N_{t-1}|N_t] = \theta y_t^*(I_t, N_t)$ is continuously increasing in I_t and N_t.

(e) The optimal safety-stock $\Delta_t^*(I_t, N_t) := x_t^*(I_t, N_t) - \bar{V}_t + p_t^*(I_t, N_t) - \gamma(N_t)$ is continuously increasing in I_t and continuously decreasing in N_t.

Theorem 2.4.4 generalizes Theorems 2.4.1 and 2.4.3 to the setting with high starting inventory (i.e., $I_t > x_t(N_t)$). More specifically, Theorem 2.4.4(a) shows that if $\eta = 0$ (i.e., all customers who are in the network will leave in the next period), the value function in each period t, $v_t(I_t, N_t)$ is supermodular in (I_t, N_t). This is because, a larger network size leads to a larger potential demand and, thus, a higher marginal value of inventory.

Analogously, the optimal expected demand, $y_t^*(I_t, N_t)$, and the optimal expected network size in the next period are all increasing in the network size N_t for all (I_t, N_t). As a consequence, if the network size is larger, the firm increases the order-up-to level, $x_t^*(I_t, N_t)$, to match demand with supply, and charges a higher sales price, $p_t^*(I_t, N_t)$, to exploit the better market condition. Since the expected demand is higher with a larger network size, the optimal safety-stock $\Delta_t^*(I_t, N_t)$ is decreasing in N_t. Theorem 2.4.4 also yields how the starting inventory level I_t influences the optimal policy when it is above the base-stock level. We show that, in this case, a higher starting inventory level prompts the firm to increase the safety stock and, to match supply with demand, charge a lower sales price.

Theorem 2.4.3 shows that network externalities impact the optimal joint pricing and inventory policy of the firm through the current size of the associated network. We proceed to directly analyze the impact of network externalities by comparing the optimal policy in an inventory system with network externalities with that in an inventory system without.

Theorem 2.4.5 *Assume that two inventory systems are identical except that one with network externalities function $\gamma(\cdot)$ and the other with $\hat{\gamma}(\cdot)$, where $\gamma(0) = \hat{\gamma}(0) = \gamma_0$ and $\hat{\gamma}(N_t) \geq \gamma(N_t) \equiv \gamma_0$ for all $N_t \geq 0$, i.e., the inventory system with function $\gamma(\cdot)$ exhibits no network externalities. Moreover, let $\hat{r}_n(n) = r_n(n) = rn$ for some constant $r \geq 0$. For each period t and each network size $N_t \geq 0$, the following statements hold: (a) $\hat{y}_t(N_t) \geq y_t(N_t)$; (b) $\hat{x}_t(N_t) \geq x_t(N_t)$; (c) There exists a threshold $\mathfrak{N}_t \geq 0$, such that $\hat{p}_t(N_t) \leq p_t(N_t)$ for $N_t \leq \mathfrak{N}_t$, whereas $\hat{p}_t(N_t) \geq p_t(N_t)$ for $N_t \geq \mathfrak{N}_t$.*

Network externalities lead to a higher potential demand for the inventory system, because social customers in the network can attract new potential customers to buy the product. Hence, as shown in Theorem 2.4.5(a,b), the presence of network externalities gives rise to a higher expected demand and, thus, drives the firm to increase the base-stock level in each period t (i.e., $\hat{y}_t(N_t) \geq y_t(N_t)$ and $\hat{x}_t(N_t) \geq x_t(N_t)$). Theorem 2.4.5(c) characterizes the impact of network externalities upon the firm's pricing policy: The optimal list-price with network externalities, $\hat{p}_t(N_t)$, may be either higher or lower than that without, $p_t(N_t)$. More specifically, if the network size is sufficiently small (i.e., below the threshold $\mathfrak{N}_t$), $\hat{p}_t(N_t) \leq p_t(N_t)$. Otherwise, the network size is sufficiently large (i.e., above the threshold $\mathfrak{N}_t$) and $\hat{p}_t(N_t) \geq p_t(N_t)$. Under network externalities, the firm faces the tradeoff between decreasing the sales price to induce high future demands and increasing the sales price to exploit the better market condition. When the current network size is small ($N_t \leq \mathfrak{N}_t$), the firm should put higher weight on inducing future demands, so the optimal price is lower with network externalities. Otherwise, $N_t \geq \mathfrak{N}_t$, generating current profits outweighs inducing future demands, and, hence, the optimal price is higher with network externalities. In short, Theorem 2.4.5(c) reveals that, because of the tradeoff between generating current profits and inducing future demands, network externalities can have some subtle implications on the pricing policy of the firm.

We now characterize the evolution of the optimal price and inventory decisions over the planning horizon. As shown in the following theorem, when the market is stationary, network externalities motivate the firm to set lower sales prices and higher base-stock levels at the beginning of the planning horizon.

Theorem 2.4.6 *Assume that $\bar{V}_t = \bar{V}_{t-1}$ for all t. For each $t = T, T-1, \cdots, 2$ and any network size $N \geq 0$, we have (a) $x_t(N) \geq x_{t-1}(N)$, (b) $y_t(N) \geq y_{t-1}(N)$, and (c) $p_t(N) \leq p_{t-1}(N)$.*

When the willingness-to-pay of the customers is stationary, Theorem 2.4.6 characterizes the evolutions of the optimal base-stock level, expected demand, and sales price under network externalities. More specifically, we show that, with the same network size N_t (and, thus, the same potential market size), the optimal expected demand, $y_t(N_t)$, and the optimal base-stock level, $x_t(N_t)$, is decreasing over the planning horizon, whereas the optimal sales price, $p_t(N_t)$, is increasing throughout the planning horizon. Under network externalities, the firm should put more weight on inducing future demands at the beginning of the planning horizon and turn to generating the current profits as the sales season approaches the end. Hence, it is optimal for the firm to offer discounts and attract more customers to purchase the product and join the network at the early stage of a sales season, and to charge a higher price to exploit the current market towards the end of the planning horizon. To match demand with supply, with the same potential market size, the optimal base-stock level is decreasing over the planning horizon. Theorem 2.4.6 is consistent with the commonly used introductory price strategy under which price discounts are offered at the introductory stage of a product. For example, when

Microsoft introduced the 500 GB Xbox 360 into the India video game market, it charged a surprisingly low introductory price of \$313.9 (see, e.g., [108]). When the customer valuation is not stationary (i.e., $\bar{V}_t$ is not equal to $\bar{V}_{t-1}$), the introductory price strategy may not necessarily be optimal. This is because, if the customer valuation is higher at the beginning of the sales season, the firm may charge a higher price to exploit the customer preference as opposed to offering discounts to induce future demands.

In our joint pricing and inventory management model with network externalities, the firm has two sources of profits: (i) selling the product, and (ii) the service fees collected from the associated network. A natural question to ask is how should the firm balance these two profit-generating sources? The following theorem addresses this question by characterizing how the marginal profit from the associated network influence the optimal policy.

Theorem 2.4.7 *Assume that two inventory systems are identical except that one with network profit function $\hat{r}_n(\cdot)$, and the other with $r_n(\cdot)$, where $\hat{r}'_n(N) \geq r'_n(N)$ for all $N \geq 0$. For each period t and any $N_t \geq 0$, we have: (a) $\hat{x}_t(N_t) \geq x_t(N_t)$, (b) $\hat{p}_t(N_t) \leq p_t(N_t)$, and (c) $\hat{y}_t(N_t) \geq y_t(N_t)$.*

Theorem 2.4.7 sheds lights on how different firms should balance the two profit sources. More specifically, Theorem 2.4.7 shows that if the associated network has a higher profit margin (i.e., $r'_n(\cdot)$ is larger), the network externalities of the product are stronger and, as a consequence, the firm should price down and increase the potential demand to exploit the more intensive network externalities. To match demand with supply, the firm also increases the base-stock level with a higher profit margin of the associated network. Theorem 2.4.7 implies that for a product with high intrinsic customer valuations and a low margin of the associated network (e.g., iPhone), the firm charges a premium price for the product; whereas if the product has low intrinsic valuations from the customers due to, e.g., fierce market competition, but the margin of the associated network is high (e.g., Xbox), the firm charges a price with a low margin for the product so as to exploit network externalities.

In summary, network externalities have several important managerial implications upon the joint pricing and inventory policy of the firm. Most importantly, network externalities create another layer of complexity in balancing the tradeoff between generating current profit and inducing future demands. To exploit network externalities, the firm should dynamically adjust its price increasing in the current network size. Moreover, network externalities give rise to higher expected demand and, hence, drive the firm to increase the base-stock level in each period. Since network externalities create the tension between generating current profits and inducing future demands, the optimal sales price with network externalities is lower than that without when the network size is small (to induce high future demands), and is higher than that without when the network size is big (to generate high current profits). From the intertemporal perspective, the firm should put more weight on inducing future demands at the early stage of a sales season than at later stages. Thus, if the customer valuation is stationary, the firm should employ the introductory price strategy that offers early purchase discounts to induce high future demands. Finally, the firm needs to trade off between generating profits from the product and from the associated network as well. With higher marginal profits of the associated network, the firm should decrease the sales price to exploit the more intensive network externalities.

2.5 Effective Strategies to Exploit Network Externalities

In this section, we study two effective strategies to exploit network externalities: (a) the price discrimination strategy and (b) the network expanding promotion strategy. Both strategies adopt the uniform idea that, the firm employs an additional leverage (price or promotion) to separate generating current profits and inducing future demands through network externalities.

2.5.1 Price Discrimination

In this subsection, we study the price discrimination strategy that is commonly used in practice under network externalities. More specifically, since only social customers will join the associated network of the product and exert network externalities over potential buyers in the future, the firm can better exploit network externalities by price discriminating different customer segments in favor of social customers. For example, in 2015, Microsoft offered price discounts for Xbox One buyers who commit to signing up for the Xbox Live Gold membership for at least one year (see, e.g., [85]).

In period t, in stead of announcing a single price p_t, the firm under the price discrimination strategy offers a price menu to customers: $(p_t^s, p_t^i) \in [\underline{p}, \bar{p}] \times [\underline{p}, \bar{p}]$, where p_t^s is the unit price of the product *with* the network sign-up commitment, and p_t^i is the unit price of the product *without* any network service subscription commitment. If $p_t^s > p_t^i$, all customers will take the price p_t^i and the model is reduced to the based model studied in Section 2.4. Hence, without loss of generality, we assume that $p_t^s \le p_t^i$. In this case, social customers will take the price p_t^s and, as committed, join the associated network, whereas individual customers will take the price p_t^i without joining the associated network. Thus, in period t, the demand from the social customers is given by $D_t^s(p_t^s, N_t) := \theta(\bar{V}_t - p_t^s + \gamma(N_t) + \xi_t)$, and that from the individual customers is given by $D_t^i(p_t^i, N_t) := (1-\theta)(\bar{V}_t - p_t^i + \gamma(N_t) + \xi_t)$. The network size at the beginning of period $t-1$ is, thus, given by $N_{t-1} = D_t^s(p_t^s, N_t) + \eta N_t + \epsilon_t$.

We define

$v_t^d(I_t, N_t) :=$ the maximum expected discounted profits with price discrimination in periods $t, \cdots, 1$, when starting period t with an inventory level I_t and network size N_t;

and $(x_t^{d*}(I_t, N_t), p_t^{s*}(I_t, N_t), p_t^{i*}(I_t, N_t))$ as the optimal pricing and inventory policy. As in the base model, we assume that, in the last period (period 1), the excess inventory is salvaged with unit value c, and the backlogged demand is filled with ordering cost c, i.e., $v_0^d(I_0, N_0) = cI_0$ for any (I_0, N_0). Employing similar dynamic programming and sample path analysis methods, we characterize the optimal policy in the model with price discrimination in the following lemma.

Lemma 5 *Define a sequence of functions $\{\pi_t^d(N_t) : t = T, T-1, \cdots, 1\}$ and a sequence of pricing and inventory policies $\{(x_t^d(N_t), p_t^s(N_t), p_t^i(N_t)) : t = T, T-1, \cdots, 1\}$ as follows:*

$$\begin{aligned}
\pi_t^d(N_t) &= \max_{(x_t, p_t^s, p_t^i) \in \mathcal{F}_d} J_t^d(x_t, p_t^s, p_t^i, N_t), \qquad (2.10)\\
\textit{where } J_t^d(x_t, p_t^s, p_t^i, N_t) &= \theta R_t(p_t^s, N_t) + (1-\theta) R_t(p_t^i, N_t)\\
&\quad + \Lambda(x_t - \bar{V}_t + \theta p_t^s + (1-\theta) p_t^i - \gamma(N_t))\\
&\quad + \beta x_t + G_t^d(\theta(\bar{V}_t - p_t^s + \gamma(N_t)) + \eta N_t),\\
\textit{with } G_t^d(y) &:= \mathbb{E}\{r_n(y + \theta\xi_t + \epsilon_t) + \alpha \pi_{t-1}^d(y + \theta\xi_t)\}, \ \pi_0^d(\cdot) \equiv 0,\\
\textit{and } (x_t^d(N_t), p_t^s(N_t), p_t^i(N_t)) &:= \operatorname{argmax}_{(x_t, p_t^s, p_t^i) \in \mathcal{F}_d} J_t^d(x_t, p_t^s, p_t^i, N_t).
\end{aligned}$$

(a) $\pi_t^d(\cdot)$ *is concave, continuously differentiable, and increasing in* N_t. $J_t^d(\cdot,\cdot,\cdot,\cdot)$ *is jointly concave and continuously differentiable in* (x_t, p_t^s, p_t^i, N_t).

(b) *If* $I_t \leq x_t^d(N_t)$, $(x_t^{d*}(I_t,N_t), p_t^{s*}(I_t,N_t), p_t^{i*}(I_t,N_t)) = (x_t^d(N_t), p_t^s(N_t), p_t^i(N_t))$ *and* $v_t^d(I_t,N_t) = cI_t + \pi_t^d(N_t)$*; otherwise,* $x_t^{d*}(I_t,N_t) = I_t$. *If* $I_T \leq x_T^d(N_T)$, $(x_t^{d*}(I_t,N_t), p_t^{s*}(I_t,N_t), p_t^{i*}(I_t,N_t)) = (x_t^d(N_t), p_t^s(N_t), p_t^i(N_t))$ *for all* t *and* (I_t, N_t) *with probability 1.*

Lemma 5 demonstrates that a network-size-dependent base-stock/list-prices policy is optimal in the model with price discrimination. As in the base model, after normalizing the value of current inventory, the state space dimension of the dynamic program can be reduced to 1. Moreover, with probability 1, the optimal policy is independent of the starting inventory level in each period, as long as the initial inventory level I_T is below the optimal period-T base-stock level $x_T^d(N_T)$.

We remark that Theorems 2.4.3, 2.4.5, 2.4.6 and 2.4.7 can be generalized to the model with price discrimination. Hence, the impact of network externalities upon the optimal pricing and inventory policy is similar in the model with price discrimination to that in the base model. To characterize the impact of the price discrimination strategy, we directly compare the optimal policy and profit in the model with price discrimination with that in the base model.

Theorem 2.5.1 *Assume that two inventory systems are identical except that one with the price discrimination strategy and the other without. For each period* t*, we have (a) if* $\gamma'(\cdot) > 0$ *and* $p_t^i(N_t) > \underline{p}$, $p_t^s(N_t) < p_t^i(N_t)$*; (b)* $p_t(N_t) \leq p_t^i(N_t)$ *for all* N_t*; and (c)* $\pi_t^d(N_t) \geq \pi_t(N_t)$ *for all* N_t*, where the inequality is strict if* $p_t^i(N_t) > p_t^s(N_t)$*. Moreover, if* $\gamma(\cdot) \equiv \gamma_0$, $\pi_t^d(N_t) = \pi_t(N_t)$ *and* $p_t^i(N_t) = p_t^s(N_t) = p_t(N_t)$ *for all* N_t.

Theorem 2.5.1 sheds lights on the impact of the price discrimination strategy upon the firm's optimal pricing policy and the optimal profit. More specifically, Theorem 2.5.1(a) shows that, as long as network externalities are prevalent in the market (i.e., $\gamma'(\cdot) > 0$) and the optimal price for individual customers is not binding from below (i.e., $p_t^i(N_t) > \underline{p}$), the firm should charge a strictly lower price for the social customers than that for the individual customers. Under the price discrimination strategy, the firm can induce high future demands by charging a low price for the social customers, and generate current profits by a high price for the individual customers. Theorem 2.5.1(b) shows that, in each period t, the optimal price for customers without price discrimination is dominated by that for the individual customers with price discrimination. Without price discrimination, the firm should both generate the current profits and induce the future demands with the single price charged to all customers, so this price is lower than that for individual customers with price discrimination, which has the sole role of generating the current profits. In Theorem 2.5.1(c), we demonstrate that the price discrimination strategy is beneficial to the firm with network externalities. Without network externalities, however, the firm should charge a single price to all customers in each period. An important implication of Theorem 2.5.1 is that, under the price discrimination strategy, the firm earns a higher profit because it can (partially) separate generating current profits and inducing future demands, the former with the price for the individual customers and the latter with the price for the social customers.

2.5.2 Network Expanding Promotion

Since the willingness-to-pay of the customers in each period is increasing in the size of the associated network, the firm may launch network expanding promotion campaigns to enlarge the network size and, hence, increase its profitability. The network expanding promotion strategy is commonly used in practice for products with network externalities. For example, in February 2015, Microsoft discounted the 12-month Xbox Live Gold membership by 33 percent to both expand the size of Xbox Live and promote the sales of Xbox One (see, e.g., [153]). In the case where the associated network is an online communication network (i.e., $r_n(\cdot) \equiv 0$), network expanding promotion is the effort and investment the firm makes in social media marketing to attract customers to create and share the messages about the product in the network (i.e., through the electronic word-of-mouth). As an example, in October 2014, Apple bought Twitter's Promoted Trend at a daily cost of $200,000 to engage Twitter users for the new iPad Air 2 launch (see, e.g., [93]).

To model the network expanding promotion of the firm, let n_t be the number of customers who join the associated service network in period t in addition to the social customers who purchase the product. The total cost of attracting n_t customers into the network is $c_n(n_t)$, where $c_n(\cdot)$ is a continuously differentiable and convexly increasing function of n_t with $c_n(0) = 0$. Note that the network expanding promotion do not change the inventory dynamics of the firm, but they do have some impacts on the network size dynamics. More specifically, with network expanding promotion, the network size at the beginning of period $t-1$ is given by: $N_{t-1} = \theta D_t(p_t, N_t) + \eta N_t + n_t + \epsilon_t$.

We define

$$v_t^p(I_t, N_t) := \text{the maximum expected discounted profits with network expanding promotion in periods } t, t-1, \cdots, 1, \text{ when starting period } t \text{ with an inventory level } I_t \text{ and network size } N_t;$$

and $(x_t^{p*}(I_t, N_t), p_t^{p*}(I_t, N_t), n_t^*(I_t, N_t))$ as the optimal pricing and inventory policy. As in the base model, we assume that, in the last period (period 1), the excess inventory is salvaged with unit value c, and the backlogged demand is filled with ordering cost c, i.e., $v_0^p(I_0, N_0) = cI_0$ for any (I_0, N_0). Employing similar dynamic programming and sample path analysis methods, we characterize the optimal policy in the model with network expanding promotion in the following lemma.

Lemma 6 *Define a sequence of functions $\{\pi_t^p(N_t) : t = T, T-1, \cdots, 1\}$ and a sequence of pricing and inventory policies $\{(x_t^p(N_t), p_t^p(N_t), n_t(N_t)) : t = T, T-1, \cdots, 1\}$ as follows:*

$$\begin{aligned}
\pi_t^p(N_t) &= \max_{(x_t,p_t,n_t)\in\mathcal{F}_p} J_t^p(x_t, p_t, n_t, N_t), \qquad (2.11)\\
\textit{where } J_t^p(x_t, p_t, n_t, N_t) &= R_t(p_t, N_t) + \beta x_t + \Lambda(x_t - \bar{V}_t + p_t - \gamma(N_t)) - c_n(n_t) \\
&\quad + G_t^p(\theta(\bar{V}_t - p_t + \gamma(N_t)) + \eta N_t + n_t),\\
\textit{with } G_t^p(y) &:= \mathbb{E}\{r_n(y + \theta\xi_t + \epsilon_t) + \alpha\pi_{t-1}^p(y + \theta\xi_t)\},\ \pi_0^p(\cdot) \equiv 0,\\
\textit{and } (x_t^p(N_t), p_t^p(N_t), n_t(N_t)) &:= \text{argmax}_{(x_t,p_t,n_t)\in\mathcal{F}_p} J_t^p(x_t, p_t, n_t, N_t).
\end{aligned}$$

(a) $\pi_t^p(\cdot)$ is concave, continuously differentiable, and increasing in N_t. $J_t^p(\cdot,\cdot,\cdot,\cdot)$ is jointly concave and continuously differentiable in (x_t, p_t, n_t).

(b) If $I_t \le x_t^p(N_t)$, $(x_t^{p*}(I_t, N_t), p_t^{p*}(I_t, N_t), n_t^*(I_t, N_t)) = (x_t^p(N_t), p_t^p(N_t), n_t(N_t))$ and $v_t^p(I_t, N_t) = cI_t + \pi_t^p(N_t)$; otherwise, $x_t^{p*}(I_t, N_t) = I_t$. If $I_T \le x_T^p(N_T)$, $(x_t^{p*}(I_t, N_t), p_t^{p*}(I_t, N_t), n_t^*(I_t, N_t)) = (x_t^p(N_t), p_t^p(N_t), n_t(N_t))$ for all t and (I_t, N_t) with probability 1.

Lemma 6 demonstrates that a network-size-dependent base-stock/list-price/promotion policy is optimal in the model with price discrimination. By normalizing the value of current inventory, we can reduce the state space dimension of the dynamic program to 1.

With probability 1, the optimal policy is independent of the starting inventory level in each period, as long as the initial inventory level I_T is below the optimal period-T base-stock level in the first period $x_T^p(N_T)$.

As in the model with price discrimination, Theorems 2.4.3, 2.4.5, 2.4.6, and 2.4.7 can be generalized to the model with network expanding promotion. We now demonstrate the effectiveness [ineffectiveness] of network expanding promotion in the model with [without] network externalities.

Theorem 2.5.2 *(a) Let* $0 < \iota < 1$, *and* $\bar{S}(N) := \sup\{\Delta : \mathbb{P}(N_{t-1} \geq \Delta | N_t = N) \geq \iota\}$. *If*

$$(1-\iota)[r_n'(\bar{S}(N)) + \alpha(\underline{p} - c)\gamma'(\bar{S}(N))] > c_n'(0), \tag{2.12}$$

then $n_t^*(I_t, N) > 0$ *for all* I_t. *Moreover,* $\bar{S}(N)$ *is continuously increasing in* N *and, for each* $0 < \iota < 1$, *there exists an* $N^*(\iota) \geq 0$, *such that* (2.12) *holds for all* $N < N^*(\iota)$.

(b) If $\gamma(\cdot) \equiv \gamma_0$ *and* $(\sum_{\tau=0}^{t-1} (\alpha\eta)^\tau) r_n'(0) \leq c_n'(0)$, $n_t^*(I_t, N_t) \equiv 0$ *for all* I_t *and* $N_t \geq 0$.

Theorem 2.5.2 characterizes the dichotomy on when the firm should offer network expanding promotion. More specifically, Theorem 2.5.2(a) shows that, when either (i) the intensity of network externalities is sufficiently strong or (ii) the associated service network is sufficiently profitable (as characterized by inequality (2.12)), it is optimal for the firm to offer network expanding promotion to customers as long as the current network size is sufficiently low (i.e., $n_t^*(I_t, N_t) > 0$ if $N_t \leq N^*(\iota)$). The intuition behind Theorem 2.5.2(a) is that, if a lower bound of the marginal value of offering network expanding promotion, $(1-\iota)[r_n'(\bar{S}(N)) + \alpha(\underline{p} - c)\gamma'(\bar{S}(N))]$, dominates its marginal cost $c_n'(0)$, the firm should offer network expanding promotion to customers. Here, $\bar{S}(N)$ can be interpreted as the threshold such that, conditioned on $N_t = N$, the probability that the network size in period $t-1$ exceeds $\bar{S}(N)$ is smaller than ι, regardless of the pricing strategy the firm employs. Hence, network expanding promotion are effective in exploiting network externalities, especially when N_t and, thus, the potential demand is low. On the other hand, Theorem 2.5.2(b) shows that if network externalities do not exist (i.e., $\gamma(\cdot) \equiv 0$) and the associated service network is not sufficiently profitable (i.e., $(\sum_{\tau=0}^{t-1} (\alpha\eta)^\tau) r_n'(0) \leq c_n'(0)$), it is optimal for the firm not to offer any network expanding promotion.

Next, we study the impact of network expanding promotion upon the firm's optimal policy.

Theorem 2.5.3 *Assume that two inventory systems are identical except that one with network expanding promotion and the other without. For each period t and each network size $N_t \geq 0$, the following statements hold: (a) $p_t^p(N_t) \geq p_t(N_t)$; (b) $y_t^p(N_t) \leq y_t(N_t)$; (c) $x_t^p(N_t) \leq x_t(N_t)$; and (d) $\pi_t^p(N_t) \geq \pi_t(N_t)$, where the inequality is strict if $n_t(N_t) > 0$.*

Theorem 2.5.3 highlights how the firm should adjust its price and inventory policy with network expanding promotion. More specifically, we show in Theorem 2.5.3(a) that, with the same network size (and, hence, the same potential market size), the firm should charge a higher sales price with network expanding promotion. Since both the sales price and the network expanding promotion helps induce future demands via network externalities, the adoption of network expanding promotion allows the firm to increase the sales price to generate higher profit in the current period. As a result, the optimal expected demand and the optimal base-stock level are lower with market expanding promotion. In Theorem 2.5.3(d), we show that network expanding promotion can improve the profitability of the firm.

To summarize, network expanding promotion helps the firm exploit network externalities by boosting the network size in each period. In particular, network expanding promotion facilitates the firm to induce future demands with network expanding promotion, while generating higher current profits with a higher sales price. The firm should offer network expanding promotion when the intensity of network externalities is sufficiently strong or the associated service network is sufficiently profitable.

2.6 Numerical Studies

This section reports a set of numerical studies that quantify the profit loss of ignoring network externalities. We also propose and quantitatively evaluate some easy-to-implement heuristics in the presence of network externalities. Our numerical results demonstrate that (1) ignoring network externalities and, thus, employing a myopic pricing and inventory policy leads to staggering profit losses when the network externalities intensity, the social customer proportion, or the carry-through rate of network size is high;

and (2) the firm can achieve low optimality gaps and effectively exploit network externalities with heuristic policies that take into account the demand induction opportunities in the *near future* only.

Throughout our numerical studies, we assume that the maximum intrinsic valuation $\bar{V}_t$ is stationary and equals 30 for each period t. The planning horizon length is $T = 20$. The network externalities function is $\gamma(N_t) = kN_t$ $(k \geq 0)$. The parameter k measures the network externalities intensity. The larger the k, the more intensive network externalities the firm faces. Hence, the demand in each period t is $D_t(p_t, N_t) = 30 + kN_t - p_t + \xi_t$, where $\{\xi_t\}_{t=1}^T$ follow *i.i.d.* normal distributions with mean 0 and standard deviation $\sigma = 2$. Note that with the linear network externalities function $\gamma(\cdot)$, Assumption 2.3.1 does not hold. This slight deviation from our analytical model, however, does not influence the insights obtained in this section. For simplicity, we assume the random perturbation in the market size dynamics ϵ_t is degenerate, i.e., $\epsilon_t = 0$ with probability 1. We set the discount factor $\alpha = 0.99$, the unit procurement cost $c = 8$, the unit holding cost $h = 1$, the unit backlogging cost $b = 10$, and the feasible price range $[\underline{p}, \bar{p}] = [0, 34]$. In the evaluation of the expected profits, we take $I_t = 0$ as the reference initial inventory level and $N_t = 0$ as the reference initial network size.

2.6.1 Impact of Network Externalities

This subsection numerically studies the impact of network externalities upon the firm's profitability under different values of network externalities intensity k, social customer proportion θ, and carry-through rate of network size η. We evaluate the profit of the firm which ignores the tradeoff between generating current profits and inducing future demands in the presence of network externalities. More specifically, we assume that the firm adopts the *myopic policy* in each period t, i.e., it adopts the pricing and inventory policy that maximizes the expected current-period profit without taking into account future demand-inducing opportunities. Equivalently, the firm employs the optimal final-period policy, $(x_1^*(\cdot,\cdot), p_1^*(\cdot,\cdot))$, throughout the planning horizon. Let V_m be the expected profit under the myopic policy, and V^* be optimal expected profit. Thus, the metric of interest is

$\lambda_m := \dfrac{V^* - V_m}{V^*} \times 100\%$, which evaluates the profit loss of ignoring network externalities.

We conduct the numerical experiments under the parameters $t = 5, 10, 15, 20$, $k = 0.2, 0.5, 0.8$, $\theta = 0.2, 0.5, 0.8$, and $\eta = 0.2, 0.5, 0.8$.

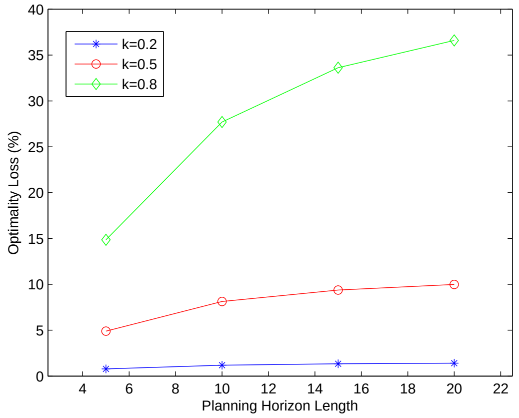

Figure 2.1. **Value of λ_m: $\theta = 0.5$, $\eta = 0.5$**

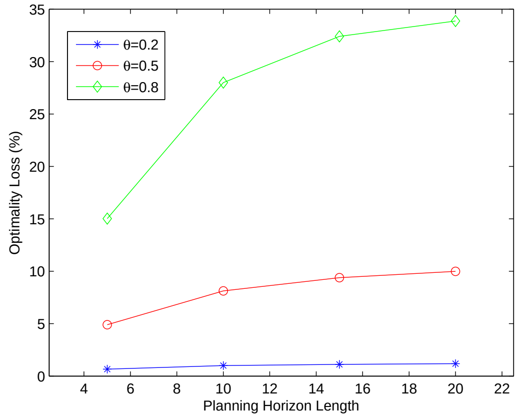

Figure 2.2. **Value of λ_m: $k = 0.5$, $\eta = 0.5$**

Figures 2.1 - 2.3 summarize the results of our numerical study on the impact of ignoring network externalities upon the firm's profitability. Our results reveal that, when the future demand-inducing opportunity of network externalities is ignored, the firm incurs a significant profit loss, which is at least 4.90% and can be as high as 36.60%, as long as the network externalities intensity k, the proportion of social customers θ, and the network size carry-through rate η are not too low (greater than 0.2 in our numerical case). If k, θ, and η are higher, the current operations decisions have greater impact upon future network sizes, thus leading to more intensive tradeoff between generating current profits and inducing future demands. Therefore, adopting the myopic policy results in significant losses if k, θ, and η are not too low. Another important implication of Figures 2.1 - 2.3 is that, if k, θ, and η are not too low, the profit loss of ignoring network externalities may be significant even when the planning horizon length is short (i.e., $t = 5$). This calls for caution that the firm under network externalities should not overlook the tradeoff between generating current profits and inducing future demands even for a short sales horizon.

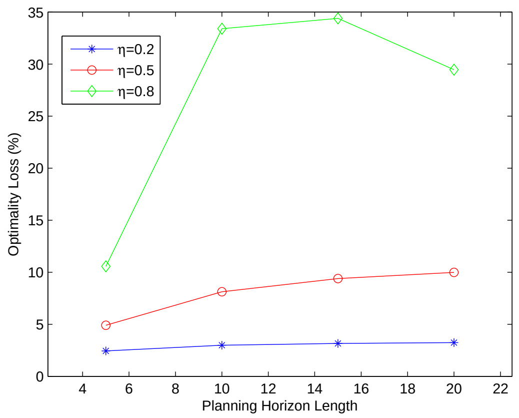

Figure 2.3. **Value of λ_m: $k = 0.5$, $\theta = 0.5$**

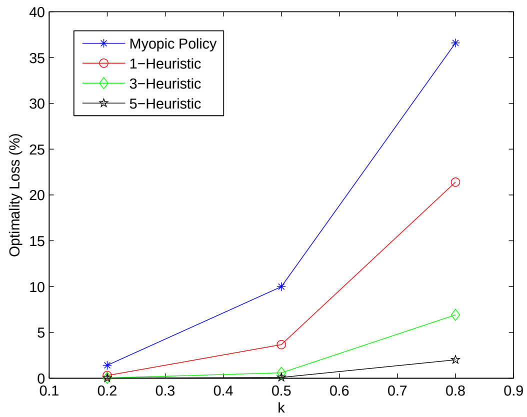

Figure 2.4. **Value of λ_m and λ_h^i: $\theta = 0.5$, $\eta = 0.5$**

2.6.2 Effective Heuristic Policies under Network Externalities

In this subsection, we propose some easy-to-implement heuristic policies and explore when these heuristics effectively leverage network externalities. As shown in Section 2.6.1, the myopic policy may have a poor performance because it ignores the opportunity of inducing future demands via network externalities. Thus, we consider the heuristic policies that balance generating current profits and inducing demands in the *near future* (within 5 periods) through network externalities. More specifically, in each period t, the firm dynamically maximizes the expected total discounted profit in the moving time window from period t to period $t+i$ $(i = 1, 3, 5)$. We call the heuristic policy to maximize the profit in the moving time window of length i as the $i-$heuristic $(i = 1, 3, 5)$. Clearly, obtaining the $i-$heuristic $(i = 1, 3, 5)$ only involves solving a dynamic program with planing horizon length $i + 1$, and is, thus, computationally light. Hence, the $i-$heuristic policy $(i = 1, 3, 5)$ is easy to implement. Let V_h^i be the expected total profit under the $i-$heuristic policy. We have $V^* \geq V_h^5 \geq V_h^3 \geq V_h^1 \geq V_m$. The metric of interest is

$\lambda_h^i := \dfrac{V^* - V_h^i}{V^*} \times 100\%$ which measures the optimality gap of the $i-$heuristic policy

$(i = 1, 3, 5)$. We conduct the numerical experiments under the parameters $t = 20$, $k = 0.2, 0.5, 0.8$, $\theta = 0.2, 0.5, 0.8$, and $\eta = 0.2, 0.5, 0.8$.

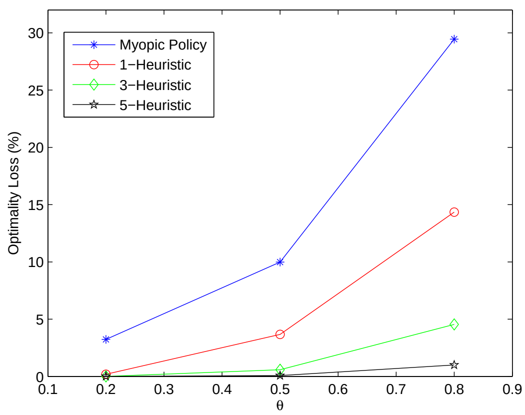

Figure 2.5. **Value of** λ_m **and** λ_h^i**:** $k = 0.5$**,** $\eta = 0.5$

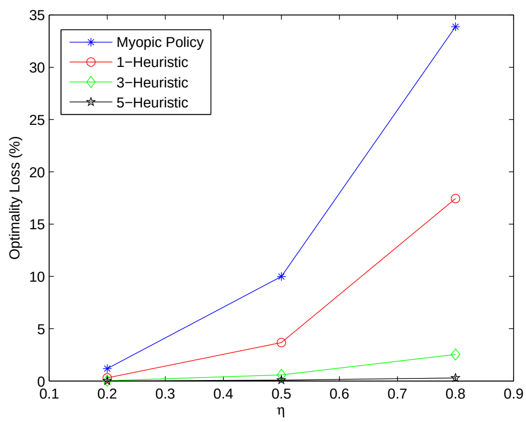

Figure 2.6. **Value of** λ_m **and** λ_h^i**:** $k = 0.5$ $\theta = 0.5$

Figures 2.4 - 2.6 summarize the results of our numerical study on the performance of $i-$heuritic policies $(i = 1, 3, 5)$. The results show that, compared with the myopic policy that completely ignores the future demand-inducing opportunities, the $i-$heuristics ($i = 1, 3, 5$) significantly improve the profitability of the firm in the presence of network externalities. In particular, the $5-$heuristic leads a very low profit loss compared with the optimal policy (no more than 2%, in contrast to the more-than-30% optimality gap of the myopic policy). Therefore, the firm can effectively exploit network externalities by slightly looking into the future and balancing the tradeoff between generating current profits and inducing *near future* demands. Moreover, as shown in Figures 2.4 - 2.6, if the network externalities intensity k, the social customer proportion θ, or the carry-through rate of network size η is higher, the $i-$heuristic policies are more valuable relative to the myopic policy. As k, θ, or η increases, the tradeoff between generating current profits and inducing future demands becomes more intensive, and, thus, the forward-looking $i-$heuristics can deliver higher values to the firm compared with the myopic policy. We have also performed numerical analysis for the $i-$heuristic policies with $i > 5$. These more forward-looking heuristic policies cannot generate significantly better performances over the $5-$heuristic policy. This further demonstrates that, to exploit network externalities, it suffices for the firm to balance generating current profits and inducing demands in the *near future*. Finally, we remark that our numerical results are robust and continue to hold in the settings where the planning horizon length T is greater than 20 and/or the market non-stationary (i.e., the maximum intrinsic valuation $\bar{V}_t$ varies with time t). For concision, we only present the results for the case where $T = 20$ and the market is stationary in this chapter.

2.7 Summary

This is the first paper in the literature to study the joint pricing and inventory management model under network externalities. To model network externalities, we assume that there is an online service or communication network associated with the product, and the customers' willingness-to-pay is increasing in the size of this network. Moreover, in each period, a fraction of the customers (i.e., the social customers) who purchase the product would join the network and exert network externalities over potential customers in the future. The firm may directly generate profits from the network via, e.g., service subscription fees. Therefore, in each period, the firm faces the tradeoff between generating current profits and inducing future demands via network externalities.

We show that the optimal policy is a network-size-dependent base-stock/list-price policy. Moreover, we demonstrate that, with probability 1, the inventory dynamics do not influence the optimal policy of the firm. As a consequence, the state space dimension of the dynamic program can be reduced to one by normalizing the current inventory value. Such state space dimension reduction greatly facilitates the analysis and enables us to deliver sharper insights from our model. Our analysis reveals that the firm needs to balance the tradeoff between generating current profits and inducing future demands through network externalities. Under network externalities, since the current demand is stochastically increasing in the network size, the optimal base-stock level and the optimal sales price are increasing in the network size as well. Network externalities lead to higher potential demands and, thus, higher base-stock levels. The optimal sales price, however, may not necessarily increase with the presence of network externalities. This is because, with network externalities, the firm should decrease the sales price to exploit the increased network externalities when the network size is small, and increase the sales price to exploit the better market condition when the network size is large. From the intertemporal perspective, the firm should put more weight on inducing future demands at the early stage of a sales season than at later stages. Thus, when the market is stationary, the firm employs the introductory price strategy that offers early purchase discounts to induce high future demands at the beginning of the sale season. Moreover, the firm needs to trade off between generating profit from the product and from the associated network. With a higher marginal profit of the associated network, the firm should decrease the sales price to exploit the more intensive network externalities.

Our analysis demonstrates the effectiveness of the price discrimination strategy and the network expanding promotion strategy in exploiting network externalities. Both strategies facilitate the firm to (partially) separate generating current profits and inducing future demands through network externalities with an additional leverage (price or promotion). Under the price discrimination strategy, the firm generates a higher current profit with a higher price for individual customers, and induces higher future demands with a lower price for social customers. Network expanding promotion should be employed when the intensity of network externalities is sufficiently strong or the associated service network is sufficiently profitable. Moreover, the firm offers network expanding promotion to induce future demands through network externalities, while generating a higher current profit with an increased price of the product.

We perform extensive numerical studies to characterize (a) the impact of ignoring network externalities, and (b) the value of some easy-to-implement heuristic policies to exploit network externalities. Our numerical results show that the profit loss of ignoring network externalities is significant, especially when the network externalities intensity, the social customer proportion, or the network size carry-through rate is high. In this scenario, the tradeoff between generating current profits and inducing future demands is most intensive, so the firm should by no means myopically optimize its current profit. On the other hand, the heuristic policies that dynamically maximize the expected profit in a moving time window of no more than 5 periods achieve low profit losses relative to the optimal policy. Hence, to leverage network externalities, it suffices for the firm to balance generating current profits and inducing demands in the *near future*.

3. Operations Impact of Network Externalities: Dynamic Competition Setting

3.1 Introduction

[1]In today's competitive and unstable market environment, it is prevalent that modern firms compete not only on generating current profits, but also on winning future market shares (see, e.g., [106]). The current decisions of all competing firms in the market not only determine their respective current profits, but also significantly influence their future demands. We refer to such inter-temporal dependence of future demands on the current decisions as market size dynamics. Under market size dynamics, myopically optimizing the current profit may lead to significant loss of future demands, and hurt the firm's profit in the long run. Therefore, the competing firms face an important tradeoff between generating current profits and inducing future demands, which we refer to as the *exploitation-induction tradeoff*.

Among others, we focus on two main drivers of the aforementioned exploitation-induction tradeoff: (a) The future demand is positively correlated with the current service level, which we refer to as the *service effect*; and (b) the future demand is positively correlated with the current demand, which we refer to as the *network effect*.

The service effect is driven by the well-recognized phenomenon that the past service experience of a customer significantly impacts his/her future purchasing decisions (see, e.g., [29, 2]). A poor service (e.g., a low fill rate of a customer's orders) generally diminishes the goodwill of a customer, thus leading to lower future orders from this customer ([1]). Moreover, it is widely observed in practice that stockouts can adversely impact future demands (see, e.g., [11, 84]). In the face of a stockout experience, a natural reaction of a customer is to order fewer items and/or switch the seller in a subsequent purchasing execution (see, e.g., [77, 131]). Therefore, good [poor] past services of a firm are likely to induce high [low] demands in the future.

[1]This chapter is based on the author's earlier work [191].

The network effect, also known as network externalities, refers to the general phenomenon that a customer's utility of purchasing a product is increasing in the number of other customers buying the same product (see, e.g., [66]). Under the network effect, a higher current demand of a firm leads to more adoptions of its product, thus increasing the utility of purchasing its product for future customers and boosting future demands. There are three major mechanisms that give rise to the network effect: (a) the direct effect, under which an increase in the adoption of a product leads to a direct increase in the value of this product for other users (see, e.g., [102]); (b) the indirect effect, under which an increase in the adoption of a product enhances the value of its complementary products or services, which in turn increases the value of the original product (see, e.g., [37]); and (c) the social effect, under which the value of a product is influenced by the social interactions of its customers with their peers (see, e.g., [36]).

In the highly inter-correlated and competitive market of the current era, the service effect and the network effect reinforce each other. This is because the fast development of information technology enables customers to easily learn the information (on, e.g., quality, service, popularity, etc.) of any product through communications with their friends and/or the customer reviews on online reviewing platforms and social media. Thus, the higher the current demand of a firm, the more information about its service quality will be released to the public, and, hence, the higher impact its service quality will have upon future demands. Moreover, the current service level of a firm impacts the future demands of itself as well as its competitors, because customers are likely to patronage the firms with good past service and abandon those with poor past service based on either their own purchasing experience or the social learning process.

The primary goal of this chapter is to develop a model that can provide insights on how the exploitation-induction tradeoff impacts the equilibrium market behavior under both the service effect and network effect. To this end, we study a periodic-review dynamic competition model, in which firms in a retail market compete under a Markov game over a finite planning horizon. The random demand of each firm in each period is determined by its market size and the current sales prices and promotional efforts of all competing firms. The promotional effort (e.g., advertising, product innovation, and/or after sales service) of a firm boosts the current demand of itself and diminishes that of its competitors. The key feature of our model is that the market sizes of the competing firms are stochastically evolving throughout the planning horizon, and their evolutions are driven by the service effect and the network effect. More specifically, to capture the market size dynamics, we assume that the future market size of each firm is stochastically increasing in its current service level and demand, and stochastically decreasing in the current service levels of its competitors. Taking the market size dynamics into consideration, each firm chooses its promotional effort, sales price, and inventory stocking quantity in each decision period, with an attempt to balance generating current profits and inducing future demands in the dynamic and competitive market. We study two competitions: (a) the simultaneous competition, under which the firms simultaneously make their promotion, price, and inventory decisions in each period; and (b) the promotion-first competition, under which the firms first make their promotional efforts and, after observing the promotion decisions in the market, choose their sales prices and inventory levels in each period.

Conducting a dynamic game analysis, we make two main contributions in this chapter: (a) We study a dynamic competition model with the inter-temporal influences of *current* decisions over *future* demands, and characterize the pure strategy Markov perfect equilibrium under both the simultaneous competition and the promotion-first competition; (b) we identify several important managerial implications of the exploitation-induction tradeoff upon the equilibrium market behavior of the dynamic competition under the service effect and the network effect.

We use the Markov perfect equilibrium paradigm to analyze our dynamic competition model, because the competing firms need to adaptively adjust their strategies based on their inventory levels and market sizes in each period. The analytical characterization of Markov perfect equilibria in a dynamic oligopoly with planning horizon length greater than two is, in general, prohibitively difficult (see, e.g., [132]). To characterize the equilibrium market outcome in our model, we employ the linear separability approach (see, e.g., [131]) and show that, under both the simultaneous competition and the promotion-first competition, the equilibrium profit of each firm in each period is linearly separable in its own inventory level and market size. Such linear separability greatly facilitates the analysis and enables us to characterize the pure strategy Markov perfect equilibrium under both competitions. Moreover, under both competitions, the pure strategy Markov perfect equilibrium has the nice feature that the equilibrium strategy of each firm only depends on the private information (i.e., inventory level and market size) of itself, but not on that of its competitors. Under the simultaneous competition, the subgame played by the competing firms in each period can be decomposed into a two-stage competition, in which the firms compete jointly on promotional effort and sales price in the first stage, and on service level in the second. Under the promotion-first competition, the subgame in each period can be decomposed into a three-stage competition, in which the firms compete on promotional effort in the first stage, on sales price in the second, and on service level in the third. Under both competitions, each stage of the subgame in each period has a pure strategy Nash equilibrium, thus ensuring the existence of a pure strategy Markov perfect equilibrium in the Markov game. We also provide mild sufficient conditions under which the Markov perfect equilibrium is unique under each competition.

Under both the simultaneous and the promotion-first competitions, the market size dynamics significantly impact the equilibrium behaviors of the competing firms via the exploitation-induction tradeoff. This tradeoff is quantified by the linear coefficient of market size for each firm in each period. The higher the market size coefficient, the more intensive the exploitation-induction tradeoff for the respective firm in the previous period. We identify three effective strategies under the service effect and the network effect: (a) improving promotional efforts, (b) offering price discounts, and (c) elevating service levels. These strategies are grounded on the uniform idea that, to balance the exploitation-induction tradeoff, the competing firms can induce higher future demands at the cost of reduced current margins. Our analysis demonstrates how the strength of the service effect and network effect impacts the equilibrium market outcome. Under stronger service and network effects, the exploitation-induction tradeoff is more intensive, so the competing firms make more promotional efforts, offer heavier price discounts, and maintain higher service levels. When the market is stationary, the intensity of the exploitation-induction tradeoff decreases over the sales season under both competitions. Hence, the equilibrium sales prices are increasing, whereas the equilibrium promotional efforts and service levels are decreasing, over the planning horizon.

Our analysis reveals two interesting differences between the simultaneous competition and the promotion-first competition under market size dynamics. First, under the simultaneous competition, the competing firms need to balance the exploitation-induction tradeoff inter-temporally, whereas, under the promotion-first competition, they have to balance this tradeoff both inter-temporally and intra-temporally. Second, we identify a new driving force for the "fat-cat" effect (i.e., in each period, the equilibrium promotional efforts may be higher under the promotion-first competition than those under the simultaneous competition): The exploitation-induction tradeoff is more intensive in the promotion-first competition than in the simultaneous competition, thus prompting the firms to make more promotional efforts under the promotion-first competition.

The rest of this Chapter is organized as follows. We position this chapter in the related literature in Section 3.2. Section 3.3 introduces the model setup. We analyze the simultaneous competition model in Section 3.4, and the promotion-first competition model in Section 3.5. We compare the equilibrium outcomes in these two competitions in Section 3.6. Section 3.7 concludes this chapter. All proofs are relegated to Appendix B.1.

3.2 Related Research

Our work is related to several streams of research in the literature. The literature on the phenomenon that the current service level impacts future demands is rich. For example, [147, 148] first studies the inventory management model, in which future demands are adversely affected by current poor service levels. [1] consider the dynamic capacity allocation problem of a supplier, whose customers remember past service. [2] propose a dynamic behavioral model to study the retention and service relationship management with the effect of past service experiences on future service quality expectations. The impact of current service on future demands has also been analyzed in a competitive environment. [92] investigate a dynamic customer service competition, in which the duopoly firms compete by investing in capacity with a fixed total number of customers. [114] study a dynamic inventory duopoly model, in which inventory is perishable and customers may defect to a competitor. [131] generalize this model to the setting with non-perishable inventory and the setting in which the firms may attract dissatisfied customers from the competition. [82] investigates the supplier competition model, in which each customer switches among suppliers based on her past service quality experience. [84] study an inventory competition, in which each customer learns about a firm's service level from her previous shopping experience, and makes her potential patronage decision among different firms accordingly. The contribution of this chapter to this literature is that we characterize the equilibrium market behavior in the joint promotional effort, sales price, and service level competition under the service effect.

The optimal pricing strategy under network externalities has received considerable attention in the economics and marketing literature. [61] characterize the optimal non-linear pricing strategy for a network product with heterogenous customers. [188] examine the equilibrium dynamic pricing strategies of an incumbent and a later entrant under network externalities. [19] consider the optimal dynamic monopoly pricing under network externalities and show that the equilibrium prices increase as time passes. [28] study the optimal pricing strategy in a network with a given network structure and characterize the relationship between optimal prices and consumers' centrality. We contribute to this stream of literature by analyzing the impact of network externalities upon the competing firms' operations decisions (i.e., the inventory policies) in a dynamic competition.

This chapter is also related to the extensive literature on dynamic pricing and inventory management. This literature diverges into two lines of research: (i) the monopoly model, in which a single firm maximizes its total expected profit over a finite or infinite planning horizon, and (ii) the competition model, in which multiple firms play a noncooperative game to maximize their respective expected per-period profits over an infinite planning horizon. The literature on the monopoly model of joint pricing and inventory management is very rich. [70] give a general treatment of this problem and show the optimality of the base-stock list-price policy. [47, 48, 49] study the joint pricing and inventory management problem with fixed ordering costs for the finite horizon, infinite horizon, and continuous review models. [52] characterize the optimal policy in the joint pricing and inventory control model with fixed ordering costs and lost sales. [96] identify a general condition under which (s, S)-type policies are optimal for a stationary joint pricing and inventory control model with fixed ordering costs. [112] study the joint pricing and inventory management problem with the random yield risk, and show that such risk drives the firm to charge a higher price in each period. The joint pricing and inventory control problem with periodic review and positive leadtime is extremely difficult. For this problem, [136] and [46] characterize the monotonicity properties of the optimal price and inventory policy for nonperishable and perishable products, respectively. We refer interested readers to [50] for a comprehensive review on the monopoly models of joint pricing and inventory management.

The research on the competition model of dynamic pricing and inventory management is also abundant. Under deterministic demands, [21] study the EOQ model of a two-echelon distribution system, characterize the equilibrium pricing and replenishment strategies of the competing retailers under both Bertrand and Cournot competitions, and identify the perfect coordination mechanisms therein. [22] address infinite-horizon models for oligopolies with competing retailers under price-sensitive uncertain demand. [23] develop a stochastic general equilibrium inventory model, in which retailers compete on both sales price and service level throughout an infinite horizon. [25] generalize this model to a decentralized supply chain setting, and characterize the perfect coordinating mechanisms under price and service competition. Our work differs from this line of literature in that we study the exploitation-induction tradeoff with the service effect and the network effect in a dynamic and competitive market. To this end, we adopt the Markov perfect equilibrium (i.e., the closed-loop equilibrium) in a finite-horizon model as opposed to the commonly used stationary strategy equilibrium (i.e., the open-loop equilibrium) in an infinite-horizon model.

Finally, from the methodological perspective, our work is related to the literature on the analysis of Markov perfect equilibrium in dynamic competition models. Markov perfect equilibrium is prevalent in the economics literature on dynamic oligopoly models (see, e.g, [122, 69, 57]). In the operations management literature, this equilibrium concept has been widely adopted to study the equilibrium behaviors in dynamic games. Employing the linear separability approach, [92, 114, 131] characterize the Markov perfect equilibrium in dynamic duopoly models with market size dynamics, and [5] analyze the structure of the pure strategy Markov perfect equilibria in a dynamic inventory competition with subscriptions. A similar approach based on the separability of player decisions and probability transition functions has been used by [6] to study a joint pricing and advertising competition, and by [130] to study a multi-period inventory competition. Due to limited technical tractability, the analysis of Markov perfect equilibrium in nonlinear and nonseparable dynamic games is scarce. [120] characterize the Markov perfect equilibrium price strategy in a finite-horizon dynamic Bertrand competition with fixed capacities. [117] numerically compute the Markov perfect equilibrium in an infinite-horizon model, in which a supplier allocates its limited capacity to competing retailers. [132] give conditions under which the stationary infinite-horizon equilibrium is also a Markov perfect equilibrium in the context of inventory duopolies. This chapter adopts the linear separability approach to characterize the pure strategy Markov perfect equilibrium of a dynamic joint promotion, price, and inventory competition under both the service effect and the network effect, and analyze the exploitation-induction tradeoff therein.

3.3 Model

Consider an industry with N competing retail firms, which serve the market with partially substitutable products over a $T-$period planning horizon, labeled backwards as $\{T, T-1, \cdots, 1\}$. In each period t, each firm i selects a promotional effort $\gamma_{i,t} \in [0, \bar{\gamma}_{i,t}]$, which represents the effort the firm makes in advertising, product innovation, and/or after-sales service to promote the demand of its product in the current period. We assume that, in any period t, the total promotional investment cost of each firm i is proportional to its realized demand in period t, $D_{i,t}$, and given by $\nu_{i,t}(\gamma_{i,t})D_{i,t}$. The per-unit demand cost rate, $\nu_{i,t}(\cdot)$, is a non-negative, convexly increasing, and twice continuously differentiable function of the promotional effort $\gamma_{i,t}$, with $\nu_{i,t}(0) = 0$. Before the demand is realized in period t, each firm i selects a sales price $p_{i,t} \in [\underline{p}_{i,t}, \bar{p}_{i,t}]$ and adjusts its inventory level to $x_{i,t}$. We assume that the excess demand of each firm is fully backlogged. In summary, each firm i makes three decisions at the beginning of any period t: (i) the promotional effort $\gamma_{i,t}$, (ii) the sales price $p_{i,t}$, and (iii) the inventory level $x_{i,t}$.

The demand of each firm i in any period t depends on the entire vector of promotional efforts $\gamma_t := (\gamma_{1,t}, \gamma_{2,t}, \cdots, \gamma_{N,t})$ and the entire vector of sales prices $p_t := (p_{1,t}, p_{2,t}, \cdots, p_{N,t})$ in period t. We denote the demand of firm i as $D_{i,t}(\gamma_t, p_t)$. More specifically, we base our analysis on the following multiplicative form of $D_{i,t}(\cdot, \cdot)$:

$$D_{i,t}(\gamma_t, p_t) = \Lambda_{i,t} d_{i,t}(\gamma_t, p_t) \xi_{i,t}, \tag{3.1}$$

where $\Lambda_{i,t} > 0$ is the market size of firm i in period t, $d_{i,t}(\gamma_t, p_t) > 0$ captures the impact of γ_t and p_t on firm i's demand in period t, and $\xi_{i,t}$ is a positive continuous random variable with a connected support. Let $F_{i,t}(\cdot)$ be the *c.d.f.* and $\bar{F}_{i,t}(\cdot)$ be the *c.c.d.f.* of $\xi_{i,t}$. The market size $\Lambda_{i,t}$ is observable by firm i at the beginning of period t through the pre-order sign-ups and/or subscriptions before the release of its product in period t. The random perturbation term $\xi_{i,t}$ is independent of the market size vector $\Lambda_t := (\Lambda_{1,t}, \Lambda_{2,t}, \cdots, \Lambda_{N,t})$, the sales price vector p_t, and the promotional effort vector γ_t. Moreover, $\{\xi_{i,t} : t =$

$T, T-1, \cdots, 1\}$ are independently distributed for each i. Without loss of generality, we normalize $\mathbb{E}[\xi_{i,t}] = 1$ for each i and any t, i.e., $\mathbb{E}[D_{i,t}(\gamma_t, p_t)] = \Lambda_{i,t} d_{i,t}(\gamma_t, p_t)$. Therefore, $d_{i,t}(\gamma_t, p_t)$ can be viewed as the normalized expected demand of firm i in period t.

We assume that $d_{i,t}(\cdot,\cdot)$ is twice continuously differentiable on $[0, \bar{\gamma}_{1,t}] \times [0, \bar{\gamma}_{2,t}] \times \cdots \times [0, \bar{\gamma}_{N,t}] \times [\underline{p}_{1,t}, \bar{p}_{1,t}] \times [\underline{p}_{2,t}, \bar{p}_{2,t}] \times \cdots \times [\underline{p}_{N,t}, \bar{p}_{N,t}]$, and satisfies the following monotonicity properties:

$$\frac{\partial d_{i,t}(\gamma_t, p_t)}{\partial \gamma_{i,t}} > 0, \ \frac{\partial d_{i,t}(\gamma_t, p_t)}{\partial \gamma_{j,t}} < 0, \ \frac{\partial d_{i,t}(\gamma_t, p_t)}{\partial p_{i,t}} < 0, \ \text{and} \ \frac{\partial d_{i,t}(\gamma_t, p_t)}{\partial p_{j,t}} > 0, \ \text{for all } j \neq i. \tag{3.2}$$

In other words, an increase in a firm's promotional effort increases the current-period demand of itself, and decreases the demands of its competitors. On the other hand, an increase in a firm's sales price decreases the demand of itself, and increases the demands of its competitors. Moreover, we assume that $d_{i,t}(\cdot,\cdot)$ is log-separable, i.e., $d_{i,t}(\gamma_t, p_t) = \psi_{i,t}(\gamma_t)\rho_{i,t}(p_t)$, where $\psi_{i,t}(\cdot)$ and $\rho_{i,t}(\cdot)$ are positive and twice-continuously differentiable. Inequalities (3.2) imply that

$$\frac{\partial \psi_{i,t}(\gamma_t)}{\partial \gamma_{i,t}} > 0, \ \frac{\partial \psi_{i,t}(\gamma_t)}{\partial \gamma_{j,t}} < 0, \ \frac{\partial \rho_{i,t}(p_t)}{\partial p_{i,t}} < 0, \ \text{and} \ \frac{\partial \rho_{i,t}(p_t)}{\partial p_{j,t}} > 0, \ \text{for all } j \neq i.$$

For technical tractability, we assume that $\psi_{i,t}(\cdot)$ and $\rho_{i,t}(\cdot)$ satisfy the log increasing differences and the diagonal dominance conditions, i.e., for any t, all i and $j \neq i$,

$$\frac{\partial^2 \log \psi_{i,t}(\gamma_t)}{\partial \gamma_{i,t}^2} < 0, \ \frac{\partial^2 \log \psi_{i,t}(\gamma_t)}{\partial \gamma_{i,t} \partial \gamma_{j,t}} \geq 0, \text{ and } \left|\frac{\partial^2 \log \psi_{i,t}(\gamma_t)}{\partial \gamma_{i,t}^2}\right| > \sum_{j \neq i} \frac{\partial^2 \log \psi_{i,t}(\gamma_t)}{\partial \gamma_{i,t} \partial \gamma_{j,t}}; \tag{3.3}$$

$$\frac{\partial^2 \log \rho_{i,t}(p_t)}{\partial p_{i,t}^2} < 0, \ \frac{\partial^2 \log \rho_{i,t}(p_t)}{\partial p_{i,t} \partial p_{j,t}} \geq 0, \ \text{and } \left|\frac{\partial^2 \log \rho_{i,t}(p_t)}{\partial p_{i,t}^2}\right| > \sum_{j \neq i} \frac{\partial^2 \log \rho_{i,t}(p_t)}{\partial p_{i,t} \partial p_{j,t}}. \tag{3.4}$$

The log increasing differences and the diagonal dominance assumptions are not restrictive, and can be satisfied by a large set of commonly used demand models in the economics and operations management literature, such as the linear, logit, Cobb-Douglas, and CES demand functions (see, e.g., [124, 22, 23]).

The expected fill rate of firm i in period t, $z_{i,t}$, is given by

$$z_{i,t} = \frac{\mathbb{E}[x_{i,t}^+ \wedge D_{i,t}(\gamma_t, p_t)]}{\mathbb{E}[D_{i,t}(\gamma_t, p_t)]} = \frac{\mathbb{E}[(\Lambda_{i,t} d_{i,t}(\gamma_t, p_t) y_{i,t})^+ \wedge (\Lambda_{i,t} d_{i,t}(\gamma_t, p_t)\xi_{i,t})]}{\Lambda_{i,t} d_{i,t}(\gamma_t, p_t)} = \mathbb{E}(y_{i,t}^+ \wedge \xi_{i,t}),$$

where $y_{i,t} := \frac{x_{i,t}}{\Lambda_{i,t} d_{i,t}(\gamma_t, p_t)}$ and $a \wedge b := \min\{a, b\}$ for any $a, b \in \mathbb{R}$. Thus, $z_{i,t}$ is concavely increasing in $y_{i,t}$ for all $y_{i,t} \geq 0$. Moreover, $z_{i,t} = 0$ if $y_{i,t} \leq 0$, and $z_{i,t} \uparrow 1$, if $y_{i,t} \to +\infty$.

The key feature of our model is that current promotion, pricing, and inventory decisions impact upon future demands via the service effect and the network effect. To model these two effects, we assume that the market size of each firm in the next period is given by the following functional form:

$$\Lambda_{i,t-1} = \eta_{i,t}(z_t, D_{i,t}, \Lambda_{i,t}, \Xi_{i,t}) = \Lambda_{i,t}\Xi^1_{i,t} + \alpha_{i,t}(z_t)D_{i,t}\Xi^2_{i,t}, \tag{3.5}$$

where $\Xi^1_{i,t}$ is a positive random variable representing the market size changes driven by exogenous factors such as economic environment. Let $\mu_{i,t} := \mathbb{E}[\Xi^1_{i,t}] > 0$. The term $\alpha_{i,t}(z_t)D_{i,t}\Xi^2_{i,t}$ summarizes the service effect and the network effect. Specifically, $\alpha_{i,t}(\cdot) \geq 0$ is a continuously differentiable function with

$$\frac{\partial \alpha_{i,t}(z_t)}{\partial z_{i,t}} \geq 0, \text{ and } \frac{\partial \alpha_{i,t}(z_t)}{\partial z_{j,t}} \leq 0, \text{ for all } j \neq i,$$

and $\Xi^2_{i,t}$ is a nonnegative random variable with $\mathbb{E}[\Xi^2_{i,t}] = 1$. $\Xi^2_{i,t}$ captures the random perturbations in the market size changes driven by the service effect and the network effect. We refer to $\{\alpha_{i,t}(\cdot) : 1 \leq i \leq N, T \geq t \geq 1\}$ as the market size evolution functions. Moreover, for technical tractability, we assume that $\alpha_{i,t}(\cdot)$ is additively separable, i.e.,

$$\alpha_{i,t}(z_t) = \kappa_{ii,t}(z_{i,t}) - \sum_{j \neq i} \kappa_{ij,t}(z_{j,t}),$$

where $\kappa_{ii,t}(\cdot) > 0$ is concave, increasing and continuously differentiable in $z_{i,t}$, and $\kappa_{ij,t}(\cdot) \geq 0$ is continuously increasing in $z_{j,t}$ for all $j \neq i$. Since $\alpha_{i,t}(\cdot) \geq 0$ for all z_t, $\kappa_{ii,t}(0) - \sum_{j \neq i} \kappa_{ij,t}(1) \geq 0$. Let $\eta_t(\cdot,\cdot,\cdot,\cdot) := (\eta_{1,t}(\cdot,\cdot,\cdot,\cdot), \eta_{2,t}(\cdot,\cdot,\cdot,\cdot), \cdots, \eta_{N,t}(\cdot,\cdot,\cdot,\cdot))$ denote the market size vector in the next period.

The evolution of the market sizes, (3.5), has several important implications. First, the future market size of each firm depends on its current market size in a Markovian fashion. Thus, the dynamic competition model in this chapter falls into the regime of Markov games. Second, although the service level of each firm does not influence the current demand of any firm due to the unobservability of the firms' inventory information to customers, it will impact the firms' future demands. This phenomenon is driven by the service effect. The higher the service level of a firm, the better service experience the customers have with this firm in the current period, and the more customers will patronage this firm in the future. Analogously, if the service levels of a firm's competitors increase, customers will be more likely to purchase from its competitors in the future. Therefore, the future demand of each firm is stochastically increasing in the current service level of this firm and stochastically decreasing in the current service level of any of its competitors. Hence, the inventory decision of each firm has the demand-inducing value driven by the service effect. Third, the future demand of each firm is positively correlated with the current demand of this firm. This phenomenon is driven by the network effect. If the realized current demand of a firm is higher, potential customers can get higher utilities if purchasing from this firm, thus giving rise to higher future demand. Because of the network effect, the sales price and promotional effort not only affect the current demand, but also influence future demands. Fourth, the service effect and the network effect reinforce each other. More specifically, the impact of current service levels upon future market sizes is higher with higher realized current demands. With the explosive growth of online social media, customers could easily learn the service qualities of all firms through social learning. As a consequence, higher current demands lead to more intensive social interactions among customers, and, hence, magnify the impact of current service levels on future demands.

We introduce the following model primitives:

δ_i = discount factor of firm i for revenues and costs in future periods, $0 < \delta_i \leq 1$,

$w_{i,t}$ = per-unit wholesales price paid by firm i in period t,

$b_{i,t}$ = per-unit backlogging cost paid by firm i in period t,

$h_{i,t}$ = per-unit holding cost paid by firm i in period t.

Without loss of generality, we assume the following inequalities hold for each i and t:

$b_{i,t} > w_{i,t} - \delta_i w_{i,t-1}$: the backlogging penalty is higher than the saving from delaying an order to the next period for each firm in any period, so that no firm will backlog all of its demand,

$h_{i,t} > \delta_i w_{i,t-1} - w_{i,t}$: the holding cost is sufficiently high so that no firm will place a speculative order.

$\bar{p}_{i,t} > \delta_i w_{i,t-1} + b_{i,t} + \nu_{i,t}(\bar{\gamma}_{i,t})$: positive margin for backlogged demand with highest price and promotional effort.

We define the normalized expected holding and backlogging cost function for firm i in period t:

$$L_{i,t}(y_{i,t}) := \mathbb{E}\{h_{i,t}(y_{i,t} - \xi_{i,t})^+ + b_{i,t}(y_{i,t} - \xi_{i,t})^-\}, \text{ where } y_{i,t} \in \mathbb{R}. \tag{3.6}$$

The state of the Markov game is given by:

$$I_t = (I_{1,t}, I_{2,t}, \cdots, I_{N,t}) = \text{the vector for the starting inventories of all firms in period } t,$$
$$\Lambda_t = (\Lambda_{1,t}, \Lambda_{2,t}, \cdots, \Lambda_{N,t}) = \text{the vector for the market sizes of all firms in period } t.$$

We use $\mathcal{S} := \mathbb{R}^N \times \mathbb{R}^N_+$ to denote the state space of each firm i in the dynamic competition.

To characterize how the market size dynamics (i.e., the service effect and the network effect) impact the equilibrium market outcome, we consider the Markov perfect equilibrium (MPE) in our dynamic competition model. An MPE satisfies two conditions: (a) in each period t, each firm i's promotion, price, and inventory strategy depends on the history of the game only through the current period state variables (I_t, Λ_t), and (b) in each period t, the strategy profile generates a Nash equilibrium in the associated proper subgame. In other words, MPE is a closed-loop equilibrium that satisfies subgame perfection in each period. Because of its simplicity and consistency with rationality, MPE is widely used in dynamic competition models in the economics (e.g., [122]) and operations management (e.g., [131]) literature.

A major technical challenge to characterize the MPE in a dynamic inventory competition model is that when the starting inventories are higher than the equilibrium order-up-to levels, the model becomes illy behaved and analytically intractable (see, e.g., [132]). This issue is worsened under endogenous pricing decisions [25]. To overcome this technical challenge, we make the following assumption throughout our analysis.

Assumption 3.3.1 *At the beginning of each period t, each firm i is allowed to sell (potentially part of) its onhand inventory to its supplier at the current-period per-unit wholesale price $w_{i,t}$.*

Assumption 3.3.1 is imposed to circumvent the aforementioned technical challenge. As will be clear by our subsequent analysis, with this assumption, the equilibrium profit of each firm i in each period t is linearly separable in its starting inventory level $I_{i,t}$ and market size $\Lambda_{i,t}$. Assumption 3.3.1 enables us to eliminate the influence of current inventory decision of any firm upon the future equilibrium behavior of the market, so as to single out and highlight the exploitation-induction tradeoff with the service effect and the network effect. Assumption 3.3.1 applies when the retail firms have such great market power that they can reach an agreement with their respective suppliers on the return policy with full price refund. [25], among others, also make this assumption to characterize the MPE in an infinite-horizon joint price and service level competition model. With Assumption 3.3.1, we can define the action space of each firm i in each period t: $\mathcal{A}_{i,t}(I_{i,t}) := [0, \bar{\gamma}_{i,t}] \times [\underline{p}_{i,t}, \bar{p}_{i,t}] \times [\min\{0, I_{i,t}\}, +\infty)$.

3.4 Simultaneous Competition

In this section, we study the simultaneous competition (SC) model where each firm i simultaneously chooses a combined promotion, price, and inventory strategy in any period t. This model applies to the scenarios where the market expanding efforts (e.g., advertising, trade-in programs, etc.) take effect instantaneously, so, in essence, the promotional effort and sales price decisions are made simultaneously in each period. Our analysis in this section focuses on characterizing the pure strategy MPE and providing insights on the impact of the exploitation-induction tradeoff in the SC model.

3.4.1 Equilibrium Analysis

In this subsection, we show that the simultaneous competition model has a pure strategy MPE. Moreover, we characterize a sufficient condition on the per-unit demand cost rate of promotional effort, $\nu_{i,t}(\cdot)$, under which the MPE is unique. Without loss of generality, we assume that, at the end of the planning horizon, each firm i salvages all the on-hand inventory and fulfills all the backlogged demand at unit wholesale price $w_{i,0} \geq 0$. The payoff function of each firm i is given by:

$$\mathbb{E}\{\sum_{t=1}^{T} \delta_i^{T-t}[p_{i,t}D_{i,t}(\gamma_t, p_t) - w_{i,t}(x_{i,t} - I_{i,t}) - h_{i,t}(x_{i,t} - D_{i,t}(\gamma_t, p_t))^+$$
$$-b_{i,t}(x_{i,t} - D_{i,t}(\gamma_t, p_t))^- - \nu_{i,t}(\gamma_{i,t})D_{i,t}(\gamma_t, p_t)] + \delta_i^T w_{i,0}I_{i,0}|I_T, \Lambda_T\}, \quad (3.7)$$

s.t. $\quad I_{i,t-1} = x_{i,t} - D_{i,t}(\gamma_t, p_t)$ for each t,

and $\quad \Lambda_{i,t-1} = \Lambda_{i,t}\Xi_{i,t}^1 + \alpha_{i,t}(z_t)D_{i,t}(\gamma_t, p_t)\Xi_{i,t}^2$ for each t.

Under an MPE, each firm i should try to maximize its expected payoff in each subgame (i.e., in each period t) conditioned on the realized inventory levels and market sizes in period t, (I_t, Λ_t):

$$\begin{aligned} & \mathbb{E}\{\sum_{\tau=1}^{t} \delta_i^{t-\tau}[p_{i,\tau}D_{i,\tau}(\gamma_\tau, p_\tau) - w_{i,\tau}(x_{i,\tau} - I_{i,\tau}) - h_{i,\tau}(x_{i,\tau} - D_{i,\tau}(\gamma_\tau, p_\tau))^+ \\ & -b_{i,\tau}(x_{i,\tau} - D_{i,\tau}(\gamma_\tau, p_\tau))^- - \nu_{i,\tau}(\gamma_{i,\tau})D_{i,\tau}(\gamma_\tau, p_\tau)] + \delta_i^t w_{i,0}I_{i,0}|I_t, \Lambda_t\}, \quad (3.8) \\ \text{s.t.} \quad & I_{i,\tau-1} = x_{i,\tau} - D_{i,\tau}(\gamma_\tau, p_\tau) \text{ for each } \tau,\ t \geq \tau \geq 1, \\ \text{and} \quad & \Lambda_{i,\tau-1} = \Lambda_{i,\tau}\Xi^1_{i,\tau} + \alpha_{i,\tau}(z_\tau)D_{i,\tau}(\gamma_\tau, p_\tau)\Xi^2_{i,\tau} \text{ for each } \tau,\ t \geq \tau \geq 1. \end{aligned}$$

A (pure) Markov strategy profile in the SC model $\sigma^{sc} := \{\sigma^{sc}_{i,t}(\cdot,\cdot) : 1 \leq i \leq N, T \geq t \geq 1\}$ prescribes each firm i's combined promotion, price, and inventory strategy in each period t, where $\sigma^{sc}_{i,t}(\cdot,\cdot) := (\gamma^{sc}_{i,t}(\cdot,\cdot), p^{sc}_{i,t}(\cdot,\cdot), x^{sc}_{i,t}(\cdot,\cdot))$ is a Borel measurable mapping from $\mathcal{S}$ to $\mathcal{A}_{i,t}(I_{i,t})$. We use $\sigma^{sc}_t := \{\sigma^{sc}_{i,t}(\cdot,\cdot) : 1 \leq i \leq N, T \geq t \geq 1\}$ to denote the pure strategy profile in the induced subgame in period t, which prescribes each firm i's (pure) strategy from period t till the end of the planning horizon.

To evaluate the expected payoff of each firm i in each period t for any given Markov strategy profile σ^{sc} in the simultaneous competition, let

$V_{i,t}(I_t, \Lambda_t|\sigma^{sc}_t) =$ the total expected discounted profit of firm i in periods $t, t-1, \cdots, 1, 0$, when starting period t with the state variable (I_t, Λ_t) and the firms play strategy σ^{sc}_t in periods $t, t-1, \cdots, 1$.

Thus, by backward induction, $V_{i,t}(\cdot,\cdot|\sigma^{sc}_t)$ satisfies the following recursive scheme for each firm i in each period t:

$$V_{i,t}(I_t, \Lambda_t|\sigma^{sc}_t) = J_{i,t}(\gamma^{sc}_t(I_t, \Lambda_t), p^{sc}_t(I_t, \Lambda_t), x^{sc}_t(I_t, \Lambda_t), I_t, \Lambda_t|\sigma^{sc}_{t-1}),$$

where $\gamma^{sc}_t(\cdot,\cdot) = (\gamma^{sc}_{1,t}(\cdot,\cdot), \gamma^{sc}_{2,t}(\cdot,\cdot), \cdots, \gamma^{sc}_{N,t}(\cdot,\cdot))$ is the period t promotional effort vector prescribed by σ^{sc}, $p^{sc}_t(\cdot,\cdot) = (p^{sc}_{1,t}(\cdot,\cdot), p^{sc}_{2,t}(\cdot,\cdot), \cdots, p^{sc}_{N,t}(\cdot,\cdot))$ is the period t sales price vector prescribed by σ^{sc}, $x^{sc}_t(\cdot,\cdot) = (x^{sc}_{1,t}(\cdot,\cdot), x^{sc}_{2,t}(\cdot,\cdot), \cdots, x^{sc}_{N,t}(\cdot,\cdot))$ is the period t post-delivery inventory vector prescribed by σ^{sc},

$$\begin{aligned} J_{i,t}(\gamma_t, p_t, x_t, I_t, \Lambda_t|\sigma^{sc}_{t-1}) =& \mathbb{E}\{p_{i,t}D_{i,t}(\gamma_t, p_t) - w_{i,t}(x_{i,t} - I_{i,t}) - h_{i,t}(x_{i,t} - D_{i,t}(\gamma_t, p_t))^+ \\ & - b_{i,t}(x_{i,t} - D_{i,t}(\gamma_t, p_t))^- - \nu_{i,t}(\gamma_{i,t})D_{i,t}(\gamma_t, p_t) \\ & + \delta_i V_{i,t-1}(x_t - D_t(\gamma_t, p_t), \eta_t(z_t, D_t(\gamma_t, p_t), \Lambda_t, \Xi_t)|\sigma^{sc}_{t-1})|I_t, \Lambda_t\}, \end{aligned} \quad (3.9)$$

and $V_{i,0}(I_t, \Lambda_t) = w_{i,0} I_{i,0}$. We now formally define the pure strategy MPE in the SC model.

Definition 3.4.1 *A (pure) Markov strategy* $\sigma^{sc*} = \{(\gamma_{i,t}^{sc*}(\cdot,\cdot), p_{i,t}^{sc*}(\cdot,\cdot), x_{i,t}^{sc*}(\cdot,\cdot)) : 1 \leq i \leq N, T \geq t \geq 1\}$ *is a pure strategy MPE in the SC model if and only if, for each firm* i*, each period* t*, and each state variable* (I_t, Λ_t)*,*

$$\begin{aligned}&(\gamma_{i,t}^{sc*}(I_t, \Lambda_t), p_{i,t}^{sc*}(I_t, \Lambda_t), x_{i,t}^{sc*}(I_t, \Lambda_t))\\ =&\text{argmax}_{(\gamma_{i,t}, p_{i,t}, x_{i,t}) \in \mathcal{A}_{i,t}(I_{i,t})} \{J_{i,t}([\gamma_{i,t}, \gamma_{-i,t}^{sc*}(I_t, \Lambda_t)], [p_{i,t}, p_{-i,t}^{sc*}(I_t, \Lambda_t)],\\ &[x_{i,t}, x_{-i,t}^{sc*}(I_t, \Lambda_t)], I_t, \Lambda_t | \sigma_{t-1}^{sc*})\}.\end{aligned} \tag{3.10}$$

By Definition 3.4.1, a (pure) Markov strategy profile in the SC model is a pure strategy MPE if it satisfies subgame perfection in each period t. Definition 3.4.1 does not guarantee the existence of an MPE, σ^{sc*}, in the SC model. In Theorem 3.4.1, below, we will show a pure strategy MPE always exists in the SC model. Moreover, under a mild additional assumption on $\nu_{i,t}(\cdot)$, the SC model has a unique pure strategy MPE. By Definition 3.4.1, the equilibrium strategy for firm i in period t, $(\gamma_{i,t}^{sc*}(\cdot,\cdot), p_{i,t}^{sc*}(\cdot,\cdot), x_{i,t}^{sc*}(\cdot,\cdot))$, may depend on the state vector of its competitors $(I_{-i,t}, \Lambda_{-i,t})$. In practice, however, each firm i's starting inventory level $I_{i,t}$ and market size $\Lambda_{i,t}$ are generally its private information that is not accessible by its competitors in the market. We will show that the equilibrium strategy profile of each firm i in each period t is only contingent on its own realized state variables $(I_{i,t}, \Lambda_{i,t})$, but independent of its competitors' private information $(I_{-i,t}, \Lambda_{-i,t})$. The following theorem characterizes the existence and the uniqueness of MPE in the SC model.

Theorem 3.4.1 *The following statements hold for the SC model:*

(a) There exists a pure strategy MPE $\sigma^{sc*} = \{(\gamma_{i,t}^{sc*}(\cdot,\cdot), p_{i,t}^{sc*}(\cdot,\cdot), x_{i,t}^{sc*}(\cdot,\cdot)) : 1 \leq i \leq N, T \geq t \geq 1\}$.

(b) For each pure strategy MPE, σ^{sc*}*, there exists a series of vectors* $\{\beta_t^{sc} : T \geq t \geq 1\}$*, where* $\beta_t^{sc} = (\beta_{1,t}^{sc}, \beta_{2,t}^{sc}, \cdots, \beta_{N,t}^{sc})$ *with* $\beta_{i,t}^{sc} > 0$ *for each* i *and* t*, such that*

$$V_{i,t}(I_t, \Lambda_t | \sigma_t^{sc*}) = w_{i,t} I_{i,t} + \beta_{i,t}^{sc} \Lambda_{i,t}, \text{ for each firm } i \text{ and each period } t. \tag{3.11}$$

(c) If the following two conditions simultaneously hold for each i *and* t*:*

(i) $\nu'_{i,t}(\cdot) \le 1$ *for all* $\gamma_{i,t} \in [0, \bar{\gamma}_{i,t}]$*; and*

(ii) $\nu''_{i,t}(\gamma_{i,t})(p_{i,t} - \delta w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \underline{c}_{i,t}) + [\nu'_{i,t}(\gamma_{i,t})]^2 \ge \nu'_{i,t}(\gamma_{i,t})$ *for all* $p_{i,t} \in [\underline{p}_{i,t}, \bar{p}_{i,t}]$ *and* $\gamma_{i,t} \in [0, \bar{\gamma}_{i,t}]$*, where*

$$\underline{c}_{i,t} := \max\{(\delta_i w_{i,t-1} - w_{i,t}) y_{i,t} - L_{i,t}(y_{i,t}) : y_{i,t} \ge 0\},$$

σ^{sc*} *is the unique MPE in the SC model. In particular, if* $\nu_{i,t}(\gamma_{i,t}) = \gamma_{i,t}$*, conditions (i) and (ii) are satisfied.*

Theorem 3.4.1(a) demonstrates the existence of a pure strategy MPE in the simultaneous competition model. Moreover, in Theorem 3.4.1(b), we show that, for each pure strategy MPE σ^{sc*}, the corresponding profit function of each firm i in each period t is linearly separable in its starting inventory level $I_{i,t}$ and market size $\Lambda_{i,t}$. We refer to the constant $\beta^{sc}_{i,t}$ as the SC market size coefficient of firm i in period t. As we will show later, the SC market size coefficient measures the intensity of the exploitation-induction tradeoff. The larger the $\beta^{sc}_{i,t}$, the more intensive the exploitation-induction tradeoff for firm i in the previous period $t+1$. Theorem 3.4.1(b) also implies that the equilibrium profit of each firm i in each period t only depends on the state variables of itself $(I_{i,t}, \Lambda_{i,t})$, but not on those of its competitors $(I_{-i,t}, \Lambda_{-i,t})$. Theorem 3.4.1(c) characterizes a sufficient condition for the uniqueness of an MPE in the SC model. In particular, if the promotional effort $\gamma_{i,t}$ refers to the actual monetary payment of promotional investment per-unit demand for each firm i in each period t (i.e., $\nu_{i,t}(\gamma_{i,t}) = \gamma_{i,t}$ for each i and t), there exists a unique MPE in the SC model. For the rest of this chapter, we assume that conditions (i) and (ii) are satisfied for each i and t and, hence, the SC model has a unique pure strategy MPE σ^{sc*}.

The linear separability of $V_{i,t}(\cdot,\cdot|\sigma^{sc*}_t)$ (i.e., Theorem 3.4.1(b)) enables us to characterize the MPE in the SC model. Plugging (3.11) into the objective function of firm i in period t, by $x_{i,t} = \Lambda_{i,t} d_{i,t}(\gamma_t, p_t) y_{i,t}$ and $z_{i,t} = \mathbb{E}(y^+_{i,t} \wedge \xi_{i,t})$, we have:

$$
\begin{aligned}
J_{i,t}(\gamma_t, p_t, x_t, I_t, \Lambda_t|\sigma_{t-1}^{sc*}) =& \mathbb{E}\{p_{i,t}D_{i,t}(\gamma_t, p_t) - w_{i,t}(x_{i,t} - I_{i,t}) - h_{i,t}(x_{i,t} - D_{i,t}(\gamma_t, p_t))^+ \\
& - b_{i,t}(x_{i,t} - D_{i,t}(\gamma_t, p_t))^- - \nu_{i,t}(\gamma_{i,t})D_{i,t}(\gamma_t, p_t) \\
& + \delta_i V_{i,t-1}(x_t - D_t(\gamma_t, p_t), \eta_t(z_t, D_t(\gamma_t, p_t), \Lambda_t, \Xi_t)|\sigma_{t-1}^{sc*})|I_t, \Lambda_t\} \\
=& \mathbb{E}\{p_{i,t}\Lambda_{i,t}d_{i,t}(\gamma_t, p_t)\xi_{i,t} - w_{i,t}(y_{i,t}\Lambda_{i,t}d_{i,t}(\gamma_t, p_t) - I_{i,t}) \\
& - h_{i,t}(y_{i,t}\Lambda_{i,t}d_{i,t}(\gamma_t, p_t) - \Lambda_{i,t}d_{i,t}(\gamma_t, p_t)\xi_{i,t})^+ \\
& - b_{i,t}(y_{i,t}\Lambda_{i,t}d_{i,t}(\gamma_t, p_t) - \Lambda_{i,t}d_{i,t}(\gamma_t, p_t)\xi_{i,t})^- \\
& - \nu_{i,t}(\gamma_{i,t})\Lambda_{i,t}d_{i,t}(\gamma_t, p_t)\xi_{i,t} + \delta_i w_{i,t-1}(y_{i,t}\Lambda_{i,t}d_{i,t}(\gamma_t, p_t) \\
& - \Lambda_{i,t}d_{i,t}(\gamma_t, p_t)\xi_{i,t}) \\
& + \delta_i \beta_{i,t-1}^{sc}(\Lambda_{i,t}\Xi_{i,t}^1 + \alpha_{i,t}(z_t)\Lambda_{i,t}d_{i,t}(\gamma_t, p_t)\xi_{i,t}\Xi_{i,t}^2)|I_t, \Lambda_t\} \\
=& w_{i,t}I_{i,t} + \Lambda_{i,t}\{\delta_i\beta_{i,t-1}^{sc}\mu_{i,t} \\
& + \psi_{i,t}(\gamma_t)\rho_{i,t}(p_t)[p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{sc}(y_t)]\}, \\
\text{where } \pi_{i,t}^{sc}(y_t) =& (\delta_i w_{i,t-1} - w_{i,t})y_{i,t} - L_{i,t}(y_{i,t}) + \delta_i\beta_{i,t-1}^{sc}(\kappa_{ii,t}(\mathbb{E}[y_{i,t}^+ \wedge \xi_{i,t}]) \\
& - \sum_{j\neq i}\kappa_{ij,t}(\mathbb{E}[y_{j,t}^+ \wedge \xi_{j,t}])), \\
\text{and } \beta_{i,0}^{sc} :=& 0 \text{ for each } i.
\end{aligned}
\tag{3.12}
$$

We observe from (3.12) that the payoff function of each firm i in the subgame of period t has a nested structure. Hence, the subgame of period t can be decomposed into two stages, where the firms compete jointly on promotion and price in the first stage, and on inventory in the second stage. Since the service level of each firm i, as measured by the expected fill rate $z_{i,t}$, is increasing in the inventory decision $y_{i,t}$, we refer to the second-stage competition as the service level competition hereafter. By backward induction, we first study the second-stage service level competition. Let $\mathcal{G}_t^{sc,2}$ be the $N-$player noncooperative game that represents the second-stage service level competition in period t, where player i has payoff function $\pi_{i,t}^{sc}(\cdot)$ and feasible action set $\mathbb{R}$. The following proposition characterizes the Nash equilibrium of the game $\mathcal{G}_t^{sc,2}$.

Proposition 3.4.1 *For each period t, the second-stage service level competition $\mathcal{G}_t^{sc,2}$ has a unique pure strategy Nash equilibrium y_t^{sc*}. Moreover, for each i, $y_{i,t}^{sc*} > 0$ is the unique solution to the following equation:*

$$(\delta_i w_{i,t-1} - w_{i,t}) - L'_{i,t}(y_{i,t}^{sc*}) + \delta_i \beta_{i,t-1}^{sc} \bar{F}_{i,t}(y_{i,t}^{sc*}) \kappa'_{ii,t}(\mathbb{E}(y_{i,t}^{sc*} \wedge \xi_{i,t})) = 0. \tag{3.13}$$

Proposition 3.4.1 demonstrates the existence and uniqueness of a pure strategy Nash equilibrium of the second-stage service level competition. Moreover, $y_{i,t}^{sc*}$ can be obtained by solving the first-order condition $\partial_{y_{i,t}} \pi_{i,t}^{sc}(y_t^{sc*}) = 0$. Let $\pi_t^{sc*} := (\pi_{1,t}^{sc*}, \pi_{2,t}^{sc*}, \cdots, \pi_{N,t}^{sc*})$ be the equilibrium payoff vector of the second-stage service level competition in period t, where $\pi_{i,t}^{sc*} = \pi_{i,t}^{sc}(y_t^{sc*})$. For each i and t, let

$$\Pi_{i,t}^{sc}(\gamma_t, p_t) := \psi_{i,t}(\gamma_t)\rho_{i,t}(p_t)[p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{sc*}]. \tag{3.14}$$

We define an $N-$player noncooperative game $\mathcal{G}_t^{sc,1}$ to represent the first-stage joint promotion and price competition in period t, where player i has payoff function $\Pi_{i,t}^{sc}(\cdot,\cdot)$ and feasible action set $[0, \bar{\gamma}_{i,t}] \times [\underline{p}_{i,t}, \bar{p}_{i,t}]$. We characterize the Nash equilibrium of the game $\mathcal{G}_t^{sc,1}$ in the following proposition.

Proposition 3.4.2 *For each period t, following statements hold:*

(a) The first-stage joint promotion and price competition, $\mathcal{G}_t^{sc,1}$, is a log-supermodular game.

(b) The game $\mathcal{G}_t^{sc,1}$ has a unique pure strategy Nash equilibrium $(\gamma_t^{sc*}, p_t^{sc*})$, which is the unique serially undominated strategy of $\mathcal{G}_t^{sc,1}$.

(c) The Nash equilibrium of $\mathcal{G}_t^{sc,1}$ is the unique solution to the following system of equations: For each i

$$\frac{\partial_{\gamma_{i,t}}\psi_{i,t}(\gamma_t^{sc*})}{\psi_{i,t}(\gamma_t^{sc*})} - \frac{\nu'_{i,t}(\gamma_{i,t}^{sc*})}{p_{i,t}^{sc*} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}^{sc*}) + \pi_{i,t}^{sc*}} \begin{cases} \leq 0, & \textit{if } \gamma_{i,t}^{sc*} = 0, \\ = 0, & \textit{if } \gamma_{i,t}^{sc*} \in (0, \bar{\gamma}_{i,t}), \quad \textit{and,} \\ \geq 0 & \textit{if } \gamma_{i,t}^{sc*} = \bar{\gamma}_{i,t}; \end{cases}$$

$$\frac{\partial_{p_{i,t}}\rho_{i,t}(p_t^{sc*})}{\rho_{i,t}(p_t^{sc*})} + \frac{1}{p_{i,t}^{sc*} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}^{sc*}) + \pi_{i,t}^{sc*}} \begin{cases} \leq 0, & \textit{if } p_{i,t}^{sc*} = \underline{p}_{i,t}, \\ = 0, & \textit{if } p_{i,t}^{sc*} \in (\underline{p}_{i,t}, \bar{p}_{i,t}), \\ \geq 0 & \textit{if } p_{i,t}^{sc*} = \bar{p}_{i,t}. \end{cases} \tag{3.15}$$

(d) Let $\Pi_t^{sc*} := (\Pi_{1,t}^{sc*}, \Pi_{2,t}^{sc*}, \cdots, \Pi_{N,t}^{sc*})$ be the equilibrium payoff vector of the first-stage joint promotion and price competition in period t, where $\Pi_{i,t}^{sc*} = \Pi_{i,t}^{sc}(\gamma_t^{sc*}, p_t^{sc*})$. We have $\Pi_{i,t}^{sc*} > 0$ for all i.

Proposition 3.4.2 shows that the first-stage joint promotion and price competition $\mathcal{G}_t^{sc,1}$ is a log-supermodular game, and has a unique pure strategy Nash equilibrium $(\gamma_t^{sc*}, p_t^{sc*})$. The unique Nash equilibrium, $(\gamma_t^{sc*}, p_t^{sc*})$, is determined by (i) the serial elimination of strictly dominated strategies, or (ii) the system of first-order conditions (3.15). Under equilibrium, by Proposition 3.4.2(d) and the objective function of period t, (3.12), each firm i earns a positive normalized expected total discounted profit, $\Lambda_{i,t}(\delta_i \beta_{i,t-1}^{sc} \mu_{i,t} + \Pi_{i,t}^{sc*})$, in the subgame of period t. Summarizing Theorem 3.4.1, Proposition 3.4.1 and Proposition 3.4.2, we have the following theorem that sharpens the characterization of the MPE in the SC model.

Theorem 3.4.2 *For each period t, the following statements hold:*

(a) For each i, $\beta_{i,t}^{sc} = \delta_i \beta_{i,t-1}^{sc} \mu_{i,t} + \Pi_{i,t}^{sc*}$.

(b) Under the unique (pure strategy) MPE σ^{sc*}, the policy of firm i is given by

$$(\gamma_{i,t}^{sc*}(I_t, \Lambda_t), p_{i,t}^{sc*}(I_t, \Lambda_t), x_{i,t}^{sc*}(I_t, \Lambda_t)) = (\gamma_{i,t}^{sc*}, p_{i,t}^{sc*}, \Lambda_{i,t} y_{i,t}^{sc*} \rho_{i,t}(p_t^{sc*}) \psi_{i,t}(\gamma_t^{sc*})). \quad (3.16)$$

Theorem 3.4.2(a) recursively computes the SC market size coefficient vectors $\{\beta_t^{sc} : T \geq t \geq 1\}$. Theorem 3.4.2(b) demonstrates that, under the MPE σ^{sc*}, each firm i's joint promotion, price, and inventory policy in each period t only depends on its own state variables $(I_{i,t}, \Lambda_{i,t})$, but not on those of its competitors $(I_{-i,t}, \Lambda_{-i,t})$, which are not accessible to firm i in general. Thus, for each firm i in each period t, its equilibrium strategy has the attractive feature that the strategy depends on its accessible information only.

In some of our analysis below, we will consider a special case of the SC model, where the market is symmetric, i.e., all competing firms have identical characteristics. We use the subscript "s" to denote the case of symmetric market. In this case, for all i, j, and t, let $\rho_{s,t}(\cdot) := \rho_{i,t}(\cdot)$, $\psi_{s,t}(\cdot) := \psi_{i,t}(\cdot)$, $\nu_{s,t}(\cdot) := \nu_{i,t}(\cdot)$, $\alpha_{s,t}(\cdot) := \alpha_{i,t}(\cdot)$, $\kappa_{sa,t}(\cdot) := \kappa_{ii,t}(\cdot)$, $\kappa_{sb,t}(\cdot) := \kappa_{ij,t}(\cdot)$, $w_{s,t} := w_{i,t}$, $h_{s,t} := h_{i,t}$, $b_{s,t} := b_{i,t}$, $\mu_{s,t} := \mu_{i,t}$, and $\delta_s := \delta_i$. Thus, let $L_{s,t}(\cdot) := L_{i,t}(\cdot)$ for each i. As shown in Theorem 3.4.1, there exists a unique pure strategy

MPE in the symmetric SC model, which we denote as σ_s^{sc*}. The following proposition is a corollary of Theorems 3.4.1-3.4.2.

Proposition 3.4.3 *The following statements hold for the symmetric SC model:*

(a) For each $t = T, T-1, \cdots, 1$, there exists a constant $\beta_{s,t}^{sc} > 0$, such that

$$V_{i,t}(I_t, \Lambda_t | \sigma_{s,t}^{sc*}) = w_{s,t} I_{i,t} + \beta_{s,t}^{sc} \Lambda_{i,t}, \text{ for all } i.$$

(b) In each period t, the second-stage service level competition $\mathcal{G}_{s,t}^{sc,2}$ is symmetric, with the payoff function for each firm i given by

$$\pi_{i,t}^{sc}(y_t) = (\delta_s w_{s,t-1} - w_{s,t}) y_{i,t} - L_{s,t}(y_{i,t}) + \delta_s \beta_{s,t-1}^{sc} (\kappa_{sa,t}(\mathbb{E}[y_{i,t}^+ \wedge \xi_{i,t}]) - \sum_{j \neq i} \kappa_{sb,t}(\mathbb{E}[y_{j,t}^+ \wedge \xi_{j,t}])).$$

Moreover, $\mathcal{G}_{s,t}^{sc,2}$ has a unique pure strategy Nash equilibrium which is symmetric, so we use $y_{s,t}^{sc*}$ [$\pi_{s,t}^{sc*}$] to denote the equilibrium strategy [payoff] of each firm in $\mathcal{G}_{s,t}^{sc,2}$.

(c) In each period t, the first-stage joint promotion and price competition $\mathcal{G}_{s,t}^{sc,1}$ is symmetric, with the payoff function for each firm i given by

$$\Pi_{i,t}^{sc}(\gamma_t, p_t) = \psi_{s,t}(\gamma_t) \rho_{s,t}(p_t) [p_{i,t} - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma_{i,t}) + \pi_{s,t}^{sc*}].$$

Moreover, $\mathcal{G}_{s,t}^{sc,1}$ has a unique pure strategy Nash equilibrium $(\gamma_{ss,t}^{sc*}, p_{ss,t}^{sc*})$ which is symmetric (i.e., $\gamma_{ss,t}^{sc*} = (\gamma_{s,t}^{sc*}, \gamma_{s,t}^{sc*}, \cdots, \gamma_{s,t}^{sc*})$ for some $\gamma_{s,t}^{sc*}$ and $p_{ss,t}^{sc*} = (p_{s,t}^{sc*}, p_{s,t}^{sc*}, \cdots, p_{s,t}^{sc*})$ for some $p_{s,t}^{sc*}$).

(d) Under the unique pure strategy MPE, σ_s^{sc*}, the policy of firm i in period t is

$$(\gamma_{i,t}^{sc*}(I_t, \Lambda_t), p_{i,t}^{sc*}(I_t, \Lambda_t), x_{i,t}^{sc*}(I_t, \Lambda_t)) = (\gamma_{s,t}^{sc*}, p_{s,t}^{sc*}, \Lambda_{i,t} y_{s,t}^{sc*} \rho_{s,t}(p_{ss,t}^{sc*}) \psi_{s,t}(\gamma_{ss,t}^{sc*})),$$

for each (I_t, Λ_t).

Proposition 3.4.3 characterizes the MPE, σ_s^{sc*}, and the market size coefficients, $\{\beta_{s,t}^{sc} : T \geq t \geq 1\}$, in the symmetric SC model. Proposition 3.4.3 shows that, in the symmetric SC model, all competing firms set the same promotional effort, sales price, and service level in each period under equilibrium, whereas the equilibrium market outcome may vary in different periods.

3.4.2 Exploitation-Induction Tradeoff

In this subsection, we study how the market size dynamics (i.e., the service effect and the network effect) influence the equilibrium market outcome in the SC model. We focus on the managerial implications of the exploitation-induction tradeoff in a dynamic and competitive market.

To begin with, we characterize the impact of the market size coefficient vectors $\{\beta_t^{sc} : T \geq t \geq 1\}$ upon the equilibrium market outcome. The following theorem serves as the building block of our subsequent analysis of the exploitation-induction tradeoff in the SC model.

Theorem 3.4.3 *For each period t, the following statements hold:*

(a) For each i and $j \neq i$, $y_{i,t}^{sc*}$ is continuously increasing in $\beta_{i,t-1}^{sc}$ and independent of $\beta_{j,t-1}^{sc}$.

(b) For each i and $j \neq i$, $\pi_{i,t}^{sc*}$ is continuously increasing in $\beta_{i,t-1}^{sc}$ and continuously decreasing in $\beta_{j,t-1}^{sc}$.

(c) If the SC model is symmetric, $\gamma_{s,t}^{sc*}$ is continuously increasing in $\pi_{s,t}^{sc*}$, whereas $p_{s,t}^{sc*}$ is continuously decreasing in $\pi_{s,t}^{sc*}$.

(d) If the SC model is symmetric and $\psi_{s,t}(\cdot)$ and $\rho_{s,t}(\cdot)$ satisfy the following monotonicity condition

$$\sum_{i=1}^{N} \frac{\partial \psi_{s,t}(\gamma_t)}{\partial \gamma_{i,t}} > 0, \text{ for all } \gamma_t, \text{ and } \sum_{i=1}^{N} \frac{\partial \rho_{s,t}(p_t)}{\partial p_{i,t}} < 0, \text{ for all } p_t, \tag{3.17}$$

$\beta_{s,t}^{sc}$ is continuously increasing in $\pi_{s,t}^{sc*}$.

(e) If the SC model is symmetric and $\pi_{s,t}^{sc*}$ is increasing in $\beta_{s,t-1}^{sc}$, $\gamma_{s,t}^{sc*}$ is continuously increasing in $\beta_{s,t-1}^{sc}$, whereas $p_{s,t}^{sc*}$ is continuously decreasing in $\beta_{s,t-1}^{sc}$.

(f) In the symmetric SC model, if the monotonicity condition (3.17) holds and $\pi_{s,t}^{sc*}$ is increasing in $\beta_{s,t-1}^{sc}$, $\beta_{s,t}^{sc}$ is continuously increasing in $\beta_{s,t-1}^{sc}$.

Theorem 3.4.3 shows that the market size coefficients $\{\beta_{i,t}^{sc} : 1 \leq i \leq N, T \geq t \geq 1\}$ quantify the intensity of the exploitation-induction tradeoff in the SC model. More specifically, if $\beta_{i,t-1}^{sc}$ is larger, firm i faces stronger exploitation-induction tradeoff in period

t. Therefore, to balance this strengthened tradeoff and to induce high future demands, each firm should improve service quality, decrease sales price, and increase promotional effort, as shown in parts (a) and (e) of Theorem 3.4.3. Moreover, Theorem 3.4.3(f) characterizes the relationship between the exploitation-induction tradeoffs in different periods, demonstrating that if the exploitation-induction tradeoff is more intensive in the next period, it is also stronger in the current period under a mild condition. The monotonicity condition (3.17) implies that a uniform increase of all N firms' promotional efforts leads to an increase in the demand of each firm, and a uniform price increase by all N firms gives rise to a decrease in the demand of each firm. This condition is commonly used in the literature (see, e.g., [23, 9]), and often referred to as the "dominant diagonal" condition for linear demand models. The assumption that $\pi_{s,t}^{sc*}$ is increasing in $\beta_{s,t-1}^{sc}$ is not restrictive either. In Lemma 26 in Appendix B.2, we give some sufficient conditions for this assumption. More specifically, Lemma 26 implies that $\pi_{s,t}^{sc*}$ is increasing in $\beta_{s,t-1}^{sc}$ if one of the following conditions holds: (i) The adverse effect of a firm's competitors' service upon its future market size is not strong; (ii) the network effect is sufficiently strong; or (iii) both the service effect and the network effect are sufficiently strong.

Now we consider a benchmark case without the service effect and the network effect. We use " $\tilde{}$ " to denote this model. Thus, in the benchmark model, the market size evolution function $\tilde{\alpha}_{i,t}(\cdot) \equiv 0$ for each firm i and each period t. Without the service effect and the network effect, the current promotion, price, and service level decisions of any firm will not influence the future demands. Therefore, the competing firms can focus on generating current profits in each period without considering inducing future demands, i.e., the exploitation-induction tradeoff is absent in this benchmark case. To characterize the impact of the service effect and the network effect upon the equilibrium outcome, the following theorem compares the Nash equilibria in $\mathcal{G}_t^{sc,2}$ and $\tilde{\mathcal{G}}_t^{sc,2}$, and the Nash equilibria in $\mathcal{G}_t^{sc,1}$ and $\tilde{\mathcal{G}}_t^{sc,1}$.

Theorem 3.4.4 *(a) For each firm i and each period t, $y_{i,t}^{sc*} \geq \tilde{y}_{i,t}^{sc*}$, $z_{i,t}^{sc*} \geq \tilde{z}_{i,t}^{sc*}$, and $\pi_{i,t}^{sc*} \geq \tilde{\pi}_{i,t}^{sc*}$.*

(b) Consider the symmetric SC model. For each period t, the following statements hold:

(i) $\gamma_{s,t}^{sc*} \geq \tilde{\gamma}_{s,t}^{sc*}$ and, thus, $\gamma_{i,t}^{sc*}(I_t, \Lambda_t) \geq \tilde{\gamma}_{i,t}^{sc*}(I_t, \Lambda_t)$ for all i and all (I_t, Λ_t).

(ii) $p_{s,t}^{sc*} \leq \tilde{p}_{s,t}^{sc*}$ and, thus, $p_{i,t}^{sc*}(I_t, \Lambda_t) \leq \tilde{p}_{i,t}^{sc*}(I_t, \Lambda_t)$ for all i and all (I_t, Λ_t).

(iii) If the monotonicity condition (3.17) *holds, we have* $x^{sc*}_{i,t}(I_t, \Lambda_t) \geq \tilde{x}^{sc*}_{i,t}(I_t, \Lambda_t)$ *for all* i *and all* (I_t, Λ_t).

Theorem 3.4.4 highlights the impact of market size dynamics upon the equilibrium market outcome. Specifically, Theorem 3.4.4(a) shows that, under the service effect and the network effect, each firm i should set a higher service level in each period t. In the symmetric SC model, Theorem 3.4.4(b-i) shows that each firm should increase its promotional effort in each period under the service effect and the network effect, in order to induce higher future demands. Analogously, Theorem 3.4.4(b-ii) shows that the service effect and the network effect give rise to lower equilibrium sales price of each firm in each period. Under the monotonicity condition (3.17), Theorem 3.4.4(b-i,ii) implies that the equilibrium expected demand of each firm in each period is higher under the service effect and the network effect. As a consequence, to match supply with the current demand and to induce future demands with the service effect, each firm should increase its base stock level in each period under the service effect and the network effect, as shown in Theorem 3.4.4(b-iii).

Theorem 3.4.4 identifies effective strategies for firms to balance the exploitation-induction tradeoff under both the service effect and the network effect. In this case, the competing firms have to tradeoff generating current profits and inducing future demands. To balance the exploitation-induction trade-off, the firms can employ three strategies to exploit the service effect and the network effect: (a) elevating service levels, (b) offering price discounts, and (c) improving promotional efforts. Elevating service levels does not lead to a higher current demand, but helps the firm induce higher future demands via the service effect. Offering price discounts and improving promotional efforts do not increase the current profits but give rise to higher current demands and, thus, induce higher future demands via the network effect. In a nutshell, the uniform idea of all three strategies is that, to balance the exploitation-induction tradeoff under the service effect and the network effect, the competing firms should induce higher future demands at the cost of reduced current margins.

To deliver sharper insights on the managerial implications of the exploitation-induction tradeoff, we confine ourselves to the symmetric SC model for the rest of this section. The following theorem characterizes how the intensities of the service effect and the network effect influence the equilibrium market outcome in the symmetric SC model.

Theorem 3.4.5 *Let two symmetric SC models be identical except that one with market size evolution functions* $\{\hat{\alpha}_{s,t}(\cdot)\}_{T\geq t\geq 1}$ *and the other with* $\{\alpha_{s,t}(\cdot)\}_{T\geq t\geq 1}$*. Assume that, for each period* t*, (i) the monotonicity condition* (3.17) *holds, and (ii)* $\kappa_{sb,t}(\cdot) \equiv \kappa^0_{sb,t}$ *for some constant* $\kappa^0_{sb,t}$*.*

(a) If $\hat{\alpha}_{s,t}(z_t) \geq \alpha_{s,t}(z_t)$ *for each period* t *and each* z_t*, we have, for each period* t*,* $\hat{\beta}^{sc}_{s,t} \geq \beta^{sc}_{s,t}$*,* $\hat{\gamma}^{sc*}_{s,t} \geq \gamma^{sc*}_{s,t}$*, and* $\hat{p}^{sc*}_{s,t} \leq p^{sc*}_{s,t}$*. Thus, for each period* t*,* $\hat{\gamma}^{sc*}_{i,t}(I_t, \Lambda_t) \geq \gamma^{sc*}_{i,t}(I_t, \Lambda_t)$ *and* $\hat{p}^{sc*}_{i,t}(I_t, \Lambda_t) \leq p^{sc*}_{i,t}(I_t, \Lambda_t)$ *for all* i *and all* $(I_t, \Lambda_t) \in \mathcal{S}$*.*

(b) If, for each period t*,* $\hat{\alpha}_{s,t}(z_t) \geq \alpha_{s,t}(z_t)$ *for all* z_t *and* $\hat{\kappa}'_{sa,t}(z_{i,t}) \geq \kappa'_{sa,t}(z_{i,t}) \geq 0$ *for all* $z_{i,t}$*, we have, for each period* t*,* $\hat{\beta}^{sc}_{s,t} \geq \beta^{sc}_{s,t}$*,* $\hat{\gamma}^{sc*}_{s,t} \geq \gamma^{sc*}_{s,t}$*,* $\hat{p}^{sc*}_{s,t} \leq p^{sc*}_{s,t}$*, and* $\hat{y}^{sc*}_{s,t} \geq y^{sc*}_{s,t}$*. Thus, for each period* t*,* $\hat{\gamma}^{sc*}_{i,t}(I_t, \Lambda_t) \geq \gamma^{sc*}_{i,t}(I_t, \Lambda_t)$*,* $\hat{p}^{sc*}_{i,t}(I_t, \Lambda_t) \leq p^{sc*}_{i,t}(I_t, \Lambda_t)$*, and* $\hat{x}^{sc*}_{i,t}(I_t, \Lambda_t) \geq x^{sc*}_{i,t}(I_t, \Lambda_t)$ *for all* i *and all* $(I_t, \Lambda_t) \in \mathcal{S}$*.*

Theorem 3.4.5 sharpens Theorem 3.4.4 by showing that if the intensities of the network effect and the service effect (captured by the magnitudes of $\alpha_{s,t}(\cdot)$ and $\kappa'_{sa,t}(\cdot)$, respectively) are higher, the exploitation-induction tradeoff becomes stronger. To balance the strengthened exploitation-induction tradeoff, each firm should increase its promotional effort, decrease its sales price, and improve its service level in each period. More specifically, Theorem 3.4.5(a) shows that a higher intensity of the network effect (i.e., larger $\alpha_{s,t}(\cdot)$) drives all the firms to make more promotional efforts and charge lower sales prices. Theorem 3.4.5(b) further suggests that higher intensities of both the network effect and the service effect (i.e., larger $\alpha_{s,t}(\cdot)$ and $\kappa'_{sa,t}(\cdot)$) prompt all the firms to make more promotional efforts, charge lower sales prices, and maintain higher service levels. Stronger service effect and network effect intensify the exploitation-induction tradeoff, thus driving the firms to put more weight on inducing future demands than on exploiting the current market. Therefore, to effectively balance the exploitation-induction tradeoff, all the firms should carefully examine the intensities of the service effect and the network effect.

Next, we analyze the exploitation-induction tradeoff from an inter-temporal perspective. Under the service effect and the network effect, how should the competing firms adjust their promotion, price, and service strategies throughout the sales season to balance the exploitation-induction tradeoff? To address this question, we characterize the evolution of the equilibrium market outcome in the stationary and symmetric SC model. In this model, the model primitives, demand functions, and market size evolution functions are identical for all firms throughout the planning horizon. In addition, the random perturbations in market demands and market size evolution are *i.i.d.* throughout the planning horizon. The following theorem characterizes the evolution of the equilibrium promotion, price, and service strategy in the stationary and symmetric SC model.

Theorem 3.4.6 *Consider the stationary and symmetric SC model. Assume that, for each period t, (i) the monotonicity condition* (3.17) *holds, and (ii)* $\pi^{sc*}_{s,t}$ *is increasing in* $\beta^{sc}_{s,t-1}$*. For each period t, the following statements hold:*

(a) $\beta^{sc}_{s,t} \geq \beta^{sc}_{s,t-1}$, $\gamma^{sc*}_{s,t} \geq \gamma^{sc*}_{s,t-1}$, $p^{sc*}_{s,t} \leq p^{sc*}_{s,t-1}$, *and* $y^{sc*}_{s,t} \geq y^{sc*}_{s,t-1}$.

(b) $\gamma^{sc*}_{i,t}(I,\Lambda) \geq \gamma^{sc*}_{i,t-1}(I,\Lambda)$, $p^{sc*}_{i,t}(I,\Lambda) \leq p^{sc*}_{i,t-1}(I,\Lambda)$, *and* $x^{sc*}_{i,t}(I,\Lambda) \geq x^{sc*}_{i,t-1}(I,\Lambda)$ *for each i and each* $(I,\Lambda) \in \mathcal{S}$.

Theorem 3.4.6 sheds light on how to balance the exploitation-induction tradeoff from an inter-temporal perspective. More specifically, we show that, if the market is symmetric and stationary, the exploitation-induction tradeoff is more intensive (i.e., $\beta^{sc}_{s,t}$ is larger) at the early stage of the sales season. Moreover, the equilibrium sales price is increasing, whereas the equilibrium promotional effort and service level are decreasing, over the planning horizon. The service effect and the network effect have greater impacts upon future demands (and, hence, future profits) when the remaining planning horizon is longer. Therefore, to adaptively balance the exploitation-induction tradeoff throughout the sales season, all the firms increase their sales prices and decrease their promotional efforts and service levels towards the end of the sales season. Our analysis justifies the widely used introductory price and promotion strategy with which firms offer discounts and launch promotional campaigns at the beginning of a sales season to attract more early purchases (see, e.g., [38, 134, 65]).

To summarize, under the service effect and the network effect, the competing firms have to trade off between generating current profits and inducing future demands. To effectively balance the exploitation-induction tradeoff, the firms should (a) increase promotional efforts, (b) offer price discounts, and (c) improve service levels. Moreover, the exploitation-induction tradeoff is more intensive (a) with stronger service effect and network effect, or (b) at the early stage of the sales season.

3.5 Promotion-First Competition

In this section, we consider the promotion-first competition (PF) model, i.e., in each period t, each firm i first selects its promotional effort and then, after observing the current-period promotional efforts of all firms, chooses a combined sales price and service level strategy. This model is suitable for the scenario in which the stickiness of market expanding choices is much higher than that of sales price and service level choices. For example, due to the long leadtime for technology development, decisions on research and development effort are usually made well in advance of sales price and service level decisions.

Employing the linear separability approach, we will show that, in the PF model, the firms engage in a three-stage competition in each period, the first stage on promotional effort, the second on sales price, and the last on service level. We will also demonstrate that the exploitation-induction tradeoff has more involved managerial implications in the PF model than its implications in the SC model. In the SC model, the competing firms balance the exploitation-induction tradeoff inter-temporally, whereas the firms in the PF model balance this tradeoff both inter-temporally and intra-temporally.

For tractability, we make the following additional assumption throughout this section:

$$\rho_{i,t}(p_t) = \phi_{i,t} - \theta_{ii,t}p_{i,t} + \sum_{j \neq i} \theta_{ij,t}p_{j,t}, \text{ for each } i \text{ and } t, \tag{3.18}$$

where $\phi_{i,t}, \theta_{ii,t} > 0$ and $\theta_{ij,t} \geq 0$ for each i, j, and t. Moreover, we assume that the diagonal dominance conditions hold for each $\rho_{i,t}(\cdot)$, i.e., for each i and t, $\theta_{ii,t} > \sum_{j \neq i} \theta_{ij,t}$ and $\theta_{ii,t} > \sum_{j \neq i} \theta_{ji,t}$. In addition, we make the same assumption as [9] as follows:

Assumption 3.5.1 *For each i and t, the minimum [maximum] allowable price $\underline{p}_{i,t}$ [$\bar{p}_{i,t}$] is sufficiently low [high] so that it will have no impact on the equilibrium market behavior.*

We will give a sufficient condition for Assumption 3.5.1 in the discussion after Proposition 3.5.2.

3.5.1 Equilibrium Analysis

In this subsection, we use the linear separability approach to characterize the pure strategy MPE in the PF model. In this model, a (pure) Markov strategy profile of firm i in period t is given by $\sigma_{i,t}^{pf} = (\gamma_{i,t}^{pf}(\cdot,\cdot), p_{i,t}^{pf}(\cdot,\cdot,\cdot), x_{i,t}^{pf}(\cdot,\cdot,\cdot))$, where $\gamma_{i,t}^{pf}(I_t, \Lambda_t)$ prescribes the promotional effort given the state variable (I_t, Λ_t), and $(p_{i,t}^{pf}(I_t, \Lambda_t, \gamma_t), x_{i,t}^{pf}(I_t, \Lambda_t, \gamma_t))$ prescribes the sales price and the post-delivery inventory level, given the state variable (I_t, Λ_t) and the current period promotional effort vector γ_t. Let $\gamma_t^{pf}(\cdot,\cdot) := (\gamma_{1,t}^{pf}(\cdot,\cdot), \cdots, \gamma_{N,t}^{pf}(\cdot,\cdot))$, $p_t^{pf}(\cdot,\cdot,\cdot) := (p_{1,t}^{pf}(\cdot,\cdot,\cdot), \cdots, p_{N,t}^{pf}(\cdot,\cdot,\cdot))$, and $x_t^{pf}(\cdot,\cdot,\cdot) := (x_{1,t}^{pf}(\cdot,\cdot,\cdot), \cdots, x_{N,t}^{pf}(\cdot,\cdot,\cdot))$. We use σ_t^{pf} to denote the (pure) strategy profile of all firms in the subgame of period t, which prescribes their (pure) strategies from period t to the end of the planning horizon.

To evaluate the expected payoff of each firm i in each period t for any given Markov strategy profile σ^{pf} in the PF model, let

$V_{i,t}(I_t, \Lambda_t | \sigma_t^{pf}) =$ the total expected discounted profit of firm i in periods $t, \cdots, 0$, when starting period t with the state variable (I_t, Λ_t) and the firms play strategy σ_t^{pf} in periods $t, t-1, \cdots, 1$.

Thus, by backward induction, $V_{i,t}(\cdot,\cdot|\sigma_t^{pf})$ satisfies the following recursive scheme for each firm i and each period t:

$$V_{i,t}(I_t, \Lambda_t|\sigma_t^{pf}) = J_{i,t}(\gamma_t^{pf}(I_t, \Lambda_t), p_t^{pf}(I_t, \Lambda_t, \gamma_t^{pf}(I_t, \Lambda_t)), x_t^{pf}(I_t, \Lambda_t, \gamma_t^{pf}(I_t, \Lambda_t)), I_t, \Lambda_t|\sigma_{t-1}^{pf}),$$

where

$$\begin{aligned} J_{i,t}(\gamma_t, p_t, x_t, I_t, \Lambda_t|\sigma_{t-1}^{pf}) = & \mathbb{E}\{p_{i,t} D_{i,t}(\gamma_t, p_t) - w_{i,t}(x_{i,t} - I_{i,t}) - h_{i,t}(x_{i,t} - D_{i,t}(\gamma_t, p_t))^+ \\ & - b_{i,t}(x_{i,t} - D_{i,t}(\gamma_t, p_t))^- - \nu_{i,t}(\gamma_{i,t}) D_{i,t}(\gamma_t, p_t) \\ & + \delta_i V_{i,t-1}(x_t - D_t(\gamma_t, p_t), \eta_t(z_t, D_t(\gamma_t, p_t), \Lambda_t, \Xi_t)|\sigma_{t-1}^{pf})|I_t, \Lambda_t\}, \end{aligned} \tag{3.19}$$

and $V_{i,0}(I_t, \Lambda_t) = w_{i,0} I_{i,0}$. We now define the pure strategy MPE in the PF model.

Definition 3.5.1 *A (pure) Markov strategy* $\sigma^{pf*} = \{(\gamma_{i,t}^{pf*}(\cdot,\cdot), p_{i,t}^{pf*}(\cdot,\cdot,\cdot), x_{i,t}^{pf*}(\cdot,\cdot,\cdot)) : 1 \le i \le N, T \ge t \ge 1\}$ *is a pure strategy MPE in the PF model if and only if, for each firm* i*, period* t*, and state variable* $(I_t, \Lambda_t) \in \mathcal{S}$*,*

$$
\begin{aligned}
&(p_{i,t}^{pf*}(I_t, \Lambda_t, \gamma_t), x_{i,t}^{pf*}(I_t, \Lambda_t, \gamma_t)) \\
=&\text{argmax}_{p_{i,t} \in [\underline{p}_{i,t}, \bar{p}_{i,t}], x_{i,t} \ge \min\{0, I_{i,t}\}} [J_{i,t}(\gamma_t, [p_{i,t}, p_{-i,t}^{pf*}(I_t, \Lambda_t, \gamma_t)], [x_{i,t}, x_{-i,t}^{pf*}(I_t, \Lambda_t, \gamma_t)], I_t, \Lambda_t | \sigma_{t-1}^{pf*})], \\
&\textit{for all } \gamma_t\textit{; and} \\
&\gamma_{i,t}^{pf*}(I_t, \Lambda_t) \\
=&\text{argmax}_{\gamma_{i,t}} [J_{i,t}([\gamma_{i,t}, \gamma_{-i,t}^{pf*}(I_t, \Lambda_t)], p_t^{pf*}(I_t, \Lambda_t, [\gamma_{i,t}, \gamma_{-i,t}^{pf*}(I_t, \Lambda_t)]), \\
&x_t^{pf*}(I_t, \Lambda_t, [\gamma_{i,t}, \gamma_{-i,t}^{pf*}(I_t, \Lambda_t)]), I_t, \Lambda_t | \sigma_{t-1}^{pf*})].
\end{aligned}
\tag{3.20}
$$

Definition 3.5.1 suggests that a pure strategy MPE in the PF model is a (pure) Markov strategy profile that satisfies subgame perfection in each stage of the competition in each period t. The following theorem shows that there exists a pure strategy MPE in the PF model.

Theorem 3.5.1 *The following statements hold for the PF model:*

(a) There exists a pure strategy MPE $\sigma^{pf*} = \{(\gamma_{i,t}^{pf*}(\cdot,\cdot), p_{i,t}^{pf*}(\cdot,\cdot,\cdot), x_{i,t}^{pf*}(\cdot,\cdot,\cdot)) : 1 \le i \le N, T \ge t \ge 1\}$.

(b) For each pure strategy MPE σ^{pf*}*, there exists a series of vectors* $\{\beta_t^{pf} : T \ge t \ge 1\}$*, where* $\beta_t^{pf} = (\beta_{1,t}^{pf}, \beta_{2,t}^{pf}, \cdots, \beta_{N,t}^{pf})$ *with* $\beta_{i,t}^{pf} > 0$ *for each* i *and* t*, such that*

$$
V_{i,t}(I_t, \Lambda_t | \sigma_t^{pf*}) = w_{i,t} I_{i,t} + \beta_{i,t}^{pf} \Lambda_{i,t}, \ \textit{for each } i, \ t, \textit{ and } (I_t, \Lambda_t) \in \mathcal{S}. \tag{3.21}
$$

(c) If $\nu_{i,t}(\gamma_{i,t}) = \gamma_{i,t}$ *for each* i *and* t*,* σ^{pf*} *is the unique MPE in the PF model.*

Theorem 3.5.1 demonstrates the existence of a pure strategy MPE in the PF model. As in the SC model, in Theorem 3.5.1(b), we show that, for each pure strategy MPE σ^{pf*}, the associated profit function of each firm i in each period t is linearly separable in its own starting inventory level $I_{i,t}$ and market size $\Lambda_{i,t}$. We refer to the constant $\beta_{i,t}^{pf}$ as the PF market size coefficient of firm i in period t, which measures the exploitation-induction tradeoff intensity in the PF model. Theorem 3.5.1(c) shows that the MPE in the PF model is unique if $\nu_{i,t}(\gamma_{i,t}) = \gamma_{i,t}$, i.e., the promotional effort $\gamma_{i,t}$ is the actual per-unit demand market expanding expenditure of firm i in period t. For the rest of this section, we assume that $\nu_{i,t}(\gamma_{i,t}) = \gamma_{i,t}$ for each i and t, and, hence, σ^{pf*} is the unique pure strategy MPE in the PF model. We use $\{\beta_t^{pf} : T \geq t \geq 1\}$ to denote the PF market size coefficient associated with σ^{pf*} hereafter.

The linear separability of $V_{i,t}(\cdot, \cdot|\sigma_t^{pf*})$ enables us to have a sharper characterization of MPE in the PF model. As in the SC model, we can rewrite the objective function of firm i in period t as follows.

$$\begin{aligned}
J_{i,t}(\gamma_t, p_t, x_t, I_t, \Lambda_t|\sigma_{t-1}^{pf*}) =& \mathbb{E}\{p_{i,t}D_{i,t}(\gamma_t, p_t) - w_{i,t}(x_{i,t} - I_{i,t}) - h_{i,t}(x_{i,t} - D_{i,t}(\gamma_t, p_t))^+ \\
& - b_{i,t}(x_{i,t} - D_{i,t}(\gamma_t, p_t))^- - \nu_{i,t}(\gamma_{i,t})D_{i,t}(\gamma_t, p_t) \\
& + \delta_i V_{i,t-1}(x_t - D_t(\gamma_t, p_t), \eta_t(z_t, D_t(\gamma_t, p_t), \Lambda_t, \Xi_t)|\sigma_{t-1}^{pf*})|I_t, \Lambda_t\} \\
=& \mathbb{E}\{p_{i,t}\Lambda_{i,t}d_{i,t}(\gamma_t, p_t)\xi_{i,t} - w_{i,t}(y_{i,t}\Lambda_{i,t}d_{i,t}(\gamma_t, p_t) - I_{i,t}) \\
& - h_{i,t}(y_{i,t}\Lambda_{i,t}d_{i,t}(\gamma_t, p_t) - \Lambda_{i,t}d_{i,t}(\gamma_t, p_t)\xi_{i,t})^+ \\
& - b_{i,t}(y_{i,t}\Lambda_{i,t}d_{i,t}(\gamma_t, p_t) - \Lambda_{i,t}d_{i,t}(\gamma_t, p_t)\xi_{i,t})^- \\
& - \nu_{i,t}(\gamma_{i,t})\Lambda_{i,t}d_{i,t}(\gamma_t, p_t)\xi_{i,t} + \delta_i w_{i,t-1}(y_{i,t}\Lambda_{i,t}d_{i,t}(\gamma_t, p_t) \\
& - \Lambda_{i,t}d_{i,t}(\gamma_t, p_t)\xi_{i,t}) \\
& + \delta_i\beta_{i,t-1}^{pf}(\Lambda_{i,t}\Xi_{i,t}^1 + \alpha_{i,t}(z_t)\Lambda_{i,t}d_{i,t}(\gamma_t, p_t)\xi_{i,t}\Xi_{i,t}^2)|I_t, \Lambda_t\} \\
=& w_{i,t}I_{i,t} + \Lambda_{i,t}\{\delta_i\beta_{i,t-1}^{pf}\mu_{i,t} \\
& + \psi_{i,t}(\gamma_t)\rho_{i,t}(p_t)[p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{pf}(y_t)]\}, \\
\text{where } \pi_{i,t}^{pf}(y_t) =& (\delta_i w_{i,t-1} - w_{i,t})y_{i,t} - L_{i,t}(y_{i,t}) + \delta_i\beta_{i,t-1}^{pf}(\kappa_{ii,t}(\mathbb{E}[y_{i,t}^+ \wedge \xi_{i,t}]) \\
& - \sum_{j\neq i}\kappa_{ij,t}(\mathbb{E}[y_{j,t}^+ \wedge \xi_{j,t}])), \\
\text{and } \beta_{i,0}^{pf} :=& 0 \text{ for each } i.
\end{aligned} \tag{3.22}$$

We observe from (3.22) that, in the PF model, the payoff function of each firm i in each period t has a nested structure. Hence, the competition in each period t can be decomposed into three stages: In the first stage, the firms compete on promotional effort; in the second stage, they compete on sales price; in the third stage, they compete on service level. By backward induction, we start the equilibrium analysis with the third-stage service level competition. Let $\mathcal{G}_t^{pf,3}$ be the $N-$player noncooperative game that represents the third-stage service level competition in period t, where player i has the payoff function

$\pi^{pf}_{i,t}(\cdot)$ and the feasible action set $\mathbb{R}$. The following proposition characterizes the Nash equilibrium of the game $\mathcal{G}^{pf,3}_t$.

Proposition 3.5.1 *For each period t, the third-stage service level competition $\mathcal{G}^{pf,3}_t$ has a unique pure strategy Nash equilibrium y^{pf*}_t. Moreover, for each i, $y^{pf*}_{i,t} > 0$ is the unique solution to the following equation:*

$$(\delta_i w_{i,t-1} - w_{i,t}) - L'_{i,t}(y^{pf*}_{i,t}) + \delta_i \beta^{pf}_{i,t-1} \bar{F}_{i,t}(y^{pf*}_{i,t}) \kappa'_{ii,t}(\mathbb{E}(y^{pf*}_{i,t} \wedge \xi_{i,t})) = 0. \tag{3.23}$$

Proposition 3.5.1 characterizes the unique pure strategy Nash equilibrium of the third-stage service level competition. Moreover, $y^{pf*}_{i,t}$ is the solution to the first-order condition $\partial_{y_{i,t}} \pi^{pf}_{i,t}(y^{pf*}_t) = 0$. Let $\pi^{pf*}_t := (\pi^{pf*}_{1,t}, \pi^{pf*}_{2,t}, \cdots, \pi^{pf*}_{N,t})$ be the equilibrium payoff vector of the third-stage service level competition in period t, where $\pi^{pf*}_{i,t} = \pi^{pf}_{i,t}(y^{pf*}_t)$. For each i and t, let

$$\Pi^{pf,2}_{i,t}(p_t|\gamma_t) := \rho_{i,t}(p_t)(p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{pf*}_{i,t}). \tag{3.24}$$

Therefore, given the outcome of the first-stage promotion competition, γ_t, we can define an $N-$player noncooperative game $\mathcal{G}^{pf,2}_t(\gamma_t)$ to represent the second-stage price competition in period t, where player i has the payoff function $\Pi^{pf,2}_{i,t}(\cdot|\gamma_t)$ and the feasible action set $[\underline{p}_{i,t}, \bar{p}_{i,t}]$. We define A_t as an $N \times N$ matrix with entries defined by $A_{ii,t} := 2\theta_{ii,t}$ and $A_{ij,t} := -\theta_{ij,t}$ where $i \neq j$. By Lemma 24(a) in Appendix B.1, A_t is invertible. Let $f_t(\gamma_t)$ be an $N-$dimensional vector with $f_{i,t}(\gamma_t) := \phi_{i,t} + \theta_{ii,t}(\delta_i w_{i,t-1} + \nu_{i,t}(\gamma_{i,t}) - \pi^{pf*}_{i,t})$. We characterize the Nash equilibrium of the game $\mathcal{G}^{pf,2}_t(\gamma_t)$ in the following proposition.

Proposition 3.5.2 *For each period t and any given γ_t, the following statements hold:*

(a) The second-stage price competition $\mathcal{G}^{pf,2}_t(\gamma_t)$ has a unique pure strategy Nash equilibrium $p^{pf*}_t(\gamma_t)$.

(b) $p^{pf*}_t(\gamma_t) = A^{-1}_t f_t(\gamma_t)$. Moreover, $p^{pf*}_{i,t}(\gamma_t)$ is continuously increasing in $\gamma_{j,t}$ for each i and j.

(c) Let $\Pi^{pf*,2}_t(\gamma_t) := (\Pi^{pf*,2}_{1,t}(\gamma_t), \Pi^{pf*,2}_{2,t}(\gamma_t), \cdots, \Pi^{pf*,2}_{N,t}(\gamma_t))$ be the equilibrium payoff vector of the second-stage price competition in period t, where $\Pi^{pf*,2}_{i,t}(\gamma_t) = \Pi^{pf,2}_{i,t}(p^{pf*}_t(\gamma_t)|\gamma_t)$. We have $\Pi^{pf*,2}_{i,t}(\gamma_t) = \theta_{ii,t}(p^{pf*}_{i,t}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{pf*}_{i,t})^2 > 0$ for all i.

Proposition 3.5.2 shows that, for any given promotional effort vector γ_t, the second-stage price competition $\mathcal{G}_t^{pf,2}(\gamma_t)$ has a unique pure strategy Nash equilibrium $p_t^{pf*}(\gamma_t) = A_t^{-1} f_t(\gamma_t)$. By Proposition 3.5.2(b), we have $p_{i,t}^{pf*}(\mathbf{0}) \leq p_{i,t}^{pf*}(\gamma_t) \leq p_{i,t}^{pf*}(\bar{\gamma}_t)$ for each i and γ_t, where $\mathbf{0}$ is an N-dimensional vector with each entry equal to 0 and $\bar{\gamma}_t := (\bar{\gamma}_{1,t}, \bar{\gamma}_{2,t}, \cdots, \bar{\gamma}_{N,t})$. Thus, a sufficient condition for Assumption 3.5.1 is that $\underline{p}_{i,t} \leq p_{i,t}^{pf*}(\mathbf{0})$ and $\bar{p}_{i,t} \geq p_{i,t}^{pf*}(\bar{\gamma}_t)$ for all i and t.

Now we study the first-stage promotion competition in period t. Let

$$\Pi_{i,t}^{pf,1}(\gamma_t) := \Pi_{i,t}^{pf*,2}(\gamma_t)\psi_{i,t}(\gamma_t) = \theta_{ii,t}(p_{i,t}^{pf*}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{pf*})^2 \psi_{i,t}(\gamma_t). \quad (3.25)$$

Thus, we can define an $N-$player noncooperative game $\mathcal{G}_t^{pf,1}$ to represent the first-stage promotion competition in period t, where player i has the payoff function $\Pi_{i,t}^{pf,1}(\cdot)$ and the feasible action set $[0, \bar{\gamma}_{i,t}]$. We characterize the Nash equilibrium of the game $\mathcal{G}_t^{pf,1}$ in the following proposition.

Proposition 3.5.3 *For each period t, the following statements hold:*

(a) The first-stage promotion competition $\mathcal{G}_t^{pf,1}$ is a log-supermodular game.

(b) There exists a unique pure strategy Nash equilibrium in the game $\mathcal{G}_t^{pf,1}$, which is the unique serially undominated strategy of $\mathcal{G}_t^{pf,1}$.

(c) The unique Nash equilibrium of $\mathcal{G}_t^{pf,1}$, γ_t^{pf*}, is the solution to the following system of equations: for each i,

$$\frac{\partial_{\gamma_{i,t}} \psi_{i,t}(\gamma_t^{pf*})}{\psi_{i,t}(\gamma_t^{pf*})} - \frac{2(1 - \theta_{ii,t}(A_t^{-1})_{ii})\nu'_{i,t}(\gamma_{i,t}^{pf*})}{p_{i,t}^{pf*}(\gamma_t^{pf*}) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}^{pf*}) + \pi_{i,t}^{pf*}} \begin{cases} \leq 0, & \text{if } \gamma_{i,t}^{pf*} = 0, \\ = 0, & \text{if } \gamma_{i,t}^{pf*} \in (0, \bar{\gamma}_{i,t}), \\ \geq 0 & \text{if } \gamma_{i,t}^{pf*} = \bar{\gamma}_{i,t}. \end{cases} \quad (3.26)$$

(d) Let $\Pi_t^{pf*,1} := (\Pi_{1,t}^{pf*,1}, \Pi_{2,t}^{pf*,1}, \cdots, \Pi_{N,t}^{pf*,1})$ be the equilibrium payoff vector associated with γ_t^{pf*}, i.e., $\Pi_{i,t}^{pf*,1} = \Pi_{i,t}^{pf,1}(\gamma_t^{pf*})$ for each i. We have $\Pi_{i,t}^{pf*,1} > 0$ for all i.

As shown in Proposition 3.5.3, in the PF model, the first-stage promotion competition in period t is a log-supermodular game and has a unique pure strategy Nash equilibrium. Moreover, the unique Nash equilibrium promotional effort vector γ_t^{pf*} can be determined by (i) the serial elimination of strictly dominated strategies, or (ii) the system of first-order conditions (3.26).

The following theorem summarizes Theorem 3.5.1 and Propositions 3.5.1-3.5.3, and characterizes the MPE in the PF model.

Theorem 3.5.2 *For each period t, the following statements hold:*

(a) For each i, $\beta^{pf}_{i,t} = \delta_i \beta^{pf}_{i,t-1}\mu_{i,t} + \Pi^{pf*,1}_{i,t}$.

(b) Under the unique pure strategy MPE σ^{pf*}, the policy of firm i in period t is given by

$$\begin{aligned}&(\gamma^{pf*}_{i,t}(I_t,\Lambda_t), p^{pf*}_{i,t}(I_t,\Lambda_t,\gamma_t), x^{pf*}_{i,t}(I_t,\Lambda_t,\gamma_t))\\ =&(\gamma^{pf*}_{i,t}, p^{pf*}_{i,t}(\gamma_t), \Lambda_{i,t} y^{pf*}_{i,t}\rho_{i,t}(p^{pf*}_t(\gamma_t))\psi_{i,t}(\gamma_t)).\end{aligned} \tag{3.27}$$

In particular, for any (I_t, Λ_t), the associated (pure strategy) equilibrium price and inventory decisions of firm i are $p^{pf*}_{i,t}(\gamma^{pf*}_t)$ and $\Lambda_{i,t} y^{pf*}_{i,t}\rho_{i,t}(p^{pf*}_t(\gamma^{pf*}_t))\psi_{i,t}(\gamma^{pf*}_t)$, respectively.

Theorem 3.5.2(a) recursively determines the PF market size coefficient vectors, $\{\beta^{pf}_t : T \geq t \geq 1\}$, associated with the unique pure strategy MPE σ^{pf*}. Theorem 3.5.2(b) demonstrates that, under the unique pure strategy MPE σ^{pf*}, each firm i's promotion, price, and inventory decisions in each period t depend on its private information (i.e., $(I_{i,t}, \Lambda_{i,t})$) only, but not on that of its competitors (i.e., $(I_{-i,t}, \Lambda_{-i,t})$). Hence, the unique pure strategy MPE in the PF model has the attractive feature that the strategy of each firm is contingent on its accessible information only.

As in the SC model, we will perform some of our analysis below with the symmetric PF model, where all firms have identical characteristics. We use the subscript "s" to denote the case of symmetric market in the PF model. In this case, $\rho_{s,t}(p_t) = \phi_{s,t} - \theta_{sa,t} p_{i,t} + \sum_{j\neq i}\theta_{sb,t} p_{j,t}$ for some nonnegative constants $\phi_{s,t}$, $\theta_{sa,t}$, and $\theta_{sb,t}$, where $\theta_{sa,t} > (N-1)\theta_{sb,t}$. We use σ^{pf*}_s to denote the unique pure strategy MPE in the symmetric PF model. The following proposition characterizes σ^{pf*}_s in the PF model.

Proposition 3.5.4 *The following statements hold for the symmetric PF model.*

(a) For each $t = T, T-1, \cdots, 1$, there exists a constant $\beta^{pf}_{s,t} > 0$, such that

$$V_{i,t}(I_t, \Lambda_t|\sigma^{pf*}_{s,t}) = w_{s,t} I_{i,t} + \beta^{pf}_{s,t}\Lambda_{i,t}, \text{ for all } i.$$

(b) In each period t, the third-stage service level competition $\mathcal{G}^{pf,3}_{s,t}$ is symmetric, with the payoff function for each firm i given by

$$\pi^{pf}_{i,t}(y_t) =(\delta_s w_{s,t-1} - w_{s,t})y_{i,t} - L_{s,t}(y_{i,t}) + \delta_s \beta^{pf}_{s,t-1}(\kappa_{sa,t}(\mathbb{E}[y^+_{i,t} \wedge \xi_{i,t}]) - \sum_{j\neq i} \kappa_{sb,t}(\mathbb{E}[y^+_{j,t} \wedge \xi_{j,t}])).$$

Moreover, $\mathcal{G}^{pf,3}_{s,t}$ has a unique pure strategy Nash equilibrium, which is symmetric, so we use $y^{pf*}_{s,t}$ [$\pi^{pf*}_{s,t}$] to denote the equilibrium strategy [payoff] of each firm in $\mathcal{G}^{pf,3}_{s,t}$.

(c) In each period t, the second-stage price competition $\mathcal{G}^{pf,2}_{s,t}(\gamma_t)$ is symmetric if $\gamma_{i,t} = \gamma_{j,t}$ for all $1 \leq i, j \leq N$. In this case, $\mathcal{G}^{pf,2}_{s,t}(\gamma_t)$ has a unique pure strategy Nash equilibrium $p^{pf*}_{ss,t}(\gamma_t)$, which is symmetric (i.e., $p^{pf*}_{ss,t}(\gamma_t) = (p^{pf*}_{s,t}(\gamma_t), p^{pf*}_{s,t}(\gamma_t), \cdots, p^{pf*}_{s,t}(\gamma_t))$ for some $p^{pf*}_{s,t}(\gamma_t) \in [\underline{p}_{s,t}, \bar{p}_{s,t}]$).

(d) In each period t, the first-stage promotion competition $\mathcal{G}^{pf,1}_{s,t}$ is symmetric. Moreover, $\mathcal{G}^{pf,1}_{s,t}$ has a unique pure strategy Nash equilibrium $\gamma^{pf*}_{ss,t}$, which is symmetric (i.e., $\gamma^{pf*}_{ss,t} = (\gamma^{pf*}_{s,t}, \gamma^{pf*}_{s,t}, \cdots, \gamma^{pf*}_{s,t})$ for some $\gamma^{pf*}_{s,t} \in [0, \bar{\gamma}_{s,t}]$).

(e) Under the unique pure strategy MPE σ^{pf*}_s, the policy of firm i in period t is

$$\begin{aligned}&(\gamma^{pf*}_{i,t}(I_t, \Lambda_t), p^{pf*}_{i,t}(I_t, \Lambda_t, \gamma_t), x^{pf*}_{i,t}(I_t, \Lambda_t, \gamma_t)) \\ =&(\gamma^{sc*}_{s,t}, p^{pf*}_{i,t}(\gamma_t), \Lambda_{i,t} y^{pf*}_{s,t} \rho_{s,t}(p^{pf*}_t(\gamma_t))\psi_{s,t}(\gamma_t)),\end{aligned}$$

for all (I_t, Λ_t) and γ_t. In particular, for each firm i and any (I_t, Λ_t), the equilibrium price is $p^{pf*}_{s,t}(\gamma^{pf*}_{ss,t})$, and the equilibrium post-delivery inventory level is $\Lambda_{i,t} y^{pf*}_{s,t} \rho_{s,t}(p^{pf*}_{ss,t}(\gamma^{pf*}_{ss,t}))\psi_{s,t}(\gamma^{pf*}_{ss,t})$.

Proposition 3.5.4 shows that, in the symmetric PF model, all competing firms make the same promotional effort, charge the same sales price, and maintain the same service level in each period. The PF market size coefficient is also identical for all firms in each period.

3.5.2 Exploitation-Induction Tradeoff

In this subsection, we study how the exploitation-induction tradeoff impacts the equilibrium market outcome in the PF model. As in the SC model, we first characterize the impact of the PF market size coefficient vectors, $\{\beta^{pf}_t : T \geq t \geq 1\}$.

Theorem 3.5.3 *For each period t, the following statements hold:*

(a) *For each i and $j \neq i$, $y_{i,t}^{pf*}$ is continuously increasing in $\beta_{i,t-1}^{pf}$ and independent of $\beta_{j,t-1}^{pf}$.*

(b) *For each i and $j \neq i$, $\pi_{i,t}^{pf*}$ is continuously increasing in $\beta_{i,t-1}^{pf}$ and continuously decreasing in $\beta_{j,t-1}^{pf}$.*

(c) *For each i, j, and γ_t , $p_{i,t}^{pf*}(\gamma_t)$ is continuously decreasing in $\pi_{j,t}^{pf*}$.*

(d) *If the PF model is symmetric, $\gamma_{s,t}^{pf*}$ is continuously increasing in $\pi_{s,t}^{pf*}$. If, in addition, the monotonicity condition (3.17) holds, $\beta_{s,t}^{pf}$ is continuously increasing in $\pi_{s,t}^{pf*}$ as well.*

(e) *If the PF model is symmetric and $\pi_{s,t}^{pf*}$ is increasing in $\beta_{s,t-1}^{pf}$, $\gamma_{s,t}^{pf*}$ is continuously increasing in $\beta_{s,t-1}^{pf}$, whereas $p_{i,t}^{pf*}(\gamma_t)$ is continuously decreasing in $\beta_{s,t-1}^{pf}$. If, in addition, the monotonicity condition (3.17) holds, $\beta_{s,t}^{pf}$ is continuously increasing in $\beta_{s,t-1}^{pf}$ as well.*

Theorem 3.5.3 demonstrates that the market size coefficients $\{\beta_{i,t}^{pf} : 1 \leq i \leq N, T \geq t \geq 1\}$ quantify the intensity of the exploitation-induction tradeoff in the PF model. More specifically, a larger $\beta_{i,t-1}^{pf}$ implies more intensive exploitation-induction tradeoff of firm i in period t.

As in the SC model, we use "~" to denote the benchmark case without the service effect and the network effect, where the market size evolution function $\tilde{\alpha}_{i,t}(\cdot) \equiv 0$ for each firm i and each period t. Thus, the exploitation-induction tradeoff is absent in this benchmark model, and it suffices for the firms to myopically maximize their current-period profits. The following theorem characterizes the impact of the service effect and the network effect in the PF model.

Theorem 3.5.4 (a) *For each firm i and each period t, $y_{i,t}^{pf*} \geq \tilde{y}_{i,t}^{pf*}$, $z_{i,t}^{pf*} \geq \tilde{z}_{i,t}^{pf*}$, and $\pi_{i,t}^{pf*} \geq \tilde{\pi}_{i,t}^{pf*}$.*

(b) *For each firm i and each period t, $p_{i,t}^{pf*}(\gamma_t) \leq \tilde{p}_{i,t}^{pf*}(\gamma_t)$ for all γ_t. Moreover, if the PF model is symmetric and (3.17) holds, $x_{i,t}^{pf*}(I_t, \Lambda_t, \gamma_t) \geq \tilde{x}_{i,t}^{pf*}(I_t, \Lambda_t, \gamma_t)$ for all i, t, $(I_t, \Lambda_t) \in \mathcal{S}$, and $\gamma_t \in [0, \bar{\gamma}_{s,t}]^N$.*

(c) Consider the symmetric PF model. For each period t, $\gamma_{s,t}^{pf*} \geq \tilde{\gamma}_{s,t}^{pf*}$. *Thus,* $\gamma_{i,t}^{pf*}(I_t, \Lambda_t) \geq \tilde{\gamma}_{i,t}^{pf*}(I_t, \Lambda_t)$ *for all* i *and all* $(I_t, \Lambda_t) \in \mathcal{S}$.

Consistent with Theorem 3.4.4(a), Theorem 3.5.4(a) shows that, the service effect and the network effect drive the competing firms to maintain higher service levels in the PF model. Theorem 3.5.4(b) reveals the impact of the exploitation-induction tradeoff upon the competing firms' price and inventory strategy in the PF model. Specifically, given any outcome of the first-stage promotion competition γ_t, in the second-stage price competition, each firm i should charge a lower sales price under the service effect and the network effect, so as to exploit the network effect and induce higher future demands. Moreover, in each period t, the equilibrium post-delivery inventory levels contingent on any realized promotional effort vector γ_t are also higher in the PF model under the service effect and the network effect. Theorem 3.5.4(c) sheds light on how the exploitation-induction tradeoff influences the equilibrium promotion strategies under the service effect and the network effect. In the symmetric PF model, the equilibrium promotional effort of each firm i in each period t is higher under the service effect and the network effect.

Note that, in the PF model, the equilibrium price and inventory outcomes under the service effect and the network effect, $p_{s,t}^{pf*}(\gamma_{ss,t}^{pf*})$ and $x_{i,t}^{pf*}(I_t, \Lambda_t, \gamma_{ss,t}^{pf*})$, may be either higher or lower than those without market size dynamics, $\tilde{p}_{ss,t}^{pf*}(\tilde{\gamma}_{s,t}^{pf*})$ and $\tilde{x}_{i,t}^{pf*}(I_t, \Lambda_t, \tilde{\gamma}_{ss,t}^{pf*})$. This phenomenon contrasts with the equilibrium market outcomes in the SC model, where the equilibrium sales price [post-delivery inventory level] of each firm in each period is lower [higher] under the service effect and the network effect (i.e., Theorem 3.4.4(b-i,iii)). This discrepancy is driven by the fact that, in the PF model, each firm observes the promotion decisions of its competitors before making its pricing decision. Hence, under the service effect and the network effect, the competing firms may either decrease the sales prices to induce future demands or increase the sales prices to exploit the better market condition from the increased promotional efforts (recall that $\gamma_{s,t}^{pf*} \geq \tilde{\gamma}_{s,t}^{pf*}$). In general, either effect may dominate, so we do not have a general monotonicity relationship between either the equilibrium price outcomes (i.e., $p_{s,t}^{pf*}(\gamma_{ss,t}^{pf*})$ and $\tilde{p}_{s,t}^{pf*}(\tilde{\gamma}_{ss,t}^{pf*})$) or the equilibrium inventory outcomes (i.e., $x_{i,t}^{pf*}(I_t, \Lambda_t, \gamma_{ss,t}^{pf*})$ and $\tilde{x}_{i,t}^{pf*}(I_t, \Lambda_t, \tilde{\gamma}_{ss,t}^{pf*})$). Therefore, the exploitation-induction tradeoff in the PF model is more involved than that in the SC model. The competing firms only need to trade off between generating current profits and inducing future demands intertemporally in the SC model, whereas they need to balance this tradeoff both inter-temporally and intra-temporally in the PF model.

To deliver sharper insights on the managerial implications of the exploitation-induction tradeoff, we confine ourselves to the symmetric PF model for the rest of this section.

Theorem 3.5.5 *Let two symmetric PF models be identical except that one with market size evolution functions* $\{\hat{\alpha}_{s,t}(\cdot)\}_{T\geq t\geq 1}$ *and the other with* $\{\alpha_{s,t}(\cdot)\}_{T\geq t\geq 1}$*. Assume that, for each period* t*, (i) the monotonicity condition* (3.17) *holds, and (ii)* $\kappa_{sb,t}(\cdot)\equiv\kappa^0_{sb,t}$ *for some constant* $\kappa^0_{sb,t}$*.*

(a) If $\hat{\alpha}_{s,t}(z_t)\geq\alpha_{s,t}(z_t)$ *for each period* t *and all* z_t*, we have, for each period* t*,* $\hat{\beta}^{pf}_{s,t}\geq\beta^{pf}_{s,t}$*,* $\hat{p}^{pf*}_{i,t}(\gamma_t)\leq p^{pf*}_{i,t}(\gamma_t)$ *for all* i *and* $\gamma_t\in[0,\bar{\gamma}_{s,t}]^N$*, and* $\hat{\gamma}^{pf*}_{s,t}\geq\gamma^{pf*}_{s,t}$*. Thus, for each period* t*,* $\hat{p}^{pf*}_{i,t}(I_t,\Lambda_t,\gamma_t)\leq p^{pf*}_{i,t}(I_t,\Lambda_t,\gamma_t)$ *and* $\hat{\gamma}^{pf*}_{i,t}(I_t,\Lambda_t)\geq\gamma^{pf*}_{i,t}(I_t,\Lambda_t)$ *for all* i*,* $(I_t,\Lambda_t)\in\mathcal{S}$*, and* $\gamma_t\in[0,\bar{\gamma}_{s,t}]^N$*.*

(b) If, for each period t*,* $\hat{\alpha}_{s,t}(z_t)\geq\alpha_{s,t}(z_t)$ *for all* z_t *and* $\hat{\kappa}'_{sa,t}(z_{i,t})\geq\kappa'_{sa,t}(z_{i,t})\geq 0$ *for all* $z_{i,t}$*, we have, for each period* t*,* $\hat{\beta}^{pf}_{s,t}\geq\beta^{pf}_{s,t}$*,* $\hat{y}^{pf*}_{s,t}\geq y^{pf*}_{s,t}$*,* $\hat{p}^{pf*}_{i,t}(\gamma_t)\leq p^{pf*}_{i,t}(\gamma_t)$*, and* $\hat{\gamma}^{pf*}_{s,t}\geq\gamma^{pf*}_{s,t}$*. Thus, for each period* t*,* $\hat{p}^{pf*}_{i,t}(I_t,\Lambda_t,\gamma_t)\leq p^{pf*}_{i,t}(I_t,\Lambda_t,\gamma_t)$*,* $\hat{x}^{pf*}_{i,t}(I_t,\Lambda_t,\gamma_t)\geq x^{pf*}_{i,t}(I_t,\Lambda_t,\gamma_t)$*,* $\hat{\gamma}^{pf*}_{i,t}(I_t,\Lambda_t)\geq\gamma^{pf*}_{i,t}(I_t,\Lambda_t)$ *for all* i*,* $(I_t,\Lambda_t)\in\mathcal{S}$*, and* $\gamma_t\in[0,\bar{\gamma}_{s,t}]^N$*.*

Theorem 3.5.5(a) shows that, in the symmetric PF model, higher intensity of the network effect (i.e., larger $\alpha_{s,t}(\cdot)$) drives all the competing firms to make more promotional efforts and charge lower sales prices for each observed promotion vector. Moreover, if the intensities of both the network effect and the service effect (i.e., the magnitudes of $\alpha_{s,t}(\cdot)$ and $\kappa'_{sa,t}(\cdot)$) are higher, Theorem 3.5.5(b) demonstrates that all the competing firms are prompted to maintain higher service levels as well. Therefore, in the PF model, the exploitation-induction tradeoff is stronger with more intensive service effect and network effect.

Theorem 3.5.6 *Consider the stationary symmetric PF model. Assume that, for each period* t*, (i) the monotonicity condition* (3.17) *holds, and (ii)* $\pi^{pf*}_{s,t}$ *is increasing in* $\beta^{pf}_{s,t-1}$*. For each period* t*, the following statements hold:*

(a) $\beta^{pf}_{s,t}\geq\beta^{pf}_{s,t-1}$*,* $y^{pf*}_{s,t}\geq y^{pf*}_{s,t-1}$*,* $p^{pf*}_{s,t}(\gamma)\leq p^{pf*}_{s,t-1}(\gamma)$ *for each* $\gamma\in[0,\bar{\gamma}_s]^N$*, and* $\gamma^{pf*}_{s,t}\geq\gamma^{pf*}_{s,t-1}$*.*

(b) $p^{pf*}_{i,t}(I,\Lambda,\gamma) \le p^{pf*}_{i,t-1}(I,\Lambda,\gamma)$*,* $x^{pf*}_{i,t}(I,\Lambda,\gamma) \ge x^{pf*}_{i,t-1}(I,\Lambda,\gamma)$*, and* $\gamma^{pf*}_{i,t}(I,\Lambda) \ge \gamma^{pf*}_{i,t-1}(I,\Lambda)$ *for each* i*,* $(I,\Lambda) \in \mathcal{S}$*, and* $\gamma \in [0,\bar{\gamma}_{s,t}]^N$*.*

Analogous to Theorem 3.4.6, Theorem 3.5.6 justifies the widely used introductory price and promotion strategy. More specifically, this result shows that if the market is stationary and symmetric in the PF model, the competing firms should decrease the promotional efforts (i.e., $\gamma^{pf*}_{s,t}$) and service levels (i.e., $y^{pf*}_{s,t}$), and increase the sales prices contingent on any realized promotional efforts (i.e., $p^{pf*}_{s,t}(\gamma_t)$), over the planning horizon. Hence, Theorem 3.5.6 suggests that, in the PF model, the exploitation-induction tradeoff is more intensive at the early stage of the sales season than at later stages.

To conclude this section, we remark that, because of the aforementioned intra-temporal exploitation-induction tradeoff under the promotion-first competition, Theorems 3.5.5-3.5.6 cannot give the monotone relationships on the equilibrium outcomes of each firm i's sales price (i.e., $p^{pf*}_{i,t}(I_t,\Lambda_t,\gamma^{pf*}_{ss,t})$) and post-deliver inventory level (i.e., $x^{pf*}_{i,t}(I_t,\Lambda_t,\gamma^{pf*}_{ss,t})$).

3.6 Comparison of the Two Competition Models

As demonstrated above, the exploitation-induction tradeoff is more involved in the PF model than that in the SC model. In this section, we compare the unique MPE in the SC model and that in the PF model, and discuss how the exploitation-induction tradeoff impacts the equilibrium market outcomes under different competitions.

Theorem 3.6.1 *Consider the symmetric SC and PF models. Assume that, for each period* t*, (i) the demand function* $\rho_{i,t}(\cdot)$ *is linear and given by* (3.18)*, (ii)* $\nu_{i,t}(\gamma_{i,t}) = \gamma_{i,t}$*, (iii) the monotonicity condition* (3.17) *holds, (iv) Assumption 3.5.1 holds, (v)* $\pi^{sc*}_{s,t}$ *is increasing in* $\beta^{sc}_{s,t-1}$*, and (vi)* $\pi^{pf*}_{s,t}$ *is increasing in* $\beta^{pf}_{s,t-1}$*. The following statements hold:*

(a) If $\beta^{pf}_{s,t-1} \ge \beta^{sc}_{s,t-1}$*,* $y^{pf*}_{s,t} \ge y^{sc*}_{s,t}$ *and* $\gamma^{pf*}_{s,t} \ge \gamma^{sc*}_{s,t}$*.*

(b) For each period t*, there exists an* $\epsilon_t \in [0, \frac{1}{N-1}]$*, such that, if* $\theta_{sb,t} \le \epsilon_t \theta_{sa,t}$*, we have*

(i) $\beta^{pf}_{s,t} \ge \beta^{sc}_{s,t}$ *and, thus,* $V_{i,t}(I_t,\Lambda_t|\sigma^{pf*}_t) \ge V_{i,t}(I_t,\Lambda_t|\sigma^{sc*}_t)$ *for each firm* i *and all* $(I_t,\Lambda_t) \in \mathcal{S}$*;*

(ii) $y^{pf*}_{s,t} \ge y^{sc*}_{s,t}$*;*

(iii) $\gamma^{pf*}_{s,t} \ge \gamma^{sc*}_{s,t}$*.*

Theorem 3.6.1 shows that, if the product differentiation is sufficiently high (as captured by the condition that $\theta_{sb,t} \leq \epsilon_t \theta_{sa,t}$), the PF competition leads to stronger exploitation-induction tradeoff (i.e., $\beta_{s,t}^{pf} \geq \beta_{s,t}^{sc}$). As a consequence, the competing firms should set higher service levels and promotional efforts in the PF model. Compared with the simultaneous competition, the promotion-first competition enables the firm to responsively adjust their sales prices in accordance to the market condition and their competitors' promotion strategies. If the product differentiation is sufficiently high, such pricing flexibility gives rise to higher expected profits of all firms and more intensive exploitation-induction tradeoff in the PF model.

Theorem 3.6.1 also reveals the "fat-cat" effect in our dynamic competition model: When the price decisions are made after observing the promotional efforts in each period, the firms tend to "overinvest" in promotional efforts. As shown in the literature (e.g., [78, 9]), one driving force for this phenomenon is that, under the PF competition, the firms can charge higher prices in the subsequent price competition with increased promotional efforts in each period. Theorem 3.6.1 identifies a new driving force for the "fat-cat" effect: The firms under the PF competition make more promotional efforts to balance the more intensive exploitation-induction tradeoff therein. Therefore, our analysis delivers a new insight to the literature that the exploitation-induction tradeoff may give rise to the "fat-cat" effect in dynamic competition.

3.7 Summary

This chapter studies a dynamic joint promotion, price, and service competition model, in which current decisions influence future demands through the service effect and the network effect. Our model highlights an important tradeoff in a dynamic and competitive market: the tradeoff between generating current profits and inducing future demands (i.e., the exploitation-induction tradeoff). We characterize the impact of the exploitation-induction tradeoff upon the equilibrium market outcome under the service effect and the network effect, and identify the effective strategies to balance this tradeoff under dynamic competition.

We employ the linear separability approach to characterize the pure strategy MPE both in the SC model and in the PF model. An important feature of the MPE in both models is that the equilibrium strategy of each firm in each period only depends on the private inventory and market size information of itself, but not on that of its competitors. Moreover, the exploitation-induction tradeoff is more intensive if the service effect and the network effect are stronger; and this trade-off decreases over the planning horizon. The exploitation-induction tradeoff is more involved in the PF model than in the SC model. This is because the competing firms need to balance this tradeoff both inter-temporally and intra-temporally in the PF model, whereas they only need to balance it inter-temporally in the SC model. More specifically, in the SC model, to effectively balance the exploitation-induction tradeoff, the firms should (a) increase promotional efforts, (b) offer price discounts, and (c) improve service levels. In the PF model, the firms should increase promotional efforts under the service effect and the network effect. Given the same promotional effort in the first stage competition, the firms need to decrease their sales prices under the network effect. However, with an increased promotional effort in the first stage competition, the equilibrium sales prices in the second stage competition may either decrease to increase. Analogously, the equilibrium post-delivery inventory levels may either decrease or increase in the PF model under the service effect and the network effect. Finally, we identify the "fat-cat" effect in our dynamic competition model: If the product differentiation is sufficiently high, under the MPE, the firms make more promotional efforts in the PF model than in the SC model. The driving force of this phenomenon is that the exploitation-induction tradeoff is more intensive under the promotion-first competition than under the simultaneous competition.

4. Trade-in Remanufacturing, Strategic Customer Behavior and Government Subsidies

4.1 Introduction

[1]It is a common practice for firms to offer trade-in rebates to recycle used products. For example, Apple offers both in-store and online trade-in programs, which allow customers to exchange their used iPhones, iPads, and Macs for credits to purchase new ones ([13]). Analogously, Amazon allows Kindle owners to trade in their old products for newer versions at a discount price ([55]). More examples on the adoption of trade-in rebates to collect cores for remanufacturing have been reported in industries like furniture, carpets, and power tools, etc. (see [142]).

Recycling used products through trade-in rebates has been lauded for its various benefits. From the economic perspective, the return product flow from trade-in rebates serves as an important source for generating revenue and reducing costs. With the recycled products, firms can recover the residual values by either remanufacturing them into new ones or reusing their components and materials. Following the literature (e.g., [142]), throughout the chapter we use the term *remanufacturing* to represent the general revenue-generating process through recycling and recovering used products. In practice, the revenue-generating/cost-saving effect of trade-in based remanufacturing could be significant. Xerox, which partly bases its remanufacturing on trade-in returns, has saved several hundred million dollars each year, which accounts for 40%-65% of the company's manufacturing costs ([146]). From the strategic perspective, trade-in rebates may improve firm profitability by elevating customer switching costs ([105]), discouraging second-hand markets ([109]), increasing purchase frequency ([166]), and reducing inefficiencies arising from the lemon problem ([141]). From the environmental perspective, trade-in rebates encourage customers to return used products, thus generating less waste and disposals. In particular, the electronics market is featured with frequent product introductions and generates more than one million tons of so-called e-wastes each year ([140]). Using trade-

[1]This chapter is based on the author's earlier work [193].

in rebates, Apple collected more than 40,000 tons of e-wastes in 2014, which account for more than 75 percent of the products they sold seven years earlier (see [12]).

It has been empirically verified that customers exhibit forward-looking behaviors in the electronics market due to frequent product introductions ([140]). In particular, when the firm offers trade-in rebates, strategic customer behavior naturally arises, because customers can anticipate a possible price discount in the *future* if making a purchase *now* ([166, 80, 141]). Moreover, advances in information technology enable customers to easily obtain product and price information. For example, Kayak launched the price forecast service to help customers decide when to book a flight ([68]). As a consequence, strategic customer behavior has become more prevalent in today's business world. Although strategic customer behavior has been widely acknowledged in the literature, it is not clear how such behavior would affect the economic and environmental benefits of trade-in remanufacturing.

Governments around the world have made tremendous efforts to promote recycling and remanufacturing used products. One commonly used strategy is to provide subsidies for remanufacturing. For instance, in January 2015, the Chinese government released a policy to subsidize the use of remanufactured vehicle engines and transmissions ([44]). Analogously, the Chinese government established a special fund in 2011 to provide subsidies to companies engaged in the recycling and recovering of waste electrical and electronic products (e.g., [174]). As another example, a recent report backed by the Scottish government and Zero Waste Scotland (ZWS) concluded that Scotland was in a unique position to develop a circular economy and called for government subsidies to help boost closed loop recycling, reuse, bio-refining, and remanufacturing ([161]). In the literature, the effects of government subsidies for remanufacturing/trade-in remanufacturing have been studied in settings without explicitly modeling customer behaviors (e.g., [128, 118]). Despite its importance, the question of how the government should design the subsidization policy under strategic customer behavior to induce the social optimum has not been thoroughly explored.

The primary goal of this chapter is to analyze how strategic customer behavior influences the value of trade-in remanufacturing from the perspectives of the firm, the environment, and the government. For this purpose, we develop a two-period model in which a profit-maximizing firm sells two generations of a product in an *ex-ante* uncertain market.

To highlight the impact of strategic customer behavior, we consider two scenarios, one with strategic customers and the other with myopic customers. Strategic customers make their purchasing decisions based on both *current* and anticipated *future* utilities, whereas myopic customers make decisions based on *current* utilities only. In the first period, the firm sells the first-generation product in the market. In the second period, the firm sells the second-generation product to new customers (who have not purchased in the first period); meanwhile the firm offers trade-in rebates through which repeat customers (who have purchased in the first period) exchange used products for new second-generation ones at a discounted price. The firm generates revenue by remanufacturing the recycled products. This remanufacturing process also reduces the (negative) environmental impact of the business, because it decreases energy and raw material consumption, as well as waste disposal. We model the government as a policy-maker whose subsidy/tax policy may affect the firm's pricing and production strategy as well as the customers' purchasing decisions. The objective of the government is to maximize the social welfare, i.e., the sum of firm profit and customer surplus less environmental impact.

We find that strategic customer behavior has important implications on the practice of trade-in remanufacturing. First, under trade-in remanufacturing, the firm can earn a higher profit with strategic customers than with myopic customers if the revenue-generating effect of remanufacturing is sufficiently strong. In other words, strategic customer behavior may improve firm profit, which is in contrast with the commonly believed notion that strategic customer behavior hurts firm profit. When the firm employs trade-in remanufacturing, strategic customers will anticipate the *future* trade-in rebate (i.e., price discount) in the second period, which depends on the additional value generated by remanufacturing. Note that a deeper discount in the second period will induce a higher willingness-to-pay in the first period. Thus, strategic customers may be willing to pay a higher first-period price than myopic customers if the revenue-generating effect of remanufacturing is strong enough, which allows the firm to extract a higher profit. This implies that when early purchases (of strategic customers) can be induced by the additional benefits (i.e., the trade-in option and the deep discount brought by remanufacturing), a firm may benefit from strategic customer behavior. Without trade-in remanufacturing, however, strategic customer behavior always hurts the firm's profit, as reported in the literature.

Second, with strategic customers, the adoption of trade-in remanufacturing may create a tension between firm profitability and environmental sustainability. Trade-in rebate essentially offers an early purchase reward and thus can deliver additional value by exploiting the forward-looking behavior of strategic customers. As a result, trade-in remanufacturing is more valuable to the firm with strategic customers than with myopic customers. However, the early-purchase inducing effect of trade-in remanufacturing also prompts the firm to increase production quantities significantly under strategic customer behavior. The increased production quantities may outweigh the environmental advantage of remanufacturing unless the unit environmental benefit of remanufacturing is very high. Hence, trade-in remanufacturing generally hurts the environment with strategic customers. Moreover, we find that trade-in remanufacturing decreases customer surplus, and consequently, the social welfare may decrease as well. Therefore, our results call for caution on the adoption of trade-in remanufacturing under strategic customer behavior, because it is likely to be severely detrimental to the environment and the society.

With myopic customers, however, trade-in manufacturing generally benefits the environment. The price discrimination effect of trade-in rebates increases the expected unit profit from new customers in the second period. This effect drives the firm to decrease the first-period production quantity and thus increase the potential second-period market size of new customers. As long as the unit environmental benefit of remanufacturing is not too low, trade-in remanufacturing induces lower production quantities and, thus, benefits the environment. Therefore, for the scenario with myopic customers, the tension between firm and environment does not exist in general.

The tension between firm profitability and environmental sustainability under strategic customer behavior motivates us to study how government intervention can achieve the socially optimal outcome. Specifically, we focus on the subsidization policy the government can use to promote the activities of used products recycling (e.g., trade-in rebates, remanufacturing, and take-backs; see [128, 170, 161]). An intuitive policy observed in practice is to subsidize the firm/customers for selling/purchasing remanufactured products. However, we find that subsidizing remanufactured products alone actually hurts the environment and is not sufficient to achieve the social optimum. This cautions the policy-makers about how to promote remanufacturing through subsidization. With either strategic or myopic customers, in order to induce the social optimum, it suffices for the government to use a simple linear subsidy/tax scheme for the sales of both product generations and remanufacturing. In addition, if the total unit economic and environmental value of remanufacturing is low, the government should provide more subsidies to the firm with strategic customers than with myopic customers, and vice versa.

The rest of the chapter is organized as follows. In Section 4.2, we position this chapter in the related literature. The base model and the equilibrium analysis are presented in Section 4.3. In Section 4.4, we analyze the impact of trade-in remanufacturing upon the firm and the environment. In Section 4.5, we characterize the government policy that can induce the social optimum. This chapter concludes with Section 4.6. All proofs are given in Appendix C.2.

4.2 Related Research

This chapter builds upon two streams of research in the literature: (1) remanufacturing and closed-loop supply chain management, and (2) strategic customer behavior.

There is a rapidly growing stream of literature on remanufacturing and closed-loop supply chain management. Comprehensive reviews of this literature are given by [91] and [154]. Several papers study the optimal inventory policy with return flows of used products; see, e.g., [167, 163], and [87]. These papers focus on characterizing the cost-minimizing inventory policy in a system with exogenously given demand rate, price, and remanufacturability. More recently, researchers start to explicitly model some strategic issues related to remanufacturing, such as used product acquisition, demand segmentation, product cannibalization, and competition. [146] study the optimal reverse channel structure for the collection of used products from customers. [74] analyze the competition between new and remanufactured products (i.e., the cannibalization effect) and characterize the optimal recovery strategy. When remanufacturability is an endogenous decision, [59] investigate a joint pricing and production technology selection problem of a manufacturer who sells a remanufacturable product to heterogeneous customers. Under the cannibalization effect of remanufactured products, [75] study the competition between an original equipment manufacturer (OEM) and an independent operator who only sells remanufactured products. [14] show that remanufacturing could serve as a marketing strategy to target the customers in the green segment and, hence, enhance the profitability of the OEM. [133] characterize the optimal relicensing strategy of an

OEM to mitigate the cannibalization effect in the secondary market. [81] study how the rate of product innovation affects the firm's reuse and remanufacturing decisions. [90] investigate the quality design and environmental consequences of green consumerism with remanufacturing. There are papers that address behaviorial issues related to remanufacturing such as how the remanufactured products affect the customer valuation of new products ([3]). Government regulations on remanufacturing have also been studied in the literature; see, e.g., [118]. [56] study the impact of demand uncertainty on government subsidies for green technology adoption. The impact of trade-in rebates has also received some attention in the remanufacturing literature. For example, [142] examine the value of price discrimination for new and repeat customers with differentiated ages (and qualities) of the returned products.

The impact of strategic customer behavior has received an increasing amount of attention in the operations management literature. [149] provide a comprehensive review on customer behavior models in revenue management and auctions. [20] show that rational customers drive a monopolist firm to charge a lower price for any given state in each period. [157] characterizes the optimal pricing strategy with a heterogenous group of strategic customers. When customers are forward-looking, [17] study the optimal single mark-down timing with finite inventories. In a newsvendor model where customers anticipate the likelihood of stockout before deciding whether to make a purchase, [58] and [159, 160] study the impact of strategic customer behavior on newsvendor profit, supply chain performance, and the role of product availability in inducing demand, respectively. [115] propose the effective capacity rationing strategy to induce early purchases with strategic customers. [40, 41] and [176] demonstrate how quick response can be employed to mitigate strategic customer behavior. [99] study opaque selling and last-minute selling with strategic customers in a revenue management framework. In a cheap talk framework, [8] show that, though nonverifiable, the availability information improves the profit of a service firm and the expected utility of its customers. [7] further demonstrate that a single retailer providing availability information on its own cannot create any credibility with homogeneous customers. [54] investigate the integrated information and pricing strategy with strategic customers and the customer preorders before product release. [135] demonstrates how vertical product differentiability helps mitigate strategic customer behavior. Recently, there are papers addressing the optimal strategy with multiple product introductions and strategic customer behavior. For example, in the presence of strategic consumers, [113] characterize the optimal product rollover strategies, whereas [116] study the new product launch strategy.

There are a few papers that investigate trade-in rebates with forward-looking customers. [166] show that, under strategic customer behavior, trade-ins can serve as a mechanism to achieve price commitment. [80] study the monopoly pricing of overlapping generations of a durable good with and without a second-hand market. In an infinite-horizon model setting, [141] demonstrate that trade-in rebates can alleviate the inefficiencies arising from the lemon problem.

This chapter contributes to the aforementioned streams of research by studying the interaction between trade-in remanufacturing and strategic customer behavior, and how such interaction affects the economic and environmental values of trade-in remanufacturing. We demonstrate that strategic customer behavior may benefit the firm, but give rise to a tension between firm profitability and environmental sustainability under trade-in remanufacturing. In addition, we characterize how the government can achieve the social optimum, using a simple linear subsidy/tax scheme with either strategic or myopic customers.

4.3 Model and Equilibrium Analysis

4.3.1 Model Setup

We consider a monopoly firm (he) in the market who sells a product to customers (she) in a two-period sales horizon. In the first period, the firm produces the first-generation product at a unit production cost c_1. The potential market size X, which is the total number of potential customers, is *ex-ante* uncertain. The customers are infinitesimal, each requesting at most one unit of the product in any period. Demand uncertainty is a common feature with new product introduction, but the firm can obtain more accurate demand information as the market matures. Hence, in the second period, the market uncertainty is resolved so the realized market size X becomes known to the firm. Without loss of generality, we assume that $X > 0$, with a distribution function $F(\cdot)$ and density function $f(\cdot) = F'(\cdot)$.

A customer's valuation V for the first-generation product is independently drawn from a continuous distribution with a distribution function $G(\cdot)$ supported on $[\underline{v}, \bar{v}]$ $(0 \leq \underline{v} < \bar{v})$. We call the customer with product valuation V the type-V customer. At the beginning of the sales horizon, each customer only knows the distribution of her own valuation $G(\cdot)$, but not the realization V. This assumption captures the customers' uncertainties about the quality, and fits the situation where the product is brand new. In the second period, all customers observe their own type V. For the customers who purchased the product in period 1, they learn their type V by consuming the product. For the customers who did not purchase the product in period 1, they learn its quality and fit (thus, their type V) through social learning platforms (e.g., Facebook and Amazon customer review systems). Hence, the customers are homogeneous *ex ante* (i.e., at the beginning of period 1), but heterogeneous *ex post* (i.e., at the beginning of period 2). This is a common setting in the models concerning strategic customer behavior (see, e.g., [175, 158]). We assume that the valuation distribution $G(\cdot)$ has an increasing failure rate, i.e., $g(v)/\bar{G}(v)$ is increasing in v, where $g(\cdot) = G'(\cdot)$ is the density function and $\bar{G}(\cdot) = 1 - G(\cdot)$. This is a mild assumption and can be satisfied by most commonly used distributions. Let $\mu := \mathbb{E}(V) > c_1$, i.e., in expectation, a customer's valuation exceeds the production cost.

The firm offers an upgraded version of the product in period 2. This is a customary practice for product categories like consumer electronics, home appliances, and furniture. A type-V customer has a valuation of $(1 + \alpha)V$ for the upgraded second-generation product, where $\alpha \geq 0$ is exogenously given and captures the innovation level (e.g., the improved features) of the upgraded product. Accordingly, let the production cost of the second-generation product be c_2. To model the product depreciation, we take the approach of [166]: If a type-V customer has already bought the product in period 1, her valuation of consuming the used product in period 2 is $(1-k)V$, where $k \in [0, 1]$ refers to the depreciation factor. Specifically, if $k = 0$, the product is completely durable; if $k = 1$, the product is completely useless after the first period (either the product is worn out or the technology is obsolete). Therefore, the willingness-to-pay of a type-V customer in period 2 is $(1+\alpha)V$ if she *did not* purchase the product in period 1 (i.e., a new customer), and is $(1 + \alpha)V - (1 - k)V = (k + \alpha)V$ if she *purchased* the product in period 1 (i.e., a repeat customer).

As widely recognized in the literature, the firm can generate revenue from remanufacturing by reusing the materials and components of the recycled products (see [146, 142]). We now model the revenue-generating effect of remanufacturing. There are two types of remanufacturing in our model. First, the firm recycles the unsold first-generation products at the end of period 1. The recycled leftover inventory in the first period is remanufactured and can generate a net per-unit revenue r_1 ($r_1 < c_1$) for the firm. That is, in the base model we assume no excess inventory is carried over to the second period. This assumption applies when the inventory holding cost is sufficiently high or the firm does not want to dilute the sales of the newer generation product, which is usually the case in the electronics market. Moreover, this assumption facilitates the technical tractability of our model. Our results can be extended to the setting wherein the firm may hold leftover inventory and offer both product generations in the second period. The second type of remanufacturing is by using the returned products in period 2, i.e., customers who bought the product in period 1 can trade the old product for a second-generation one at a discounted price in period 2. The net revenue of remanufacturing from a used product in period 2 is r_2 ($r_2 < c_2$). Following [146], we assume that all remanufactured products are upgraded to the quality standards of new ones, so that consumers cannot distinguish them from newly made products. Relaxing this assumption will not affect our qualitative results.

The environmental impact of the product is the aggregate (negative) lifetime impact of the product on the environment. The total environmental impact is the production quantity of the product multiplied by the per-unit impact (see, e.g., [162, 4]). Let $\kappa_1 > 0$ denote the unit environmental impact of the first-generation product. Analogously, we denote $\kappa_2 > 0$ as the unit environmental impact of the second-generation product. Such impact may refer to the use of natural resources, emission of harmful gases, and generation of solid wastes. Moreover, κ_1 and κ_2 can be estimated by the conventional life-cycle analysis (see, e.g., [4]). To model the environmental benefit of remanufacturing, let ι_1 ($\iota_1 < \kappa_1$) be the unit environmental benefit of recycling the first-period leftover inventory, and ι_2 ($\iota_2 < \kappa_2$) be that of recycling the used products through trade-in rebates. Here, ι_1 and ι_2 refer to the reductions in both the production environmental impact in period 2 and the end-of-use and end-of-life product disposal, by recycling and reusing the materials and components of the first-generation products. To capture the environmental advantage of the second-generation product, we assume that $\kappa_1 - \iota_t \geq \kappa_2$ ($t = 1, 2$), i.e., the total environmental impact of the first-generation product dominates that of the second-generation product even if the end-of-use/end-of-life first-generation products are recycled and remanufactured.

The sequence of events unfolds as follows. At the beginning of period 1, the firm announces the price p_1 and decides the production quantity Q_1. Each customer observes p_1, but not Q_1, and makes her decision whether to order the product or to wait until period 2. The first-period demand $X_1 \leq X$ is then realized, the firm collects his first-period revenue, and all customers stay in the market. Note that X_1 is determined by the collective effect of all customers' purchasing behaviors. If $X_1 \leq Q_1$, any customer who requests a product can get one in period 1. Otherwise, $X_1 > Q_1$, then the Q_1 products are randomly allocated to the demand, and $X_1 - Q_1$ customers have to wait due to the limited availability. At the end of period 1, the firm recycles and remanufactures the leftover inventory. At the beginning of period 2, the firm learns the realized total market size X, and each individual customer learns her type V. The firm then announces the price p_2^n for new customers as well as the trade-in price $p_2^r \leq p_2^n$ ($p_2^n - p_2^r$ is the trade-in rebate); all new customers decide whether to purchase the second-generation product, whereas all repeat customers decide whether to trade their used products in for new second-generation ones. Finally, the firm produces the second-generation products, recycles and remanufactures the used products from repeat customers, and collects the second-period revenue.

For notational convenience, we will use $\mathbb{E}[\cdot]$ to denote the expectation operation, $x \wedge y$ to denote the minimum of two numbers x and y, and $\epsilon_1 \overset{d}{=} \epsilon_2$ to denote that two random variables ϵ_1 and ϵ_2 follow the same distribution. The scenario with myopic customers will be denoted with "~".

4.3.2 Equilibrium Analysis

We consider two scenarios, one with strategic customers and the other with myopic customers. Strategic customers maximize their total expected surplus over the two-period horizon, whereas myopic customers maximize their expected current-period surplus in each period. In both scenarios, the firm seeks to maximize his total expected profit over the entire horizon. For expositional convenience, we assume there is a common discount factor for the firm and the customers in period 2, denoted by $\delta \in (0, 1]$. To highlight the impact of strategic customer behavior upon the economic and environmental values of trade-in remanufacturing, we assume that the customers are either purely strategic or completely myopic. In reality, the actual customer behavior may take a form between these two extremes. Our model can be easily adapted to capture this situation by fixing the discount factor of the firm at δ, and allowing the discount factor of the customers δ_c to vary in the interval $[0, \delta]$. The higher the δ_c, the greater the customers' concern about future utilities, and thus the more strategic they are. In particular, $\delta_c = \delta$ ($\delta_c = 0$) corresponds to the scenario with purely strategic (myopic) customers.

To characterize the game outcome, we adopt the rational expectation (RE) equilibrium concept. The RE equilibrium was proposed by [129] and has been widely used in the operations management literature (e.g., [159, 160, 40, 41]). Using backward induction, we start with the decisions of the two parties in period 2. There are $X_2^n = X - (X_1 \wedge Q_1)$ new customers and $X_2^r = X_1 \wedge Q_1$ repeat customers in the market. Note that, since period 2 is the final period, strategic and myopic customers exhibit the same purchasing behavior therein. Hence, regardless of customer behavior, the firm should adopt the same pricing strategy in period 2 as well. Given (X_2^n, X_2^r), let $p_2^n(X_2^n, X_2^r)$ and $Q_2^n(X_2^n, X_2^r)$ be the equilibrium price and production quantity for new customers in period 2. Analogously, $p_2^r(X_2^n, X_2^r)$ and $Q_2^r(X_2^n, X_2^r)$ are the equilibrium trade-in price and production quantity for repeat customers, respectively. Correspondingly, we denote $\pi_2(X_2^n, X_2^r)$ as the equilibrium second-period profit of the firm.

Lemma 7 *(a) For any (X_2^n, X_2^r), $p_2^n(X_2^n, X_2^r) \equiv p_2^{n*}$ and $p_2^r(X_2^n, X_2^r) \equiv p_2^{r*}$, where*

$$p_2^{n*} = \text{argmax}_{p_2^n \geq 0}(p_2^n - c_2)\bar{G}\left(\frac{p_2^n}{1+\alpha}\right) \text{ and } p_2^{r*} = \text{argmax}_{p_2^r \geq 0}(p_2^r - c_2 + r_2)\bar{G}\left(\frac{p_2^r}{k+\alpha}\right).$$

Moreover, $p_2^{r*} < p_2^{n*}$ if and only if $k < 1$ or $r_2 > 0$.

(b) For any (X_2^n, X_2^r), $Q_2^n(X_2^n, X_2^r) = \bar{G}\left(\frac{p_2^{n*}}{1+\alpha}\right) X_2^n$, and $Q_2^r(X_2^n, X_2^r) = \bar{G}\left(\frac{p_2^{r*}}{k+\alpha}\right) X_2^r$.

(c) There exist two positive constants β_n^* and β_r^*, such that $\pi_2(X_2^n, X_2^r) = \beta_n^* X_2^n + \beta_r^* X_2^r$ for all (X_2^n, X_2^r), where

$$\beta_n^* = \max_{p_2^n \geq 0}(p_2^n - c_2)\bar{G}\left(\frac{p_2^n}{1+\alpha}\right) \text{ and } \beta_r^* = \max_{p_2^r \geq 0}(p_2^r - c_2 + r_2)\bar{G}\left(\frac{p_2^r}{k+\alpha}\right).$$

Lemma 7 characterizes the equilibrium pricing and production strategy of the firm in period 2. Specifically, both the equilibrium price for new customers and the equilibrium trade-in price are independent of the realized market size (X_2^n, X_2^r). Hence, the equilibrium production quantity for new (repeat) customers is a fixed fraction of the corresponding market size X_2^n (X_2^r) in period 2. As long as the used product is not completely useless to customers in period 2 (i.e., $k < 1$) or remanufacturing generates a positive revenue (i.e., $r_2 > 0$), the firm offers positive trade-in rebates to repeat customers. Moreover, the equilibrium profit of the firm in period 2, $\pi_2(X_2^n, X_2^r)$, is linearly separable in X_2^n and X_2^r.

We now analyze the firm's and the customers' decisions in period 1. We begin with the customers' purchasing behavior. A strategic customer forms beliefs about the first-period product availability probability $\mathfrak{a}$, the second-period price for new customers $\mathfrak{p}_2^n$, and the second-period trade-in price $\mathfrak{p}_2^r$, where $\mathfrak{a}$, $\mathfrak{p}_2^n$, and $\mathfrak{p}_2^r$ are all nonnegative random variables. Based on the belief vector $(\mathfrak{a}, \mathfrak{p}_2^n, \mathfrak{p}_2^r)$ and the observed first-period price p_1, she computes the expected utility of making an immediate purchase, $\mathcal{U}_p := \mathfrak{a}(\mathbb{E}[V] + \delta\mathbb{E}[(k+\alpha)V - \mathfrak{p}_2^r]^+ - p_1) + (1-\mathfrak{a})\delta\mathbb{E}[(1+\alpha)V - \mathfrak{p}_2^n]^+$, and the expected utility of waiting, $\mathcal{U}_w := \delta\mathbb{E}[(1+\alpha)V - \mathfrak{p}_2^n]^+$. Hence, the first-period reservation price of a strategic customer, ξ_r, is given by $\xi_r := \max\{p_1 : \mathcal{U}_p \geq \mathcal{U}_w\}$, and she will make a purchase in period 1 if and only if $p_1 \leq \xi_r$. The decision-making process of a myopic customer is simpler because she does not form beliefs about the first-period availability and second-period prices, but bases her purchasing decision on the current utility only. Hence, the first-period reservation price for a myopic customer equals her expected valuation of the product, i.e., $\tilde{\xi}_r = \mathbb{E}[V] = \mu$. Following the standard approach in the marketing ([175]) and the strategic customer behavior ([159, 41]) literature with homogeneous customers, we assume that all customers will make a purchase in period 1 if p_1 equals their reservation prices (ξ_r for strategic customers and $\tilde{\xi}_r$ for myopic customers). Thus, with strategic (myopic) customers, the first-period demand, X_1, is given by $X_1 = X \cdot \mathbf{1}_{\{p_1 \leq \xi_r\}}$ ($X_1 = X \cdot \mathbf{1}_{\{p_1 \leq \tilde{\xi}_r\}}$).

Next, we consider the firm's problem in period 1. The firm does not know the exact reservation price of strategic (myopic) customers ξ_r ($\tilde{\xi}_r$), but forms a belief $\mathfrak{r}_1$ ($\tilde{\mathfrak{r}}_1$) about it. To maximize his expected profit, the firm sets the first-period price p_1 ($\tilde{p}_1$) equal to the expected reservation price $\mathfrak{r}_1$ ($\tilde{\mathfrak{r}}_1$), which is the highest price (the firm believes) strategic (myopic) customers are willing to pay in the first period. Thus, the firm believes that the first-period demand $X_1 = X$. Thus, the second-period market size of new customers is $X_2^n = (X - Q_1)^+$, and that of repeat customers is $X_2^r = X \wedge Q_1$. Moreover, the firm sets the first-period production quantity Q_1 to maximize the total expected profit with strategic (myopic) customers $\Pi_f(Q_1)$ ($\tilde{\Pi}_f(Q_1)$), where $\Pi_f(Q_1) = \mathbb{E}\{p_1(X \wedge Q_1) - c_1 Q_1 + r_1(Q_1 - X)^+ + \delta\pi_2(X_2^n, X_2^r)\}$ and $\tilde{\Pi}_f(Q_1) = \mathbb{E}\{\tilde{p}_1(X \wedge Q_1) - c_1 Q_1 + r_1(Q_1 - X)^+ + \delta\pi_2(X_2^n, X_2^r)\}$, with $p_1 = \mathfrak{r}_1$, $\tilde{p}_1 = \tilde{\mathfrak{r}}_1$, $X_2^n = (X - Q_1)^+$, and $X_2^r = X \wedge Q_1$. Finally, under the RE equilibrium, all beliefs are rationally formulated and thus consistent with the actual outcomes.

Let $(p_1^*, Q_1^*, \xi_r^*, \mathfrak{r}^*, \mathfrak{a}^*, \mathfrak{p}_2^{n*}, \mathfrak{p}_2^{r*})$ and $(\tilde{p}_1^*, \tilde{Q}_1^*, \tilde{\xi}_r^*, \tilde{\mathfrak{r}}^*)$ be the RE equilibria in the market with strategic and myopic customers, respectively. For concision, the formal definitions of the RE equilibria in both scenarios are given in Appendix C.1. To characterize the RE equilibrium, we define two auxiliary variables $m_1^* := \mu + \delta[\beta_r^* - \beta_n^* + \mathbb{E}((k+\alpha)V - p_r^*)^+ - \mathbb{E}((1+\alpha)V - p_n^*)^+]$ and $\tilde{m}_1^* := \mu + \delta(\beta_r^* - \beta_n^*)$. As will be clear in our subsequent analysis, m_1^* ($\tilde{m}_1^*$) is the first-period effective marginal revenue with strategic (myopic) customers, which summarizes the impact of the second-period decisions on the first-period firm profit. Based on Lemma 7, we can characterize the RE equilibrium market outcome in the scenario with either strategic or myopic customers.

Theorem 4.3.1 *(a) With strategic customers, an RE equilibrium $(p_1^*, Q_1^*, \xi_r^*, \mathfrak{r}^*, \mathfrak{p}_2^{n*}, \mathfrak{p}_2^{r*}, \mathfrak{a}^*)$ exists with (i) $p_1^* = \mu + \delta[\mathbb{E}((k+\alpha)V - p_2^{r*})^+ - \mathbb{E}((1+\alpha)V - p_2^{n*})^+]$; and (ii) $Q_1^* = \bar{F}^{-1}(\frac{c_1 - r_1}{m_1^* - r_1})$. Moreover, all RE equilibria give rise to the identical expected total profit of the firm, $\Pi_f^* = (m_1^* - r_1)\mathbb{E}(X \wedge Q_1^*) - (c_1 - r_1)Q_1^* + \delta\beta_n^*\mathbb{E}(X)$.*

(b) With myopic customers, an RE equilibrium $(\tilde{p}_1^*, \tilde{Q}_1^*, \tilde{\xi}_r^*, \tilde{\mathfrak{r}}^*)$ exists with (i) $\tilde{p}_1^* = \mu$; and (ii) $\tilde{Q}_1^* = \bar{F}^{-1}(\frac{c_1 - r_1}{\tilde{m}_1^* - r_1})$. Moreover, all RE equilibria give rise to the identical expected total profit of the firm, $\tilde{\Pi}_f^* = (\tilde{m}_1^* - r_1)\mathbb{E}(X \wedge \tilde{Q}_1^*) - (c_1 - r_1)\tilde{Q}_1^* + \delta\beta_n^*\mathbb{E}(X)$.

Theorem 4.3.1(a) and (b) characterize the RE equilibrium market outcomes in the scenarios with strategic and myopic customers, respectively. In each scenario, the first-period price equals the corresponding expected reservation price of the customers, and the first-period production quantity can be determined by the solution of a corresponding newsvendor problem. In equilibrium, the total environmental impact should be the difference between the total environmental impact of production/disposal and the total environmental benefit of remanufacturing. Hence, the equilibrium environmental impact with strategic customers is $I_e^* = \mathbb{E}\{\kappa_1 Q_1^* + \delta\kappa_2(Q_2^n(X_2^{n*}, X_2^{r*}) + Q_2^r(X_2^{n*}, X_2^{r*})) - \iota_1(Q_1^* - X)^+ - \delta\iota_2 Q_2^r(X_2^{n*}, X_2^{r*})\}$, where $X_2^{n*} = (X - Q_1^*)^+$ and $X_2^{r*} = X \wedge Q_1^*$; whereas that with myopic customers is $\tilde{I}_e^* = \mathbb{E}\{\kappa_1 \tilde{Q}_1^* + \delta\kappa_2(Q_2^n(\tilde{X}_2^{n*}, \tilde{X}_2^{r*}) + Q_2^r(\tilde{X}_2^{n*}, \tilde{X}_2^{r*})) - \iota_1(\tilde{Q}_1^* - X)^+ - \delta\iota_2 Q_2^r(\tilde{X}_2^{n*}, \tilde{X}_2^{r*})\}$, where $\tilde{X}_2^{n*} = (X - \tilde{Q}_1^*)^+$ and $\tilde{X}_2^{r*} = X \wedge \tilde{Q}_1^*$.

4.4 Impact of Trade-in Remanufacturing

In this section, we analyze the impact of trade-in remanufacturing on the firm and the environment under different customer behaviors (i.e., strategic or myopic customers). Our focus is on how strategic customer behavior influences the economic and environmental values of trade-in remanufacturing.

To facilitate our comparison, we first introduce a benchmark model where the firm does not offer trade-in rebates to customers. As a consequence, the firm cannot recycle used products for remanufacturing in period 2. We call this the No Trade-in Remanufacturing (NTR) model, which is denoted by the superscript "u" hereafter. We use $p_2^u(X_2^n, X_2^r)$ to denote the equilibrium second-period pricing strategy of the firm in the NTR model, which does not depend on customer behavior. As in the base model, the firm forms a belief about the customers' expected willingness-to-pay in the first period, and bases his price and production decisions on this belief. The customers, on the other hand, form beliefs about the product availability and the second-period price, and time their purchases. Again, the formal definitions of the RE equilibrium in the NTR model are given in Appendix C.1. By the same argument in the proof of Theorem 4.3.1, we can show that an RE equilibrium exists with either strategic or myopic customers in the NTR model. Let (p_1^{u*}, Q_1^{u*}) denote the equilibrium first-period price and production decisions of the firm with strategic customers, and $(\tilde{p}_1^{u*}, \tilde{Q}_1^{u*})$ denote those with myopic customers in the NTR model. Accordingly, the associated equilibrium expected profit of the firm (environmental impact) is denoted by Π_f^{u*} (I_e^{u*}) in the scenario with strategic customers, and by $\tilde{\Pi}_f^{u*}$ ($\tilde{I}_e^{u*}$) in the scenario with myopic customers.

Let $\Pi_f^u(Q_1)$ ($\tilde{\Pi}_f^u(Q_1)$) be the expected profit of the firm with strategic (myopic) customers to produce Q_1 products in the period 1 in the NTR model. We characterize the objective functions $\Pi_f(\cdot)$, $\tilde{\Pi}_f(\cdot)$, $\Pi_f^u(\cdot)$, and $\tilde{\Pi}_f^u(\cdot)$ in the following lemma.

Lemma 8 *The objective functions are given by* $\Pi_f(Q_1) = (m_1^* - r_1)\mathbb{E}(X \wedge Q_1) - (c_1 - r_1)Q_1 + \delta\beta_2^{n*}\mathbb{E}(X)$, $\tilde{\Pi}_f(Q_1) = (\tilde{m}_1^* - r_1)\mathbb{E}(X \wedge Q_1) - (c_1 - r_1)Q_1 + \delta\beta_2^{n*}\mathbb{E}(X)$,

$$\Pi_f^u(Q_1) = (m_1^u(Q_1) - r_1)\mathbb{E}(X \wedge Q_1) - (c_1 - r_1)Q_1 + \delta\mathbb{E}\{(p_2^u(X_2^n, X_2^r) - c_2)\bar{G}\left(\frac{p_2^u(X_2^n, X_2^r)}{1+\alpha}\right)X\},$$

and

$$\tilde{\Pi}^u_f(Q_1) = (\tilde{m}^u_1(Q_1)-r_1)\mathbb{E}(X\wedge Q_1)-(c_1-r_1)Q_1+\delta\mathbb{E}\{(p^u_2(X^n_2,X^r_2)-c_2)\bar{G}\left(\frac{p^u_2(X^n_2,X^r_2)}{1+\alpha}\right)X\},$$

where $X^n_2 = (X-Q_1)^+$, $X^r_2 = X\wedge Q_1$. *The expressions of* $m^u_1(\cdot)$ *and* $\tilde{m}^u_1(\cdot)$ *are given in Appendix C.2.*

Lemma 8 implies that, in the NTR model, the effective first-period marginal revenue is production-quantity-dependent, and given by $m^u_1(\cdot)$ in the scenario with strategic customers and by $\tilde{m}^u_1(\cdot)$ with myopic customers. The economic interpretation of $m^u_1(Q_1)$ ($\tilde{m}^u_1(Q_1)$) is that, when the first-period production quantity is Q_1, it measures the additional expected marginal revenue to sell the product in period 1 over that in period 2 with strategic (myopic) customers. Hence, the higher the $m^u_1(Q_1)$ and $\tilde{m}^u_1(Q_1)$, the more profitable it is for the firm to sell the first-generation product in the NTR model with strategic and myopic customers, respectively. In other words, $m^u_1(\cdot)$ and $\tilde{m}^u_1(\cdot)$ capture the willingness-to-produce of the firm in period 1. Without loss of generality, we focus on the case where $m^u_1(\cdot) > 0$ and $\tilde{m}^u_1(\cdot) > 0$ for all $Q_1 \geq 0$, i.e., the firm can gain a positive revenue to sell the first-generation product. Otherwise, the firm will not produce or sell anything in period 1.

4.4.1 Impact on Firm Profit

This subsection investigates the value of trade-in remanufacturing to the firm. To begin with, we characterize the role of strategic customer behavior, depending on whether the firm adopts trade-in remanufacturing or not.

Theorem 4.4.1 *(a) Under trade-in remanufacturing, let* $e^* := \mathbb{E}((k+\alpha)V - p^{r*}_2)^+ - \mathbb{E}((1+\alpha)V - p^{n*}_2)^+$. *Then, we have (i)* $p^*_1 > \tilde{p}^*_1$ *if and only if* $e^* > 0$, *(ii)* $Q^*_1 > \tilde{Q}^*_1$ *if and only if* $e^* > 0$, *and (iii)* $\Pi^*_f > \tilde{\Pi}^*_f$ *if and only if* $e^* > 0$. *Moreover, there exists a threshold* $\bar{r} \geq \frac{1-k}{1+\alpha}c_2$, *such that* $e^* > 0$ *if and only if* $r_2 > \bar{r}$.

(b) Under no trade-in remanufacturing, we have (i) $p^{u*}_1 \leq \tilde{p}^{u*}_1$, *where the inequality is strict if* $k < 1$, *(ii)* $Q^{u*}_1 \leq \tilde{Q}^{u*}_1$, *and (iii)* $\Pi^{u*}_f \leq \tilde{\Pi}^{u*}_f$, *where the inequality is strict if* $k < 1$ *and* $\tilde{Q}^{u*}_1 > 0$.

Under trade-in remanufacturing, Theorem 4.4.1(a) compares the equilibrium outcomes with different customer behaviors. We find that the key to this comparison is the difference between the expected surplus of a repeat customer and that of a new customer in period 2 (i.e., e^*). With strategic customers, the firm charges a higher first-period price, sets a higher first-period production level, and earns a higher total expected profit, if and only if the expected second-period surplus of a repeat customer is higher than that of a new customer (i.e, $e^* > 0$). In particular, the presence of strategic customer behavior benefits the firm if the revenue-generating effect of remanufacturing is strong enough (i.e., $r_2 > \bar{r}$). This result is in sharp contrast with the well-established notion in the literature that strategic customer behavior hurts a firm's profit (e.g., [17, 159]). Trade-in remanufacturing leads to a price discount for repeat customers in period 2, which can be perceived by strategic customers when deciding whether to make a purchase in period 1. This discount outweighs the benefit of strategic waiting if the revenue generated from remanufacturing is sufficiently high (i.e., $r_2 > \bar{r}$). In this case, the presence of forward-looking behavior will enable the firm to charge a higher price, produce more, and thus earn a higher profit. We emphasize that both the trade-in option and the revenue-generating effect of remanufacturing are essential for the firm to benefit from strategic customer behavior: The former induces strategic customers to anticipate the price discount for repeat customers, whereas the latter brings in the additional benefit that guarantees a deep discount so that strategic customers are willing to pay an even higher first-period price than myopic customers. In contrast, Theorem 4.4.1(b) shows that, without trade-in remanufacturing, the firm always suffers from strategic customer behavior, as reported in the existing literature.

Theorem 4.4.1 suggests that the presence of strategic customer behavior will make trade-in remanufacturing more attractive to the firm. Next, we study how trade-in remanufacturing influences the profit and the pricing strategy of the firm under different customer behaviors. The following theorem compares the equilibrium prices and profits in the NTR model and those in the base model with either strategic or myopic customers.

Theorem 4.4.2 *(a) In period 2, $p_2^u(X_2^n, X_2^r)$ is increasing in X_2^n and decreasing in X_2^r. Moreover, for any (X_2^n, X_2^r), $p_2^{r*} \leq p_2^u(X_2^n, X_2^r) \leq p_2^{n*}$, where the inequalities are strict if $k < 1$ and $X_2^n, X_2^r > 0$.*

(b) With strategic customers, we have (i) $p_1^{u*} \leq p_1^*$, where the inequality is strict if $p_2^{r*} < p_2^{n*}$; and (ii) $\Pi_f^{u*} \leq \Pi_f^*$, where the inequality is strict if $p_2^{r*} < p_2^{n*}$ and $Q_1^* > 0$.

(c) With myopic customers, we have (i) $\tilde{p}_1^{u*} = \tilde{p}_1^*$; and (ii) $\tilde{\Pi}_f^{u*} \leq \tilde{\Pi}_f^*$, where the inequality is strict if $p_2^{r*} < p_2^{n*}$ and $\tilde{Q}_1^{u*} > 0$.

Theorem 4.4.2 shows that the equilibrium second-period price without trade-in remanufacturing, $p_2^u(\cdot,\cdot)$, is bounded from below by the equilibrium second-period trade-in price p_2^{r*}, and from above by the equilibrium second-period price for new customers p_2^{n*}. Hence, under trade-in remanufacturing, the expected utility of strategic customers to make a purchase in the first period increases (i.e., $\delta\mathbb{E}[(k+\alpha)V - p_2^{r*}]^+ \geq \delta\mathbb{E}[(k+\alpha)V - p_2^u(X_2^n, X_2^r)]^+$), whereas the benefit of waiting decreases decreases (i.e., $\delta\mathbb{E}[(1+\alpha)V - p_2^{n*}]^+ \leq \delta\mathbb{E}[(1+\alpha)V - p_2^u(X_2^n, X_2^r)]^+$. This implies that trade-in remanufacturing makes strategic customers more willing to purchase immediately than to wait until period 2. Therefore, trade-in remanufacturing enables the firm to exploit the forward-looking behavior of strategic customers and thus induces early purchases from them. With myopic customers, however, trade-in remanufacturing does not have an early-purchase inducing effect because myopic customers do not care about their future surplus. This result is also consistent with the finding in the durable product literature that the secondary market gives rise to greater resale value of a durable product and thus can increase the sales of the new product upfront (see, e.g., [94, 169]).

From Theorem 4.4.2, we can see there are three beneficial effects of trade-in remanufacturing that may improve firm profit: (a) the revenue-generating effect of remanufacturing, (b) the price discrimination effect of trade-in rebates, i.e., the differentiated prices for new and repeat customers helps the firm exploit the customer segmentation in period 2, and (c) the early-purchase inducing effect of trade-in rebates, i.e., the price discount to repeat customers enables the firm to exploit the forward-looking behavior of strategic customers by offering early-purchase rewards. The first two effects benefit the firm with either strategic or myopic customers, whereas the third effect improves the firm's profit with strategic customers only. In the following, we conduct extensive numerical experiments to quantify the third effect, and deliver insights on how strategic customer behavior influences the value of trade-in remanufacturing to the firm.

The design of the numerical study is as follows. Let the customer valuation V follow a uniform distribution on $[0, 1]$ ($\mu = \mathbb{E}(V) = 0.5$). The discount factor is $\delta = 0.95$, the unit environmental impact of the first-generation product is $\kappa_1 = 1$, and the unit environmental impact of the second-generation product is $\kappa_2 = 0.75$. To focus on the impact of customer behaviors, we set $r_1 = r_2 = 0$ (i.e., there is no revenue-generating effect associated with remanufacturing), and the unit environmental benefits of recycling/remanufacturing to be $\iota_1 = 0$ and $\iota_2 = 0.3$ (these two values will be useful when studying the environmental impact in Section 4.2). The unit production cost of the first-generation product is $c_1 \in \{0.05, 0.1, 0.15, 0.2, 0.25\}$. The innovation level of the second-generation product is $\alpha \in \{0, 0.05, 0.1, 0.15, 0.2\}$, and the unit production cost of the second-generation product is $c_2 = 0.25(1+\alpha) \in \{0.25, 0.2625, 0.275, 0.2875, 0.3\}$. We consider the depreciation factor $k \in \{0.3, 0.4, 0.5, 0.6, 0.7\}$. The demand X follows a gamma distribution with mean 100 and coefficient of variation $CV(X)$ taking values from the set $\{0.5, 0.6, 0.7, 0.8, 0.9\}$. Thus, we have a total of 625 parameter combinations that cover a wide range of reasonable problem scenarios. The above problem scenarios form a subset of the extensive experiments we have conducted. Our numerical findings are very robust. For concision, we will only present the results for the parameter combinations listed above.

We calculate the expected profit for each scenario with either strategic or myopic customers both in the base model, $(\Pi_f^*, \tilde{\Pi}_f^*)$ and in the NTR model, $(\Pi_f^{u*}, \tilde{\Pi}_f^{u*})$. The two metrics of interest are: $\gamma_s := (\Pi_f^* - \Pi_f^{u*})/\Pi_f^{u*} \times 100\%$, and $\gamma_m := (\tilde{\Pi}_f^* - \tilde{\Pi}_f^{u*})/\tilde{\Pi}_f^{u*} \times 100\%$, i.e., γ_s (γ_m) refers to the relative profit improvement of trade-in remanufacturing with strategic (myopic) customers. We evaluate γ_s and γ_m under the 625 parameter combinations and report that, under each combination, γ_s is significantly higher than γ_m. More specifically, γ_s is at least 5.8% and can be as high as 61.6%, with an average of 30.2%; whereas γ_m ranges from 0.008% to 11.7%, with an average of 3.1%. We give the summary statistics of γ_s and γ_m in Table 4.1.

	Min	5th percentile	Median	95th percentile	Max	Mean	Stan. Dev.
γ_s	5.8	11.3	28.3	55.8	61.6	30.2	13.1
γ_m	0.008	0.22	2.5	8.1	11.7	3.1	2.5

Table 4.1
Summary Statistics: Firm Profit (%)

Our numerical results deliver an important message on the economic value of trade-in remanufacturing: Trade-in remanufacturing delivers a much higher value to the firm with strategic customers than with myopic customers (γ_s is significantly higher than γ_m for each problem instance). Recall that, with myopic customers, trade-in remanufacturing only has the benefits of revenue-generating and price discrimination, whereas, with strategic customers, this strategy has the additional value of inducing early purchases. Therefore, these results indicate that the value of trade-in remanufacturing to the firm mainly comes from the early-purchase inducing effect of trade-in rebates to exploit strategic customer behavior, rather than from the revenue-generating effect of remanufacturing or the price discrimination effect of trade-in rebates to exploit customer segmentation.

4.4.2 Impact on Environment and Customer Surplus

Our next goal is to examine the environmental value of trade-in remanufacturing under different customer behaviors. We first characterize how trade-in remanufacturing influences the effective first-period marginal revenue and production quantities of the firm.

Theorem 4.4.3 *Assume $k < 1$.*

(a) With strategic customers, we have (i) $m_1^u(Q_1)$ is decreasing in Q_1; (ii) $m_1^u(Q_1) < m_1^*$ for all Q_1; (iii) $Q_1^{u*} \leq Q_1^*$, where the inequality is strict if $Q_1^* > 0$.

(b) With myopic customers, we have (i) $\tilde{m}_1^u(Q_1)$ is increasing in Q_1; (ii) for each $r_2 < \bar{r}_2$[2], there exists a threshold $\bar{Q}(r_2)$ increasing in r_2, such that $\tilde{m}_1^u(Q_1) \leq \tilde{m}_1^*$ for all $Q_1 \leq \bar{Q}(r_2)$, and $\tilde{m}_1^u(Q_1) > \tilde{m}_1^*$ for all $Q_1 > \bar{Q}(r_2)$; (iii) for each $r_2 < \bar{r}_2$, there exists a threshold $\bar{c}_1(r_2) > 0$, such that $Q_1^{u*} > Q_1^*$ if $c_1 \leq \bar{c}_1(r_2)$.

Theorem 4.4.3 provides an interesting comparison between the scenarios of strategic and myopic customers: With strategic customers, trade-in remanufacturing always increases the first-period production quantity of the firm, whereas it may prompt the firm to produce less with myopic customers. More specifically, Theorem 4.4.3(a) shows that, under strategic customer behavior, the effective marginal revenue with trade-in remanufacturing always dominates that without (i.e., $m_1^u(\cdot) < m_1^*$). As a result, the firm produces more in period 1 under trade-in remanufacturing. Theorem 4.4.3(b), however, suggests that, with myopic customers, trade-in remanufacturing may give rise to a lower first-period effective marginal revenue if the production quantity is large (i.e.,

[2]The expression of $\bar{r}_2$ is given in Appendix C.2.

$\tilde{m}_1^u(Q_1) > \tilde{m}_1^*$ if $Q_1 > \bar{Q}(r_2)$), thus driving the firm to lower the first-period production quantity if the first-period unit production cost is low (i.e., $c_1 \leq \bar{c}_1(r_2)$). Recall from Theorem 4.4.2 that trade-in remanufacturing increases the first-period willingness-to-pay of strategic customers, which, in turn, drives the firm to produce more in period 1. Such early-purchase and, thus, early-production inducing effects of trade-in remanufacturing, however, are absent with myopic customers. In the scenario of myopic customers, on the other hand, the price discrimination effect of trade-in remanufacturing improves the unit profit generated from the new customers in period 2, thus leading to a lower effective first-period marginal revenue if the revenue-generating effect of remanufacturing is not too strong (i.e., $r_2 < \bar{r}_2$). As a consequence, the firm decreases the first-period production quantity to increase the second-period market size of new customers.

Theorem 4.4.3 demonstrates the contrasting effects of trade-in remanufacturing on production quantities under different customer behaviors. How does trade-in remanufacturing affect the environment? The answer is given in the next theorem.

Theorem 4.4.4 *(a) With strategic customers, there exists a threshold $\bar{\iota}_2^u > 0$, such that $I_e^* \geq I_e^{u*}$ if $\iota_2 \leq \bar{\iota}_2^u$.*

(b) Assume that $r_2 < \bar{r}_2$ and $c_1 \leq \bar{c}_1(r_2)$. With myopic customers, there exists a threshold $\tilde{\iota}_2^u < \kappa_2$, such that $\tilde{I}_e^{u*} \geq \tilde{I}_e^*$ if $\iota_2 \geq \tilde{\iota}_2^u$.

When customers are strategic, trade-in rebates encourage them to recycle the used first-generation products more frequently, so they also purchase the product more frequently. In this scenario, trade-in remanufacturing leads to a worsened outcome for the environment if the unit environmental benefit of remanufacturing is not high enough to justify the early-production inducing effect (i.e., $\iota_2 \leq \bar{\iota}_2^u$ in Theorem 4.4.4(a)). When the customers are myopic and the unit production cost is sufficiently low, trade-in remanufacturing motivates the firm to produce less in period 1 (see Theorem 4.4.3(b)). Hence, trade-in remanufacturing helps improve the environment as long as the unit environmental benefit of remanufacturing is not too low (i.e., $\iota_2 \geq \tilde{\iota}_2^u$ in Theorem 5(b)). Theorem 4.4.4 reveals the significant impact of customer behavior on the environmental value of trade-in remanufacturing. With strategic customers, the adoption of trade-in remanufacturing is likely to be detrimental to the environment, whereas, with myopic customers, adopting trade-in remanufacturing may benefit both the firm and the environment. Some papers in the literature (e.g., [59, 81, 90]) have also established that remanufacturing may increase the production quantity and thus environmental impact. Our work, however, demonstrates that the environmental impact of trade-in remanufacturing depends critically on customer behavior.

We now numerically illustrate the environmental value of trade-in remanufacturing. We employ the same numerical setup as Section 4.4.1. Recall that I_e^* ($\tilde{I}_e^*$) is the expected environmental impact for the scenario with strategic (myopic) customers in the base model, and I_e^{u*} ($\tilde{I}_e^{u*}$) is that in the NTR model. We are interested in the following two metrics: $\eta_s := (I_e^* - I_e^{u*})/I_e^{u*} \times 100\%$, and $\eta_m := (\tilde{I}_e^* - \tilde{I}_e^{u*})/\tilde{I}_e^{u*} \times 100\%$, i.e., η_s (η_m) refers to the relative change of the environmental impact when adopting trade-in remanufacturing with strategic (myopic) customers.

We evaluate η_s and η_m under the 625 parameter combinations and obtain the following numerical findings: (i) Under each parameter combination, η_s is significantly higher than η_m; and (ii) For most of the parameter combinations, $\eta_s > 0$ but $\eta_m < 0$. Specifically, η_s takes values from -1.2% to 171.9%, with an average of 49.2%; whereas η_m ranges from -10.2% to 4.5%, with an average of -5.0%. Moreover, $\eta_s < 0$ (i.e., trade-in remanufacturing benefits the environment with strategic customers) for 10 out of the 625 (i.e., 1.6%) problem instances we examine, whereas $\eta_m < 0$ (i.e., trade-in remanufacturing benefits the environment with myopic customers) for 585 out of the 625 (i.e., 93.6%) problem instances. Table 4.2 summarizes the statistics of η_s and η_m.

	Min	5th percentile	Median	95th percentile	Max	Mean	Stan. Dev.
η_s	-1.2	2.0	37.8	117.8	171.9	49.2	41.4
η_m	-10.2	-8.5	-5.5	0.51	4.5	-5.0	2.7

Table 4.2
Summary Statistics: Environmental Impact (%)

Table 2 confirms that trade-in remanufacturing generally leads to much higher environmental impact with strategic customers than with myopic customers (η_s is significantly higher than η_m). Though beneficial to the firm (see Table 4.1), the early-purchase inducing effect of trade-in remanufacturing gives rise to much higher production quantities under strategic customer behavior, and thus leads to a much worse outcome from the environmental perspective. Therefore, strategic customer behavior has opposing effects on the value of trade-in remanufacturing to the firm and the environment: It makes this strategy more attractive to the firm, but less desirable to the environment.

The above results suggest that trade-in remanufacturing may create a tension between firm profitability and environmental sustainability with strategic customers, but benefits both the firm and the environment with myopic customers. Since η_s is significantly larger than zero for most of the numerical cases we examine, trade-in remanufacturing is detrimental to the environment for a large set of reasonable problem instances under strategic customer behavior. Hence, in general, the early-purchase inducing effect dominates the environmental benefit of remanufacturing with strategic customers. Under strategic customer behavior, the firm significantly benefits from trade-in remanufacturing, but the environment significantly suffers from this strategy (i.e., $\gamma_s > 0$ and, in general, $\eta_s > 0$). With myopic customers, however, both the firm and the environment would benefit from the adoption of trade-in remanufacturing (i.e., $\gamma_m > 0$ and, in general, $\eta_m < 0$).

Although an increased production quantity means more pressure on the environment, it also increases the consumption level of the product. To conclude this section, we explore how trade-in remanufacturing impacts the total customer surplus under different customer behaviors. We use S_c^* ($\tilde{S}_c^*$) and S_c^{u*} ($\tilde{S}_c^{u*}$) to denote the equilibrium total customer surplus for the scenarios with strategic (myopic) customers in the base model and the NTR model, respectively.

Theorem 4.4.5 *(a) In the base model, we have* $S_c^* = \delta\mathbb{E}[((1+\alpha)V - p_2^{n*})^+X]$ *and* $\tilde{S}_c^* = \delta\mathbb{E}[((1+\alpha)V - p_2^{n*})^+(X - \tilde{Q}_1^*)^+] + \delta\mathbb{E}[((k+\alpha)V - p_2^{r*})^+(X \wedge \tilde{Q}_1^*)]$.

(b) In the NTR model, we have $S_c^{u*} = \delta\mathbb{E}[((1+\alpha)V - \mathfrak{p}_2^{u*})^+X]$ *and* $\tilde{S}_c^{u*} = \delta\mathbb{E}[((1+\alpha)V - \mathfrak{p}_2^{u*})^+(X - \tilde{Q}_1^{u*})^+] + \delta\mathbb{E}[((k+\alpha)V - \mathfrak{p}_2^{u*})^+(X \wedge \tilde{Q}_1^{u*})]$.

(c) We have the following relationship on the customer surpluses of strategic customers: $S_c^* \leq S_c^{u*}$*, where the inequality is strict if* $k < 1$ *and* $Q_1^{u*} > 0$.

Theorem 4.4.5(a) and (b) compute the total customer surpluses in the base model and the NTR model. Moreover, in Theorem 4.4.5(c), we demonstrate that, with strategic customers, the total customer surplus always decreases with the adoption of trade-in remanufacturing. This is because, with strategic customers, the total customer surplus only depends on the (perceived) price for new customers in period 2, which is higher under the adoption of trade-in remanufacturing (see Theorem 4.4.2(a)). By Theorem 4.4.3(a), one may argue that, under strategic customer behavior, trade-in remanufacturing increases production quantities and thus increases the customer surplus. Theorem 4.4.5(c), however, shows that the total customer surplus actually decreases in this scenario. Hence, under strategic customer behavior, trade-in remanufacturing gives rise to higher production quantities without improving the customer surplus. Further, the social welfare (i.e., firm profit plus customer surplus less environmental impact) is likely to decrease under trade-in remanufacturing as well. This has been confirmed in the numerical study we explored in this section.

To summarize, customer behavior plays an important role in the economic and environmental values of trade-in remanufacturing. With myopic customers, trade-in remanufacturing benefits both the firm and the environment. With strategic customers, trade-in remanufacturing would be even more beneficial to the firm; however, meanwhile it may hurt the environment, decrease customer surplus, and possibly lower social welfare. Therefore, it is important for firms and policy-makers to understand customer behavior when making decisions related to trade-in remanufacturing.

4.5 Social Optimum and Government Intervention

As shown in Section 4.4, adopting trade-in remanufacturing may create a tension between firm profitability and environmental sustainability under strategic customer behavior. In this section, we analyze how a policy-maker (e.g., the government) can design the public policy to resolve this tension and maximize the social welfare under different customer behaviors.

To characterize the socially optimal outcome, we assume that the government can set the prices and production levels, with an objective to maximize the social welfare. Let W_s denote the social welfare, which is defined by the expected profit of the firm Π_f, plus the expected customer surplus S_c, net the expected environmental impact I_e, i.e.,

$$W_s = \Pi_f + S_c - I_e.$$

By backward induction, we start with the second-period pricing and production problem. As in the base model, strategic and myopic customers exhibit the same purchasing behavior in period 2. For any given realized market size in period 2 (X_2^n, X_2^r), we use

$(p^n_{s,2}(X^n_2, X^r_2), p^r_{s,2}(X^n_2, X^r_2))$ to denote the equilibrium pricing strategy, and $(Q^n_{s,2}(X^n_2, X^r_2), Q^r_{s,2}(X^n_2, X^r_2))$ to denote the equilibrium production strategy. Correspondingly, we denote $w_2(X^n_2, X^r_2)$ as the equilibrium second-period social welfare.

Lemma 9 *(a) For any (X^n_2, X^r_2), $p^n_{s,2}(X^n_2, X^r_2) \equiv p^{n*}_{s,2}$ and $p^r_{s,2}(X^n_2, X^r_2) \equiv p^{r*}_{s,2}$, where $p^{n*}_{s,2} = c_2 + \kappa_2$ and $p^{r*}_{s,2} = c_2 - r_2 + \kappa_2 - \iota_2$. Hence, $p^{n*}_{s,2} > p^{r*}_{s,2}$ if and only if $r_2 > 0$ or $\iota_2 > 0$.*

(b) For any (X^n_2, X^r_2), $Q^n_{s,2}(X^n_2, X^r_2) = \bar{G}\left(\frac{p^{n*}_{s,2}}{1+\alpha}\right) X^n_2$, and $Q^r_{s,2}(X^n_2, X^r_2) = \bar{G}\left(\frac{p^{r*}_{s,2}}{k+\alpha}\right) X^r_2$.

(c) There exist two positive constants $\beta^*_{s,n}$ and $\beta^*_{s,r}$, such that $w_2(X^n_2, X^r_2) = \beta^*_{s,n} X^n_2 + \beta^*_{s,r} X^r_2$ for all (X^n_2, X^r_2), where $\beta^*_{s,n} = \mathbb{E}[(1+\alpha)V - p^{n*}_{s,2}]^+$ and $\beta^*_{s,r} = \mathbb{E}[(k+\alpha)V - p^{r*}_{s,2}]^+$.

Lemma 9 implies that, with either strategic or myopic customers, the socially optimal second-period pricing strategy takes the form that the prices for new and repeat customers are equal to the respective net unit production cost plus the net unit environmental impact (i.e., $p^{n*}_{s,2} = c_2 + \kappa_2$ and $p^{r*}_{s,2} = c_2 - r_2 + \kappa_2 - \iota_2$). Moreover, the equilibrium social welfare is linear in the realized market size (X^n_2, X^r_2).

In period 1, strategic customers base their purchasing decisions on their rational expectations, whereas myopic customers decide whether to make a purchase by comparing the current price and the expected valuation. Let $(p^*_{s,1}, Q^*_{s,1})$ denote the equilibrium first-period price and production quantity with strategic customers, and $(\tilde{p}^*_{s,1}, \tilde{Q}^*_{s,1})$ denote those with myopic customers. As in the base model and the NTR model, we introduce the first-period effective marginal welfare with either strategic or myopic customers, $m^*_{s,1} = \tilde{m}^*_{s,1} := \mu + \delta[\beta^*_{s,r} - \beta^*_{s,n}]$. The following lemma characterizes the social welfare maximizing equilibrium outcomes.

Lemma 10 *(a) With strategic customers, we have (i) $p^*_{s,1} = m^*_{s,1}$; (ii) $Q^*_{s,1} = \bar{F}^{-1}(\frac{c_1+\kappa_1-r_1-\iota_1}{m^*_{s,1}-r_1-\iota_1})$; and (iii) the equilibrium expected social welfare is $W^*_s = (m^*_{s,1} - r_1 - \iota_1)\mathbb{E}(X \wedge Q^*_{s,1}) - (c_1 + \kappa_1 - r_1 - \iota_1)Q^*_{s,1} + \delta\beta^*_{s,n}\mathbb{E}[X]$.*

(b) With myopic customers, we have (i) $\tilde{p}^*_{s,1} = \mu$; (ii) $\tilde{Q}^*_1 = \bar{F}^{-1}(\frac{c_1+\kappa_1-r_1-\iota_1}{\tilde{m}^*_{s,1}-r_1-\iota_1})$; and (iii) the equilibrium expected social welfare is $\tilde{W}^*_s = (\tilde{m}^*_{s,1} - r_1 - \iota_1)\mathbb{E}(X \wedge \tilde{Q}^*_{s,1}) - (c_1 + \kappa_1 - r_1 - \iota_1)\tilde{Q}^*_{s,1} + \delta\beta^*_{s,n}\mathbb{E}[X]$.

(c) Let $e_s^* := \beta_{s,r}^* - \beta_{s,n}^*$. Then, we have (i) $p_{s,1}^* \geq \tilde{p}_{s,1}^*$ if and only if $e_s^* \geq 0$; (ii) $Q_{s,1}^* = \tilde{Q}_{s,1}^*$; and (iii) $W_s^* = \tilde{W}_s^*$.

Since the social planner needs to balance firm profit, customer surplus, and environmental impact, whereas the firm maximizes its own profit only, the social-welfare-maximizing equilibrium outcome may be quite different from the profit-maximizing one, as shown by comparing Lemma 10 with Theorem 4.3.1. In particular, if the unit environmental impacts, κ_1 and κ_2, are sufficiently large, the social planner will set lower production quantities than the firm will do to limit the total environmental impacts. Lemma 10(c) characterizes how different customer behaviors influence the social-welfare-maximizing RE equilibrium outcome. Specifically, we show that the expected optimal social welfare with strategic customers is the same as that with myopic customers, and so is the optimal first-period production quantity. The equilibrium first-period price, however, depends on customer behavior. If the expected surplus of a repeat customer dominates that of a new customer (i.e., $e_s^* \geq 0$), the equilibrium first-period price is higher with strategic customers. Otherwise, $e_s^* < 0$, the equilibrium first-period price is higher with myopic customers. We notice that e_s^* is the counterpart of e^* (see Theorem 4.4.1), both of which characterize the additional expected utility of a repeat customer over a new one in period 2.

We now analyze how the government, whose objective is to maximize the expected social welfare W_s, could induce the firm, whose objective is to maximize his expected profit Π_f, to set the socially optimal prices and production quantities under different customer behaviors. A commonly-observed government subsidization policy is to subsidize the firm or customers for the remanufactured products (see, e.g., [128, 44]). To model this subsidization policy, we assume that the government offers the firm a per-unit subsidy s_r for remanufacturing leftover inventory and used products. The per-unit subsidy to the firm is without loss of generality, because all results and qualitative insights in this section continue to hold with the per-unit subsidy to customers, and the proportional subsidy[3] to either the firm or the customers. For expositional ease, we take the approach of per-unit subsidy to the firm.

[3]The proportional subsidy refers to the government subsidization scheme under which the unit subsidy is proportional to (e.g., 10% of) the sales price/trade-in price.

We first study how the government subsidization policy for remanufactured products would influence the equilibrium outcome in the following theorem.

Theorem 4.5.1 *(a) For any (X_2^n, X_2^r), we have (i) p_2^{r*} is decreasing in s_r; and (ii) $Q_2^r(X_2^n, X_2^r)$ is increasing in s_r.*

(b) With strategic customers, we have (i) p_1^* is increasing in s_r; (ii) Q_1^* is increasing in s_r; (iii) Π_f^* is increasing in s_r; and (iv) I_e^* is increasing in s_r.

(c) With myopic customers, we have (i) $\tilde{p}_1^*$ is independent of s_r; (ii) $\tilde{Q}_1^*$ is increasing in s_r; (iii) $\tilde{\Pi}_f^*$ is increasing in s_r; and (iv) $\tilde{I}_e^*$ is increasing in s_r.

One of the main goals for the government to subsidize remanufacturing is to improve the environment (see [44, 161]). Theorem 4.5.1(b,c), however, suggests that if the government only subsidizes for remanufacturing (i.e., $s_r > 0$), the environment actually suffers from this subsidization policy with either strategic or myopic customers (i.e., I_e^* and $\tilde{I}_e^*$ are increasing in s_r). This result follows from the rationale that subsidizing remanufactured products not only promotes the adoption of remanufacturing, but also increases the production levels of the first-generation product, which is the least environmentally friendly product version. The environment thus suffers from the increased production levels under the subsidization for remanufacturing alone. Therefore, the government should be careful about designing the subsidization policy, because haphazard subsidization for remanufacturing may result in an undesired outcome.

Motivated by the discrepancy between the intention and outcome of a commonly used government subsidization policy for remanufacturing, we consider an alternative more general government policy that subsidizes for/taxes on the production of both generation products and remanufacturing. Some other comprehensive government subsidization policies on production, recycling, remanufacturing, and trade-in rebates are discussed in, e.g., [186, 118], and [170]. The goal of such government subsidization programs is to promote the development of remanufacturing, curb pollution, and stimulate consumption. We assume that government subsidies (taxes) are provided (charged) for the sales of both generation products, and recycling/remanufacturing the leftover inventory and used products. Specifically, let $s_g := (s_1, s_2, s_r)$ denote the subsidy/tax scheme the government adopts. The government offers the firm a per-unit subsidy s_1 for sales of the first-generation product, a per-unit subsidy s_2 for sales of the second-generation product, and a per-unit subsidy s_r for remanufacturing. If $s_i < 0$ ($i = 1, 2, r$), the firm taxes on the sales of the product or remanufacturing leftover inventory and used products. In particular, we remark that the aforementioned most common government subsidization policy for remanufacturing alone is a special case of this general subsidy/tax scheme with $s_1 = 0$, $s_2 = 0$, and $s_r > 0$.

We now analyze how the government should design the linear subsidy/tax scheme to induce the socially optimal outcome under different customer behaviors.

Theorem 4.5.2 *(a) With strategic customers, there exists a unique linear subsidy/tax scheme $s_g^* = (s_1^*, s_2^*, s_r^*)$, under which the socially optimal RE equilibrium outcome is achieved. Moreover, we have (i) s_2^* is the unique solution to $p_{s,2}^{n*} = \mathrm{argmax}_{p_2^n \geq 0}\{(p_2^n + s_2 - c_2)\bar{G}(\frac{p_2^n}{1+\alpha})\}$; (ii) s_r^* is the unique solution to $p_{s,2}^{r*} = \mathrm{argmax}_{p_2^r \geq 0}\{(p_2^r + s_r + s_2^* - c_2 + r_2)\bar{G}(\frac{p_2^r}{k+\alpha})\}$; (iii) s_1^* is the unique solution to $\frac{c_1+\kappa_1-r_1-\iota_1-s_r^*}{m_{s,1}^*-r_1-s_r^*} = \frac{c_1-r_1}{m_1^s(s_1)-r_1}$, where $m_1^s(s_1) := s_1 + m_{s,1}^* + \delta[(\kappa_2 + s_2^* + s_r^* - \iota_2)\bar{G}(\frac{p_{s,2}^{r*}}{k+\alpha}) - (\kappa_2 + s_2^*)\bar{G}(\frac{p_{s,2}^{n*}}{1+\alpha})]$; (iv) s_1^* is decreasing in κ_1, s_2^* is decreasing in κ_2, and s_r^* is increasing in ι_2; and (v) there exists a threshold vector $(\bar{\kappa}_1^s, \bar{\kappa}_2^s, \bar{\iota}_2^s)$, such that $s_1^* \geq 0$ if and only if $\kappa_1 \leq \bar{\kappa}_1^s$, $s_2^* \geq 0$ if and only if $\kappa_2 \leq \bar{\kappa}_2^s$, and $s_r^* \geq 0$ if and only if $\iota_2 \geq \bar{\iota}_2^s$.*

(b) With myopic customers, there exists a unique linear subsidy/tax scheme $\tilde{s}_g^* = (\tilde{s}_1^*, \tilde{s}_2^*, \tilde{s}_r^*)$, under which the socially optimal RE equilibrium outcome is achieved. Moreover, we have (i) $\tilde{s}_2^* = s_2^*$; (ii) $\tilde{s}_r^* = s_r^*$; (iii) $\tilde{s}_1^*$ is the unique solution to $\frac{c_1+\kappa_1-r_1-\iota_1-s_r^*}{\tilde{m}_{s,1}^*-r_1-s_r^*} = \frac{c_1-r_1}{\tilde{m}_1^s(s_1)-r_1}$, where $\tilde{m}_1^s(s_1) := s_1 + \mu + \delta[(\kappa_2 + s_2^* + s_r^* - \iota_2)\bar{G}(\frac{p_{s,2}^{r*}}{k+\alpha}) - (\kappa_2 + s_2^*)\bar{G}(\frac{p_{s,2}^{n*}}{1+\alpha})]$; (iv) $\tilde{s}_1^*$ is decreasing in κ_1; and (v) there exists a threshold $\bar{\tilde{\kappa}}_1^s$, such that $\tilde{s}_1^* \geq 0$ if and only if $\kappa_1 \leq \bar{\tilde{\kappa}}_1^s$.

(c) We have (i) $s_1^* \geq \tilde{s}_1^*$ if and only if $e_s^* \leq 0$; and (ii) $\bar{\kappa}_1^s \geq \bar{\tilde{\kappa}}_1^s$ if and only if $e_s^* \leq 0$, where e_s^* is defined in Lemma 10(c).

Theorem 4.5.2 demonstrates that the government can use a simple linear subsidy/tax scheme to induce the socially optimal outcome in the scenarios with either strategic or myopic customers. The linear subsidy/tax policy s_g helps control the margin of the firm and the willingness-to-pay of the customers. Hence, the government can use this incentive scheme to regulate the market and ensure the firm sets the socially optimal prices and production quantities with either strategic or myopic customers. More specifically, in both scenarios, the government should provide a combined subsidy/tax scheme for the sales of both product generations and the recycle of leftover inventory and used products. Since some components in s_g^* and $\tilde{s}_g^*$ may be negative, it is possible that the government taxes the firm on some product versions to discourage their sales. This phenomenon results from the government's goal of balancing the tradeoff between firm profit, customer surplus, and environmental impact. In particular, with either strategic or myopic customers, the government subsidizes more for (taxes less on) the sales of one product version if its unit environmental impact increases. Analogously, more subsidies (less taxes) should be provided for (charged on) remanufacturing if its unit environmental benefit is higher.

Comparing the scenarios with strategic and myopic customers (i.e., Theorem 4.5.2(c)) sheds light on how different customer behaviors influence the optimal government subsidy policy. We find that the optimal subsidy/tax rates for the second-generation product and remanufacturing are independent of whether the customers are strategic or myopic (i.e., $\tilde{s}_2^* = s_2^*$ and $\tilde{s}_r^* = s_r^*$). The optimal subsidy/tax rate for the first-generation product, however, is sensitive to customer behavior. The government should provide a higher subsidy/lower tax for sales of the first-generation product with strategic customers than with myopic customers (i.e., $s_1^* \geq \tilde{s}_1^*$) if and only if, in period 2, the expected surplus of a new customer dominates that of a repeat customer (i.e., $e_s^* \leq 0$). If $e_s^* \leq 0$, strategic customers are reluctant to make an immediate purchase, so, to regulate the market with strategic customers, the government should provide more subsidies for the sales of the first-generation product to induce early purchases. On the other hand, if $e_s^* > 0$, a repeat customer has higher expected surplus in period 2, and thus strategic customers are more willing to purchase the product immediately in period 1. In this case, to discourage strategic customers from overconsumption in period 1, the government offers less subsidies for the sales of the first-generation product with strategic customers than it does with myopic customers. The rationale behind the dichotomy in Theorem 4.5.2(c) is that, with the adoption of trade-in remanufacturing, strategic customers anticipate both the purchasing option as a new customer and the trade-in option as a repeat customer. Depending on which option has a higher expected utility, a strategic customer may have a higher or lower willingness-to-pay than a myopic customer does. Hence, the government may provide higher or lower incentives in period 1 to encourage or discourage the early purchases of strategic customers accordingly.

Based on Theorem 4.5.2, we now compare the total government costs of the optimal subsidy/tax scheme under different customer behaviors. For any subsidy/tax scheme s_g, we denote $C_g(s_g)$ ($\tilde{C}_g(s_g)$) as the associated expected total government cost under RE equilibrium with strategic (myopic) customers. Define $C_g^* := C_g(s_g^*)$ and $\tilde{C}_g^* := \tilde{C}_g(\tilde{s}_g^*)$ as the social-welfare-maximizing government costs with strategic and myopic customers, respectively.

Theorem 4.5.3 *(a)* $C_g^* - \tilde{C}_g^* = (s_1^* - \tilde{s}_1^*)\mathbb{E}(X \wedge Q_{s,1}^*)$.

(b) $C_g^* \geq \tilde{C}_g^*$ *if and only if* $e_s^* \leq 0$. *Moreover, there exists a threshold* $\bar{\mathcal{V}}_2 > 0$, *such that* $e_s^* \leq 0$, *if and only if* $r_2 + \iota_2 \leq \bar{\mathcal{V}}_2$.

Theorem 4.5.3 compares the social-welfare-maximizing government costs in scenarios with strategic and myopic customers. Specifically, we show that the total cost to regulate a market with strategic customers is higher than with myopic customers whenever the socially optimal subsidy for the first-generation product with strategic customers dominates that with myopic customers (i.e., $s_1^* \geq \tilde{s}_1^*$). Equivalently, according to Theorem 4.5.3(b), it costs the government more to regulate a market with strategic customers if the expected surplus of a new customer dominates that of a repeat customer in period 2 (i.e., $e_s^* \leq 0$). In this case, more subsidies should be provided to incentivise the more reluctant strategic customers to make an early purchase in period 1. Another implication of Theorem 4.5.3(b) is that if the total unit economic and environmental value of remanufacturing, $r_2 + \iota_2$, is sufficiently low (i.e., below the threshold $\bar{\mathcal{V}}_2$), the total government cost is lower with strategic customers. Therefore, our analysis delivers the new insight to the literature that strategic customer behavior has a negative (positive) impact upon the government if the total economic and environmental value of remanufacturing is low (high).

4.6 Summary

In this chapter, we develop an analytical model to study how different customer behaviors influence the economic and environmental values of trade-in remanufacturing. From the firm's perspective, we show that trade-in remanufacturing is generally much more valuable with strategic customers than with myopic customers. This is because a trade-in rebate essentially offers an early purchase reward and thus can deliver additional value by exploiting the forward-looking behavior of strategic customers. In particular, with the adoption of trade-in remanufacturing, strategic customer behavior may help increase the firm's profit, which contrasts the common belief in the literature that strategic customer behavior hurts firm profit. In the trade-in remanufacturing setting, the price discount in the second period increases with the revenue generated from remanufacturing; thus when the revenue-generating effect is strong enough, the willingness-to-pay of strategic customers in the first period could be even higher than that of myopic customers, which allows the firm to extract more profit with strategic customers.

From the environmental perspective, trade-in remanufacturing decreases the unit environmental impact, but increases the production quantities through the early-purchase inducing effect with strategic customers. Moreover, under strategic customer behavior, adopting trade-in remanufacturing may decrease the customer surplus and social welfare. Hence, with strategic customers, caution is needed on the adoption of trade-in remanufacturing, because it could be detrimental to the environment and the society. With myopic customers, however, trade-in remanufacturing leads to a lower first-period production quantity in general. Our results indicate that customer behavior plays an important role in the value of trade-in remanufacturing. Specifically, with strategic customers, trade-in remanufacturing may create a tension between firm profitability and environmental sustainability; but, with myopic customers, it generally benefits both the firm and the environment.

To resolve the above tension caused by trade-in remanufacturing, we also study how the government should design a regulatory policy to balance firm profit, customer surplus, and environmental impact. A commonly observed policy is to subsidize the remanufactured products. However, we find that despite its intention to protect the environment, such a policy fails to achieve the social optimum and is actually harmful to the environment. To achieve the socially optimal outcome, we show that it suffices for the government to employ a simple linear incentive scheme. This scheme imposes either subsidy or tax on the sales of both product generations as well as the remanufactured products: A subsidy (tax) should be applied if the environmental impact of the product is sufficiently low (high).

5. Pricing and Inventory Management under the Scarcity Effect of Inventory

5.1 Introduction

[1]In the operations management literature, joint pricing and inventory management has received extensive attention. A key assumption in the existing models in this stream of literature is that demand, though random, is independent of inventory (e.g., [70]), so that sales and, hence, revenue link to inventory only through the stockout effect.

In quite a few industries (e.g., automobile, electronics and luxury products, etc.), however, we have observed strong empirical and anecdotal evidence that demand may be correlated with the amount of inventory carried by retailers. A high inventory level sometimes promotes sales because it creates a strong visual impact (the *billboard effect*) and signals abundant potential availability, both of which can make the item more desirable and increase the chance of customer purchase. On the other hand, it is also commonly observed in practice that an ample inventory conveys to the customers the message that the item is of low popularity and quality, thus inducing low demand.

The negative correlations between demand and inventory are well supported by psychological and economic theories as well as rich anecdotal observations and empirical data. The phenomena that a low inventory level may increase and a high inventory level may decrease demand are often referred to as the "*scarcity effect*" of inventory. Three major mechanisms drive the *scarcity effect* of inventory: (1) inventory level signals the quality and popularity of a product; (2) inventory level implies the stockout risk of a product; and (3) inventory level reveals the pricing strategy the retailer will employ. We now discuss these three mechanisms in detail.

First, it has been well established in psychological commodity theory that supply scarcity increases the attractiveness of a product to customers ([30]). This notion has been tested and refined by various experiments with respect to a large scope of product categories (e.g., food, wine and book) by, e.g., [187], [182] and [178]. The desirability

[1]This chapter is based on the author's earlier work [189]

of the product is enhanced by scarce inventory, because customers are likely to infer product quality and popularity from its inventory level. A lower inventory level signals more consumption by other customers and, hence, the product is more popular and of higher quality. On the other hand, observing a high inventory, a customer naturally believes that the item has many units because no one wants to buy it. Some recent marketing (e.g., [156]) and operations management (e.g., [180]) papers employ game theoretic models to demonstrate that the scarcity strategy can effectively signal to the customers the high quality of a product, thus creating a "hot product". Empirical results regarding the *scarcity effect* of inventory upon demand in automobile industry can also be found in, e.g., [33] and [39].

Second, a low inventory level spreads a sense of urgency among customers that soon the product will be sold out and potential buyers will be put on a wait-list. Such backlogging risk motivates customers to make an immediate purchase instead of searching for better options. A high inventory, however, grants customers the luxury of waiting and searching, thus lowering the current demand. Similar mechanism also drives the search behavior that a low inventory of one product type discourages a customer to search for better types ([42]). Knowingly limiting the availability of a product, the retailer can induce "buying frenzies" among uninformed customers and set a higher price ([60]).

Third, as shown in pricing and revenue management literature (e.g., [70], [83]), retailers increase their sales prices when inventories are low. Therefore, customers infer from a low inventory level that it is unreasonable to expect a lower price and would like to purchase the item immediately (see, e.g., [17]). On the other hand, a high inventory level suggests that the sales price will be more likely to decrease and, hence, encourages customers to wait before buying. Carefully making use of this mechanism, the retailer can enjoy the benefits of inducing customers to purchase early at high prices ([115]). A similar idea has also been adopted in the advance selling literature (e.g., [175]), which shows that firms may limit its capacity for advance selling to credibly signal its pricing strategy to customers.

Along with the rich theoretical and empirical justifications of the *scarcity effect* of inventory, practitioners have extensively adopted this idea in their marketing strategies. [64] and [31] document that the "scarcity strategy", in which the supply of products is deliberately limited, has already become a basic tactic for marketers to promote their sales. An increasing number of automobile manufacturers create significant levels of scarcity and make a long list of hard-to-get car models over years (see [33]). Albeit facing thousands of customers who had signed up in the wait-lists, none of the manufacturers rushed to accelerate its production ([184]). Likewise, [123] documents that the BMW Mini Cooper promotes its line by limiting its supply and letting the potential owners wait for, on average, two and half months before they own their new cars. The limited distribution strategy has helped the demand of Mini Cooper take off since its reintroduction in the US market. Similar promotional strategy also appears in the electronics market, especially at the introduction stage of a new product generation. Fans have been excited by the long wait to get Sony Play Stations ([184]), Nintendo Game Boys ([183]) and Apple iPads 2 ([150]).

In this chapter, we study the dynamic pricing and inventory management model under the *scarcity effect* of inventory. The stochastic demand is modeled as a decreasing function of the sales price and the customer-accessible inventory level at the beginning of each decision epoch. Unmet demand is fully backlogged to the next period. The wait-lists observed or spread through “word-of-mouth” successfully signal the high quality and popularity of the product and attract more customers (see, e.g., [31] and [64]). From the strategic perspective, joint pricing and inventory decisions effectively deliver the information regarding the quality and popularity of the product. Specifically, pricing flexibility induces more strategic behavior of customers (e.g., waiting for potential price discount), which further strengthens the *scarcity effect* of inventory, because customers may anticipate the price changes based on current inventory (see, e.g., [115]).

We develop a unified joint price and inventory management model that incorporates both inventory withholding and inventory disposal to deal with the *scarcity effect*. Under the inventory withholding policy, the firm displays only part of its inventory and withholds the rest in a warehouse not observable by customers, so as to induce higher potential demand. Analogously, with inventory disposal, the firm can dispose its unnecessary excess inventory with some salvage value. Both inventory withholding and disposal may incur a cost. We show that a customer-accessible-inventory-dependent order-up-to/dispose-down-to/display-up-to list-price policy is optimal. Moreover, the order-up-to/display-up-to and list-price levels are decreasing in the customer-accessible inventory level. When the *scarcity effect* of inventory is sufficiently strong, the firm should display no positive inventory so that every customer must wait before getting the product. In this case, the strong *scarcity effect* creates more opportunities than risks, so the firm can proactively take advantage of it and induce more demand by making customers wait (e.g., the marketing strategy of BMW).

When it is too costly to withhold or dispose inventory, the unified model is reduced to the model without inventory withholding or the model without inventory disposal, both of which deliver sharper insights. In the model without inventory withholding/disposal, we show that the inventory-dependent demand increases the overstocking risk and, thus, lowers the optimal sales prices and order-up-to levels. With higher operational flexibility (a higher salvage value or the inventory withholding opportunity), however, the firm deals with the *scarcity effect* of inventory more effectively and, hence, increases its sales prices and order-up-to/display-up-to levels. In short, inventory disposal/withholding benefits the firm by enhancing its operational flexibility and agility.

We also generalize the unified model by incorporating responsive inventory reallocation, which allows the firm to reallocate (with a cost) its inventory between display and warehouse after demand realizes. In this case, the firm can keep a low inventory and better hedge against risks of the demand uncertainty and the *scarcity effect* of inventory.

We perform extensive numerical studies to demonstrate (a) the robustness of our analytical results, (b) the impact of the *scarcity effect* upon the profitability of the firm, and (c) the value of dynamic pricing under the *scarcity effect* of inventory. Our numerical results show that the analytical characterizations of the optimal policies in our model are robust and hold in all of our numerical experiments. Both the profit loss of ignoring the *scarcity effect* and the value of dynamic pricing under the *scarcity effect* are significant, and increase in the intensity of the *scarcity effect* and/or demand variability. This is because: (1) the *scarcity effect* decreases the future demand and magnifies future demand variability; and (2) dynamic pricing facilitates the firm to induce higher future demand and dampen future demand variability. In addition, a longer planning horizon increases the impact of the *scarcity effect*, and decreases the value of dynamic pricing.

To conclude this section, we summarize our main contributions as follows: (1) To the best of our knowledge, we are the first to study the joint pricing and inventory management under the *scarcity effect* of inventory. We characterize the optimal policy in a general unified model and generalize our results to the model with responsive inventory reallocation. (2) We analyze how the *scarcity effect* of inventory impacts the firm's optimal price and inventory policies and study the effect of operational flexibilities on the firm's optimal decisions under the *scarcity effect*. (3) We identify the rationale of the phenomenon that firms with intense *scarcity effect* deliberately make their customers wait before getting the product. (4) We numerically study the profit loss of ignoring the *scarcity effect* and the value of dynamic pricing under the *scarcity effect*.

The rest of the chapter is organized as follows. In Section 5.2, we position this chapter in the related literature. Section 5.3 presents the basic formulation, notations and assumptions of our model. In Section 5.4, we propose and analyze the unified model. Section 5.5 discusses the additional results and insights in two important special cases (the model without inventory withholding and the model without inventory disposal). Section 5.6 generalizes the unified model to the model with responsive reallocation. Section 5.7 reports our numerical findings. We conclude this chapter by summarizing our findings and discussing a possible extension in Section 5.8. All proofs are relegated to Appendix D.1.

5.2 Related Research

This chapter is mainly related to two lines of research in the literature: (1) inventory management with inventory-dependent demand and (2) optimal joint pricing and inventory policy.

There is a large body of literature on inventory-dependent demand. We refer interested readers to [177] for a comprehensive review. The dependence of demand on inventory is usually modeled in two ways in the literature: (1) potential demand is increasing in the inventory level after replenishment; and (2) potential demand is decreasing in the inventory level before replenishment (leftover inventory from the previous period).

The first approach to model inventory-dependent demand assumes that demand increases with inventory (the *billboard effect*). [86] study a periodic review inventory model, in which the random demand in each period is increasing in the inventory level after replenishment. [58] consider a single-period newsvendor model where demand is decreasing in price and positively correlated with inventory level. Several other operations management and marketing papers also assume that demand depends on the instantaneous (after replenishment) inventory level, in particular via the shelf-space effect. We refer interested readers to, e.g., [171, 172], [34, 35], [119], [18] and [53].

The other effect of inventory upon demand, as discussed in Section 5.1, is the *scarcity effect*. That is, high leftover inventory (i.e., inventory at the beginning of the period before replenishment) negatively influences the potential demand. In the psychological commodity theory literature, [30] argues that supply scarcity increases the attractiveness of a product, which has been tested by numerous experiments in, e.g., [187, 178]. [156, 180] use game theoretic models to show that the firm can use the scarcity strategy to signal the high quality of a product. [17], among others, demonstrate that customers may strategically wait for price discounts when observing a high inventory. [115] propose an effective pricing scheme to induce customers to make early purchases under a revenue management framework. The idea that supply condition can signal the potential pricing strategy and the product quality has also been adopted in the advance selling literature (e.g., [175]). [33, 39] conduct empirical studies to show that the *scarcity effect* of inventory upon demand prevails in automobile industry.

To the best of our knowledge, [145] is the only paper in inventory management literature that incorporates the *scarcity effect* of inventory (called "wait-list effect" in that paper) and assumes that potential demand is a decreasing function of leftover inventory. They show the optimality of understocking and propose the inventory withholding strategy. This chapter generalizes [145] in the following aspects. (1) We introduce a unified model that encompasses dynamic pricing, inventory withholding and inventory disposal, and explicitly captures the interaction between price, inventory and demand. In particular, we analytically show the impact of inventory-dependent demand on the firm's pricing policy, whereas [145] do not allow price adjustment during planning horizon and numerically test the improvement of inventory-withholding policy under different price elasticities of demand. We also numerically show that the value of dynamic pricing under the *scarcity effect* of inventory is significant and increases with the *scarcity effect* intensity and/or demand variability. (2) Because of the endogenous pricing decision introduced to the dynamic program, the analysis of our model is more involved and requires a different approach. (3) Two special cases of our unified model (i.e., the model without inventory withholding and the model without inventory disposal) demonstrate that inventory withholding and inventory disposal help mitigate the overage risk of inventory-dependent demand. (4) In addition to the understocking and inventory withholding policy proposed in [145], our model suggests three other strategies to dampen the negative effect of inventory-dependent demand: (a) price reduction, (b) inventory disposal, and (c) responsive inventory reallocation. (5) We show that when the *scarcity effect* of inventory is sufficiently strong, the firm should display no positive inventory and let every customer wait. To sum up, this chapter generalizes the model in [145] and strengthens its results and insights.

There is an extensive literature on dynamic pricing and inventory control under general stochastic demand. [70] study the inventory system in a periodic review model, where the firm faces price-dependent demand in each decision period and unsatisfied demand is fully backlogged. A list-price order-up-to policy is shown to be optimal. This line of literature has grown rapidly since [70]. For example, [47, 48, 49] analyze the joint pricing and inventory control problem with fixed ordering cost and show the optimality of (s, S, p) policy for finite horizon, infinite horizon and continuous review models. [52] study the joint pricing and inventory control problem under lost sales. In the case of a single unreliable supplier, [112, 73] show that supply uncertainty drives the firm to charge higher prices under random yield and random capacity, respectively. [51] take into consideration costly price adjustments in joint pricing and inventory management. When the replenishment leadtime is positive, the joint pricing and inventory control problem under periodic review is extremely difficult, and [136] partially characterize the structure of the optimal policy. We refer interested readers to [50] for a comprehensive survey on joint pricing and inventory control models. The major difference of this chapter from this stream of research is that we take into account inventory-dependent demand and show that the *scarcity effect* of inventory drives the firm to order less/dispose more/withhold inventory and charge a lower sales price. To the best of our knowledge, only [58, 35] have studied the joint pricing and inventory control problem with inventory-dependent demand. However, both papers consider a single period model where demand is increasing in the available inventory after replenishment.

5.3 Model Formulation

We specify our unified model, notations and assumptions in this section. Consider a firm which faces random demand and periodically makes pricing and inventory decisions in a $T-$period planning horizon, labeled backwards as $\{T, T-1, \cdots, 1\}$. The firm stores its on-hand inventory in two locations, one with customer-accessible inventory to satisfy and stimulate demand, and the other as a warehouse to withhold inventory that is unobservable to customers. The firm can either replenish or dispose inventory, and it can also reallocate its on-hand inventory between the customer-accessible storage and the warehouse. If the firm places an order, the replenished inventory is delivered to the warehouse, after which the firm decides how much inventory to reallocate to the customer-accessible storage. On the other hand, if the firm disposes its on-hand inventory, it first ships inventory, if any, from the customer-accessible storage to the warehouse, and then chooses the disposal quantity.

In each period, the sequence of events unfolds as follows: At the beginning of each period, the firm reviews its total and customer-accessible leftover inventories from last period, simultaneously chooses the order/disposal and reallocation quantities and the sales price, pays the ordering and reallocation costs, and receives the disposal salvages. The ordering and reallocation lead times are assumed to be zero so that the replenished and reallocated inventories are received immediately. Inventory disposal is also executed at once. The demand then realizes and the revenue is collected. At the end of the decision period, the holding and backlogging costs are paid, and the total and customer-accessible inventories are carried over to the beginning of the next period.

The state of the system is given by:

$I_t^a =$ the starting customer-accessible inventory level before replenishment/disposal /reallocation in period t, $t = T, T-1, \cdots, 1$, where the superscript 'a' refers to "customer-accessible";

$I_t =$ the starting total inventory level before replenishment/disposal/reallocation in period t, $t = T, T-1, \cdots, 1$.

Note that, the amount of inventory the firm withholds in the warehouse is $I_t - I_t^a \geq 0$. We introduce the following notation to denote the decisions of the firm:

p_t = the sales price charged in period t, $t = T, T-1, \cdots, 1$;

x_t^a = the customer-accessible inventory level after replenishment/disposal/reallocation but before demand realizes in period t, $t = T, T-1, \cdots, 1$;

x_t = the total inventory level after replenishment/disposal/reallocation but before demand realizes in period t, $t = T, T-1, \cdots, 1$.

We assume that the price p_t is bounded from above by the maximum allowable price $\bar{p}$ and from below by the minimum allowable price $\underline{p}$. Without loss of generality, we also assume that the customer-accessible inventory storage capacity of the firm is K_a ($0 < K_a \leq +\infty$), whereas the warehouse capacity is infinite. In other words, the customer-accessible inventory level after replenishment/disposal/reallocation cannot exceed K_a in each period, i.e., $x_t^a \leq K_a$ for all $t = T, T-1, \cdots, 1$. Following the "no-artificial wait-list" notion (see [145]), we assume that the firm cannot decrease its customer-accessible-inventory level if a wait-list already exists, i.e., $x_t \geq x_t^a \geq \min\{I_t^a, 0\}$.

We introduce the following model primitives:

α = discount factor of revenues and costs in future periods, $0 < \alpha \leq 1$;

c = purchasing cost per unit ordered;

s = salvage value per unit disposed;

b = backlogging cost per unit backlogged at the end of a period;

h_a = holding cost per unit stocked and accessible to customers at the end of a period;

h_w = holding cost per unit stocked in the warehouse at the end of a period;

r_d = unit reallocation fee from the warehouse to the customer-accessible storage;

r_w = unit reallocation fee from the customer-accessible storage to the warehouse.

Without loss of generality, we assume the following inequalities hold:

$b > (1-\alpha)(r_d + c)$: the backlogging penalty is higher than the saving from delaying an order to the next period, so that the firm will not backlog all of its demand;

$c > s$: unit procurement cost dominates the unit salvage value;

$\underline{p} > \alpha(c + r_d) + b$: positive margin for backlogged demand.

Note that although we assume that the parameters and demand are stationary throughout the planning horizon, the structural results in this chapter remain valid when the parameters and demand distributions are time-dependent.

As discussed in Section 5.1, we assume that demand in period t, D_t, depends negatively on the prevailing price and customer-accessible inventory level at the beginning of this period according to a general stochastic functional form: $D_t = \delta(p_t, I_t^a, \epsilon_t)$, where ϵ_t is a random term with a known continuous distribution and a connected support. $\delta(\cdot, \cdot, \epsilon_t)$ is a twice continuously differentiable function strictly decreasing in p_t and decreasing in I_t^a for any ϵ_t. We base our analysis of the problem on the following demand form:

$$\delta(p_t, I_t, \epsilon_t) = (d(p_t) + \gamma(I_t^a))\epsilon_t^m + \epsilon_t^a, \text{ where } \mathbb{E}\{\epsilon_t^a\} = 0 \text{ and } \mathbb{E}\{\epsilon_t^m\} = 1. \tag{5.1}$$

We assume that ϵ_t's are *i.i.d.* random vectors with ϵ_t^a supported on $[\underline{a}, \overline{a}]$ and ϵ_t^m supported on $[\underline{m}, \overline{m}]$ ($\underline{m} \geq 0$). At least one of the two random variables (ϵ_t^a and ϵ_t^m) follows a continuous distribution (i.e., $\underline{a} \neq \overline{a}$ or $\underline{m} \neq \overline{m}$), which ensures that D_t follows a non-degenerate continuous distribution supported on the interval: $[(d(p_t)+\gamma(I_t^a))\underline{m}+\underline{a}, (d(p_t)+\gamma(I_t^a))\overline{m}+\overline{a}]$, for any (p_t, I_t^a). Note that the above demand model is quite general and includes as special cases several demand models from the existing literature. For example, when $\epsilon_t^m = 1$ with probability 1, the demand model is reduced to the additive demand model; if $\epsilon_t^a = 0$ with probability 1, it is reduced to the multiplicative demand model (as a generalized version of the one proposed in [145]); and if $\gamma(\cdot) \equiv 0$, the demand model is reduced to the standard price-dependent demand model (as the one proposed in [47]). The term $d(p_t)$ summarizes the impact of price on demand in period t. As assumed above, $d(\cdot)$ is strictly decreasing in p_t. In some market where competition is fierce and the firm has no pricing power, the price is exogenously fixed at p_0 and the price induced demand is fixed at $d_0 = d(p_0)$. The term $\gamma(I_t^a)$, which is a decreasing function of I_t^a, captures the *scarcity effect* of inventory on demand. Hereafter, we refer to $\gamma(\cdot)$ as the scarcity function, and $\gamma'(\cdot)$ as the intensity of *scarcity effect*. The dependence of demand on inventory is measured by $\gamma'(\cdot)$. i.e., the smaller the $\gamma'(\cdot)$, the more intensive the potential demand depends on the customer-accessible inventory level. When demand is independent of inventory, $\gamma(I_t^a) \equiv \gamma_0$ for all customer-accessible inventory level I_t^a. Note that our demand model generalizes the one in [145] in the sense that our model also captures the impact of endogenous sales price on demand.

Since $d(\cdot)$ is strictly decreasing in p_t, we assume $p(d_t)$ be its strictly decreasing inverse. For the convenience of our analysis, we change the decision variable from p_t to $d_t \in [\underline{d}, \bar{d}]$, where $\underline{d} = d(\bar{p})$ and $\bar{d} = d(\underline{p})$. To avoid the unrealistic case where demand becomes negative, we assume that $\underline{d} + \gamma(K_a) \geq 0$ to ensure that $\mathbb{E}\{D_t\} = d_t + \gamma(I_t^a) \geq 0$ for any $d_t \in [\underline{d}, \bar{d}]$ and $I_t^a \leq K_a$. We impose the following three assumptions throughout our analysis.

Assumption 5.3.1 *$p(\cdot)$ is twice continuously differentiable and concavely decreasing in d_t, with $p'(d_t) < 0$ for $d_t \in [\underline{d}, \bar{d}]$. In addition, $p(d_t)d_t$ is concave in d_t.*

The concavity of $p(d_t)d_t$ in d_t suggests the decreasing marginal revenue with respect to demand, which is a standard assumption in joint pricing and inventory management literature, see, e.g., [47, 112, 136]. For a more comprehensive discussion on decreasing marginal revenue assumptions, see [196]. The concavity of $p(\cdot)$ implies that the demand is more price-sensitive when sales prices are higher. This is also a common assumption in the literature, see, e.g., [70].

As [145], we also assume that demand is concavely decreasing in the customer-accessible leftover inventory:

Assumption 5.3.2 *$\gamma(\cdot)$ is concavely decreasing and twice continuously differentiable. In addition,*

$$\lim_{I_t^a \to -\infty} \gamma'(I_t^a) = 0 \text{ and } \lim_{I_t^a \to -\infty} \gamma(I_t^a) = \gamma_0.$$

The concavity of $\gamma(\cdot)$ refers to the phenomenon that a higher customer-accessible leftover inventory level has a greater marginal effect on potential demand. However, when the backlogged demand is very high, its value of stimulating high potential demand is limited, because $\gamma(\cdot)$ is bounded from above. In other words, the impact of inventory on demand is small under a large backorder volume so demand does not increase to infinity. Therefore, the firm cannot induce arbitrarily high demand by creating an arbitrarily long wait-list. The underlying intuition of the boundedness of $\gamma(\cdot)$ is that the high demand induced by a long wait-list is canceled out by the impatience it arouses.

Assumption 5.3.3 *Let*

$$R(d_t, I_t^a) := (p(d_t) - b - \alpha(c + r_d))(d_t + \gamma(I_t^a)). \tag{5.2}$$

$R(d_t, I_t^a)$ is jointly concave in (d_t, I_t^a) on its domain.

Assumption 5.3.3 is imposed mainly for technical tractability, because it is required to establish the joint concavity of the objective and value functions in each period (see the discussions after Lemma 14). Note that $R(d_t, I_t^a)$ is the expected difference between the revenue and the total cost (i.e., the procuring, displaying and backlogging costs) to satisfy the current demand in the next period, when the firm holds a customer-accessible inventory I_t^a and charges a sales price $p(d_t)$. The joint concavity of $R(\cdot, \cdot)$ implies that the expected difference between the revenue and the total cost to meet the current demand in the next period has decreasing marginal values with respect to both the expected price-induced demand and customer-accessible inventory level. The joint concavity of $R(\cdot, \cdot)$ is stronger than the concavity of expected revenue (Assumption 5.3.1), because it also captures the impact of inventory-dependent demand upon revenue, procurement cost, reallocation cost and backlogging cost. We discuss this assumption in detail in the following subsection.

5.3.1 Discussions on Assumption 5.3.3

Assumption 5.3.3 is essential to show the analytical results in this chapter. We first characterize the necessary and sufficient condition for Assumption 5.3.3:

Lemma 11 *$R(d_t, I_t^a)$ is jointly concave in (d_t, I_t^a) on its domain if and only if*

$$(p''(d_t)(d_t + \gamma(I_t^a)) + 2p'(d_t))(p(d_t) - b - \alpha(c + r_d))\gamma''(I_t^a) \geq (p'(d_t)\gamma'(I_t^a))^2, \qquad (5.3)$$

for all $d_t \in [\underline{d}, \bar{d}]$ and $I_t^a \leq K_a$.

Condition (5.3) is complicated and somewhat difficult to understand. Hence, we give the following simpler necessary condition for Assumption 5.3.3 to hold.

Lemma 12 *If $R(\cdot, \cdot)$ is jointly concave on its domain, then we have:*

(a) For any I_t^a such that $\gamma''(I_t^a) = 0$, $\gamma'(I_t^a) = 0$ as well. Therefore, there exists a threshold $I^* \leq K_a$ (I^* may be $-\infty$), such that $\gamma'(I_t^a) \begin{cases} < 0, & \text{if } I_t^a > I^*, \\ = 0, & \text{otherwise,} \end{cases}$ *and*

$$\gamma''(I_t^a) \begin{cases} < 0, & \text{if } I_t^a > I^*, \\ = 0, & \text{otherwise.} \end{cases}$$

(b) *There exists an* $0 < M < +\infty$, *such that, for any* $I_t^a \le K_a$, $(\gamma'(I_t^a))^2 \le -M\gamma''(I_t^a)$.

Lemma 12(a) shows that, if Assumption 5.3.3 is satisfied, there exists a threshold inventory level I^*, such that there is no *scarcity effect* for all customer accessible inventory level below this threshold and the scarcity function is strictly decreasing and strictly concave for all customer accessible inventory level above this threshold. Lemma 12(b) proves that $R(\cdot,\cdot)$ is jointly concave only if, for all I_t^a, compared with $|\gamma'(I_t^a)|$, $|\gamma''(I_t^a)|$ is sufficiently big. In other words, in the region where the *scarcity effect* exists (i.e., $\gamma'(I_t^a) < 0$), the curvature of the function $\gamma(\cdot)$ should be sufficiently big. This condition is not restrictive and, for example, can be satisfied by the commonly used power or exponential families of *scarcity functions*. We remark that, mathematically, Lemma 12(a) is a corollary of Lemma 12(b). Next, we show that the necessary condition characterized in Lemma 12(b) is also sufficient to some extent.

Lemma 13 *If there exists an* $0 < M < +\infty$, *such that, for any* $I_t^a \le K_a$, $(\gamma'(I_t^a))^2 \le -M\gamma''(I_t^a)$, *the following statements hold:*

(a) *For any inverse demand curve* $p(\cdot)$, *there exists a threshold* $\delta^* < +\infty$, *such that, for any* $\delta \ge \delta^*$, *with* $\hat{p}_\delta(\cdot) := p(\cdot) + \delta$, $\hat{R}_\delta(d_t, I_t^a) := (\hat{p}_\delta(d_t) - b - \alpha(c + r_d))(d_t + \gamma(I_t^a))$ *is jointly concave in* (d_t, I_t^a) *for* $d_t \in [\underline{d}, \bar{d}]$ *and* $I_t^a \le K_a$.

(b) *Suppose that* $p''(\cdot) \neq 0$ *for any* $d_t \in [\underline{d}, \bar{d}]$. *For any scarcity function* $\gamma(\cdot)$, *there exists a threshold* $\varsigma^* < +\infty$, *such that, for any* $\varsigma \ge \varsigma^*$, *with* $\hat{\gamma}_\varsigma(\cdot) := \gamma(\cdot) + \varsigma$, $\hat{R}_\varsigma(d_t, I_t^a) := (p(d_t) - b - \alpha(c + r_d))(d_t + \hat{\gamma}_\varsigma(I_t^a))$ *is jointly concave in* (d_t, I_t^a) *for* $d_t \in [\underline{d}, \bar{d}]$ *and* $I_t^a \le K_a$.

Lemma 13 demonstrates that, as long as the condition characterized in Lemma 12(b) on the *scarcity function*, $\gamma(\cdot)$, is satisfied, $R(\cdot,\cdot)$ is jointly concave on its domain if (a) the sales price of the product, $p(\cdot)$, is sufficiently high relative to the inverse of price sensitivity, $|p'(\cdot)|$; or (b) the price is not linear in demand, and the *scarcity effect* driven demand, $\gamma(\cdot)$, is sufficiently high relative to the *scarcity intensity*, $|\gamma'(\cdot)|$. These sufficient conditions have a clear economic interpretation: the price elasticity of demand (i.e., $|\frac{\mathrm{d}d_t/d_t}{\mathrm{d}p_t/p_t}|$) is sufficiently high relative to the *inventory elasticity of demand* (defined as $|\frac{\mathrm{d}\gamma/\gamma}{\mathrm{d}I_t^a/I_t^a}|$). In practice, this condition is not restrictive. Compared with the primary demand leverage (i.e., the sales price), the customer accessible inventory (through the *scarcity effect*) has less impact upon the potential demand, because not every customer cares about the backlogging risk of a product, but everyone cares about its price. Therefore, Assumption 5.3.3 can be satisfied under a mild condition with economic interpretation.

Finally, when Assumption 5.3.3 does not hold (i.e., $R(\cdot, \cdot)$ is not jointly concave), we have conducted extensive numerical experiments to test the robustness of our analytical results. Our numerical results verify that the analytical characterizations of the optimal policies in our model are robust and hold for non-concave $R(\cdot, \cdot)$'s in all of our experiments. In particular, Lemma 12 implies that when the scarcity function $\gamma(\cdot)$ contains a linear and strictly decreasing piece, $R(\cdot, \cdot)$ is not jointly concave. We present our numerical experiments for this case in Section 5.7.1.

5.4 Unified Model

In this section, we propose a unified model to analyze the joint pricing and inventory replenishment/disposal/reallocation problem when the firm faces random demand which is negatively correlated with the customer-accessible leftover inventory. We characterize the structure of the optimal pricing and inventory policy and give sufficient conditions under which the firm does not (a) dispose its on-hand inventory, (b) withhold any inventory, (c) reallocate its customer-accessible inventory to the warehouse, or (d) display any positive inventory to customers.

This model is suitable for the case where the firm can both withhold its on-hand inventory in its private warehouse not observable by customers (e.g., clothing and electronics markets) and dispose it (e.g., in the hi-tech industry, the evolution of product generation is so fast that the retailers/manufacturers have to sell excess old versions at a significantly discounted price). When potential demand is negatively correlated with the customer-accessible leftover inventory, the firm faces greater overage risk, because a high customer-accessible leftover inventory not only incurs a high holding cost but also suppresses potential demand. Both inventory withholding and inventory disposal policies enable the firm to strategically keep a low customer-accessible inventory, so as to induce high potential demand and mitigate the overstocking risk. Hence, we incorporate inventory withholding and inventory disposal into our unified model.

The unified model is quite general and can be reduced to several specific models that are of interest on their own. For example, we show that if the warehouse holding cost h_w is sufficiently large, the unified model is reduced to the one without inventory withholding, which is discussed in detail in Section 5.5.1. Besides, if the disposal salvage value s is sufficiently low, the unified model is reduced to the one without inventory disposal, which is discussed in detail in Section 5.5.2.

To formulate the planning problem as a dynamic program, let:

$V_t(I_t^a, I_t)$ = the maximum expected discounted profits in periods $t, t-1, \cdots, 1$, when starting period t with a customer-accessible inventory level I_t^a and a total inventory level I_t.

Without loss of generality, we assume that the excess inventory in the last period (period 1) is discarded without any salvage value, i.e., $V_0(I_0^a, I_0) = 0$, for any (I_0^a, I_0).

The optimal value functions satisfy the following recursive scheme:

$$V_t(I_t^a, I_t) = r_d I_t^a + cI_t + \max_{(x_t^a, x_t, d_t) \in F(I_t^a)} J_t(x_t^a, x_t, d_t, I_t^a, I_t), \tag{5.4}$$

where $F(I_t^a) := \{(x_t^a, x_t, d_t) : x_t^a \in [\min\{I_t^a, 0\}, K_a], x_t \geq x_t^a, d_t \in [\underline{d}, \bar{d}]\}$ denotes the set of feasible inventory and pricing decisions, and

$$\begin{aligned}
J_t(x_t^a, x_t, d_t, I_t^a, I_t) &= -r_d I_t^a - cI_t + p(d_t)\mathbb{E}[\delta(p(d_t), I_t^a, \epsilon_t)] - c(x_t - I_t)^+ + s(x_t - I_t)^- \\
&\quad -r_d(x_t^a - I_t^a)^+ - r_w(x_t^a - I_t^a)^- - h_w(x_t - x_t^a) \\
&\quad -\mathbb{E}\{h_a(x_t^a - \delta(p(d_t), I_t^a, \epsilon_t))^+ + b(x_t^a - \delta(p(d_t), I_t^a, \epsilon_t))^-\} \\
&\quad +\alpha\mathbb{E}\{V_{t-1}(x_t^a - \delta(p(d_t), I_t^a, \epsilon_t), x_t - \delta(p(d_t), I_t^a, \epsilon_t))\} \\
&= p(d_t)(d_t + \gamma(I_t^a)) - (c - s)(x_t - I_t)^- - (h_w + c)x_t \\
&\quad -(r_d + r_w)(x_t^a - I_t^a)^- + (h_w - r_d)x_t^a \\
&\quad +\mathbb{E}[(b + \alpha r_d)(x_t^a - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a) \\
&\quad +\alpha c(x_t - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a)] \\
&\quad +\mathbb{E}\{\alpha[V_{t-1}(x_t^a - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a, x_t - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a) \\
&\quad -r_d(x_t^a - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a) - c(x_t - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a)] \\
&\quad -(b + h_a)(x_t^a - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a)^+\} \\
&= (p(d_t) - \alpha(c + r_d) - b)(d_t + \gamma(I_t^a)) - (c - s)(x_t - I_t)^- \\
&\quad -(r_d + r_w)(x_t^a - I_t^a)^- - (h_w + (1 - \alpha)c)x_t \\
&\quad +(h_w + b - (1 - \alpha)r_d)x_t^a \\
&\quad +\mathbb{E}\{-(h_a + b)(x_t^a - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a)^+ \\
&\quad +\alpha[V_{t-1}(x_t^a - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a, x_t - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a) \\
&\quad -r_d(x_t^a - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a) - c(x_t - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a)]\} \\
&= R(d_t, I_t^a) - \theta(x_t - I_t)^- - (r_d + r_w)(x_t^a - I_t^a)^- - \psi x_t + \phi x_t^a \\
&\quad +\mathbb{E}\{G_t(x_t^a - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a, x_t - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a)\}, \qquad (5.5)
\end{aligned}$$

$$\begin{aligned}
\text{where } G_t(x, y) &:= -(b + h_a)x^+ + \alpha(V_{t-1}(x, y) - r_d x - cy), \\
\theta &:= c - s = \text{the unit loss of inventory disposal}, \qquad (5.6) \\
\psi &:= h_w + (1 - \alpha)c \\
&= \text{the unit cost of replenishing and holding inventory in the warehouse}, \\
\phi &:= h_w + b - (1 - \alpha)r_d \\
&= \text{the unit saving of reallocating warehouse inventory to the customer-accessible storage.}
\end{aligned}$$

We use $(x_t^{a*}(I_t^a, I_t), x_t^*(I_t^a, I_t), d_t^*(I_t^a, I_t))$ to denote the maximizer in (5.4), which stands for the optimal policy in period t, with customer-accessible inventory level I_t^a and total inventory level I_t. To characterize the structure of the optimal inventory replenishment/disposal/reallocation and pricing policies, we define the following optimizers: $(x_t^a(I_t^a), d_t(I_t^a))$ and $(\tilde{x}_t^a(I_t^a), \tilde{x}_t(I_t^a), \tilde{d}_t(I_t^a))$. Let

$$\begin{aligned}(x_t^a(I_t^a), d_t(I_t^a)) \quad &:= \quad \mathrm{argmax}_{x_t^a \in [\min\{I_t^a, 0\}, K_a], d_t \in [\underline{d}, \bar{d}]} R(d_t, I_t^a) + \beta x_t^a \\ & \quad\quad + \mathbb{E}[G_t(x_t^a - \delta(p(d_t), I_t^a, \epsilon_t), x_t^a - \delta(p(d_t), I_t^a, \epsilon_t))],\end{aligned} \tag{5.7}$$

$$\text{where} \qquad \beta \quad := \quad b - (1-\alpha)(c + r_d) > 0. \tag{5.8}$$

$x_t^a(I_t^a)$ is the optimal order-up-to inventory level, if the firm procures positive inventory and displays all of its on-hand inventory to customers, whereas $d_t(I_t^a)$ is the optimal expected price-induced demand in this case. Let

$$\begin{aligned}&(\tilde{x}_t^a(I_t^a), \tilde{x}_t(I_t^a), \tilde{d}_t(I_t^a)) \\ :=&\mathrm{argmax}_{(x_t^a, x_t, d_t) \in F(I_t^a)} \{R(d_t, I_t^a) + (\theta - \psi)x_t - (r_d + r_w)(x_t^a - I_t^a)^- + \phi x_t^a \\ &+ \mathbb{E}\{G_t(x_t^a - \delta(p(d_t), I_t^a, \epsilon_t), x_t - \delta(p(d_t), I_t^a, \epsilon_t))\}\}.\end{aligned} \tag{5.9}$$

When the firm disposes its on-hand inventory, $\tilde{x}_t^a(I_t^a)$ is the optimal display-up-to inventory level and $\tilde{x}_t(I_t^a)$ is the optimal dispose-down-to inventory level, whereas $\tilde{d}_t(I_t^a)$ is the optimal expected price-induced demand. The following lemma establishes the properties of the two optimizers:

Lemma 14 *For each $t = T, T-1, \cdots, 1$, the following statements hold:*

(a) $J_t(x_t^a, x_t, d_t, I_t^a, I_t)$ is jointly concave and continuously differentiable in $(x_t^a, x_t, d_t, I_t^a, I_t)$ except for a set of measure zero; for any fixed (I_t^a, I_t), $J_t(\cdot, \cdot, \cdot, I_t^a, I_t)$ is strictly jointly concave in (x_t^a, x_t, d_t).

(b) $V_t(I_t^a, I_t)$ is jointly concave and continuously differentiable in (I_t^a, I_t), whereas $V_t(I_t^a, I_t) - r_d I_t^a - cI_t$ is decreasing in I_t^a and I_t.

Lemma 14 proves that the objective function in each period is jointly concave and almost everywhere differentiable and the value function is jointly concave and continuously differentiable. Moreover, the second half of Lemma 14(b) implies that the normalized value function, $V_t(I_t^a, I_t) - r_d I_t^a - cI_t$, is decreasing in both the customer-accessible inventory level I_t^a and the total inventory level I_t, which generalizes Proposition 5.1 in [145]. We remark that the joint concavity of $R(\cdot,\cdot)$ on its entire domain is necessary to prove that the objective functions $J_t(\cdot,\cdot,\cdot,I_t^a,I_t)$ and the value functions $V_t(\cdot,\cdot)$ are jointly concave, which is essential to analytically establish other structural results in this chapter. We can easily find examples in which $R(\cdot,\cdot)$ fails to be jointly concave (e.g., $\gamma(\cdot)$ contains a linear and strictly decreasing piece) and leads to non-concave $J_t(\cdot,\cdot,\cdot,I_t^a,I_t)$'s and $V_t(\cdot,\cdot)$'s. In this case, we are unable to analytically show the structural results in this chapter (e.g., Theorem 5.4.1 and Theorem 5.5.1). In Section 5.7.1, we numerically test whether the structure of the optimal policy characterized in our theoretical model still holds. With the help of Lemma 14, we characterize the structural properties of the optimal policy in the unified model as follows:

Theorem 5.4.1 *For $t = T, T-1, \cdots, 1$, the following statements hold:*

(a) $x_t^a(I_t^a) \le \tilde{x}_t(I_t^a)$. Moreover, let $q_t^*(I_t^a, I_t) := x_t^*(I_t^a, I_t) - I_t$ denote the optimal order/disposal quantity and we have:

$$q_t^*(I_t^a, I_t) \begin{cases} > 0 & \text{if } I_t < x_t^a(I_t^a), \\ = 0 & \text{if } x_t^a(I_t^a) \le I_t \le \tilde{x}_t(I_t^a), \\ < 0 & \text{otherwise}, \end{cases}$$

i.e., it is optimal to order if and only if $I_t < x_t^a(I_t^a)$ and to dispose if and only if $I_t > \tilde{x}_t(I_t^a)$.

(b) If $I_t < x_t^a(I_t^a)$, $x_t^{a*}(I_t^a, I_t) = x_t^*(I_t^a, I_t) = x_t^a(I_t^a)$, $d_t^*(I_t^a, I_t) = d_t(I_t^a)$, i.e., it is optimal to order and display up to $x_t^a(I_t^a)$ and charge a list-price $p(d_t(I_t^a))$.

(c) If $I_t > \tilde{x}_t(I_t^a)$, $(x_t^{a*}(I_t^a, I_t), x_t^*(I_t^a, I_t), d_t^*(I_t^a, I_t)) = (\tilde{x}_t^a(I_t^a), \tilde{x}_t(I_t^a), \tilde{d}_t(I_t^a))$, i.e., it is optimal to dispose the total inventory level down to $\tilde{x}_t(I_t^a)$, display $\tilde{x}_t^a(I_t^a)$, and charge a list-price $p(\tilde{d}_t(I_t^a))$.

(d) If $I_t \in [x_t^a(I_t^a), \tilde{x}_t(I_t^a)]$, $x_t^*(I_t^a, I_t) = I_t$, i.e., it is optimal to keep the total inventory level.

(e) $x_t^a(I_t^a)$ is continuously decreasing in I_t^a, whereas $d_t(I_t^a)$ is continuously increasing in I_t^a.

Theorem 5.4.1 generalizes Proposition 5 in [145] by characterizing the structure of the optimal policy in our unified model. We show that a customer-accessible-inventory-dependent order-up-to/dispose-down-to/display-up-to list-price policy is optimal. The optimal policy is characterized by two thresholds: the ordering threshold $x_t^a(I_t^a)$ and the disposal threshold $\tilde{x}_t(I_t^a)$, both of which depend on the customer-accessible inventory level, I_t^a. If the total inventory level, I_t, is below the ordering threshold, i.e., $I_t < x_t^a(I_t^a)$, the firm should order-up-to this threshold, display all of its on-hand inventory to customers, and charge a customer-accessible-inventory-dependent list-price $p(d_t(I_t^a))$. If the total inventory level is higher than the disposal threshold, i.e., $I_t > \tilde{x}_t(I_t^a)$, the firm should dispose-down-to this threshold, display part of its on-hand inventory, $\tilde{x}_t^a(I_t^a)$, to customers, and charge a customer-accessible-inventory-dependent list-price $p(\tilde{d}_t(I_t^a))$. If the total inventory level is between the above two thresholds, i.e., $I_t \in [x_t^a(I_t^a), \tilde{x}_t(I_t^a)]$, the firm should keep its total net inventory and display part of it to customers. In particular, Theorem 5.4.1(b) implies that if it is optimal to order, the firm should not withhold anything. Order-and-withhold policy is dominated by displaying the same amount of inventory to customers but not ordering the inventory that will be withheld (so no inventory will be withheld). This is intuitive, because the marginal cost of order-and-withhold is at least $c + h_w$ (procurement cost and holding cost in the warehouse), while the marginal benefit of inventory withholding is at most αc (saving from the purchasing cost in the next period). Moreover, part (e) of Theorem 5.4.1 demonstrates that as the excess customer-accessible inventory level increases, lower demand is induced and the firm has a greater incentive to turn it over, both of which give rise to lower optimal order-up-to levels and optimal sales prices.

The excess inventory of the firm generally has three impacts on the performance of the system: (1) satisfying future demand, (2) incurring holding costs and (3) inducing/suppressing potential demand, the first with positive marginal value and the other two with negative marginal values. Hence, after normalizing the first effect ($V_t(I_t^a, I_t) - r_d I_t^a - cI_t$), the value-to-go function of the firm is decreasing in its customer-accessible inventory level and total inventory level. To better deal with the intertwined tradeoff between these three effects, the firm can adopt dynamic pricing, inventory withholding and inventory disposal strategies. As suggested in Theorem 5.4.1, the firm needs to price the product in accordance to the customer-accessible inventory level so as to better control the *scarcity effect* of demand. Theorem 5.4.1 also shows that when the total inventory is high, the firm should withhold and dispose its on-hand inventory, which saves holding costs and mitigates the risk of suppressing potential demand. On the other hand, the opportunity to redisplay the withheld inventory in the warehouse enables the firm to satisfy potential demand without discouraging it. In short, combining dynamic pricing, inventory withholding and inventory disposal policies helps the firm better match supply and demand and greatly enhances its profitability.

We proceed to analyze how the model primitives influence the firm's optimal operational decisions, such as inventory disposal, inventory withholding, and inventory display.

Theorem 5.4.2 *The following statements hold:*

(a) If $h_w \geq \alpha c - s$, $\tilde{x}_t(I_t^a) = \tilde{x}_t^a(I_t^a)$ *for any* $t = T, T-1, \cdots, 1$.

(b) There exists an $s_* < c$, *such that, if* $s \leq s_*$, $\tilde{x}_t(I_t^a) = +\infty$ *for any* $I_t^a \leq K_a$ *and* $t = T, T-1, \cdots, 1$.

(c) If $\inf_{I_t^a < K_a} \gamma'(I_t^a) \geq -M$, *for some* $M < +\infty$, *there exists an* $r_* < +\infty$, *such that, if* $r_w \geq r_*$, $\tilde{x}_t^a(I_t^a) \geq I_t^a$, *for any* $I_t^a \leq K_a$ *and* $t = T, T-1, \cdots, 1$. *On the other hand, if* $\inf_{I_t^a < K_a} \gamma'(I_t^a) = -\infty$, *for any* $r_w > 0$, *there exists a threshold* $I_t^*(r_w) < K_a$, *such that, if* $I_t^a \geq I_t^*(r_w)$, $\tilde{x}_t^a(I_t^a) < I_t^a$, *for any* $t = T, T-1, \cdots, 2$.

(d) Let $\iota < 1$, *and* $\bar{D} := \sup\{\Delta : P(D_t \geq \Delta) \geq \iota\}$, *i.e., the probability that the demand in period* t *exceeds* $\bar{D}$ *is smaller than* ι, *regardless of the policy the firm employs. If*

$$\alpha(\underline{p} - b - \alpha(c + r_d) + \underline{m}\beta)(1 - \iota)\gamma'(-\bar{D}) + (r_d + r_w + \phi) \leq 0, \tag{5.10}$$

then $x_t^{a*}(I_t^a, I_t) \leq 0$ *for any* $I_t^a \leq K_a$, I_t, *and* $t = T, T-1, \cdots, 1$.

Theorem 5.4.2(a) shows that, when the warehouse holding cost is sufficiently high ($h_w \geq \alpha c - s$), the firm should display all of its on-hand inventory to customers. Part (b) demonstrates that, when inventory disposal is sufficiently costly ($s \leq s_*$), the firm would rather not dispose any of its inventory, regardless of its total inventory level. When the condition in part (a) [part (b)] holds, the unified model is reduced to the model without inventory withholding [inventory disposal], which generates additional insights and is thoroughly discussed in Section 5.5.1 [Section 5.5.2]. Theorem 5.4.2(c) reveals that the optimal inventory reallocation balances the tradeoff between saving the current reallocation cost and stimulating future demand. More specifically, if the intensity of *scarcity effect* is bounded, the firm should not reallocate its inventory from the customer-accessible storage to the warehouse, as long as the reallocation fee is sufficiently high. Otherwise (i.e., the intensity of *scarcity effect* is unbounded), the firm should always withhold part of its inventory in the warehouse, if the excess customer-accessible inventory level is high enough.

Theorem 5.4.2(d) shows that when the demand-stimulating effect/*scarcity effect* of inventory is sufficiently strong (characterized by (5.10)), the backlogging cost incurred by the wait-list is dominated by the revenue generated by the *scarcity effect*. Therefore, the firm should not display any positive inventory, and every customer has to join a wait-list before receiving the product. This analytical result justifies the marketing strategy adopted by, e.g., BMW, in which the availability of Mini Cooper is intentionally limited and more customers are attracted by its wait-list.

5.5 Additional Results in Two Special Cases

In this section, we study two important special cases of our unified model that are of interest on their own: the model without inventory withholding and the model without inventory disposal. As shown in Theorem 5.4.2, when it is too expensive to withhold [dispose] inventory, it is optimal for the firm not to withhold [dispose] any inventory. These two special cases deliver new results and sharper insights on the impact of the inventory-dependent demand upon the firm's pricing and inventory decisions. We also characterize how the operational flexibilities (e.g., an increase in the salvage value and the inventory withholding opportunity) facilitate the firm to mitigate the additional overage risk caused by inventory-dependent demand.

5.5.1 Without Inventory Withholding

In some circumstances, the firm cannot store its inventory in the warehouse, due to, e.g., too costly withholding or too inconvenient transportation. For instance, car dealers usually display all of its automobiles in the store, because withholding and redisplaying the inventory is too costly and inconvenient. In this subsection, we confine our analysis to the model without inventory withholding. In this model, since no inventory is stored in the warehouse, the state space dimension is reduced to one, and such reduction offers new results and sharper insights on how the inventory-dependent demand influences the firm's optimal decisions. More specifically, we demonstrate that the *scarcity effect* of inventory increases the overstocking risk and, thus, drives the firm to set a lower order-up-to level and charge a lower sales price. On the other hand, when the firm is blessed with a higher disposal flexibility (i.e., a higher salvage value), it has more capacity to mitigate such overage risk by getting rid of its surplus inventory. We show that the firm with a higher salvage value sets higher order-up-to levels and sales prices.

To formulate the planning problem as a dynamic program, let:

$V_t^s(I_t^a)$ = the maximum expected discounted profits in periods $t, t-1, \cdots, 1$, when starting period t with a customer-accessible inventory level I_t^a.

Since no inventory is withheld in the warehouse in this model, $I_t^a = I_t$, and we don't need to record the total inventory level I_t. Therefore, the state space dimension is reduced to one. Similarly, we will not incur the warehouse inventory holding cost (h_w), the redisplay cost (r_d), and the withholding cost (r_w) in this model. The superscript 's' refers to "single location storage".

Without loss of generality, we assume the excess inventory in the last period (period 1) is discarded without any salvage value, i.e., $V_0^s(I_0^a) = 0$, for any $I_0^a \le K_a$. The value functions satisfy the following recursive scheme:

$$V_t^s(I_t^a) = cI_t^a + \max_{(x_t^a, d_t) \in F^s(I_t^a)} J_t^s(x_t^a, d_t, I_t^a),$$

where $F^s(I_t^a) := [\min\{0, I_t^a\}, K_a] \times [\underline{d}, \bar{d}]$ denotes the set of feasible order-up-to/dispose-down-to levels and expected price-induced demand, and

$$\begin{aligned} J_t^s(x_t^a, d_t, I_t^a) =& p(d_t)\mathbb{E}[\delta(p(d_t), I_t^a, \epsilon_t)] + s(x_t^a - I_t^a)^- - c(x_t^a - I_t^a)^+ - cI_t^a \\ & - \mathbb{E}[b(x_t^a - \delta(p(d_t), I_t^a, \epsilon_t))^- + h_a(x_t^a - \delta(p(d_t), I_t^a, \epsilon_t))^+] \\ & + \alpha\mathbb{E}[V_{t-1}^s(x_t^a - \delta(p(d_t), I_t^a, \epsilon_t))]. \end{aligned}$$

Following the algebraic manipulation similar to that in (5.5), we obtain:

$$\begin{aligned} J_t^s(x_t^a, d_t, I_t^a) =& R^s(d_t, I_t^a) + \beta^s x_t^a - \theta(x_t^a - I_t^a)^- + \mathbb{E}[G_t^s(x_t^a - \delta(p(d_t), I_t^a, \epsilon_t))], \\ \text{where } R^s(d_t, I_t^a) :=& (p(d_t) - b - \alpha c)(d_t + \gamma(I_t^a)), \\ G_t^s(y) :=& -(b + h_a)y^+ + \alpha[V_{t-1}^s(y) - cy], \\ \beta^s :=& b - (1-\alpha)c, \end{aligned} \tag{5.11}$$

and θ is defined in (5.6). Note that, under Assumption 5.3.3, $R^s(d_t, I_t^a) = R(d_t, I_t^a) + \alpha r_d(d_t + \gamma(I_t^a))$ is jointly concave on its domain.

As a corollary of Theorem 5.4.1, the optimal policy in the model without inventory withholding is an inventory-dependent order-up-to/dispose-down-to list-price policy, as shown below:

Theorem 5.5.1 *Consider a model without inventory withholding. For each $t = T, T-1, \cdots, 1$, the following statements hold:*

(a) $g_t^s(x_t^a, d_t, I_t^a) := \mathbb{E}[G_t^s(x_t^a - \delta(p(d_t), I_t^a, \epsilon_t))]$ is jointly concave and continuously differentiable in (x_t^a, d_t, I_t^a) if $x_t^a \neq I_t^a$; for any fixed I_t^a, $g_t^s(\cdot, \cdot, I_t^a)$ is strictly concave.

(b) $V_t^s(I_t^a)$ is concave in I_t^a. $V_t^s(I_t^a) - cI_t^a$ is decreasing and continuously differentiable in I_t^a.

(c) $J_t^s(\cdot, \cdot, I_t^a)$ is strictly concave for any fixed I_t^a, and there exists a unique $(x_t^{s*}(I_t^a), d_t^{s*}(I_t^a))$ such that

$$(x_t^{s*}(I_t^a), d_t^{s*}(I_t^a)) = \text{argmax}_{(x_t, d_t) \in F^s(I_t^a)} J_t^s(x_t^a, d_t, I_t^a).$$

(d) Let $q_t^{s*}(I_t^a) = x_t^{s*}(I_t^a) - I_t^a$ denote the optimal order/disposal quantity. There exist two threshold inventory levels I_t^H and I_t^L ($I_t^L < I_t^H$), such that,

$$q_t^{s*}(I_t^a) \begin{cases} > 0 & \text{if } I_t^a < I_t^L, \\ = 0 & \text{if } I_t^L \leq I_t^a \leq I_t^H, \\ < 0 & \text{otherwise,} \end{cases}$$

i.e., the firm should order if its inventory level I_t^a is less than the lower threshold I_t^L, dispose if it is more than the higher threshold I_t^H, and not order or dispose if it is between the two thresholds.

(e) If $I_t^a < I_t^L$ or $I_t^a > I_t^H$, the optimal order-up-to/dispose-down-to level $x_t^{s*}(I_t^a)$ is decreasing in I_t^a. If $I_t^L \leq I_t^a \leq I_t^H$, the optimal inventory after replenishment/disposal is increasing in I_t^a.

(f) The optimal price-induced-demand $d_t^{s*}(I_t^a)$ is increasing in I_t^a.

Theorem 5.5.1 implies that, when the firm cannot withhold its on-hand inventory, the optimal policy is to order when the customer-accessible inventory level is low (below I_t^L), to dispose when it is high (above I_t^H), and not to adjust when it is between the two thresholds. The optimal order-up-to/dispose-down-to and list-price levels are customer-accessible-inventory-dependent. As shown in Theorem 5.5.1, when the customer-accessible inventory level is higher, both order-up-to/dispose-down-to levels and sales prices are lower, because a high customer-accessible inventory level suppresses potential demand and the firm has a strong incentive to turn it over.

We proceed to analyze how the *scarcity effect* of inventory impacts the optimal pricing and inventory policies. Compared with the model in which demand is independent of inventory, when potential demand is negatively correlated with customer-accessible leftover inventory levels, the marginal value of on-hand inventory decreases and the firm suffers from the demand reduction caused by a high inventory level. As a result, the firm should order less/dispose more to mitigate the additional overstocking risk caused by the *scarcity effect* of inventory. At the same time, to better catch the sales opportunity, it is optimal to underprice the product so as to attract more customers. Moreover, in a market where the firm has little power to set the sales price, we are able to prove a sharper result that with a more intensive *scarcity effect*, the firm should keep a lower inventory level after replenishment/disposal. The following theorem formalizes these intuitions.

Theorem 5.5.2 *Consider a model without inventory withholding. Assume* $D_t = \delta(d_t, I_t^a, \epsilon_t)$ *and* $\hat{D}_t = \hat{\delta}(d_t, I_t^a, \epsilon_t)$ *with inventory dependent term* $\gamma(I_t^a)$ *and* $\hat{\gamma}(I_t^a)$, *respectively. We also assume that the demand is of additive form (i.e.,* $\epsilon_t^m = 1$ *with probability 1). The following statements hold:*

(a) Assume that $\hat{\gamma}(I_t^a) = \gamma_0 = \lim_{x\to-\infty}\gamma(x)$ *for all* $I_t^a \le K_a$*, i.e.,* $\hat{D}_t$ *does not depend on the customer-accessible inventory level. We have that* $I_t^L \le \hat{I}_t^L$, $I_t^H \le \hat{I}_t^H$, $x_t^{s*}(I_t^a) \le \hat{x}_t^{s*}(I_t^a)$ *and* $d_t^{s*}(I_t^a) \ge \hat{d}_t^{s*}(I_t^a)$ *for all* $I_t^a \le K_a$.

(b) Assume that $\gamma'(I_t^a) \le \hat{\gamma}'(I_t^a)$ *for all* $I_t^a \le K_a$ *and that*

$$\lim_{I_t^a\to-\infty}\gamma(I_t^a) = \lim_{I_t^a\to-\infty}\hat{\gamma}(I_t^a) = \gamma_0.$$

Let $\underline{p} = \bar{p} = p_0$ *and* $d_0 = d(p_0)$. *We have* $I_t^L \le \hat{I}_t^L$, $I_t^H \le \hat{I}_t^H$ *and* $x_t^{s*}(I_t^a) \le \hat{x}_t^{s*}(I_t^a)$ *for all* $I_t^a \le K_a$.

As a generalization of Theorem 3.2 in [145] to the model with dynamic pricing and inventory disposal, Theorem 5.5.2 shows that the firm should understock and underprice the product under the *scarcity effect* of inventory. In Theorem 5.5.2, we need the additive demand assumption, i.e., $\epsilon_t^m = 1$ almost surely. The additive demand model is widely applied in the joint pricing and inventory control literature (see, e.g., [112, 73, 136]), mostly because it enhances the technical tractability and facilitates the analysis. To show Theorem 5.5.2 and other comparisons between the optimizers in different models (Theorems 5.5.3 - 5.5.5 below), we need to iteratively establish the comparisons between the derivatives of value functions. The additive demand form is necessary to link the monotonicity relationship between optimizers and that between derivatives. All results in this chapter, except Theorems 5.5.2 - 5.5.5, hold for the more general demand form introduced in (5.1).

Efficiently disposing surplus inventory protects the firm from the demand-suppressing effect of inventory. As the salvage value increases, the cost of inventory disposal decreases, and the firm has greater disposal flexibility. We characterize how the salvage value impacts the optimal pricing and inventory decisions in the following theorem:

Theorem 5.5.3 *Consider a model without inventory withholding. For any* $t = T, T-1, \cdots, 1$, *assume that the demand is of additive form (i.e.,* $\epsilon_t^m = 1$ *with probability 1), and* $s < \hat{s}$.

(a) $\partial_{I_t^a} \hat{V}_t^s(I_t^a) \geq \partial_{I_t^a} V_t^s(I_t^a)$.

(b) $\hat{I}_t^L \geq I_t^L$.

(c) $\hat{x}_t^{s*}(I_t^a) \geq x_t^{s*}(I_t^a)$ *and, hence,* $\hat{q}_t^{s*}(I_t^a) \geq q_t^{s*}(I_t^a)$ *for all* $I_t^a \leq \hat{I}_t^H$.

(d) $\hat{d}_t^{s*}(I_t^a) \leq d_t^{s*}(I_t^a)$.

Theorem 5.5.3(a) shows that the marginal value of on-hand inventory increases in the salvage value. Parts (b) - (d) demonstrate that with a higher salvage value, the firm should set higher ordering thresholds, order-up-to levels, and sales prices. On one hand, recall from Theorem 5.5.2 that the inventory-dependent demand strengthens overstocking risk by suppressing potential demand so that both optimal order-up-to/disposal-down-to levels and optimal sales prices are lower in the model with inventory-dependent demand than those in the model with inventory-independent demand. On the other hand, however, Theorem 5.5.3 demonstrates that increased operational flexibility (i.e., a higher salvage value) mitigates the demand loss driven by a high customer-accessible inventory level and, hence, with higher disposal flexibility, the firm is able to set higher order-up-to levels and sales prices to win more profit.

5.5.2 Without Inventory Disposal

The model without inventory disposal applies to the cases where the inventory is either too expensive or too inconvenient to dispose. For example, in the automobile industry, the unsold cars of the last year model is too costly to dispose. In other industries like chemical engineering, products are often so environmentally unfriendly that they cannot be disposed arbitrarily. The model without inventory disposal has a simpler optimal policy structure (customer-accessible-inventory-dependent order-up-to/display-up-to list-price policy) and, like the model without inventory withholding, delivers sharper insights regarding the impacts of inventory-dependent demand and inventory withholding policy. More specifically, we show that inventory-dependent demand motivates the firm to order less and charge a lower sales price, whereas the inventory withholding policy helps mitigate the overage risk and increases the optimal order-up-to levels and sales prices.

As a counterpart of Theorem 5.5.2, the following theorem shows that inventory-dependent demand drives down the optimal order-up-to levels and sales prices in the model without inventory disposal:

Theorem 5.5.4 *Consider a model without inventory disposal. For any $t = T, T-1, \cdots, 1$, assume that $r_d = r_w = 0$, and $h_w \geq h_a$, i.e., reallocation is costless and it is more costly to store the inventory in the warehouse. In addition, assume that $D_t = \delta(d_t, I_t^a, \epsilon_t)$ and $\hat{D}_t = \hat{\delta}(d_t, I_t^a, \epsilon_t)$ with inventory dependent term $\gamma(I_t^a)$ and $\hat{\gamma}(I_t^a)$, respectively, where $\hat{\gamma}(I_t^a) = \gamma_0 = \lim_{x\to-\infty}\gamma(x)$ for all $I_t^a \leq K_a$, i.e., $\hat{D}_t$ does not depend on the customer-accessible inventory level. Further assume that, the demand is of additive form (i.e., $\epsilon_t^m = 1$ with probability 1). We have:*

(a) The firm in the system with demand $\hat{D}_t$ should not withhold any inventory.

(b) $x_t^a(I_t^a) \leq \hat{x}_t^{s*}(I_t^a)$ and $d_t(I_t^a) \geq \hat{d}_t^{s*}(I_t^a)$ for all $I_t^a \leq K_a$.

Inventory withholding policy enables the firm to better control demand by intentionally making part of its inventory unavailable to its customers. Hence, inventory withholding policy can stabilize the demand process and increase the optimal order-up-to levels and sales prices, as shown below:

Theorem 5.5.5 *Consider a model without inventory disposal. For any* $t = T, T-1, \cdots, 1$, *assume that the demand is of additive form (i.e.,* $\epsilon_t^m = 1$ *with probability 1),* $r_d = r_w = 0$ *(i.e., reallocation is costless). If* $I_t = I_t^a$, *we have* $x_t^a(I_t^a) \geq x_t^{s*}(I_t^a)$ *for* $I_t^a \leq \max\{I_t^a : x_t^a(I_t^a) \geq I_t^a\}$, *and* $d_t^*(I_t^a, I_t) \leq d_t^{s*}(I_t^a)$ *for* $I_t^a \leq K_a$.

Note that, Theorem 6 needs the assumption that inventory reallocation is costless ($r_d = r_w = 0$), because this assumption is necessary to reduce the state space dimension in its proof. We also assume $r_d = r_w = 0$ for Theorem 7, mainly for expositional convenience and the results still hold under the general condition that $r_d, r_w \geq 0$.

To summarize, inventory withholding and inventory disposal have similar strategic implications in dealing with inventory-dependent demand. The firm employs these strategies to hedge against the overage risk caused by the *scarcity effect* of inventory and stimulate more potential demand.

5.6 Responsive Inventory Reallocation

In our previous analysis, we assume that the firm can withhold and redisplay inventory only at the beginning of the decision epoch before the demand realizes. In this subsection, we relax this assumption by allowing the firm to responsively reallocate its on-hand inventory after the demand realization. The responsive inventory reallocation enables the firm to optimize its inventory policy after the demand uncertainty realizes, so that the supply and demand are better matched and the tradeoff between meeting current and inducing potential demand is better balanced. Note that when responsive inventory reallocation is allowed, the firm should not reallocate its inventory before the demand realizes.

At the beginning of each period, the firm chooses its inventory replenishment/disposal quantity and the sales price. The demand then realizes, after which the firm decides the inventory reallocation quantities between the warehouse and the customer-accessible storage.

To formulate the planning problem as a dynamic program, let

$V_t^r(I_t^a, I_t) =$ the maximum expected discounted profits in periods $t, t-1, \cdots, 1$, when starting period t with a customer-accessible inventory level I_t^a and a total inventory level I_t,

where the superscript 'r' refers to "responsive inventory reallocation". Without loss of generality, we assume the excess inventory in the last period (period 1) is discarded without any salvage value, i.e., $V_0^r(I_0^a, I_0) = 0$, for any (I_0^a, I_0).

We first analyze the optimal reallocation policy in period t. Assume that the order-up-to/dispose-down-to level set by the firm before the demand realization is x_t and the realized demand is D_t. The optimal display-up-to level, $x_t^{ra*}(I_t^a, x_t, D_t)$, after inventory reallocation, is given by:

$$\begin{aligned}
&x_t^{ra*}(I_t^a, x_t, D_t) \\
&= \text{argmax}_{\min\{0, I_t^a - D_t\} \le x_t^a \le x_t - D_t} \{-r_d(x_t^a - I_t^a + D_t)^+ - r_w(x_t^a - I_t^a + D_t)^- - b x_t^{a-} \\
&- h_a x_t^{a+} - h_w(x_t - x_t^a - D_t) + \alpha V_{t-1}^r(x_t^a, x_t - D_t)\}
\end{aligned}$$

Hence, the optimal value functions satisfy the following recursive scheme:

$$\begin{aligned}
V_t^r(I_t^a, I_t) = &\max_{(x_t, d_t) \in F^r(I_t^a)} \{p(d_t)\mathbb{E}\{\delta(p(d_t), I_t^a, \epsilon_t)\} - c(x_t - I_t)^+ + s(x_t - I_t)^- \\
&+ \mathbb{E}_{D_t}\{\max_{\min\{0, I_t^a - D_t\} \le x_t^a \le \min\{K_a, x_t - D_t\}} \{-r_d(x_t^a - I_t^a + D_t)^+ - r_w(x_t^a - I_t^a + D_t)^- \\
&- b x_t^{a-} - h_a x_t^{a+} - h_w(x_t - x_t^a - D_t) + \alpha V_{t-1}^r(x_t^a, x_t - D_t)\}\}\},
\end{aligned}$$

where $F^r(I_t^a) := \{(x_t, d_t) : x_t \ge \min\{I_t^a, 0\}, d_t \in [\underline{d}, \bar{d}]\}$. Following the algebraic manipulation similar to that in Equation (5.5), we have:

$$\begin{aligned}
V_t^r(I_t^a, I_t) = &r_d I_t^a + c I_t + \max_{(x_t, d_t) \in F^r(I_t^a)} \{R(d_t, I_t^a) + r_d(d_t + \gamma(I_t^a)) - \theta(x_t - I_t)^- - \psi x_t \\
&+ \mathbb{E}_{D_t}\{\max_{\min\{D_t, I_t^a\} \le y_t^a \le \min\{x_t, K_a + D_t\}} \{-(r_d + r_w)(y_t^a - I_t^a)^- + \phi y_t^a \\
&+ G_t^r(y_t^a - D_t, x_t - D_t)\}\}\}, \\
\text{with } G_t^r(x, y) := &-(h_a + b)x^+ + \alpha[V_{t-1}^r(x, y) - r_d x - c y].
\end{aligned} \tag{5.12}$$

Comparing the value functions (5.12) and (5.4), it is immediate that by postponing the reallocation decision till after demand realization, the firm achieves a higher expected total profit. In the following theorem, we characterize the optimal inventory replenishment/disposal/reallocation and pricing policy in the model with responsive inventory reallocation:

Theorem 5.6.1 *The following statements hold for $t = T, T-1, \cdots, 1$:*

(a) $V_t^r(I_t^a, I_t)$ is jointly concave and continuously differentiable in (I_t^a, I_t), whereas the normalized value function $V_t^r(I_t^a, I_t) - r_d I_t^a - cI_t$ is decreasing in I_t^a and I_t.

(b) For any given x_t and realized D_t, $v_t^r(y_t^a|I_t^a, x_t, D_t) := -(r_d + r_w)(y_t^a - I_t^a)^- + \phi y_t^a + G_t^r(y_t^a - D_t, x_t - D_t)$ is concave in y_t^a. Therefore, the optimal customer-accessible-inventory level is:

$$x_t^{ra*}(I_t^a, x_t, D_t) = \text{argmax}_{\min\{D_t, I_t^a\} \le y_t^a \le \min\{x_t, K_a + D_t\}} \{v_t^r(y_t^a|I_t^a, x_t, D_t)\} - D_t.$$

(c) There exist two customer-accessible-inventory-level-dependent thresholds, $x_t^r(I_t^a)$ and $\tilde{x}_t^r(I_t^a)$ ($x_t^r(I_t^a) \le \tilde{x}_t^r(I_t^a)$), such that it is optimal to order up to $x_t^r(I_t^a)$ if and only if $I_t < x_t^r(I_t^a)$, to dispose down to $\tilde{x}_t^r(I_t^a)$, if and only if $I_t > \tilde{x}_t^r(I_t^a)$, and to keep the total inventory level otherwise. Moreover, there exist two customer-accessible-inventory-level-dependent sales prices $p(d_t^r(I_t^a))$ and $p(\tilde{d}_t^r(I_t^a))$, such that it is optimal to charge a sales price $p(d_t^r(I_t^a))$ if $I_t \le x_t^r(I_t^a)$, and to charge a sales price $p(\tilde{d}_t^r(I_t^a))$ if $I_t \ge \tilde{x}_t^r(I_t^a)$.

Theorem 5.6.1(a) proves the joint concavity and continuous differentiability of the optimal value functions. Part (b) shows that, in each period, the optimal reallocation policy is obtained by solving a one-dimensional convex optimization after the demand realizes. Consistent with Theorem 5.4.1, part (c) of Theorem 5.6.1 proves that it is optimal to order if the total inventory level is low ($I_t < x_t^r(I_t^a)$), and to dispose if it is high ($I_t > \tilde{x}_t^r(I_t^a)$), and to keep the starting inventory level otherwise. Compared with Theorem 5.4.1, which characterizes optimal policy in the unified model, Theorem 5.6.1 demonstrates that it is possible that the firm order-and-withholds some inventory under the optimal responsive inventory reallocation policy, because, in this case, the firm is blessed with the flexibility to reallocate inventory after the demand uncertainty is resolved.

As in Theorem 5.4.2, we can show that if the warehouse holding cost, h_w, is high enough, it is optimal not to hold any inventory in the warehouse; if the salvage value,

s, is low enough, it is optimal not to dispose anything; and if the reallocation fee to withhold inventory, r_w, is high enough, it is optimal to not reallocate any customer-accessible inventory to the warehouse.

5.7 Numerical Studies

This section reports a set of numerical studies that (a) verify the robustness of our analytical results when Assumption 5.3.3 does not hold; (b) quantify the profit loss of ignoring the *scarcity effect* of inventory when making the pricing and inventory decisions; and (c) quantitatively evaluate the benefit of dynamic pricing in the presence of the *scarcity effect*. Our numerical results demonstrate that (1) the structural results developed in our theoretical model are robust and hold for a large set of non-concave $R(\cdot, \cdot)$ functions; (2) the impact of the *scarcity effect* is significant and it is higher when the scarcity intensity, demand variability, and/or planning horizon length increase; and (3) the value of dynamic pricing under the *scarcity effect* is significant and it is higher under higher scarcity intensity, demand variability and/or shorter planning horizon.

Throughout our numerical studies, we assume that the firm can neither withhold nor dispose its on-hand inventory for two reasons: (a) to have a clear illustration of the optimal policy structure in a model where Assumption 5.3.3 does not hold; and (b) to single out and highlight the impact of the focal operational elements (i.e., the *scarcity effect* of inventory and the dynamic pricing strategy). We also assume that the demand in each period is of the additive form, i.e., $\epsilon_t^m = 1$ almost surely and $D_t = d_t + \gamma(I_t^a) + \epsilon_t^a$. Let $\{\epsilon_t^a\}_{t=1}^T$ follow *i.i.d.* normal distributions with mean 0 and standard deviation σ. The inverse demand function is linear with slope -1, i.e., $p(d_t) = p_0 - d_t$. We set the discount factor $\alpha = 0.95$, the unit holding cost $h = 1$, and the unit backlogging cost $b = 10$.

5.7.1 Optimal Policy Structure with Non-concave $R(\cdot, \cdot)$ Functions

In this subsection, we numerically examine whether the structural results in our theoretical model are robust when Assumption 5.3.3 does not hold, i.e., $R(\cdot, \cdot)$ is not jointly concave. We have performed extensive numerical experiments to test the robustness of our analytical results. In all our numerical experiments, although Assumption 5.3.3 is violated, the characterizations of the optimal policy by our theoretical analysis (i.e., Theorem 5.4.1, Theorem 5.5.1, and Theorem 5.5.2) continue to hold. More specifically, our numerical results verify that (a) the inventory-dependent order-up-to/list-price policy is optimal and the order-up-to level is decreasing in the starting inventory level; (b) the optimal sales price [price-induced demand] is decreasing [increasing] in the starting inventory level; and (c) compared with an inventory system without the *scarcity effect*, the firm with the *scarcity effect* sets lower order-up-to levels and lower sales prices. Therefore, the structural results of our theoretical model are robust and hold for non-concave $R(\cdot,\cdot)$ functions in all our numerical experiments.

Note that from Lemma 12(a) that if the scarcity function $\gamma(\cdot)$ contains a linear and strictly decreasing piece, $R(\cdot,\cdot)$ is not jointly concave. Hence, we report our numerical results for the case where $\gamma(I_t^a) = \begin{cases} \gamma_0 - \exp(\eta I_t^a), & \text{for } I_t^a \le 0, \\ \gamma_0 - 1 - \eta I_t^a, & \text{for } 0 < I_t^a \le K_a, \end{cases}$ with $\eta > 0$. It's clear that $\gamma(\cdot)$ is concavely decreasing and continuously differentiable in I_t^a for all $I_t^a \le K_a$, but $R(\cdot,\cdot)$ is not jointly concave in the region $\{(d_t, I_t^a) : d_t \in [\underline{d}, \bar{d}], I_t^a \in [0, K_a]\}$. We have performed extensive numerical experiments which test many combinations of different values of p_0, γ_0, c, η, σ, $\underline{d}$, $\bar{d}$, K_a, and t. In all the scenarios we examine, the predictions of the optimal policy by our theoretical analysis (i.e., Theorem 5.4.1, Theorem 5.5.1, and Theorem 5.5.2) continue to hold without Assumption 5.3.3. Figures 5.1 - 5.2 illustrate the optimal order-up-to level and price-induced demand with the parameter values $p_0 = 30$, $\gamma_0 = 9$, $c = 8$, $\eta = 0.5$, $\sigma = 2$, $[\underline{d}, \bar{d}] = [6, 12]$, $K_a = 18$, and $t = 20$.

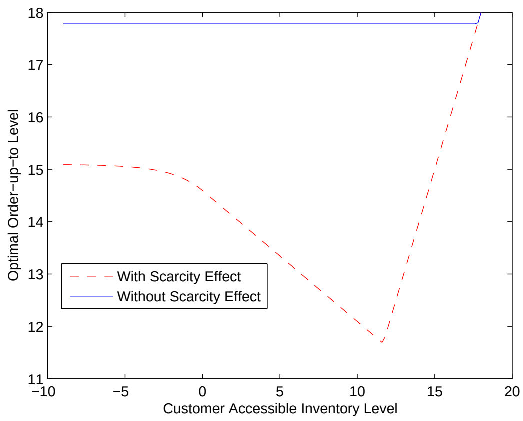

Figure 5.1. **Optimal Ordering-up-to Level**

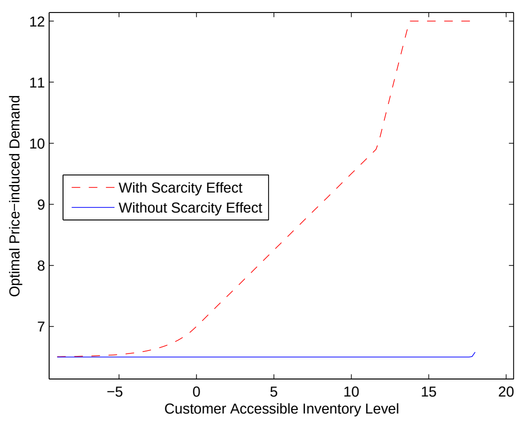

Figure 5.2. **Optimal Price-induced Demand**

5.7.2 Impact of Scarcity Effect

This subsection numerically studies the impact of the *scarcity effect* of inventory upon the firm's profitability by quantifying the profit loss of ignoring this effect under different levels of *scarcity effect* intensity, demand variability and planning horizon length. As in Section 5.7.1, we assume that $\gamma(I_t^a) = \begin{cases} \gamma_0 - \exp(\eta I_t^a), & \text{for } I_t^a \leq 0, \\ \gamma_0 - 1 - \eta I_t^a, & \text{for } 0 < I_t^a \leq K_a, \end{cases}$ where $\eta > 0$. Note that η represents the *scarcity effect* intensity of the inventory system: the larger the η, the more intense the *scarcity effect*. We need to evaluate the profit of a firm which ignores the *scarcity effect*, $\tilde{V}$. To compute $\tilde{V}$, we first numerically obtain the optimal policy in an inventory system without the *scarcity effect* and then evaluate the total profits of this policy in an inventory system with the *scarcity effect*. We also evaluate the optimal profit of a firm under the *scarcity effect*, V^*. In the evaluation of V^* and $\tilde{V}$, we take $I_t^a = 0$ as the reference customer-accessible inventory level. The metric of interest is

$$\lambda_{scarcity} := \frac{V^* - \tilde{V}}{V^*}, \text{ under different values of } \eta, \sigma \text{ and } t.$$

Our numerical experiments are conducted under the following values of parameters: $p_0 = 21$, $\gamma_0 = 4$, $c = 4$, $\eta = 0.35, 0.4, 0.45, 0.5, 0.55$, $\sigma = 1, 2, 3$, $[\underline{d}, \bar{d}] = [6, 12]$, $K_a = 18$, and $t = 5, 10$.

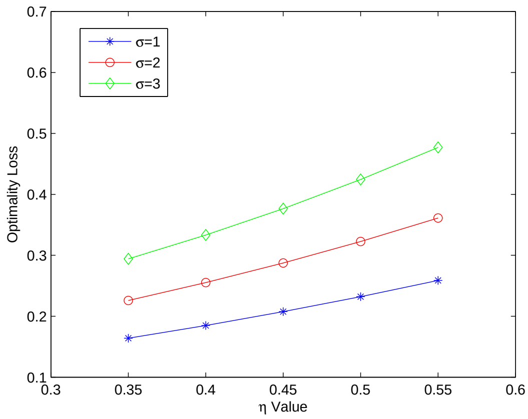

Figure 5.3. **Value of $\lambda_{scarcity}$:** $t = 5$

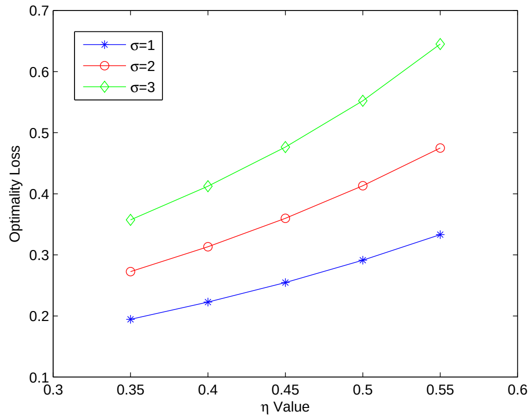

Figure 5.4. **Value of $\lambda_{scarcity}$:** $t = 10$

Figures 5.3 - 5.4 summarize the results of our numerical study on the impact of the *scarcity effect* upon the firm's profitability. Our results reveal that, when the *scarcity* *effect* is ignored, all numerical experiments exhibit a significant profit loss, which is at least 16.41% and can be as high as 64.52%. Moreover, the impact of *scarcity effect* is increasing in the scarcity intensity, demand variability, and planning horizon length. The *scarcity effect* has two effects upon the firm's profitability: (a) it decreases future demand, and (b) it increases demand variability, because the variability of potential demand is intensified by that of the past demand via the *scarcity effect*. Hence, with higher scarcity intensity [demand variability], the first [second] effect lowers more profit of the firm. The comparison between Figure 5.3 and Figure 5.4 implies that the impact of the *scarcity effect* accumulates over time, so the profit loss of ignoring the *scarcity effect* is higher under a longer planning horizon. In short, the *scarcity effect* of inventory matters significantly to the firm's profitability when the *scarcity effect* intensity and demand variability is high, and the planning horizon is long. Our numerical finding confirms the result in [145] that the profit loss is increasing in the *scarcity effect* intensity. On the other hand, our numerical finding on the impact of demand variability contrasts that in [145], which shows that the profit loss of ignoring the *scarcity effect* is decreasing in demand variability. In their experiments, the potential demand is convexly decreasing in the leftover inventory level, so higher demand variability increases the expected potential demand and, thus, the firm's profitability under the *scarcity effect*.

5.7.3 Value of Dynamic Pricing

In this subsection, we numerically explore the value of dynamic pricing under the *scarcity effect* of inventory with different levels of *scarcity effect* intensity, demand variability and planning horizon length. As in Sections 5.7.1 - 5.7.2, we assume that $\gamma(I_t^a) = \begin{cases} \gamma_0 - \exp(\eta I_t^a), & \text{for } I_t^a \leq 0, \\ \gamma_0 - 1 - \eta I_t^a, & \text{for } 0 < I_t^a \leq K_a, \end{cases}$ where $\eta > 0$. We need to evaluate the profit of a firm, which adopts the optimal static pricing strategy, $\hat{V}$. To compute $\hat{V}$, we first evaluate the total profit of an inventory system for any fixed price p_t in each t, and then maximize over p_t to select the optimal static price. Consistent with V^*, $\hat{V}$ is evaluated at the reference customer-accessible inventory level $I_t^a = 0$. The metric of interest is

$$\lambda_{pricing} := \frac{V^* - \hat{V}}{\hat{V}}, \text{ under different values of } \eta, \sigma \text{ and } t.$$

Our numerical experiments are conducted under the following values of parameters: $p_0 = 21$, $\gamma_0 = 4$, $c = 4$, $\eta = 0.35, 0.4, 0.45, 0.5, 0.55$, $\sigma = 1, 2, 3$, $[\underline{d}, \bar{d}] = [6, 12]$, $K_a = 18$, and $t = 5, 10$.

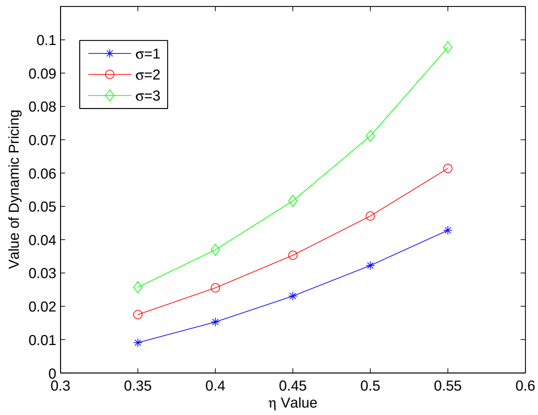

Figure 5.5. **Value of $\lambda_{pricing}$: $t = 5$**

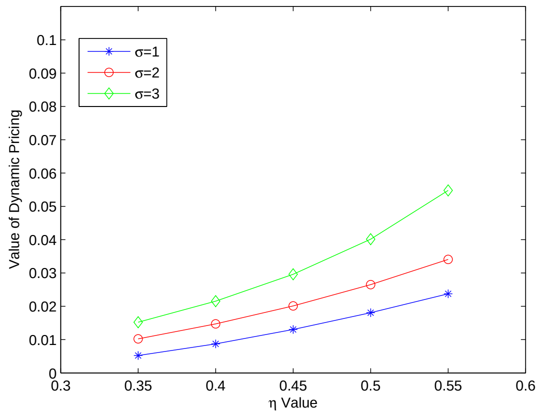

Figure 5.6. **Value of $\lambda_{pricing}$: $t = 10$**

Figures 5.5 - 5.6 summarize the results of our numerical study on the value of dynamic pricing. The results show that the value of dynamic pricing is significant in the presence of the *scarcity effect*. Federgruen and Heching (1999) document that the profit improvement of dynamic pricing in a 5-period model is between $0.46\% - 2.24\%$, when the coefficient of variation for demand varies between 0.7 and 1.4. The numerical experiments of Figure 5.5 report a much higher profit improvement (between $0.91\% - 9.78\%$) of dynamic pricing in a 5-period model with the coefficient of variation of demand between 0.11 and 0.33. Thus, the *scarcity effect* of inventory gives rise to significantly higher value of dynamic pricing. The value of dynamic pricing is driven by the following three effects: (a) it achieves better match between supply and demand; (b) it helps induce higher future demand; and (c) it dampens future demand variability. While effect (a) also improves the performance of an inventory system without the *scarcity effect*, effects (b) and (c) have their impact only upon a firm with the *scarcity effect*. Therefore, the value of dynamic pricing is significantly increased by the *scarcity effect*. Moreover, with higher *scarcity effect* intensity [demand variability], effect (b) [(c)] enhances the firm's profitability more significantly. The comparison between Figure 5.5 and Figure 5.6 implies that the value of dynamic pricing decreases over time. This is consistent with the findings in Federgruen and Heching (1999) that the optimal dynamic pricing policy converges to the optimal static pricing policy, as the planning horizon length goes to infinity. In short, the value of dynamic pricing under the *scarcity effect* of inventory is most significant when the intensity of *scarcity effect* and demand variability is high, and the planning horizon length is moderate.

To conclude this section, we remark that all the numerical results and insights in this section are robust and hold for (a) the general demand form, Equation (5.1), and (b) a large variety of different inverse demand functions (i.e., $p(\cdot)$) and scarcity functions (i.e., $\gamma(\cdot)$) that give rise to concave or non-concave $R(\cdot,\cdot)$ functions.

5.8 Summary and Extension

We conclude this chapter with a summary of the main results and managerial insights derived from our model and some thoughts on a possible direction of future research.

This chapter is the first in the literature to study the joint pricing and inventory management model under the *scarcity effect* of inventory. Demand is modeled as a decreasing stochastic function of both price and customer-accessible inventory level. We propose a unified model in which the firm has several operational flexibilities to hedge against the risk of the stochastic inventory-dependent demand: (a) dynamic pricing, through which the firm can dynamically adjust its sales price; (b) inventory withholding, through which the firm can withhold part of its inventory from customers; and (c) inventory disposal, through which the firm can dispose part of its surplus inventory. We show that a customer-accessible-inventory-dependent order-up-to/dispose-down-to/display-up-to list-price policy is optimal. The order-up-to/display-up-to and list-price levels are decreasing in the customer-accessible inventory level, because of the negative dependence of demand on inventory. When the *scarcity effect* of inventory is sufficiently strong, the firm can strategically benefit from the *scarcity effect* by displaying no positive inventory and making every customer wait, because the revenue generated by the strong *scarcity effect* dominates the backlogging cost of the wait-list.

When the warehouse holding cost [salvage value] is sufficiently high [low], it is too costly to withhold [dispose] inventory, and the unified model is reduced to the model without inventory withholding [disposal]. The model without inventory withholding [disposal] generates additional results and sharper insights. In the model without inventory withholding/disposal, we show that optimal sales prices and order-up-to levels are lower under the *scarcity effect* of inventory than those under inventory-independent demand. Higher operational flexibility (a higher salvage value or the inventory withholding opportunity), however, helps the firm hedge against the overstocking risk and, hence, drives the firm to set higher order-up-to/display-up-to levels and sales prices.

In addition, responsive inventory reallocation is another effective way to deal with the *scarcity effect* of inventory. The reallocation flexibility after demand realization enables the firm to better hedge against the demand uncertainty and balance the tradeoff between meeting current demand and inducing potential demand. In this case, since the firm can reallocate its on-hand inventory after demand realizes, it may be optimal to order-and-withhold when the realized demand is small.

We perform extensive numerical studies to demonstrate (a) the robustness of our analytical results, (b) the impact of the *scarcity effect* upon the profit of the firm, and (c) the value of dynamic pricing under the *scarcity effect* of inventory. Our numerical results show that the analytical characterizations of the optimal policies in our model are robust and hold for non-concave $R(\cdot,\cdot)$ functions in all our experiments. The impact of *scarcity effect* upon the firm's profit is two-fold: (a) it decreases future demand, and (b) it increases demand variability. Hence, the profit loss of ignoring the *scarcity effect* is higher under higher scarcity intensity (via effect (a)), higher demand variability (via effect (b)), and longer planning horizon (via both effects). The value of dynamic pricing under the *scarcity effect* is three-fold: (a) it better matches supply and demand; (b) it helps induce higher future demand; and (c) it dampens future demand variability. Effect (b) [(c)] leads to higher value of dynamic pricing under higher scarcity intensity [demand variability]. Moreover, the optimal dynamic pricing policy converges to the optimal static pricing policy as the planning horizon length goes to infinity, so the value of dynamic pricing decreases over time.

Finally, we remark that all the analytical results in this chapter can be easily extended to the infinite horizon discounted model with the standard argument that demonstrates the preservation of the structural properties as the planning horizon length goes to infinity.

In this subsection, we propose a possible extension of our work: the analysis of the model that encompasses both the *scarcity effect* and the *promotional effect* of inventory.

As discussed in Section 5.2, the displayed inventory has both the service and the promotional effects (see, e.g., [34, 35]), because a higher customer-accessible inventory level creates a stronger visual impact and customers infer a greater chance to get the product. In the literature, this phenomenon is also called the *billboard effect* and the *shelf-space effect* (e.g. [39, 18, 53]).

It is interesting to analyze the model which incorporates both the *scarcity effect* of pre-replenishment inventory and the *promotional effect* of post-replenishment inventory. More specifically, we assume that the demand in period t, $D_t = \delta(p_t, I_t^a, x_t^a, \epsilon_t) = (d(p_t) + \gamma_1(I_t^a) + \gamma_2(x_t^a))\epsilon_t^m + \epsilon_t^a$, where $\gamma_1(\cdot)$ is a decreasing function of pre-replenishment customer-accessible inventory level I_t^a, and $\gamma_2(\cdot)$ is an increasing function of post-replenishment customer-accessible inventory level x_t^a. As before, assume that $d(\cdot)$ is a strictly decreasing function of sales price p_t, $\mathbb{E}\{\epsilon_t^m\} = 1$ and $\mathbb{E}\{\epsilon_t^a\} = 0$.

It is challenging to characterize the optimal joint pricing and inventory management policy under this generalized inventory-dependent demand. In particular, the effect of inventory on the firm's profitability is more involved and it is unclear how to strike a balance between the overage and underage risks in this model. We will explore this problem in our future research.

6. Comparative Statics Analysis Method for Joint Pricing and Inventory Management Models

6.1 Introduction

[1]Comparative statics analysis is integral to studying an inventory management system under dynamic pricing, because it delivers important insights regarding how the system should optimally respond to changes in the exogenous market condition and/or internal state over the planning horizon. For instance, a firm under an uncertain market environment often faces the conundrum that whether it should increase or decrease the sales price and order-up-to level under a higher procurement cost. Analogously, it is also important to modify the price and inventory policies in accordance to firm-level strategic changes like contracting with an additional supplier or expanding the target customer segments. As an essential tool in economics, engineering and operations management, comparative statics analysis offers a systematic method to study these challenges that are both common and essential in inventory management models under dynamic pricing.

More specifically, we consider the optimal pricing and replenishment policies in a general periodic-review joint pricing and inventory management model with multiple customer segments and supply channels under a fluctuating market environment. The firm replenishes its inventory from a portfolio of supply channels with different cost functions. The cost function of each supply channel is determined by a supply-channel-dependent reference procurement cost (e.g., the raw material procurement cost in each supply channel). The customer market is segmented into several independent classes with different demand functions. The firm charges a sales price to each demand segment in each decision period. The demand function of each demand segment is determined by a demand-segment-dependent market size. Both the reference procurement cost of each supply channel and the market size of each demand segment evolve according to an underlying exogenous Markov process. Hence, our joint pricing and inventory management model captures three important features in today's competitive and unstable market: demand

[1]This chapter is based on the author's earlier work [192].

segmentation, supply diversification, and market environment fluctuation. In this quite general dynamic pricing and inventory management model, comparative statics analysis plays an essential role in the characterization of (a) the optimal pricing and inventory policies, and (b) the impact of market environment fluctuation, demand segmentation, and supply diversification upon the optimal sales prices and order quantities.

There are two standard methods to perform comparative statics analysis in the economics and operations management literature: (a) the implicit function theorem (IFT) approach, and (b) the monotone comparative statics (MCS) approach.

The IFT approach characterizes the derivative of the optimizer with respect to the parameters by applying the implicit function theorem to the first-order condition. In order to apply this approach, it is clear from the assumptions of the implicit function theorem that (a) the objective function needs to be twice continuously differentiable with respect to the complete vector of decision variables and parameters, and (b) the Hessian of the objective function with respect to the decision variables at the optimizer needs to be non-degenerate. In our general joint pricing and inventory management model, as we will show later, condition (a), in general, is not satisfied, whereas condition (b) is very difficult to check. Moreover, the IFT approach is not scalable, i.e., the analytical characterization of the derivatives via the implicit function theorem soon becomes intractable as the number of demand segments and supply channels increases. See, e.g., [27, 168]. In short, the IFT approach is not effective in performing comparative statics analysis in our model.

The MCS approach studies the impact of a parameter change on the marginal value of decision variables for objective functions defined on lattices. The MCS approach is very powerful in comparative statics analysis, because it does not require any regularity assumption regarding the objective function. In order to apply the MCS approach, the objective function needs to satisfy a certain form of complementarity conditions (e.g., joint supermodularity or, more generally, the single crossing property). Another feature of the MCS approach is that, *all* of the optimal decision variables should be monotone (in strong set order) in parameters. In our joint pricing and inventory management model, either the joint supermodularity or the single crossing property is very difficult, if not impossible, to establish in each decision epoch. Moreover, as we will show later, it is possible in our model that *only part* of the optimal decision variables (i.e., sales prices and order quantities) are monotone in the market parameters. Hence, the MCS approach does not apply to our model.

The limitations of the IFT and MCS approaches motivate us to develop a new method for the comparative statics analysis of our general joint pricing and inventory management model with demand segmentation, supply diversification, and market environment fluctuation. The new method provides rigorous proofs of comparative statics analysis and structural properties in our model. More specifically, our method proves the desired comparative statics results by contradiction and carefully analyzes how changes in parameter values impact the marginal value of each decision variable (i.e., the first-order partial derivative of the objective function). We identify a simple yet powerful lemma which translates the monotonicity relationship between the optimizers into that between the partial derivatives of the objective function under different parameter values. Our comparative statics method employs this lemma and some model-specific structural properties (e.g., the concavity of the objective function and/or the supermodularity of the objective function in one decision variable and one parameter) to construct a contradiction by iteratively linking the monotone relationship between the optimizers and that between the partial derivatives of the objective function. Note that the structural properties needed by our approach are weaker than those required by the IFT and MCS approaches (e.g., second-order continuous differentiability and complementarity). The lemma also enables us to make componentwise comparisons between the optimizers under different parameter values, because the monotonicity of the objective function's partial derivative with respect to one decision variable at the optimizer of interest is independent of the values of other decision variables. Hence, unlike the IFT approach, our new method is scalable; and unlike the MCS approach, our new method enables us to perform comparative statics analysis in a model where *only part* of the optimal decision variables are monotone in the parameter.

To perform comparative statics analysis in each decision epoch of our general joint pricing and inventory management model, we integrate our new method with the standard backward induction argument to iteratively link the comparison between optimizers and that between partial derivatives of the value functions and objective functions. We characterize the optimal joint pricing and ordering policy for an arbitrary number of demand segments and supply channels as a threshold policy, under which there exists a market-environment-dependent threshold for each demand segment [supply channel] such that it is optimal to sell to [order from] this segment [channel] if and only if the starting inventory level is above [below] its corresponding threshold. Moreover, both the optimal sales price for each demand segment and the optimal order quantity through each supply channel are decreasing in the starting inventory level of the firm. We also show that the optimal sales prices and order quantities are increasing in the market size. When the reference procurement costs of some supply channels increase, the firm increases the sales price in each demand segment, and the order quantities from the supply channels with unchanged reference procurement costs. Each firm's optimal order quantity may not be monotone in its own reference procurement cost. Serving a new demand segment drives the firm to increase its sales prices and order quantities, whereas expanding the supply pool has the opposite effect: it prompts the firm to decrease its sales prices and order quantities.

Our method is robust and applicable to comparative statics analysis in some other settings. For example, we consider joint price and effort competition games in which an arbitrary number of firms compete on price and effort level. More specifically, we study two competition models: (a) the effort-level-first competition where the firms first compete on effort and then on price, and (b) the simultaneous competition where the firms simultaneously compete on price and effort. Each firm's demand is increasing in the total effort level of all firms. As we will show later, the IFT approach is not scalable whereas the complementarity conditions required by the MCS approach are not satisfied, so the standard IFT and MCS approaches do not work in this model. Our new comparative statics method enables us to prove the existence and uniqueness of the equilibrium in the effort-level-first competition. In both competition models, we show that the equilibrium total effort level, and the equilibrium sales price and demand volume of each firm are increasing in the market index of any firm. We also identify the fat-cat effect in this setting, i.e., the equilibrium total effort level and the equilibrium price and demand of each firm in the effort-level-first competition are higher than their counterparts in the simultaneous competition.

To sum up, we propose a new method for comparative statics analysis in a general joint pricing and inventory management model with demand segmentation, supply diversification, and market environment fluctuation. This new comparative statics method requires less restrictive assumptions than the standard IFT and MCS approaches. Moreover, our method makes componentwise comparisons between optimizers with different parameter values, so it is well scalabile, and is capable of performing comparative statics analysis in a model where *some of* the optimal decision variables are *not* monotone in the parameter. The proposed method also applies to the comparative statics analysis in some other settings where the standard approaches do not work.

The rest of the chapter is organized as follows. We position this chapter in the related literature in Section 6.2. Section 6.3 presents our new method for comparative statics analysis in joint pricing and inventory management models. In Section 6.4, we apply the proposed method to study a general joint pricing and inventory management model with demand segmentation, supply diversification, and market environment fluctuation. Section 6.5 demonstrates the applicability of our method in competition games. We conclude this chapter by summarizing our method and findings in Section 6.6. Most of the proofs are relegated to Appendix E.1. Throughout this chapter, we use ∂ to denote the derivative operator of a single variable function, and ∂_x to denote the partial derivative operator of a multi-variable function with respect to variable x. For any multivariate continuously differentiable function $f(x_1, x_2, \cdots, x_n)$ and $\tilde{x} := (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)$ in $f(\cdot)$'s domain, we use $\partial_{x_i} f(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)$ to denote $\partial_{x_i} f(x_1, x_2, \cdots, x_n)|_{x=\tilde{x}}$ for any i. For any two n-dimensional vectors $v = (v_1, v_2, \cdots, v_n)$ and $\hat{v} = (\hat{v}_1, \hat{v}_2, \cdots, \hat{v}_n)$, we use $\hat{v} > v$ to denote that $\hat{v}_i \geq v_i$ for each $1 \leq i \leq n$, and $\hat{v} \neq v$.

6.2 Related Research

This chapter is built upon two streams of literature: (a) the method and application of comparative statics analysis, and (b) dynamic pricing and inventory management.

Comparative statics analysis is formalized in the economics literature by [95] and [144], where the classical IFT approach is introduced. The MCS approach is first established by [164]. He shows that the maximizer of a supermodular function is increasing in the parameters in the strong set order sense. [126] derive a necessary and sufficient condition (i.e., quasi-supermodularity and the single crossing property) for the solution set of an optimization problem to be monotone in parameters. [15] generalizes this result to stochastic optimization problems and characterizes necessary and sufficient conditions, based on the properties of utility functions and probability distributions, for comparative statics predictions to hold. [45] establish a new preservation property of supermodularity in a class of two-dimensional parametric optimization problems, where the feasible sets may not be lattices. Comparative statics analysis in game theoretic models has also been extensively studied in the literature (e.g., [124, 125, 24]). This literature mainly focuses on supermodular games. We refer interested readers to the monograph by [165] that coherently synthesizes the theory and applications of the MCS approach.

There is extensive application of comparative statics analysis in the operations management literature. See, e.g., [151, 152, 98, 71, 111]. The majority of the papers in this stream of research apply the IFT and MCS approaches to establish comparative statics results and the structural properties of their models. [27] is a notable exception that develops a novel analytical approach for the comparative statics analysis in multi-product, multi-resource newsvendor networks with responsive pricing. In their setting, the IFT approach is prohibitively difficult, whereas their new approach exploits the relationship between convex stochastic orders and dual variables, and does not suffer from the curse of dimensionality. This chapter contributes to this line of research by developing a new analytical method for comparative statics analysis in a general joint pricing and inventory management model. The major strength of the proposed method lies in the following three aspects: (a) it does not need the conditions required by the IFT and MCS approaches that are restrictive in dynamic pricing and inventory management models (e.g., second-order continuous differentiability and complementarity); (b) it does not suffer from the curse of dimensionality; and (c) it is amenable for comparative statics analysis in a model where *only part* of the optimal decision variables are monotone in the parameter.

This work is also related to the growing literature on the dynamic pricing and inventory management problem under general stochastic demand. [70] provide a general treatment of this problem, and show the optimality of a list-price/order-up-to policy. This line of literature has grown rapidly since [70]. For example, [47, 48, 49] analyze the joint pricing and inventory control problem with fixed set-up cost, and show that (s, S, p) policy is optimal for finite horizon, infinite horizon and continuous review models. [52] and [96] study the joint pricing and inventory control problem under lost sales. In the case of a single unreliable supplier with random yield, [112] show that supply uncertainty drives the firm to charge a higher price. When the replenishment leadtime is positive, the joint pricing and inventory control problem under periodic review is extremely difficult.

For this problem, [136] partially characterize the structure of the optimal policy, whereas [26] develop a simple heuristic that resolves the computational complexity. Several papers in this stream of literature also take into consideration consumer behaviors. [179] study the joint pricing and inventory management model in which customers bid for units of a firm's product over an infinite horizon. [97] characterize the optimal pricing and production policy under customer subscription and retention/attrition. [110] establish the concavity of the objective function in the nested logit model, and apply this model to analyze the joint pricing and inventory management problem with multiple products. The literature on the joint pricing and inventory management problem under a fluctuating market environment is scarce. To the best of our knowledge, [173] is the only paper which studies the dynamic pricing and inventory management problem under fluctuating procurement costs. We refer interested readers to [50] for a comprehensive survey on joint pricing and inventory control models. This chapter contributes to this stream of research by developing a new analytical method for the comparative statics analysis in a general joint pricing and inventory management model with demand segmentation, supply diversification, and market environment fluctuation.

6.3 A New Comparative Statics Method

In this section, we first give an example to illustrate why the IFT and MCS approaches are not applicable for comparative statics analysis in our general joint pricing and inventory management model. We then develop a new analytical method for comparative statics analysis therein.

6.3.1 An Illustrative Example

In this subsection, we give an example that clearly illustrates why the IFT and MCS approaches do not apply to the general joint pricing and inventory management model with demand segmentation, supply diversification, and market environment fluctuation.

Let $f_i(y_i)$ be a first-order continuously differentiable and strictly concave function on $\mathcal{Y}_i = [A_i, B_i]$ for $i = 1, 2, \cdots, p$, and $g_i(y_i|\gamma)$ be continuously differentiable and strictly concave in y_i and submodular in (y_i, γ), where $y_i \in \mathcal{Y}_i = [A_i, B_i]$ and $\gamma \in \Gamma \subset [\underline{\gamma}, \bar{\gamma}]$, for each $i = p+1, p+2, \cdots, p+q$. Moreover, assume that $h(y_0|\gamma)$ is continuously differentiable and concave in y_0 and supermodular in (y_0, γ), where $y_0 \in \mathbb{R}$ and $\gamma \in \Gamma$. Let λ_i be a positive constant for any $i = 1, 2, \cdots, p+q$. Consider the following optimization problem:

$$
\begin{aligned}
(y_1^*(\gamma), y_2^*(\gamma), \cdots, y_p^*(\gamma), y_{p+1}^*(\gamma), \cdots, y_{p+q}^*(\gamma)) =& \text{argmax}_{y \in \mathcal{Y}} F(y|\gamma), \\
\text{s.t. } F(y|\gamma) =& \sum_{i=1}^{p} f_i(y_i) + \sum_{i=p+1}^{p+q} g_i(y_i|\gamma) + h(\sum_{i=1}^{p+q} \lambda_i y_i|\gamma), \\
\mathcal{Y} =& \mathcal{Y}_1 \times \mathcal{Y}_2 \times \cdots \times \mathcal{Y}_{p+q}, \gamma \in \Gamma.
\end{aligned}
\tag{6.1}
$$

The objective function $F(\cdot|\gamma)$ in the optimization problem (6.1) is strictly jointly concave in $y = (y_1, y_2, \cdots, y_{p+q})$. Hence, the optimizer $y^*(\gamma) = (y_1^*(\gamma), y_2^*(\gamma), \cdots, y_{p+q}^*(\gamma))$ is well defined and unique. As we will show in Section 6.4, several optimization problems in our general joint pricing and inventory management model can be reduced to a convex program similar to (6.1). A natural question on (6.1) is: how does $y^*(\gamma)$ change with γ? We have the following lemma that addresses this question.

Lemma 15 *For the optimization problem defined in* (6.1), $y_i^*(\gamma)$ *is increasing in* γ, *for all* $1 \le i \le p$.

Lemma 15 characterizes the impact of the parameter γ upon the optimizer $\{y_i^*(\gamma)\}_{1 \le i \le p}$. Note that Lemma 15 does not give any comparative statics result for $\{y_i^*(\gamma)\}_{p+1 \le i \le p+q}$, because it is easy to construct functions $\{f_i(\cdot), g_i(\cdot|\cdot), h(\cdot|\cdot)\}$ and feasible set $\mathcal{Y} \times \Gamma$, such that $y_i^*(\cdot)$ is not monotone in γ for some $p+1 \le i \le p+q$. See Appendix E.2 for an example. We also remark that the assumption that $f_i(\cdot)$'s and $g_i(\cdot|\gamma)$'s are strictly concave is mainly for expositional convenience. When this assumption is relaxed to that they are weakly concave, the optimizers may not be unique and we select the lexicographically smallest one. In this case, Lemma 15 still holds and our new comparative statics method is also valid.

We now explain in detail why the IFT and MCS approaches cannot be used to prove Lemma 15. The first issue related to the IFT approach is that $F(\cdot|\cdot)$ may not be twice continuously differentiable on its domain. For example, if Γ is finite, $F(\cdot|\cdot)$ is not twice continuously differentiable. Now we assume that Γ is an interval and $F(\cdot|\cdot)$ is twice continuously differentiable. For any optimizer $y^*(\gamma)$ that lies in the interior of $\mathcal{Y} \times \Gamma$, the implicit function theorem implies that $y^*(\gamma)$ is continuously differentiable in γ, and the derivative is given by:

$$\frac{\mathrm{d}y^*}{\mathrm{d}\gamma} = -\Omega^{-1}V, \tag{6.2}$$

where Ω is a $(p+q)\times(p+q)$ matrix with $\Omega_{i,j} = \lambda_i\lambda_j\partial^2_{y_0}h(\sum_{l=1}^{p+q}\lambda_l y_l^*(\gamma)|\gamma)$ for all $i\neq j$, $\Omega_{i,i} = f_i''(y_i^*(\gamma)) + (\lambda_i)^2\partial^2_{y_0}h(\sum_{l=1}^{p+q}\lambda_l y_l^*(\gamma)|\gamma)$ for $1\leq i\leq p$, and $\Omega_{i,i} = \partial^2_{y_i}g_i(y_i^*(\gamma)|\gamma) + (\lambda_i)^2\partial^2_{y_0}h(\sum_{l=1}^{p+q}\lambda_l y_l^*(\gamma)|\gamma)$ for $p+1\leq i\leq p+q$, and V is a $(p+q)$-vector with $V_i = \lambda_i\partial_{y_0}\partial_\gamma h(\sum_{l=1}^{p+q}\lambda_l y_l^*(\gamma)|\gamma)$ for $1\leq i\leq p$, and
$V_i = \partial_{y_i}\partial_\gamma g_i(y_i^*(\gamma)|\gamma) + \lambda_i\partial_{y_0}\partial_\gamma h(\sum_{l=1}^{p+q}\lambda_l y_l^*(\gamma)|\gamma)$ for $p+1\leq i\leq p+q$. As p and q increase, however, computing Ω^{-1} in (6.2) suffers from the curse of dimensionality, and, in general, we are unable to characterize the sign of $\mathrm{d}y_i^*/\mathrm{d}\gamma$ for each i. When $y^*(\gamma)$ is on the boundary of $\mathcal{Y}$ (i.e., some of the constraints are binding), the IFT approach can be generalized to the perturbation analysis approach (see, e.g., [76]), which, again, determines the sign of the derivative of $y^*(\gamma)$ with respect to γ by characterizing the inverse of Hessian, and, hence, suffers from the curse of dimensionality. Thus, it is very difficult, if not impossible, to perform comparative statics analysis in (6.1) by the IFT approach.

The MCS approach also fails to conduct the comparative statics analysis in (6.1). More specifically, it's clear that $F(\cdot|\cdot)$ is not jointly supermodular in (y,γ), nor does it satisfy the single crossing property in $(y;\gamma)$. By [126], in order to apply the MCS approach, it is necessary that the objective function should satisfy the single crossing property with respect to the decision vector and the parameter. Moreover, the MCS approach, when applicable, always gives a comparative statics prediction for *all* the decision variables of an optimization problem (see, e.g., [165]). In our optimization problem (6.1), however, it is easy to specify functions $\{f_i(\cdot), g_i(\cdot|\cdot), h(\cdot|\cdot)\}$ and feasible set $\mathcal{Y}\times\Gamma$, such that $y_i^*(\cdot)$ is not monotone in γ for some $p+1\leq i\leq p+q$, as shown by the example in Appendix E.2. Therefore, the MCS approach does not apply to the comparative statics analysis in (6.1).

6.3.2 Proof of Lemma 15 with Our New Method

Since the standard IFT and MCS approaches are not applicable to conducting comparative statics analysis in (6.1), we develop a new method to prove Lemma 15. Before presenting the proof of Lemma 15 and our new method in detail, we introduce a lemma that plays a key role therein:

Lemma 16 *Let $F_i(z, Z_i)$ be a first-order differentiable function in (z, Z_i) for $i = 1, 2$, where $z \in [\underline{z}, \bar{z}]$ ($\underline{z}$ and $\bar{z}$ might be infinite) and $Z_i \in \mathcal{Z}_i$, where $\mathcal{Z}_i$ is the feasible set of Z_i. For $i = 1, 2$, let*

$$(z_i^*, Z_i^*) := \text{argmax}_{(z, Z_i) \in [\underline{z}, \bar{z}] \times \mathcal{Z}_i} F_i(z, Z_i),$$

be the optimizer of $F_i(\cdot, \cdot)$. If $z_1^* < z_2^*$, we have: $\partial_z F_1(z_1^*, Z_1^*) \leq \partial_z F_2(z_2^*, Z_2^*)$.

Proof. $z_1^* < z_2^*$, so $\underline{z} \leq z_1^* < z_2^* \leq \bar{z}$. Hence, $\partial_z F_1(z_1^*, Z_1^*) \begin{cases} = 0 & \text{if } z_1^* > \underline{z}, \\ \leq 0 & \text{if } z_1^* = \underline{z}; \end{cases}$ and $\partial_z F_2(z_2^*, Z_2^*) \begin{cases} = 0 & \text{if } z_2^* < \bar{z}, \\ \geq 0 & \text{if } z_2^* = \bar{z}, \end{cases}$ i.e., $\partial_z F_1(z_1^*, Z_1^*) \leq 0 \leq \partial_z F_2(z_2^*, Z_2^*)$. *Q.E.D.*

Lemma 16 is straightforward, but it is a powerful tool in our new comparative statics method, as illustrated by the proof of Lemma 15:

Proof of Lemma 15. We show by contradiction, i.e., we derive a contradiction under the assumption that $y_i^*(\gamma) > y_i^*(\hat{\gamma})$ for some $1 \leq i \leq p$ and $\gamma < \hat{\gamma}$. Without loss of generality, we choose $i = 1$, i.e.,

$$y_1^*(\gamma) > y_1^*(\hat{\gamma}). \tag{6.3}$$

Denote $y_0^*(\gamma) := \sum_{j=1}^{p+q} \lambda_j y_j^*(\gamma)$ for all $\gamma \in \Gamma$. Lemma 16 implies that $\partial_{y_1} F(y^*(\gamma)|\gamma) \geq \partial_{y_1} F(y^*(\hat{\gamma})|\hat{\gamma})$, i.e.,

$$\begin{aligned} \partial_{y_1} f_1(y_1^*(\gamma)) + \lambda_1 \partial_{y_0} h(y_0^*(\gamma)|\gamma) = \partial_{y_1} F(y^*(\gamma)|\gamma) \geq & \partial_{y_1} F(y^*(\hat{\gamma})|\hat{\gamma}) \\ = & \partial_{y_1} f_1(y_1^*(\hat{\gamma})) + \lambda_1 \partial_{y_0} h(y_0^*(\hat{\gamma})|\hat{\gamma}). \end{aligned} \tag{6.4}$$

The strict concavity of $f_1(\cdot)$ yields that $\partial_{y_1} f_1(y_1^*(\gamma)) < \partial_{y_1} f_1(y_1^*(\hat{\gamma}))$. Hence,

$$\partial_{y_0} h(y_0^*(\gamma)|\gamma) > \partial_{y_0} h(y_0^*(\hat{\gamma})|\hat{\gamma}). \tag{6.5}$$

Since $h(\cdot|\cdot)$ is supermodular in (y_0, γ) and concave in y_0, $y_0^*(\gamma) < y_0^*(\hat{\gamma})$. Therefore,

$$\text{there exists a } j,\ 2 \leq j \leq p + q, \text{ such that } y_j^*(\gamma) < y_j^*(\hat{\gamma}). \tag{6.6}$$

If $2 \leq j \leq p$, we invoke Lemma 16 again, so $y_j^*(\gamma) < y_j^*(\hat{\gamma})$ implies that $\partial_{y_j} F(y^*(\gamma)|\gamma) \leq \partial_{y_j} F(y^*(\hat{\gamma})|\hat{\gamma})$, i.e.,

$$\begin{aligned} \partial_{y_j} f_j(y_j^*(\gamma)) + \lambda_j \partial_{y_0} h(y_0^*(\gamma)|\gamma) = \partial_{y_j} F(y^*(\gamma)|\gamma) \leq & \partial_{y_j} F(y^*(\hat{\gamma})|\hat{\gamma}) \\ = & \partial_{y_j} f_j(y_j^*(\hat{\gamma})) + \lambda_j \partial_{y_0} h(y_0^*(\hat{\gamma})|\hat{\gamma}). \end{aligned}$$

Since $\partial_{y_0} h(y_0^*(\gamma)|\gamma) > \partial_{y_0} h(y_0^*(\hat{\gamma})|\hat{\gamma})$ by (6.5), $\partial_{y_j} f_j(y_j^*(\gamma)) < \partial_{y_j} f_j(y_j^*(\hat{\gamma}))$. Since $f_j(\cdot)$ is strictly concave, $y_j^*(\gamma) < y_j^*(\hat{\gamma})$ implies that

$$\partial_{y_j} f_j(y_j^*(\gamma)) > \partial_{y_j} f_j(y_j^*(\hat{\gamma})), \text{ which contradicts } \partial_{y_j} f_j(y_j^*(\gamma)) < \partial_{y_j} f_j(y_j^*(\hat{\gamma})). \tag{6.7}$$

Analogously, if $p + 1 \le j \le p + q$ in (6.6), Lemma 16 implies that $\partial_{y_j} F(y^*(\gamma)|\gamma) \le \partial_{y_j} F(y^*(\hat{\gamma})|\hat{\gamma})$, i.e.,

$$\begin{aligned}\partial_{y_j} g_j(y_j^*(\gamma)|\gamma) + \lambda_j \partial_{y_0} h(y_0^*(\gamma)|\gamma) = \partial_{y_j} F(y^*(\gamma)|\gamma) \le & \partial_{y_j} F(y^*(\hat{\gamma})|\hat{\gamma}) \\ = & \partial_{y_j} g_j(y_j^*(\hat{\gamma})|\hat{\gamma}) + \lambda_j \partial_{y_0} h(y_0^*(\hat{\gamma})|\hat{\gamma}).\end{aligned}$$

Since $\partial_{y_0} h(y_0^*(\gamma)|\gamma) > \partial_{y_0} h(y_0^*(\hat{\gamma})|\hat{\gamma})$ by (6.5), $\partial_{y_j} g_j(y_j^*(\gamma)|\gamma) < \partial_{y_j} g_j(y_j^*(\hat{\gamma})|\hat{\gamma})$. Since $g_j(\cdot|\cdot)$ is submodular in (y_j, γ) and strictly concave in y_j, $y_j^*(\gamma) < y_j^*(\hat{\gamma})$ implies that

$$\partial_{y_j} g_j(y_j^*(\gamma)|\gamma) > \partial_{y_j} g_j(y_j^*(\hat{\gamma})|\hat{\gamma}), \text{ which contradicts } \partial_{y_j} g_j(y_j^*(\gamma)|\gamma) < \partial_{y_j} g_j(y_j^*(\hat{\gamma})|\hat{\gamma}). \tag{6.8}$$

Combining the contradictions of (6.7) and (6.8), we have $y_1^*(\gamma) \le y_1^*(\hat{\gamma})$. Repeat the above argument for $1 < i \le p$, it follows that $y_i^*(\gamma) \le y_i^*(\hat{\gamma})$ for all $1 \le i \le p$. *Q.E.D.*

As we can see from the proof of Lemma 15, our new method employs Lemma 16 to make componentwise comparisons between the optimizers under different parameter values. More specifically, the method consists of five steps: STEP (A). For each of the focal decision variable with some potential comparative statics result, we first assume, to the contrary, that the comparative statics prediction of this decision variable is *reversed* for some parameter values (e.g., inequality (6.3) in the proof of Lemma 15). STEP (B). We invoke Lemma 16 to characterize some monotone relationships of the *partial derivatives* of the objective function with respect to this decision variable at these parameter values (e.g., inequality (6.4) in the proof of Lemma 15). STEP (C). Using some model specific properties of the objective function (e.g., the supermodularity in one decision variable and the parameter, componentwise concavity, and first-order differentiability), such monotone relationships of the partial derivatives can be translated *back* into the monotone relationship of *another optimal decision variable* at the given parameter values (e.g., inequality (6.6) in the proof of Lemma 15). STEP (D). Repeating steps (B) - (C), we employ Lemma 16 to *iteratively* establish the monotone relationship of partial derivatives and that of some other optimal decision variables at the given parameter values. This iterative procedure is stopped when either (i) the desired comparative statics result for the focal decision variable is proved by contradiction (e.g., inequalities (6.7) and (6.8) in the proof of Lemma 15), or (ii) no further monotone relationship can be established (e.g., the case in which we assume that $y_i^*(\gamma) > y_i^*(\hat{\gamma})$ or $y_i^*(\gamma) < y_i^*(\hat{\gamma})$ for $\gamma < \hat{\gamma}$ and $p+1 \leq i \leq p+q$, see Appendix E.2). Step (e). We repeat the same iterative procedure, i.e., steps (a) - (d), for each focal decision variable to obtain its corresponding comparative statics result.

Note that there are two stopping conditions for the iterative procedure in Step (d). When the stopping condition (ii) applies, by our experience, it is very likely that there exist some model specifications such that the desired comparative statics result for the focal decision variable does not hold. For example, in the optimization problem (6.1), no contradiction can be reached under any monotone comparative statics prediction of $y_i^*(\cdot)$ $(p+1 \leq i \leq p+q)$ with respect to γ for general $\{f_i(\cdot), g_i(\cdot|\cdot), h(\cdot|\cdot)\}$ functions. In Appendix E.2, we discuss in detail on how the iterative procedure in Step (d) is stopped without reaching a contradiction in this case, and give an example in which $y_i^*(\gamma)$ $(p+1 \leq i \leq p+q)$ is not monotone in γ. Hence, our new method not only helps prove the comparative statics results when they exist, but also helps identify cases in which comparative statics results do not hold for some decision variables.

Our method proves the desired comparative statics result by contradiction. The essence is to construct a contradiction by iteratively linking the monotone relationship between the optimizers and that between the partial derivatives. Though simple, Lemma 16 plays a crucial role in this process, because, in Step (b), it translates the monotonicity of the focal decision variable (in the parameter) into that of the partial derivative of the objective function with respect to this decision variable at the optimizing point. Hence, in Step (d), Lemma 16 enables us to iteratively link the monotone relationship of optimizers and that of partial derivatives, which is the key to establish a contradiction in our method. The main benefit of Lemma 16 is that the monotonicity of the partial derivatives with respect to the focal decision variable is irrelevant to the values of other decision variables at the optimizing point. This benefit allows us to perform comparative statics analysis componentwisely in Step (e). Hence, our method enables us to perform comparative statics analysis in a model where only part of the optimal decision variables are monotone in the parameter, and it is scalable. The componentwise comparison between the optimizers is also the key difference between our method and the IFT and

MCS approaches, both of which involve the analysis of some properties of the objective function in terms of the whole decision vector (e.g., the Hessian and/or the joint supermodularity of the objective function). Moreover, since the objective functions in Lemma 16, $F_i(\cdot, \cdot)$ $(i = 1, 2)$, can be completely different, our method can be used to compare the optimal decisions in different models. See, e.g., the proofs of Theorems 6.4.7, 6.4.8, and 6.5.5 in Appendix E.1.

Although our method is fundamentally different from the IFT and MCS approaches, it shares some similarity with these two standard approaches. As the IFT approach, the proposed method studies the first-order (KKT) condition at the optimizer of interest. Hence, the objective function needs to satisfy the first-order continuous differentiability condition, but not necessarily the second-order continuous differentiability condition. Analogous to the MCS approach, our new method analyzes the impact of the parameter upon the marginal value of each decision variable in detail, so that we can translate the monotonicity of partial derivatives with respect to one decision variable back into the monotonicity of another optimal decision variable. Thus, in order to obtain a contradiction (and a comparative statics result), our method requires the objective function to be supermodular in the parameter and each of the focal decision variables (e.g., $F(y|\gamma)$ is supermodular in (y_i, γ) for each $1 \leq i \leq p$ in our example), but not necessarily jointly supermodular or satisfying the single crossing property. The above two condition relaxations enhance the applicability of our method in the general joint pricing and inventory management model with demand segmentation, supply diversification, and market environment fluctuation, where the second-order continuous differentiability and/or, in particular, the joint supermodularity of the objective function in each decision epoch are hard, if not impossible, to establish. In the next section, we discuss in detail how the new method facilitates the comparative statics analysis in this model. We also demonstrate the applicability of our method in game theoretic models of joint price and effort competition in Section 6.5.

6.4 Application of the New Comparative Statics Method in a General Joint Pricing and Inventory Management Model

In this section, we employ our new comparative statics method to study a general joint pricing and inventory management model with demand segmentation, supply diversification, and market environment fluctuation. The comparative statics analysis is essential to studying this model, because it enables us to characterize the optimal pricing and replenishment policy, and the impact of demand segmentation, supply diversification, and market environment fluctuation therein. The analysis in this section demonstrates the applicability of our new method in joint pricing and inventory management models.

6.4.1 Model

We consider a T-period joint pricing and inventory management model, in which a firm replenishes inventory from a portfolio of supply channels and serves multiple demand segments with endogenous sales prices. The firm maximizes its total discounted profit over the planning horizon by optimizing its joint pricing and inventory policy in each period. The periods are indexed backwards as $\{T, T-1, \cdots, 1\}$ and the discount factor is denoted as $\alpha \in (0,1)$.

We assume that the customer market is completely segmented, i.e., each customer in the market unambiguously belongs to a specific demand segment. Complete segmentation applies to the settings where customers are classified based on the differences in, e.g., (a) geographic area, (b) the need for product feature, (c) socioeconomic attributes, and (d) business sector (in B2B market) (see, e.g., [10, 139]). There are n demand segments in the market, and we denote them as $\mathcal{N} := \{1, 2, \cdots, n\}$. In period t, the firm selects a vector of prices, $p_t = (p_t^1, p_t^2, \cdots, p_t^n)$, for different demand segments. More specifically, for each $i \in \mathcal{N}$, $p_t^i \in [p_{\min}^i, p_{\max}^i]$ is the sales price for customers in segment i, where $p_{\min}^i > 0$ $[p_{\max}^i \leq +\infty]$ is the minimum [maximum] allowable price for this segment. We use $\Lambda_t^i > 0$ to denote the expected maximum demand (i.e., the market size) from segment i in period t. Let $\Lambda_t := (\Lambda_t^1, \Lambda_t^2, \cdots, \Lambda_t^n)$ be the market size vector. Since customers are completely segmented, the demand from segment i is independent of the sales price in segment j $(i \neq j)$. Specifically, we assume that, given the sales price p_t^i and the market size Λ_t^i, the demand from segment i in period t is given as follows:

$$D_t^i(p_t^i, \Lambda_t^i) = \Lambda_t^i d^i(p_t^i)\varsigma_t + \epsilon_t^i. \tag{6.9}$$

In (6.9), $d^i(p_t^i)$ denotes the probability that an arriving customer in segment i will make a purchase when facing a sales price p_t^i, where $d^i(\cdot)$ is a strictly decreasing function of p_t^i. A typical example of this specification is the independent reservation price model (e.g., s[83]). ς_t is the demand-segment-independent multiplicative market size perturbation, which represents the common demand shock (e.g., global economic changes) on each segment. We assume that $\{\varsigma_t\}_{t=T}^1$ are *i.i.d.* positive random variables independent of Λ_t with mean 1. The additive random perturbation term ϵ_t^i captures all other uncertainties not explicitly considered in this model. We assume that $\{\epsilon_t^i\}_{t=T}^1$ are *i.i.d.* continuous random variables independent of Λ_t and ς_t with mean 0. Hence, $D_t^i(p_t^i, \Lambda_t^i)$ follows a continuous distribution for any given (p_t^i, Λ_t^i) and $i \in \mathcal{N}$. We use $D_t(p_t, \Lambda_t) = (D_t^1(p_t^1, \Lambda_t^1), D_t^2(p_t^2, \Lambda_t^2), \cdots, D_t^n(p_t^n, \Lambda_t^n))$ to denote the demand vector for all demand segments, with the sales price vector p_t and the market size vector Λ_t in period t. Given (p_t, Λ_t), the accumulative demand from all segments in period t is given by:

$$D_t^a(p_t, \Lambda_t) = \sum_{i \in \mathcal{N}} D_t^i(p_t^i, \Lambda_t^i) = (\sum_{i=1}^{n} \Lambda_t^i d^i(p_t^i))\varsigma_t + \epsilon_t, \tag{6.10}$$

where the superscript 'a' refers to "accumulative", and $\epsilon_t := \sum_{i=1}^n \epsilon_t^i$ represents the accumulative additive perturbation in period t.

For each i, since $d^i(p_t^i)$ is strictly decreasing, it has a strictly decreasing inverse $p^i(\cdot)$ that maps from $[d_{\min}^i, d_{\max}^i]$ to $[p_{\min}^i, p_{\max}^i]$, where $d_{\min}^i = d^i(p_{\max}^i) = 0$ and $d_{\max}^i = d^i(p_{\min}^i) \leq 1$. We view the purchasing probability vector $d_t := (d_t^1, d_t^2, \cdots, d_t^n)$, instead of the sales price vector p_t, as the decision variable in each period. Without loss of generality, we assume that $d_{\max}^i = d_{\max} \leq 1$ for any $i \in \mathcal{N}$, i.e., the maximum expected purchasing probability is the same for every demand segment. Since $d^i(p_{\max}^i) = 0$, our model endogenizes the option that, for any $i \in \mathcal{N}$, the firm can choose not to sell to demand segment i by charging a prohibitively high sales price $p_{\max}^i$. We impose the following assumption throughout our analysis:

Assumption 6.4.1 *For each demand segment $i \in \mathcal{N}$, $R^i(d_t^i) := p^i(d_t^i)d_t^i$ is continuously differentiable and concave in $d_t^i \in [0, d_{\max}]$.*

Note that the strict monotonicity of $p^i(\cdot)$, together with the concavity of $R^i(\cdot)$, suggests that $R^i(\cdot)$ is strictly concave in d_t^i for each $i \in \mathcal{N}$. We remark that, when there is only one demand segment ($n = 1$), our demand model is reduced to the most commonly studied demand model in the joint pricing and inventory management literature. See, e.g., [47, 50, 189].

The firm sources from a portfolio of m supply channels, which is denoted as $\mathcal{M} = \{1, 2, \cdots, m\}$. In period t, the firm selects a vector of order quantities, $q_t = (q_t^1, q_t^2, \cdots, q_t^m)$, from different supply channels. More specifically, the firm orders $q_t^j \geq 0$ from supply channel j and pays a cost $C^j(q_t^j|c_t^j)$, where $C^j(\cdot|c_t^j)$ is the cost function of supply channel j when the reference procurement cost is c_t^j, and $C^j(0|c_t^j) = 0$ for all $j \in \mathcal{M}$. The reference procurement cost c_t^j is an index for the actual procurement cost of supply channel j, which is independent of the firm's pricing and inventory policy. For example, c_t^j can be viewed as the unit procurement cost of the raw material in supply channel j. Since an increase in c_t^j increases the marginal cost of sourcing from supply channel j, we assume that $C^j(\cdot|\cdot)$ is supermodular in (q_t^j, c_t^j) for any $j \in \mathcal{M}$. We use $c_t := (c_t^1, c_t^2, \cdots, c_t^m)$ to denote the reference procurement cost vector in period t. Moreover, we assume that there exists diseconomy of scale to source from each supply channel, i.e., $C^j(\cdot|c_t^j)$ is convexly increasing in q_t^j for each $j \in \mathcal{M}$. In reality, this assumption applies when the supply channel is capacitated, so that orders exceeding the standard capacity are charged a higher rate for the additional outsourcing costs and/or overtime labor costs (see [155]). The assumption of convex ordering cost is necessary to prove the convexity [concavity] of the optimal cost [profit] function in a multi-period model, and common in the inventory management literature (see, e.g., [181, 50]). Without loss of generality, we assume that $C^j(\cdot|c_t^j)$ is continuously differentiable in q_t^j for any $q_t^j \geq 0$. For expositional ease (i.e., to ensure the uniqueness of the optimizer), we assume that $C^j(\cdot|c_t^j)$ is strictly convex for each j. This assumption is made without loss of generality. If we relax this assumption to that $C^j(\cdot|c_t^j)$'s are weakly convex, all results in this section continue to hold with more tedious proofs, as long as we select the lexicographically smallest optimizer in each decision epoch. Consistent with most of the joint pricing and inventory management models in the literature, we assume that the replenishment leadtime to source from any supply channel is 0. Finally, we remark that the firm employs the supply diversification strategy to hedge against: (a) the procurement cost fluctuation risk caused by the volatility of c_t, and (b) the diseconomy of scale for each supply channel.

The firm operates under a fluctuating market environment with stochastically varying market sizes Λ_t and reference procurement costs c_t. Let the $(n+m)$-vector $\theta_t := (\Lambda_t, c_t)$ be the state of the market environment in period t. We assume that θ_t evolves according to an exogenous Markov process throughout the planning horizon. Let $\Lambda_t^{-i} := (\Lambda_t^1, \cdots, \Lambda_t^{i-1}, \Lambda_t^{i+1}, \cdots, \Lambda_t^n)$ and $c_t^{-j} := (c_t^1, \cdots, c_t^{j-1}, c_t^{j+1}, \cdots, c_t^m)$. We assume that, for any $i \in \mathcal{N}$ [$j \in \mathcal{M}$], conditioned on Λ_t^i [c_t^j], Λ_{t-1}^i [c_{t-1}^j] is independent of (Λ_t^{-i}, c_t) [(Λ_t, c_t^{-j})], i.e., Λ_t^i $[c_t^j]$ is a sufficient statistic for Λ_{t-1}^i $[c_{t-1}^j]$. Hence, the dynamics of θ_t can be represented as $\Lambda_{t-1}^i = \xi_t^{\Lambda,i}(\Lambda_t^i)$ and $c_{t-1}^j = \xi_t^{c,j}(c_t^j)$, where $\mathbb{E}\{\xi_t^{\Lambda,i}(\Lambda_t^i)|\theta_t\}, \mathbb{E}\{\xi_t^{c,j}(c_t^j)|\theta_t\} < +\infty$. We further assume that, if $\hat{\Lambda}_t^i > \Lambda_t^i$ $[\hat{c}_t^j > c_t^j]$, $\xi_t^{\Lambda,i}(\hat{\Lambda}_t^i) \geq_{s.d.} \xi_t^{\Lambda,i}(\Lambda_t^i)$ $[\xi_t^{c,j}(\hat{c}_t^j) \geq_{s.d.} \xi_t^{c,j}(c_t^j)]$, where $\geq_{s.d.}$ denotes the first-order stochastic dominance. This is an intuitive assumption, since a higher current market size is more likely to give rise to a higher market size in the next period, and the same is true for the reference procurement cost. Moreover, we assume that, for any given θ_t, $\xi_t^{\Lambda,i}(\Lambda_t^i)$ and $\xi_t^{c,j}(c_t^j)$ are independent of ϵ_t and ς_t.

The sequence of events in each period unfolds as follows. At the beginning of period t, the firm reviews its inventory level I_t and the realized state of market environment θ_t. The firm then simultaneously decides the sales price for each demand segment and the order quantity from each supply channel, and pays the total procurement cost $\sum_{j\in\mathcal{M}} C^j(q_t^j|c_t^j)$. The orders are received immediately, after which the price-dependent stochastic demand vector $D_t(p_t, \Lambda_t)$ realizes. The firm then collects revenue from the realized demand. Unmet demand is fully backlogged and excess inventory is fully carried over to the next period. Finally, the firm pays $H(z)$ for the inventory holding and backlogging cost for z units of ending net inventory, where $H(\cdot)$ is a convex function with $H(0) = 0$ and $H(\cdot) > 0$ otherwise. Moreover, we assume that $H(\cdot)$ satisfies the Lipchitz continuity with the Lipchitz constant c_H, i.e., for any $z_1, z_2 \in \mathbb{R}$, $|H(z_1) - H(z_2)| \leq c_H|z_1 - z_2|$. Note that although the demand, cost, and inventory penalty functions are assumed to be stationary for expositional convenience, the structural results in this section remain valid when they are time-dependent.

To formulate the planning problem as a dynamic program, let

$V_t(I_t|\theta_t)$ = the maximum expected discounted total profit in periods $t, t-1, \cdots, 0$, when the starting inventory level in period t is I_t and the realized market environment state is θ_t.

Without loss of generality, we assume that excess inventory at the end of the planning horizon is discarded without any salvage value, i.e., $V_0(I_0|\theta_0) = 0$. The optimal value functions satisfy the following recursive scheme:

$$V_t(I_t|\theta_t) = \max_{(d_t,q_t)\in\mathcal{F}} J_t(d_t, q_t, I_t|\theta_t), \tag{6.11}$$

$$\begin{aligned}
\text{where } \mathcal{F} &:= \{(d_t, q_t) : \forall i \in \mathcal{N}, d_t^i \in [0, d_{\max}], \forall j \in \mathcal{M}, q_t^j \geq 0\}, \text{ and} \quad (6.12) \\
J_t(d_t, q_t, I_t|\theta_t) &:= \mathbb{E}\{\sum_{i\in\mathcal{N}} p^i(d_t^i) D_t^i(p_t^i(d_t^i), \Lambda_t^i) - \sum_{j\in\mathcal{M}} C^j(q_t^j|c_t^j) \\
&\quad -H(I_t + \sum_{j\in\mathcal{M}} q_t^j - D_t^a(p(d_t), \Lambda_t)) \\
&\quad +\alpha V_{t-1}(I_t + \sum_{j\in\mathcal{M}} q_t^j - D_t^a(p(d_t), \Lambda_t)|\theta_{t-1})|\theta_t\} \\
&= (\sum_{i\in\mathcal{N}} \Lambda_t^i R^i(d_t^i)) - \sum_{j\in\mathcal{M}} C^j(q_t^j|c_t^j) \\
&\quad +\mathbb{E}_{\varsigma_t}\{\Psi_t(I_t + \sum_{j\in\mathcal{M}} q_t^j - (\sum_{i\in\mathcal{N}} \Lambda_t^i d_t^i)\varsigma_t|\theta_t)\}, \quad (6.13) \\
\text{with } \Psi_t(z|\theta_t) &:= \mathbb{E}_{\theta_{t-1},\epsilon_t}\{-H(z - \epsilon_t) + \alpha V_{t-1}(z - \epsilon_t|\theta_{t-1})|\theta_t\}. \quad (6.14)
\end{aligned}$$

Therefore, for each period t, the firm's profit-maximizing problem is to select a joint pricing and replenishment policy $(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t)) \in \mathcal{F}$ to maximize $J_t(d_t, q_t, I_t|\theta_t)$, with starting inventory level I_t and market environment state θ_t. We use $x_t := I_t + \sum_{j\in\mathcal{M}} q_t^j$ to denote the total order-up-to level, and $x_t^*(I_t, \theta_t) := I_t + \sum_{j\in\mathcal{M}} q_t^{j*}(I_t, \theta_t)$ to denote the optimal total order-up-to level. Moreover, let $\mathcal{N}_t^*(I_t, \theta_t) := \{i \in \mathcal{N} : d_t^{i*}(I_t, \theta_t) > 0\}$ and $\mathcal{M}_t^*(I_t, \theta_t) := \{j \in \mathcal{N} : q_t^{j*}(I_t, \theta_t) > 0\}$, i.e., $\mathcal{N}_t^*(I_t, \theta_t)$ is the optimal set of active demand segments to which the firm sells, and $\mathcal{M}_t^*(I_t, \theta_t)$ is the optimal set of active supply channels from which the firm orders.

To conclude this subsection, we characterize some preliminary concavity and differentiability properties of the value and objective functions in the following lemma.

Lemma 17 *For $t = T, T-1, \cdots, 1$ and any given (I_t, θ_t), the following statements hold:*

(a) $\Psi_t(\cdot|\theta_t)$ is concave and continuously differentiable in z.

(b) $J_t(\cdot, \cdot, I_t|\theta_t)$ is strictly jointly concave and continuously differentiable in (d_t, q_t).

(a) $V_t(\cdot|\theta_t)$ is concave and continuously differentiable in I_t.

It follows immediately from Lemma 17 that the optimal joint pricing and ordering policy $(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t))$ is well-defined and unique in the feasible set $\mathcal{F}$.

6.4.2 Comparative Statics Analysis with Our New Method

First we observe that, by Equation (6.13), the objective function in each period $J_t(\cdot, \cdot, I_t|\theta_t)$ is of the similar form to our illustrative optimization problem (i.e., Equation (6.1)) in Section 6.3. Therefore, following the same argument as the discussion in Section 6.3.1, the standard IFT and MCS approaches are generally not applicable to comparative statics analysis in our general joint pricing and inventory management model with demand segmentation, supply diversification and fluctuating market environment. Therefore, we employ our new comparative statics method to study this model. Moreover, Lemma 15 applies to the proofs of several comparative statics results in this subsection, including Theorem 6.4.1 and Theorems 6.4.4-6.4.6. To begin with, we apply our new comparative statics method to characterize the optimal policy structure in the following theorem.

Theorem 6.4.1 (Optimal policy structure.) *For $t = T, T-1, \cdots, 1$ and any given θ_t, the following statements hold:*

(a) For each $i \in \mathcal{N}$, $d_t^{i*}(I_t, \theta_t)$ is continuously increasing in I_t. Moreover, there exists a threshold $I_t^{d,i}(\theta_t) < +\infty$, such that it is optimal to serve demand segment i, if and only if $I_t > I_t^{d,i}(\theta_t)$, i.e., $d_t^{i*}(I_t, \theta_t) \begin{cases} > 0, & I_t > I_t^{d,i}(\theta_t), \\ = 0, & \text{otherwise.} \end{cases}$ Moreover, $\mathcal{N}_t^*(I_t, \theta_t) \subset \mathcal{N}_t^*(\hat{I}_t, \theta_t)$ for all $I_t < \hat{I}_t$.

(b) For each $j \in \mathcal{M}$, $q_t^{j*}(I_t, \theta_t)$ is continuously decreasing in I_t. Moreover, there exists a threshold $I_t^{q,j}(\theta_t) < +\infty$, such that it is optimal to order from supply channel j if and only if $I_t < I_t^{q,j}(\theta_t)$, i.e., $q_t^{j*}(I_t, \theta_t) \begin{cases} > 0, & I_t < I_t^{q,j}(\theta_t), \\ = 0, & \text{otherwise.} \end{cases}$ Moreover, $\mathcal{M}_t^*(\hat{I}_t, \theta_t) \subset \mathcal{M}_t^*(I_t, \theta_t)$ for all $I_t < \hat{I}_t$.

(c) $x_t^*(I_t, \theta_t)$ is continuously increasing in I_t.

Theorem 6.4.1 shows that, in each period, the optimal policy is a state-dependent threshold policy. More specifically, for each demand segment $i \in \mathcal{N}$ [supply channel $j \in \mathcal{M}$], the firm should sell to this segment [order from this channel] if and only if the starting inventory level I_t is above [below] the corresponding threshold $I_t^{d,i}(\theta_t)$ [$I_t^{q,j}(\theta_t)$]. This optimal policy structure is characterized by employing our new method to establish the monotonicity of the optimal sales price/order quantity with respect to the starting inventory level. More specifically, both the optimal sales price for each demand segment, $p^i(d_t^{i*}(I_t, \theta_t))$, and the optimal order quantity from each supply channel, $q_t^{j*}(I_t, \theta_t)$, are decreasing in the starting inventory level, whereas the optimal total order-up-to level $x_t^*(I_t, \theta_t)$ is increasing in the starting inventory level. Consequently, the optimal set of active demand segments, $\mathcal{N}_t^*(I_t, \theta_t)$ [active supply channels, $\mathcal{M}_t^*(I_t, \theta_t)$], is increasing [decreasing], in the set inclusion order, in the starting inventory level. Theorem 6.4.1 generalizes the base-stock list-price policy in the joint pricing and inventory management literature to the general setting with demand segmentation, supply diversification, and market environment fluctuation. Due to the diseconomy of scale and supply diversification, the order-up-to level and sales prices are inventory-dependent in this general setting. Finally, we remark that if the multiplicative random perturbation in market size is demand-segment-dependent (i.e., $D_t^i(p_t^i, \Lambda_t^i) = \Lambda_t^i d^i(p_t^i)\varsigma_t^i + \epsilon_t^i$ for $i \in \mathcal{N}$ and ς_t^i's are independent for different i's), parts (b) and (c) of Theorem 6.4.1 still hold but part (a) doesn't. It is well established in the inventory management literature that when there exist multiple multiplicative random perturbations in the system, the optimal order quantities and/or sales prices are, in general, not monotone in the starting inventory level (see [72]).

A key question in this inventory system is, for a given starting inventory level and market state, how to determine the optimal set of active demand segments, $\mathcal{N}_t^*(I_t, \theta_t)$, and the optimal set of active supply channels, $\mathcal{M}_t^*(I_t, \theta_t)$. The following theorem partially addresses this issue by comparing the optimal purchasing probabilities for different demand segments, and the optimal order quantities from different supply channels.

Theorem 6.4.2 *For $t = T, T-1, \cdots, 1$ and any given θ_t, the following statements hold:*

(a) Given $i, \hat{i} \in \mathcal{N}$, if $\partial_{d_t^i} R^i(z) \geq \partial_{d_t^{\hat{i}}} R^{\hat{i}}(z)$ for each $z \in [0, d_{\max}]$, $d_t^{i*}(I_t, \theta_t) \geq d_t^{\hat{i}*}(I_t, \theta_t)$, and $I_t^{d,i}(\theta_t) \leq I_t^{d,\hat{i}}(\theta_t)$ for any (I_t, θ_t). In particular, if $\partial_{d_t^1} R^1(z) \geq \partial_{d_t^2} R^2(z) \geq \cdots \geq \partial_{d_t^n} R^n(z)$ for each $z \in [0, d_{\max}]$, $I_t^{d,1}(\theta_t) \leq I_t^{d,2}(\theta_t) \leq \cdots \leq I_t^{d,n}(\theta_t)$ for any θ_t, and $\mathcal{N}_t^*(I_t, \theta_t) = \{1, 2, , \cdots, i^*\}$, where $i^* = \max\{i : I_t > I_t^{d,i}(\theta_t)\}$.

(b) Assume that c_t is fixed. Given $j, \hat{j} \in \mathcal{M}$, if $\partial_{q_t^j} C^j(z|c_t^j) \geq \partial_{q_t^{\hat{j}}} C^{\hat{j}}(z|c_t^{\hat{j}})$ for any $z \geq 0$, $q_t^{j*}(I_t, \theta_t) \leq q_t^{\hat{j}*}(I_t, \theta_t)$ and $I_t^{q,j}(\theta_t) \leq I_t^{q,\hat{j}}(\theta_t)$ for any I_t and Λ_t. In particular, if $\partial_{q_t^1} C^1(z|c_t^1) \geq \partial_{q_t^2} C^2(z|c_t^2) \geq \cdots \geq \partial_{q_t^m} C^m(z|c_t^m)$ for any $z \geq 0$, $I_t^{q,1}(\theta_t) \leq I_t^{q,2}(\theta_t) \leq \cdots \leq I_t^{q,m}(\theta_t)$ for any Λ_t, and $\mathcal{M}_t^*(I_t, \theta_t) = \{j^*, j^*+1, , \cdots, m\}$, where $j^* = \min\{j : I_t < I_t^{q,j}(\theta_t)\}$.

In Theorem 6.4.2, we show that the firm sells more to a demand segment with higher marginal revenue with respect to demand, and it orders more from a supply channel with lower marginal procurement cost. Moreover, when the marginal revenues with respect to demand [marginal procurement costs] for different demand segments [supply channels] have the same order for *all* purchasing probabilities [order quantities], the optimal set of active demand segments [supply channels], $\mathcal{N}_t^*(I_t, \theta_t)$ $[\mathcal{M}_t^*(I_t, \theta_t)]$, is consecutive in the marginal revenue with respect to demand [marginal procurement cost].

Next, we employ our new comparative statics method to study the impact of market fluctuation upon the firm's optimal pricing and ordering policy. In this application, we integrate our new method with the standard backward induction argument to perform comparative statics analysis in a dynamic program. More specifically, by employing Lemma 16, we iteratively link the comparison between optimizers and that between partial derivatives of the value functions and objective functions by backward induction. This treatment is necessary because the current market state also impacts future market states and, thus, the value functions in the future. For the rest of this subsection, we make the additional assumption that $\varsigma_t = 1$ with probability 1 for all t, i.e., the demand process follows an additive form. The additive demand assumption is commonly imposed in the joint pricing and inventory management literature for tractability (see, e.g., [112, 136, 189]). In our model, this assumption enables us to iteratively link the monotone relationship between the optimizers and that between the partial derivatives. For the rest of this subsection, since $\varsigma_t = 1$ with probability 1 for all t, we rewrite the objective function in period t as

$$J_t(d_t, q_t, I_t|\theta_t) = (\sum_{i\in\mathcal{N}} \Lambda_t^i R^i(d_t^i)) - \sum_{j\in\mathcal{M}} C^j(q_t^j|c_t^j) + \Psi_t(I_t + \sum_{j\in\mathcal{M}} q_t^j - (\sum_{i\in\mathcal{N}} \Lambda_t^i d_t^i)|\theta_t).$$

Moreover, we define $\Delta_t^*(I_t, \theta_t) := x_t^*(I_t, \theta_t) - (\sum_{i\in\mathcal{N}} \Lambda_t^i d_t^{i*}(I_t, \theta_t))$ as the optimal safety stock in period t with starting inventory level I_t and market state θ_t. The following theorem characterizes the impact of current market size on the optimal sales prices and order quantities.

Theorem 6.4.3 (Impact of market size.) *Assume that, for each $t = T, T-1, \cdots, 1$, $\varsigma_t = 1$ with probability 1. For any given t, let $\theta_t = (\Lambda_t, c_t)$ and $\hat{\theta}_t = (\hat{\Lambda}_t, c_t)$ with $\hat{\Lambda}_t > \Lambda_t$. For any I_t, the following statements hold:*

(a) $\partial_{I_t} V_t(I_t|\hat{\theta}_t) \geq \partial_{I_t} V_t(I_t|\theta_t)$.

(b) For each $i \in \mathcal{N}$, $d_t^{i*}(I_t, \hat{\theta}_t) \leq d_t^{i*}(I_t, \theta_t)$, $I_t^{d,i}(\hat{\theta}_t) \geq I_t^{d,i}(\theta_t)$, and, thus, $\mathcal{N}_t^*(I_t, \hat{\theta}_t) \subset \mathcal{N}_t^*(I_t, \theta_t)$.

(c) For each $j \in \mathcal{M}$, $q_t^{j*}(I_t, \hat{\theta}_t) \geq q_t^{j*}(I_t, \theta_t)$, $I_t^{q,j}(\hat{\theta}_t) \geq I_t^{q,j}(\theta_t)$, and, thus, $\mathcal{M}_t^*(I_t, \theta_t) \subset \mathcal{M}_t^*(I_t, \hat{\theta}_t)$.

(d) $x_t^*(I_t, \hat{\theta}_t) \geq x_t^*(I_t, \theta_t)$.

Theorem 6.4.3 proves that an increase in the current market size of any demand segment has the following impacts: (a) it prompts the firm to increase the sales price for each demand segment; (b) it drives the firm to order more from each supply channel; and (c) it motivates the firm to set a higher total order-up-to level. As the market size of one demand segment increases, the firm should increase its order quantities from all the supply channels to match supply with demand, so the optimal set of active supply channels is enlarged. At the same time, the firm should increase its sales prices in all demand segments, and the optimal set of active demand segments is smaller. Moreover, since the potential market size is more likely to become larger with a larger current market size, it is optimal for the firm to keep a higher total order-up-to level.

The risks and opportunities of procurement cost fluctuation have been extensively studied in [173]. In a model with one demand segment and two supply channels, the paper shows that inventory becomes more valuable under a higher current procurement cost, and the optimal sales price is increasing in the current procurement cost so that the firm should pass part of the cost fluctuation risk to its customers. In Theorem 6.4.4 below, we generalize these results to our joint pricing and inventory management model with demand segmentation, supply diversification, and market environment fluctuation. More specifically, we show that, with a higher reference procurement cost of any supply channel, the marginal value of inventory is higher, and the firm charges a higher sales price in each demand segment. As a result, the demand in each segment and the optimal set of active segments are decreasing in the reference procurement cost of any supply channel.

On the other hand, [173] show that the impact of cost on the firm's replenishment policy is more involved, because the current procurement cost also summarizes the information on future costs. When facing a higher current procurement cost, the firm faces the tradeoff between ordering less to save current cost and ordering more to speculate on higher future costs. Numerical studies in [173] demonstrate that the optimal order quantities may not be monotone in the current procurement cost when the firm orders its inventory either from a spot market or through a forward-buying contract. In our model, the optimal order quantity from a supply channel continues to be non-monotone in its own reference procurement cost. However, we are able to show, in the following theorem, that as the reference procurement costs of one or more supply channels increase, the optimal order quantities and ordering thresholds of the supply channels with unchanged reference procurement costs increase as well.

Theorem 6.4.4 (Impact of current reference procurement cost.) *Assume that, for each* $t = T, T-1, \cdots, 1$, $\varsigma_t = 1$ *with probability 1. For any given* t, *let* $\theta_t = (\Lambda_t, c_t)$ *and* $\hat{\theta}_t = (\Lambda_t, \hat{c}_t)$ *with* $\hat{c}_t > c_t$. *For any* I_t, *the following statements hold:*

(a) $\partial_{I_t} V_t(I_t|\hat{\theta}_t) \geq \partial_{I_t} V_t(I_t|\theta_t)$.

(b) For each $i \in \mathcal{N}$, $d_t^{i*}(I_t, \hat{\theta}_t) \leq d_t^{i*}(I_t, \theta_t)$, $I_t^{d,i}(\hat{\theta}_t) \geq I_t^{d,i}(\theta_t)$, *and, thus,* $\mathcal{N}_t^*(I_t, \hat{\theta}_t) \subset \mathcal{N}_t^*(I_t, \theta_t)$.

(c) If $\hat{c}_t^j = c_t^j$, $q_t^{j*}(I_t, \hat{\theta}_t) \geq q_t^{j*}(I_t, \theta_t)$ *and* $I_t^{q,j}(\hat{\theta}_t) \geq I_t^{q,j}(\theta_t)$.

In addition to the current market condition, the firm should also take into account the future market trend to achieve the long-run optimality. Our new comparative statics method enables us to offer insights on the optimal responses of the firm to potential changes in the future market condition. We first study the impact of future market size trend on the firm's optimal decisions.

Theorem 6.4.5 (Impact of market size trend.) *Assume that, for each* $t = T, T-1, \cdots, 1$, $\varsigma_t = 1$ *with probability 1. Let the two systems be equivalent except that* $\hat{\xi}_t^{\Lambda,i}(\Lambda_t^i) \geq_{s.d.} \xi_t^{\Lambda,i}(\Lambda_t^i)$ *for any* t, $i \in \mathcal{N}$, *and* Λ_t. *For any* t *and* (I_t, θ_t), *the following statements hold:*

(a) $\partial_{I_t} \hat{V}_t(I_t|\theta_t) \geq \partial_{I_t} V_t(I_t|\theta_t)$.

(b) For each $i \in \mathcal{N}$, $\hat{d}_t^{i*}(I_t, \theta_t) \leq d_t^{i*}(I_t, \theta_t)$, $\hat{I}_t^{d,i}(\theta_t) \geq I_t^{d,i}(\theta_t)$, *and, thus,* $\hat{\mathcal{N}}_t^*(I_t, \theta_t) \subset \mathcal{N}_t^*(I_t, \theta_t)$.

(c) For each $j \in \mathcal{M}$, $\hat{q}_t^{j*}(I_t, \theta_t) \geq q_t^{j*}(I_t, \theta_t)$, $\hat{I}_t^{q,j}(\theta_t) \geq I_t^{q,j}(\theta_t)$, *and, thus,* $\mathcal{M}_t^*(I_t, \theta_t) \subset \hat{\mathcal{M}}_t^*(I_t, \theta_t)$.

(d) $\hat{x}_t^*(I_t, \theta_t) \geq x_t^*(I_t, \theta_t)$ *and* $\hat{\Delta}_t^*(I_t, \theta_t) \geq \Delta_t^*(I_t, \theta_t)$*.*

Theorem 6.4.5 shows that, under a higher market size trend for any demand segment, it is optimal to charge higher sales prices to all demand segments and, thus, sell to a smaller set of segments. On the other hand, a higher market size trend implies higher future demand, so the firm should order more from all supply channels, expand the set of active supply channels, and set a higher safety stock to hold more inventory for future consumption.

As shown by [173], a higher procurement cost trend increases the marginal value of inventory and prompts the firm to increase its order quantities both from the spot market and through the forward-buying contract so as to save the future cost. A higher safety stock should also be kept. In addition, the firm should raise its sales price to consume its inventory in the most profitable way. In our general model, we show that, when the reference procurement cost trend in one system is higher than that in the other, all of the comparative statics results in [173] continue to hold for each demand segment and supply channel. In addition, with a higher cost trend, the optimal set of active demand segments [supply channels] is smaller [larger].

Theorem 6.4.6 (Impact of cost trend.) *Assume that, for each* $t = T, T-1, \cdots, 1$, $\varsigma_t = 1$ *with probability 1. Let the two systems be equivalent except that* $\hat{\xi}_t^{c,j}(c_t^j) \geq_{s.d.} \xi_t^{c,j}(c_t^j)$ *for any* t, $j \in \mathcal{M}$ *and* c_t*. For any* t *and* (I_t, θ_t)*, the following statements hold:*

(a) $\partial_{I_t} \hat{V}_t(I_t|\theta_t) \geq \partial_{I_t} V_t(I_t|\theta_t)$*.*

(b) For each $i \in \mathcal{N}$, $\hat{d}_t^{i*}(I_t, \theta_t) \leq d_t^{i*}(I_t, \theta_t)$, $\hat{I}_t^{d,i}(\theta_t) \geq I_t^{d,i}(\theta_t)$*, and, thus,* $\hat{\mathcal{N}}_t^*(I_t, \theta_t) \subset \mathcal{N}_t^*(I_t, \theta_t)$*.*

(c) For each $j \in \mathcal{M}$, $\hat{q}_t^{j*}(I_t, \theta_t) \geq q_t^{j*}(I_t, \theta_t)$, $\hat{I}_t^{q,j}(\theta_t) \geq I_t^{q,j}(\theta_t)$*, and, thus,* $\mathcal{M}_t^*(I_t, \theta_t) \subset \hat{\mathcal{M}}_t^*(I_t, \theta_t)$*.*

(d) $\hat{x}_t^*(I_t, \theta_t) \geq x_t^*(I_t, \theta_t)$ *and* $\hat{\Delta}_t^*(I_t, \theta_t) \geq \Delta_t^*(I_t, \theta_t)$*.*

In addition, our new method enables us to perform comparative statics analysis for the optimal decisions in different models with non-parameterizable changes. More specifically, we employ our method to characterize the impact of sales and procurement flexibilities (i.e., additional demand segments and supply channels) upon the firm's optimal pricing and replenishment policy. When the firm is blessed with the opportunity to sell to additional demand segments, the marginal value of inventory increases, and the firm should charge higher prices in the original segments. Moreover, the firm should increase its replenishment quantities from all supply channels and expand the set of active supply channels, so as to match supply with the higher demand from a larger pool of segments. These intuitions are formalized in the following theorem.

Theorem 6.4.7 (IMPACT OF ADDITIONAL DEMAND SEGMENTS.) *Assume that, for each* $t = T, T-1, \cdots, 1$, $\varsigma_t = 1$ *with probability 1. Let the two systems be equivalent except for* $\mathcal{N} \subset \hat{\mathcal{N}}$. *For* $t = T, T-1, \cdots, 1$, *and any* (I_t, θ_t), *the following statements hold:*

(a) $\partial_{I_t} \hat{V}_t(I_t|\theta_t) \geq \partial_{I_t} V_t(I_t|\theta_t)$.

(b) For each $i \in \mathcal{N}$, $\hat{d}_t^{i*}(I_t, \theta_t) \leq d_t^{i*}(I_t, \theta_t)$, $\hat{I}_t^{d,i}(\theta_t) \geq I_t^{d,i}(\theta_t)$, *and, thus,* $(\hat{\mathcal{N}}_t^*(I_t, \theta_t) \cap \mathcal{N}) \subset \mathcal{N}_t^*(I_t, \theta_t)$.

(c) For each $j \in \mathcal{M}$, $\hat{q}_t^{j*}(I_t, \theta_t) \geq q_t^{j*}(I_t, \theta_t)$, $\hat{I}_t^{q,j}(\theta_t) \geq I_t^{q,j}(\theta_t)$, *and, thus,* $\mathcal{M}_t^*(I_t, \theta_t) \subset \hat{\mathcal{M}}_t^*(I_t, \theta_t)$.

(d) $\hat{x}_t^*(I_t, \theta_t) \geq x_t^*(I_t, \theta_t)$.

On the other hand, the supply diversification strategy enables the firm to hedge against the procurement cost fluctuation risk and the diseconomy of scale of the supply channels. By sourcing from a larger supply pool, the firm enjoys more procurement flexibility, and orders less from each of the original supply channels. Moreover, the marginal value of inventory is smaller with a larger supply pool, and, to match supply with demand, the firm should set lower sales prices in all demand segments and sell to more segments.

Theorem 6.4.8 (IMPACT OF ADDITIONAL SUPPLY CHANNELS.) *Assume that, for each* $t = T, T-1, \cdots, 1$, $\varsigma_t = 1$ *with probability 1. Let the two systems be equivalent except for* $\mathcal{M} \subset \hat{\mathcal{M}}$. *For* $t = T, T-1, \cdots, 1$, *and any* (I_t, θ_t), *the following statements hold:*

(a) $\partial_{I_t} \hat{V}_t(I_t|\theta_t) \leq \partial_{I_t} V_t(I_t|\theta_t)$.

(b) For each $i \in \mathcal{N}$, $\hat{d}_t^{i*}(I_t, \theta_t) \geq d_t^{i*}(I_t, \theta_t)$, $\hat{I}_t^{d,i}(\theta_t) \leq I_t^{d,i}(\theta_t)$, *and, thus,* $\mathcal{N}_t^*(I_t, \theta_t) \subset \hat{\mathcal{N}}_t^*(I_t, \theta_t)$.

(c) *For each* $j \in \mathcal{M}$, $\hat{q}_t^{j*}(I_t, \theta_t) \le q_t^{j*}(I_t, \theta_t)$, $\hat{I}_t^{q,j}(\theta_t) \le I_t^{q,j}(\theta_t)$, *and, thus,* $(\hat{\mathcal{M}}_t^*(I_t, \theta_t) \cap \mathcal{M}) \subset \mathcal{M}_t^*(I_t, \theta_t)$.

To sum up, comparative statics analysis is essential in our general joint pricing and inventory management model with demand segmentation, supply diversification, and market environment fluctuation. Although the standard IFT and MCS approaches do not apply to this complex model, our new comparative statics method enables us to characterize its optimal policy as a state-dependent threshold policy, and to analyze the impact of market fluctuation and operational flexibilities upon the optimal policy.

6.5 Application of the New Comparative Statics Method in a Competition Model

In this section, we apply our new method to comparative statics analysis in a joint price and effort competition model, which, as we will show, cannot be conducted with the standard IFT and MCS approaches. We remark that the notations in this section are independent of those in Sections 6.3 and 6.4.

More specifically, we consider an oligopoly industry of N competing firms. Each firm offers a partially substitutable product with unit production cost $c_i > 0$. Each firm i selects a sales price $p_i \in [p_i^{\min}, p_i^{\max}]$ and an effort level $y_i \ge 0$. We assume that, for each i, $p_i^{\min} = c_i$. In addition, we make the same assumption as [9] that $p_i^{\max}$ is sufficiently large so that it has no impact on the equilibrium behavior. Each firm i can exert effort y_i (on, e.g., R&D or advertising) to increase its demand. We use $Y := \sum_{i=1}^{N} y_i$ to denote the total effort level. Let $p := (p_1, p_2, \cdots, p_N)$ be the vector of sales prices and $y := (y_1, y_2, \cdots, y_N)$ be the vector of effort levels. For any decision vector (p, y), the demand for firm i is given in the following quasi-separable form:

$$\lambda_i(p, y, \theta_i) = (\theta_i + f(Y) - b_i p_i + \sum_{j \neq i} \beta_{ij} p_j)^+, \tag{6.15}$$

where $\theta_i \in [\theta^{\min}, \theta^{\max}]$ is the firm-dependent market index, capturing other impact factors on demand beyond price and effort (e.g., brand image). It is commonly assumed in the R&D and research joint venture literature (e.g., [104]) that the total effort level in the industry has an accumulative effect upon the demand of each firm. We model this accumulative effect by $f(Y)$, and assume that $f(\cdot)$ is a strictly increasing, continuously differentiable, and strictly concave function that is bounded from above by $M < +\infty$. As [9], we assume that $\lambda_i(p, y, \theta_i) > 0$ whenever $p_i = p_i^{\min} = c_i$, i.e., each firm i can have a positive demand under zero profit margin, regardless of the competing firms' price decisions and all firms' effort decisions. We remark that although we do not assume $\lambda_i(p, y, \theta_i) > 0$ for all feasible (p, y), we will show that the equilibrium demand of each firm is positive. We also make the standard assumption that $b_i, \beta_{ij} > 0$ for all i, j, and that the dominant diagonal condition holds, i.e., $b_i > \sum_{j \neq i} \beta_{ij} > 0$ and $b_i > \sum_{j \neq i} \beta_{ji} > 0$. The interpretation of the dominant diagonal condition is that an uniform price increase of all firms cannot result in a demand increase in any firm, and that a price increase of any firm cannot result in an increase in the total demand of the industry (see, also, [9]).

Consistent with the standard assumption in the economics literature ([121]), we assume that, for each firm i, the cost of exerting effort y_i is $C_i(y_i)$, where $C_i(\cdot)$ is an increasing, strictly convex, and continuously differentiable function. Thus, the profit of firm i in this joint price and effort competition is given by:

$$\Pi_i(p, y|\theta) = (p_i - c_i)\lambda_i(p, y, \theta_i) - C_i(y_i), \tag{6.16}$$

where $\theta := (\theta_1, \theta_2, \cdots, \theta_N)$ represents the market index vector.

Depending on the industry dynamics, two competition models are considered: (a) the effort-level-first competition (EF), and (b) the simultaneous competition (SC). In the effort-level-first competition, the firms first choose their effort levels (on, e.g., R&D or advertising), and then select the sales prices in the second stage. In the simultaneous competition, the firms make effort and price decisions simultaneously. In the next two subsections, we employ our new comparative statics method to characterize the equilibrium in these two competition models, and study how the equilibrium prices and effort levels change with the market index vector θ. Finally, in Section 6.5.3, we compare the equilibrium decisions in these two competition models.

6.5.1 Effort-Level-First Competition

In this subsection, we study the effort-level-first competition model. In this model, the firms engage in a two-stage game, in which they compete on market expanding effort in the first stage and on sales price in the second stage. This model is suitable for the scenario in which the stickiness of market expanding effort choices is much higher than that of sales price choices. For example, due to the long leadtime for technology development, decisions on R&D effort are usually made well in advance of price decisions.

To analyze this two-stage game, we begin with the price competition in the second stage. In this stage, the effort level in the first stage y is observable by all firms. Let A be an $N \times N$ matrix with $A_{ii} = 2b_i$, $A_{ij} = -\beta_{ij}$ for $i \neq j$, $a(Y, \theta)$ be a column vector with $a_i(Y, \theta) = \theta_i + f(Y) = \theta_i + f(\sum_{j=1}^{N} y_j)$, and κ be a column vector with $\kappa_i = b_i c_i$. Given any effort level y, the equilibrium price, $p^*(y, \theta)$, in the second-stage competition is characterized by the following theorem.

Theorem 6.5.1 (SECOND-STAGE PRICE COMPETITION.) *For a given effort level vector y, the following statements hold:*

(a) The equilibrium in the second-stage price competition is unique and given by $p^*(y, \theta) = A^{-1}(a(Y, \theta) + \kappa)$, with $p_i^*(y, \theta) > p_i^{\min} = c_i$. The unique equilibrium demand for firm i is given by $\lambda_i^*(y, \theta) = b_i(p_i^*(y, \theta) - c_i) > 0$. Hence, for any i, $p_i^*(y, \theta)$ and $\lambda_i^*(y, \theta)$ depend on the effort level vector y only through the total effort level Y.

(b) $p_i^*(y, \theta)$ is strictly increasing in Y, with $\partial_{y_j} p_i^*(y, \theta) = \partial_Y p_i^*(y, \theta) = (\sum_{l=1}^{N} (A^{-1})_{il}) f'(Y)$ for each i and j. Hence, $\lambda_i^*(y, \theta)$ is strictly increasing in Y and, thus, y_j for any i and j.

Theorem 6.5.1 proves that, given any effort level vector in the first-stage, the second-stage price competition has a unique equilibrium. Moreover, under the equilibrium, each firm achieves a positive profit margin and a positive demand. Both the equilibrium price and demand of each firm are strictly increasing in the total effort level. Higher effort level in the first stage increases the market demand, and motivates each firm to charge a higher sales price.

Based on Theorem 6.5.1, we study the first-stage competition, in which the firms choose their effort levels. Since $p^*(y, \theta)$ depends on y only through the total effort level Y, we use $p^*(Y, \theta)$ to denote the equilibrium price in the second-stage competition. As a result, the equilibrium demand can be represented as $\lambda^*(Y, \theta)$, with $\lambda_i^*(Y, \theta) = b_i(p_i^*(Y, \theta) - c_i)$. Plugging $p^*(Y, \theta)$ and $\lambda^*(Y, \theta)$ into (6.16), we obtain the objective functions in the first-stage game:

$$\pi_i(y|\theta) = b_i(p_i^*(Y, \theta) - c_i)^2 - C_i(y_i), \text{ for } i = 1, 2, \cdots, N. \tag{6.17}$$

Following [104], we make the following assumption on $f(\cdot)$.

Assumption 6.5.1 $(f(x) - f_0)^2$ *is concave in* x *for* $x \geq 0$*, where* $f_0 := f(0)$.

Assumption 6.5.1 guarantees the concavity of the objective functions in the first-stage effort competition and, thus, the existence of an equilibrium. This assumption is the counterpart of Assumption 3 in [104]. With the help of Assumption 6.5.1, we characterize the equilibrium of the first-stage effort competition in the following theorem.

Theorem 6.5.2 (Effort-level-first competition.) *Under Assumption 6.5.1, the following statements hold:*

(a) *Given any* θ*, the first-stage effort competition has a unique equilibrium* $y^*_{EF}(\theta)$.

(b) *Let* $Y^*_{EF}(\theta) := \sum_{i=1}^N y^*_{EF,i}(\theta)$ *be the equilibrium total effort level in the first-stage competition.* $p^*(Y^*_{EF}(\theta), \theta)$ *is the unique associated equilibrium price vector and* $\lambda^*(Y^*_{EF}(\theta), \theta)$ *is the unique associated equilibrium demand vector in the second-stage competition, where* $p^*(\cdot, \cdot)$ *and* $\lambda^*(\cdot, \cdot)$ *are given in Theorem 6.5.1(a).*

Theorem 6.5.2 shows that the two-stage effort-level-first competition has a unique subgame perfect equilibrium. The proof of Theorem 6.5.2 heavily relies on our new comparative statics method, which enables us to establish the monotone relationship that the equilibrium effort level of each firm, $y^*_{EF,i}(\theta)$, is decreasing in the equilibrium total effort level $Y^*_{EF}(\theta)$. Such monotonicity, together with the identity that $\sum_{i=1}^N y^*_{EF,i}(\theta) = Y^*_{EF}(\theta)$, guarantees the uniqueness of the equilibrium in the first-stage effort competition.

A natural question in this competition is how the market index θ influences the equilibrium price $p^*(Y^*_{EF}(\theta), \theta)$ and equilibrium effort $y^*_{EF}(\theta)$. Intuition suggests that, under a better market condition, the firms should decrease their market expanding efforts to save costs. The following theorem shows that this intuition is reversed in the effort-level-first competition.

Theorem 6.5.3 (Impact of market index.) *Under Assumption 6.5.1, the following statements hold:*

(a) $Y^*_{EF}(\theta)$ *is increasing in* θ_i *for any* i.

(b) $p^*_i(Y^*_{EF}(\theta), \theta)$ *and* $\lambda^*_i(Y^*_{EF}(\theta), \theta)$ *are increasing in* θ_j *for any* i *and* j.

Theorem 6.5.3 shows that the equilibrium total effort level $Y^*_{EF}(\theta)$ is increasing in each market index, θ_i. As a result, the equilibrium sales price and demand of each firm are increasing in the market index of any firm. Note that Theorem 6.5.3 cannot be proved by the standard IFT and MCS approaches. It is possible that $f(\cdot)$, and hence, the objective function $\pi_i(\cdot|\cdot)$, are not twice continuously differentiable. Even if $\pi_i(\cdot|\cdot)$ is twice continuously differentiable in (y, θ) for each i, calculating the inverse of the Hessian in the first-stage effort competition can be prohibitively difficult when N is large. Therefore, the IFT approach has poor scalability, and it is very difficult, if not impossible, to prove Theorem 6.5.3 by the IFT approach. On the other hand, although $\pi_i(\cdot|\cdot)$ is supermodular in y_i and θ_j for any i and j, it is not jointly supermodular in (y, θ_j). Hence, the complementarity conditions required in supermodular games (see [124]) do not hold, and the MCS approach does not apply to this model. We employ our new comparative statics method to prove Theorem 6.5.3(a). We assume, to the contrary, that $Y^*_{EF}(\theta)$ is decreasing in θ_i for some i, and construct a contradiction with the iterative procedure in Section 6.3.2. This approach exploits the supermodularity of $\pi_i(\cdot|\cdot)$ in y_i and θ_j for any i and j, but does not require the joint supermodularity of $\pi_i(\cdot|\cdot)$ in (y, θ_j). Part (b) of Theorem 6.5.3 follows directly from part (a).

6.5.2 Simultaneous Competition

In some scenarios, the market expanding effort (on, e.g., advertising) takes effect instantaneously. Hence, decisions on effort can be made at the same time as price decisions. In this scenario, the firms engage in a simultaneous price and effort competition. Specifically, each firm i simultaneously selects (p_i, y_i) to maximize $\Pi_i(p, y|\theta)$ defined by (6.16). The next theorem characterizes the equilibrium and the impact of market index upon the equilibrium in the simultaneous competition.

Theorem 6.5.4 (Simultaneous competition.) *Under Assumption 6.5.1, the following statements hold:*

(a) Given any θ, the simultaneous competition has a unique equilibrium $(p^*_{SC}(\theta), y^*_{SC}(\theta))$, which satisfies the system of equations:

$$\begin{aligned} p^*_{SC}(\theta) &= p^*(y^*_{SC}(\theta), \theta) \\ &= A^{-1}(a(Y^*_{SC}(\theta), \theta) + \kappa), \end{aligned} \quad (6.18)$$

$$(p^*_{SC,i}(\theta) - c_i)f'(Y^*_{SC}(\theta)) - C'_i(y^*_{SC,i}(\theta)) \begin{cases} = 0, & \text{if } y^*_{SC,i}(\theta) > 0, \\ \leq 0, & \text{otherwise,} \end{cases} \quad (6.19)$$
$$\text{for all } i = 1, 2, \cdots, N,$$

where $Y^*_{SC}(\theta) = \sum_{i=1}^N y^*_{SC,i}(\theta)$. Conversely, the system of equations (6.18) *and* (6.19) *has a unique solution, which is the equilibrium of the simultaneous competition. The equilibrium demand is given by $\lambda^*_{SC}(\theta) = (\lambda^*_{SC,1}(\theta), \lambda^*_{SC,2}(\theta), \cdots, \lambda^*_{SC,N}(\theta))$, where $\lambda^*_{SC,i}(\theta) = b_i(p^*_{SC,i}(\theta) - c_i)$. Moreover, for any i, $p^*_{SC,i}(\theta) > c_i$ and $\lambda^*_{SC,i}(\theta) > 0$.*

(b) $Y^*_{SC}(\theta)$ is increasing in θ_i for any i. Moreover, $p^*_{SC,i}(\theta)$ and $\lambda^*_{SC,i}(\theta)$ are increasing in θ_j for any i and j.

Under Assumption 6.5.1, Theorem 6.5.4(a) proves the existence and uniqueness of the equilibrium in the simultaneous price and effort competition. Moreover, we show that, under the equilibrium, each firm earns a positive profit margin and a positive demand in the simultaneous competition. Note that Assumption 6.5.1 does not guarantee the joint concavity of $\Pi_i(p, y|\theta)$ in (p_i, y_i). Hence, we cannot use the standard approach to prove the existence of an equilibrium in the simultaneous competition. Instead, we show that the system of equations (6.18) and (6.19) has a unique solution, which is an equilibrium of the simultaneous competition. We prove the uniqueness of the equilibrium by showing that any equilibrium of the simultaneous competition must satisfy the system of equations (6.18) and (6.19).

In Theorem 6.5.4(b), we employ our new comparative statics method to show that, in the simultaneous competition, the equilibrium total effort level, and the equilibrium sales price and demand volume of each firm are increasing in the market index of any firm. This result is consistent with its counterpart in the effort-level-first competition (i.e., Theorem 6.5.3).

6.5.3 A Comparison of Equilibria in the Two Competition Models

In this subsection, we compare the equilibrium in the effort-level-first competition (characterized in Theorem 6.5.2) and that in the simultaneous competition (characterized in Theorem 6.5.4). We summarize the comparison results in the following theorem:

Theorem 6.5.5 *Under Assumption 6.5.1, the following statements hold:*

(a) $p_i^*(y_{EF}^*(\theta), \theta) \geq p_{SC,i}^*(\theta)$ *for any* i *and* θ*.*

(b) $Y_{EF}^*(\theta) \geq Y_{SC}^*(\theta)$ *for any* θ*.*

(c) $\lambda_i^*(Y_{EF}^*(\theta), \theta) \geq \lambda_{SC,i}^*(\theta)$ *for any* i *and* θ*.*

Theorem 6.5.5 shows that, in the effort-level-first competition, the firms exert higher total market expanding effort to dampen the subsequent price competition, which results in higher equilibrium price and demand of each firm than their counterparts in the simultaneous competition. This phenomenon (i.e., the "fat-cat effect", see [78]) has also been identified by [9] in a joint price and service level competition setting. They show that the equilibrium sales prices, demand volumes, and service levels are higher in the service-level-first competition model than those in the simultaneous competition model. To prove this result, [9] inductively show that, for each k, the k^{th} iteration of the tatônnement scheme for the service-level-first competition model is higher, in price and service level, than that for the simultaneous competition model. Since the joint price and service competition games in [9] are supermodular, the tatônnement scheme can generate the minimum equilibria and, thus, their monotone relationship in the two competitions. In our model, however, neither the effort-level-first competition nor the simultaneous competition is supermodular, so we employ our new comparative statics method to prove Theorem 6.5.5. We first prove part (b) by employing the iterative procedure in Section 6.3.2 to construct a contradiction under the (incorrect) assumption that $Y_{EF}^*(\theta) < Y_{SC}^*(\theta)$ for some θ. Parts (a) and (c) follow directly from part (b) by Theorem 6.5.1 and Theorem 6.5.4.

To conclude this section, we remark that all of the comparative statics results on the equilibrium total effort level in Theorems 6.5.3 - 6.5.5 cannot be generalized to ones on the equilibrium effort level of each firm. This is because, in both competition models, although the objective function of each firm i is supermodular in y_i and θ_j for any i

and j, it is not jointly supermodular in (y, θ_j). In other words, some firms may free-ride on a higher effort level of their competitors, and thus, decrease their own effort levels. When this effect dominates the effort-prompting effect of a better market condition, the equilibrium effort levels of some firms may be decreasing in the market indices.

6.6 Summary

In this chapter, we consider a general joint pricing and inventory management model with demand segmentation, supply diversification, and market environment fluctuation. In this model, comparative statics analysis is integral to characterizing its optimal policy and analyzing the impact of demand segmentation, supply diversification, and market environment fluctuation upon the optimal policy. The standard comparative statics methods (i.e., the IFT and MCS approaches) do not apply, because (a) the second-order continuous differentiability and complementarity conditions are not satisfied in our model, (b) the IFT approach is not scalable, and (c) some of the optimal decision variables are not monotone in the parameter (i.e., the MCS approach does not work in this case).

Therefore, we develop a new comparative statics method. Our new method employs a simple but powerful lemma (Lemma 16) and some model-specific properties (e.g., the supermodularity in one decision variable and the parameter and the componentwise concavity of the objective function) to iteratively link the comparison between optimizers and that between the partial derivatives of objective functions, so as to construct contradictions under the assumption that the desired comparative statics results are reversed. Lemma 16 enables us to make componentwise comparisons of the optimal decision variables at different parameters, which is the essential difference between our method and the standard approaches. The componentwise comparison between optimizers facilitates the scalability of our method and its application in a model where only *part* of the optimizers are monotone in the parameter. We remark that when a contradiction cannot be reached using our new method, a counterexample of the original comparative statics prediction, in general, can be found. Hence, the proposed method can also help identify cases in which comparative statics results do not hold for some decision variables.

Though fundamentally different, our new method shares some similarity with the standard IFT and MCS approaches. Analogous to the IFT approach, the proposed method studies the first-order (KKT) condition at the optimizer of interest. Hence, our method requires the objective function be first-order continuously differentiable, but not necessarily second-order continuously differentiable. Following the idea of the MCS approach, our method carefully examines the impact of the parameter upon the marginal values of the decision variables, so that we can translate the monotonicity of partial derivatives back into the monotonicity of another decision variable at the optimizer. Thus, to reach a contradiction (and hence, a comparative statics result), our method requires the objective function be supermodular in the parameter and each of the focal decision variables, but not necessarily jointly supermodular or satisfying the single crossing property. The above two condition relaxations enhance the applicability of our method in the general joint pricing and inventory management model, where the second-order continuous differentiability and joint supermodularity of the objective function in each decision epoch are hard, if not impossible, to establish.

We employ our new method to analyze the joint pricing and inventory management model under demand segmentation, supply diversification, and market environment fluctuation. Our new comparative statics method enables us to characterize the optimal joint pricing and ordering policy for an arbitrary number of demand segments and supply channels as a threshold policy, under which there exists a market environment dependent threshold for each demand segment [supply channel] such that it is optimal to sell to [order from] this segment [channel] if and only if the starting inventory level is above [below] its corresponding threshold. The optimal sales price for each demand segment and the optimal order quantity from each supply channel are decreasing in the starting inventory level of the firm, and increasing in the market size of any demand segment. When the reference procurement costs of some supply channels increase, the firm increases the sales price in each demand segment, and the order quantities from the supply channels with unchanged reference procurement costs. Each firm's optimal order quantity may not be monotone in its own reference procurement cost. Expanding the set of demand segments drives the firm to increase its sales price in each demand segment and order quantity from each supply channel, whereas expanding the supply pool decreases the optimal sales prices and order quantities.

To demonstrate the applicability of our new comparative statics method in other settings, we employ it to study joint price and effort competition games, in which the total effort level has a positive impact upon each firm's demand. More specifically, we consider two competition models: (a) the firms compete on effort in the first stage and on price in the second stage; and (b) the firms simultaneously compete on price and effort. The standard IFT and MCS approaches are not amenable for the comparative statics analysis in this setting, because the IFT approach has poor scalability, and the complementarity conditions required by the MCS approach are not satisfied. We apply our new method to show the existence and uniqueness of equilibrium in the effort-level-first competition. We prove that, in both competition models, the equilibrium total effort level, and the equilibrium price and demand of each firm are increasing in the market index of any firm. We also demonstrate the fat-cat effect in this setting, i.e., the sequential decision making gives rise to a higher total effort level and a higher price and demand of each firm in the effort-level-first competition than those in the simultaneous competition.

In summary, our new method enables us to perform comparative statics analysis in a general joint pricing and inventory management model and a joint price and effort competition model. Standard IFT and MCS approaches are not amenable for both settings. Our new method makes componentwise comparisons between the focal decision variables under different parameter values, so it is capable of performing comparative statics analysis in a model where some of the decision variables are non-monotone, and it is scalable. Hence, our new method is promising for comparative statics analysis in other operations management models.

7. Concluding Remarks

This dissertation focuses on the impact of some new market trends (such as social networks, sustainability concerns, and customer behaviors) upon a firm's pricing and inventory policies. Our results demonstrate that these emerging trends lead to interesting new tradeoffs and, hence, would significantly influence a firm's operations decisions. On the other hand, the firm can adopt innovative pricing and inventory strategies to exploit these market trends and substantially improve its profit. To facilitate the analysis, we develop an effective comparative statics analysis method for a general class of joint pricing and inventory management models.

The combined pricing and inventory policy is inarguably a very important operations decision for any firm that delivers physical products to customers. We believe there are several promising avenues for future research related to this topic. Instead of digging into the details, we focus on the high-level discussions of future research directions.

Multi-item inventory systems. The dissertation only studies the pricing and inventory policy of a single-product model. While this setting is interesting and relevant by itself, multi-item inventory models would better capture the situation of a retailer in the e-commerce market. For a retailer on an online e-commerce platform like Amazon, it generally holds and sells inventories of different products. So the retailer needs to jointly manage the pricing, sourcing, storing, and delivery strategies of all its products. With multiple products handled together, the key issue the firm faces is how to allocate the capital, transportation, and human capacities among different products. Among others, it is interesting to study how the dynamic pricing flexibility would complement the firm's capacity allocation strategy, and alleviate its capacity constraint pressure.

Information asymmetry. In this dissertation, we assume in our model that information is symmetric to everyone. If the market exhibits information asymmetry between the firm and the customers, we need to employ dynamic mechanism design techniques to study the optimal pricing and inventory control policy therein. From the application perspective, introducing information asymmetry well captures the current market trends of, e.g., sharing economy, social networks, and online auctions. An enriched joint pricing and inventory management model with information asymmetry enables us to characterize the operational impact of these new marketplace innovations, and study the role of information in market design issues.

Data-driven integrated pricing and inventory optimization. In this dissertation, the decision maker (i.e., the firm) have full knowledge about the demand function and demand distribution. In reality, however, this is not necessarily the case, since the demand function and distribution may not be available to the firm. In this situation, the firm should collect the previous demand data and employ some data-driven methods to do the prediction and prescription simultaneously. It is interesting to develop some data-driven algorithms to optimize the joint pricing and inventory control policy in an online manner. The objective is to achieve the maximum expected profit under the full demand information assumption asymptotically. The key issue without knowing the demand distribution is to balance the well-known exploration-exploitation tradeoff under the integrated pricing and inventory management framework.

To sum up, the integrated pricing and inventory control problem is of both theoretical interest and practical relevance. This dissertation's main contribution is to establish new models and methods to study the impact of new market trends on the joint dynamic pricing and inventory policy of a firm. We also hope the dissertation would help inspire future research on this topic.

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APPENDICES

A. Appendix for Chapter 2

A.1 Proofs of Statements

We use ∂ to denote the derivative operator of a single variable function, ∂_x to denote the partial derivative operator of a multi-variable function with respect to variable x, and $1_{\{\cdot\}}$ to denote the indicator function. For any multivariate continuously differentiable function $f(x_1, x_2, \cdots, x_n)$ and $\tilde{x} := (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)$ in $f(\cdot)$'s domain, $\forall i$, we use $\partial_{x_i} f(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)$ to denote $\partial_{x_i} f(x_1, x_2, \cdots, x_n)|_{x=\tilde{x}}$. The following lemma is used throughout our proofs.

Lemma 18 *Let $F_i(z, Z)$ be a continuously differentiable and jointly concave function in (z, Z) for $i = 1, 2$, where $z \in [\underline{z}, \bar{z}]$ ($\underline{z}$ and $\bar{z}$ might be infinite) and $Z \in \mathbb{R}^n$. For $i = 1, 2$, let $(z_i, Z_i) := \text{argmax}_{(z,Z)} F_i(z, Z)$ be the optimizers of $F_i(\cdot, \cdot)$. If $z_1 < z_2$, we have: $\partial_z F_1(z_1, Z_1) \leq \partial_z F_2(z_2, Z_2)$.*

Proof: $z_1 < z_2$, so $\underline{z} \leq z_1 < z_2 \leq \bar{z}$. Hence, $\partial_z F_1(z_1, Z_1) \begin{cases} = 0 & \text{if } z_1 > \underline{z}, \\ \leq 0 & \text{if } z_1 = \underline{z}; \end{cases}$ and $\partial_z F_2(z_2, Z_2) \begin{cases} = 0 & \text{if } z_2 < \bar{z}, \\ \geq 0 & \text{if } z_2 = \bar{z}, \end{cases}$ i.e., $\partial_z F_1(z_1, Z_1) \leq 0 \leq \partial_z F_2(z_2, Z_2)$. *Q.E.D.*

Proof of Lemma 1: Since $\gamma(\cdot)$ is twice continuously differentiable, $R_t(\cdot, \cdot)$ is twice continuously differentiable, and jointly concave in (p_t, N_t) if and only if the Hessian of $R_t(\cdot, \cdot)$ is negative semi-definite, i.e., $\partial^2_{p_t} R_t(p_t, N_t) \leq 0$, and $\partial^2_{p_t} R_t(p_t, N_t) \partial^2_{N_t} R_t(p_t, N_t) \geq (\partial_{p_t} \partial_{N_t} R_t(p_t, N_t))^2$, where $\partial^2_{p_t} R_t(p_t, N_t) = -2$, $\partial^2_{N_t} R_t(p_t, N_t) = (p_t - b - \alpha c)\gamma''(N_t)$, and $\partial_{p_t} \partial_{N_t} R_t(p_t, N_t) = \gamma'(N_t)$. Hence, $R_t(\cdot, \cdot)$ is jointly concave on $[\underline{p}, \bar{p}] \times [0, +\infty)$ if and only if $-2(p_t - b - \alpha c)\gamma''(N_t) \geq (\gamma'(N_t))^2$ for all (p_t, N_t). Since $-2(p_t - b - \alpha c)\gamma''(N_t) \geq -2(\underline{p} - b - \alpha c)\gamma''(N_t)$, $-2(p_t - b - \alpha c)\gamma''(N_t) \geq (\gamma'(N_t))^2$ for all (p_t, N_t) if and only if $-2(\underline{p} - b - \alpha c)\gamma''(N_t) \geq (\gamma'(N_t))^2$ for all $N_t \geq 0$. This proves Lemma 1. *Q.E.D.*

Proof of Lemma 2: We prove parts (a) - (c) together by backward induction.

We first show, by backward induction that if $v_{t-1}(I_{t-1}, N_{t-1}) - cI_{t-1}$ is jointly concave in (I_{t-1}, N_{t-1}), decreasing in I_{t-1}, and increasing in N_{t-1}, (i) $\Psi_t(\cdot, \cdot)$ is jointly concave in (x, y), decreasing in x, and increasing in y; (ii) $J_t(\cdot, \cdot, \cdot)$ is jointly concave in (x_t, p_t, N_t); and (iii) $v_t(I_t, N_t) - cI_t$ is jointly concave in (I_t, N_t), decreasing in I_t, and increasing in N_t. It is clear that $v_0(I_0, N_0) - cI_0 = 0$ is jointly concave, decreasing in I_0, and increasing in N_0. Hence, the initial condition holds.

Assume that $v_{t-1}(I_{t-1}, N_{t-1}) - cI_{t-1}$ is jointly concave in (I_{t-1}, N_{t-1}), decreasing in I_{t-1}, and increasing in N_{t-1}. Because $r_n(\cdot)$ is concavely increasing, $\mathbb{E}[r_n(y + \theta\xi_t + \epsilon_t)]$ is concavely increasing in y. Since concavity and monotonicity are preserved under expectation, $\Psi_t(\cdot, \cdot)$ is jointly concave in (x, y), decreasing in x, and increasing in y. Analogously, $\Lambda(x)$ is concavely decreasing in x. We now verify that $\Psi_t(x_t - \bar{V}_t + p_t - \gamma(N_t), \theta(\bar{V}_t - p_t + \gamma(N_t)) + \eta N_t)$ is jointly concave in (x_t, p_t, N_t) and increasing in N_t. Since $\gamma(\cdot)$ is increasing in N_t, whereas $\Psi_t(x, y)$ is decreasing in x and increasing in y, $\Psi_t(x_t - \bar{V}_t + p_t - \gamma(N_t), \theta(\bar{V}_t - p_t + \gamma(N_t)) + \eta N_t)$ is increasing in N_t. Let $\lambda \in [0, 1]$, $x_* = \lambda x_t + (1 - \lambda)\hat{x}_t$, $p_* = \lambda p_t + (1 - \lambda)\hat{p}_t$, and $N_* = \lambda N_t + (1 - \lambda)\hat{N}_t$, we have:

$$\begin{aligned}
& \lambda \Psi_t(x_t - \bar{V}_t + p_t - \gamma(N_t), \theta(\bar{V}_t - p_t + \gamma(N_t)) + \eta N_t) \\
& +(1 - \lambda)\Psi_t(\hat{x}_t - \bar{V}_t + \hat{p}_t - \gamma(\hat{N}_t), \theta(\bar{V}_t - \hat{p}_t + \gamma(\hat{N}_t)) + \eta \hat{N}_t) \\
\leq \; & \Psi_t(x_* - \bar{V}_t + p_* - \lambda\gamma(N_t) - (1 - \lambda)\gamma(\hat{N}_t), \theta(\bar{V}_t - p_* + \lambda\gamma(N_t) + (1 - \lambda)\gamma(\hat{N}_t)) + \eta N_*) \\
\leq \; & \Psi_t(x_* - \bar{V}_t + p_* - \gamma(N_t^*), \theta(\bar{V}_t - p_* + \gamma(N_t^*) + \eta N_*),
\end{aligned}$$

where the first inequality follows from the joint concavity of $\Psi_t(\cdot, \cdot)$, and the second from the concavity of $\gamma(\cdot)$, and that $\Psi_t(\cdot, \cdot)$ is decreasing in x and increasing in y. It's clear that $\Lambda(x) = \mathbb{E}\{-(h + b)(x - \xi_t)^+\}$ is concavely decreasing in x. Hence, similar argument to the case of $\Psi_t(x_t - \bar{V}_t + p_t - \gamma(N_t), \theta(\bar{V}_t - p_t + \gamma(N_t)) + \eta N_t)$ implies that $\Lambda(x_t - \bar{V}_t + p_t - \gamma(N_t))$ is jointly concave in (x_t, p_t, N_t) and increasing in N_t. By Assumption 2.3.1, $R_t(p_t, N_t)$ is jointly concave in (p_t, N_t). Moreover, since $\gamma(\cdot)$ is increasing in N_t, $R_t(p_t, N_t)$ is increasing in N_t as well. Hence, by (2.5),

$$\begin{aligned}
J_t(x_t, p_t, N_t) \; = \; & R_t(p_t, N_t) + \beta x_t + \Lambda(x_t - \bar{V}_t + p_t - \gamma(N_t)) \\
& + \Psi_t(x_t - \bar{V}_t + p_t - \gamma(N_t), \theta(\bar{V}_t - p_t + \gamma(N_t)) + \eta N_t)
\end{aligned}$$

is jointly concave in (x_t, p_t, N_t) and increasing in N_t.

Since concavity is preserved under maximization (e.g., [32] Section 3.2.5), the joint concavity of $v_t(\cdot, \cdot)$ follows directly from that of $J_t(\cdot, \cdot, \cdot)$. Note that for any $\hat{I}_t > I_t$, $\mathcal{F}(\hat{I}_t) \subseteq \mathcal{F}(I_t)$. Thus,

$$\begin{aligned}
v_t(\hat{I}_t, N_t) - c\hat{I}_t \; & = \max_{(x_t, p_t) \in \mathcal{F}(\hat{I}_t)} J_t(x_t, p_t, N_t) \\
& \leq \max_{(x_t, p_t) \in \mathcal{F}(I_t)} J_t(x_t, p_t, N_t) \\
& = v_t(I_t, N_t) - cI_t.
\end{aligned}$$

Hence, $v_t(I_t, N_t) - cI_t$ is decreasing in I_t. Since $J_t(x_t, p_t, N_t)$ is increasing in N_t for any (x_t, p_t, N_t), for any $\hat{N}_t > N_t$,

$$\begin{aligned}
v_t(I_t, \hat{N}_t) - cI_t \; & = \max_{(x_t, p_t) \in \mathcal{F}(I_t)} J_t(x_t, p_t, \hat{N}_t) \\
& \geq \max_{(x_t, p_t) \in \mathcal{F}(I_t)} J_t(x_t, p_t, N_t) \\
& = v_t(I_t, N_t) - cI_t.
\end{aligned}$$

Thus, $v_t(I_t, N_t) - cI_t$ is increasing in N_t.

Second, we show that by backward induction, that if $v_{t-1}(\cdot, \cdot)$ is continuously differentiable, $\Psi_t(\cdot, \cdot)$, $J_t(\cdot, \cdot, \cdot)$, and $v_t(\cdot, \cdot)$ are continuously differentiable as well. For $t = 0$, $v_0(I_0, N_0) = cI_0$ is clearly continuously differentiable. Thus, the initial condition holds.

If $v_{t-1}(\cdot, \cdot)$ is continuously differentiable, $\Psi_t(\cdot, \cdot)$ is continuously differentiable with partial derivatives given by

$$\partial_x \Psi_t(x, y) = \mathbb{E}\{\alpha[\partial_{I_t} v_{t-1}(x - \xi_t, y + \theta\xi_t + \epsilon_t) - c]\}, \tag{A.1}$$

$$\partial_y \Psi_t(x, y) = \mathbb{E}\{r_n'(y + \theta\xi_t + \epsilon_t) + \alpha[\partial_{N_{t-1}} v_{t-1}(x - \xi_t, y + \theta\xi_t + \epsilon_t)\}, \tag{A.2}$$

where the exchangeability of differentiation and expectation is easily justified using the canonical argument (see, e.g., Theorem A.5.1 in [63], the condition of which can be easily verified observing the continuity of the partial derivatives of $v_{t-1}(\cdot,\cdot)$, and that the distributions of ξ_t and ϵ_t are continuous.). Moreover, since ξ_t is continuously distributed, $\Lambda(\cdot)$ is continuously differentiable. Since $R_t(\cdot,\cdot)$ is continuously differentiable, by (2.5), $J_t(\cdot,\cdot,\cdot)$ is continuously differentiable. If $I_t \neq x_t(N_t)$, the continuous differentiability of $v_t(\cdot,\cdot)$ follows immediately from that of $J_t(\cdot,\cdot,\cdot)$ and the envelope theorem. To complete the proof, it suffices to check that, for all $N_t \geq 0$, the left and right partial derivatives of the first variable at $(x_t(N_t), N_t)$, $\partial_{I_t} v_t(x_t(N_t)-, N_t)$ and $\partial_{I_t} v_t(x_t(N_t)+, N_t)$ are equal. By the envelope theorem,

$$\begin{cases} \partial_{I_t} v_t(x_t(N_t)-, N_t) = & c, \\ \partial_{I_t} v_t(x_t(N_t)+, N_t) = & c + \beta + \partial_x \Lambda(x_t(N_t) - y_t(N_t)) + \partial_x \Psi_t(x_t(N_t) - y_t(N_t), \theta y_t(N_t) + \eta N_t). \end{cases}$$

The first-order condition with respect to x_t implies that

$$\beta + \partial_x \Lambda(x_t(N_t) - y_t(N_t)) + \partial_x \Psi_t(x_t(N_t) - y_t(N_t), \theta y_t(N_t) + \eta N_t) = 0.$$

Therefore, $\partial_{I_t} v_t(x_t(N_t)-, N_t) = \partial_{I_t} v_t(x_t(N_t)+, N_t) = c$. This completes the induction and, thus, the proof of Lemma 2. *Q.E.D.*

Proof of Theorem 2.4.1: Parts (a)-(b) follow immediately from the joint concavity of $J_t(\cdot,\cdot,N_t)$ in (x_t, p_t) for any $N_t \geq 0$.

We now show **part (c)** by backward induction. More specifically, we prove that if $x_{t-1}(N_{t-1}) > 0$ for all $N_{t-1} \geq 0$, $x_t(N_t) > 0$ for all $N_t \geq 0$. Since $v_0(I_0, N_0) = cI_0$, $\Psi_1(x,y) = \mathbb{E}[r_n(y+\theta\xi_1)] + \alpha\mathbb{E}\{v_0(x - \xi_1, y + \theta\xi_1 + \epsilon_1) - cx\} = \mathbb{E}[r_n(y+\theta\xi_1)]$. Since $D_1 \geq 0$ with probability 1, $\partial_x \Lambda(-\bar{V}_1 + p_1 - \gamma(N_1)) = 0$ for all $p_1 \in [\underline{p}, \bar{p}]$ and $N_1 \geq 0$. Hence, for any $p_1 \in [\underline{p}, \bar{p}]$ and $N_1 \geq 0$,

$$\partial_{x_1} J_1(0, p_1, N_1) = \beta - \partial_x \Lambda(-\bar{V}_1 + p_1 - \gamma(N_1)) = \beta > 0.$$

Hence, $x_1(N_1) > 0$ for any $N_1 \geq 0$. Thus, the initial condition is satisfied.

Now we assume that $x_{t-1}(N_{t-1}) > 0$ for all $N_{t-1} \geq 0$ and $x_t(\tilde{N}_t) \leq 0$ for some $\tilde{N}_t \geq 0$. Thus, $I_{t-1} = x_t(\tilde{N}_t) - D_t(p_t(\tilde{N}_t), \tilde{N}_t) \leq 0 < x_{t-1}(\tilde{N}_{t-1})$ almost surely, where $\tilde{N}_{t-1} = \theta D_t(p_t(\tilde{N}_t), \tilde{N}_t) + \eta \tilde{N}_t + \epsilon_t$. Thus, by **part (a)**, $\partial_{I_{t-1}} v_{t-1}(I_{t-1}, N_{t-1}) = c$ almost surely, when conditioned on $N_t = \tilde{N}_t$. Hence, conditioned on $N_t = \tilde{N}_t$, $\partial_x \Psi_t(x,y) = \alpha\mathbb{E}\{\partial_{I_{t-1}} v_{t-1}(I_{t-1}, \tilde{N}_{t-1}) - c | N_t = \tilde{N}_t\} = c - c = 0$, when (x_t, p_t) lies in the neighborhood of $(x_t(\tilde{N}_t), p_t(\tilde{N}_t))$. As discussed above, since $x_t(\tilde{N}_t) \leq 0$, $\partial_x \Lambda(-\bar{V}_t + p_t - \gamma(\tilde{N}_1)) = 0$ for all $p_t \in [\underline{p}, \bar{p}]$. Hence, for any $p_t \in [\underline{p}, \bar{p}]$,

$$\partial_{x_t} J_t(0, p_t, \tilde{N}_t) = \beta - \partial_x \Lambda(-\bar{V}_t + p_t - \gamma(\tilde{N}_1)) = \beta > 0.$$

Hence, $x_t(\tilde{N}_t) > 0$, which contradicts the assumption that $x_t(\tilde{N}_t) \leq 0$ is the optimizer of (2.7) when $N_t = \tilde{N}_t$. Therefore, $x_t(N_t) > 0$ for all $N_t \geq 0$. This completes the induction and, thus, the proof of **part (c)**. *Q.E.D.*

Proof of Lemma 3: We show **Parts (a)-(b)** together by backward induction. We first show that

Parts (a)-(b) hold for $t = 1$. Since Part (a) automatically holds for $t = 1$, we only need to check Part (b). Because $v_0(I_0, N_0) = cI_0$, $\Psi_1(x, y) = \mathbb{E}[r_n(y + \theta\xi_t + \epsilon_t)]$. Thus, taking the transformation $x_1 = \Delta_1 + \bar{V}_1 - p_1 + \gamma(N_1)$,

$$\begin{aligned}
J_1(x_1, p_1, N_1) =& R_1(p_1, N_1) + \beta x_1 + \Lambda(x_1 - \bar{V}_1 + p_1 - \gamma(N_1)) + \mathbb{E}[r_n(\theta(\bar{V}_1 - p_1 + \gamma(N_1)) + \eta N_1)] \\
=& (p_1 - b - \alpha c)(\bar{V}_1 - p_1 + \gamma(N_1)) + \beta(\Delta_1 + \bar{V}_1 - p_1 + \gamma(N_1)) + \beta\Delta_1 + \Lambda(\Delta_1) \\
& + \mathbb{E}[r_n(\theta(\bar{V}_1 - p_1 + \gamma(N_1)) + \eta N_1)] \\
=& (p_1 - c)(\bar{V}_1 - p_1 + \gamma(N_1)) + \beta\Delta_1 + \Lambda(\Delta_1) + \mathbb{E}[r_n(\theta(\bar{V}_1 - p_1 + \gamma(N_1)) + \eta N_1)].
\end{aligned}$$

Therefore, the optimal joint price and safety-stock $(p_1(N_1), \Delta_1(N_1))$ can be determined by

$$p_1(N_1) = \text{argmax}_{p_1 \in [\underline{p}, \bar{p}]}\{(p_1 - c)(\bar{V}_1 - p_1 + \gamma(N_1)) + \mathbb{E}[r_n(\theta(\bar{V}_1 - p_1 + \gamma(N_1)) + \eta N_1)]\},$$

and

$$\Delta_1(N_1) = \Delta^* = \text{argmax}_{\Delta}\{\beta\Delta + \Lambda(\Delta)\},$$

respectively. Hence, $x_1(N_1) = \Delta_1(N_1) + y_1(N_1) = \Delta^* + y_1(N_1)$. We have thus shown Parts (a)-(b) for $t = 1$.

We now show that if Parts (a)-(b) hold for period $t - 1$, they also hold for period t. First, taking the transformation $x_t = \Delta_t + \bar{V}_t - p_t + \gamma(N_t)$,

$$\begin{aligned}
J_t(x_t, p_t, N_t) =& R_t(p_t, N_t) + \beta x_t + \Lambda(x_t - \bar{V}_1 + p_t - \gamma(N_t)) \\
& + \Psi_t(x_t - \bar{V}_t + p_t - \gamma(N_t), \theta(\bar{V}_t - p_t + \gamma(N_t)) + \eta N_t) \\
=& R_t(p_t, N_t) + \beta(\Delta_t + \bar{V}_t - p_t + \gamma(N_t)) + \Lambda(\Delta_t) + \Psi_t(\Delta_t, \theta(\bar{V}_t - p_t + \gamma(N_t)) + \eta N_t) \\
=& (p_t - c)(\bar{V}_t - p_t + \gamma(N_t)) + \beta\Delta_t + \Lambda(\Delta_t) + \Psi_t(\Delta_t, \theta(\bar{V}_t - p_t + \gamma(N_t)) + \eta N_t).
\end{aligned}$$

Let $(p_t(N_t), \Delta_t(N_t))$ be the optimal price and safety-stock with network size N_t. We now show that $\Delta_t(N_t) \leq \Delta^*$. If, to the contrary, $\Delta_t(N_t) > \Delta^*$, Lemma 18 yields that

$$\begin{aligned}
& \partial_\Delta[(p_t(N_t) - c)(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + \beta\Delta_t(N_t) + \Lambda(\Delta_t(N_t)) \\
& + \Psi_t(\Delta_t(N_t), \theta(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + \eta N_t)] \\
\geq\ & \partial_\Delta[\beta\Delta + \Lambda(\Delta)],
\end{aligned}$$

i.e.,

$$\beta + \Lambda'(\Delta_t(N_t)) + \partial_x\Psi_t(\Delta_t(N_t), \theta(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + \eta N_t) \geq \beta + \Lambda'(\Delta^*).$$

The concavity of $\Lambda(\cdot)$ implies that $\Lambda'(\Delta_t(N_t)) \leq \Lambda'(\Delta^*)$. Moreover, since $\Psi_t(x, y)$ is decreasing in x, $\partial_x\Psi_t(\Delta_t(N_t), \theta(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + \eta N_t) \leq 0$. Therefore, $\Lambda'(\Delta_t(N_t)) = \Lambda'(\Delta^*)$ and $\partial_x\Psi_t(\Delta_t(N_t), \theta(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + \eta N_t) = 0$. Thus, by the first-order condition with respect to Δ_t, $(p_t(N_t), \Delta^*)$ is also the optimal price and safety-stock level, which is strictly lexicographically smaller than $(p_t(N_t), \Delta_t(N_t))$. This contradicts the assumption that we select the lexicographically smallest optimizer in each period. Hence, $\Delta_t(N_t) \leq \Delta^*$ for all $N_t \geq 0$.

We now show that $\mathbb{P}[x_t(N_t) - D_t(p_t(N_t), N_t) \leq x_{t-1}(N_{t-1})] = 1$ for all N_t and N_{t-1}. Note that, with probability 1,

$$\begin{aligned} x_t(N_t) - D_t(p_t(N_t), N_t) = \Delta_t(N_t) - \xi_t \leq \Delta^* - \xi_t &= x_{t-1}(N_{t-1}) - y_{t-1}(N_{t-1}) - \xi_t \\ &= x_{t-1}(N_{t-1}) - D_{t-1}(p_{t-1}(N_{t-1}), N_{t-1}), \end{aligned}$$

where the inequality follows from $\Delta^* \leq \Delta_t(N_t)$, second equality from the hypothesis induction that $x_{t-1}(N_{t-1}) = y_{t-1}(N_{t-1}) + \Delta^*$ for all $N_{t-1} \geq 0$, and the last equality from $\xi_{t-1} \overset{d}{=} \xi_t$ and the identity that $D_{t-1}(p_{t-1}(N_{t-1}), N_{t-1}) = y_{t-1}(N_{t-1}) + \xi_{t-1}$. Because $D_{t-1}(p_{t-1}(N_{t-1}), N_{t-1}) \geq 0$ with probability 1,

$$x_t(N_t) - D_t(p_t(N_t), N_t) \leq x_{t-1}(N_{t-1}) - D_{t-1}(p_{t-1}(N_{t-1}), N_{t-1}) \leq x_{t-1}(N_{t-1})$$

with probability 1, i.e., part (a) follows for period t.

Next, we show that $\Delta_t(N_t) = \Delta^*$. Observe that $\mathbb{P}[x_t(N_t) - D_t(p_t(N_t), N_t) \leq x_{t-1}(N_{t-1})] = 1$ implies that $\partial_x \Psi_t(\Delta_t(N_t), \theta(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + \eta N_t) = 0$ and, thus, $\partial_{x_t} J_t(\Delta^* + y_t(N_t), p_t(N_t), N_t) = 0$. Since $J_t(\cdot, \cdot, N_t)$ is jointly concave, the first-order condition with respect to x_t yields that $\Delta_t(N_t) = \Delta^*$ for all $N_t \geq 0$. This completes the induction and, thus, the proof of Lemma 3. *Q.E.D.*

Proof of Lemma 4:. By parts (a) and (b) of Theorem 2.4.1, if $I_t \leq x_t(N_t)$,

$$v_t(I_t, N_t) = cI_t + \pi_t(N_t),$$

where

$$\pi_t(N_t) := \max\{J_t(x_t, p_t, N_t) : x_t \geq 0, p_t \in [\underline{p}, \bar{p}]\}.$$

By Lemma 2, $\pi_t(\cdot)$ is concavely increasing and continuously differentiable in N_t.

By Lemma 3(a), for each $N_t \geq 0$, $x_t(N_t) - D_t(p_t(N_t), N_t) \leq x_{t-1}(N_{t-1})$ with probability 1. Since $v_{t-1}(I_{t-1}, N_{t-1}) = cI_{t-1} + \pi_{t-1}(N_{t-1})$ for all $I_{t-1} \leq x_{t-1}(N_{t-1})$,

$$\begin{aligned} v_{t-1}(x_t(N_t) - D_t(p_t(N_t), N_t), \theta D_t(p_t(N_t), N_t) + \eta N_t) =& c[x_t(N_t) - D_t(p_t(N_t), N_t)] \\ &+ \pi_{t-1}(\theta D_t(p_t(N_t), N_t) + \eta N_t) \end{aligned}$$

with probability 1. Taking expectation with respect to ξ_t and ϵ_t, we have, for all $N_t \geq 0$ and $x_t \leq x_t(N_t)$,

$$\begin{aligned} &\Psi_t(x_t - \bar{V}_t + p_t(N_t) - \gamma(N_t), \theta(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + \eta N_t) \\ =&\mathbb{E}[r_n(\theta(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + \eta N_t + \theta\xi_t + \epsilon_t)] + \alpha\mathbb{E}[\pi_{t-1}(\theta(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + \eta N_t + \theta\xi_t + \epsilon_t)]. \end{aligned}$$

Therefore, for all $N_t \geq 0$, if $x_t \leq \Delta^* + y_t(N_t)$,

$$J_t(x_t, p_t, N_t) = R_t(p_t, N_t) + \beta x_t - \theta q_t + \Lambda(x_t - \bar{V}_t + p_t(N_t) - \gamma(N_t)) + G_t(\theta(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + \eta N_t),$$

where $G_t(y) := \mathbb{E}[r_n(y + \theta\xi_t + \epsilon_t)] + \alpha\mathbb{E}[v_{t-1}(y + \theta\xi_t + \epsilon_t)]$.

Finally, it remains to show that $(x_t(N_t), p_t(N_t))$ maximizes the right-hand side of (2.8). Note that Theorem 2.4.1(c) and Lemma 3(a) imply that, if $I_t \leq x_t(N_t)$, with probability 1, $I_\tau \leq x_\tau(N_\tau)$ for all $\tau = t, t-1, \cdots, 1$ and, hence, $\{(x_\tau(N_\tau), p_\tau(N_\tau))\}_{\tau=t,t-1,\cdots,1}$ is the optimal policy in periods $t, t-1, \cdots, 1$.

In particular, $(x_t(N_t), p_t(N_t))$ maximizes the total expected discounted profit given that the firm adopts $\{(x_\tau(N_\tau), p_\tau(N_\tau))\}$ for $\tau = t-1, \cdots, 1$. It's straightforward to check that if the firm adopts the policy $\{(x_\tau(N_\tau), p_\tau(N_\tau))\}$ for $\tau = t-1, \cdots, 1$; and orders-up-to x_t and charges p_t in period t, the total expected discounted profit of the firm in period t is given by the right-hand side of (2.8). Since $(x_t(N_t), p_t(N_t))$ maximizes the total expected discounted profit in period t, it also maximizes the right-hand side of (2.8) for each t. This proves Lemma 4. *Q.E.D.*

Proof of Theorem 2.4.2: By Theorem 2.4.1(c) and Lemma 3(a), if $I_T \le x_T(N_t)$, $I_t \le x_t(N_t)$ for all $t = T, T-1, \cdots, 1$ with probability 1. Therefore, by Theorem 2.4.1(a), $(x_t^*(I_t, N_t), p_t^*(I_t, N_t)) = (x_t(N_t), p_t(N_t))$ with probability 1 if $I_T \le x_T(N_T)$. The characterization of $(x_t(N_t), p_t(N_t))$ follows immediately from Lemma 4 and its discussions. *Q.E.D.*

The following lemma is used throughout the rest of our proofs.

Lemma 19 *For each period t and any network size $N_t \ge 0$, the following statements hold.*

(a) $J_t(x_t(N_t), p_t(N_t), N_t) = L_t(p_t(N_t), N_t) + \beta\Delta^* + \Lambda(\Delta^*)$, *where* $L_t(p_t, N_t) := (p_t - c)(\bar{V}_t - p_t + \gamma(N_t)) + G_t(\theta(\bar{V}_t - p_t + \gamma(N_t)) + \eta N_t)$, *and Δ^* is the optimal safety stock characterized in Lemma 3(b). Hence,* $p_t(N_t) = \text{argmax}_{p_t \in [\underline{p}, \bar{p}]} L_t(p_t, N_t)$.

(b) $J_t(x_t(N_t), p_t(N_t), N_t) = K_t(y_t(N_t), N_t) + \beta\Delta^* + \Lambda(\Delta^*)$, *where* $K_t(y_t, N_t) := (\bar{V}_t + \gamma(N_t) - y_t - c)y_t + G_t(\theta y_t + \eta N_t)$. *Hence,* $y_t(N_t) = \text{argmax}_{y_t \in [\underline{y}_t(N_t), \bar{y}_t(N_t)]} K_t(y_t, N_t)$, *where* $\underline{y}_t(N_t) = \bar{V}_t + \gamma(N_t) - \bar{p}$ *and* $\bar{y}_t(N_t) = \bar{V}_t + \gamma(N_t) - \underline{p}$.

(c) Let $m_t(N_t) := \theta y_t(N_t) + \eta N_t$ *be the optimal expected network size in period $t-1$, given the current network size N_t. We have* $J_t(x_t(N_t), p_t(N_t), N_t) = M_t(m_t(N_t), N_t) + \beta\Delta^* + \Lambda(\Delta^*)$, *where* $M_t(m_t, N_t) := (\bar{V}_t + \gamma(N_t) - \frac{m_t - \eta N_t}{\theta} - c)\frac{(m_t - \eta N_t)}{\theta} + G_t(m_t)$. *Hence,* $m_t(N_t) = \text{argmax}_{m_t \in [\underline{m}_t(N_t), \bar{m}_t(N_t)]} M_t(m_t, N_t)$, *where* $\underline{m}_t(N_t) = \theta \underline{y}_t(N_t) + \eta N_t$ *and* $\bar{m}_t(N_t) = \theta \bar{y}_t(N_t) + \eta N_t$.

Proof of Lemma 19: Part (a). By Lemma 3(b), $x_t(N_t) - y_t(N_t) = \Delta^*$ for all $N_t \ge 0$. By Lemma 4, for all N_t,

$$\begin{aligned} J_t(x_t(N_t), p_t(N_t), N_t) =& R_t(p_t(N_t), N_t) + \beta x_t(N_t) + \Lambda(x_t(N_t) - \bar{V}_t + p_t(N_t) - \gamma(N_t)) \\ &+ G_t(\theta(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + \eta N_t). \end{aligned}$$

Therefore,

$$\begin{aligned} J_t(x_t(N_t), p_t(N_t), N_t) =& R_t(p_t(N_t), N_t) + \beta(\Delta^* + \bar{V}_t - p_t(N_t) + \gamma(N_t)) + \Lambda(\Delta^*) \\ &+ G_t(\theta(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + \eta N_t) \\ =& (p_t(N_t) - c)(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + G_t(\theta(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + \eta N_t) \\ &+ \beta\Delta^* + \Lambda(\Delta^*) \\ =& L_t(p_t(N_t), N_t) + \beta\Delta^* + \Lambda(\Delta^*), \end{aligned} \tag{A.3}$$

where $L_t(p_t, N_t) := (p_t - c)(\bar{V}_t - p_t + \gamma(N_t)) + G_t(\theta(\bar{V}_t - p_t + \gamma(N_t)) + \eta N_t)$. Since $(x_t(N_t), p_t(N_t))$ maximizes $J_t(\cdot, \cdot, N_t)$, $p_t(N_t) = \text{argmax}_{p_t \in [\underline{p}, \bar{p}]} L_t(p_t, N_t)$. This proves part (a).

Part (b). Since $y_t(N_t) = \bar{V}_t - p_t(N_t) + \gamma(N_t)$, $p_t(N_t) = \bar{V}_t - y_t(N_t) + \gamma(N_t)$. Plug this into (A.3), we have

$$\begin{aligned} J_t(x_t(N_t), p_t(N_t), N_t) =& (p_t(N_t) - c)(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + G_t(\theta(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + \eta N_t) \\ & + \beta\Delta^* + \Lambda(\Delta^*) \\ =& (\bar{V}_t - y_t(N_t) + \gamma(N_t) - c)y_t(N_t) + G_t(\theta y_t(N_t) + \eta N_t) + \beta\Delta^* + \Lambda(\Delta^*) \\ =& K_t(y_t(N_t), N_t) + \beta\Delta^* + \Lambda(\Delta^*), \end{aligned}$$

where $K_t(y_t, N_t) := (\bar{V}_t + \gamma(N_t) - y_t - c)y_t + G_t(\theta y_t + \eta N_t)$. Since $(x_t(N_t), p_t(N_t))$ maximizes $J_t(\cdot, \cdot, N_t)$, $y_t(N_t) = \text{argmax}_{y_t \in [\underline{y}_t(N_t), \bar{y}_t(N_t)]} K_t(y_t, N_t)$. The expressions of $\underline{y}_t(N_t)$ and $\bar{y}_t(N_t)$ follow directly from the identity $y_t(N_t) = \bar{V}_t - p_t(N_t) + \gamma(N_t)$ and $p_t \in [\underline{p}, \bar{p}]$. This proves part (b).

Part (c). Observe that $m_t(N_t) = \theta y_t(N_t) + \eta N_t$ and $\theta > 0$ imply that $p_t(N_t) = \bar{V}_t - y_t(N_t) + \gamma(N_t) = \bar{V}_t + \gamma(N_t) - \frac{m_t(N_t) - \eta N_t}{\theta}$. Plug this into (A.3), we have

$$\begin{aligned} J_t(x_t(N_t), p_t(N_t), N_t) =& (p_t(N_t) - c)(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + G_t(\theta(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + \eta N_t) \\ & + \beta\Delta^* + \Lambda(\Delta^*) \\ =& (\bar{V}_t + \gamma(N_t) - \frac{m_t(N_t) - \eta N_t}{\theta} - c)\frac{m_t(N_t) - \eta N_t}{\theta} + G_t(m_t(N_t)) + \beta\Delta^* \\ & + \Lambda(\Delta^*) \\ =& M_t(m_t(N_t), N_t) + \beta\Delta^* + \Lambda(\Delta^*), \end{aligned}$$

where $M_t(m_t, N_t) := (\bar{V}_t + \gamma(N_t) - \frac{m_t - \eta N_t}{\theta} - c)\frac{(m_t - \eta N_t)}{\theta} + G_t(m_t)$. Since $(x_t(N_t), p_t(N_t))$ maximizes $J_t(\cdot, \cdot, N_t)$, $m_t(N_t) = \text{argmax}_{m_t \in [\underline{m}_t(N_t), \bar{m}_t(N_t)]} M_t(m_t, N_t)$. The expressions of $\underline{m}_t(N_t)$ and $\bar{m}_t(N_t)$ follow directly from the identity $m_t(N_t) = \theta y_t(N_t) + \eta N_t$ and that $y_t \in [\underline{y}_t(N_t), \bar{y}_t(N_t)]$. This establishes part (c). *Q.E.D.*

Proof of Theorem 2.4.3: Part (a). We first show $p_t(\hat{N}_t) \geq p_t(N_t)$. By Lemma 19(a) $p_t(N_t) = \text{argmax}_{p_t} L_t(p_t, N_t)$ and $p_t(\hat{N}_t) = \text{argmax}_{p_t} L_t(p_t, \hat{N}_t)$. Hence, it suffices to show that $L_t(\cdot, \cdot)$ is supermodular in (p_t, N_t). Since $\partial_{p_t} L_t(p_t, N_t) = \bar{V}_t + \gamma(N_t) - 2p_t + c - \theta G'_t(\theta(\bar{V}_t - p_t + \gamma(N_t)) + \eta N_t)$, Since $G_t(\cdot)$ is concave, $\partial_{p_t} L_t(p_t, N_t)$ is increasing in N_t. Hence, $L_t(\cdot, \cdot)$ is supermodular in (p_t, N_t) and, thus, $p_t(\hat{N}_t) \geq p_t(N_t)$ for all $\hat{N}_t > N_t$ (See [165]).

Part (b). We now show that $\mathbb{E}[N_{t-1}|\hat{N}_t] = m_t(\hat{N}_t) = \theta y_t(\hat{N}_t) + \eta \hat{N}_t \geq \mathbb{E}[N_{t-1}|N_t] = m_t(N_t) = \theta y_t(N_t) + \eta N_t$. By Lemma 19(c), $m_t(\hat{N}_t) = \text{argmax}_{m_t} M_t(m_t(\hat{N}_t), \hat{N}_t)$ and $m_t(N_t) = \text{argmax}_{m_t} M_t(m_t(N_t), N_t)$. To show that $m_t(\hat{N}_t) \geq m_t(N_t)$, it suffices to prove that $M_t(\cdot, \cdot)$ is supermodular in (m_t, N_t) and the feasible set $\{(m_t, N_t) : m_t \in [\underline{m}(N_t), \bar{m}_t(N_t)]\}$ is a lattice. Direct computation yields that

$$\partial_{m_t} M_t(m_t, N_t) = \frac{1}{\theta}(\bar{V}_t + \gamma(N_t) - \frac{m_t - \eta N_t}{\theta} - c) - \frac{m_t - \eta N_t}{\theta^2} + G'_t(m_t).$$

Since $\theta > 0$, $\partial_{m_t} M_t(m_t, N_t)$ is increasing in N_t. Thus, $M_t(\cdot,\cdot)$ is supermodular in (m_t, N_t). Since $\underline{m}_t(N_t)$ and $m_t(N_t)$ are continuously increasing in N_t, the feasible set $\{(m_t, N_t) : m_t \in [\underline{m}(N_t), \bar{m}_t(N_t)]\}$ is a lattice. Hence, $m_t(\hat{N}_t) \geq m_t(N_t)$ for all $\hat{N}_t > N_t$.

Part (c). Since $\gamma(\hat{N}_t) = \gamma(N_t)$, $p_t(\hat{N}_t) \geq p_t(N_t)$ implies that $y_t(\hat{N}_t) = \bar{V}_t - p_t(\hat{N}_t) + \gamma(\hat{N}_t) \leq \bar{V}_t - p_t(N_t) + \gamma(N_t) = y_t(N_t)$. Moreover, by Lemma 3(b), $x_t(\hat{N}_t) = \Delta^* + y_t(\hat{N}_t) \leq \Delta^* + y_t(N_t) = x_t(N_t)$.

Part (d). Since $\eta = 0$ and $m_t(\hat{N}_t) \geq m_t(N_t)$, $y_t(\hat{N}_t) = \frac{m_t(\hat{N}_t)}{\theta} \geq \frac{m_t(N_t)}{\theta} = y_t(N_t)$. By Lemma 3(b), $x_t(\hat{N}_t) = \Delta^* + y_t(\hat{N}_t) \geq \Delta^* + y_t(N_t) = x_t(N_t)$. *Q.E.D.*

Proof of Theorem 2.4.4: Part (a). We show part (a) by backward induction. More specifically, we show that if $\eta = 0$ and $v_{t-1}(\cdot,\cdot)$ is supermodular in (I_{t-1}, N_{t-1}), $v_t(\cdot,\cdot)$ is supermodular in (I_t, N_t). Since $v_0(I_0, N_0) = cI_0$, the initial condition is satisfied.

Since supermodularity is preserved under expectation, $\Psi_t(x, y) = \mathbb{E}\{r_n(y + \theta\xi_t + \epsilon_t) + \alpha[v_{t-1}(x - \xi_t, y + \theta\xi_t + \epsilon_t) - cx]\}$ is supermodular in (x, y). Let $y_t = \bar{V}_t - p_t + \gamma(N_t)$. Observe that

$$\begin{aligned} J_t(x_t, p_t, N_t) =& R_t(p_t, N_t) + \beta x_t + \Lambda(x_t - \bar{V}_t + p_t - \gamma(N_t)) + \Psi_t(x_t - \bar{V}_t + p_t - \gamma(N_t), \theta(\bar{V}_t - p_t + \gamma(N_t))) \\ =& (\bar{V}_t + \gamma(N_t) - y_t - \alpha c - b)y_t + \beta x_t + \Lambda(x_t - y_t) + \Psi_t(x_t - y_t, \theta y_t). \end{aligned}$$

Hence,

$$v_t(I_t, N_t) = cI_t + \max_{(x_t, y_t) \in \mathcal{F}'_t(I_t, N_t)} \{(\bar{V}_t + \gamma(N_t) - y_t - \alpha c - b)y_t + \beta x_t + \Lambda(x_t - y_t) + \Psi_t(x_t - y_t, \theta y_t)\},$$

where $\mathcal{F}'_t(I_t, N_t) := \{(x_t, y_t) : x_t \geq I_t, y_t \in [\bar{V}_t + \gamma(N_t) - \bar{p}, \bar{V}_t + \gamma(N_t) - \underline{p}]\}$. Because $\gamma(\cdot)$ is increasing in N_t, $\Lambda(\cdot)$ is concave, and $\Psi_t(\cdot,\cdot)$ is concave and supermodular, $(\bar{V}_t + \gamma(N_t) - y_t - \alpha c - b)y_t + \beta x_t + \Lambda(x_t - y_t) + \Psi_t(x_t - y_t, \theta y_t)$ is supermodular in (x_t, y_t, N_t). Moreover, it's straightforward to verify that the feasible set $\{(x_t, y_t, I_t, N_t) : N_t \geq 0, (x_t, y_t) \in \mathcal{F}'_t(I_t)\}$ is a lattice in $\mathbb{R}^4$. Therefore, $v_t(I_t, N_t)$ is supermodular in (I_t, N_t). This completes the induction and, thus, the proof of part (a).

Part (b). The continuity results in parts (b)-(e) all follow from the joint concavity and continuous differentiability of $J_t(\cdot,\cdot,\cdot)$ in (x_t, p_t, N_t). Since $x_t^*(I_t, N_t) = \max\{I_t, x_t(N_t)\}$, $x_t^*(I_t, N_t)$ is increasing in I_t. Moreover, because the objective function $(\bar{V}_t + \gamma(N_t) - y_t - \alpha c - b)y_t + \beta x_t + \Lambda(x_t - y_t) + \Psi_t(x_t - y_t, \theta y_t)$ is supermodular in (x_t, y_t, N_t), $x_t^*(I_t, N_t)$ is increasing in N_t as well. This proves part (b).

Part (c). If $I_t \leq x_t(N_t)$, $p_t^*(I_t, N_t) = p_t(N_t)$, which is independent of I_t. If $I_t > x_t(N_t)$, $x_t^*(I_t, N_t) = I_t$ and, thus,

$$J_t(x_t^*(I_t, N_t), p_t, N_t) = R_t(p_t, N_t) + \beta I_t + \Lambda(I_t - \bar{V}_t + p_t - \gamma(N_t)) + \Psi_t(I_t - \bar{V}_t + p_t - \gamma(N_t), \theta(\bar{V}_t - p_t + \gamma(N_t))). \tag{A.4}$$

Since $\Lambda(\cdot)$ is concave and $\Psi_t(\cdot,\cdot)$ is concave and supermodular, $J_t(x_t^*(I_t, N_t), p_t, N_t)$ is submodular in (I_t, p_t). Hence, $p_t^*(I_t, N_t)$ is decreasing in I_t for all (I_t, N_t). By Theorem 2.4.3(d), if $I_t \leq x_t(N_t)$, $p_t^*(I_t, N_t) = p_t(N_t)$ is increasing in N_t. If $I_t > x_t(N_t)$, we observe from (A.4) that $J_t(x_t^*(I_t, N_t), p_t, N_t)$ is supermodular in (p_t, N_t). Hence, $p_t^*(I_t, N_t)$ is increasing in N_t for all (I_t, N_t). This proves part (c).

Part (d). If $I_t \leq x_t(N_t)$, $y_t^*(I_t, N_t) = y_t(N_t)$, which is independent of I_t. If $I_t > x_t(N_t)$, $x_t^*(I_t, N_t) = I_t$ and, thus,

$$J_t(x_t^*(I_t, N_t), p_t, N_t) = (\bar{V}_t + \gamma(N_t) - y_t - \alpha c - b)y_t + \beta I_t + \Lambda(I_t - y_t) + \Psi_t(I_t - y_t, \theta y_t).$$

Since $\Lambda(\cdot)$ is concave and $\Psi_t(\cdot,\cdot)$ is concave and supermodular, $J_t(x_t^*(I_t,N_t),p_t,N_t)$ is supermodular in (I_t,y_t) and its domain is a sublattice of $\mathbb{R}^2$. Hence, $y_t^*(I_t,N_t)$ is increasing in I_t for all (I_t,N_t). By Theorem 2.4.3(d), if $I_t \le x_t(N_t)$, $y_t^*(I_t,N_t) = y_t(N_t)$ is increasing in N_t. If $I_t > x_t(N_t)$, $x_t^*(I_t,N_t) = I_t$ and, thus, $J_t(x_t^*(I_t,N_t),p_t,N_t) = (\bar{V}_t + \gamma(N_t) - y_t - \alpha c - b)y_t + \beta I_t + \Lambda(I_t - y_t) + \Psi_t(I_t - y_t, \theta y_t)$. The supermodularity of $J_t(x_t^*(I_t,N_t),p_t,N_t)$ in (y_t,N_t) follows directly from that $\gamma(\cdot)$ is increasing in N_t. Moreover, the feasible set $\{(y_t,N_t) : y_t \in [\bar{V}_t + \gamma(N_t) - \bar{p}, \bar{V}_t + \gamma(N_t) - \underline{p}]\}$ is clearly a sublattice of $\mathbb{R}^2$. Therefore, $y_t^*(I_t,N_t)$ is increasing in N_t for all (I_t,N_t). This proves part (d).

Part (e). If $I_t \le x_t(N_t)$, by Theorem 2.4.1(c), $\Delta_t^*(I_t,N_t) = \Delta^*$ is independent of I_t and N_t. If $I_t > x_t(N_t)$, since $I_t - \Delta_t = y_t$,

$$J_t(x_t^*(I_t,N_t),p_t,N_t) = (\bar{V}_t + \gamma(N_t) + \Delta_t - I_t - \alpha c - b)(I_t - \Delta_t) + \beta I_t + \Lambda(\Delta_t) + \Psi_t(\Delta_t, \theta(I_t - \Delta_t)).$$

Since $\Psi_t(\cdot,\cdot)$ is concave and supermodular, $J_t(x_t^*(I_t,N_t),p_t,N_t)$ is supermodular in (I_t,Δ_t). Moreover, the feasible set $\{(I_t,\Delta_t) : \Delta_t \in [I_t - \bar{V}_t - \gamma(N_t) + \underline{p}, I_t - \bar{V}_t - \gamma(N_t) + \bar{p}]\}$ is clearly a sublattice of $\mathbb{R}^2$. Hence, $\Delta_t^*(I_t,N_t)$ is increasing in I_t for all (I_t,N_t). Moreover, since $\Delta_t^*(I_t,N_t) = I_t - y_t^*(I_t,N_t)$, by part (d), $\Delta_t^*(I_t,N_t)$ is decreasing in N_t. This proves part (e). *Q.E.D.*

Proof of Theorem 2.4.5: Part (a). Since $\gamma(\cdot) \equiv \gamma_0$ and $r_n'(n) \equiv r$, the optimal policy of the firm $(p_t(\cdot),x_t(\cdot))$ is independent of the current network size N_t. Hence, $\hat{\pi}_t'(N_t) \ge \pi_t'(N_t) \equiv 0$ for all t and $N_t \ge 0$. We now show that $\hat{y}_t(N_t) \ge y_t(N_t)$ for all $N_t \ge 0$. Note that $\hat{\pi}_{t-1}'(N_{t-1}) \ge \pi_{t-1}'(N_{t-1})$ for all $N_{t-1} \ge 0$ implies that

$$\hat{G}_t'(y) = \mathbb{E}\{r_n'(y+\theta\xi_t+\epsilon_t) + \alpha\hat{\pi}_{t-1}'(y+\theta\xi_t+\epsilon_t)\} \ge \mathbb{E}\{r_n'(y+\theta\xi_t+\epsilon_t) + \alpha\pi_{t-1}'(y+\theta\xi_t+\epsilon_t)\} = G_t'(y),$$

for all y. By Lemma 3(b), $\hat{x}_t(N_t) = \hat{y}_t(N_t) + \Delta^*$, $x_t(N_t) = y_t(N_t) + \Delta^*$, $\hat{y}_t(N_t) = \bar{V}_t - \hat{p}_t(N_t) + \hat{\gamma}(N_t)$, and $y_t(N_t) = \bar{V}_t - p_t(N_t) + \gamma(N_t)$. By Lemma 19(b), we have $\hat{J}_t(\hat{x}_t(N_t),\hat{p}_t(N_t),N_t) = \hat{K}_t(\hat{y}_t(N_t),N_t) + \beta\Delta^* + \Lambda(\Delta^*)$ and $J_t(x_t(N_t),p_t(N_t),N_t) = K_t(y_t(N_t),N_t) + \beta\Delta^* + \Lambda(\Delta^*)$.

We now show $\hat{y}_t(N_t) \ge y_t(N_t)$. Assume, to the contrary, that $\hat{y}_t(N_t) < y_t(N_t)$. Lemma 18 yields that $\partial_{y_t}\hat{K}_t(\hat{y}_t(N_t),N_t) \le \partial_{y_t}K_t(y_t(N_t),N_t)$, i.e.,

$$-2\hat{y}_t(N_t) + \hat{\gamma}(N_t) + \theta\hat{G}_t'(\theta\hat{y}_t(N_t) + \eta N_t) \le -2y_t(N_t) + \gamma(N_t) + \theta G_t'(\theta y_t(N_t) + \eta N_t).$$

Because $\hat{G}_t'(\cdot) \ge G_t'(\cdot)$ and $\hat{y}_t(N_t) < y_t(N_t)$, the concavity of $\hat{G}_t(\cdot)$ and $G_t(\cdot)$ implies that $\hat{G}_t'(\theta\hat{y}_t(N_t) + \eta N_t) \ge G_t'(\theta y_t(N_t) + \eta N_t)$. Since $\hat{\gamma}(N_t) \ge \gamma(N_t)$, we have $-2\hat{y}_t(N_t) \le -2y_t(N_t)$, which contradicts the assumption that $\hat{y}_t(N_t) < y_t(N_t)$. Hence, $\hat{y}_t(N_t) \ge y_t(N_t)$. This completes the proof of part (a).

Part (b). By Lemma 3(b) and part (a), $\hat{x}_t(N_t) = \hat{y}_t(N_t) + \Delta^* \ge y_t(N_t) + \Delta^* = x_t(N_t)$ for all $N_t \ge 0$. This proves part (b).

Part (c). We first show that $\hat{p}_t(0) \le p_t(0)$. Observe that $\hat{p}_t(0) = \bar{V}_t + \hat{\gamma}(0) - \hat{y}_t(0)$ and $p_t(0) = \bar{V}_t + \gamma(0) - y_t(0)$. By part (a), $\hat{y}_t(0) \ge y_t(0)$. Moreover, since $\hat{\gamma}(0) = \gamma(0) = \gamma_0$, $\hat{p}_t(0) \le p_t(0)$. Since $\gamma(\cdot) \equiv \gamma_0$, $p_t(N_t) \equiv p_t(0)$. Moreover, Theorem 2.4.3(a) implies that $\hat{p}_t(N_t)$ is increasing in N_t. The joint concavity of $\hat{J}_t(\cdot,\cdot,\cdot)$ implies that $\hat{p}_t(N_t)$ is continuously increasing in N_t. Thus, let $\mathfrak{N}_t$ be the smallest N_t such that $\hat{p}_t(N_t) \ge p_t(N_t) = p_t(0)$. The monotonicity of $\hat{p}_t(\cdot)$ then suggests that $\hat{p}_t(N_t) \le p_t(N_t)$ if

$N_t \leq \mathfrak{N}_t$, and $\hat{p}_t(N_t) \geq p_t(N_t)$ if $N_t \geq \mathfrak{N}_t$. This proves part (c). *Q.E.D.*

Proof of Theorem 2.4.6: We show Theorem 2.4.6 by backward induction. More specifically, we show that if $\bar{V}_t = \bar{V}_{t-1}$ and $\pi'_{t-1}(N) \geq \pi'_{t-2}(N)$ for all $N \geq 0$, (i) $y_t(N) \geq y_{t-1}(N)$ for all $N \geq 0$, (ii) $p_t(N) \leq p_{t-1}(N)$ for all $N \geq 0$, (iii) $x_t(N) \geq x_{t-1}(N)$ for all $N \geq 0$, and (iv) $\pi'_t(N) \geq \pi'_{t-1}(N)$ for all $N \geq 0$. Since $\pi'_1(N) \geq \pi'_0(N) \equiv 0$ for all N, the initial condition is satisfied.

Note that $\pi'_{t-1}(N) \geq \pi'_{t-2}(N)$ for all $N \geq 0$ implies that

$$G'_t(y) = \mathbb{E}\{r'_n(y+\theta\xi_t+\epsilon_t)+\alpha\pi'_{t-1}(y+\theta\xi_t+\epsilon_t)\} \geq \mathbb{E}\{r'_n(y+\theta\xi_t+\epsilon_t)+\alpha\pi'_{t-2}(y+\theta\xi_t+\epsilon_t)\} = G'_{t-1}(y),$$

for all y. By Lemma 19(b), $J_t(x_t(N), p_t(N), N) = K_t(y_t(N), N) + \beta\Delta^* + \Lambda(\Delta^*)$ and $J_{t-1}(x_{t-1}(N), p_{t-1}(N), N) = K_{t-1}(y_{t-1}(N), N) + \beta\Delta^* + \Lambda(\Delta^*)$.

We first prove that $y_t(N) \geq y_{t-1}(N)$ for all N. Assume, to the contrary, that $y_t(N) < y_{t-1}(N)$ for some N. Lemma 18 implies that $\partial_{y_t} K_t(y_t(N), N) \leq \partial_{y_{t-1}} K_{t-1}(y_{t-1}(N), N)$, i.e.,

$$-2y_t(N) + \gamma(N) + \theta G'_t(\theta y_t(N) + \eta N) \leq -2y_{t-1}(N) + \gamma(N) + \theta G'_{t-1}(\theta y_{t-1}(N) + \eta N).$$

Because $G'_t(\cdot) \geq G'_{t-1}(\cdot)$ for all y and $y_t(N) < y_{t-1}(N)$, the concavity of $G_t(\cdot)$ and $G_{t-1}(\cdot)$ implies that $G'_t(\theta y_t(N) + \eta N) \geq G'_{t-1}(\theta y_{t-1}(N) + \eta N)$. Thus, we have $-2y_t(N) \leq -2y_{t-1}(N)$, which contradicts the assumption that $y_t(N) < y_{t-1}(N)$. Hence, $y_t(N) \geq y_{t-1}(N)$ for all $N \geq 0$. By Theorem 2.4.1(c), it follows immediately that $x_t(N) = y_t(N) + \Delta^* \geq y_{t-1}(N) + \Delta^* = x_{t-1}(N)$ for all $N \geq 0$.

Next, we show that $p_t(N) \leq p_{t-1}(N)$ for all $N \geq 0$. By Lemma 19(a), $J_t(x_t(N), p_t(N), N) = L_t(p_t(N), N) + \beta\Delta^* + \Lambda(\Delta^*)$ and $J_{t-1}(x_{t-1}(N), p_{t-1}(N), N) = L_{t-1}(p_{t-1}(N), N) + \beta\Delta^* + \Lambda(\Delta^*)$. Assume, to the contrary, that $p_t(N) > p_{t-1}(N)$ for some N. Lemma 18 implies that $\partial_{p_t} L_t(p_t(N), N) \geq \partial_{p_{t-1}} L_{t-1}(p_{t-1}(N), N)$, i.e.,

$$\begin{aligned} & -2p_t(N) + \bar{V}_t + c + \gamma(N) - \theta G'_t(\theta(\bar{V}_t - p_t(N) + \gamma(N)) + \eta N) \\ \geq & -2p_{t-1}(N) + \bar{V}_{t-1} + c + \gamma(N) - \theta G'_{t-1}(\theta(\bar{V}_{t-1} - p_{t-1}(N) + \gamma(N)) + \eta N). \end{aligned}$$

Because $G'_t(\cdot) \geq G'_{t-1}(\cdot)$ for all y and $p_t(N) > p_{t-1}(N)$, the concavity of $G_t(\cdot)$ and $G_{t-1}(\cdot)$ implies that $G'_t(\theta(\bar{V}_t - p_t(N) + \gamma(N)) + \eta N) \geq G'_{t-1}(\theta(\bar{V}_{t-1} - p_{t-1}(N) + \gamma(N)) + \eta N)$. Since $\bar{V}_t = \bar{V}_{t-1}$, we have $-2p_t(N) \geq -2p_{t-1}(N)$, which contradicts the assumption that $p_t(N) > p_{t-1}(N)$. Hence, $p_t(N) \leq p_{t-1}(N)$ for all $N \geq 0$.

Finally, to complete the induction, we show that $\pi'_t(N) \geq \pi'_{t-1}(N)$ for all N. By the envelope theorem,

$$\pi'_t(N) = (p_t(N) - c)\gamma'(N) + (\eta + \theta\gamma'(N))G'_t(\theta(\bar{V}_t - p_t(N) + \gamma(N)) + \eta N),$$

and

$$\pi'_{t-1}(N) = (p_{t-1}(N) - c)\gamma'(N) + (\eta + \theta\gamma'(N))G'_{t-1}(\theta(\bar{V}_{t-1} - p_{t-1}(N) + \gamma(N)) + \eta N),$$

If $p_t(N) = p_{t-1}(N)$, $\pi'_t(N) \geq \pi'_{t-1}(N)$ follows immediately from $\gamma'(N) \geq 0$ and $G'_t(\cdot) \geq G'_{t-1}(\cdot)$. If $p_t(N) < p_{t-1}(N)$, Lemma 18 yields that $\partial_{p_t} L_t(p_t(N), N) \leq \partial_{p_{t-1}} L_{t-1}(p_{t-1}(N), N)$, i.e.,

$$\begin{aligned} & -2p_t(N) + \bar{V}_t + c + \gamma(N) - \theta G'_t(\theta(\bar{V}_t - p_t(N) + \gamma(N)) + \eta N) \\ \leq & -2p_{t-1}(N) + \bar{V}_{t-1} + c + \gamma(N) - \theta G'_{t-1}(\theta(\bar{V}_{t-1} - p_{t-1}(N) + \gamma(N)) + \eta N). \end{aligned}$$

Hence, by $\bar{V}_t = \bar{V}_{t-1}$,

$$\begin{aligned} & p_t(N) + \theta G_t'(\theta(\bar{V}_t - p_t(N) + \gamma(N)) + \eta N) \\ \geq & p_{t-1}(N) + \theta G_{t-1}'(\theta(\bar{V}_{t-1} - p_{t-1}(N) + \gamma(N)) + \eta N) + (p_{t-1}(N) - p_t(N)). \end{aligned} \tag{A.5}$$

Since $\theta > 0$, $p_t(N) < p_{t-1}(N)$ implies that $G_t'(\theta(\bar{V}_t - p_t(N) + \gamma(N)) + \eta N) \geq G_{t-1}'(\theta(\bar{V}_{t-1} - p_{t-1}(N) + \gamma(N)) + \eta N)$. Therefore,

$$\begin{aligned} \pi_t'(N) - \pi_{t-1}'(N) = & [(p_t(N) - p_{t-1}(N)) + \theta(G_t'(\theta(\bar{V}_t - p_t(N) + \gamma(N)) + \eta N) \\ & - G_{t-1}'(\theta(\bar{V}_{t-1} - p_{t-1}(N) + \gamma(N)) + \eta N))]\gamma'(N) \\ & + \eta(G_t'(\theta(\bar{V}_t - p_t(N) + \gamma(N)) + \eta N) - G_{t-1}'(\theta(\bar{V}_{t-1} - p_{t-1}(N) + \gamma(N)) + \eta N)) \\ \geq & 0. \end{aligned}$$

Hence, $\pi_t'(N) \geq \pi_{t-1}'(N)$ for all N. This completes the induction and, thus, the proof of Theorem 2.4.6. *Q.E.D.*

Proof of Theorem 2.4.7: We show Theorem 2.4.7 by backward induction. More specifically, we show that if $\hat{\pi}_{t-1}'(\cdot) \geq \pi_{t-1}'(\cdot)$ for all $N_{t-1} \geq 0$ and $\hat{r}_n'(\cdot) \geq r_n'(\cdot)$ for all $N \geq 0$, (i) $\hat{y}_t(N_t) \geq y_t(N_t)$ for all $N_t \geq 0$; (ii) $\hat{x}_t(N_t) \geq x_t(N_t)$ for all $N_t \geq 0$; (iii) $\hat{p}_t(N_t) \leq p_t(N_t)$ for all $N_t \geq 0$; and (iv) $\hat{\pi}_t'(N_t) \geq \pi_t'(N_t)$ for all $N_t \geq 0$. Since $\hat{\pi}_0'(\cdot) = \pi_0'(\cdot) \equiv 0$, the initial condition is satisfied.

Note that $\hat{\pi}_{t-1}'(N_{t-1}) \geq \pi_{t-1}'(N_{t-1})$ for all $N_{t-1} \geq 0$ and $\hat{r}_n'(\cdot) \geq r_n'(\cdot)$ for all $N \geq 0$ imply that

$$\hat{G}_t'(y) = \mathbb{E}\{\hat{r}_n'(y + \theta\xi_t + \epsilon_t) + \alpha\hat{\pi}_{t-1}'(y + \theta\xi_t + \epsilon_t)\} \geq \mathbb{E}\{r_n'(y + \theta\xi_t + \epsilon_t) + \alpha\pi_{t-1}'(y + \theta\xi_t + \epsilon_t)\} = G_t'(y),$$

for all y. By Lemma 19(a), $\hat{J}_t(\hat{x}_t(N_t), \hat{p}_t(N_t), N_t) = \hat{K}_t(\hat{y}_t(N_t), N_t) + \beta\Delta^* + \Lambda(\Delta^*)$ and $J_t(x_t(N_t), p_t(N_t), N_t) = K_t(y_t, N_t) + \beta\Delta^* + \Lambda(\Delta^*)$.

We first show that $\hat{y}_t(N_t) \geq y_t(N_t)$. Assume, to the contrary, that $\hat{y}_t(N_t) < y_t(N_t)$ for some N_t. Lemma 18 yields that $\partial_{y_t}\hat{K}_t(\hat{y}_t(N_t), N_t) \leq \partial_{y_t}K_t(y_t(N_t), N_t)$, i.e.,

$$-2\hat{y}_t(N_t) + \gamma(N_t) + \theta\hat{G}_t'(\theta\hat{y}_t(N_t) + \eta N_t) \leq -2y_t(N_t) + \gamma(N_t) + \theta G_t'(\theta y_t(N_t) + \eta N_t).$$

Because $\hat{G}_t'(\cdot) \geq G_t'(\cdot)$ and $\hat{y}_t(N_t) < y_t(N_t)$, the concavity of $\hat{G}_t(\cdot)$ and $G_t(\cdot)$ implies that $\hat{G}_t'(\theta\hat{y}_t(N_t) + \eta N_t) \geq G_t'(\theta y_t(N_t) + \eta N_t)$. Hence, we have $-2\hat{y}_t(N_t) \leq -2y_t(N_t)$, which contradicts the assumption that $\hat{y}_t(N_t) < y_t(N_t)$. Thus, $\hat{y}_t(N_t) \geq y_t(N_t)$ and, hence, $\hat{x}_t(N_t) = \hat{y}_t(N_t) + \Delta^* \geq y_t(N_t) + \Delta^* = x_t(N_t)$.

Next, we show that $\hat{p}_t(N_t) \leq p_t(N_t)$. By Lemma 19(a), $\hat{J}_t(\hat{x}_t(N_t), \hat{p}_t(N_t), N_t) = \hat{L}_t(\hat{p}_t(N_t), N_t) + \beta\Delta^* + \Lambda(\Delta^*)$ and $J_t(x_t(N_t), p_t(N_t), N_t) = L_t(p_t(N_t), N_t) + \beta\Delta^* + \Lambda(\Delta^*)$ Assume, to the contrary, that $\hat{p}_t(N_t) > p_t(N_t)$ for some N_t. Lemma 18 implies that $\partial_{p_t}\hat{L}_t(\hat{p}_t(N_t), N_t) \geq \partial_{p_t}L_t(p_t(N_t), N_t)$, i.e.,

$$\begin{aligned} & -2\hat{p}_t(N_t) + \bar{V}_t + c + \gamma(N_t) - \theta\hat{G}_t'(\theta(\bar{V}_t - \hat{p}_t(N_t) + \gamma(N_t)) + \eta N_t) \\ \geq & -2p_t(N_t) + \bar{V}_t + c + \gamma(N_t) - \theta G_t'(\theta(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + \eta N_t). \end{aligned}$$

Because $\hat{G}_t'(\cdot) \geq G_t'(\cdot)$ for all y and $\hat{p}_t(N_t) > p_t(N_t)$, the concavity of $\hat{G}_t(\cdot)$ and $G_t(\cdot)$ implies that $\hat{G}_t'(\theta(\bar{V}_t - \hat{p}_t(N_t) + \gamma(N_t)) + \eta N_t) \geq G_t'(\theta(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + \eta N_t)$. We have $-2\hat{p}_t(N_t) \geq -2p_t(N_t)$, which contradicts the assumption that $\hat{p}_t(N_t) > p_t(N_t)$. Hence, $\hat{p}_t(N_t) \geq p_t(N_t)$ for all $N_t \geq 0$.

Finally, to complete the induction, we show that $\hat{\pi}_t'(N_t) \geq \pi_t'(N_t)$ for all N_t. By the envelope theorem,

$$\hat{\pi}_t'(N_t) = (\hat{p}_t(N_t) - c)\gamma'(N_t) + (\eta + \theta\gamma'(N_t))\hat{G}_t'(\theta(\bar{V}_t - \hat{p}_t(N_t) + \gamma(N_t)) + \eta N_t),$$

and

$$\pi_t'(N_t) = (p_t(N_t) - c)\gamma'(N_t) + (\eta + \theta\gamma'(N_t))G_t'(\theta(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + \eta N_t),$$

If $\hat{p}_t(N_t) = p_t(N_t)$, $\hat{\pi}_t'(N_t) \geq \pi_t'(N_t)$ follows immediately from $\gamma'(N) \geq 0$ and $\hat{G}_t'(\cdot) \geq G_t'(\cdot)$ for all y. If $\hat{p}_t(N_t) < p_t(N_t)$, Lemma 18 yields that $\partial_{p_t}\hat{L}_t(\hat{p}_t(N_t), N_t) \leq \partial_{p_t}L_t(p_t(N_t), N_t)$, i.e.,

$$\begin{aligned} & -2\hat{p}_t(N_t) + \bar{V}_t + c + \gamma(N_t) - \theta\hat{G}_t'(\theta(\bar{V}_t - \hat{p}_t(N_t) + \gamma(N_t)) + \eta N_t) \\ \leq & -2p_t(N_t) + \bar{V}_t + c + \gamma(N_t) - \theta G_t'(\theta(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + \eta N_t). \end{aligned}$$

Hence,

$$\begin{aligned} & \hat{p}_t(N_t) + \theta\hat{G}_t'(\theta(\bar{V}_t - \hat{p}_t(N_t) + \gamma(N_t)) + \eta N_t) \\ \geq & p_t(N_t) + \theta G_t'(\theta(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + \eta N_t) + (p_t(N_t) - \hat{p}_t(N_t)). \end{aligned} \tag{A.6}$$

Since $\theta > 0$, $\hat{p}_t(N_t) < p_t(N_t)$ implies that $\hat{G}_t'(\theta(\bar{V}_t - \hat{p}_t(N_t) + \gamma(N_t)) + \eta N_t) \geq G_t'(\theta(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + \eta N_t)$. Therefore, by (A.6),

$$\begin{aligned} \hat{\pi}_t'(N_t) - \pi_t'(N_t) = & [(\hat{p}_t(N_t) - p_t(N_t)) + \theta(\hat{G}_t'(\theta(\bar{V}_t - \hat{p}_t(N_t) + \gamma(N_t)) + \eta N_t) \\ & - G_t'(\theta(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + \eta N_t))]\gamma'(N_t) \\ & + \eta(\hat{G}_t'(\theta(\bar{V}_t - \hat{p}_t(N_t) + \gamma(N_t)) + \eta N_t) - G_t'(\theta(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + \eta N_t)) \\ \geq & 0. \end{aligned}$$

Hence, $\hat{\pi}_t'(N_t) \geq \pi_t'(N_t)$ for all N_t. This completes the induction and, thus, the proof of Theorem 2.4.7. *Q.E.D.*

Proof of Lemma 5: Part (a). The concavity, differentiability, and monotonicity of $\pi_t^d(\cdot)$ and $J_t^d(\cdot,\cdot,\cdot,\cdot)$ follow from the same backward induction argument as the proof of Lemma 2. Hence, we omit the proof of part (a) for brevity.

Part (b). The optimal value function $v_t^d(I_t, N_t)$ satisfies the following recursive scheme:

$$v_t^d(I_t, N_t) = cI_t + \max_{(x_t, p_t^s, p_t^i) \in \mathcal{F}_d(I_t)} J_t^d(x_t, p_t^s, p_t^i, N_t), \tag{A.7}$$

where $\mathcal{F}_d(I_t) := \{(x_t, p_t^s, p_t^i) \in [I_t, +\infty) \times [\underline{p}, \bar{p}] \times [\underline{p}, \bar{p}] : p_t^s \leq p_t^i\}$ denotes the set of feasible decisions with price discrimination and

$$\begin{aligned} J_t^d(x_t, p_t^s, p_t^i, N_t) \quad = \quad & \theta R_t(p_t^s, N_t) + (1-\theta)R_t(p_t^i, N_t) + \beta x_t + \Lambda(x_t - \bar{V}_t + \theta p_t^s + (1-\theta)p_t^i - \gamma(N_t)) \\ & + \Psi_t^d(x_t - \bar{V}_t + \theta p_t^s + (1-\theta)p_t^i - \gamma(N_t), \theta(\bar{V}_t - p_t^s + \gamma(N_t)) + \eta N_t), \qquad \text{(A.8)} \\ \text{with } \Psi_t^d(x, y) \quad := \quad & \mathbb{E}\{\alpha[r_n(y + \theta\xi_t + \epsilon_t) + v_{t-1}^d(x - \xi_t, y + \theta\xi_t + \epsilon_t) - cx]\}. \end{aligned}$$

The derivation of (A.8) is as follows:

$$
\begin{aligned}
J_t^d(x_t, p_t^s, p_t^i, N_t) &:= -cI_t + \mathbb{E}\{p_t^s D_t^s(p_t^s, N_t) + p_t^i D_t^i(p_t^i, N_t) - c(x_t - I_t) - h(x_t - D_t^s(p_t^s, N_t) \\
&\quad -D_t^i(p_t^i, N_t))^+ - b(x_t - D_t^s(p_t^s, N_t) - D_t^i(p_t^i, N_t))^- + r_n(D_t^s(p_t^s, N_t) + \eta N_t + \epsilon_t) \\
&\quad +\alpha v_{t-1}^d(x_t - D_t^s(p_t^s, N_t) - D_t^i(p_t^i, N_t), D_t^s(p_t^s, N_t) + \eta N_t + \epsilon_t)|N_t\}, \\
&= \theta(p_t^s - \alpha c - b)(\bar{V}_t - p_t^s + \gamma(N_t)) + (1-\theta)(p_t^i - \alpha c - b)(\bar{V}_t - p_t^i + \gamma(N_t)) \\
&\quad +(b - (1-\alpha)c)x_t \\
&\quad +\mathbb{E}\{r_n(D_t^s(p_t^s, N_t) + \eta N_t + \epsilon_t) - (h+b)(x_t - \bar{V}_t + \theta p_t^s + (1-\theta)p_t^i - \gamma(N_t) - \xi_t)^+ \\
&\quad +\alpha[v_{t-1}^d(x_t - \bar{V}_t + \theta p_t^s + (1-\theta)p_t^i - \gamma(N_t) - \xi_t, \\
&\quad \theta(\bar{V}_t - p_t^s + \gamma(N_t) + \xi_t) + \eta N_t + \epsilon_t) \\
&\quad -c(x_t - \bar{V}_t + \theta p_t^s + (1-\theta)p_t^i - \gamma(N_t) - \xi_t)]|N_t\} \\
&= \theta R_t(p_t^s, N_t) + (1-\theta)R_t(p_t^i, N_t) + \beta x_t + \Lambda(x_t - \bar{V}_t + \theta p_t^s + (1-\theta)p_t^i - \gamma(N_t)) \\
&\quad +\Psi_t^d(x_t - \bar{V}_t + \theta p_t^s + (1-\theta)p_t^i - \gamma(N_t), \theta(\bar{V}_t - p_t^s + \gamma(N_t)) + \eta N_t).
\end{aligned}
$$

We use $(\hat{x}_t^d(N_t), \hat{p}_t^s(N_t), \hat{p}_t^i(N_t))$ to denote the unconstrained maximizer of (A.8). The same argument as the proof of Lemma 2 yields that $J_t^d(\cdot,\cdot,\cdot,\cdot)$ is jointly concave in (x_t, p_t^s, p_t^i, N_t). Hence, if $I_t \le \hat{x}_t^d(N_t)$, $(x_t^{d*}(I_t, N_t), p_t^{s*}(I_t, N_t), p_t^{i*}(I_t, N_t)) = (\hat{x}_t^d(N_t), \hat{p}_t^s(N_t), \hat{p}_t^i(N_t))$; otherwise $(I_t > \hat{x}_t(N_t))$, $x_t^{d*}(I_t, N_t) = I_t$.

The same argument as the proof of Lemma 3 implies that $\mathbb{P}[\hat{x}_t^d(N_t) - D_t^s(\hat{p}_t^s(N_t), N_t) - D_t^i(\hat{p}_t^i(N_t), N_t) \le \hat{x}_{t-1}^d(N_{t-1})] = 1$. Hence, the same argument as the proof of Lemma 4 yields that $(\hat{x}_t^d(N_t), \hat{p}_t^s(N_t), \hat{p}_t^i(N_t)) = (x_t^d(N_t), p_t^s(N_t), p_t^i(N_t))$ for all $N_t \ge 0$. Thus, if $I_T \le x_T^d(N_T)$, $I_t \le x_t^d(N_t)$ for all t with probability 1. Hence, part (b) follows. *Q.E.D.*

The following lemma is a counterpart of Lemma 19 in the model with price discrimination.

Lemma 20 *For each period t and any network size $N_t \ge 0$, the following statements hold.*

(a) $x_t^d(N_t) = y_t^s(N_t) + y_t^i(N_t) + \Delta^*$, *where Δ^* is the optimal safety stock characterized in Lemma 3(b).*

(b) $J_t^d(x_t^d(N_t), p_t^s(N_t), p_t^i(N_t), N_t) = L_t^s(p_t^s(N_t), N_t) + (1-\theta)\mathcal{R}_t(p_t^i(N_t), N_t) + \beta\Delta^* + \Lambda(\Delta^*)$, *where* $L_t^s(p_t, N_t) := \theta(p_t - c)(\bar{V}_t - p_t + \gamma(N_t)) + G_t^d(\theta(\bar{V}_t - p_t + \gamma(N_t)) + \eta N_t)$, *and* $\mathcal{R}_t(p_t, N_t) := (p_t - c)(\bar{V}_t - p_t + \gamma(N_t))$. *Hence,* $p_t^s(N_t) = \operatorname{argmax}_{p_t^s \in [\underline{p}, \bar{p}]} L_t^s(p_t, N_t)$ *and* $p_t^i(N_t) = \operatorname{argmax}_{p_t^i \in [\underline{p}, \bar{p}]} \mathcal{R}_t(p_t^i, N_t)$.

(c) $J_t^d(x_t^d(N_t), p_t^s(N_t), p_t^i(N_t), N_t) = K_t^s(y_t^s(N_t), N_t) + (1-\theta)\mathcal{R}_t(p_t^i(N_t), N_t) + \beta\Delta^* + \Lambda(\Delta^*)$, *where* $K_t^s(y_t, N_t) := (\bar{V}_t + \gamma(N_t) - \frac{y_t}{\theta} - c)y_t + G_t^d(y_t + \eta N_t)$. *Hence,* $y_t^s(N_t)\operatorname{argmax}_{[y_t^s \in [\underline{y}_t^s(N_t), \bar{y}_t^s(N_t)]} K_t^s(y_t, N_t)$, *where* $\underline{y}_t^s(N_t) := \theta(\bar{V}_t + \gamma(N_t) - \bar{p})$ *and* $\bar{y}_t^s(N_t) := \theta(\bar{V}_t + \gamma(N_t) - \underline{p})$.

(d) Let $m_t^s(N_t) := y_t^s(N_t) + \eta N_t$ *be the optimal expected network size in period $t-1$, given the current network size N_t. We have* $J_t^d(x_t^d(N_t), p_t^s(N_t), p_t^i(N_t), N_t) = M_t^s(m_t^s(N_t), N_t) + (1-\theta)\mathcal{R}_t(p_t^i(N_t), N_t) + \beta\Delta^* + \Lambda(\Delta^*)$, *where* $M_t^s(m_t, N_t) := (\bar{V}_t + \gamma(N_t) - \frac{m_t - \eta N_t}{\theta} - c)(m_t - \eta N_t) + G_t^d(m_t)$. *Hence,* $m_t^s(N_t)\operatorname{argmax}_{m_t^s \in [\underline{m}_t^s(N_t), \bar{m}_t^s(N_t)]} M_t^s(m_t^s, N_t)$, *where* $\underline{m}_t^s(N_t) := \underline{y}_t^s(N_t) + \eta N_t$ *and* $\bar{m}_t^s(N_t) := \bar{y}_t^s(N_t) + \eta N_t$.

Proof of Lemma 20: Part (a). Part (a) follows from the same argument as the proof of Lemma 3(b), so we omit its proof for brevity.

Part (b). By part (a), $x_t^d(N_t) - y_t^s(N_t) - y_t^i(N_t) = \Delta^*$ for all $N_t \geq 0$. By the Bellman equation 2.10, for all N_t,

$$\begin{aligned} J_t^d(x_t^d(N_t), p_t^s(N_t), p_t^i(N_t), N_t) =& \theta R_t(p_t^s(N_t), N_t) + (1-\theta) R_t(p_t^i(N_t), N_t) + \beta x_t^d(N_t) \\ &+ \Lambda(x_t^d(N_t) - y_t^s(N_t) - y_t^i(N_t)) \\ &+ G_t^d(\theta(\bar{V}_t - p_t^s(N_t) + \gamma(N_t)) + \eta N_t). \end{aligned}$$

Therefore,

$$\begin{aligned} J_t^d(x_t^d(N_t), p_t^s(N_t), p_t^i(N_t), N_t) =& \theta R_t(p_t^s(N_t), N_t) + (1-\theta) R_t(p_t^i(N_t), N_t) \\ &+ \beta(\Delta^* + \bar{V}_t - \theta p_t^s(N_t) - (1-\theta) p_t^i(N_t) + \gamma(N_t)) + \Lambda(\Delta^*) \\ &+ G_t^d(\theta(\bar{V}_t - p_t^s(N_t) + \gamma(N_t)) + \eta N_t) \\ =& \theta(p_t^s(N_t) - c)(\bar{V}_t - p_t^s(N_t) + \gamma(N_t)) + (1-\theta)\mathcal{R}_t(p_t^i(N_t), N_t) \\ &+ G_t^d(\theta(\bar{V}_t - p_t^s(N_t) + \gamma(N_t)) + \eta N_t) + \beta\Delta^* + \Lambda(\Delta^*) \\ =& L_t^s(p_t^s(N_t), N_t) + (1-\theta)\mathcal{R}_t(p_t^i(N_t), N_t) + \beta\Delta^* + \Lambda(\Delta^*), \end{aligned} \tag{A.9}$$

where $L_t^s(p_t, N_t) := \theta(p_t - c)(\bar{V}_t - p_t + \gamma(N_t)) + G_t^d(\theta(\bar{V}_t - p_t + \gamma(N_t)) + \eta N_t)$, and $\mathcal{R}_t(p_t, N_t) := (p_t - c)(\bar{V}_t - p_t + \gamma(N_t))$. Since $(x_t^d(N_t), p_t^s(N_t), p_t^i(N_t))$ maximizes $J_t^d(\cdot, \cdot, \cdot, N_t)$ for all N_t, $p_t^s(N_t) = \text{argmax}_{p_t^s \in [\underline{p}, \bar{p}]} L_t^s(p_t^s, N_t)$ and $p_t^i(N_t) = \text{argmax}_{p_t^i \in [\underline{p}, \bar{p}]} \mathcal{R}_t(p_t^i, N_t)$. This proves part (b).

Part (c). Since $y_t^s(N_t) = \theta(\bar{V}_t - p_t^s(N_t) + \gamma(N_t))$ and $\theta > 0$, $p_t^s(N_t) = \bar{V}_t - \frac{y_t^s(N_t)}{\theta} + \gamma(N_t)$. Plug this into (A.9), we have

$$\begin{aligned} J_t^d(x_t^d(N_t), p_t^s(N_t), p_t^i(N_t), N_t) =& \theta(p_t^s(N_t) - c)(\bar{V}_t - p_t^s(N_t) + \gamma(N_t)) + (1-\theta)\mathcal{R}_t(p_t^i(N_t), N_t) \\ &+ G_t^d(\theta(\bar{V}_t - p_t^s(N_t) + \gamma(N_t)) + \eta N_t) + \beta\Delta^* + \Lambda(\Delta^*) \\ =& (\bar{V}_t - \frac{y_t^s(N_t)}{\theta} + \gamma(N_t) - c) y_t^s(N_t) + (1-\theta)\mathcal{R}_t(p_t^i(N_t), N_t) \\ &+ G_t^d(y_t^s(N_t) + \eta N_t) + \beta\Delta^* + \Lambda(\Delta^*) \\ =& K_t^s(y_t^s(N_t), N_t) + (1-\theta)\mathcal{R}_t(p_t^i(N_t), N_t) + \beta\Delta^* + \Lambda(\Delta^*), \end{aligned}$$

where $K_t^s(y_t, N_t) := (\bar{V}_t - \frac{y_t}{\theta} + \gamma(N_t) - c) y_t + G_t^d(y_t + \eta N_t)$. Since $(x_t^d(N_t), p_t^s(N_t), p_t^i(N_t))$ maximizes $J_t^d(\cdot, \cdot, \cdot, N_t)$ for all N_t, $y_t^s(N_t) = \text{argmax}_{y_t^s \in [\underline{y}_t^s(N_t), \bar{y}_t^s(N_t)]} K_t^s(y_t^s, N_t)$. The expressions of $\underline{y}_t^s(N_t)$ and $\bar{y}_t^s(N_t)$ follow immediately from the identity $y_t^s = \theta(\bar{V}_t - p_t^s + \gamma(N_t))$ and that $p_t^s \in [\underline{p}, \bar{p}]$. This proves part (c).

Part (d). Observe that $m_t^s(N_t) = y_t^s(N_t) + \eta N_t$ and $\theta > 0$ imply that $p_t^s(N_t) = \bar{V}_t - \frac{y_t^s(N_t)}{\theta} + \gamma(N_t) = \bar{V}_t + \gamma(N_t) - \frac{m_t^s(N_t)-\eta N_t}{\theta}$. Plug this into (A.9), we have

$$\begin{aligned}
J_t^d(x_t^d(N_t), p_t^s(N_t), p_t^i(N_t), N_t) =& \theta(p_t^s(N_t) - c)(\bar{V}_t - p_t^s(N_t) + \gamma(N_t)) + (1-\theta)\mathcal{R}_t(p_t^i(N_t), N_t) \\
&+ G_t^d(\theta(\bar{V}_t - p_t^s(N_t) + \gamma(N_t)) + \eta N_t) + \beta\Delta^* + \Lambda(\Delta^*) \\
=& (\bar{V}_t + \gamma(N_t) - \frac{m_t^s(N_t) - \eta N_t}{\theta} - c)(m_t^s(N_t) - \eta N_t) + G_t^d(m_t^s(N_t)) \\
&+ \beta\Delta^* + \Lambda(\Delta^*) \\
=& M_t^s(m_t^s(N_t), N_t) + (1-\theta)\mathcal{R}_t(p_t^i(N_t), N_t) + \beta\Delta^* + \Lambda(\Delta^*),
\end{aligned}$$

where $M_t^s(m_t, N_t) := (\bar{V}_t + \gamma(N_t) - \frac{m_t - \eta N_t}{\theta} - c)(m_t - \eta N_t) + G_t^d(m_t)$. Since $(x_t^d(N_t), p_t^s(N_t), p_t^i(N_t))$ maximizes $J_t^d(\cdot,\cdot,\cdot,N_t)$ for all N_t, $m_t^s(N_t) = \text{argmax}_{m_t^s \in [\underline{m}_t^s(N_t), \bar{m}_t^s(N_t)]} M_t^s(m_t^s, N_t)$. The expressions of $\underline{m}_t^s(N_t)$ and $\bar{m}_t^s(N_t)$ follow immediately from the identity $m_t^s = y_t^s + \eta N_t$ and that $y_t^s \in [\underline{y}_t^s(N_t), \bar{y}_t^s(N_t)]$. This establishes part (d). *Q.E.D.*

Proof of Theorem 2.5.1: Part (a). Direct computation yields that $\partial_{p_t} L_t^s(p_t, N_t) = \theta[-2p_t - c + \bar{V}_t + \gamma(N_t) - \partial_y G_t^d(\theta(\bar{V}_t - p_t + \gamma(N_t)) + \eta N_t)]$ and $\partial_{p_t}\mathcal{R}_t(p_t, N_t) = -2p_t - c + \bar{V}_t + \gamma(N_t)$. Since $p_t^i(N_t) > \underline{p}$, the first order condition with respect to p_t implies that $\partial_{p_t}\mathcal{R}_t(p_t^i(N_t), N_t) = 0$, i.e., $-2p_t^i(N_t) - c + \bar{V}_t + \gamma(N_t) = 0$. Hence, $\partial_{p_t} L_t^s(p_t^i(N_t), N_t) = -\theta G_t^d(\theta(\bar{V}_t - p_t^i(N_t) + \gamma(N_t)) + \eta N_t)$. Since $\gamma'(\cdot) > 0$ for all $N_t \geq 0$, $G_t^d(\theta(\bar{V}_t - p_t^i(N_t) + \gamma(N_t)) + \eta N_t) > 0$. Moreover, $\theta > 0$ implies that $\partial_{p_t} L_t^s(p_t^i(N_t), N_t) < 0$. Because $L_t^s(\cdot, N_t)$ is concave in p_t and $p_t^i(N_t) > \underline{p}$, $p_t^s(N_t) = \text{argmax}_{p_t \in [\underline{p}, \bar{p}]} L_t^s(p_t, N_t) < p_t^i(N_t)$. This proves part (a).

Part (b). Assume, to the contrary, that $p_t(N_t) > p_t^i(N_t)$. Lemma 18 yields that $\partial_{p_t} L_t(p_t(N_t), N_t) \geq \partial_{p_t}\mathcal{R}_t(p_t^i(N_t), N_t)$, i.e., $\theta[-2p_t(N_t) - c + \bar{V}_t + \gamma(N_t) - G_t'(\theta(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + \eta N_t)] \geq \theta[-2p_t^i(N_t) - c + \bar{V}_t + \gamma(N_t)]$. $G_t'(\cdot) \geq 0$ implies that $p_t(N_t) \leq p_t^i(N_t)$, which contradicts the assumption that $p_t(N_t) > p_t^i(N_t)$. This proves part (b).

Part (c). Observe that, if $p_\tau^i(\cdot) = p_\tau^s(\cdot) = p_\tau(\cdot)$ for each $\tau \leq t$ and any $N_\tau \geq 0$, $\pi_t^d(N_t) = \pi_t(N_t)$ for all $N_t \geq 0$. Hence, $\pi_t(\cdot)$ is a lower bound for $\pi_t^d(\cdot)$. Now assume that $p_t^i(N_\tau) > p_t^s(N_t)$. Because $p_t^s(\cdot)$ and $p_t^i(\cdot)$ are the lexicographically smallest optimizers, we must have $\pi_t^d(N_t) > \pi_t(N_t)$. Otherwise there are two policies (one with $p_t^s(N_t) = p_t^i(N_t)$ and the other with $p_t^s(N_t) < p_t^i(N_t)$) that are lexicographically different but generate the same optimal profit, which contradicts that the optimal policy $(x_t^d(N_t), p_t^s(N_t), p_t^i(N_t))$ is the lexicographically smallest optimizer. On the other hand, if $\gamma(\cdot) \equiv \gamma_0$ and $r(\cdot) \equiv 0$ for all $N_t \geq 0$, $\partial_{N_t}\pi_t(\cdot) \equiv 0$ for all t and, hence, $p_t^i(\cdot) = p_t^s(\cdot)$ for all t and $N_t \geq 0$. Moreover, since $p_t(\cdot)$ is the optimal pricing policy if the firm charges a single price to all customers in each period t, the optimal price discrimination strategy should be $p_t^i(\cdot) = p_t^s(\cdot) = p_t(\cdot)$ for each t. Hence, $\pi_t^d(N_t) = \pi_t(N_t)$ for all $N_t \geq 0$. This proves part (c). *Q.E.D.*

Proof of Lemma 6: Part (a). Part (a) follows from the same argument as the proof of Lemma 2, so we omit its proof for brevity.

Part (b). The optimal value function $v_t^p(I_t, N_t)$ satisfies the following recursive scheme:

$$v_t^p(I_t, N_t) = cI_t + \max_{(x_t,p_t,n_t)\in\mathcal{F}_p(I_t)} J_t^p(x_t, p_t, n_t, N_t), \tag{A.10}$$

where $\mathcal{F}_p(I_t) := [I_t, +\infty) \times [\underline{p}, \bar{p}] \times [0, +\infty)$ denotes the set of feasible decisions and

$$\begin{aligned} J_t^p(x_t, p_t, n_t, N_t) &= R_t(p_t, N_t) + \beta x_t + \Lambda(x_t - \bar{V}_t + p_t - \gamma(N_t)) - c_n(n_t) \\ &\quad + \Psi_t^p(x_t - \bar{V}_t + p_t - \gamma(N_t), \theta(\bar{V}_t - p_t + \gamma(N_t)) + \eta N_t + n_t), \\ \text{with } \Psi_t^p(x, y) &:= \mathbb{E}\{r_n(y + \theta\xi_t + \epsilon_t) + \alpha v_{t-1}^p(x - \xi_t, y + \theta\xi_t + \epsilon_t) - cx\}. \end{aligned} \tag{A.11}$$

The derivation of (A.11) is given as follows:

$$\begin{aligned} J_t^p(x_t, p_t, n_t, N_t) &:= -cI_t + \mathbb{E}\{p_t D_t(p_t, N_t) - c(x_t - I_t) - h(x_t - D_t(p_t, N_t))^+ - b(x_t - D_t(p_t, N_t))^- \\ &\quad + r_n(\theta D_t(p_t, N_t) + \eta N_t + n_t + \epsilon_t) - c_n(n_t) \\ &\quad + \alpha v_{t-1}^p(x_t - D_t(p_t, N_t), \theta D_t(p_t, N_t) + \eta N_t + n_t + \epsilon_t)|N_t\}, \\ &= (p_t - \alpha c - b)(\bar{V}_t - p_t + \gamma(N_t)) + (b - (1-\alpha)c)x_t - c_n(n_t) \\ &\quad + \mathbb{E}\{r_n(\theta(\bar{V}_t - p_t + \gamma(N_t) + \xi_t) + \eta N_t + n_t + \epsilon_t) \\ &\quad - (h + b)(x_t - \bar{V}_t + p_t - \gamma(N_t) - \xi_t)^+ \\ &\quad + \alpha[v_{t-1}^p(x_t - \bar{V}_t + p_t - \gamma(N_t) - \xi_t, \theta(\bar{V}_t - p_t + \gamma(N_t) + \xi_t) + \eta N_t + n_t + \epsilon_t) \\ &\quad - c(x_t - \bar{V}_t + p_t - \gamma(N_t) - \xi_t)]|N_t\} \\ &= R_t(p_t, N_t) + \beta x_t + \Lambda(x_t - \bar{V}_t + p_t - \gamma(N_t)) - c_n(n_t) \\ &\quad + \Psi_t^p(x_t - \bar{V}_t + p_t - \gamma(N_t), \theta(\bar{V}_t - p_t + \gamma(N_t)) + \eta N_t + n_t). \end{aligned}$$

We use $(\hat{x}_t^p(N_t), \hat{p}_t^p(N_t), \hat{n}_t(N_t))$ as the unconstrained optimizer of (A.11). The same argument as the proof of Lemma 2 yields that $J_t^d(\cdot,\cdot,\cdot,\cdot)$ is jointly concave in (x_t, p_t, n_t, N_t). Hence, if $I_t \le \hat{x}_t^p(N_t)$, $(x_t^{p*}(I_t, N_t), p_t^{p*}(I_t, N_t), n_t^*(I_t, N_t)) = (\hat{x}_t^p(N_t), \hat{p}_t^p(N_t), \hat{n}_t(N_t))$; otherwise $(I_t > \hat{x}_t^p(N_t))$ $x_t^{p*}(I_t, N_t) = I_t$.

The same argument as the proof of Lemma 3 implies that $\mathbb{P}[\hat{x}_t^p(N_t) - D_t(\hat{p}_t^p(N_t), N_t) \le \hat{x}_{t-1}^p(N_{t-1})] = 1$. Hence, the same argument as the proof of Lemma 4 yields that $(\hat{x}_t^p(N_t), \hat{p}_t^p(N_t), \hat{n}_t(N_t)) = (x_t^p(N_t), p_t^p(N_t), n_t(N_t))$ for all $N_t \ge 0$. Thus, if $I_T \le x_T^p(N_T)$, $I_t \le x_t^p(N_t)$ for all t with probability 1. Hence, part (b) follows. *Q.E.D.*

The following lemma is a counterpart of Lemma 19 in the model with network expanding promotion.

Lemma 21 *For each period t and any network size $N_t \ge 0$, the following statements hold.*

(a) $x_t^p(N_t) = y_t^p(N_t) + \Delta^*$, *where Δ^* is the optimal safety stock characterized in Lemma 3(b).*

(b) $J_t^p(x_t^p(N_t), p_t^p(N_t), n_t(N_t), N_t) = L_t^p(p_t^p(N_t), n_t(N_t), N_t) + \beta\Delta^* + \Lambda(\Delta^*)$, *where* $L_t^p(p_t, n_t, N_t) := (p_t - c)(\bar{V}_t - p_t + \gamma(N_t)) - c_n(n_t) + G_t^p(\theta(\bar{V}_t - p_t + \gamma(N_t)) + \eta N_t + n_t)$.
Hence, $(p_t^p(N_t), n_t(N_t)) = \text{argmax}_{(p_t,n_t)\in[\underline{p},\bar{p}]\times[0,+\infty)} L_t^p(p_t, n_t, N_t)$.

(c) $J_t^p(x_t^p(N_t), p_t^p(N_t), n_t(N_t), N_t) = K_t^p(y_t^p(N_t), n_t(N_t), N_t) + \beta\Delta^* + \Lambda(\Delta^*)$, *where* $K_t^p(y_t, n_t, N_t) := (\bar{V}_t + \gamma(N_t) - \frac{y_t}{\theta} - c)y_t - c_n(n_t) + G_t^d(\theta y_t + \eta N_t + n_t)$.

Hence, $(y^p_t(N_t), n_t(N_t)) = \text{argmax}_{(y_t,n_t)\in\{(y_t,n_t):y_t\in[y^s_t\in \underline{y}_t(N_t),\bar{y}_t(N_t)]\}} K^p_t(y_t, n_t, N_t)$*, where* $\underline{y}_t(N_t)$ *and* $\bar{y}_t(N_t)$ *are defined in Lemma 19(b).*

(d) *Let* $m^p_t(N_t) := \theta y^p_t(N_t) + \eta N_t$ *be the optimal expected network size in period* $t-1$*, given the current network size* N_t*. We have* $J^p_t(x^p_t(N_t), p^p_t(N_t), n_t(N_t), N_t) = M^p_t(m^p_t(N_t), n_t(N_t), N_t) + \beta\Delta^* + \Lambda(\Delta^*)$*, where* $M^p_t(m_t, n_t, N_t) := (\bar{V}_t + \gamma(N_t) - \frac{m_t-\eta N_t}{\theta} - c)\frac{(m_t-\eta N_t)}{\theta} - c_n(n_t) + G^p_t(m_t+n_t)$*. Hence,* $(m^p_t(N_t), n_t(N_t)) = \text{argmax}_{(m_t n_t)\in\{(m_t,n_t):m_t\in[\underline{m}_t(N_t),\bar{m}_t(N_t)]\}} M^p_t(m_t, n_t, N_t)$*, where* $\underline{m}_t(N_t)$ *and* $\bar{m}_t(N_t)$ *are defined in Lemma 19(c).*

Proof of Lemma 21: Part (a). Part (a) follows from the same argument as the proof of Lemma 3(b), so we omit its proof for brevity.

Part (b). By part (a), $x^p_t(N_t) - y^p_t(N_t) = \Delta^*$ for all $N_t \geq 0$. By the Bellman equation 2.11, for all N_t,

$$\begin{aligned}J^p_t(x^p_t(N_t), p^p_t(N_t), n_t(N_t), N_t) =& R_t(p^p_t(N_t), N_t) + \beta x^p_t(N_t) - c_n(n_t(N_t))\\ &+ \Lambda(x^p_t(N_t) - y_t(N_t))\\ &+ G^p_t(\theta(\bar{V}_t - p^p_t(N_t) + \gamma(N_t)) + n_t(N_t) + \eta N_t).\end{aligned}$$

Therefore,

$$\begin{aligned}J^p_t(x^p_t(N_t), p^p_t(N_t), n_t(N_t), N_t) =& R_t(p^p_t(N_t), N_t) + \beta x^p_t(N_t) - c_n(n_t(N_t))\\ &+ \Lambda(x^p_t(N_t) - y_t(N_t))\\ &+ G^p_t(\theta(\bar{V}_t - p^p_t(N_t) + \gamma(N_t)) + n_t(N_t) + \eta N_t)\\ =&(p^p_t(N_t) - c)(\bar{V}_t - p^s_t(N_t) + \gamma(N_t)) - c_n(n_t(N_t))\\ &+ G^p_t(\theta(\bar{V}_t - p^p_t(N_t) + \gamma(N_t)) + \eta N_t + n_t(N_t)) + \beta\Delta^* + \Lambda(\Delta^*)\\ =& L^p_t(p^p_t(N_t), n_t(N_t), N_t) + \beta\Delta^* + \Lambda(\Delta^*),\end{aligned} \tag{A.12}$$

where $L^p_t(p_t, n_t, N_t) := (p_t - c)(\bar{V}_t - p_t + \gamma(N_t)) - c_n(n_t) + G^p_t(\theta(\bar{V}_t - p_t + \gamma(N_t)) + \eta N_t + n_t)$. Since $(x^p_t(N_t), p^p_t(N_t), n_t(N_t))$ maximizes $J^p_t(\cdot,\cdot,\cdot,N_t)$ for all N_t,
$(p^p_t(N_t), n_t(N_t)) = \text{argmax}_{(p_t,n_t)\in[\underline{p},\bar{p}]\times[0,+\infty)} L^p_t(p_t, n_t, N_t)$. This proves part (b).

Part (c). Since $y^p_t(N_t) = \bar{V}_t - p^p_t(N_t) + \gamma(N_t)$, $p^p_t(N_t) = \bar{V}_t - y^p_t(N_t) + \gamma(N_t)$. Plug this into (A.12), we have

$$\begin{aligned}J^p_t(x^p_t(N_t), p^p_t(N_t), n_t(N_t), N_t) =& R_t(p^p_t(N_t), N_t) + \beta x^p_t(N_t) - c_n(n_t(N_t))\\ &+ \Lambda(x^p_t(N_t) - y_t(N_t))\\ &+ G^p_t(\theta(\bar{V}_t - p^p_t(N_t) + \gamma(N_t)) + n_t(N_t) + \eta N_t)\\ =&(\bar{V}_t - y^p_t(N_t) + \gamma(N_t) - c)y^p_t(N_t) - c_n(n_t(N_t))\\ &+ G^p_t(\theta y^p_t(N_t) + \eta N_t + n_t(N_t)) + \beta\Delta^* + \Lambda(\Delta^*)\\ =& K^p_t(y^p_t(N_t), n_t(N_t), N_t) + \beta\Delta^* + \Lambda(\Delta^*),\end{aligned}$$

where $K^p_t(y_t, N_t) := (\bar{V}_t - y_t + \gamma(N_t) - c)y_t - c_n(n_t) + G^d_t(\theta y_t + \eta N_t + n_t)$. Since $(x^p_t(N_t), p^p_t(N_t), n_t(N_t))$ maximizes $J^p_t(\cdot,\cdot,\cdot,N_t)$ for all N_t,

$(y_t^p(N_t), n_t(N_t)) = \text{argmax}_{(y_t,n_t)\in\{(y_t,n_t):y_t[\underline{y}_t(N_t),\bar{y}_t(N_t)]\}} K_t^p(y_t, n_t, N_t)$. This proves part (c).

Part (d). Observe that $m_t^p(N_t) = \theta y_t^p(N_t) + \eta N_t$ and $\theta > 0$ imply that $p_t^p(N_t) = \bar{V}_t - y_t^p(N_t) + \gamma(N_t) = \bar{V}_t + \gamma(N_t) - \frac{m_t^p(N_t)-\eta N_t}{\theta}$. Plug this into (A.12), we have

$$\begin{aligned}
J_t^p(x_t^p(N_t), p_t^p(N_t), n_t(N_t), N_t) =& R_t(p_t^p(N_t), N_t) + \beta x_t^p(N_t) - c_n(n_t(N_t)) \\
&+ \Lambda(x_t^p(N_t) - y_t(N_t)) \\
&+ G_t^p(\theta(\bar{V}_t - p_t^p(N_t) + \gamma(N_t)) + n_t(N_t) + \eta N_t) \\
=& (\bar{V}_t + \gamma(N_t) - \frac{m_t^p(N_t) - \eta N_t}{\theta} - c)\frac{(m_t^p(N_t) - \eta N_t)}{\theta} \\
&+ G_t^p(m_t^p(N_t) + n_t(N_t)) + \beta\Delta^* + \Lambda(\Delta^*) \\
=& M_t^p(m_t^p(N_t), n_t(N_t), N_t) + \beta\Delta^* + \Lambda(\Delta^*),
\end{aligned}$$

where $M_t^p(m_t, N_t) := (\bar{V}_t + \gamma(N_t) - \frac{m_t-\eta N_t}{\theta} - c)\frac{(m_t-\eta N_t)}{\theta} - c_n(n_t) + G_t^p(m_t)$. Since $(x_t^p(N_t), p_t^p(N_t), n_t(N_t))$ maximizes $J_t^p(\cdot,\cdot,\cdot,N_t)$ for all N_t, $(m_t^p(N_t), n_t(N_t)) = \text{argmax}_{(m_t,n_t)\in\{(m_t,n_t):m_t\in[\underline{m}_t^s(N_t),\bar{m}_t^s(N_t)]\}} M_t^p(m_t, n_t, N_t)$. This establishes part (d). *Q.E.D.*

Proof of Theorem 2.5.2: Part (a). We first show that if (2.12) holds, $n_t^*(I_t, N) > 0$ for all I_t. Observe that, since $\partial_y \Psi_{t-1}^p(x, y) \geq 0$,

$$\partial_{N_{t-1}} v_{t-1}^p(I_{t-1}, N_{t-1}) \geq (\underline{p} - b - \alpha c)\gamma'(N_{t+1}) - \gamma'(N_{t-1})\Lambda'(w_{t-1}^*),$$

where $w_{t-1}^* = x_{t-1}^*(I_{t-1}, N_{t-1}) - y_{t-1}^*(I_{t-1}, N_{t-1})$. The first-order condition with respect to x_{t-1} yields that $\Lambda'(w_{t-1}^*) \leq -\beta$. Thus,

$$\partial_{N_{t-1}} v_{t-1}(I_{t-1}, N_{t-1}) \geq (\underline{p} - c)\gamma'(N_{t-1}). \tag{A.13}$$

Therefore, for any $x_t \geq I_t$ and $p_t \in [\underline{p}, \bar{p}]$,

$$\begin{aligned}
\partial_{n_t} J_t^p(x_t, p_t, 0, N) \geq& \mathbb{E}\{r_n'(N_{t-1}) + \alpha\partial_{N_{t-1}} v_{t-1}^p(x_t - D_t(p_t, N), N_{t-1})|N_t = N\} - c_n'(0) \\
\geq& \alpha\mathbb{E}\{r_n'(N_{t-1}) + (\underline{p} - c)\gamma'(N_{t-1})|N_t = N\} - c_n'(0) \\
\geq& (1 - \iota)[r_n'(\bar{S}(N)) + \alpha(\underline{p} - c)\gamma'(\bar{S}(N))] - c_n'(0) \\
>& 0,
\end{aligned} \tag{A.14}$$

where the second inequality follows from (A.13), and the fourth from the assumption (2.12). The third inequality of (A.14) follows from the following inequality:

$$\begin{aligned}
\mathbb{E}[r_n'(N_{t-1}) + \alpha(\underline{p} - c)\gamma'(N_{t-1})|N_t = N] &= \mathbb{E}_{N_{t-1}\geq\bar{S}(N)}[r_n'(N_{t-1}) + \alpha(\underline{p} - c)\gamma'(N_{t-1})|N_t = N] \\
&\quad + \mathbb{E}_{N_{t-1}<\bar{S}(N)}[r_n'(N_{t-1}) + \alpha(\underline{p} - c)\gamma'(N_{t-1})|N_t = N] \\
&\geq 0 + \mathbb{E}_{N_{t-1}<\bar{S}(N)}[r_n'(\bar{S}(N)) + \alpha(\underline{p} - c)\gamma'(\bar{S}(N))] \\
&\geq (1 - \iota)[r_n'(\bar{S}(N)) + \alpha(\underline{p} - c)\gamma'(\bar{S}(N))],
\end{aligned}$$

where the first inequality follows from the concavity of $r_n(\cdot)$ and $\gamma(\cdot)$, and the second from the definition of $\bar{S}(N)$. The inequality (A.14) yields that $n_t^*(I_t, N) > 0$ for all I_t.

Since $\gamma(\cdot)$ is continuously increasing in N_t, $\bar{S}(N)$ is continuously increasing in N. The concavity of $r_n(\cdot)$ and $\gamma(\cdot)$ implies that $r_n'(\bar{S}(N))$ and $\gamma'(\bar{S}(N))$ are continuously decreasing in N. Therefore, let

$$N^*(\iota) := \max\{N \geq 0 : (1-\iota)[r_n'(\bar{S}(N)) + \alpha(\underline{p} - c)\gamma'(\bar{S}(N))] > c_n'(0)\}.$$

We have (2.12) holds for all $N < N^*(\iota)$. This completes the proof of part (a).

Part (b). Since $\gamma(\cdot) \equiv \gamma_0$ and $r_n(\cdot)$ is concavely increasing in N_t, $\partial_{N_{t-1}} v_{t-1}^p(I_{t-1}, N_{t-1}) \leq \partial_{N_{t-1}} v_t^p(I_{t-1}, 0) \leq (\sum_{\tau=1}^{t-1}(\alpha^{\tau-1}\eta^\tau) r_n'(0)$. Thus, if $(\sum_{\tau=0}^{t-1}(\alpha\eta)^\tau) r_n'(0) \leq c_n'(0)$,

$$\begin{aligned}
\partial_{n_t} J_t^p(x_t, p_t, n_t, N_t) \leq & \mathbb{E}\{r_n'(n_t) + \alpha\partial_{N_{t-1}} v_{t-1}^p(x_t - D_t(p_t, N), N_{t-1} + n_t)|N_t\} - c_n'(0) \\
\leq & r_n'(0) + \alpha(\sum_{\tau=1}^{t-1}(\alpha^{\tau-1}\eta^\tau)) r_n'(0) - c_n'(0) \\
\leq & (\sum_{\tau=0}^{t-1}(\alpha\eta)^\tau) r_n'(0) - c_n'(0) \\
\leq & 0.
\end{aligned}$$

Hence, $n_t^*(I_t, N_t) = 0$ for all (I_t, N_t). This completes the proof of part (b). *Q.E.D.*

Proof of Theorem 2.5.3: Parts (a)-(c). We prove parts (a)-(c) together by backward induction. More specifically, we show that if $\partial_{N_{t-1}}\pi_{t-1}^p(\cdot) \leq \partial_{N_{t-1}}\pi_{t-1}(\cdot)$ for all $N_{t-1} \geq 0$, (i) $p_t^p(N_t) \geq p_t(N_t)$, (ii) $y_t^p(N_t) \leq y_t(N_t)$, (iii) $x_t^p(N_t) \leq x_t(N_t)$, and (iv) $\partial_{N_t}\pi_t^p(\cdot) \leq \partial_{N_t}\pi_t(\cdot)$ for all $N_t \geq 0$. Since $\partial_{N_0}\pi_0^p(\cdot) = \partial_{N_0}\pi_0(\cdot) \equiv 0$, the initial condition is satisfied.

We first show that $y_t^p(N_t) \leq y_t(N_t)$. Note that $\partial_{N_{t-1}}\pi_{t-1}^p(N_{t-1}) \leq \partial_{N_{t-1}}\pi_{t-1}(N_{t-1})$ for all $N_{t-1} \geq 0$ implies that

$$\begin{aligned}
\partial_y G_t^p(y) = & \mathbb{E}\{r_n'(y + \theta\xi_t + \epsilon_t) + \alpha\partial_{N_{t-1}}\pi_{t-1}^p(y + \theta\xi_t + \epsilon_t)\} \\
\leq & \mathbb{E}\{r_n'(y + \theta\xi_t + \epsilon_t) + \alpha\partial_{N_{t-1}}\pi_{t-1}(y + \theta\xi_t + \epsilon_t)\} \\
= & \partial_y G_t(y),
\end{aligned}$$

for all y. By Lemma 19(b) and Lemma 21(c), $J_t^p(x_t^p(N_t), p_t^p(N_t), n_t(N_t), N_t) = K_t^p(y_t^p(N_t), n_t(N_t), N_t) + \beta\Delta^* + \Lambda(\Delta^*)$ and $J_t(x_t(N_t), p_t(N_t), N_t) = K_t(y_t(N_t), N_t) + \beta\Delta^* + \Lambda(\Delta^*)$. Assume, to the contrary, that $y_t^p(N_t) > y_t(N_t)$ for some N_t. Lemma 18 yields that $\partial_{y_t} K_t^p(y_t^p(N_t), n_t(N_t), N_t) \geq \partial_{y_t} K_t(y_t(N_t), N_t)$, i.e.,

$$-2y_t^p(N_t) + \gamma(N_t) + \theta\partial_y G_t^p(\theta y_t^p(N_t) + \eta N_t + n_t(N_t)) \geq -2y_t(N_t) + \gamma(N_t) + \theta\partial_y G_t(\theta y_t(N_t) + \eta N_t).$$

Since $y_t^p(N_t) > y_t(N_t)$,

$$\partial_y G_t^p(\theta y_t^p(N_t) + \eta N_t + n_t(N_t)) > \partial_y G_t(\theta y_t(N_t) + \eta N_t). \tag{A.15}$$

Because $\partial_y G_t(\cdot) \leq \partial_y G_t^p(\cdot)$ and $y_t^p(N_t) > y_t(N_t)$, the concavity of $G_t^p(\cdot)$ and $G_t(\cdot)$ implies that $\theta y_t^p(N_t) + \eta N_t + n_t(N_t) < \theta y_t(N_t) + \eta N_t$. However, $n_t(N_t) \geq 0$ and $y_t^p(N_t) > y_t(N_t)$ imply that $\theta y_t^p(N_t) + \eta N_t + n_t(N_t) > \theta y_t(N_t) + \eta N_t$, which forms a contradiction. Thus, $y_t^p(N_t) \leq y_t(N_t)$ for all $N_t \geq 0$. Hence,

$x_t^p(N_t) = y_t^p(N_t) + \Delta^* \leq y_t(N_t) + \Delta^* = x_t(N_t)$ and $p_t^p(N_t) = \bar{V}_t - y_t^p(N_t) + \gamma(N_t) \geq \bar{V}_t - y_t(N_t) + \gamma(N_t) = p_t(N_t)$.

Finally, to complete the induction, we show that $\partial_{N_t}\pi_t^p(N_t) \geq \partial_{N_t}\pi_t(N_t)$ for all $N_t \geq 0$. By the envelope theorem,

$$\partial_{N_t}\pi_t^p(N_t) = (p_t^p(N_t) - c)\gamma'(N_t) + (\eta + \theta\gamma'(N_t))\partial_y G_t^p(\theta(\bar{V}_t - p_t^p(N_t) + \gamma(N_t)) + \eta N_t + n_t(N_t)),$$

and

$$\partial_{N_t}\pi_t(N_t) = (p_t(N_t) - c)\gamma'(N_t) + (\eta + \theta\gamma'(N_t))\partial_y G_t(\theta(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + \eta N_t).$$

If $p_t^p(N_t) = p_t(N_t)$, $\partial_{N_t}\pi_t^p(N_t) \leq \partial_{N_t}\pi_t(N_t)$ follows immediately from $\gamma'(N) \geq 0$ and $\partial_y G_t^p(\cdot) \leq \partial_y G_t(\cdot)$ for all y.

If $p_t^p(N_t) > p_t(N_t)$, Lemma 18 yields that $\partial_{p_t} L_t^p(p_t^p(N_t), n_t(N_t), N_t) \geq \partial_{p_t} L_t(p_t(N_t), N_t)$, i.e.,

$$\begin{aligned}&-2p_t^p(N_t) + \bar{V}_t + c + \gamma(N_t) - \theta\partial_y G_t^p(\theta(\bar{V}_t - p_t^p(N_t) + \gamma(N_t)) + \eta N_t + n_t(N_t))\\ \geq &-2p_t(N_t) + \bar{V}_t + c + \gamma(N_t) - \theta\partial_y G_t(\theta(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + \eta N_t).\end{aligned}$$

Hence,

$$\begin{aligned}&p_t^p(N_t) - c + \theta\partial_y G_t^p(\theta(\bar{V}_t - p_t^p(N_t) + \gamma(N_t)) + \eta N_t + n_t(N_t))\\ \leq &p_t(N_t) - c + \theta\partial_y G_t(\theta(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + \eta N_t) + (p_t(N_t) - p_t^p(N_t)).\end{aligned} \tag{A.16}$$

Since $\theta > 0$, $p_t^p(N_t) > p_t(N_t)$ implies that $\partial_y G_t^p(\theta(\bar{V}_t - p_t^p(N_t) + \gamma(N_t)) + \eta N_t + n_t(N_t)) \leq \partial_y G_t(\theta(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + \eta N_t)$. Therefore,

$$\begin{aligned}\partial_{N_t}\pi_t^p(N_t) - \partial_{N_t}\pi_t(N_t) =&[(p_t^p(N_t) - p_t(N_t)) + \theta(\partial_y G_t^p(\theta(\bar{V}_t - p_t^p(N_t) + \gamma(N_t)) + \eta N_t + n_t(N_t))\\ &- \partial_y G_t(\theta(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + \eta N_t))]\gamma'(N_t)\\ &+ \eta(\partial_y G_t^p(\theta(\bar{V}_t - p_t^p(N_t) + \gamma(N_t)) + \eta N_t + n_t(N_t))\\ &- \partial_y G_t(\theta(\bar{V}_t - p_t(N_t) + \gamma(N_t)) + \eta N_t))\\ \leq&0.\end{aligned}$$

Hence, $\partial_{N_t}\pi_t^p(N_t) \leq \partial_{N_t}\pi_t(N_t)$ for all $N_t \geq 0$. This completes the induction and, thus, the proof of parts (a)-(c).

Part (d). Note that $\pi_t(\cdot)$ is the normalized optimal profit with the Bellman equation (2.9) and feasible decision set $\{(x_t, p_t, n_t) : x_t \geq 0, p_t \in [\underline{p}, \bar{p}], n_t = 0\} \subset \mathcal{F}_p$, which is the feasible decision set associated with the profit $\pi_t^p(\cdot)$. Thus, $\pi_t^p(N_t) \geq \pi_t(N_t)$ for all t and any $N_t \geq 0$. If $n_t(N_t) > 0$, we must have $\pi_t^p(N_t) > \pi_t(N_t)$. Otherwise there are two lexicographically different policies (one with $n_t(N_t) = 0$ and the other with $n_t(N_t) > 0$) that generate the same optimal normalized profit $\pi_t(N_t)$. This contradicts the assumption that the lexicographically smallest policy is selected. Thus, $\pi_t^p(N_t) > \pi_t(N_t)$, which establishes part (d). *Q.E.D.*

A.2 More Conditions on Assumption 2.3.1

Assumption 2.3.1 is essential to show the analytical results in this paper. Thus, we characterize the conditions under which this assumption is satisfied in the following lemma.

Lemma 22 *The following statements hold:*

(a) If $R_t(\cdot,\cdot)$ is jointly concave on its domain, then we have:

(i) For any N_t such that $\gamma'(N_t)=0$, $\gamma''(N_t)=0$ as well. Thus, there exists a threshold $N^*\geq 0$, such that $\gamma'(N_t)\begin{cases}>0, & \text{if } N_t<N^*,\\ =0, & \text{otherwise,}\end{cases}$ *and* $\gamma_l''(N_t)\begin{cases}<0, & \text{if } N_t<N^*,\\ =0, & \text{otherwise.}\end{cases}$

(ii) There exists a constant $0<M<+\infty$ such that, for any $N_t\geq 0$, $(\gamma'(N_t))^2\leq -M\gamma_l''(N_t)$.

(b) If there exists a constant $0<M<+\infty$ such that, for any $N_t\geq 0$, $(\gamma'(N_t))^2\leq -M\gamma_l''(N_t)$, then we have:

(i) There exists a threshold $\delta^*<+\infty$ such that, for any $\delta\geq\delta^*$, with $\bar{V}_t^\delta:=\bar{V}_t+\delta$, $\underline{p}^\delta=\underline{p}+\delta$, and $\bar{p}^\delta=\bar{p}+\delta$, $R_t^\delta(p_t,N_t):=(p_t-b-\alpha c)(\bar{V}_t^\delta-p_t+\gamma(N_t))$ is jointly concave in (p_t,N_t) for $p_t\in[\underline{p}^\delta,\bar{p}^\delta]$ and $N_t\geq 0$.

(ii) For any network externalities function $\gamma(\cdot)$, there exists an threshold $0<\varsigma^*<+\infty$ such that, for any $\varsigma\geq\varsigma^*$, with $\gamma^\varsigma(\cdot):=\gamma^\varsigma(\cdot)/\varsigma$, $R_t^\varsigma(p_t,N_t):=(p_t-b-\alpha c)(\bar{V}_t-p_t+\gamma^\varsigma(N_t))$ is jointly concave in (p_t,N_t) for $p_t\in[\underline{p},\bar{p}]$ and $N_t\geq 0$.

Part (a) characterizes a simpler necessary condition for the joint concavity of $R_t(\cdot,\cdot)$. It implies that $R_t(\cdot,\cdot)$ is jointly concave only if, for all N_t, $|\gamma''(N_t)|$ is sufficiently big compared with $\gamma'(N_t)$. In other words, in the region where network externalities exist (i.e., $\gamma'(N_t)>0$), the curvature of $\gamma(\cdot)$ should be sufficiently big. Part (b) shows that if the necessary condition characterized by part (a) is satisfied, $R_t(\cdot,\cdot)$ is jointly concave if (i) p_t is sufficiently big relative to the expected demand $\bar{V}_t-p_t+\gamma(N_t)$; or (ii) $\gamma'(\cdot)$ is sufficiently small. Hence, the necessary conditions characterized in part (a) are also sufficient to some extent. The sufficient conditions in part (b) have a clear economic interpretation: the price elasticity of demand (i.e., $|(\,\mathrm{d}\mathbb{E}[D_t(p_t,N_t)]/\mathbb{E}[D_t(p_t,N_t)])/(\,\mathrm{d}p_t/p_t)|$) is sufficiently big relative to the network size elasticity of demand (i.e., $|(\,\mathrm{d}\mathbb{E}[D_t(p_t,N_t)]/\mathbb{E}[D_t(p_t,N_t)])/(\,\mathrm{d}N_t/N_t)|$). This condition is generally satisfied in practice, because, compared with the primary demand leverage (i.e., sales price), network externalities have less impact upon demand in general.

Proof of Lemma 22: Part (a-i). If $\gamma''(N_t)=0$, the left-hand-side of (2.3) equals 0. Moreover, the right-hand-side of (2.3) is greater than or equal to 0 to ensure the joint concavity of $R_t(\cdot,\cdot)$ (see Lemma 1). Hence, the right-hand-side of (2.3) has to be 0. Thus, $\gamma'(N_t)=0$ for this case. For the second half of **part (a-i)**, it suffices to show that if $\gamma'(N^0)=0$ then $\gamma'(N_t)=0$ for all $N_t\geq N^0$. Since $\gamma''(N_t)\leq 0$ for all $N_t\geq 0$, $\gamma'(N_t)\leq\gamma'(N^0)=0$. On the other hand, $\gamma'(N_t)\geq 0$ for all $N_t\geq 0$. Thus, $\gamma'(N_t)=0$ for all $N_t\geq N^0$.

Part (a-ii). By part (a-i), for any N_t, $\gamma''(N_t)=0$, $\gamma'(N_t)=0$ as well. Thus, $(\gamma'(N_t))^2\leq -M\gamma''(N_t)$ for any $0<M<+\infty$. We now consider the case $\gamma''(N_t)<0$. By (2.3), define $M:=2(\underline{p}-\alpha c-b)>0$, the joint concavity of $R_t(\cdot,\cdot)$ implies that $-M\gamma''(N_t)\geq(\gamma'(N_t))^2$. This establishes part (a).

Part (b-i). By Lemma 1, $R_t^{\delta}(\cdot,\cdot)$ is jointly concave if and only if $-2(\underline{p}^{\delta} - \alpha c - b)\gamma''(N_t) \geq (\gamma'(N_t))^2$ for all $N_t \geq 0$. We define $\delta^* := \frac{M}{2} - \underline{p} + \alpha c + b$. Hence, if $\delta \geq \delta^*$, $-2(\underline{p}^{\delta} - \alpha c - b) \leq -M$. Therefore,

$$-2(\underline{p}^{\delta} - \alpha c - b)\gamma''(N_t) \geq -M\gamma''(N_t) \geq (\gamma'(N_t))^2$$

for all $N_t \geq 0$, where the last inequality follows from the assumption that $-M\gamma''(N_t) \geq (\gamma'(N_t))^2$ for all $N_t \geq 0$. Part (b-i) follows.

Part (b-ii). Note that $\partial_{N_t}\gamma^{\varsigma}(N_t) = \gamma'(N_t)/\varsigma$ and $\partial^2_{N_t}\gamma^{\varsigma}(N_t) = \gamma''(N_t)/\varsigma$ for any $\varsigma > 0$ and $N_t \geq 0$. Thus, by Lemma 1, $R_t^{\varsigma}(\cdot,\cdot)$ is jointly concave if and only if

$$-2(\underline{p} - \alpha c - b)\frac{\gamma''(N_t)}{\varsigma} \geq \frac{(\gamma'(N_t))^2}{\varsigma^2} \Longleftrightarrow -2\varsigma(\underline{p} - \alpha c - b)\gamma''(N_t) \geq (\gamma'(N_t))^2.$$

Define $\varsigma^* = \frac{M}{2(\underline{p} - \alpha c - b)} > 0$. We have, if $\varsigma \geq \varsigma^*$,

$$-2\varsigma(\underline{p} - \alpha c - b)\gamma''(N_t) \geq -M\gamma''(N_t) \geq (\gamma'(N_t))^2,$$

where the last inequality follows from the assumption that $-M\gamma''(N_t) \geq (\gamma'(N_t))^2$ for all N_t. Hence, $R_t^{\varsigma}(\cdot,\cdot)$ is jointly concave if $\varsigma \geq \varsigma^*$. *Q.E.D.*

B. Appendix for Chapter 3

B.1 Proofs of Statements

We use ∂ to denote the derivative operator of a single variable function, and ∂_x to denote the partial derivative operator of a multi-variable function with respect to variable x. For any multivariate continuously differentiable function $f(x_1, x_2, \cdots, x_n)$ and $\tilde{x} := (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)$ in $f(\cdot)$'s domain, $\forall i$, we use $\partial_{x_i} f(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)$ to denote $\partial_{x_i} f(x_1, x_2, \cdots, x_n)|_{x=\tilde{x}}$. The following lemma is used throughout our proof.

Lemma 23 *Let $G_i(z, Z)$ be a continuously differentiable function in (z, Z), where $z \in [\underline{z}, \bar{z}]$ ($\underline{z}$ and $\bar{z}$ might be infinite) and $Z \in \mathbb{R}^{n_i}$ for $i = 1, 2$. For $i = 1, 2$, let $(z_i, Z_i) := \mathrm{argmax}_{(z,Z)} G_i(z, Z)$ be the optimizers of $G_i(\cdot, \cdot)$. If $z_1 < z_2$, we have: $\partial_z G_1(z_1, Z_1) \leq \partial_z G_2(z_2, Z_2)$.*

Proof: $z_1 < z_2$, so $\underline{z} \leq z_1 < z_2 \leq \bar{z}$. Hence, $\partial_z G_1(z_1, Z_1) \begin{cases} = 0 & \text{if } z_1 > \underline{z}, \\ \leq 0 & \text{if } z_1 = \underline{z}; \end{cases}$

and $\partial_z G_2(z_2, Z_2) \begin{cases} = 0 & \text{if } z_2 < \bar{z}, \\ \geq 0 & \text{if } z_2 = \bar{z}, \end{cases}$ i.e., $\partial_z G_1(z_1, Z_1) \leq 0 \leq \partial_z G_2(z_2, Z_2)$. *Q.E.D.*

Proof of Theorems 3.4.1-3.4.2 and Propositions 3.4.1-3.4.2: We show Theorem 3.4.1, Proposition 3.4.1, Proposition 3.4.2, and Theorem 3.4.2 together by backward induction. More specifically, we show that, if $V_{i,t-1}(I_{t-1}, \Lambda_{t-1}|\sigma_{t-1}^{sc*}) = w_{i,t-1} I_{i,t-1} + \beta_{i,t-1}^{sc} \Lambda_{i,t-1}$ for all i, (a) Proposition 3.4.1 holds for period t, (b) Proposition 3.4.2 holds for period t, (c) there exists a Markov strategy profile $\{(\gamma_{i,t}^{sc*}(\cdot,\cdot), p_{i,t}^{sc*}(\cdot,\cdot), x_{i,t}^{sc*}(\cdot,\cdot)) : 1 \leq i \leq N\}$ which forms a Nash equilibrium in the subgame of period t, (d) under conditions (i) and (ii) in Theorem 3.4.1(c), the Nash equilibrium in the subgame of period t, $\{(\gamma_{i,t}^{sc*}(\cdot,\cdot), p_{i,t}^{sc*}(\cdot,\cdot), x_{i,t}^{sc*}(\cdot,\cdot)) : 1 \leq i \leq N\}$, is unique, and (e) there exists a positive vector β_t^{sc}, such that $V_{i,t}(I_t, \Lambda_t|\sigma_t^{sc*}) = w_{i,t} I_{i,t} + \beta_{i,t}^{sc} \Lambda_{i,t}$ for all i. Because $V_{i,0}(I_0, \Lambda_0) = w_{i,0} I_{i,0}$ for all i, the initial condition is satisfied.

Since $V_{i,t-1}(I_{t-1}, \Lambda_{t-1}|\sigma_{t-1}^{sc*}) = w_{i,t-1} I_{i,t-1} + \beta_{i,t-1}^{sc} \Lambda_{i,t-1}$ for all i, Equation (3.12) implies that the objective function of player i in $\mathcal{G}_t^{sc,2}$ is

$$\pi_{i,t}^{sc}(y_t) = (\delta_i w_{i,t-1} - w_{i,t}) y_{i,t} - L_{i,t}(y_{i,t}) + \delta_i \beta_{i,t-1}^{sc} (\kappa_{ii,t}(\mathbb{E}[y_{i,t}^+ \wedge \xi_{i,t}]) - \sum_{j \neq i} \kappa_{ij,t}(\mathbb{E}[y_{j,t}^+ \wedge \xi_{j,t}])).$$

Thus, for any given strategy of other players $y_{-i,t}$, player i maximizes the following univariate function:

$$\zeta_{i,t}^{sc}(y_{i,t}) := (\delta_i w_{i,t-1} - w_{i,t}) y_{i,t} - L_{i,t}(y_{i,t}) + \delta_i \beta_{i,t-1}^{sc} \kappa_{ii,t}(\mathbb{E}[y_{i,t}^+ \wedge \xi_{i,t}]).$$

If $y_{i,t} < 0$, $(y_{i,t} - \xi_{i,t})^+ = 0$, $(y_{i,t} - \xi_{i,t})^- = \xi_{i,t} - y_{i,t}$, and, thus, $-L_{i,t}(y_{i,t}) = -b_{i,t}\mathbb{E}(\xi_{i,t} - y_{i,t}) = -b_{i,t} + b_{i,t}y_{i,t}$. Moreover, $y_{i,t} < 0$ implies that $\delta_i\beta^{sc}_{i,t-1}\kappa_{ii,t}(\mathbb{E}[y^+_{i,t} \wedge \xi_{i,t}]) \equiv \delta_i\beta^{sc}_{i,t-1}\kappa_{ii,t}(0)$. Hence, if $y_{i,t} < 0$,

$$\zeta^{sc}_{i,t}(y_{i,t}) = -b_{i,t} + (\delta_i w_{i,t-1} - w_{i,t} + b_{i,t})y_{i,t} + \delta_i\beta^{sc}_{i,t-1}\kappa_{ii,t}(0).$$

Because $b_{i,t} > w_{i,t} - \delta_i w_{i,t-1}$, $\zeta^{sc}_{i,t}(\cdot)$ is strictly increasing in $y_{i,t}$ for $y_{i,t} \leq 0$.

Observe that $-L_{i,t}(\cdot)$ is concave and continuously differentiable in $y_{i,t}$. Since $\mathbb{E}(y^+_{i,t}\wedge\xi_{i,t})$ is concavely increasing and continuously differentiable in $y_{i,t}$ for $y_{i,t} \geq 0$, and $\kappa_{ii,t}(\cdot)$ is concavely increasing and continuously differentiable, $\kappa_{ii,t}(\mathbb{E}[y^+_{i,t} \wedge \xi_{i,t}])$ is concavely increasing and continuously differentiable in $y_{i,t}$ for $y_{i,t} \geq 0$. Hence, $\zeta^{sc}_{i,t}(\cdot)$ is concave and continuously differentiable in $y_{i,t}$ for $y_{i,t} \geq 0$. Observe that

$$\partial_{y_{i,t}}\zeta^{sc}_{i,t}(0+) = \delta_i w_{i,t-1} - w_{i,t} + b_{i,t} + \delta_i\beta^{sc}_{i,t-1}\bar{F}_{i,t}(0)\kappa'_{ii,t}(\mathbb{E}(0\wedge\xi_{i,t})) = \delta_i w_{i,t-1} - w_{i,t} + b_{i,t} + \delta_i\beta^{sc}_{i,t-1}\kappa'_{ii,t}(0) > 0,$$

where the inequality follows from $\delta_i w_{i,t-1} - w_{i,t} + b_{i,t} > 0$ and $\kappa'_{ii,t}(0) \geq 0$. Therefore, the optimizer of $\zeta^{sc}_{i,t}(\cdot)$, $y^{sc*}_{i,t}$, is the solution to the first-order condition: $\partial_{y_{i,t}}\zeta^{sc}_{i,t}(y^{sc*}_{i,t}) = 0$, or, equivalently,

$$(\delta_i w_{i,t-1} - w_{i,t}) - L'_{i,t}(y^{sc*}_{i,t}) + \delta_i\beta^{sc}_{i,t-1}\bar{F}_{i,t}(y^{sc*}_{i,t})\kappa'_{ii,t}(\mathbb{E}(y^{sc*}_{i,t} \wedge \xi_{i,t})) = 0.$$

Because $\xi_{i,t}$ is continuously distributed, $y^{sc*}_{i,t}$ is unique for each i. Moreover, $y^{sc*}_{i,t} > 0$ and $\zeta^{sc}_{i,t}(y^{sc*}_{i,t}) > \zeta^{sc}_{i,t}(0) = -b_{i,t} + \delta_i\beta^{sc}_{i,t-1}\kappa_{ii,t}(0)$ for each i.

We now show that Proposition 3.4.2 holds for period t. Since $\zeta^{sc}_{i,t}(y^{sc*}_{i,t}) > \zeta^{sc}_{i,t}(0) = -b_{i,t} + \delta_i\beta^{sc}_{i,t-1}\kappa_{ii,t}(0)$ and $\alpha_{i,t}(z_t) \geq \kappa_{ii,t}(0) - \sum_{j\neq i}\kappa_{ij,t}(1) \geq 0$, we have $\pi^{sc*}_{i,t} > \zeta^{sc}_{i,t}(0) - \delta_i\beta^{sc}_{i,t-1}\sum_{j\neq i}\kappa_{ij,t}(1) \geq -b_{i,t}$. Observe that

$$\bar{p}_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{sc*}_{i,t} > \bar{p}_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\bar{\gamma}_{i,t}) - b_{i,t} > 0.$$

Thus, if $p_{i,t} = \bar{p}_{i,t}$, $p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{sc*}_{i,t} > 0$. Therefore, each firm i could at least earn a positive payoff of $(\bar{p}_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\bar{\gamma}_{i,t}) - b_{i,t})\underline{\epsilon}_{i,t}$ by charging the maximum allowable price $\bar{p}_{i,t}$, where

$$\underline{\epsilon}_{i,t} := \min\{\psi_{i,t}(\gamma_t)\rho_{i,t}(p_t) : \gamma_t \in [0, \bar{\gamma}_{1,t}] \times \cdots \times [0, \bar{\gamma}_{N,t}] \times [\underline{p}_{1,t}, \bar{p}_{1,t}] \times \cdots \times [\underline{p}_{N,t}, \bar{p}_{N,t}]\} > 0.$$

Let

$$\bar{\epsilon}_{i,t} := \max\{\psi_{i,t}(\gamma_t)\rho_{i,t}(p_t) : \gamma_t \in [0, \bar{\gamma}_{1,t}] \times \cdots \times [0, \bar{\gamma}_{N,t}] \times [\underline{p}_{1,t}, \bar{p}_{1,t}] \times \cdots \times [\underline{p}_{N,t}, \bar{p}_{N,t}]\} \geq \underline{\epsilon}_{i,t}.$$

Hence, we can restrict the feasible action set of firm i in $\mathcal{G}^{sc,1}_t$ to

$$\begin{aligned}\mathcal{A}^{sc,1}_{i,t} := \{(\gamma_{i,t}, p_{i,t}) \in [0, \bar{\gamma}_{i,t}] \times [\underline{p}_{i,t}, \bar{p}_{i,t}] : p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{sc*}_{i,t} \\ \geq \frac{(\bar{p}_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\bar{\gamma}_{i,t}) - b_{i,t})\underline{\epsilon}_{i,t}}{\bar{\epsilon}_{i,t}} > 0\},\end{aligned}$$

which is a nonempty and complete sublattice of $\mathbb{R}^2$. Thus, $\Pi^{sc}_{i,t}(\gamma_t, p_t) > 0$ and

$$\log(\Pi^{sc}_{i,t}(\gamma_t, p_t)) = \log(p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{sc*}_{i,t}) + \log(\psi_{i,t}(\gamma_t)) + \log(\rho_{i,t}(p_t)) \tag{B.1}$$

is well-defined on $\mathcal{A}^{sc,1}_{i,t}$. Because $\rho_{i,t}(\cdot)$ and $\psi_{i,t}(\cdot)$ satisfy (3.3) and (3.4), for each i and $j \neq i$, we have

$$\frac{\partial^2 \log(\Pi^{sc}_{i,t}(\gamma_t, p_t))}{\partial\gamma_{i,t}\partial p_{i,t}} = \frac{\partial^2 \log(p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{sc*}_{i,t})}{\partial\gamma_{i,t}\partial p_{i,t}} = \frac{\nu'_{i,t}(\gamma_{i,t})}{(p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{sc*}_{i,t})^2} \geq 0,$$

$$\frac{\partial^2 \log(\Pi^{sc}_{i,t}(\gamma_t,p_t))}{\partial\gamma_{i,t}\partial p_{j,t}} = 0,\ \frac{\partial^2 \log(\Pi^{sc}_{i,t}(\gamma_t,p_t))}{\partial\gamma_{i,t}\partial\gamma_{j,t}} = \frac{\partial^2 \log(\psi_{i,t}(\gamma_t))}{\partial\gamma_{i,t}\partial\gamma_{j,t}} \geq 0,$$
$$\frac{\partial^2 \log(\Pi^{sc}_{i,t}(\gamma_t,p_t))}{\partial p_{i,t}\partial\gamma_{j,t}} = 0,\ \text{and}\ \frac{\partial^2 \log(\Pi^{sc}_{i,t}(\gamma_t,p_t))}{\partial p_{i,t}\partial p_{j,t}} = \frac{\partial^2 \log(\rho_{i,t}(p_t))}{\partial p_{i,t}\partial p_{j,t}} \geq 0.$$

Hence, $\mathcal{G}^{sc,1}_t$ is a log-supermodular game and, thus, has pure strategy Nash equilibria which are the smallest and largest undominated strategies (see Theorem 5 in [124]).

Next, we show that if conditions (i) and (ii) in Theorem 3.4.1(c) hold, the Nash equilibrium of $\mathcal{G}^{sc,1}_t$ is unique. First, we show that under conditions (i) and (ii) in Theorem 3.4.1(c),

$$\frac{\partial^2 \log \Pi^{sc}_{i,t}(\gamma_t,p_t)}{\partial p^2_{i,t}} < 0,\ |\frac{\partial^2 \log \Pi^{sc}_{i,t}(\gamma_t,p_t)}{\partial p^2_{i,t}}| > \sum_{j\neq i}\frac{\partial^2 \log(\Pi^{sc}_{i,t}(\gamma_t,p_t))}{\partial p_{i,t}\partial p_{j,t}} + \sum_{j=1}^{N}\frac{\partial^2 \log(\Pi^{sc}_{i,t}(\gamma_t,p_t))}{\partial p_{i,t}\partial\gamma_{j,t}}, \quad \text{(B.2)}$$

$$\frac{\partial^2 \log \Pi^{sc}_{i,t}(\gamma_t,p_t)}{\partial \gamma^2_{i,t}} < 0,\ \text{and}\ |\frac{\partial^2 \log \Pi^{sc}_{i,t}(\gamma_t,p_t)}{\partial \gamma^2_{i,t}}| > \sum_{j\neq i}\frac{\partial^2 \log(\Pi^{sc}_{i,t}(\gamma_t,p_t))}{\partial \gamma_{i,t}\partial\gamma_{j,t}} + \sum_{j=1}^{N}\frac{\partial^2 \log(\Pi^{sc}_{i,t}(\gamma_t,p_t))}{\partial\gamma_{i,t}\partial p_{j,t}}. \quad \text{(B.3)}$$

Note that, by (B.1) and (3.4),

$$\frac{\partial^2 \log \Pi^{sc}_{i,t}(\gamma_t,p_t)}{\partial p^2_{i,t}} = \frac{\partial^2 \log \rho_{i,t}(p_t)}{\partial p^2_{i,t}} - \frac{1}{(p_{i,t}-\delta_i w_{i,t-1}-\nu_{i,t}(\gamma_{i,t})+\pi^{sc*}_{i,t})^2} < 0,$$

and

$$|\frac{\partial^2 \log \Pi^{sc}_{i,t}(\gamma_t,p_t)}{\partial p^2_{i,t}}| = |\frac{\partial^2 \log \rho_{i,t}(p_t)}{\partial p^2_{i,t}}| + \frac{1}{(p_{i,t}-\delta_i w_{i,t-1}-\nu_{i,t}(\gamma_{i,t})+\pi^{sc*}_{i,t})^2}.$$

Since $\frac{\partial^2 \log(\Pi^{sc}_{i,t}(\gamma_t,p_t))}{\partial p_{i,t}\partial\gamma_{j,t}} = 0$ for $j \neq i$, and

$$\frac{\partial^2 \log(\Pi^{sc}_{i,t}(\gamma_t,p_t))}{\partial p_{i,t}\partial\gamma_{i,t}} = \frac{\nu'_{i,t}(\gamma_{i,t})}{(p_{i,t}-\delta_i w_{i,t-1}-\nu_{i,t}(\gamma_{i,t})+\pi^{sc*}_{i,t})^2},$$

we have

$$\begin{aligned}
|\frac{\partial^2 \log \Pi^{sc}_{i,t}(\gamma_t,p_t)}{\partial p^2_{i,t}}| &= |\frac{\partial^2 \log \rho_{i,t}(p_t)}{\partial p^2_{i,t}}| + \frac{1}{(p_{i,t}-\delta_i w_{i,t-1}-\nu_{i,t}(\gamma_{i,t})+\pi^{sc*}_{i,t})^2} \\
&> \sum_{j\neq i}\frac{\partial^2 \log(\Pi^{sc}_{i,t}(\gamma_t,p_t))}{\partial p_{i,t}\partial p_{j,t}} + \frac{\nu'_{i,t}(\gamma_{i,t})}{(p_{i,t}-\delta_i w_{i,t-1}-\nu_{i,t}(\gamma_{i,t})+\pi^{sc*}_{i,t})^2} \\
&= \sum_{j\neq i}\frac{\partial^2 \log(\Pi^{sc}_{i,t}(\gamma_t,p_t))}{\partial p_{i,t}\partial p_{j,t}} + \sum_{j=1}^{N}\frac{\partial^2 \log(\Pi^{sc}_{i,t}(\gamma_t,p_t))}{\partial p_{i,t}\partial\gamma_{j,t}},
\end{aligned}$$

where the inequality follows from (3.4) and condition (i). Hence, (B.2) holds for all i and all (γ_t, p_t).

Since $\nu''_{i,t}(\cdot) \geq 0$ and (3.3), we have

$$\frac{\partial^2 \log \Pi^{sc}_{i,t}(\gamma_t,p_t)}{\partial \gamma^2_{i,t}} = \frac{\partial^2 \log \psi_{i,t}(\gamma_t)}{\partial \gamma^2_{i,t}} - \frac{\nu''_{i,t}(\gamma_t)(p_{i,t}-\delta_i w_{i,t-1}-\nu_{i,t}(\gamma_{i,t})+\pi^{sc*}_{i,t})+(\nu'_{i,t}(\gamma_t))^2}{(p_{i,t}-\delta_i w_{i,t-1}-\nu_{i,t}(\gamma_{i,t})+\pi^{sc*}_{i,t})^2} < 0,$$

and

$$|\frac{\partial^2 \log \Pi^{sc}_{i,t}(\gamma_t,p_t)}{\partial \gamma^2_{i,t}}| = |\frac{\partial^2 \log \psi_{i,t}(\gamma_t)}{\partial \gamma^2_{i,t}}| + \frac{\nu''_{i,t}(\gamma_t)(p_{i,t}-\delta_i w_{i,t-1}-\nu_{i,t}(\gamma_{i,t})+\pi^{sc*}_{i,t})+(\nu'_{i,t}(\gamma_t))^2}{(p_{i,t}-\delta_i w_{i,t-1}-\nu_{i,t}(\gamma_{i,t})+\pi^{sc*}_{i,t})^2}.$$

Since $\frac{\partial^2 \log(\Pi^{sc}_{i,t}(\gamma_t,p_t))}{\partial\gamma_{i,t}\partial p_{j,t}} = 0$ for $j \neq i$, and

$$\frac{\partial^2 \log(\Pi^{sc}_{i,t}(\gamma_t,p_t))}{\partial\gamma_{i,t}\partial p_{i,t}} = \frac{\nu'_{i,t}(\gamma_{i,t})}{(p_{i,t}-\delta_i w_{i,t-1}-\nu_{i,t}(\gamma_{i,t})+\pi^{sc*}_{i,t})^2},$$

we have

$$
\begin{aligned}
|\frac{\partial^2 \log \Pi_{i,t}^{sc}(\gamma_t, p_t)}{\partial \gamma_{i,t}^2}| &= |\frac{\partial^2 \log \psi_{i,t}(\gamma_t)}{\partial \gamma_{i,t}^2}| + \frac{\nu_{i,t}''(\gamma_t)(p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{sc*}) + (\nu_{i,t}'(\gamma_t))^2}{(p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{sc*})^2} \\
&> \sum_{j \neq i} \frac{\partial^2 \log(\Pi_{i,t}^{sc}(\gamma_t, p_t))}{\partial \gamma_{i,t} \partial \gamma_{j,t}} + \frac{\nu_{i,t}''(\gamma_t)(p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \underline{c}_{i,t}) + (\nu_{i,t}'(\gamma_t))^2}{(p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{sc*})^2} \\
&\geq \sum_{j \neq i} \frac{\partial^2 \log(\Pi_{i,t}^{sc}(\gamma_t, p_t))}{\partial \gamma_{i,t} \partial \gamma_{j,t}} + \frac{\nu_{i,t}'(\gamma_{i,t})}{(p_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{sc*})^2} \\
&= \sum_{j \neq i} \frac{\partial^2 \log(\Pi_{i,t}^{sc}(\gamma_t, p_t))}{\partial \gamma_{i,t} \partial \gamma_{j,t}} + \sum_{j=1}^{N} \frac{\partial^2 \log(\Pi_{i,t}^{sc}(\gamma_t, p_t))}{\partial \gamma_{i,t} \partial p_{j,t}},
\end{aligned}
$$

where the first inequality follows from (3.4) and $\pi_{i,t}^{sc*} \geq \underline{c}_{i,t}$, and the second from condition (ii). Hence, (B.3) holds for all i and all (γ_t, p_t).

We now show that if (B.2) and (B.3) hold, $\mathcal{G}_t^{sc,1}$ has a unique Nash equilibrium. Recall that the set of Nash equilibria in $\mathcal{G}_t^{sc,1}$ forms a complete lattice (see Theorem 2 in [194]). If, to the contrary, there exist two distinct equilibria (γ_t^*, p_t^*) and $(\hat{\gamma}_t^*, \hat{p}_t^*)$, where $\hat{p}_{i,t}^* \geq p_{i,t}^*$ for all i and $\hat{\gamma}_{j,t}^* \geq \gamma_{j,t}^*$ for all j, with the inequality being strict for some i or j. If, for some i, $\hat{p}_{i,t}^* > p_{i,t}^*$, $\hat{p}_{i,t}^* - p_{i,t}^* \geq \hat{p}_{l,t}^* - p_{l,t}^*$ for all l, and $\hat{p}_{i,t}^* - p_{i,t}^* \geq \hat{\gamma}_{l,t}^* - \gamma_{l,t}^*$ for all l, without loss of generality, we assume that $i = 1$. Lemma 23 suggests that

$$
\partial_{p_{1,t}} \log(\Pi_{1,t}^{sc}(\hat{\gamma}_t^*, \hat{p}_t^*)) \geq \partial_{p_{1,t}} \log(\Pi_{1,t}^{sc}(\gamma_t^*, p_t^*)). \tag{B.4}
$$

On the other hand, by Newton-Leibniz formula, we have

$$
\begin{aligned}
& \partial_{p_{1,t}} \log(\Pi_{1,t}^{sc}(\hat{\gamma}_t^*, \hat{p}_t^*)) - \partial_{p_{1,t}} \log(\Pi_{1,t}^{sc}(\gamma_t^*, p_t^*)) \\
= & \int_{s=0}^{1} [\sum_{j=1}^{N} (\hat{p}_{j,t}^* - p_{j,t}^*) \frac{\partial^2 \log(\Pi_{1,t}^{sc}((1-s)\gamma_t^* + s\hat{\gamma}_t^*, (1-s)p_t^* + s\hat{p}_t^*))}{\partial p_{1,t} \partial p_{j,t}} \\
& + \sum_{j=1}^{N} (\hat{\gamma}_{j,t}^* - \gamma_{j,t}^*) \frac{\partial^2 \log(\Pi_{1,t}^{sc}((1-s)\gamma_t^* + s\hat{\gamma}_t^*, (1-s)p_t^* + s\hat{p}_t^*))}{\partial p_{1,t} \partial \gamma_{j,t}}] \, \mathrm{d}s \\
\leq & \int_{s=0}^{1} [\sum_{j=1}^{N} (\hat{p}_{1,t}^* - p_{1,t}^*) \frac{\partial^2 \log(\Pi_{1,t}^{sc}((1-s)\gamma_t^* + s\hat{\gamma}_t^*, (1-s)p_t^* + s\hat{p}_t^*))}{\partial p_{1,t} \partial p_{j,t}} \\
& + \sum_{j=1}^{N} (\hat{p}_{1,t}^* - p_{1,t}^*) \frac{\partial^2 \log(\Pi_{1,t}^{sc}((1-s)\gamma_t^* + s\hat{\gamma}_t^*, (1-s)p_t^* + s\hat{p}_t^*))}{\partial p_{1,t} \partial \gamma_{j,t}}] \, \mathrm{d}s \\
< & \; 0,
\end{aligned}
$$

where the first inequality follows from $\hat{p}_{1,t}^* - p_{1,t}^* \geq \hat{p}_{l,t}^* - p_{l,t}^*$ for all l and $\hat{p}_{1,t}^* - p_{1,t}^* \geq \hat{\gamma}_{l,t}^* - \gamma_{l,t}^*$ for all l, and the second from $\hat{p}_{1,t}^* - p_{1,t}^* > 0$ and (B.2). This contradicts (B.4).

If, for some j, $\hat{\gamma}_{j,t}^* > \gamma_{j,t}^*$, $\hat{\gamma}_{j,t}^* - \gamma_{j,t}^* \geq \hat{p}_{l,t}^* - p_{l,t}^*$ for all l, and $\hat{\gamma}_{j,t}^* - \gamma_{j,t}^* \geq \hat{\gamma}_{l,t}^* - \gamma_{l,t}^*$ for all l, without loss of generality, we assume that $j = 1$. Lemma 23 suggests that

$$
\partial_{\gamma_{1,t}} \log(\Pi_{1,t}^{sc}(\hat{\gamma}_t^*, \hat{p}_t^*)) \geq \partial_{\gamma_{1,t}} \log(\Pi_{1,t}^{sc}(\gamma_t^*, p_t^*)). \tag{B.5}
$$

On the other hand, by Newton-Leibniz formula, we have

$$
\begin{aligned}
& \partial_{\gamma_{1,t}} \log(\Pi^{sc}_{1,t}(\hat{\gamma}^*_t, \hat{p}^*_t)) - \partial_{\gamma_{1,t}} \log(\Pi^{sc}_{1,t}(\gamma^*_t, p^*_t)) \\
= & \int_{s=0}^{1} [\sum_{j=1}^{N} (\hat{\gamma}^*_{j,t} - \gamma^*_{j,t}) \frac{\partial^2 \log(\Pi^{sc}_{1,t}((1-s)\gamma^*_t + s\hat{\gamma}^*_t, (1-s)p^*_t + s\hat{p}^*_t))}{\partial\gamma_{1,t}\partial\gamma_{j,t}} \\
& + \sum_{j=1}^{N} (\hat{p}^*_{j,t} - p^*_{j,t}) \frac{\partial^2 \log(\Pi^{sc}_{1,t}((1-s)\gamma^*_t + s\hat{\gamma}^*_t, (1-s)p^*_t + s\hat{p}^*_t))}{\partial\gamma_{1,t}\partial p_{j,t}}] \,\mathrm{d}s \\
\leq & \int_{s=0}^{1} [\sum_{j=1}^{N} (\hat{\gamma}^*_{1,t} - \gamma^*_{1,t}) \frac{\partial^2 \log(\Pi^{sc}_{1,t}((1-s)\gamma^*_t + s\hat{\gamma}^*_t, (1-s)p^*_t + s\hat{p}^*_t))}{\partial\gamma_{1,t}\partial\gamma_{j,t}} \\
& + \sum_{j=1}^{N} (\hat{\gamma}^*_{1,t} - \gamma^*_{1,t}) \frac{\partial^2 \log(\Pi^{sc}_{1,t}((1-s)\gamma^*_t + s\hat{\gamma}^*_t, (1-s)p^*_t + s\hat{p}^*_t))}{\partial\gamma_{1,t}\partial p_{j,t}}] \,\mathrm{d}s \\
< & \; 0,
\end{aligned}
$$

where the first inequality follows from $\hat{\gamma}^*_{1,t} - \gamma^*_{1,t} \geq \hat{p}^*_{l,t} - p^*_{l,t}$ for all l and $\hat{\gamma}^*_{1,t} - \gamma^*_{1,t} \geq \hat{\gamma}^*_{l,t} - \gamma^*_{l,t}$ for all l, and the second from $\hat{\gamma}^*_{1,t} - \gamma^*_{1,t} > 0$ and (B.3). This contradicts (B.5). Therefore, the Nash equilibrium in $\mathcal{G}^{sc,1}_t$ is unique, if conditions (i) and (ii) in Theorem 3.4.1(c) hold.

If $\nu_{i,t}(\gamma_{i,t}) = \gamma_{i,t}$, we have $\nu'_{i,t}(\gamma_{i,t}) = 1$ and $\nu''_{i,t}(\gamma_{i,t}) = 0$ for all $\gamma_{i,t} \in [0, \bar{\gamma}_{i,t}]$. Thus, if $\nu_{i,t}(\gamma_{i,t}) = \gamma_{i,t}$, conditions (i) and (ii) in Theorem 3.4.1(c) hold.

Note that for any $\lambda \in [0,1]$ and $(\gamma_{i,t}, p_{i,t}), (\hat{\gamma}_{i,t}, \hat{p}_{i,t}) \in [0, \bar{\gamma}_{1,t}] \times [0, \bar{\gamma}_{2,t}] \times \cdots \times [0, \bar{\gamma}_{N,t}] \times [\underline{p}_{1,t}, \bar{p}_{1,t}] \times [\underline{p}_{2,t}, \bar{p}_{2,t}] \times \cdots \times [\underline{p}_{N,t}, \bar{p}_{N,t}]$,

$$
\begin{aligned}
& \lambda \log(\hat{p}_{i,t} - \delta_i w_{i,t} - \nu_{i,t}(\hat{\gamma}_{i,t}) + \pi^{sc*}_{i,t}) + (1-\lambda)\log(p_{i,t} - \delta_i w_{i,t} - \nu_{i,t}(\gamma_{i,t}) + \pi^{sc*}_{i,t}) \\
\leq & \log(\lambda\hat{p}_{i,t} + (1-\lambda)p_{i,t} - \delta_i w_{i,t} - \lambda\nu_{i,t}(\hat{\gamma}_{i,t}) - (1-\lambda)\nu_{i,t}(\gamma_{i,t}) + \pi^{sc*}_{i,t}) \\
\leq & \log(\lambda\hat{p}_{i,t} + (1-\lambda)p_{i,t} - \delta_i w_{i,t} - \nu_{i,t}(\lambda\hat{\gamma}_{i,t} + (1-\lambda)\gamma_{i,t}) + \pi^{sc*}_{i,t}),
\end{aligned}
$$

where the first inequality follows from the concavity of $\log(\cdot)$, and the second from that $\log(\cdot)$ is an increasing function and $\nu_{i,t}(\cdot)$ is a convex function. Thus, $\log(p_{i,t} - \delta_i w_{i,t} - \nu_{i,t}(\gamma_{i,t}) + \pi^{sc*}_{i,t})$ is jointly concave in $(\gamma_{i,t}, p_{i,t})$. Hence, the diagonal dominance condition (3.3) and (3.4) implies that $\log(\Pi^{sc}_{i,t}(\gamma_t, p_t))$ is jointly concave in $(\gamma_{i,t}, p_{i,t})$ for any given $(\gamma_{-i,t}, p_{-i,t})$. Therefore, the first-order conditions with respect to $\gamma_{i,t}$ and $p_{i,t}$ is the necessary and sufficient condition for $(\gamma^{sc*}_t, p^{sc*}_t)$ to be the unique Nash equilibrium in $\mathcal{G}^{sc,1}_t$. Since

$$
\partial_{\gamma_{i,t}} \log(\Pi^{sc}_{i,t}(\gamma_t, p_t)) = \frac{\partial_{\gamma_{i,t}}\psi_{i,t}(\gamma_t)}{\psi_{i,t}(\gamma_t)} - \frac{\nu'_{i,t}(\gamma_t)}{p_{i,t} - \delta w_{i,t} - \nu_{i,t}(\gamma_{i,t}) + \pi^{sc*}_{i,t}},
$$

and

$$
\partial_{p_{i,t}} \log(\Pi^{sc}_{i,t}(\gamma_t, p_t)) = \frac{\partial_{p_{i,t}}\rho_{i,t}(p_t)}{\rho_{i,t}(p_t)} + \frac{1}{p_{i,t} - \delta w_{i,t} - \nu_{i,t}(\gamma_{i,t}) + \pi^{sc*}_{i,t}},
$$

the Nash equilibrium of $\mathcal{G}^{sc,1}_t$ is a solution to the system of equations (3.15). Since $\mathcal{G}^{sc,1}_t$ has a unique equilibrium, (3.15) has a unique solution, which coincides with the unique pure strategy Nash equilibrium of $\mathcal{G}^{sc,1}_t$. As shown above, for all i,

$$
\Pi^{sc}_{i,t}(\gamma^{sc*}_t, p^{sc*}_t) \geq (\bar{p}_{i,t} - \delta_i w_{i,t-1} - \nu_{i,t}(\bar{\gamma}_{i,t}) - b_{i,t})\underline{\epsilon}_{i,t} > 0.
$$

Hence, $\Pi^{sc*}_{i,t} = \Pi^{sc}_{i,t}(\gamma^{sc*}_t, p^{sc*}_t) > 0$ for all i.

Next, we show that $\{(\gamma_{i,t}^{sc*}, p_{i,t}^{sc*}, \Lambda_{i,t} y_{i,t}^{sc*} \rho_{i,t}(p_t^{sc*})\psi_{i,t}(\gamma_t^{sc*})) : 1 \leq i \leq N\}$ is an equilibrium in the subgame of period t. Since $y_{i,t}^{sc*} > 0$, $\Lambda_{i,t} y_{i,t}^{sc*} \rho_{i,t}(p_t^{sc*})\psi_{i,t}(\gamma_t^{sc*}) > 0$ for all i. Therefore, regardless of the starting inventory in period t, $I_{i,t}$, firm i could adjust its inventory to $x_{i,t}^{sc*}(I_t, \Lambda_t) = \Lambda_{i,t} y_{i,t}^{sc*} \rho_{i,t}(p_t^{sc*})\psi_{i,t}(\gamma_t^{sc*})$. Thus, by Propositions 3.4.1-3.4.2, $\{(\gamma_{i,t}^{sc*}, p_{i,t}^{sc*}, \Lambda_{i,t} y_{i,t}^{sc*} \rho_{i,t}(p_t^{sc*})\psi_{i,t}(\gamma_t^{sc*})) : 1 \leq i \leq N\}$ forms an equilibrium in the subgame of period t. In particular, if conditions (i) and (ii) hold, $\{(\gamma_{i,t}^{sc*}, p_{i,t}^{sc*}, \Lambda_{i,t} y_{i,t}^{sc*} \rho_{i,t}(p_t^{sc*})\psi_{i,t}(\gamma_t^{sc*})) : 1 \leq i \leq N\}$ is the unique equilibrium in the subgame of period t.

Next, we show that there exists a positive vector $\beta_t^{sc} = (\beta_{1,t}^{sc}, \beta_{2,t}^{sc}, \cdots, \beta_{N,t}^{sc})$, such that $V_{i,t}(I_t, \Lambda_t|\sigma_t^{sc*}) = w_{i,t} I_{i,t} + \beta_{i,t}^{sc}\Lambda_{i,t}$. By (3.12), we have that $V_{i,t}(I_t, \Lambda_t|\sigma_t^{sc*}) = J_{i,t}(\gamma_t^{sc*}, p_t^{sc*}, \Lambda_{i,t} y_{i,t}^{sc*} \rho_{i,t}(p_t^{sc*})\psi_{i,t}(\gamma_t^{sc*}), I_t, \Lambda_t|\sigma_{t-1}^{sc*}) = w_{i,t} I_{i,t} + (\sigma_i \beta_{i,t-1}^{sc}\mu_{i,t} + \Pi_{i,t}^{sc*})\Lambda_{i,t}$. Since $\beta_{i,t-1}^{sc} \geq 0$ and $\Pi_{i,t}^{sc*} > 0$, $\beta_{i,t}^{sc} = \delta_i \beta_{i,t-1}^{sc}\mu_{i,t} + \Pi_{i,t}^{sc*} > 0$. This completes the induction and, thus, the proof of Theorem 3.4.1, Proposition 3.4.1, Proposition 3.4.2, and Theorem 3.4.2. *Q.E.D.*

Proof of Proposition 3.4.3: By Theorems 3.4.1-3.4.2, and Propositions 3.4.1-3.4.2, it suffices to show that, if there exists a constant $\beta_{s,t-1}^{sc} \geq 0$, such that $V_{i,t-1}(I_{t-1}, \Lambda_{t-1}|\sigma_{t-1}^{sc*}) = w_{s,t} I_{i,t-1} + \beta_{s,t-1}^{sc}\Lambda_{i,t-1}$ for all i, we have: (a) the unique Nash equilibrium in $\mathcal{G}_t^{sc,2}$ is symmetric, i.e., $y_{i,t}^{sc*} = y_{j,t}^{sc*}$ for all i, j; (b) the unique Nash equilibrium in $\mathcal{G}_t^{sc,1}$ is symmetric, i.e., $(\gamma_{i,t}^{sc*}, p_{i,t}^{sc*}) = (\gamma_{j,t}^{sc*}, p_{j,t}^{sc*})$ for all $i \neq j$, and (c) there exists a constant $\beta_{s,t}^{sc} > 0$, such that $V_{i,t}(I_t, \Lambda_t|\sigma_{s,t}^{sc*}) = w_{s,t} I_{i,t} + \beta_{s,t}^{sc}\Lambda_{i,t}$ for all i. Since $V_{i,0}(I_t, \Lambda_t) = w_{s,0} I_{i,0}$ for all i, the initial condition is satisfied with $\beta_{s,0}^{sc} = 0$.

Since $V_{i,t-1}(I_{t-1}, \Lambda_{t-1}|\sigma_{t-1}^{sc*}) = w_{s,t} I_{i,t-1} + \beta_{s,t}^{sc}\Lambda_{i,t-1}$ for all i, by (3.12),

$$\pi_{i,t}^{sc}(y_t) = (\delta_s w_{s,t-1} - w_{s,t})y_{i,t} - L_{s,t}(y_{i,t}) + \delta_s \beta_{s,t-1}^{sc}(\kappa_{sa,t}(\mathbb{E}(y_{i,t}^+ \wedge \xi_{i,t})) - \sum_{j \neq i} \kappa_{sb,t}(\mathbb{E}(y_{j,t}^+ \wedge \xi_{j,t}))).$$

Hence, $\zeta_{i,t}^{sc}(y_{i,t}) = (\delta_s w_{s,t-1} - w_{s,t})y_{i,t} - L_{s,t}(y_{i,t}) + \delta_s \beta_{s,t-1}^{sc}\kappa_{sa,t}(\mathbb{E}(y_{i,t}^+ \wedge \xi_{i,t}))$. Thus, $\zeta_{i,t}^{sc}(\cdot) \equiv \zeta_{j,t}^{sc}(\cdot)$ for all i and j. Therefore, for all i and j,

$$y_{i,t}^{sc*} = \text{argmax}_y \zeta_{i,t}^{sc}(y) = \text{argmax}_y \zeta_{j,t}^{sc}(y) = y_{j,t}^{sc*}$$

and, hence,

$$\pi_{i,t}^{sc*} = \pi_{i,t}^{sc}(y_t^{sc*}) = \pi_{j,t}^{sc}(y_t^{sc*}) = \pi_{j,t}^{sc*}.$$

We denote $y_{s,t}^{sc*} = y_{i,t}^{sc*}$ for each i, and $\pi_{s,t}^{sc*} = \pi_{i,t}^{sc*}$ for each i. Observe that, the objective functions of $\mathcal{G}_t^{sc,1}$,

$$\{\Pi_{i,t}^{sc}(\gamma_t, p_t) = \rho_{s,t}(p_t)\psi_{s,t}(\gamma_t)[p_{i,t} - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma_{i,t}) + \pi_{s,t}^{sc*}] : 1 \leq i \leq N\}$$

are symmetric. Hence, if there exists an asymmetric Nash equilibrium $(\gamma_t^{sc*}, p_t^{sc*})$, there exists another Nash equilibrium $(\underline{\gamma}_t^{sc*}, \underline{p}_t^{sc*}) \neq (\gamma_t^{sc*}, p_t^{sc*})$, where $\underline{\gamma}_t^{sc*}$ is a permutation of γ_t^{sc*} and $\underline{p}_t^{sc*}$ is a permutation of p_t^{sc*}. This contradicts the uniqueness of the Nash equilibrium in $\mathcal{G}_t^{sc,1}$. Thus, the unique Nash equilibrium in $\mathcal{G}_t^{sc,1}$ is symmetric. Hence, $\Pi_{i,t}^{sc*} = \Pi_{i,t}^{sc}(\gamma_{ss,t}^{sc*}, p_{ss,t}^{sc*}) = \rho_{s,t}(p_{ss,t}^{sc*})\psi_{s,t}(\gamma_{ss,t}^{sc*})[p_{s,t}^{sc*} - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma_{s,t}^{sc*}) + \pi_{s,t}^{sc*}] = \Pi_{j,t}^{sc}(\gamma_{ss,t}^{sc*}, p_{ss,t}^{sc*}) = \Pi_{j,t}^{sc*}$, which is positive. Thus, we denote the payoff of each firm i as $\Pi_{s,t}^{sc*}$. By Theorem 3.4.2(a),

$$\beta_{i,t}^{sc} = \delta_s \beta_{s,t-1}^{sc}\mu_{s,t} + \Pi_{i,t}^{sc*} = \delta_s \beta_{s,t-1}^{sc}\mu_{s,t} + \Pi_{j,t}^{sc*} = \beta_{j,t}^{sc} > 0.$$

Thus, we denote the SC market size coefficient of each firm i as $\beta^{sc}_{s,t}$. This completes the induction and, thus, the proof of Proposition 3.4.3. *Q.E.D.*

Proof of Theorem 3.4.3: Part (a). Clearly, by (3.13), $y^{sc*}_{i,t}$ is independent of $\beta^{sc}_{j,t-1}$ for all $j \neq i$. Moreover, because

$$\frac{\partial^2 \zeta^{sc}_{i,t}(y_{i,t})}{\partial y_{i,t}\partial \beta^{sc}_{i,t-1}} = \begin{cases} \delta_i \bar{F}_{i,t}(y_{i,t})\kappa'_{ii,t}(\mathbb{E}(y_{i,t}\wedge \xi_{i,t})) \geq 0, \text{ if } y_{i,t} \geq 0; \\ 0, \text{ otherwise}, \end{cases}$$

$\zeta^{sc}_{i,t}(y_{i,t})$ is supermodular in $(y_{i,t}, \beta^{sc}_{i,t-1})$. Therefore, $y^{sc*}_{i,t} = \text{argmax}_{y_{i,t}\in\mathbb{R}}\zeta^{sc}_{i,t}(y_{i,t})$ is increasing in $\beta^{sc}_{i,t-1}$. The continuity of $y^{sc*}_{i,t}$ in $\beta^{sc}_{i,t-1}$ follows directly from the continuous differentiability of $\zeta^{sc}_{i,t}(\cdot)$ in $(y_{i,t}, \beta^{sc}_{i,t-1})$. This completes the proof of part (a).

Part (b). Note that, by part (a), $\sum_{l\neq i}\kappa_{il,t}(\mathbb{E}((y^{sc*}_{l,t})^+ \wedge \xi_{l,t}))$ is independent of $\beta^{sc}_{i,t-1}$ and continuously increasing in $\beta^{sc}_{j,t-1}$ for $j \neq i$. Moreover,

$$\zeta^{sc}_{i,t}(y_{i,t}) = (\delta_i w_{i,t-1} - w_{i,t})y_{i,t} - L_{i,t}(y_{i,t}) + \delta_i \beta^{sc}_{i,t-1}\kappa_{ii,t}(\mathbb{E}[y^+_{i,t}\wedge \xi_{i,t}])$$

is continuously increasing in $\beta^{sc}_{i,t-1}$ and independent of $\beta^{sc}_{j,t-1}$ for all $j \neq i$. Thus,

$$\pi^{sc*}_{i,t} = [\max_{y_{i,t}\geq 0}\zeta^{sc}_{i,t}(y_{i,t})] - \sum_{j\neq i}\kappa_{ij,t}(\mathbb{E}(y^{sc*}_{j,t}\wedge \xi_{j,t}))$$

is continuously increasing in $\beta^{sc}_{i,t-1}$ and continuously decreasing in $\beta^{sc}_{j,t-1}$ for all $j \neq i$. This completes the proof of part (b).

Part (c). We denote the objective function of each firm i in $\mathcal{G}^{sc,1}_{s,t}$ as $\Pi^{sc}_{i,t}(\cdot,\cdot|\pi^{sc*}_{s,t})$ to capture the dependence of the objective functions on $\pi^{sc*}_{s,t}$. The unique symmetric Nash equilibrium in $\mathcal{G}^{sc,1}_{s,t}$ is denoted as $(\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t}), p^{sc*}_{ss,t}(\pi^{sc*}_{s,t}))$, where $\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t}) = (\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}), \gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}), \cdots, \gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))$ and $p^{sc*}_{ss,t}(\pi^{sc*}_{s,t}) = (p^{sc*}_{s,t}(\pi^{sc*}_{s,t}), p^{sc*}_{s,t}(\pi^{sc*}_{s,t}), \cdots, p^{sc*}_{s,t}(\pi^{sc*}_{s,t}))$. It suffices to show that, if $\bar{\pi}^{sc*}_{s,t} > \pi^{sc*}_{s,t}$, $\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) \geq \gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})$, and $p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) \leq p^{sc*}_{s,t}(\pi^{sc*}_{s,t})$.

We first show that $p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) \leq p^{sc*}_{s,t}(\pi^{sc*}_{s,t})$ for all $\bar{\pi}^{sc*}_{s,t} > \pi^{sc*}_{s,t}$. Assume, to the contrary, that $p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) > p^{sc*}_{s,t}(\pi^{sc*}_{s,t})$. Lemma 23 implies that
$\partial_{p_{1,t}}\log(\Pi^{sc}_{1,t}(\gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t}), p^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})|\bar{\pi}^{sc*}_{s,t})) \geq \partial_{p_{1,t}}\log(\Pi^{sc}_{1,t}(\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t}), p^{sc*}_{ss,t}(\pi^{sc*}_{s,t})|\pi^{sc*}_{s,t}))$, i.e.,

$$\begin{aligned} &\partial_{p_{1,t}}\log\rho_{s,t}(p^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})) + \frac{1}{p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})) + \bar{\pi}^{sc*}_{s,t}} \\ \geq &\partial_{p_{1,t}}\log\rho_{s,t}(p^{sc*}_{ss,t}(\pi^{sc*}_{s,t})) + \frac{1}{p^{sc*}_{s,t}(\pi^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})) + \pi^{sc*}_{s,t}}. \end{aligned} \tag{B.6}$$

By (3.4) and Newton-Leibniz formula, we have

$$\begin{aligned} &\partial_{p_{1,t}}\log\rho_{s,t}(p^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})) - \partial_{p_{1,t}}\log\rho_{s,t}(p^{sc*}_{ss,t}(\pi^{sc*}_{s,t})) \\ = &\int_{s=0}^{1}[\sum_{j=1}^{N}(p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) - p^{sc*}_{s,t}(\pi^{sc*}_{s,t}))\frac{\partial^2\log\rho_{s,t}((1-s)p^{sc*}_{ss,t}(\pi^{sc*}_{s,t}) + s p^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t}))}{\partial p_{1,t}\partial p_{j,t}}]\,\mathrm{d}s \\ < &0. \end{aligned}$$

Hence, inequality (B.6) suggests that

$$p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})) + \bar{\pi}^{sc*}_{s,t} < p^{sc*}_{s,t}(\pi^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})) + \pi^{sc*}_{s,t}. \tag{B.7}$$

Since $p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) > p^{sc*}_{s,t}(\pi^{sc*}_{s,t})$ and $\bar{\pi}^{sc*}_{s,t} > \pi^{sc*}_{s,t}$, $\nu_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})) > \nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))$. Thus, $\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) > \gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})$. Lemma 23 yields that $\partial_{\gamma_{1,t}} \log(\Pi^{sc}_{1,t}(\gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t}), p^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})|\bar{\pi}^{sc*}_{s,t})) \geq \partial_{\gamma_{1,t}} \log(\Pi^{sc}_{1,t}(\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t}), p^{sc*}_{ss,t}(\pi^{sc*}_{s,t})|\pi^{sc*}_{s,t}))$, i.e.,

$$\begin{aligned}
&\partial_{\gamma_{1,t}} \log \psi_{s,t}(\gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})) - \frac{\nu'_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}))}{p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})) + \bar{\pi}^{sc*}_{s,t}} \\
\geq &\partial_{\gamma_{1,t}} \log \psi_{s,t}(\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t})) - \frac{\nu'_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))}{p^{sc*}_{s,t}(\pi^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})) + \pi^{sc*}_{s,t}}
\end{aligned} \tag{B.8}$$

Since $\nu_{s,t}(\cdot)$ is convexly increasing, $\nu'_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) \geq \nu'_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))$. Thus, inequality (B.7) implies that

$$-\frac{\nu'_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}))}{p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})) + \bar{\pi}^{sc*}_{s,t}} < -\frac{\nu'_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))}{p^{sc*}_{s,t}(\pi^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})) + \pi^{sc*}_{s,t}}.$$

Hence, (B.8) suggests that

$$\partial_{\gamma_{1,t}} \log \psi_{s,t}(\gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})) > \partial_{\gamma_{1,t}} \log \psi_{s,t}(p^{sc*}_{ss,t}(\pi^{sc*}_{s,t})). \tag{B.9}$$

By (3.3) and Newton-Leibniz formula, we have

$$\begin{aligned}
&\partial_{\gamma_{1,t}} \log \psi_{s,t}(\gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})) - \partial_{\gamma_{1,t}} \log \psi_{s,t}(p^{sc*}_{ss,t}(\pi^{sc*}_{s,t})) \\
= &\int_{s=0}^{1} [\sum_{j=1}^{N} (\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) - \gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})) \frac{\partial^2 \log \psi_{s,t}((1-s)\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t}) + s\gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t}))}{\partial \gamma_{1,t} \partial \gamma_{j,t}}] \, \mathrm{d}s \\
< &\ 0,
\end{aligned}$$

which contradicts (B.9). Therefore, for all $\bar{\pi}^{sc*}_{s,t} > \pi^{sc*}_{s,t}$, we have $p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) \leq p^{sc*}_{s,t}(\pi^{sc*}_{s,t})$.

We now show that $\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) \geq \gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})$ for all $\bar{\pi}^{sc*}_{s,t} > \pi^{sc*}_{s,t}$. Assume, to the contrary, that $\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) < \gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})$. Lemma 23 implies that $\partial_{\gamma_{1,t}} \log(\Pi^{sc}_{1,t}(\gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t}), p^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})|\bar{\pi}^{sc*}_{s,t})) \leq \partial_{\gamma_{1,t}} \log(\Pi^{sc}_{1,t}(\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t}), p^{sc*}_{ss,t}(\pi^{sc*}_{s,t})|\pi^{sc*}_{s,t}))$, i.e.,

$$\begin{aligned}
&\partial_{\gamma_{1,t}} \log \psi_{s,t}(\gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})) - \frac{\nu'_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}))}{p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})) + \bar{\pi}^{sc*}_{s,t}} \\
\leq &\partial_{\gamma_{1,t}} \log \psi_{s,t}(\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t})) - \frac{\nu'_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))}{p^{sc*}_{s,t}(\pi^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})) + \pi^{sc*}_{s,t}}.
\end{aligned} \tag{B.10}$$

By (3.3) and Newton-Leibniz formula, we have

$$\begin{aligned}
&\partial_{\gamma_{1,t}} \log \psi_{s,t}(\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t})) - \partial_{\gamma_{1,t}} \log \psi_{s,t}(\gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t})) \\
= &\int_{s=0}^{1} [\sum_{j=1}^{N} (\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}) - \gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})) \frac{\partial^2 \log \psi_{s,t}(s\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t}) + (1-s)\gamma^{sc*}_{ss,t}(\bar{\pi}^{sc*}_{s,t}))}{\partial \gamma_{1,t} \partial \gamma_{j,t}}] \, \mathrm{d}s < 0.
\end{aligned}$$

Hence, inequality (B.10) implies that

$$-\frac{\nu'_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}))}{p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})) + \bar{\pi}^{sc*}_{s,t}} < -\frac{\nu'_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))}{p^{sc*}_{s,t}(\pi^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})) + \pi^{sc*}_{s,t}}.$$

Since $\nu_{s,t}(\cdot)$ is convexly increasing, $\nu'_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})) \leq \nu'_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))$. Hence,

$$p^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t})) + \bar{\pi}^{sc*}_{s,t} < p^{sc*}_{s,t}(\pi^{sc*}_{s,t}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})) + \pi^{sc*}_{s,t}.$$

Since $\nu_{s,t}(\gamma_{s,t}^{sc*}(\bar{\pi}_{s,t}^{sc*})) \le \nu_{s,t}(\gamma_{s,t}^{sc*}(\pi_{s,t}^{sc*}))$ and $\bar{\pi}_{s,t}^{sc*} > \pi_{s,t}^{sc*}$, $p_{s,t}^{sc*}(\bar{\pi}_{s,t}^{sc*}) < p_{s,t}^{sc*}(\pi_{s,t}^{sc*})$. Lemma 23 implies that $\partial_{p_{1,t}} \log(\Pi_{1,t}^{sc}(\gamma_{ss,t}^{sc*}(\bar{\pi}_{s,t}^{sc*}), p_{ss,t}^{sc*}(\bar{\pi}_{s,t}^{sc*})|\bar{\pi}_{s,t}^{sc*})) \le \partial_{p_{1,t}} \log(\Pi_{1,t}^{sc}(\gamma_{ss,t}^{sc*}(\pi_{s,t}^{sc*}), p_{ss,t}^{sc*}(\pi_{s,t}^{sc*})|\pi_{s,t}^{sc*}))$, i.e.,

$$\begin{aligned}
&\partial_{p_{1,t}} \log \rho_{s,t}(p_{ss,t}^{sc*}(\bar{\pi}_{s,t}^{sc*})) + \frac{1}{p_{s,t}^{sc*}(\bar{\pi}_{s,t}^{sc*}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma_{s,t}^{sc*}(\bar{\pi}_{s,t}^{sc*})) + \bar{\pi}_{s,t}^{sc*}} \\
\le&\partial_{p_{1,t}} \log \rho_{s,t}(p_{ss,t}^{sc*}(\pi_{s,t}^{sc*})) + \frac{1}{p_{s,t}^{sc*}(\pi_{s,t}^{sc*}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma_{s,t}^{sc*}(\pi_{s,t}^{sc*})) + \pi_{s,t}^{sc*}}.
\end{aligned} \tag{B.11}$$

Because

$$\frac{1}{p_{s,t}^{sc*}(\bar{\pi}_{s,t}^{sc*}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma_{s,t}^{sc*}(\bar{\pi}_{s,t}^{sc*})) + \bar{\pi}_{s,t}^{sc*}} > \frac{1}{p_{s,t}^{sc*}(\pi_{s,t}^{sc*}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma_{s,t}^{sc*}(\pi_{s,t}^{sc*})) + \pi_{s,t}^{sc*}},$$

we have that

$$\partial_{p_{1,t}} \log \rho_{s,t}(p_{ss,t}^{sc*}(\bar{\pi}_{s,t}^{sc*})) < \partial_{p_{1,t}} \log \rho_{s,t}(p_{ss,t}^{sc*}(\pi_{s,t}^{sc*})). \tag{B.12}$$

By (3.4) and Newton-Leibniz formula, we have

$$\begin{aligned}
& \partial_{p_{1,t}} \log \rho_{s,t}(p_{ss,t}^{sc*}(\pi_{s,t}^{sc*})) - \partial_{p_{1,t}} \log \rho_{s,t}(p_{ss,t}^{sc*}(\bar{\pi}_{s,t}^{sc*})) \\
= & \int_{s=0}^{1} [\sum_{j=1}^{N} (p_{s,t}^{sc*}(\pi_{s,t}^{sc*}) - p_{s,t}^{sc*}(\bar{\pi}_{s,t}^{sc*})) \frac{\partial^2 \log \rho_{s,t}(s p_{ss,t}^{sc*}(\pi_{s,t}^{sc*}) + (1-s) p_{ss,t}^{sc*}(\bar{\pi}_{s,t}^{sc*}))}{\partial p_{1,t} \partial p_{j,t}}] \, \mathrm{d}s \\
< & \ 0,
\end{aligned}$$

which contradicts (B.12). Therefore, for all $\bar{\pi}_{s,t}^{sc*} > \pi_{s,t}^{sc*}$, we have $\gamma_{s,t}^{sc*}(\bar{\pi}_{s,t}^{sc*}) \le \gamma_{s,t}^{sc*}(\pi_{s,t}^{sc*})$. The continuity of $\gamma_{s,t}^{sc*}(\pi_{s,t}^{sc*})$ and $p_{s,t}^{sc*}(\pi_{s,t}^{sc*})$ in $\pi_{s,t}^{sc*}$ follows directly from that $\Pi_{i,t}^{sc}(\gamma_t, p_t|\pi_{s,t}^{sc*})$ is twice continuously differentiable and the implicit function theorem. This completes the proof of part (c).

Part (d). By Theorem 3.4.2(a), $\beta_{s,t}^{sc} = \delta_s \beta_{s,t}^{sc} \mu_{s,t} + \Pi_{s,t}^{sc*}$, it suffices to show that $\Pi_{s,t}^{sc*}(\pi_{s,t}^{sc*})$ is continuously increasing in $\pi_{s,t}^{sc*}$, where $\Pi_{s,t}^{sc*}(\pi_{s,t}^{sc*}) := \Pi_{i,t}^{sc}(\gamma_{ss,t}^{sc*}(\pi_{s,t}^{sc*}), p_{ss,t}^{sc*}(\pi_{s,t}^{sc*}))$.

Assume that $\bar{\pi}_{s,t}^{sc*} > \pi_{s,t}^{sc*}$. Since part (c) implies that $p_{s,t}^{sc*}(\bar{\pi}_{s,t}^{sc*}) \le p_{s,t}^{sc*}(\pi_{s,t}^{sc*})$ and $\gamma_{s,t}^{sc*}(\bar{\pi}_{s,t}^{sc*}) \ge \gamma_{s,t}^{sc*}(\pi_{s,t}^{sc*})$, the monotonicity condition (3.17) implies that

$$\rho_{s,t}(p_{ss,t}^{sc*}(\bar{\pi}_{s,t}^{sc*})) \ge \rho_{s,t}(p_{ss,t}^{sc*}(\pi_{s,t}^{sc*})) \text{ and } \psi_{s,t}(\gamma_{ss,t}^{sc*}(\bar{\pi}_{s,t}^{sc*})) \ge \psi_{s,t}(\gamma_{ss,t}^{sc*}(\pi_{s,t}^{sc*})). \tag{B.13}$$

If $p_{s,t}^{sc*}(\bar{\pi}_{s,t}^{sc*}) = p_{s,t}^{sc*}(\pi_{s,t}^{sc*})$ and $\gamma_{s,t}^{sc*}(\bar{\pi}_{s,t}^{sc*}) = \gamma_{s,t}^{sc*}(\pi_{s,t}^{sc*})$, by $\bar{\pi}_{s,t}^{sc*} > \pi_{s,t}^{sc*}$, we have

$$p_{s,t}^{sc*}(\bar{\pi}_{s,t}^{sc*}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma_{s,t}^{sc*}(\bar{\pi}_{s,t}^{sc*})) + \bar{\pi}_{s,t}^{sc*} > p_{s,t}^{sc*}(\pi_{s,t}^{sc*}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma_{s,t}^{sc*}(\pi_{s,t}^{sc*})) + \pi_{s,t}^{sc*}.$$

Thus,

$$\begin{aligned}
\Pi_{s,t}^{sc*}(\bar{\pi}_{s,t}^{sc*}) &= \Pi_{i,t}^{sc}(\gamma_{ss,t}^{sc*}(\bar{\pi}_{s,t}^{sc*}), p_{ss,t}^{sc*}(\bar{\pi}_{s,t}^{sc*})|\bar{\pi}_{s,t}^{sc*}) \\
&= (p_{s,t}^{sc*}(\bar{\pi}_{s,t}^{sc*}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma_{s,t}^{sc*}(\bar{\pi}_{s,t}^{sc*})) + \bar{\pi}_{s,t}^{sc*}) \rho_{s,t}(p_{ss,t}^{sc*}(\bar{\pi}_{s,t}^{sc*})) \psi_{s,t}(\gamma_{ss,t}^{sc*}(\bar{\pi}_{s,t}^{sc*})) \\
&> (p_{s,t}^{sc*}(\pi_{s,t}^{sc*}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma_{s,t}^{sc*}(\pi_{s,t}^{sc*})) + \pi_{s,t}^{sc*}) \rho_{s,t}(p_{ss,t}^{sc*}(\pi_{s,t}^{sc*})) \psi_{s,t}(\gamma_{ss,t}^{sc*}(\pi_{s,t}^{sc*})) \\
&= \Pi_{i,t}^{sc}(\gamma_{ss,t}^{sc*}(\pi_{s,t}^{sc*}), p_{ss,t}^{sc*}(\pi_{s,t}^{sc*})|\pi_{s,t}^{sc*}) \\
&= \Pi_{s,t}^{sc*}(\pi_{s,t}^{sc*}).
\end{aligned}$$

If $p_{s,t}^{sc*}(\bar{\pi}_{s,t}^{sc*}) < p_{s,t}^{sc*}(\pi_{s,t}^{sc*})$, Lemma 23 yields that $\partial_{p_{1,t}} \log(\Pi_{1,t}^{sc}(p_{ss,t}^{sc*}(\bar{\pi}_{s,t}^{sc*}), \gamma_{ss,t}^{sc*}(\bar{\pi}_{s,t}^{sc*})|\bar{\pi}_{s,t}^{sc*})) \le \partial_{p_{1,t}} \log(\Pi_{1,t}^{sc}(p_{ss,t}^{sc*}(\pi_{s,t}^{sc*}), \gamma_{ss,t}^{sc*}(\pi_{s,t}^{sc*})|\pi_{s,t}^{sc*}))$, i.e.,

$$\begin{aligned}
&\partial_{p_{1,t}} \log \rho_{s,t}(p_{ss,t}^{sc*}(\bar{\pi}_{s,t}^{sc*})) + \frac{1}{p_{s,t}^{sc*}(\bar{\pi}_{s,t}^{sc*}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma_{s,t}^{sc*}(\bar{\pi}_{s,t}^{sc*})) + \bar{\pi}_{s,t}^{sc*}} \\
\le&\partial_{p_{1,t}} \log \rho_{s,t}(p_{ss,t}^{sc*}(\pi_{s,t}^{sc*})) + \frac{1}{p_{s,t}^{sc*}(\pi_{s,t}^{sc*}) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma_{s,t}^{sc*}(\pi_{s,t}^{sc*})) + \pi_{s,t}^{sc*}}.
\end{aligned} \tag{B.14}$$

By (3.4) and Newton-Leibniz formula, we have

$$\begin{aligned}&\partial_{p_{1,t}}\log\rho_{s,t}(p^{sc*}_{ss,t}(\pi^{sc*}_{s,t}))-\partial_{p_{1,t}}\log\rho_{s,t}(p^{sc*}_{ss,t}(\bar\pi^{sc*}_{s,t}))\\ =\ &\int_{s=0}^{1}[\sum_{j=1}^{N}(p^{sc*}_{s,t}(\pi^{sc*}_{s,t})-p^{sc*}_{s,t}(\bar\pi^{sc*}_{s,t}))\frac{\partial^2\log\rho_{s,t}((1-s)p^{sc*}_{ss,t}(\bar\pi^{sc*}_{s,t})+sp^{sc*}_{ss,t}(\pi^{sc*}_{s,t}))}{\partial p_{1,t}\partial p_{j,t}}]\,\mathrm{d}s<0.\end{aligned}$$

Hence, (B.14) implies that

$$p^{sc*}_{s,t}(\bar\pi^{sc*}_{s,t})-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma^{sc*}_{s,t}(\bar\pi^{sc*}_{s,t}))+\bar\pi^{sc*}_{s,t}>p^{sc*}_{s,t}(\pi^{sc*}_{s,t})-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))+\pi^{sc*}_{s,t}.$$

Therefore,

$$\begin{aligned}\Pi^{sc*}_{s,t}(\bar\pi^{sc*}_{s,t})&=\Pi^{sc}_{i,t}(\gamma^{sc*}_{ss,t}(\bar\pi^{sc*}_{s,t}),p^{sc*}_{ss,t}(\bar\pi^{sc*}_{s,t})|\bar\pi^{sc*}_{s,t})\\&=(p^{sc*}_{s,t}(\bar\pi^{sc*}_{s,t})-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma^{sc*}_{s,t}(\bar\pi^{sc*}_{s,t}))+\bar\pi^{sc*}_{s,t})\rho_{s,t}(p^{sc*}_{ss,t}(\bar\pi^{sc*}_{s,t}))\psi_{s,t}(\gamma^{sc*}_{ss,t}(\bar\pi^{sc*}_{s,t}))\\&>(p^{sc*}_{s,t}(\pi^{sc*}_{s,t})-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))+\pi^{sc*}_{s,t})\rho_{s,t}(p^{sc*}_{ss,t}(\pi^{sc*}_{s,t}))\psi_{s,t}(\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t}))\\&=\Pi^{sc}_{i,t}(\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t}),p^{sc*}_{ss,t}(\pi^{sc*}_{s,t})|\pi^{sc*}_{s,t})\\&=\Pi^{sc*}_{s,t}(\pi^{sc*}_{s,t}).\end{aligned}$$

If $p^{sc*}_{s,t}(\bar\pi^{sc*}_{s,t})=p^{sc*}_{s,t}(\pi^{sc*}_{s,t})$ and $\gamma^{sc*}_{s,t}(\bar\pi^{sc*}_{s,t})>\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t})$, Lemma 23 yields that $\partial_{\gamma_{1,t}}\log(\Pi^{sc}_{1,t}(p^{sc*}_{ss,t}(\bar\pi^{sc*}_{s,t}),\gamma^{sc*}_{ss,t}(\bar\pi^{sc*}_{s,t})|\bar\pi^{sc*}_{s,t}))\geq\partial_{\gamma_{1,t}}\log(\Pi^{sc}_{1,t}(p^{sc*}_{ss,t}(\pi^{sc*}_{s,t}),\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t})|\pi^{sc*}_{s,t}))$, i.e.,

$$\begin{aligned}&\partial_{\gamma_{1,t}}\log\psi_{s,t}(\gamma^{sc*}_{ss,t}(\bar\pi^{sc*}_{s,t}))-\frac{\nu'_{s,t}(\gamma^{sc*}_{s,t}(\bar\pi^{sc*}_{s,t}))}{p^{sc*}_{s,t}(\bar\pi^{sc*}_{s,t})-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma^{sc*}_{s,t}(\bar\pi^{sc*}_{s,t}))+\bar\pi^{sc*}_{s,t}}\\ \geq&\partial_{\gamma_{1,t}}\log\psi_{s,t}(\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t}))-\frac{\nu'_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))}{p^{sc*}_{s,t}(\pi^{sc*}_{s,t})-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))+\pi^{sc*}_{s,t}}.\end{aligned}\tag{B.15}$$

By (3.4) and Newton-Leibniz formula, we have

$$\begin{aligned}&\partial_{\gamma_{1,t}}\log\psi_{s,t}(\gamma^{sc*}_{ss,t}(\bar\pi^{sc*}_{s,t}))-\partial_{\gamma_{1,t}}\log\psi_{s,t}(\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t}))\\ =\ &\int_{s=0}^{1}[\sum_{j=1}^{N}(\gamma^{sc*}_{s,t}(\bar\pi^{sc*}_{s,t})-\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))\frac{\partial^2\log\psi_{s,t}(s\gamma^{sc*}_{ss,t}(\bar\pi^{sc*}_{s,t})+(1-s)\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t}))}{\partial\gamma_{1,t}\partial\gamma_{j,t}}]\,\mathrm{d}s<0.\end{aligned}$$

Hence, (B.15) implies that

$$-\frac{\nu'_{s,t}(\gamma^{sc*}_{s,t}(\bar\pi^{sc*}_{s,t}))}{p^{sc*}_{s,t}(\bar\pi^{sc*}_{s,t})-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma^{sc*}_{s,t}(\bar\pi^{sc*}_{s,t}))+\bar\pi^{sc*}_{s,t}}>-\frac{\nu'_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))}{p^{sc*}_{s,t}((\pi^{sc*}_{s,t}))-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))+\pi^{sc*}_{s,t}}.\tag{B.16}$$

Since $\nu_{s,t}(\cdot)$ is convexly increasing, $\nu'_{s,t}(\gamma^{sc*}_{s,t}(\bar\pi^{sc*}_{s,t}))\geq\nu'_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))$. Hence, (B.16) implies that

$$p^{sc*}_{s,t}(\bar\pi^{sc*}_{s,t})-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma^{sc*}_{s,t}(\bar\pi^{sc*}_{s,t}))+\bar\pi^{sc*}_{s,t}>p^{sc*}_{s,t}((\pi^{sc*}_{s,t}))-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))+\pi^{sc*}_{s,t}.$$

Therefore,

$$\begin{aligned}\Pi^{sc*}_{s,t}(\bar\pi^{sc*}_{s,t})&=\Pi^{sc}_{i,t}(\gamma^{sc*}_{ss,t}(\bar\pi^{sc*}_{s,t}),p^{sc*}_{ss,t}(\bar\pi^{sc*}_{s,t})|\bar\pi^{sc*}_{s,t})\\&=(p^{sc*}_{s,t}(\bar\pi^{sc*}_{s,t})-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma^{sc*}_{s,t}(\bar\pi^{sc*}_{s,t}))+\bar\pi^{sc*}_{s,t})\rho_{s,t}(p^{sc*}_{ss,t}(\bar\pi^{sc*}_{s,t}))\psi_{s,t}(\gamma^{sc*}_{ss,t}(\bar\pi^{sc*}_{s,t}))\\&>(p^{sc*}_{s,t}(\pi^{sc*}_{s,t})-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma^{sc*}_{s,t}(\pi^{sc*}_{s,t}))+\pi^{sc*}_{s,t})\rho_{s,t}(p^{sc*}_{ss,t}(\pi^{sc*}_{s,t}))\psi_{s,t}(\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t}))\\&=\Pi^{sc}_{i,t}(\gamma^{sc*}_{ss,t}(\pi^{sc*}_{s,t}),p^{sc*}_{ss,t}(\pi^{sc*}_{s,t})|\pi^{sc*}_{s,t})\\&=\Pi^{sc*}_{s,t}(\pi^{sc*}_{s,t}).\end{aligned}$$

Thus, we have shown that, if $\bar{\pi}^{sc*}_{s,t} > \pi^{sc*}_{s,t}$, $\Pi^{sc*}_{s,t}(\bar{\pi}^{sc*}_{s,t}) > \Pi^{sc*}_{s,t}(\pi^{sc*}_{s,t})$ and, hence, by Theorem 3.4.2(a), $\beta^{sc}_{s,t}(\bar{\pi}^{sc*}_{s,t}) > \beta^{sc}_{s,t}(\pi^{sc*}_{s,t})$. The continuity of $\beta^{sc}_{s,t}$ in $\pi^{sc*}_{s,t}$ follows directly from the continuous differentiability of $\Pi^{sc}_{i,t}(\gamma_t, p_t|\pi^{sc*}_{s,t})$ in $(\gamma_t, p_t, \pi^{sc*}_{s,t})$ and the continuity of $(\gamma^{sc*}_{ss,t}, p^{sc*}_{ss,t})$ in $\pi^{sc*}_{s,t}$. This completes the proof of part (d).

Part (e). By part (c), it suffices to show that, $\pi^{sc*}_{s,t}$ is continuously increasing in $\beta^{cs}_{s,t-1}$. The monotonicity follows from the assumption, whereas the continuity follows directly from part (a) and that the compound function is continuous if each individual function is continuous. This completes the proof of part (e).

Part (f). By the proof of part (e), $\pi^{sc*}_{s,t}$ is continuously increasing in $\beta^{cs}_{s,t-1}$. By part (d), $\beta^{sc}_{s,t}$ is continuously increasing in $\beta^{sc}_{s,t-1}$. *Q.E.D.*

Proof of Theorem 3.4.4: Part (a). Because $\beta^{sc}_{i,t-1} \geq \tilde{\beta}^{sc}_{i,t-1} = 0$ for each i and t, Theorem 3.4.3(a) implies that $y^{sc*}_{i,t} \geq \tilde{y}^{sc*}_{i,t}$ for all i and t. Thus,

$$z^{sc*}_{i,t} = \mathbb{E}[(y^{sc*}_{i,t})^+ \wedge \xi_{i,t}] \geq \mathbb{E}[(\tilde{y}^{sc*}_{i,t})^+ \wedge \xi_{i,t}] = z^{sc*}_{i,t}, \text{ for all } i \text{ and } t.$$

Moreover, since $\tilde{\beta}^{sc}_{i,t-1} = 0$, $\tilde{\pi}^{sc}_{i,t}(y_t) = (\delta_i w_{i,t-1} - w_{i,t})y_{i,t} - L_{i,t}(y_{i,t})$. Moreover, if $y_{i,t} \leq 0$, $\tilde{\pi}^{sc}_{i,t}(y_t)$ is strictly increasing in $y_{i,t}$. Hence, $\tilde{\pi}^{sc*}_{i,t} = \max\{(\delta_i w_{i,t-1} - w_{i,t})y_{i,t} - L_{i,t}(y_{i,t}) : y_{i,t} \geq 0\}$. Thus,

$$\begin{aligned}
\pi^{sc*}_{i,t} =& \max\{(\delta_i w_{i,t-1} - w_{i,t})y_{i,t} - L_{i,t}(y_{i,t}) + \delta_i \beta^{sc}_{i,t-1}(\kappa_{ii,t}(\mathbb{E}[y_{i,t} \wedge \xi_{i,t}]) - \sum_{j \neq i} \kappa_{ij,t}(\mathbb{E}[y^*_{j,t} \wedge \xi_{j,t}])) : y_{i,t} \geq 0\} \\
\geq& \max\{(\delta_i w_{i,t-1} - w_{i,t})y_{i,t} - L_{i,t}(y_{i,t}) + \delta_i \beta^{sc}_{i,t-1}(\kappa_{ii,t}(0) - \sum_{j \neq i} \kappa_{ij,t}(1)) : y_{i,t} \geq 0\} \\
\geq& \max\{(\delta_i w_{i,t-1} - w_{i,t})y_{i,t} - L_{i,t}(y_{i,t}) : y_{i,t} \geq 0\} \\
=& \tilde{\pi}^{sc*}_{i,t},
\end{aligned}$$

where the first inequality follows from that $\kappa_{ii,t}(\cdot)$ is increasing in $y_{i,t}$ and $\kappa_{ij,t}(\cdot)$ is increasing in $y_{j,t}$, and the second from that $\alpha_{i,t}(\cdot) \geq 0$ for all i, t, and z_t. This proves part (a).

Part (b-i). Part (a) suggests that $\pi^{sc*}_{s,t} \geq \tilde{\pi}^{sc*}_{s,t}$ for all t. Thus, by Theorem 3.4.3(c), $\gamma^{sc*}_{s,t} \geq \tilde{\gamma}^{sc*}_{s,t}$ for all t. By Theorem 3.4.2(b), $\gamma^{sc*}_{i,t}(I_t, \Lambda_t) = \gamma^{sc*}_{s,t} \geq \tilde{\gamma}^{sc*}_{s,t} = \tilde{\gamma}^{sc*}_{i,t}(I_t, \Lambda_t)$ for all t and $(I_t, \Lambda_t) \in \mathcal{S}$. This proves part (b-i).

Part (b-ii). Part (a) suggests that $\pi^{sc*}_{s,t} \geq \tilde{\pi}^{sc*}_{s,t}$ for all t. Thus, by Theorem 3.4.3(c), $p^{sc*}_{s,t} \leq \tilde{p}^{sc*}_{s,t}$ for all t. By Theorem 3.4.2(b), $p^{sc*}_{i,t}(I_t, \Lambda_t) = p^{sc*}_{s,t} \leq \tilde{p}^{sc*}_{s,t} = \tilde{p}^{sc*}_{i,t}(I_t, \Lambda_t)$ for all t and $(I_t, \Lambda_t) \in \mathcal{S}$. This proves part (b-ii).

Part (b-iii). By Proposition 3.4.3(d), $x^{sc*}_{i,t}(I_t, \Lambda_t) = y^{sc*}_{s,t} \rho_{s,t}(p^{sc*}_{ss,t}) \psi_{s,t}(\gamma^{sc*}_{ss,t}) \Lambda_{i,t}$ and $\tilde{x}^{sc*}_{i,t}(I_t, \Lambda_t) = \tilde{y}^{sc*}_{s,t} \rho_{s,t}(\tilde{p}^{sc*}_{ss,t}) \psi_{s,t}(\tilde{\gamma}^{sc*}_{ss,t}) \Lambda_{i,t}$. Part (a) implies that $y^{sc*}_{s,t} \geq \tilde{y}^{sc*}_{s,t}$. Since, by parts (b-i) and (b-ii), $p^{sc*}_{s,t} \leq \tilde{p}^{sc*}_{s,t}$ and $\gamma^{sc*}_{s,t} \geq \tilde{\gamma}^{sc*}_{s,t}$, the monotonicity condition (3.17) yields that $\rho_{s,t}(p^{sc*}_{ss,t}) \geq \rho_{s,t}(\tilde{p}^{sc*}_{ss,t})$, and $\psi_{s,t}(\gamma^{sc*}_{ss,t}) \geq \psi_{s,t}(\tilde{\gamma}^{sc*}_{ss,t})$. Therefore, for each $(I_t, \Lambda_t) \in \mathcal{S}$,

$$x^{sc*}_{i,t}(I_t, \Lambda_t) = y^{sc*}_{s,t} \rho_{s,t}(p^{sc*}_{ss,t}) \psi_{s,t}(\gamma^{sc*}_{ss,t}) \Lambda_{i,t} \geq \tilde{y}^{sc*}_{s,t} \rho_{s,t}(\tilde{p}^{sc*}_{ss,t}) \psi_{s,t}(\tilde{\gamma}^{sc*}_{ss,t}) \Lambda_{i,t} = \tilde{x}^{sc*}_{i,t}(I_t, \Lambda_t).$$

This completes the proof of part (b-iii). *Q.E.D.*

Proof of Theorem 3.4.5: Part (a). We show part (a) by backward induction. More specifically, we show that if $\hat{\alpha}_{s,t}(z_t) \geq \alpha_{s,t}(z_t)$ for all z_t and $\hat{\beta}^{sc}_{s,t-1} \geq \beta^{sc}_{s,t-1}$, (i) $\hat{\pi}^{sc*}_{s,t} \geq \pi^{sc*}_{s,t}$, (ii) $\hat{\gamma}^{sc*}_{s,t} \geq \gamma^{sc*}_{s,t}$, (iii) $\hat{\gamma}^{sc*}_{i,t}(I_t, \Lambda_t) \geq \gamma^{sc*}_{s,t}(I_t, \Lambda_t)$ for each i and $(I_t, \Lambda_t) \in \mathcal{S}$, (iv) $\hat{p}^{sc*}_{s,t} \leq p^{sc*}_{s,t}$, (v) $\hat{p}^{sc*}_{i,t}(I_t, \Lambda_t) \leq p^{sc*}_{i,t}(I_t, \Lambda_t)$ for each i and $(I_t, \Lambda_t) \in \mathcal{S}$, and (vi) $\hat{\beta}^{sc}_{s,t} \geq \beta^{sc}_{s,t}$. Since $\hat{\beta}^{sc}_{s,0} = \beta^{sc}_{s,0} = 0$, the initial condition is satisfied.

Since $\hat{\alpha}_{s,t}(z_t) \geq \alpha_{s,t}(z_t)$ for all z_t,

$$\hat{\kappa}_{sa,t}(y_{i,t}) - (N-1)\hat{\kappa}^0_{sb,t} \geq \kappa_{sa,t}(y_{i,t}) - (N-1)\kappa^0_{sb,t} \geq 0, \text{ for all } y_{i,t} \geq 0.$$

Therefore,

$$\begin{aligned}
\hat{\pi}^{sc*}_{s,t} &= \max\{(\delta_s w_{s,t-1} - w_{s,t})y_{i,t} - L_{s,t}(y_{i,t}) + \delta_s \hat{\beta}^{sc}_{s,t-1}(\hat{\kappa}_{sa,t}(\mathbb{E}[y_{i,t} \wedge \xi_{i,t}]) - (N-1)\hat{\kappa}^0_{sb,t}) : y_{i,t} \geq 0\} \\
&\geq \max\{(\delta_s w_{s,t-1} - w_{s,t})y_{i,t} - L_{s,t}(y_{i,t}) + \delta_s \beta^{sc}_{s,t-1}(\kappa_{sa,t}(\mathbb{E}[y_{i,t} \wedge \xi_{i,t}]) - (N-1)\kappa^0_{sb,t}) : y_{i,t} \geq 0\} \\
&= \pi^{sc*}_{s,t}.
\end{aligned}$$

Since $\hat{\pi}^{sc*}_{s,t} \geq \pi^{sc*}_{s,t}$, Theorem 3.4.3(c) implies that $\hat{\gamma}^{sc*}_{s,t} \geq \gamma^{sc*}_{s,t}$ and $\hat{p}^{sc*}_{s,t} \leq p^{sc*}_{s,t}$. Thus, $\hat{\gamma}^{sc*}_{i,t}(I_t, \Lambda_t) = \hat{\gamma}^{sc*}_{s,t} \geq \gamma^{sc*}_{s,t} = \gamma^{sc*}_{i,t}(I_t, \Lambda_t)$ for each i and all $(I_t, \Lambda_t) \in \mathcal{S}$. Analogously, $\hat{p}^{sc*}_{i,t}(I_t, \Lambda_t) = \hat{p}^{sc*}_{s,t} \leq p^{sc*}_{s,t} = p^{sc*}_{i,t}(I_t, \Lambda_t)$ for each i and all $(I_t, \Lambda_t) \in \mathcal{S}$. By Theorem 3.4.3(d), $\hat{\pi}^{sc*}_{s,t} \geq \pi^{sc*}_{s,t}$ implies that $\hat{\beta}^{sc}_{s,t} \geq \beta^{sc}_{s,t}$. This completes the induction and, thus, the proof of part (a).

Part (b). By part (a), it suffices to show that, if $\hat{\alpha}_{s,t}(z_t) \geq \alpha_{s,t}(z_t)$ for all z_t, $\hat{\kappa}'_{sa,t}(z_{i,t}) \geq \kappa'_{sa,t}(z_{i,t})$ for all $z_{i,t}$, and $\hat{\beta}^{sc}_{s,t-1} \geq \beta^{sc}_{s,t-1}$, we have (i) $\hat{y}^{sc*}_{s,t} \geq y^{sc*}_{s,t}$ and (ii) $\hat{x}^{sc*}_{i,t}(I_t, \Lambda_t) \geq x^{sc*}_{i,t}(I_t, \Lambda_t)$ for each i and $(I_t, \Lambda_t) \in \mathcal{S}$.

First, we show that $\hat{y}^{sc*}_{s,t} \geq y^{sc*}_{s,t}$. If, to the contrary, $\hat{y}^{sc*}_{s,t} < y^{sc*}_{s,t}$, Lemma 23 yields that

$$\begin{aligned}
&\partial_{y_{i,t}}[(\delta_s w_{s,t-1} - w_{s,t})\hat{y}^{sc*}_{s,t} - L_{s,t}(\hat{y}^{sc*}_{s,t}) + \delta_s \hat{\beta}^{sc}_{s,t-1}(\hat{\kappa}_{sa,t}(\mathbb{E}[\hat{y}^{sc*}_{s,t} \wedge \xi_{i,t}]) - (N-1)\hat{\kappa}^0_{sb,t})] \\
\leq\ &\partial_{y_{i,t}}[(\delta_s w_{s,t-1} - w_{s,t})y^{sc*}_{s,t} - L_{s,t}(y^{sc*}_{s,t}) + \delta_s \beta^{sc}_{s,t-1}(\kappa_{sa,t}(\mathbb{E}[y^{sc*}_{s,t} \wedge \xi_{i,t}]) - (N-1)\kappa^0_{sb,t})],
\end{aligned}$$

i.e.,

$$\begin{aligned}
&(\delta_s w_{s,t-1} - w_{s,t}) - L'_{s,t}(\hat{y}^{sc*}_{s,t}) + \delta_s \hat{\beta}^{sc}_{s,t-1}\bar{F}_{s,t}(\hat{y}^{sc*}_{s,t})\hat{\kappa}'_{sa,t}(\mathbb{E}[\hat{y}^{sc*}_{s,t} \wedge \xi_{i,t}]) \\
\leq\ &(\delta_s w_{s,t-1} - w_{s,t}) - L'_{s,t}(y^{sc*}_{s,t}) + \delta_s \beta^{sc}_{s,t-1}\bar{F}_{s,t}(y^{sc*}_{s,t})\kappa'_{sa,t}(\mathbb{E}[y^{sc*}_{s,t} \wedge \xi_{i,t}]). \qquad \text{(B.17)}
\end{aligned}$$

Since $-L_{s,t}(\cdot)$ is strictly concave in $y_{i,t}$ and $\hat{y}^{sc*}_{s,t} < y^{sc*}_{s,t}$, (B.17) implies that

$$\delta_s \hat{\beta}^{sc}_{s,t-1}\bar{F}_{s,t}(\hat{y}^{sc*}_{s,t})\hat{\kappa}'_{sa,t}(\mathbb{E}[\hat{y}^{sc*}_{s,t} \wedge \xi_{i,t}]) < \delta_s \beta^{sc}_{s,t-1}\bar{F}_{s,t}(y^{sc*}_{s,t})\kappa'_{sa,t}(\mathbb{E}[y^{sc*}_{s,t} \wedge \xi_{i,t}]). \qquad \text{(B.18)}$$

However, since $\hat{\kappa}'_{sa,t}(z_{i,t}) \geq \kappa'_{sa,t}(z_{i,t})$ for all $z_{i,t}$ and $\hat{y}^{sc*}_{s,t} < y^{sc*}_{s,t}$, we have $\hat{\kappa}'_{sa,t}(\mathbb{E}[\hat{y}^{sc*}_{s,t} \wedge \xi_{i,t}]) \geq \kappa'_{sa,t}(\mathbb{E}[y^{sc*}_{s,t} \wedge \xi_{i,t}])$ and $\bar{F}_{s,t}(\hat{y}^{sc*}_{s,t}) \geq \bar{F}_{s,t}(y^{sc*}_{s,t})$. Because $\hat{\beta}^{sc}_{s,t-1} \geq \beta^{sc}_{s,t-1}$,

$$\delta_s \hat{\beta}^{sc}_{s,t-1}\bar{F}_{s,t}(\hat{y}^{sc*}_{s,t})\hat{\kappa}'_{sa,t}(\mathbb{E}[\hat{y}^{sc*}_{s,t} \wedge \xi_{i,t}]) \geq \delta_s \beta^{sc}_{s,t-1}\bar{F}_{s,t}(y^{sc*}_{s,t})\kappa'_{sa,t}(\mathbb{E}[y^{sc*}_{s,t} \wedge \xi_{i,t}]),$$

which contradicts (B.18). The inequality $\hat{y}^{sc*}_{s,t} \geq y^{sc*}_{s,t}$ then follows immediately.

Now we show that $\hat{x}^{sc*}_{i,t}(I_t, \Lambda_t) \geq x^{sc*}_{i,t}(I_t, \Lambda_t)$ for each i and $(I_t, \Lambda_t) \in \mathcal{S}$. By Proposition 3.4.3(d), $\hat{x}^{sc*}_{i,t}(I_t, \Lambda_t) = \hat{y}^{sc*}_{s,t}\rho_{s,t}(\hat{p}^{sc*}_{ss,t})\psi_{s,t}(\hat{\gamma}^{sc*}_{ss,t})\Lambda_{i,t}$ and $x^{sc*}_{i,t}(I_t, \Lambda_t) = y^{sc*}_{s,t}\rho_{s,t}(p^{sc*}_{ss,t})\psi_{s,t}(\gamma^{sc*}_{ss,t})\Lambda_{i,t}$. We have shown that $\hat{y}^{sc*}_{s,t} \geq y^{sc*}_{s,t}$. Since (3.17) holds for period t, $\rho_{s,t}(\hat{p}^{sc*}_{ss,t}) \geq \rho_{s,t}(p^{sc*}_{ss,t})$, and $\psi_{s,t}(\hat{\gamma}^{sc*}_{ss,t}) \geq \psi_{s,t}(\gamma^{sc*}_{ss,t})$. Therefore, for each i and $(I_t, \Lambda_t) \in \mathcal{S}$,

$$\hat{x}^{sc*}_{i,t}(I_t, \Lambda_t) = \hat{y}^{sc*}_{s,t} \rho_{s,t}(\hat{p}^{sc*}_{ss,t}) \psi_{s,t}(\hat{\gamma}^{sc*}_{ss,t}) \Lambda_{i,t} \geq y^{sc*}_{s,t} \rho_{s,t}(p^{sc*}_{ss,t}) \psi_{s,t}(\gamma^{sc*}_{ss,t}) \Lambda_{i,t} = x^{sc*}_{i,t}(I_t, \Lambda_t).$$

This completes the proof of part (b). *Q.E.D.*

Proof of Theorem 3.4.6: We show **parts (a)-(b)** together by backward induction. More specifically, we show that if $\beta^{sc}_{s,t-1} \geq \beta^{sc}_{s,t-2}$, (i) $y^{sc*}_{s,t} \geq y^{sc*}_{s,t-1}$, (ii) $\gamma^{sc*}_{s,t} \geq \gamma^{sc*}_{s,t-1}$, (iii) $\gamma^{sc*}_{i,t}(I, \Lambda) \geq \gamma^{sc*}_{i,t-1}(I, \Lambda)$ for each i and $(I, \Lambda) \in \mathcal{S}$, (iv) $p^{sc*}_{s,t} \leq p^{sc*}_{s,t-1}$, (v) $p^{sc*}_{i,t}(I, \Lambda) \leq p^{sc*}_{i,t-1}(I, \Lambda)$ for each i and $(I, \Lambda) \in \mathcal{S}$, (vi) $x^{sc*}_{i,t}(I, \Lambda) \geq x^{sc*}_{i,t-1}(I, \Lambda)$ for each i and $(I, \Lambda) \in \mathcal{S}$, and (vii) $\beta^{sc}_{s,t} \geq \beta^{sc}_{s,t-1}$. Since, by Proposition 3.4.3(a), $\beta^{sc}_{s,1} \geq \beta^{sc}_{s,0} = 0$. Thus, the initial condition is satisfied.

Since the model is stationary, by Theorem 3.4.3(a), $\beta^{sc}_{s,t-1} \geq \beta^{sc}_{s,t-2}$ suggests that $y^{sc*}_{s,t} \geq y^{sc*}_{s,t-1}$. Analogously, Theorem 3.4.3(e) yields that $\gamma^{sc*}_{s,t} \geq \gamma^{sc*}_{s,t-1}$ and $p^{sc*}_{s,t} \leq p^{sc*}_{s,t-1}$. Hence, $\gamma^{sc*}_{i,t}(I, \Lambda) = \gamma^{sc*}_{s,t} \geq \gamma^{sc*}_{s,t-1} = \gamma^{sc*}_{i,t-1}(I, \Lambda)$ and $p^{sc*}_{i,t}(I, \Lambda) = p^{sc*}_{s,t} \leq p^{sc*}_{s,t-1} = p^{sc*}_{i,t-1}(I, \Lambda)$ for each i and $(I, \Lambda) \in \mathcal{S}$. Because the monotonicity condition (3.17) holds, we have $\rho_{s,t}(p^{sc*}_{ss,t}) \geq \rho_{s,t-1}(p^{sc*}_{ss,t-1})$, and $\psi_{s,t}(\gamma^{sc*}_{ss,t}) \geq \psi_{s,t-1}(\gamma^{sc*}_{ss,t-1})$. Therefore, for each i and $(I, \Lambda) \in \mathcal{S}$,

$$x^{sc*}_{i,t}(I, \Lambda) = y^{sc*}_{s,t} \rho_{s,t}(p^{sc*}_{ss,t}) \psi_{s,t}(\gamma^{sc*}_{ss,t}) \Lambda_i \geq y^{sc*}_{s,t-1} \rho_{s,t-1}(p^{sc*}_{ss,t-1}) \psi_{s,t-1}(\gamma^{sc*}_{ss,t-1}) \Lambda_i = x^{sc*}_{i,t-1}(I, \Lambda).$$

Finally, $\beta^{sc}_{s,t} \geq \beta^{sc}_{s,t-1}$ follows immediately from Theorem 3.4.3(f) and $\beta^{sc}_{s,t-1} \geq \beta^{sc}_{s,t-2}$. This completes the induction and, thus, the proof of Theorem 3.4.6. *Q.E.D.*

Before presenting the proofs of the results in the PF model, we give the following lemma that is used throughout the rest of our proofs.

Lemma 24 *Let A_t be an $N \times N$ matrix with entries defined by $A_{ii,t} = 2\theta_{ii,t}$ and $A_{ij,t} = -\theta_{ij,t}$ where $i \neq j$. The following statements hold:*

(a) A_t is invertible. Moreover, $(A_t^{-1})_{ij} \geq 0$ for all $1 \leq i, j \leq N$.

(b) $\frac{1}{2} \leq \theta_{ii,t}(A_t^{-1})_{ii} < 1$.

(c) $\frac{1}{2} \leq \sum_{j=1}^{N} \theta_{jj,t}(A_t^{-1})_{ij} < 1$.

Proof: Part (a) follows from Lemma 2(a) in [24] and **Part (b)** follows from Lemma 2(c) in [24].

Part (c). Let $\mathcal{I}$ be the $N \times N$ identity matrix, B_t be the $N \times N$ matrix with

$$(B_t)_{ij} = \begin{cases} 0 & \text{if } i = j, \\ \frac{\theta_{ij,t}}{\theta_{ii,t}} & \text{if } i \neq j; \end{cases}$$

and C_t be the $N \times N$ diagonal matrix with

$$(C_t)_{ij} = \begin{cases} 2\theta_{ii,t} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Because $\theta_{ii,t} > \sum_{j\neq i}\theta_{ij,t}$, B_t is a substochastic matrix.

Observe that, $A_t = C_t(\mathcal{I}-\frac{1}{2}B_t)$ and, hence, $A_t^{-1} = (\mathcal{I}-\frac{1}{2}B_t)^{-1}C_t^{-1}$. Let $\theta_t = (\theta_{11,t}, \theta_{22,t}\cdots, \theta_{NN,t})'$ be the $N-$dimensinal vector. Thus, $\sum_{j=1}^N \theta_{jj,t}(A_t^{-1})_{ij} = (A_t^{-1}\theta_t)_i$. Moreover,

$$A_t^{-1}\theta_t = (\mathcal{I}-\frac{1}{2}B_t)^{-1}C_t^{-1}\theta_t = (\mathcal{I}-\frac{1}{2}B_t)^{-1}(C_t^{-1}\theta_t) = \frac{1}{2}(\mathcal{I}-\frac{1}{2}B_t)^{-1},$$

where the last equality follows from $C_t^{-1}\theta_t = \frac{1}{2}\mathcal{I}$. Therefore,

$$\sum_{j=1}^N \theta_{jj,t}(A_t^{-1})_{ij} = \frac{1}{2}\sum_{j=1}^N[(\mathcal{I}-\frac{1}{2}B_t)^{-1}]_{ij} = \frac{1}{2}\sum_{j=1}^N[\mathcal{I}+\sum_{l=1}^{+\infty}\left(\frac{1}{2}\right)^l (B_t)^l]_{ij},$$

where the second equality follows from the fact that $\mathcal{I}-\frac{1}{2}B_t$ is a diagonal dominant matrix. Thus, for all i, $\sum_{j=1}^N \theta_{jj,t}(A_t^{-1})_{ij} \geq \frac{1}{2}\sum_{j=1}^N \mathcal{I}_{ij} = \frac{1}{2}$. On the other hand, for all i,

$$\frac{1}{2}\sum_{j=1}^N[\mathcal{I}+\sum_{l=1}^{+\infty}\left(\frac{1}{2}\right)^l (B_t)^l]_{ij} = \frac{1}{2}\sum_{j=1}^N[\sum_{l=0}^{+\infty}\left(\frac{1}{2}\right)^l (B_t)^l]_{ij} = \frac{1}{2}\sum_{l=0}^{+\infty}[\left(\frac{1}{2}\right)^l\sum_{j=1}^N (B_t)^l_{ij}] < \frac{1}{2}\sum_{l=0}^{+\infty}\left(\frac{1}{2}\right)^l = 1,$$

where the inequality follows from that B_t is a sub-stochastic matrix. This completes the proof of part (c). *Q.E.D.*

Proof of Theorems 3.5.1-3.5.2 and Propositions 3.5.1-3.5.3: We show Theorem 3.5.1, Proposition 3.5.1, Proposition 3.5.2, Proposition 3.5.3, and Theorem 3.5.2 together by backward induction. More specifically, we show that, if $V_{i,t-1}(I_{t-1}, \Lambda_{t-1}|\sigma_{t-1}^{pf*}) = w_{i,t-1}I_{i,t-1} + \beta_{i,t-1}^{pf}\Lambda_{i,t-1}$ for all i, (a) Proposition 3.5.1 holds for period t, (b) Proposition 3.5.2 holds for period t, (c) Proposition 3.5.3 holds for period t, (d) there exists a Markov strategy profile $\{(\gamma_{i,t}^{pf*}(\cdot,\cdot), p_{i,t}^{pf*}(\cdot,\cdot,\cdot), x_{i,t}^{pf*}(\cdot,\cdot,\cdot)) : 1 \leq i \leq N\}$, which forms an equilibrium in the subgame of period t, (e) if $\nu_{i,t}(\gamma_{i,t}) = \gamma_{i,t}$ for all i and $\gamma_{i,t}$, the equilibrium in the subgame of period t, $\{(\gamma_{i,t}^{pf*}(\cdot,\cdot), p_{i,t}^{pf*}(\cdot,\cdot,\cdot), x_{i,t}^{pf*}(\cdot,\cdot,\cdot)) : 1 \leq i \leq N\}$, is unique, and (f) there exists a positive vector $\beta_t^{pf} = (\beta_{1,t}^{pf}, \beta_{2,t}^{pf}, \cdots, \beta_{N,t}^{pf})$, such that $V_{i,t}(I_t, \Lambda_t|\sigma_t^{pf*}) = w_{i,t}I_{i,t} + \beta_{i,t}^{pf}\Lambda_{i,t}$ for all i. Because $V_{i,0}(I_0, \Lambda_0) = w_{i,0}I_{i,0}$ for all i, the initial condition is satisfied.

First, we observe that Proposition 3.5.1 follows directly from the same argument as the proof of Proposition 3.4.1. We now show Proposition 3.5.2 holds in period t. Because $\partial^2_{p_{i,t}}\Pi_{i,t}^{pf,2}(p_t|\gamma_t) = -2\theta_{ii,t} < 0$, $\Pi_{i,t}^{pf,2}(\cdot, p_{-i,t}|\gamma_t)$ is strictly concave in $p_{i,t}$ for any given $p_{-i,t}$. Hence, by Theorem 1.2 in [79], $\mathcal{G}_t^{pf,2}$ has a pure strategy Nash equilibrium $p_t^{pf*}(\gamma_t)$. Since, for each i and t, $\underline{p}_{i,t}$ is sufficiently low whereas $\bar{p}_{i,t}$ is sufficiently high so that they will not affect the equilibrium behaviors of all firms, $p_t^{pf*}(\gamma_t)$ can be characterized by first-order conditions $\partial_{p_{i,t}}\Pi_{i,t}^{pf,2}(p_t^{pf*}(\gamma_t)|\gamma_t) = 0$ for each i, i.e.,

$$\begin{aligned} &-\theta_{ii,t}(p_{i,t}^{pf*}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{pf*}(\gamma_t)) + \rho_{i,t}(p_t^{pf*}(\gamma_t)) \\ =& -2\theta_{ii,t}p_{i,t}^{pf*}(\gamma_t) + \sum_{j\neq i}\theta_{ij,t}p_{j,t}^{pf*}(\gamma_t) + f_{i,t}(\gamma_t) = 0, \text{ for all } i. \end{aligned} \tag{B.19}$$

In terms of the matrix language, we have $A_t p_t^{pf*}(\gamma_t) = f_t(\gamma_t)$. By Lemma 24(a), A_t is invertible and, thus, $p_t^{pf*}(\gamma_t)$ is uniquely determined by $p_t^{pf*}(\gamma_t) = A_t^{-1}f_t(\gamma_t)$. To show that $p_{i,t}^{pf*}(\gamma_t) = \sum_j (A_t^{-1})_{ij}f_{j,t}(\gamma_t)$ is continuously increasing in $\gamma_{j,t}$, we observe that

$$\frac{\partial p_{i,t}^{pf*}(\gamma_t)}{\partial \gamma_{j,t}} = (A_t^{-1})_{ij}\theta_{jj,t}\nu'_{j,t}(\gamma_{j,t}).$$

Since, by Lemma 24(a), $(A_t^{-1})_{ij} \geq 0$ for all i and j, we have $\partial_{\gamma_{j,t}} p_{i,t}^{pf*}(\gamma_t) \geq 0$ and, thus, $p_{i,t}^{pf*}(\gamma_t)$ is continuously increasing in $\gamma_{j,t}$ for each j.

Now, we compute $\Pi_{i,t}^{pf*,2}(\gamma_t)$.

$$\begin{aligned}
\Pi_{i,t}^{pf*,2}(\gamma_t) &= \rho_{i,t}(p_t^{pf*}(\gamma_t))(p_{i,t}^{pf*}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{pf*}) \\
&= (\phi_{i,t} - \theta_{ii,t} p_{i,t}^{pf*}(\gamma_t) + \sum_{j \neq i} \theta_{ij,t} p_{j,t}^{pf*}(\gamma_t))(p_{i,t}^{pf*}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{pf*}) \\
&= (\theta_{ii,t} p_{i,t}^{pf*}(\gamma_t) - f_{i,t}(\gamma_t) + \phi_{i,t})(p_{i,t}^{pf*}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{pf*}) \\
&= \theta_{ii,t}(p_{i,t}^{pf*}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{pf*})^2,
\end{aligned}$$

where the third equality follows from (B.19) and the last from $f_{i,t}(\gamma_t) = \phi_{i,t} + \theta_{ii,t}(\delta_i w_{i,t-1} + \nu_{i,t}(\gamma_{i,t}) - \pi_{i,t}^{pf*})$. The above computation also implies that $\rho_{i,t}(p_t^{pf*}(\gamma_t)) = \theta_{ii,t}(p_{i,t}^{pf*}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{pf*})$. We now show that $\Pi_{i,t}^{pf*,2}(\gamma_t) > 0$. Note that $\Pi_{i,t}^{pf*,2}(\gamma_t) = \frac{1}{\theta_{ii,t}}[\rho_{i,t}(p_t^{pf*}(\gamma_t))]^2 > 0$, where the inequality follows from the assumption that $\rho_{i,t}(\cdot) > 0$ for all p_t. This completes the proof of Proposition 3.5.2.

Next, we show Proposition 3.5.3. Since $\Pi_{i,t}^{pf*,2}(\gamma_t) > 0$ for all γ_t, $\Pi_{i,t}^{pf,1}(\gamma_t) = \Pi_{i,t}^{pf*,2}(\gamma_t)\psi_{i,t}(\gamma_t) > 0$ and, hence, $\log(\Pi_{i,t}^{pf,1}(\cdot))$ is well defined. Therefore,

$$\log(\Pi_{i,t}^{pf,1}(\gamma_t)) = \log(\theta_{ii,t}) + 2\log(p_{i,t}^{pf*}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{pf*}) + \log(\psi_{i,t}(\gamma_t)). \tag{B.20}$$

Since

$$p_{j,t}^{pf*}(\gamma_t) = \sum_{l=1}^{N} (A_t^{-1})_{jl} f_{l,t}(\gamma_t) = \sum_{l=1}^{N} [(A_t^{-1})_{jl}(\phi_{l,t} + \theta_{ll,t}(\delta_l w_{l,t-1} + \nu_{l,t}(\gamma_{l,t}) - \pi_{l,t}^{pf*}))], \text{ for all } j,$$

by direct computation,

$$\frac{\partial^2 \log(\Pi_{i,t}^{pf,1}(\gamma_t))}{\partial\gamma_{i,t}\partial\gamma_{j,t}} = \frac{2(1 - \theta_{ii,t}(A_t^{-1})_{ii})\theta_{jj,t}(A_t^{-1})_{ij}\nu'_{i,t}(\gamma_{i,t})\nu'_{j,t}(\gamma_{j,t})}{(p_{i,t}^{pf*}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{pf*})^2} + \frac{\partial^2 \log(\psi_{i,t}(\gamma_t))}{\partial\gamma_{i,t}\partial\gamma_{j,t}}, \text{ for all } j \neq i. \tag{B.21}$$

By Lemma 24(a,b), $1 - \theta_{ii,t}(A_t^{-1})_{ii} > 0$ and $(A_t^{-1})_{ij} \geq 0$. Thus, the first term of (B.21) is non-negative. Because $\psi_{i,t}(\cdot)$ satisfies (3.3),

$$\frac{\partial^2 \log(\Pi_{i,t}^{pf,1}(\gamma_t))}{\partial\gamma_{i,t}\partial\gamma_{j,t}} \geq \frac{\partial^2 \log(\psi_{i,t}(\gamma_t))}{\partial\gamma_{i,t}\partial\gamma_{j,t}} \geq 0, \text{ for all } j \neq i.$$

and, thus, $\mathcal{G}_t^{pf,1}$ is a log-supermodular game. The feasible action set of player i, $[0, \bar{\gamma}_{i,t}]$, is a compact subset of $\mathbb{R}$. Therefore, by Theorem 2 in [194], the pure strategy Nash equilibria of $\mathcal{G}_t^{pf,1}$ is a nonempty complete sublattice of $\mathbb{R}^N$

We now show that if $\nu_{i,t}(\gamma_{i,t}) = \gamma_{i,t}$, the Nash equilibrium of $\mathcal{G}_t^{pf,1}$ is unique. We first show that

$$\frac{\partial^2 \log(\Pi_{i,t}^{pf,1}(\gamma_t))}{\partial\gamma_{i,t}^2} < 0, \text{ and } \left|\frac{\partial^2 \log(\Pi_{i,t}^{pf,1}(\gamma_t))}{\partial\gamma_{i,t}^2}\right| > \sum_{j \neq i} \frac{\partial^2 \log(\Pi_{i,t}^{pf,1}(\gamma_t))}{\partial\gamma_{i,t}\partial\gamma_{j,t}}, \text{ for all } i \text{ and } \gamma_t. \tag{B.22}$$

Since $\nu_{l,t}(\gamma_{l,t}) = \gamma_{l,t}$ for all l (i.e., $\nu'_{l,t}(\cdot) \equiv 1$ for all l), direct computation yields that

$$\frac{\partial^2 \log(\Pi_{i,t}^{pf,1}(\gamma_t))}{\partial\gamma_{i,t}^2} = \frac{\partial^2 \log(\psi_{i,t}(\gamma_t))}{\partial\gamma_{i,t}^2} - \frac{2(1 - \theta_{ii,t}(A_t^{-1})_{ii})^2}{(p_{i,t}^{pf*}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi_{i,t}^{pf*})^2}.$$

Inequality (3.3) implies that $\partial^2_{\gamma_{i,t}} \log(\psi_{i,t}(\gamma_t)) < 0$ and, thus, $\partial^2_{\gamma_{i,t}} \log(\Pi^{pf,1}_{i,t}(\gamma_t)) < 0$. Moreover,

$$|\frac{\partial^2 \log(\Pi^{pf,1}_{i,t}(\gamma_t))}{\partial \gamma^2_{i,t}}| = |\frac{\partial^2 \log(\psi_{i,t}(\gamma_t))}{\partial \gamma^2_{i,t}}| + \frac{2(1-\theta_{ii,t}(A_t^{-1})_{ii})^2}{(p^{pf*}_{i,t}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{pf*}_{i,t})^2}$$

and

$$\sum_{j\neq i} \frac{\partial^2 \log(\Pi^{pf,1}_{i,t}(\gamma_t))}{\partial \gamma_{i,t}\partial \gamma_{j,t}} = \sum_{j\neq i} \frac{\partial^2 \log(\psi_{i,t}(\gamma_t))}{\partial \gamma_{i,t}\partial \gamma_{j,t}} + \sum_{j\neq i} \frac{2(1-\theta_{ii,t}(A_t^{-1})_{ii})\theta_{jj,t}(A_t^{-1})_{ij}}{(p^{pf*}_{i,t}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{pf*}_{i,t})^2}.$$

Inequality (3.3) implies that

$$|\frac{\partial^2 \log(\psi_{i,t}(\gamma_t))}{\partial \gamma^2_{i,t}}| > \sum_{j\neq i} \frac{\partial^2 \log(\psi_{i,t}(\gamma_t))}{\partial \gamma_{i,t}\partial \gamma_{j,t}}.$$

Lemma 24(b) implies that $1-\theta_{ii,t}(A_t^{-1})_{ii} > 0$. Moreover, Lemma 24(c) suggests that $1-(A_t^{-1})_{ii}\theta_{ii,t} > \sum_{j\neq i}(A_t^{-1})_{ij}\theta_{jj,t}$ and, hence,

$$\frac{2(1-\theta_{ii,t}(A_t^{-1})_{ii})^2}{(p^{pf*}_{i,t}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{pf*}_{i,t})^2} > \sum_{j\neq i} \frac{2(1-\theta_{ii,t}(A_t^{-1})_{ii})\theta_{jj,t}(A_t^{-1})_{ij}}{(p^{pf*}_{i,t}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{pf*}_{i,t})^2}.$$

Therefore, inequality (B.22) holds for all γ_t.

Because $\mathcal{G}^{pf,1}_t$ is a log-supermodular game, by Theorem 5 in [124], if there are two distinct pure strategy Nash equilibria $\hat{\gamma}^{pf*}_t \neq \gamma^{pf*}_t$, we must have $\hat{\gamma}^{pf*}_{i,t} \geq \gamma^{pf*}_{i,t}$ for each i, with the inequality being strict for some i. Without loss of generality, we assume that $\hat{\gamma}^{pf*}_{1,t} > \gamma^{pf*}_{1,t}$ and $\hat{\gamma}^{pf*}_{1,t} - \gamma^{pf*}_{1,t} \geq \hat{\gamma}^{pf*}_{i,t} - \gamma^{pf*}_{i,t}$ for each i. Lemma 23 yields that

$$\frac{\partial \log(\Pi^{pf,1}_{1,t}(\hat{\gamma}^{pf*}_t))}{\partial \gamma_{1,t}} \geq \frac{\partial \log(\Pi^{pf,1}_{1,t}(\gamma^{pf*}_t))}{\partial \gamma_{1,t}} \tag{B.23}$$

Since $\partial_{\gamma_{1,t}}\partial_{\gamma_{i,t}} \log(\Pi^{pf,1}_{1,t}(\gamma_t))$ is Lebesgue integrable for all $i \neq 1$ and γ_t, Newton-Leibniz formula implies that

$$\begin{aligned}\frac{\partial \log(\Pi^{pf,1}_{1,t}(\hat{\gamma}^{pf*}_t))}{\partial \gamma_{1,t}} - \frac{\partial \log(\Pi^{pf,1}_{1,t}(\gamma^{pf*}_t))}{\partial \gamma_{1,t}} &= \int_{s=0}^1 \sum_{j=1}^N (\hat{\gamma}^{pf*}_{j,t} - \gamma^{pf*}_{j,t}) \frac{\partial^2 \log(\Pi^{pf,1}_{1,t}((1-s)\gamma^{pf*}_t + s\hat{\gamma}^{pf*}_t))}{\partial \gamma_{1,t}\partial \gamma_{j,t}}\, \mathrm{d}s \\ &\leq \int_{s=0}^1 \sum_{j=1}^N (\hat{\gamma}^{pf*}_{1,t} - \gamma^{pf*}_{1,t}) \frac{\partial^2 \log(\Pi^{pf,1}_{1,t}((1-s)\gamma^{pf*}_t + s\hat{\gamma}^{pf*}_t))}{\partial \gamma_{1,t}\partial \gamma_{j,t}}\, \mathrm{d}s \\ &<0,\end{aligned}$$

where the first inequality follows from $\hat{\gamma}^{pf*}_{1,t} - \gamma^{pf*}_{1,t} \geq \hat{\gamma}^{pf*}_{i,t} - \gamma^{pf*}_{i,t}$ for all i, and the second from (B.22), and $\hat{\gamma}^{pf*}_{1,t} - \gamma^{pf*}_{1,t} > 0$. This contradicts (B.23). Thus, $\mathcal{G}^{pf,1}_t$ has a unique pure strategy Nash equilibrium γ^{pf*}_t.

We now show that the unique pure strategy Nash equilibrium γ^{pf*}_t can be characterized by the system of first-order conditions (3.26). First, (B.22) implies that $\log(\Pi^{pf,1}_{i,t}(\cdot, \gamma_{-i,t}))$ is strictly concave in $\gamma_{i,t}$ for any i and any fixed $\gamma_{-i,t}$. Hence, γ^{pf*}_t must satisfy the system of first-order conditions, i.e., for each i, $\partial_{\gamma_{i,t}} \log(\Pi^{pf,1}_{i,t}(\gamma^{pf*}_t)) \leq 0$ if $\gamma^{pf*}_{i,t} = 0$; $\partial_{\gamma_{i,t}} \log(\Pi^{pf,1}_{i,t}(\gamma^{pf*}_t)) = 0$ if $\gamma^{pf*}_{i,t} \in (0, \bar{\gamma}_{i,t})$; and $\partial_{\gamma_{i,t}} \log(\Pi^{pf,1}_{i,t}(\gamma^{pf*}_t)) \geq 0$ if $\gamma^{pf*}_{i,t} = \bar{\gamma}_{i,t}$. Differentiate (B.20), and we have

$$\partial_{\gamma_{i,t}} \log(\Pi^{pf,1}_{i,t}(\gamma_t)) = \frac{\partial_{\gamma_{i,t}}\psi_{i,t}(\gamma_t)}{\psi_{i,t}(\gamma_t)} - \frac{2(1-\theta_{ii,t}(A_t^{-1})_{ii})\nu'_{i,t}(\gamma_{i,t})}{p^{pf*}_{i,t}(\gamma_t) - \delta_i w_{i,t-1} - \nu_{i,t}(\gamma_{i,t}) + \pi^{pf*}_{i,t}}.$$

Thus, γ_t^{pf*} satisfies the system of first-order conditions (3.26). Since, by Proposition 3.5.2(c), $\Pi_{i,t}^{pf*,2}(\gamma_t^{pf*}) > 0$ and $\psi_{i,t}(\gamma_t^{pf*}) > 0$, we have $\Pi_{i,t}^{pf*,1} = \Pi_{i,t}^{pf*,2}(\gamma_t^{pf*})\psi_{i,t}(\gamma_t^{pf*}) > 0$ for all i. This completes the proof of Proposition 3.5.3.

Next, we show that $\{(\gamma_{i,t}^{pf*}, p_{i,t}^{pf*}(\gamma_t), \Lambda_{i,t}y_{i,t}^{pf*}\rho_{i,t}(p_t^{pf*}(\gamma_t))\psi_{i,t}(\gamma_t)) : 1 \leq i \leq N\}$ is an equilibrium in the subgame of period t. By Proposition 3.5.1, $y_{i,t}^{pf*} > 0$, $\Lambda_{i,t}y_{i,t}^{pf*}\rho_{i,t}(p_t^{pf*}(\gamma_t))\psi_{i,t}(\gamma_t) > 0$ for all i. Therefore, regardless of the starting inventory level in period t, $I_{i,t}$, firm i could adjust its inventory to $x_{i,t}^{pf*}(I_t, \Lambda_t, \gamma_t) = \Lambda_{i,t}y_{i,t}^{pf*}\rho_{i,t}(p_t^{pf*}(\gamma_t))\psi_{i,t}(\gamma_t)$. Thus, $\{(\gamma_{i,t}^{pf*}, p_{i,t}^{pf*}(\gamma_t), \Lambda_{i,t}y_{i,t}^{pf*}\rho_{i,t}(p_t^{pf*}(\gamma_t))\psi_{i,t}(\gamma_t)) : 1 \leq i \leq N\}$ forms an equilibrium in the subgame of period t. In particular, this equilibrium is the unique one, if $\nu_{i,t}(\gamma_{i,t}) = \gamma_{i,t}$ for all i.

Finally, we show that there exists a positive vector $\beta_t^{pf} = (\beta_{1,t}^{pf}, \beta_{2,t}^{pf}, \cdots, \beta_{N,t}^{pf})$, such that $V_{i,t}(I_t, \Lambda_t|\sigma_t^{pf*}) = w_{i,t}I_{i,t} + \beta_{i,t}^{pf}\Lambda_{i,t}$. By (3.22), we have that

$$\begin{aligned} V_{i,t}(I_t, \Lambda_t|\sigma_t^{pf*}) =& J_{i,t}(\gamma_{i,t}^{pf*}, p_{i,t}^{pf*}(\gamma_t^{pf*}), \Lambda_{i,t}y_{i,t}^{pf*}\rho_{i,t}(p_t^{pf*}(\gamma_t^{pf*}))\psi_{i,t}(\gamma_t^{pf*}), I_t, \Lambda_t|\sigma_{t-1}^{pf*}) \\ =& w_{i,t}I_{i,t} + (\sigma_i\beta_{i,t-1}^{pf}\mu_{i,t} + \Pi_{i,t}^{pf*,1})\Lambda_{i,t}. \end{aligned}$$

Since $\beta_{i,t-1}^{pf} > 0$, $\beta_{i,t}^{pf} = \delta_i\beta_{i,t-1}^{pf}\mu_{i,t} + \Pi_{i,t}^{pf*,1} > 0$. This completes the induction and, thus, the proof of Theorem 3.5.1, Proposition 3.5.1, Proposition 3.5.2, Proposition 3.5.3, and Theorem 3.5.2. *Q.E.D.*

Proof of Proposition 3.5.4: By Theorems 3.5.1-3.5.2, and Propositions 3.5.1-3.5.3, it suffices to show that, if there exists a constant $\beta_{s,t-1}^{pf} \geq 0$, such that $V_{i,t-1}(I_{t-1}, \Lambda_{t-1}|\sigma_{t-1}^{pf*}) = w_{s,t}I_{i,t-1} + \beta_{s,t-1}^{pf}\Lambda_{i,t-1}$ for all i, we have: (a) the unique Nash equilibrium in $\mathcal{G}_t^{pf,3}$ is symmetric, i.e., $y_{i,t}^{pf*} = y_{j,t}^{pf*}$ for all i, j; (b) the unique Nash equilibrium in $\mathcal{G}_t^{pf,2}(\gamma_t)$ is symmetric if $\gamma_{i,t} = \gamma_{j,t}$ for all i and j, (c), the unique Nash equilibrium in $\mathcal{G}_t^{pf,1}$, γ_t^{pf*} is symmetric, and (d) there exists a constant $\beta_{s,t}^{pf} > 0$, such that $V_{i,t}(I_t, \Lambda_t|\sigma_{s,t}^{pf*}) = w_{s,t}I_{i,t} + \beta_{s,t}^{pf}\Lambda_{i,t}$ for all i. Since $V_{i,0}(I_t, \Lambda_t) = w_{i,0}I_{i,0}$ for all i, the initial condition is satisfied with $\beta_{s,0}^{pf} = 0$.

First, we observe that $y_{i,t}^{pf*} = y_{j,t}^{pf*}$ and $\pi_{i,t}^{pf*} = \pi_{j,t}^{pf*}$ for all i and j follow directly from the same proof of Proposition 3.4.3. Thus, we omit their proofs for brevity, and denote $y_{s,t}^{pf*} := y_{i,t}^{pf*}$ and $\pi_{s,t}^{pf*} = \pi_{i,t}^{pf*}$ for each firm i in $\mathcal{G}_t^{pf,3}$.

Next, we show that if $\gamma_{i,t} = \gamma_{j,t}$ for all i and j, $p_{i,t}^{pf*}(\gamma_t) = p_{j,t}^{pf*}(\gamma_t)$. Direct computation yields that, for the symmetric PF model, $\sum_{j=1}^N (A_t^{-1})_{ij}$ is independent of i. Thus, if the value of $\gamma_{j,t}$ is independent of j,

$$\begin{aligned} p_{i,t}^{pf*}(\gamma_t) =& \sum_{j=1}^N (A_t^{-1})_{ij} f_{j,t}(\gamma_t) = \sum_{j=1}^N [(A_t^{-1})_{ij}(\phi_{s,t} + \theta_{sa,t}(\delta_s w_{s,t-1} + \nu_{s,t}(\gamma_{j,t}) - \pi_{s,t}^{pf*}))] \\ =& (\phi_{s,t} + \theta_{sa,t}(\delta_s w_{s,t-1} + \nu_{s,t}(\gamma_{j,t}) - \pi_{s,t}^{pf*})) \sum_{j=1}^N (A_t^{-1})_{ij}, \end{aligned} \tag{B.24}$$

which is independent of firm i, which we denote as $p_{s,t}^{pf*}(\gamma_t)$.

Note that the objective functions of $\mathcal{G}_t^{pf,1}$,

$$\{\Pi_{i,t}^{pf,1}(\gamma_t) = \theta_{sa,t}(p_{i,t}^{pf*}(\gamma_t) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma_{i,t}) + \pi_{s,t}^{pf*})\psi_{s,t}(\gamma_t) : 1 \leq i \leq N\}$$

are symmetric. Thus, if there exists an asymmetric Nash equilibrium γ_t^{pf*}, there exists another Nash equilibrium $\underline{\gamma}_t^{pf*} \neq \gamma_t^{pf*}$, where $\underline{\gamma}_t^{pf*}$ is a permutation of γ_t^{pf*}. This contradicts the uniqueness of the Nash equilibrium in $\mathcal{G}_t^{pf,1}$. Thus, the unique Nash equilibrium in $\mathcal{G}_t^{pf,1}$ is symmetric, which we denote as $\gamma_{ss,t}^{pf*} = (\gamma_{s,t}^{pf*}, \gamma_{s,t}^{pf*}, \cdots, \gamma_{s,t}^{pf*})$. Hence,

$$\Pi_{i,t}^{pf*,1} = \Pi_{i,t}^{pf,1}(\gamma_{ss,t}^{pf*}) = \Pi_{j,t}^{pf,1}(\gamma_{ss,t}^{pf*}) = \Pi_{j,t}^{pf*,1} > 0.$$

Thus, we denote the payoff of each firm i in $\mathcal{G}_t^{pf,1}$ as $\Pi_{s,t}^{pf*,1}$. By Theorem 3.5.2(a),

$$\beta_{i,t}^{pf} = \delta_s \beta_{s,t-1}^{pf} \mu_{s,t} + \Pi_{i,t}^{pf*,1} = \delta_s \beta_{s,t-1}^{pf} \mu_{s,t} + \Pi_{j,t}^{pf*,1} = \beta_{j,t}^{pf} > 0.$$

Thus, we denote the PF market size coefficient of each firm i as $\beta_{s,t}^{pf}$. This completes the induction and, thus, the proof of Proposition 3.5.4. *Q.E.D.*

Proof of Theorem 3.5.3: Parts (a)-(b). The proof of parts (a)-(b) follows from the same argument as that of Theorem 3.4.3(a)-(b) and is, hence, omitted.

Part (c). Because

$$p_{i,t}^{pf*}(\gamma_t) = \sum_{j=1}^{N} (A_t^{-1})_{ij} f_{j,t}(\gamma_t) = \sum_{j=1}^{N} [(A_t^{-1})_{ij}(\phi_{j,t} + \theta_{jj,t}(\delta_j w_{j,t-1} + \nu_{j,t}(\gamma_{j,t}) - \pi_{j,t}^{pf*}))],$$

we have

$$\partial_{\pi_{j,t}^{pf*}} p_{i,t}^{pf*}(\gamma_t) = -\theta_{jj,t}(A_t^{-1})_{ij} \leq 0,$$

where the inequality follows from Lemma 24(a). Thus, $p_{i,t}^{pf*}(\gamma_t)$ is continuously decreasing in $\pi_{j,t}^{pf*}$ for each j. Part (c) follows.

Part (d). We denote the objective function of each firm i in $\mathcal{G}_{s,t}^{pf,1}$ as $\Pi_{i,t}^{pf,1}(\cdot|\pi_{s,t}^{pf*})$ to capture its dependence on $\pi_{s,t}^{pf*}$. The unique symmetric pure strategy Nash equilibrium in $\mathcal{G}_{s,t}^{pf,1}$ is denoted as $\gamma_{ss,t}^{pf*}(\pi_{s,t}^{pf*})$ to capture the dependence of the equilibrium on $\pi_{s,t}^{pf*}$, where
$\gamma_{ss,t}^{pf*}(\pi_{s,t}^{pf*}) = (\gamma_{s,t}^{pf*}(\pi_{s,t}^{pf*}), \gamma_{s,t}^{pf*}(\pi_{s,t}^{pf*}), \cdots, \gamma_{s,t}^{pf*}(\pi_{s,t}^{pf*}))$. We first show that, if $\bar{\pi}_{s,t}^{pf*} > \pi_{s,t}^{pf*}$, $\gamma_{s,t}^{pf*}(\bar{\pi}_{s,t}^{pf*}) \geq \gamma_{s,t}^{pf*}(\pi_{s,t}^{pf*})$.

If, to the contrary, $\gamma_{s,t}^{pf*}(\bar{\pi}_{s,t}^{pf*}) < \gamma_{s,t}^{pf*}(\pi_{s,t}^{pf*})$, Lemma 23 yields that $\partial_{\gamma_{1,t}} \log(\Pi_{1,t}^{pf,1}(\gamma_{s,t}^{pf*}(\bar{\pi}_{s,t}^{pf*})|\bar{\pi}_{s,t}^{pf*})) \leq \partial_{\gamma_{1,t}} \log(\Pi_{1,t}^{pf,1}(\gamma_{s,t}^{pf*}(\pi_{s,t}^{pf*})|\pi_{s,t}^{pf*}))$, i.e.,

$$\begin{aligned} & \partial_{\gamma_{1,t}} \log(\psi_{s,t}(\gamma_{s,t}^{pf*}(\bar{\pi}_{s,t}^{pf*})) - \frac{2(1-\theta_{sa,t}(A_t^{-1})_{ii})\nu'_{s,t}(\gamma_{s,t}^{pf*}(\bar{\pi}_{s,t}^{pf*}))}{p_{s,t}^{pf*}(\gamma_{ss,t}^{pf*}(\bar{\pi}_{s,t}^{pf*})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma_{s,t}^{pf*}(\bar{\pi}_{s,t}^{pf*})) + \bar{\pi}_{s,t}^{pf*}} \\ \leq \; & \partial_{\gamma_{1,t}} \log(\psi_{s,t}(\gamma_{s,t}^{pf*}(\pi_{s,t}^{pf*})) - \frac{2(1-\theta_{sa,t}(A_t^{-1})_{ii})\nu'_{s,t}(\gamma_{s,t}^{pf*}(\pi_{s,t}^{pf*}))}{p_{s,t}^{pf*}(\gamma_{ss,t}^{pf*}(\pi_{s,t}^{pf*})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma_{s,t}^{pf*}(\pi_{s,t}^{pf*})) + \pi_{s,t}^{pf*}}. \end{aligned}$$

Note that

$$\begin{aligned} & [p_{s,t}^{pf*}(\gamma_{ss,t}^{pf*}(\bar{\pi}_{s,t}^{pf*})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma_{s,t}^{pf*}(\bar{\pi}_{s,t}^{pf*})) + \bar{\pi}_{s,t}^{pf*}] - [p_{s,t}^{pf*}(\gamma_{ss,t}^{pf*}(\pi_{s,t}^{pf*})) - \delta_s w_{s,t-1} \\ & - \nu_{s,t}(\gamma_{s,t}^{pf*}(\pi_{s,t}^{pf*})) + \pi_{s,t}^{pf*}] \\ = & (1 - \sum_{j=1}^{N} (A_t^{-1})_{1j}\theta_{sa,t})(\nu_{s,t}(\gamma_{s,t}^{pf*}(\pi_{s,t}^{pf*})) - \nu_{s,t}(\gamma_{s,t}^{pf*}(\bar{\pi}_{s,t}^{pf*}))) + (1 - \sum_{j=1}^{N} (A_t^{-1})_{1j}\theta_{sa,t})(\bar{\pi}_{s,t}^{pf*} - \pi_{s,t}^{pf*}) \\ > & 0 \end{aligned} \tag{B.25}$$

where the inequality follows from Lemma 24(c). Thus,

$$\begin{aligned}&p^{pf*}_{s,t}(\gamma^{pf*}_{ss,t}(\bar\pi^{pf*}_{s,t})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{pf*}_{s,t}(\bar\pi^{pf*}_{s,t})) + \bar\pi^{pf*}_{s,t}\\ >&p^{pf*}_{s,t}(\gamma^{pf*}_{ss,t}(\pi^{pf*}_{s,t})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t})) + \pi^{pf*}_{s,t} > 0.\end{aligned}$$

Lemma 24(b) implies that $1-\theta_{sa,t}(A_t^{-1})_{ii} > 0$. Hence,

$$\begin{aligned}&-\frac{2(1-\theta_{sa,t}(A_t^{-1})_{ii})\nu'_{s,t}(\gamma^{pf*}_{s,t}(\bar\pi^{pf*}_{s,t}))}{p^{pf*}_{s,t}(\gamma^{pf*}_{ss,t}(\bar\pi^{pf*}_{s,t})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{pf*}_{s,t}(\bar\pi^{pf*}_{s,t})) + \bar\pi^{pf*}_{s,t}}\\ \geq&-\frac{2(1-\theta_{sa,t}(A_t^{-1})_{ii})\nu'_{s,t}(\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t}))}{p^{pf*}_{s,t}(\gamma^{pf*}_{ss,t}(\pi^{pf*}_{s,t})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t})) + \pi^{pf*}_{s,t}}.\end{aligned}$$

Thus, we have

$$\partial_{\gamma_{1,t}}\log(\psi_{s,t}(\gamma^{pf*}_{s,t}(\bar\pi^{pf*}_{s,t})) \leq \partial_{\gamma_{1,t}}\log(\psi_{s,t}(\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t})). \tag{B.26}$$

By (3.3) and Newton-Leibniz formula,

$$\begin{aligned}&\partial_{\gamma_{1,t}}\log(\psi_{1,t}(\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t}))) - \partial_{\gamma_{1,t}}\log(\psi_{1,t}(\gamma^{pf*}_{s,t}(\bar\pi^{pf*}_{s,t})))\\ =&\int_{s=0}^1\sum_{j=1}^N(\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t}) - \gamma^{pf*}_{s,t}(\bar\pi^{pf*}_{s,t}))[\frac{\partial^2\log(\psi_{s,t}(s\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t}) + (1-s)\gamma^{pf*}_{s,t}(\bar\pi^{pf*}_{s,t})))}{\partial\gamma_{1,t}\partial\gamma_{j,t}}]\,\mathrm{d}s\\ <&\ 0,\end{aligned}$$

which contradicts (B.26). Therefore, $\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t})$ is increasing in $\pi^{pf*}_{s,t}$. The continuity of $\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t})$ in $\pi^{pf*}_{s,t}$ follows directly from that $\Pi^{pf,1}_{i,t}(\gamma_t|\pi^{pf*}_{s,t})$ is twice continuously differentiable in $(\gamma_t, \pi^{pf*}_{s,t})$ and the implicit function theorem.

Next we show that if (3.17) holds, $\beta^{pf}_{s,t}(\pi^{pf*}_{s,t})$ is increasing in $\pi^{pf*}_{s,t}$. By Theorem 3.5.2(a), it suffices to show that $\Pi^{pf*,1}_{s,t}(\pi^{pf*}_{s,t}) := \Pi^{pf*,1}_{s,t}(\gamma^{pf*}_{ss,t}(\pi^{pf*}_{s,t})|\pi^{pf*}_{s,t})$ is increasing in $\pi^{pf*}_{s,t}$. Assume that $\bar\pi^{pf*}_{s,t} > \pi^{pf*}_{s,t}$. Since we have just shown $\gamma^{pf*}_{s,t}(\bar\pi^{pf*}_{s,t}) \geq \gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t})$, (3.17) implies that $\psi_{s,t}(\gamma^{pf*}_{ss,t}(\bar\pi^{pf*}_{s,t})) \geq \psi_{s,t}(\gamma^{pf*}_{ss,t}(\pi^{pf*}_{s,t}))$.

If $\gamma^{pf*}_{s,t}(\bar\pi^{pf*}_{s,t}) = \gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t})$,

$$p^{pf*}_{s,t}(\gamma^{pf*}_{ss,t}(\bar\pi^{pf*}_{s,t}))-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma^{pf*}_{s,t}(\bar\pi^{pf*}_{s,t}))+\bar\pi^{pf*}_{s,t} > p^{pf*}_{s,t}(\gamma^{pf*}_{ss,t}(\pi^{pf*}_{s,t}))-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t}))+\pi^{pf*}_{s,t},$$

and, hence,

$$\begin{aligned}\Pi^{pf*,1}_{s,t}(\bar\pi^{pf*}_{s,t}) &= \theta_{sa,t}(p^{pf*}_{s,t}(\gamma^{pf*}_{ss,t}(\bar\pi^{pf*}_{s,t})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{pf*}_{s,t}(\bar\pi^{pf*}_{s,t})) + \bar\pi^{pf*}_{s,t})^2\psi_{s,t}(\gamma^{pf*}_{ss,t}(\bar\pi^{pf*}_{s,t}))\\ &> \theta_{sa,t}(p^{pf*}_{s,t}(\gamma^{pf*}_{ss,t}(\pi^{pf*}_{s,t})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t})) + \pi^{pf*}_{s,t})^2\psi_{s,t}(\gamma^{pf*}_{ss,t}(\pi^{pf*}_{s,t}))\\ &= \Pi^{pf*,1}_{s,t}(\pi^{pf*}_{s,t}).\end{aligned}$$

If $\gamma^{pf*}_{ss,t}(\bar\pi^{pf*}_{s,t}) > \gamma^{pf*}_{ss,t}(\pi^{pf*}_{s,t})$, Lemma 23 implies that $\partial_{\gamma_{1,t}}\log(\Pi^{pf,1}_{1,t}(\gamma^{pf*}_{s,t}(\bar\pi^{pf*}_{s,t})|\bar\pi^{pf*}_{s,t})) \geq \partial_{\gamma_{1,t}}\log(\Pi^{pf,1}_{1,t}(\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t})|\pi^{pf*}_{s,t}))$, i.e.,

$$\begin{aligned}&\partial_{\gamma_{1,t}}\log(\psi_{s,t}(\gamma^{pf*}_{s,t}(\bar\pi^{pf*}_{s,t})) - \frac{2(1-\theta_{sa,t}(A_t^{-1})_{ii})\nu'_{s,t}(\gamma^{pf*}_{s,t}(\bar\pi^{pf*}_{s,t}))}{p^{pf*}_{s,t}(\gamma^{pf*}_{ss,t}(\bar\pi^{pf*}_{s,t})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{pf*}_{s,t}(\bar\pi^{pf*}_{s,t})) + \bar\pi^{pf*}_{s,t}}\\ \geq\ &\partial_{\gamma_{1,t}}\log(\psi_{s,t}(\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t})) - \frac{2(1-\theta_{sa,t}(A_t^{-1})_{ii})\nu'_{s,t}(\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t}))}{p^{pf*}_{s,t}(\gamma^{pf*}_{ss,t}(\pi^{pf*}_{s,t})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t})) + \pi^{pf*}_{s,t}}.\end{aligned}$$

By (3.3) and Newton-Leibniz formula,

$$\begin{aligned}
& \partial_{\gamma_{1,t}} \log(\psi_{1,t}(\gamma^{pf*}_{s,t}(\bar{\pi}^{pf*}_{s,t}))) - \partial_{\gamma_{1,t}} \log(\psi_{1,t}(\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t}))) \\
= & \int_{s=0}^{1} \sum_{j=1}^{N} (\gamma^{pf*}_{s,t}(\bar{\pi}^{pf*}_{s,t}) - \gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t}))[\frac{\partial^2 \log(\psi_{s,t}((1-s)\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t}) + s\gamma^{pf*}_{s,t}(\bar{\pi}^{pf*}_{s,t})))}{\partial\gamma_{1,t}\partial\gamma_{j,t}}]\,\mathrm{d}s \\
< & \ 0,
\end{aligned}$$

Hence,

$$\begin{aligned}
& -\frac{2(1-\theta_{sa,t}(A_t^{-1})_{ii})\nu'_{s,t}(\gamma^{pf*}_{s,t}(\bar{\pi}^{pf*}_{s,t}))}{p^{pf*}_{s,t}(\gamma^{pf*}_{ss,t}(\bar{\pi}^{pf*}_{s,t})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{pf*}_{s,t}(\bar{\pi}^{pf*}_{s,t})) + \bar{\pi}^{pf*}_{s,t}} \\
> & -\frac{2(1-\theta_{sa,t}(A_t^{-1})_{ii})\nu'_{s,t}(\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t}))}{p^{pf*}_{s,t}(\gamma^{pf*}_{ss,t}(\pi^{pf*}_{s,t})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t})) + \pi^{pf*}_{s,t}}.
\end{aligned}$$

Because, by Lemma 24(b) and the convexity of $\nu_{s,t}(\cdot)$, $1-\theta_{sa,t}(A_t^{-1})_{ii} > 0$ and $\nu'_{s,t}(\gamma^{pf*}_{s,t}(\bar{\pi}^{pf*}_{s,t})) \geq \nu'_{s,t}(\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t}))$, we have

$$p^{pf*}_{s,t}(\gamma^{pf*}_{ss,t}(\bar{\pi}^{pf*}_{s,t})) - \delta_s w_{s,t} - \nu_{s,t}(\gamma^{pf*}_{s,t}(\bar{\pi}^{pf*}_{s,t})) + \bar{\pi}^{pf*}_{s,t} > p^{pf*}_{s,t}(\gamma^{pf*}_{ss,t}(\pi^{pf*}_{s,t})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t})) + \pi^{pf*}_{s,t}.$$

Therefore,

$$\begin{aligned}
\Pi^{pf*,1}_{s,t}(\bar{\pi}^{pf*}_{s,t}) &= \theta_{sa,t}(p^{pf*}_{s,t}(\gamma^{pf*}_{ss,t}(\bar{\pi}^{pf*}_{s,t})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{pf*}_{s,t}(\bar{\pi}^{pf*}_{s,t})) + \bar{\pi}^{pf*}_{s,t})^2 \psi_{s,t}(\gamma^{pf*}_{ss,t}(\bar{\pi}^{pf*}_{s,t})) \\
&> \theta_{sa,t}(p^{pf*}_{s,t}(\gamma^{pf*}_{ss,t}(\pi^{pf*}_{s,t})) - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t})) + \pi^{pf*}_{s,t})^2 \psi_{s,t}(\gamma^{pf*}_{ss,t}(\pi^{pf*}_{s,t})) \\
&= \Pi^{pf*,1}_{s,t}(\pi^{pf*}_{s,t}).
\end{aligned}$$

We have, thus, shown that $\beta^{pf}_{s,t}(\pi^{pf*}_{s,t})$ is increasing in $\pi^{pf*}_{s,t}$. The continuity of $\beta^{pf}_{s,t}(\pi^{pf*}_{s,t})$ in $\pi^{pf*}_{s,t}$ follows directly from that of $\gamma^{pf*}_{s,t}(\pi^{pf*}_{s,t})$ and that $\Pi^{pf,1}_{i,t}(\gamma_t|\pi^{pf*}_{s,t})$ is continuous in $(\gamma_t, \pi^{pf*}_{s,t})$. This concludes the proof of part (d).

Part (e). By part (d), we have that $\gamma^{pf*}_{s,t}$ is continuously increasing in $\pi^{pf*}_{s,t}$ and, thus, $\beta^{pf}_{s,t-1}$. By part (c), we have that $p^{pf*}_{i,t}(\gamma_t)$ is continuously decreasing in $\pi^{pf*}_{s,t}$ and, thus, $\beta^{pf}_{s,t-1}$. Moreover, if (3.17) holds, part (d) yields that $\beta^{pf}_{s,t}$ is continuously increasing in $\pi^{pf*}_{s,t}$ and, thus, $\beta^{pf}_{s,t-1}$ as well. This completes the proof of part (e). *Q.E.D.*

Proof of Theorem 3.5.4: Part (a). Part (a) follows from the same argument as the proof of Theorem 3.4.4(a) and is, hence, omitted.

Part (b). By part (a), $\pi^{pf*}_{i,t} \geq \tilde{\pi}^{pf*}_{i,t}$ for each i. Hence, Theorem 3.5.3(c) yields that $p^{pf*}_{i,t}(\gamma_t) \leq \tilde{p}^{pf*}_{i,t}(\gamma_t)$ for each firm i and each γ_t.

When the PF model is symmetric, $\sum_{j=1}^{N}\theta_{jj,t}(A_t^{-1})_{ij}$ is independent of i. Direct computation yields that

$$\tilde{p}^{pf*}_{i,t}(\gamma_t) - p^{pf*}_{i,t}(\gamma_t) = (\sum_{j=1}^{N}\theta_{jj,t}(A_t^{-1})_{ij})(\pi^{pf*}_{s,t} - \tilde{\pi}^{pf*}_{s,t}) \geq 0, \text{ for all } \gamma_t,$$

which is independent of i. Thus, (3.17) and Newton-Leibniz formula imply that

$$\rho_{s,t}(\tilde{p}^{pf*}_t(\gamma_t)) - \rho_{s,t}(p^{pf*}_t(\gamma_t)) = \int_{s=0}^{1}\sum_{i=1}^{N}(\tilde{p}^{pf*}_{i,t}(\gamma_t) - p^{pf*}_{i,t}(\gamma_t))\frac{\partial\rho_{s,t}((1-s)p^{pf*}_t(\gamma_t) + s\tilde{p}^{pf*}_t(\gamma_t))}{\partial p_{i,t}}\,\mathrm{d}s \leq 0.$$

Hence, $\rho_{s,t}(p_t^{pf*}(\gamma_t)) \geq \rho_{s,t}(\tilde{p}_t^{pf*}(\gamma_t))$. Since $y_{s,t}^{pf*} \geq \tilde{y}_{s,t}^{pf*}$, Theorem 3.5.2(b) implies that, for any $(I_t, \Lambda_t) \in \mathcal{S}$ and $\gamma_t \in [0, \bar{\gamma}_{s,t}]^N$,

$$x_{i,t}^{pf*}(I_t, \Lambda_t, \gamma_t) = y_{s,t}^{pf*}\rho_{s,t}(p_t^{pf*}(\gamma_t))\psi_{s,t}(\gamma_t) \geq \tilde{y}_{s,t}^{pf*}\rho_{s,t}(\tilde{p}_t^{pf*}(\gamma_t))\psi_{s,t}(\gamma_t) = \tilde{x}_{i,t}^{pf*}(I_t, \Lambda_t, \gamma_t).$$

This completes the proof of part (b).

Part (c). Because $\pi_{s,t}^{pf*} \geq \tilde{\pi}_{s,t}^{pf*}$, Theorem 3.5.3(d) yields that $\gamma_{s,t}^{pf*} \geq \tilde{\gamma}_{s,t}^{pf*}$ and, hence, $\gamma_{i,t}^{pf*}(I_t, \Lambda_t) = \gamma_{s,t}^{pf*} \geq \tilde{\gamma}_{s,t}^{pf*} = \tilde{\gamma}_{s,t}^{pf*}(I_t, \Lambda_t)$ for each i and $(I_t, \Lambda_t) \in \mathcal{S}$. This completes the proof of part (c). *Q.E.D.*

Proof of Theorem 3.5.5: Part (a). We show part (a) by backward induction. More specifically, we show that if $\hat{\alpha}_{s,t}(z_t) \geq \alpha_{s,t}(z_t)$ for all z_t and $\hat{\beta}_{s,t-1}^{pf} \geq \beta_{s,t-1}^{pf}$, (i) $\hat{\pi}_{s,t}^{pf*} \geq \pi_{s,t}^{pf*}$, (ii) $\hat{p}_{s,t}^{pf*}(\gamma_t) \leq p_{s,t}^{pf*}(\gamma_t)$, (iii) $\hat{p}_{i,t}^{pf*}(I_t, \Lambda_t, \gamma_t) \leq p_{i,t}^{pf*}(I_t, \Lambda_t, \gamma_t)$ for each i, $(I_t, \Lambda_t) \in \mathcal{S}$, and $\gamma_t \in [0, \bar{\gamma}_{s,t}]^N$, (iv) $\hat{\gamma}_{s,t}^{pf*} \geq \gamma_{s,t}^{pf*}$, (v) $\hat{\gamma}_{i,t}^{pf*}(I_t, \Lambda_t) \geq \gamma_{s,t}^{pf*}(I_t, \Lambda_t)$ for each i and $(I_t, \Lambda_t) \in \mathcal{S}$, and (vi) $\hat{\beta}_{s,t}^{pf} \geq \beta_{s,t}^{pf}$. Since $\hat{\beta}_{s,0}^{pf} = \beta_{s,0}^{pf} = 0$, the initial condition is satisfied.

The same argument as the proof of Theorem 3.4.5(a) implies that $\hat{\pi}_{s,t}^{pf*} \geq \pi_{s,t}^{pf*}$. Hence, Theorem 3.5.3(c) implies that $\hat{p}_{i,t}^{pf*}(\gamma_t) \leq p_{i,t}^{pf*}(\gamma_t)$ for all i and γ_t. Thus, $\hat{p}_{i,t}^{pf*}(I_t, \Lambda_t, \gamma_t) = \hat{p}_{i,t}^{pf*}(\gamma_t) \leq p_{i,t}^{pf*}(\gamma_t) = p_{i,t}^{pf*}(I_t, \Lambda_t, \gamma_t)$ for each i, $(I_t, \Lambda_t) \in \mathcal{S}$, and $\gamma_t \in [0, \bar{\gamma}_{s,t}]^N$. Analogously, Theorem 3.5.3(d) implies that $\hat{\gamma}_{s,t}^{pf*} \geq \gamma_{s,t}^{pf*}$. Hence, $\hat{\gamma}_{i,t}^{pf*}(I_t, \Lambda_t) = \hat{\gamma}_{s,t}^{pf*} \geq \gamma_{s,t}^{pf*} = \gamma_{i,t}^{pf*}(I_t, \Lambda_t)$ for each i and all $(I_t, \Lambda_t) \in \mathcal{S}$. By Theorem 3.5.3(d), under inequality (3.17), $\hat{\pi}_{s,t}^{pf*} \geq \pi_{s,t}^{pf*}$ implies that $\hat{\beta}_{s,t}^{pf} \geq \beta_{s,t}^{pf}$. This completes the induction and, thus, the proof of part (a).

Part (b). By part (a), it suffices to show that, if $\hat{\alpha}_{s,t}(z_t) \geq \alpha_{s,t}(z_t)$ for all z_t, $\hat{\kappa}'_{sa,t}(z_{i,t}) \geq \kappa'_{sa,t}(z_{i,t})$ for all $z_{i,t}$, and $\hat{\beta}_{s,t-1}^{pf} \geq \beta_{s,t-1}^{pf}$, we have (i) $\hat{y}_{s,t}^{pf*} \geq y_{s,t}^{pf*}$ and (ii) $\hat{x}_{i,t}^{pf*}(I_t, \Lambda_t, \gamma_t) \geq x_{i,t}^{pf*}(I_t, \Lambda_t, \gamma_t)$ for each i, $(I_t, \Lambda_t) \in \mathcal{S}$, and $\gamma_t \in [0, \bar{\gamma}_{s,t}]^N$.

The same argument as the proof of Theorem 3.4.5(b) suggests that $\hat{y}_{s,t}^{pf*} \geq y_{s,t}^{pf*}$. We now show that $\hat{x}_{i,t}^{pf*}(I_t, \Lambda_t, \gamma_t) \geq x_{i,t}^{pf*}(I_t, \Lambda_t, \gamma_t)$ for each i, $(I_t, \Lambda_t) \in \mathcal{S}$, and $\gamma_t \in [0, \bar{\gamma}_{s,t}]^N$. Because the PF model is symmetric, $\sum_{j=1}^N \theta_{jj,t}(A_t^{-1})_{ij}$ is independent of i. Direct computation yields that

$$p_{i,t}^{pf*}(\gamma_t) - \hat{p}_{i,t}^{pf*}(\gamma_t) = (\sum_{j=1}^N \theta_{jj,t}(A_t^{-1})_{ij})(\hat{\pi}_{s,t}^{pf*} - \pi_{s,t}^{pf*}) \geq 0, \text{ for all } \gamma_t,$$

which is independent of i. Thus, (3.17) and Newton-Leibniz formula implies that

$$\rho_{s,t}(p_t^{pf*}(\gamma_t)) - \rho_{s,t}(\hat{p}_t^{pf*}(\gamma_t)) = \int_{s=0}^1 \sum_{i=1}^N (p_{i,t}^{pf*}(\gamma_t) - \hat{p}_{i,t}^{pf*}(\gamma_t)) \frac{\partial \rho_{s,t}((1-s)\hat{p}_t^{pf*}(\gamma_t) + s p_t^{pf*}(\gamma_t))}{\partial p_{i,t}} \, \mathrm{d}s \leq 0.$$

Hence, $\rho_{s,t}(\hat{p}_t^{pf*}(\gamma_t)) \geq \rho_{s,t}(p_t^{pf*}(\gamma_t))$ for all γ_t. Since $\hat{y}_{s,t}^{pf*} \geq y_{s,t}^{pf*}$, Theorem 3.5.2(b) implies that, for any $(I_t, \Lambda_t) \in \mathcal{S}$ and $\gamma_t \in [0, \bar{\gamma}_{s,t}]^N$,

$$\hat{x}_{i,t}^{pf*}(I_t, \Lambda_t, \gamma_t) = \hat{y}_{s,t}^{pf*}\rho_{s,t}(\hat{p}_t^{pf*}(\gamma_t))\psi_{s,t}(\gamma_t) \geq y_{s,t}^{pf*}\rho_{s,t}(p_t^{pf*}(\gamma_t))\psi_{s,t}(\gamma_t) = x_{i,t}^{pf*}(I_t, \Lambda_t, \gamma_t).$$

This completes the proof of part (b). *Q.E.D.*

Proof of Theorem 3.5.6: We show **parts (a)-(b)** together by backward induction. More specifically, we show that if $\beta_{s,t-1}^{pf} \geq \beta_{s,t-2}^{pf}$, (i) $y_{s,t}^{pf*} \geq y_{s,t-1}^{pf*}$, (ii) $p_{i,t}^{pf*}(\gamma) \leq p_{i,t-1}^{pf*}(\gamma)$ for all $\gamma \in [0, \bar{\gamma}_{s,t}]^N$,

(iii) $p^{pf*}_{i,t}(I,\Lambda,\gamma) \le p^{pf*}_{i,t-1}(I,\Lambda,\gamma)$ for each i, $(I,\Lambda) \in \mathcal{S}$, and $\gamma \in [0,\bar{\gamma}_{s,t}]^N$, (iv) $\gamma^{pf*}_{s,t} \ge \gamma^{pf*}_{s,t-1}$, (v) $\gamma^{pf*}_{i,t}(I,\Lambda) \ge \gamma^{pf*}_{i,t-1}(I,\Lambda)$ for each i and $(I,\Lambda) \in \mathcal{S}$, (vi) $x^{pf*}_{i,t}(I,\Lambda,\gamma) \ge x^{pf*}_{i,t-1}(I,\Lambda,\gamma)$ for each i, $(I,\Lambda) \in \mathcal{S}$, and $\gamma \in [0,\bar{\gamma}_{s,t}]^N$, and (vii) $\beta^{pf}_{s,t} \ge \beta^{pf}_{s,t-1}$. Since, by Theorem 3.5.2(a), $\beta^{pf}_{s,1} \ge \beta^{pf}_{s,0} = 0$. Thus, the initial condition is satisfied.

Since the model is stationary, by Theorem 3.5.3(a), $\beta^{pf}_{s,t-1} \ge \beta^{pf}_{s,t-2}$ suggests that $y^{pf*}_{s,t} \ge y^{pf*}_{s,t-1}$. Since $\pi^{pf*}_{s,t}$ is increasing in $\beta^{pf}_{s,t-1}$, $\beta^{pf}_{s,t-1} \ge \beta^{pf}_{s,t-2}$ implies that $\pi^{pf*}_{s,t} \ge \pi^{pf*}_{s,t-1}$. Theorem 3.5.3(c) yields that $p^{pf*}_{s,t}(\gamma) \le p^{pf*}_{s,t-1}(\gamma)$ for all $\gamma \in [0,\bar{\gamma}_{s,t}]^N$. Theorem 3.5.3(e) implies that $\gamma^{pf*}_{s,t} \ge \gamma^{pf*}_{s,t-1}$. Hence, $p^{pf*}_{i,t}(I,\Lambda,\gamma) = p^{pf*}_{i,t}(\gamma) \le p^{pf*}_{i,t-1}(\gamma) = p^{pf*}_{i,t-1}(I,\Lambda,\gamma)$ for each i, $(I,\Lambda) \in \mathcal{S}$, and $\gamma \in [0,\bar{\gamma}_{s,t}]^N$, and $\gamma^{pf*}_{i,t}(I,\Lambda) = \gamma^{pf*}_{s,t} \ge \gamma^{pf*}_{s,t-1} = \gamma^{pf*}_{i,t-1}(I,\Lambda)$ for each i and $(I,\Lambda) \in \mathcal{S}$. We now show that $x^{pf*}_{i,t}(I,\Lambda,\gamma) \ge x^{pf*}_{i,t-1}(I,\Lambda,\gamma)$ for each i, $(I,\Lambda) \in \mathcal{S}$, and $\gamma \in [0,\bar{\gamma}]^N$. Because the PF model is symmetric, $\sum_{j=1}^N \theta_{jj,t}(A^{-1})_{ij}$ is independent of i. Direct computation yields that

$$p^{pf*}_{i,t-1}(\gamma) - p^{pf*}_{i,t}(\gamma) = (\sum_{j=1}^N \theta_{jj}(A^{-1})_{ij})(\pi^{pf*}_{s,t} - \pi^{pf*}_{s,t-1}) \ge 0, \text{ for all } \gamma,$$

which is independent of i. Thus, (3.17) and Newton-Leibniz formula implies that

$$\rho_s(p^{pf*}_{t-1}(\gamma)) - \rho_s(p^{pf*}_t(\gamma)) = \int_{s=0}^1 \sum_{i=1}^N (p^{pf*}_{i,t-1}(\gamma) - p^{pf*}_{i,t}(\gamma)) \frac{\partial \rho_s((1-s)p^{pf*}_t(\gamma) + sp^{pf*}_{t-1}(\gamma))}{\partial p_i} \, \mathrm{d}s \le 0.$$

Hence, $\rho_s(p^{pf*}_t(\gamma)) \ge \rho_s(p^{pf*}_{t-1}(\gamma))$ for all γ. Since $y^{pf*}_{s,t} \ge y^{pf*}_{s,t-1}$, Theorem 3.5.2(b) implies that, for any $(I,\Lambda) \in \mathcal{S}$ and $\gamma \in [0,\bar{\gamma}_{s,t}]^N$,

$$x^{pf*}_{i,t}(I,\Lambda,\gamma) = y^{pf*}_{s,t}\rho_s(p^{pf*}_t(\gamma))\psi_s(\gamma_t) \ge y^{pf*}_{s,t-1}\rho_s(p^{pf*}_{t-1}(\gamma_t))\psi_{s,t}(\gamma_t) = x^{pf*}_{i,t-1}(I,\Lambda,\gamma).$$

Finally, we show that $\beta^{pf}_{s,t} \ge \beta^{pf}_{s,t-1}$. Since the model is stationary and $\pi^{pf*}_{s,t} \ge \pi^{pf*}_{s,t-1}$, $\beta^{pf}_{s,t} \ge \beta^{pf}_{s,t-1}$ follows from Theorem 3.5.3(d) immediately. This completes the induction and, thus, the proof of Theorem 3.5.6. *Q.E.D.*

Proof of Theorem 3.6.1: Part (a). Because $\beta^{pf}_{s,t-1} \ge \beta^{sc}_{s,t-1}$, $\pi^{pf*}_{s,t} \ge \pi^{sc*}_{s,t}$. The same argument as the proof of Theorem 3.4.3(a) implies that $y^{pf*}_{s,t} \ge y^{sc*}_{s,t}$.

We now show that, if $\pi^{pf*}_{s,t} \ge \pi^{sc*}_{s,t}$, $\gamma^{pf*}_{s,t} \ge \gamma^{sc*}_{s,t}$. Proposition 3.5.2 implies that $p^{pf*}_t(\gamma^{pf*}_{ss,t}) = A_t^{-1}f_t(\gamma^{pf*}_{ss,t})$. By Proposition 3.4.2, the equilibrium sales prices, $p^{sc*}_{ss,t}$, satisfy the system of first-order equations (3.15). Equivalently, $p^{sc*}_{ss,t} = A_t^{-1}f_t(\gamma^{sc*}_{ss,t})$.

We assume, to the contrary, that $\gamma^{pf*}_{s,t} < \gamma^{sc*}_{s,t}$. Lemma 23 implies that $\partial_{\gamma_{1,t}} \log(\Pi^{pf,1}_{1,t}(\gamma^{pf*}_{ss,t})) \le \partial_{\gamma_{1,t}} \log(\Pi^{sc}_{1,t}(\gamma^{sc*}_{ss,t}, p^{sc*}_{ss,t}))$, i.e.,

$$\begin{aligned}
&- \frac{2(1-\theta_{sa,t}(A_t^{-1})_{11})\nu'_{s,t}(\gamma^{pf*}_{s,t})}{\sum_{j=1}^N (A_t^{-1})_{1j}[\phi_{sa,t} + \theta_{sa,t}(\delta_s w_{s,t-1} + \nu_{s,t}(\gamma^{pf*}_{s,t}) - \pi^{pf*}_{s,t})] - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{pf*}_{s,t}) + \pi^{pf*}_{s,t}} \\
&+ \partial_{\gamma_{1,t}} \log(\psi_{s,t}(\gamma^{pf*}_{ss,t})) \\
\le &- \frac{\nu'_{s,t}(\gamma^{sc*}_{s,t})}{\sum_{j=1}^N (A_t^{-1})_{1j}[\phi_{sa,t} + \theta_{sa,t}(\delta_s w_{s,t-1} + \nu_{s,t}(\gamma^{sc*}_{s,t}) - \pi^{sc*}_{s,t})] - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}) + \pi^{sc*}_{s,t}} \\
&+ \partial_{\gamma_{1,t}} \log(\psi_{s,t}(\gamma^{sc*}_{ss,t})).
\end{aligned} \tag{B.27}$$

Inequality (3.3) and the Newton-Leibniz formula imply that

$$\begin{aligned}\partial_{\gamma_{1,t}} \log(\psi_{s,t}(\gamma_{ss,t}^{sc*})) - \partial_{\gamma_{1,t}} \log(\psi_{s,t}(\gamma_{ss,t}^{pf*})) &= \int_{s=0}^{1} \sum_{j=1}^{N} (\gamma_{s,t}^{sc*} - \gamma_{s,t}^{pf*}) [\frac{\partial^2 \log(\psi_{s,t}((1-s)\gamma_{s,t}^{pf*} + s\gamma_{s,t}^{sc*}))}{\partial \gamma_{1,t} \partial \gamma_{j,t}}] \, \mathrm{d}s \\ &< 0.\end{aligned}$$

By (B.27),

$$\begin{aligned}&-\frac{2(1-\theta_{sa,t}(A_t^{-1})_{11})\nu'_{s,t}(\gamma_{s,t}^{pf*})}{\sum_{j=1}^{N}(A_t^{-1})_{1j}[\phi_{sa,t}+\theta_{sa,t}(\delta_s w_{s,t-1}+\nu_{s,t}(\gamma_{s,t}^{pf*})-\pi_{s,t}^{pf*})]-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma_{s,t}^{pf*})+\pi_{s,t}^{pf*}} \\ &< -\frac{\nu'_{s,t}(\gamma_{s,t}^{sc*})}{\sum_{j=1}^{N}(A_t^{-1})_{1j}[\phi_{sa,t}+\theta_{sa,t}(\delta_s w_{s,t-1}+\nu_{s,t}(\gamma_{s,t}^{sc*})-\pi_{s,t}^{sc*})]-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma_{s,t}^{sc*})+\pi_{s,t}^{sc*}}.\end{aligned}$$

Lemma 24(b) suggests that $0 \le 2(1-\theta_{sa,t}(A_t^{-1})_{11})\nu'_{s,t}(\gamma_{s,t}^{pf*}) \le \nu'_{s,t}(\gamma_{s,t}^{sc*})$. Hence,

$$\begin{aligned}&\sum_{j=1}^{N}(A_t^{-1})_{1j}[\phi_{sa,t}+\theta_{sa,t}(\delta_s w_{s,t-1}+\nu_{s,t}(\gamma_{s,t}^{pf*})-\pi_{s,t}^{pf*})]-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma_{s,t}^{pf*})+\pi_{s,t}^{pf*} \\ &< \sum_{j=1}^{N}(A_t^{-1})_{1j}[\phi_{sa,t}+\theta_{sa,t}(\delta_s w_{s,t-1}+\nu_{s,t}(\gamma_{s,t}^{sc*})-\pi_{s,t}^{sc*})]-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma_{s,t}^{sc*})+\pi_{s,t}^{sc*}.\end{aligned} \tag{B.28}$$

Since $\pi_{s,t}^{pf*} \ge \pi_{s,t}^{sc*}$ and $\nu_{s,t}(\gamma_{s,t}^{pf*}) \le \nu_{s,t}(\gamma_{s,t}^{sc*})$, $\pi_{s,t}^{pf*} - \nu_{s,t}(\gamma_{s,t}^{pf*}) \ge \pi_{s,t}^{sc*} - \nu_{s,t}(\gamma_{s,t}^{sc*})$. Lemma 24(c) implies that $1 - \sum_{j=1}^{N}(A_t^{-1})_{1j}\theta_{sa,t} > 0$. Therefore,

$$\begin{aligned}&\sum_{j=1}^{N}(A_t^{-1})_{1j}[\phi_{sa,t}+\theta_{sa,t}(\delta_s w_{s,t-1}+\nu_{s,t}(\gamma_{s,t}^{pf*})-\pi_{s,t}^{pf*})]-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma_{s,t}^{pf*})+\pi_{s,t}^{pf*} \\ &= \sum_{j=1}^{N}(A_t^{-1})_{1j}(\phi_{sa,t}+\theta_{sa,t}\delta_s w_{s,t-1})-\delta_s w_{s,t-1}+(1-\sum_{j=1}^{N}(A_t^{-1})_{1j}\theta_{sa,t})(\pi_{s,t}^{pf*}-\nu_{s,t}(\gamma_{s,t}^{pf*})) \\ &\ge \sum_{j=1}^{N}(A_t^{-1})_{1j}(\phi_{sa,t}+\theta_{sa,t}\delta_s w_{s,t-1})-\delta_s w_{s,t-1}+(1-\sum_{j=1}^{N}(A_t^{-1})_{1j}\theta_{sa,t})(\pi_{s,t}^{sc*}-\nu_{s,t}(\gamma_{s,t}^{sc*})) \\ &= \sum_{j=1}^{N}(A_t^{-1})_{1j}[\phi_{sa,t}+\theta_{sa,t}(\delta_s w_{s,t-1}+\nu_{s,t}(\gamma_{s,t}^{sc*})-\pi_{s,t}^{sc*})]-\delta_s w_{s,t-1}-\nu_{s,t}(\gamma_{s,t}^{sc*})+\pi_{s,t}^{sc*},\end{aligned}$$

which contradicts the inequality (B.28). Therefore, $\gamma_{s,t}^{pf*} \ge \gamma_{s,t}^{sc*}$. This completes the proof of part (a).

Part (b). We first show, by backward induction, that, if $\theta_{sb,t} = 0$ for each t, $\beta_{s,t}^{pf} \ge \beta_{s,t}^{sc}$ for each t. Since $\beta_{s,0}^{pf} = \beta_{s,0}^{sc} = 0$, the initial condition is satisfied. Now we prove that if $\beta_{s,t-1}^{pf} \ge \beta_{s,t-1}^{sc}$ and $\theta_{sb,t} = 0$, we have $\beta_{s,t}^{pf} \ge \beta_{s,t}^{sc}$.

First, we observe that if $\theta_{sb,t} = 0$, $(A_t^{-1})_{11}\theta_{sa,t} = \frac{1}{2}$ and, thus, $2(1-\theta_{sa,t}(A_t^{-1})_{11}) = 1$. Part (a) shows that $\gamma_{s,t}^{pf*} \ge \gamma_{s,t}^{sc*}$. If $\gamma_{s,t}^{pf*} = \gamma_{s,t}^{sc*}$,

$$\begin{aligned}\Pi_{s,t}^{pf*,1} &= \theta_{sa,t}((A_t^{-1}f_t(\gamma_{ss,t}^{pf*}))_i - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma_{s,t}^{pf*}) + \pi_{s,t}^{pf*})^2 \psi_{s,t}(\gamma_{ss,t}^{pf*}) \\ &\ge \theta_{sa,t}((A_t^{-1}f_t(\gamma_{ss,t}^{sc*}))_i - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma_{s,t}^{sc*}) + \pi_{s,t}^{sc*})^2 \psi_{s,t}(\gamma_{ss,t}^{sc*}) \\ &= \Pi_{s,t}^{sc*},\end{aligned}$$

where the inequality follows from $\pi_{s,t}^{pf*} \ge \pi_{s,t}^{sc*}$.

If $\gamma^{pf*}_{s,t} > \gamma^{sc*}_{s,t}$, Lemma 23 implies that $\partial_{\gamma_{1,t}} \log(\Pi^{pf,1}_{1,t}(\gamma^{pf*}_{ss,t})) \geq \partial_{\gamma_{1,t}} \log(\Pi^{sc}_{1,t}(\gamma^{sc*}_{ss,t}, p^{sc*}_{ss,t}))$, i.e.,

$$\begin{aligned}&-\frac{2(1-\theta_{sa,t}(A_t^{-1})_{11})\nu'_{s,t}(\gamma^{pf*}_{s,t})}{(A_t^{-1}f_t(\gamma^{pf*}_{ss,t}))_1 - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{pf*}_{s,t}) + \pi^{pf*}_{s,t}} + \partial_{\gamma_{1,t}}\log(\psi_{s,t}(\gamma^{pf*}_{ss,t})) \\ \geq &-\frac{\nu'_{s,t}(\gamma^{sc*}_{s,t})}{(A_t^{-1}f_t(\gamma^{sc*}_{ss,t}))_1 - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}) + \pi^{sc*}_{s,t}} + \partial_{\gamma_{1,t}}\log(\psi_{s,t}(\gamma^{sc*}_{ss,t})).\end{aligned} \tag{B.29}$$

Inequality (3.3) and the Newton-Leibniz formula imply that

$$\partial_{\gamma_{1,t}}\log(\psi_{s,t}(\gamma^{pf*}_{ss,t})) - \partial_{\gamma_{1,t}}\log(\psi_{s,t}(\gamma^{sc*}_{ss,t})) = \int_{s=0}^{1}\sum_{j=1}^{N}(\gamma^{pf*}_{s,t} - \gamma^{sc*}_{s,t})[\frac{\partial^2 \log(\psi_{s,t}((1-s)\gamma^{sc*}_{s,t} + s\gamma^{pf*}_{s,t}))}{\partial\gamma_{1,t}\partial\gamma_{j,t}}]\, ds < 0.$$

By (B.29), we have

$$-\frac{2(1-\theta_{sa,t}(A_t^{-1})_{11})\nu'_{s,t}(\gamma^{pf*}_{s,t})}{(A_t^{-1}f_t(\gamma^{pf*}_{ss,t}))_1 - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{pf*}_{s,t}) + \pi^{pf*}_{s,t}} > -\frac{\nu'_{s,t}(\gamma^{sc*}_{s,t})}{(A_t^{-1}f_t(\gamma^{sc*}_{ss,t}))_1 - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}) + \pi^{sc*}_{s,t}}.$$

Because $2(1-\theta_{sa,t}(A_t^{-1})_{11}) = 1$ and $\gamma^{pf*}_{s,t} > \gamma^{sc*}_{s,t}$, $2(1-\theta_{sa,t}(A_t^{-1})_{11})\nu'_{s,t}(\gamma^{pf*}_{s,t}) \geq \nu'_{s,t}(\gamma^{sc*}_{s,t})$. Therefore,

$$(A_t^{-1}f_t(\gamma^{pf*}_{ss,t}))_1 - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{pf*}_{s,t}) + \pi^{pf*}_{s,t} > (A_t^{-1}f_t(\gamma^{sc*}_{ss,t}))_1 - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}) + \pi^{sc*}_{s,t} > 0.$$

By inequality (3.17), $\gamma^{pf*}_{s,t} > \gamma^{sc*}_{s,t}$ implies that $\psi_{s,t}(\gamma^{pf*}_{ss,t}) > \psi_{s,t}(\gamma^{sc*}_{ss,t})$. Thus, we have

$$\begin{aligned}\Pi^{pf*,1}_{s,t} =&\theta_{sa,t}((A_t^{-1}f_t(\gamma^{pf*}_{ss,t}))_1 - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{pf*}_{s,t}) + \pi^{pf*}_{s,t})^2\psi_{s,t}(\gamma^{pf*}_{ss,t}) \\ >&\theta_{sa,t}((A_t^{-1}f_t(\gamma^{sc*}_{ss,t}))_1 - \delta_s w_{s,t-1} - \nu_{s,t}(\gamma^{sc*}_{s,t}) + \pi^{sc*}_{s,t})^2\psi_{s,t}(\gamma^{sc*}_{ss,t}) \\ =&\Pi^{sc*}_{s,t}.\end{aligned}$$

We have thus shown that if $\beta^{pf}_{s,t-1} \geq \beta^{sc}_{s,t-1}$, $\Pi^{pf*,1}_{s,t} \geq \Pi^{sc*}_{s,t}$. By Theorem 3.4.2(a) and Theorem 3.5.2(a),

$$\beta^{pf}_{s,t} = \delta_s\beta^{pf}_{s,t-1}\mu_{s,t} + \Pi^{pf*,1}_{s,t} \geq \delta_s\beta^{sc}_{s,t-1}\mu_{s,t} + \Pi^{sc*}_{s,t} = \beta^{sc}_{s,t}.$$

This completes the induction and, by part (a), the proof of part (b) for the case $\theta_{sb,t} = 0$.

For any fixed $\theta_{sa,t}$, both $\beta^{pf}_{s,t}$ and $\beta^{sc}_{s,t}$ are continuous in $\theta_{sb,t}$. Thus, for each period t, there exists a $\epsilon_t \geq 0$, such that, if $\theta_{sb,t} \leq \epsilon_t\theta_{sa,t}$, $\beta^{pf}_{s,t} \geq \beta^{sc}_{s,t}$. It remains to show that $\epsilon_t \leq \frac{1}{N-1}$. This inequality follows from the diagonal dominance condition that $\theta_{sa,t} > (N-1)\theta_{sb,t}$. This completes the proof of part (b). *Q.E.D.*

B.2 Sufficient Conditions for the Monotonicity of $\pi^{sc*}_{s,t}$ $[\pi^{pf*}_{s,t}]$ in $\beta^{sc}_{s,t-1}$ $[\beta^{pf}_{s,t-1}]$

In this section, we give some sufficient conditions under which $\pi^{sc*}_{s,t}$ $[\pi^{pf*}_{s,t}]$ is increasing in $\beta^{sc}_{s,t-1}$ $[\beta^{pf}_{s,t-1}]$. Observe that, if $t = 1$, $\beta^{sc}_{s,t-1} = \beta^{pf}_{s,t-1} = 0$. So we only consider the case $t \geq 2$.

We define the $N-$player noncooperative game, $\mathcal{G}_{s,t}$, as the symmetric game with each player i's payoff function given by

$$\pi_{i,t}(y_t) = (\delta_s w_{s,t-1} - w_{s,t})y_{i,t} - L_{s,t}(y_{i,t}) + \delta_s\beta(\kappa_{sa,t}(\mathbb{E}[y^+_{i,t}\wedge\xi_{i,t}]) - \sum_{j\neq i}\kappa_{sb,t}(\mathbb{E}[y^+_{j,t}\wedge\xi_{j,t}])),$$

and feasible set given by $\mathbb{R}^+$. Hence, $\mathcal{G}^{sc,2}_{s,t}$ $[\mathcal{G}^{pf,3}_{s,t}]$ can be viewed as $\mathcal{G}_{s,t}$ with $\beta = \beta^{sc}_{s,t-1}$ $[\beta = \beta^{pf}_{s,t-1}]$. By Propositions 3.4.3 and 3.5.4, $\mathcal{G}_{s,t}$ has a unique symmetric pure strategy Nash equilibrium. Thus, we use

$y^*_{s,t}(\beta)$ and $\pi^*_{s,t}(\beta)$ to denote the equilibrium strategy and payoff of each player in the game $\mathcal{G}_{s,t}$ with parameter β.

Let $y^*_{s,t}(\beta;\lambda,1)$ and $\pi^*_{s,t}(\beta;\lambda,1)$ $(\lambda > 0)$ be the equilibrium strategy and payoff of each firm in $\mathcal{G}_{s,t}(\lambda,1)$, where $\mathcal{G}_{s,t}(\lambda,1)$ is identical to $\mathcal{G}_{s,t}$ except that $\alpha_{s,t}(z_t)$ is replaced with $\kappa_{sa,t}(z_{i,t})-\frac{1}{\lambda}(\sum_{j\neq i}\kappa_{sb,t}(z_{j,t}))$ in the objective function $\pi_{i,t}(\cdot)$, i.e.,

$$\pi_{i,t}(y_t) = (\delta_s w_{s,t-1} - w_{s,t})y_{i,t} - L_{s,t}(y_{i,t}) + \delta_s\beta(\kappa_{sa,t}(\mathbb{E}[y^+_{i,t}\wedge\xi_{i,t}]) - \frac{1}{\lambda}(\sum_{j\neq i}\kappa_{sb,t}(\mathbb{E}[y^+_{j,t}\wedge\xi_{j,t}]))).$$

Analogously, let $y^*_{s,t}(\beta;\lambda,2)$ and $\pi^*_{s,t}(\beta;\lambda,2)$ $(\lambda \geq 0)$ be the equilibrium strategy and payoff of each firm in $\mathcal{G}_{s,t}(\lambda,2)$, where $\mathcal{G}_{s,t}(\lambda,2)$ is identical to $\mathcal{G}_{s,t}$ except that with $\alpha_{s,t}(z_t)$ is replaced with $\alpha_{s,t}(z_t)+\lambda$ in the objective function $\pi_{i,t}(\cdot)$, i.e.,

$$\pi_{i,t}(y_t) = (\delta_s w_{s,t-1} - w_{s,t})y_{i,t} - L_{s,t}(y_{i,t}) + \delta_s\beta(\kappa_{sa,t}(\mathbb{E}[y^+_{i,t}\wedge\xi_{i,t}]) - \sum_{j\neq i}\kappa_{sb,t}(\mathbb{E}[y^+_{j,t}\wedge\xi_{j,t}]) + \lambda).$$

Finally, let $y^*_{s,t}(\beta;\lambda,3)$ and $\pi^*_{s,t}(\beta;\lambda,3)$ $(\lambda > 0)$ be the equilibrium strategy and payoff of each firm in $\mathcal{G}_{s,t}(\lambda,3)$, where $\mathcal{G}_{s,t}(\lambda,3)$ is identical to $\mathcal{G}_{s,t}$ except that $\alpha_{s,t}(z_t)$ is replaced with $\lambda\alpha_{s,t}(z_t)$ in the objective function $\pi_{s,t}(\cdot)$, i.e.,

$$\pi_{i,t}(y_t) = (\delta_s w_{s,t-1} - w_{s,t})y_{i,t} - L_{s,t}(y_{i,t}) + \delta_s\beta\lambda(\kappa_{sa,t}(\mathbb{E}[y^+_{i,t}\wedge\xi_{i,t}]) - \sum_{j\neq i}\kappa_{sb,t}(\mathbb{E}[y^+_{j,t}\wedge\xi_{j,t}])).$$

In some of our analysis below, we assume that $\alpha_{s,t}(\cdot)$ satisfies the monotonicity condition similar to (3.17),

$$\sum_{i=1}^{N}\frac{\partial\alpha_{s,t}(z_t)}{\partial z_{i,t}} > 0. \tag{B.30}$$

i.e., a uniform increase in the current expected fill rates gives rise to a higher expected market size of each firm in the next period.

First, we give a lower bound for the value of $\beta^{sc}_{s,t-1}$ and $\beta^{pf}_{s,t-1}$. By Theorem 3.4.2(a) and Theorem 3.5.2(a), $\beta^{sc}_{s,t-1} \geq \underline{\beta}_{s,t-1}$ and $\beta^{pf}_{s,t-1} \geq \underline{\beta}_{s,t-1}$, where

$$\underline{\beta}_{s,t-1} := \underline{\Pi}_{s,1}\prod_{\tau=1}^{t-1}(\delta_s\mu_{s,\tau}),$$

with $\underline{\Pi}_{s,1} := \min\{\Pi^{sc*}_{s,1}, \Pi^{pf*,1}_{s,1}\} > 0$. Thus, we assume in this section that $\beta \geq \underline{\beta}_{s,t-1} > 0$.

Let the density of $\xi_{s,t}$ be defined as $q_{s,t}(\cdot) = F'_{s,t}(\cdot)$ and its failure rate defined as $r_{s,t}(\cdot) := q_{s,t}(\cdot)/\bar{F}_{s,t}(\cdot)$. We have the following lemma on the Lipschitz continuity of $y^*_{s,t}(\beta)$ and $y^*_{s,t}(\beta;\lambda,i)$ $(i = 1,2,3)$.

Lemma 25 *If $\kappa_{sa,t}(\cdot)$ is twice continuously differentiable and the failure rate of $\xi_{s,t}$ is bounded from below by $\underline{r}_{s,t} > 0$ on its support, there exists a constant $K_{s,t} > 0$, independent of λ, i, and β, such that $|y^*_{s,t}(\hat{\beta}) - y^*_{s,t}(\beta)| \leq K_{s,t}|\hat{\beta}-\beta|$ and $|y^*_{s,t}(\hat{\beta};\lambda,i) - y^*_{s,t}(\beta;\lambda,i)| \leq K_{s,t}|\hat{\beta}-\beta|$ for all $\lambda > 0$, $i = 1,2,3$, and $\hat{\beta},\beta \geq 0$.*

Proof: Since $\kappa_{sa,t}(\cdot)$ is twice continuously differentiable, by the implicit function theorem, $y^*_{s,t}(\beta)$ and $y^*_{s,t}(\beta;\lambda,i)$ $(i=1,2,3)$ are continuously differentiable in β with the derivatives given by:

$$\frac{\partial y^*_{s,t}(\beta)}{\partial\beta}=\frac{\partial y^*_{s,t}(\beta;\lambda,1)}{\partial\beta}=\frac{\partial y^*_{s,t}(\beta;\lambda,2)}{\partial\beta}$$
$$=\frac{\delta_s\bar{F}_{s,t}(y^*_{s,t}(\beta))\kappa'_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta)\wedge\xi_{s,t}])}{L''(y^*_{s,t}(\beta))+\delta_s\beta q_{s,t}(y^*_{s,t}(\beta))\kappa'_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta)\wedge\xi_{s,t}])-\delta_s\beta\bar{F}^2_{s,t}(y^*_{s,t}(\beta))\kappa''_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta)\wedge\xi_{s,t}])},$$

and

$$\frac{\partial y^*_{s,t}(\beta;\lambda,3)}{\partial\beta}$$
$$=\frac{\lambda\delta_s\bar{F}_{s,t}(y^*_{s,t}(\beta))\kappa'_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta)\wedge\xi_{s,t}])}{L''(y^*_{s,t}(\beta))+\lambda\delta_s\beta q_{s,t}(y^*_{s,t}(\beta))\kappa'_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta)\wedge\xi_{s,t}])-\lambda\delta_s\beta\bar{F}^2_{s,t}(y^*_{s,t}(\beta))\kappa''_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta)\wedge\xi_{s,t}])}.$$

Observe that

$$\frac{\delta_s\bar{F}_{s,t}(y^*_{s,t}(\beta))\kappa'_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta)\wedge\xi_{s,t}])}{L''(y^*_{s,t}(\beta))+\delta_s\beta q_{s,t}(y^*_{s,t}(\beta))\kappa'_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta)\wedge\xi_{s,t}])-\delta_s\beta\bar{F}^2_{s,t}(y^*_{s,t}(\beta))\kappa''_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta)\wedge\xi_{s,t}])}$$
$$\leq\frac{\delta_s\bar{F}_{s,t}(y^*_{s,t}(\beta))\kappa'_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta)\wedge\xi_{s,t}])}{\delta_s\beta q_{s,t}(y^*_{s,t}(\beta))\kappa'_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta)\wedge\xi_{s,t}])}\leq\frac{1}{\underline{\beta}_{s,t-1}r_{s,t}(y^*_{s,t}(\beta))}\leq\frac{1}{\underline{\beta}_{s,t-1}\underline{r}_{s,t}},$$

where the first inequality follows from the convexity of $L_{s,t}(\cdot)$ and the concavity of $\kappa_{sa,t}(\cdot)$, the second from $\kappa'_{sa,t}(\cdot)\geq 0$, and the last from $r_{s,t}(\cdot)\geq\underline{r}_{s,t}$. Analogously, we have

$$\frac{\lambda\delta_s\bar{F}_{s,t}(y^*_{s,t}(\beta))\kappa'_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta)\wedge\xi_{s,t}])}{L''(y^*_{s,t}(\beta))+\lambda\delta_s\beta q_{s,t}(y^*_{s,t}(\beta))\kappa'_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta)\wedge\xi_{s,t}])-\lambda\delta_s\beta\bar{F}^2_{s,t}(y^*_{s,t}(\beta))\kappa''_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta)\wedge\xi_{s,t}])}$$
$$\leq\frac{\lambda\delta_s\bar{F}_{s,t}(y^*_{s,t}(\beta))\kappa'_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta)\wedge\xi_{s,t}])}{\lambda\delta_s\beta q_{s,t}(y^*_{s,t}(\beta))\kappa'_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta)\wedge\xi_{s,t}])}\leq\frac{1}{\underline{\beta}_{s,t-1}r_{s,t}(y^*_{s,t}(\beta))}\leq\frac{1}{\underline{\beta}_{s,t-1}\underline{r}_{s,t}}.$$

By the mean value theorem,

$$|y^*_{s,t}(\hat{\beta})-y^*_{s,t}(\beta)|=|\hat{\beta}-\beta||\frac{\partial y^*_{s,t}(\tilde{\beta})}{\partial\beta}|\leq K_{s,t}|\hat{\beta}-\beta|,$$

where $\tilde{\beta}$ is a real number that lies between β and $\hat{\beta}$, and $K_{s,t}:=\frac{1}{\underline{\beta}_{s,t-1}\underline{r}_{s,t}}$. The inequality $|y^*_{s,t}(\hat{\beta};\lambda,i)-y^*_{s,t}(\beta;\lambda,i)|\leq K_{s,t}|\hat{\beta}-\beta|$ for all $\lambda>0$ and $i=1,2,3$ follows from exactly the same argument. *Q.E.D.*

We remark that the assumption that the failure rate $r_{s,t}(\cdot)$ is uniformly bounded away from 0 is not a restrictive assumption, and can be satisfied by, e.g., all the distributions that satisfy (i) the increasing failure rate property, and (ii) the density $q_{s,t}(\cdot)$ being positive on the lower bound of its support. The same argument as the proof of Theorem 3.4.3(a) and Theorem 3.5.3(a) imply that, for all $\hat{\beta}>\beta$, $y^*_{s,t}(\hat{\beta})\geq y^*_{s,t}(\beta)$ and $y^*_{s,t}(\hat{\beta};\lambda,i)\geq y^*_{s,t}(\beta;\lambda,i)$ $(i=1,2,3)$. We now characterize sufficient conditions for $\pi^*_{s,t}(\beta)$ and $\pi^*_{s,t}(\beta;\lambda,i)$ $(i=1,2,3)$ to be increasing in β.

Lemma 26 *The following statements hold:*

(a) If $\kappa_{sb,t}(\cdot)\equiv\kappa^0_{sb,t}$ for some constant $\kappa^0_{sb,t}$, $\pi^*_{s,t}(\beta)$ is increasing in β.

(b) Assume that $\alpha_{s,t}(\cdot)>0$ for all z_t and that the conditions of Lemma 25 hold, we have:

(i) If $\kappa_{sb,t}(\cdot)$ is Lipschitz continuous, there exists an $M^1_{s,t}<+\infty$, such that for all $\lambda\geq M^1_{s,t}$, $\pi^*_{s,t}(\beta;\lambda,1)$ is increasing in β.

(ii) *If the monotonicity condition* (B.30) *holds, there exists an* $M_{s,t}^2 < +\infty$*, such that for all* $\lambda \geq M_{s,t}^2$*,* $\pi_{s,t}^*(\beta;\lambda,2)$ *is increasing in* β*.*

(iii) *If the monotonicity condition* (B.30) *holds, there exists an* $M_{s,t}^3 < +\infty$*, such that for all* $\lambda \geq M_{s,t}^3$*,* $\pi_{s,t}^*(\beta;\lambda,3)$ *is increasing in* β*.*

Proof: Part (a). Observe that, $\delta_s\beta\kappa_{sa,t}(\mathbb{E}[y_{i,t}^+ \wedge \xi_{i,t}])$ is increasing in β for any $y_{i,t}$. Therefore,

$$\pi_{s,t}^*(\beta) = \max\{(\delta_s w_{s,t-1} - w_{s,t})y_{i,t} - L_{s,t}(y_{i,t}) + \delta_s\beta\kappa_{sa,t}(\mathbb{E}[y_{i,t}^+ \wedge \xi_{i,t}]) - (N-1)\kappa_{sb,t}^0 : y_{i,t} \geq 0\}$$

is increasing in β. This completes the proof of part (a).

Part (b-i). Let $\hat{\beta} > \beta$, and $k_t < +\infty$ be the Lipschitz constant for $\kappa_{sb,t}(\cdot)$. Since $\alpha_{s,t}(\cdot)$ is a continuous function on a compact support, $\alpha_{s,t}(\cdot) > 0$ for all z_t implies that $\alpha_{s,t}(\cdot) \geq \underline{\alpha}_{s,t} > 0$ for some constant $\underline{\alpha}_{s,t}$. We define

$$\zeta_{i,t}(y_{i,t}) := (\delta_s w_{s,t-1} - w_{s,t})y_{i,t} - L_{s,t}(y_{i,t}) + \delta_s\beta\kappa_{sa,t}(\mathbb{E}[y_{i,t} \wedge \xi_{i,t}]).$$

By the envelope theorem,

$$\frac{\partial\zeta_{i,t}(y_{s,t}^*(\beta;\lambda,1))}{\partial\beta} = \delta_s\kappa_{sa,t}(\mathbb{E}[y_{s,t}^*(\beta;\lambda,1) \wedge \xi_{i,t}]) \geq \delta_s\underline{\alpha}_{s,t} > 0,$$

where the first inequality follows from $\kappa_{sa,t}(z_{i,t}) \geq \alpha_{s,t}(z_t) \geq \underline{\alpha}_{s,t}$. By the mean value theorem and $\hat{\beta} > \beta$,

$$\zeta_{i,t}(y_{s,t}^*(\hat{\beta};\lambda,1)) - \zeta_{i,t}(y_{s,t}^*(\beta;\lambda,1)) \geq \delta_s\underline{\alpha}_{s,t}(\hat{\beta} - \beta). \tag{B.31}$$

At the same time, since $\alpha_{s,\tau}(\cdot)$, $\rho_{s,\tau}(\cdot)$, and $\psi_{s,\tau}(\cdot)$ are all uniformly bounded from above for $\tau \leq t-1$, $\beta_{s,t-1}^{sc}$ and $\beta_{s,t-1}^{pf}$ have a uniform upper bound, which we denote as $\bar{\beta}_{s,t-1} < +\infty$. On the other hand,

$$\begin{aligned}
&\frac{\delta_s}{\lambda}(N-1)[\hat{\beta}\kappa_{sb,t}(\mathbb{E}[y_{s,t}^*(\hat{\beta};\lambda,1) \wedge \xi_{s,t}]) - \beta\kappa_{sb,t}(\mathbb{E}[y_{s,t}^*(\beta;\lambda,1) \wedge \xi_{s,t}])] \\
=&\frac{\delta_s}{\lambda}(N-1)[\hat{\beta}\kappa_{sb,t}(\mathbb{E}[y_{s,t}^*(\hat{\beta};\lambda,1) \wedge \xi_{s,t}]) - \hat{\beta}\kappa_{sb,t}(\mathbb{E}[y_{s,t}^*(\beta;\lambda,1) \wedge \xi_{s,t}]) \\
&+ \hat{\beta}\kappa_{sb,t}(\mathbb{E}[y_{s,t}^*(\beta;\lambda,1) \wedge \xi_{s,t}]) - \beta\kappa_{sb,t}(\mathbb{E}[y_{s,t}^*(\beta;\lambda,1) \wedge \xi_{s,t}])] \\
\leq&\frac{\delta_s}{\lambda}(N-1)[\bar{\beta}_{s,t-1}k_t(y_{s,t}^*(\hat{\beta};\lambda,1) - y_{s,t}^*(\beta;\lambda,1)) + (\hat{\beta} - \beta)\bar{\kappa}_{sb,t}] \\
\leq&\frac{\delta_s}{\lambda}(N-1)(\bar{\beta}_{s,t-1}k_tK_{s,t} + \bar{\kappa}_{sb,t})(\hat{\beta} - \beta),
\end{aligned} \tag{B.32}$$

where the first inequality follows from the Lipschitz continuity of $\kappa_{sb,t}(\cdot)$, $y_{s,t}^*(\hat{\beta};\lambda,1) \geq y_{s,t}^*(\beta;\lambda,1)$, and $\mathbb{E}[y_{s,t}^*(\hat{\beta};\lambda,1) \wedge \xi_{s,t}] - \mathbb{E}[y_{s,t}^*(\beta;\lambda,1) \wedge \xi_{s,t}] \leq y_{s,t}^*(\hat{\beta};\lambda,1) - y_{s,t}^*(\beta;\lambda,1)$, with $\bar{\kappa}_{sb,t} := \max\{\kappa_{sb,t}(z_{i,t}) : z_{i,t} \in [0,1]\} < +\infty$, and the second from Lemma 25. Define

$$M_{s,t}^1 := \frac{(N-1)(\bar{\beta}_{s,t-1}k_tK_{s,t} + \bar{\kappa}_{sb,t})}{\underline{\alpha}_{s,t}} < +\infty.$$

If $\lambda \geq M_{s,t}^1$,

$$\begin{aligned}
\pi_{s,t}^*(\hat{\beta};\lambda,1) - \pi_{s,t}^*(\beta;\lambda,1) &= \zeta_{i,t}(y_{s,t}^*(\hat{\beta};\lambda,1)) - \zeta_{i,t}(y_{s,t}^*(\beta;\lambda,1)) \\
&\quad -\frac{(N-1)\delta_s}{\lambda}[\hat{\beta}\kappa_{sb,t}(y_{s,t}^*(\hat{\beta};\lambda,1)) - \beta\kappa_{sb,t}(y_{s,t}^*(\beta;\lambda,1))] \\
&\geq (\delta_s\underline{\alpha}_{s,t} - \frac{\delta_s}{\lambda}(N-1)(\bar{\beta}_{s,t-1}k_tK_{s,t} + \bar{\kappa}_{sb,t}))(\hat{\beta} - \beta) \\
&\geq (\delta_s\underline{\alpha}_{s,t} - \delta_s\underline{\alpha}_{s,t})(\hat{\beta} - \beta) \\
&= 0,
\end{aligned}$$

where the first inequality follows from (B.31) and (B.32), and the second from $\lambda \geq M^1_{s,t}$. This establishes part (b-i).

Part (b-ii). Let $H_{s,t}(y_{i,t}) := (\delta_s w_{s,t-1} - w_{s,t})y_{i,t} - L_{s,t}(y_{i,t})$. Since

$$\delta_s w_{s,t-1} - w_{s,t} - h_{s,t} \leq H'_{s,t}(y_{i,t}) \leq b_{s,t} + \delta_s w_{s,t-1} - w_{s,t},$$

$H_{s,t}(\cdot)$ is Lipschitz continuous with the Lipschitz constant equal to $l_t := \max\{|\delta_s w_{s,t-1} - w_{s,t} - h_{s,t}|, |b_{s,t} + \delta_s w_{s,t-1} - w_{s,t}|\} < +\infty$. Thus,

$$H_{s,t}(y^*_{s,t}(\beta;\lambda,2)) - H_{s,t}(y^*_{s,t}(\hat{\beta};\lambda,2)) \leq l_t(y^*_{s,t}(\hat{\beta};\lambda,2) - y^*_{s,t}(\beta;\lambda,2)) \leq l_t K_{s,t}(\hat{\beta} - \beta), \tag{B.33}$$

where the second inequality follows from Lemma 25 and $y^*_{s,t}(\hat{\beta};\lambda,2) \geq y^*_{s,t}(\beta;\lambda,2)$. On the other hand,

$$\begin{aligned}
&\delta_s\hat{\beta}(\kappa_{sa,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,2) \wedge \xi_{s,t}]) - (N-1)\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,2) \wedge \xi_{s,t}]) + \lambda) \\
&\quad - \delta_s\beta(\kappa_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta;\lambda,2) \wedge \xi_{s,t}]) - (N-1)\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\beta;\lambda,2) \wedge \xi_{s,t}]) + \lambda) \\
\geq& \delta_s\hat{\beta}(\kappa_{sa,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,2) \wedge \xi_{s,t}]) - (N-1)\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,2) \wedge \xi_{s,t}]) + \lambda) \\
&\quad - \delta_s\beta(\kappa_{sa,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,2) \wedge \xi_{s,t}]) - (N-1)\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,2) \wedge \xi_{s,t}]) + \lambda) \\
\geq& \delta_s\lambda(\hat{\beta} - \beta) + \delta_s\underline{\alpha}_{s,t}(\hat{\beta} - \beta) \\
=& \delta_s(\lambda + \underline{\alpha}_{s,t})(\hat{\beta} - \beta),
\end{aligned} \tag{B.34}$$

where the first inequality follows from (B.30) and the second from the definition of $\underline{\alpha}_{s,t}$. Define

$$M^2_{s,t} := \frac{l_t K_{s,t}}{\delta_s} - \underline{\alpha}_{s,t} < +\infty.$$

If $\lambda \geq M^2_{s,t}$,

$$\begin{aligned}
\pi^*_{s,t}(\hat{\beta};\lambda,2) - \pi^*_{s,t}(\beta;\lambda,2) \quad = \quad & \delta_s\hat{\beta}(\kappa_{sa,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,2) \wedge \xi_{s,t}]) - (N-1)\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,2) \wedge \xi_{s,t}]) + \lambda) \\
& - \delta_s\beta(\kappa_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta;\lambda,2) \wedge \xi_{s,t}]) - (N-1)\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\beta;\lambda,2) \wedge \xi_{s,t}]) + \lambda) \\
& - (H_{s,t}(y^*_{s,t}(\beta;\lambda,2)) - H_{s,t}(y^*_{s,t}(\hat{\beta};\lambda,2))) \\
\geq \quad & (\delta_s\lambda + \delta_s\underline{\alpha}_{s,t} - l_t K_{s,t})(\hat{\beta} - \beta) \\
\geq \quad & (l_t K_{s,t} - \delta_s\underline{\alpha}_{s,t} + \delta_s\underline{\alpha}_{s,t} - l_t K_{s,t})(\hat{\beta} - \beta) \\
= \quad & 0,
\end{aligned}$$

where the first inequality follows from (B.33) and (B.34), and the second from $\lambda \geq M^2_{s,t}$. This establishes part (b-ii).

Part (b-iii). As shown in part (b-ii), $H_{s,t}(\cdot)$ is a Lipschitz function with the Lipschitz constant l_t. Thus,

$$H_{s,t}(y^*_{s,t}(\beta;\lambda,3)) - H_{s,t}(y^*_{s,t}(\hat{\beta};\lambda,3)) \leq l_t(y^*_{s,t}(\hat{\beta};\lambda,3) - y^*_{s,t}(\beta;\lambda,3)) \leq l_t K_{s,t}(\hat{\beta} - \beta), \tag{B.35}$$

where the second inequality follows from Lemma 25 and $y^*_{s,t}(\hat{\beta};\lambda,3) \geq y^*_{s,t}(\beta;\lambda,3)$. The monotonicity condition (B.30) and $y^*_{s,t}(\hat{\beta};\lambda,3) \geq y^*_{s,t}(\beta;\lambda,3)$ implies that

$$\begin{aligned}
& \kappa_{sa,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,3) \wedge \xi_{s,t}]) - (N-1)\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,3) \wedge \xi_{s,t}]) \\
\geq \quad & \kappa_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta;\lambda,3) \wedge \xi_{s,t}]) - (N-1)\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\beta;\lambda,3) \wedge \xi_{s,t}]).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\delta_s\hat{\beta}\lambda(\kappa_{sa,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,3)\wedge\xi_{s,t}]) - (N-1)\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,3)\wedge\xi_{s,t}])) \\
&\quad - \delta_s\beta\lambda(\kappa_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta;\lambda,3)\wedge\xi_{s,t}]) - (N-1)\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\beta;\lambda,3)\wedge\xi_{s,t}])) \\
\geq&\delta_s\hat{\beta}\lambda(\kappa_{sa,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,3)\wedge\xi_{s,t}]) - (N-1)\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,3)\wedge\xi_{s,t}])) \\
&\quad - \delta_s\beta\lambda(\kappa_{sa,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,3)\wedge\xi_{s,t}]) - (N-1)\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,3)\wedge\xi_{s,t}])) \\
\geq&\delta_s\lambda(\kappa_{sa,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,3)\wedge\xi_{s,t}]) - (N-1)\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,3)\wedge\xi_{s,t}]))(\hat{\beta}-\beta) \\
\geq&\delta_s\lambda\underline{\alpha}_{s,t}(\hat{\beta}-\beta),
\end{aligned} \tag{B.36}$$

where the last inequality follows from the definition of $\underline{\alpha}_{s,t}$. Define

$$M^3_{s,t} := \frac{l_t K_{s,t}}{\delta_s\underline{\alpha}_{s,t}} < +\infty.$$

If $\lambda \geq M^3_{s,t}$,

$$\begin{aligned}
\pi^*_{s,t}(\hat{\beta};\lambda,3) - \pi^*_{s,t}(\beta;\lambda,3) &= \delta_s\hat{\beta}\lambda(\kappa_{sa,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,3)\wedge\xi_{s,t}]) - (N-1)\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\hat{\beta};\lambda,3)\wedge\xi_{s,t}])) \\
&\quad -\delta_s\beta\lambda(\kappa_{sa,t}(\mathbb{E}[y^*_{s,t}(\beta;\lambda,3)\wedge\xi_{s,t}]) - (N-1)\kappa_{sb,t}(\mathbb{E}[y^*_{s,t}(\beta;\lambda,3)\wedge\xi_{s,t}])) \\
&\quad -(H_{s,t}(y^*_{s,t}(\beta;\lambda,3)) - H_{s,t}(y^*_{s,t}(\hat{\beta};\lambda,3))) \\
&\geq (\delta_s\lambda\underline{\alpha}_{s,t} - l_t K_{s,t})(\hat{\beta}-\beta) \\
&\geq (l_t K_{s,t} - l_t K_{s,t})(\hat{\beta}-\beta) \\
&= 0,
\end{aligned}$$

where the first inequality follows from (B.35) and (B.36), and the second from $\lambda \geq M^3_{s,t}$. This establishes part (b-iii). *Q.E.D.*

Lemma 26 has several economical interpretations. Parts (a) and (b-i) imply that, if the adverse effect of a firm's competitors' service level upon its future market size is not strong, $\pi^{sc*}_{s,t}$ $[\pi^{pf*}_{s,t}]$ is increasing in $\beta^{sc}_{s,t-1}$ $[\beta^{pf}_{s,t-1}]$. Part (b-ii) implies that if the network effect is sufficiently strong, $\pi^{sc*}_{s,t}$ $[\pi^{pf*}_{s,t}]$ is increasing in $\beta^{sc}_{s,t-1}$ $[\beta^{pf}_{s,t-1}]$. Finally, part (b-iii) implies that if the both the service effect and the network effect are sufficiently strong, $\pi^{sc*}_{s,t}$ $[\pi^{pf*}_{s,t}]$ is increasing in $\beta^{sc}_{s,t-1}$ $[\beta^{pf}_{s,t-1}]$.

C. Appendix for Chapter 4

C.1 Equilibrium Definitions

We now give the definitions of the RE equilibria in the four scenarios considered in this paper: (a) the base model with strategic customers, (b) the base model with myopic customers, (c) the NTR model with strategic customers, and (d) the NTR model with myopic customers. Let $A(Q_1) := \mathbb{E}[X \wedge Q_1]/\mathbb{E}[X]$ $(Q_1 \geq 0)$ be the availability function given the first-period production quantity Q_1 (see [160]).

Definition C.1.1 (Base model with strategic customers.) *An RE equilibrium in the base model with strategic customers consists of* $(p_1^*, Q_1^*, \xi_r^*, \mathfrak{r}^*, \mathfrak{a}^*, \mathfrak{p}_2^{n*}, \mathfrak{p}_2^{r*})$ *satisfying*

(a) $p_1^* = \mathfrak{r}^*$*;* $Q_1^* = \text{argmax}_{Q_1 \geq 0} \Pi_f(Q_1)$ *where* $\Pi_f(\cdot)$ *is given in Lemma 8;*

(b) $\xi_r^* = \mu + \delta\mathbb{E}[(k+\alpha)V - \mathfrak{p}_2^{r*}]^+ - \delta\mathbb{E}[(1+\alpha)V - \mathfrak{p}_2^{n*}]^+$*;*

(c) $\mathfrak{r}^* = \xi_r^*$*;*

(d) $\mathfrak{a}^* = A(Q_1^*)$*;* $(\mathfrak{p}_2^{n*}, \mathfrak{p}_2^{r*}) \stackrel{d}{=} (p_2^{n*}, p_2^{r*})$.

Definition C.1.2 (Base model with myopic customers.) *An RE equilibrium in the base model with myopic customers consists of* $(\tilde{p}_1^*, \tilde{Q}_1^*, \tilde{\xi}_r^*, \tilde{\mathfrak{r}}^*)$ *satisfying*

(a) $\tilde{p}_1^* = \tilde{\mathfrak{r}}^*$*;* $\tilde{Q}_1^* = \text{argmax}_{Q_1 \geq 0} \tilde{\Pi}_f(Q_1)$ *where* $\tilde{\Pi}_f(\cdot)$ *is given in Lemma 8;*

(b) $\tilde{\xi}_r^* = \mu$*;*

(c) $\tilde{\mathfrak{r}}^* = \tilde{\xi}_r^*$.

Definition C.1.3 (NTR model with strategic customers.) *An RE equilibrium in the NTR model with strategic customers consists of* $(p_1^{u*}, Q_1^{u*}, \xi_r^{u*}, \mathfrak{r}^{u*}, \mathfrak{a}^{u*}, \mathfrak{p}_2^{u*})$ *satisfying*

(a) $p_1^{u*} = \mathfrak{r}^{u*}$*;* $Q_1^{u*} = \text{argmax}_{Q_1 \geq 0} \Pi_f^u(Q_1)$*, where* $\Pi_f^u(\cdot)$ *is given in Lemma 8;*

(b) $\xi_r^{u*} = \mu + \delta\mathbb{E}[(k+\alpha)V - \mathfrak{p}_2^{u*}]^+ - \delta\mathbb{E}[(1+\alpha)V - \mathfrak{p}_2^{u*}]^+$*;*

(c) $\mathfrak{r}^{u*} = \xi_r^{u*}$*;*

(d) $\mathfrak{a}^{u*} = A(Q_1^{u*})$*;* $\mathfrak{p}_2^{u*} \stackrel{d}{=} p_2^u((X - Q_1^{u*})^+, X \wedge Q_1^{u*})$*, where* $p_2^u(\cdot,\cdot)$ *is characterized in Theorem 4.4.2(a).*

Definition C.1.4 (NTR model with myopic customers.) *An RE equilibrium in the NTR model with myopic customers consists of* $(\tilde{p}_1^{u*}, \tilde{Q}_1^{u*}, \tilde{\xi}_r^{u*}, \tilde{\mathfrak{r}}^{u*})$ *satisfying*

(a) $\tilde{p}_1^{u*} = \tilde{\mathfrak{r}}^{u*}$*;* $\tilde{Q}_1^{u*} = \text{argmax}_{Q_1 \geq 0} \tilde{\Pi}_f^u(Q_1)$ *where* $\tilde{\Pi}_f^u(\cdot)$ *is given in Lemma 8;*

(b) $\tilde{\xi}_r^{u*} = \mu$*;*

(c) $\tilde{\mathfrak{r}}^{u*} = \tilde{\xi}_r^{u*}$.

In Definitions C.1.1-C.1.4, conditions (a) and (b) follow from that the decisions are optimal given the rational beliefs, and conditions (c) and (d) follow from that the rational beliefs are consistent with actual outcomes.

C.2 Proofs of Statements

We use $h_1'(\cdot)$ to denote the derivative operator of a single variable function $h_1(\cdot)$, $\partial_x h_2(\cdot)$ to denote the partial derivative operator of a multi-variable function, $h_2(\cdot)$, with respect to variable x, and $1_{\{\cdot\}}$ to denote the indicator function. For any multivariate continuously differentiable function $h_2(x_1, x_2, \cdots, x_n)$ and $x' := (x_1', x_2', \cdots, x_n')$ in $h_2(\cdot)$'s domain, $\forall i$, we use $\partial_{x_i} h_2(x_1', x_2', \cdots, x_n')$ to denote $\partial_{x_i} h_2(x_1, x_2, \cdots, x_n)|_{x=x'}$.

Proof of Lemma 7: Part (a). Given (p_2^n, p_2^r) $(p_2^r \leq p_2^n)$, a new customer will make a purchase if and only if $(1+\alpha)V \geq p_2^n$, whereas a repeat customer will make a purchase if and only if $(k+\alpha)V \geq p_2^r$. Thus, the *ex ante* probability that a new customer will purchase the second-generation product is $\bar{G}\left(\frac{p_2^n}{1+\alpha}\right)$, whereas the probability that a repeat customer will join the trade-in program is $\bar{G}\left(\frac{p_2^r}{k+\alpha}\right)$. Therefore, conditioned on the realized market size (X_2^n, X_2^r), the expected profit of the firm in period 2 is given by:

$$\begin{aligned} \Pi_2(p_2^n, p_2^r | X_2^n, X_2^r) :=& X_2^n (p_2^n - c_2) \bar{G}\left(\frac{p_2^n}{1+\alpha}\right) + X_2^r (p_2^r - c_2 + r_2) \bar{G}\left(\frac{p_2^r}{k+\alpha}\right) \\ =& X_2^n v_2^n(p_2^n) + X_2^r v_2^r(p_2^r), \end{aligned} \tag{C.1}$$

where $v_2^n(p_2^n) := (p_2^n - c_2)\bar{G}(\frac{p_2^n}{1+\alpha})$ and $v_2^r(p_2^r) := (p_2^r - c_2 + r_2)\bar{G}(\frac{p_2^r}{k+\alpha})$. We now show that $v_2^n(\cdot)$ is quasiconcave in p_2^n, and $v_2^r(\cdot)$ is quasiconcave in p_2^r. Note that

$$\partial_{p_2^n} v_2^n(p_2^n) = -\left(\frac{p_2^n - c_2}{1+\alpha}\right) g\left(\frac{p_2^n}{1+\alpha}\right) + \bar{G}\left(\frac{p_2^n}{1+\alpha}\right)$$

and

$$\partial_{p_2^r} v_2^r(p_2^r) = -\left(\frac{p_2^r - c_2 + r_2}{k+\alpha}\right) g\left(\frac{p_2^r}{k+\alpha}\right) + \bar{G}\left(\frac{p_2^r}{k+\alpha}\right).$$

Because $g(v)/\bar{G}(v)$ is continuously increasing in v, $g(\frac{p_2^n}{1+\alpha})/\bar{G}(\frac{p_2^n}{1+\alpha})$ is continuously increasing in p_2^n and $g(\frac{p_2^r}{k+\alpha})/\bar{G}(\frac{p_2^r}{k+\alpha})$ is continuously increasing in p_2^r. Hence, $\partial_{p_2^n} v_2^n(p_2^n) = 0$ has a unique solution p_2^{n*} and $\partial_{p_2^r} v_2^r(p_2^r) = 0$ has a unique solution p_2^{r*}, where $v_2^n(\cdot)$ $[v_2^r(\cdot)]$ is strictly increasing on $[0, p_2^{n*})$ $[[0, p_2^{r*})]$ and strictly decreasing on $(p_2^{n*}, +\infty)$ $[(p_2^{r*}, +\infty)]$. Therefore, for any realized (X_2^n, X_2^r), $X_2^n v_2^n(\cdot)$ is quasiconcave in p_2^n, and $X_2^r v_2^r(\cdot)$ is quasiconcave in p_2^r. Thus, for any realized (X_2^n, X_2^r), $(p_2^n(X_2^n, X_2^r), p_2^r(X_2^n, X_2^r)) = (p_2^{n*}, p_2^{r*})$ maximizes $\Pi_2(\cdot, \cdot | X_2^n, X_2^r)$.

It remains to show that $p_2^{n*} > p_2^{r*}$ if and only if $k < 1$ or $r_2 > 0$. Note that p_2^{n*} satisfies

$$\left(\frac{p_2^{n*} - c_2}{1+\alpha}\right) \frac{g\left(\frac{p_2^{n*}}{1+\alpha}\right)}{\bar{G}\left(\frac{p_2^{n*}}{1+\alpha}\right)} = 1, \tag{C.2}$$

and p_2^{r*} satisfies

$$\left(\frac{p_2^{r*} - c_2 + r_2}{k+\alpha}\right) \frac{g\left(\frac{p_2^{r*}}{k+\alpha}\right)}{\bar{G}\left(\frac{p_2^{r*}}{k+\alpha}\right)} = 1. \tag{C.3}$$

If $k<1$ or $r_2>0$, $\frac{p_2^{n*}-c_2+r_2}{k+\alpha} > \frac{p_2^{n*}-c_2}{1+\alpha}$, and the increasing failure rate condition implies that $g\left(\frac{p_2^{n*}}{k+\alpha}\right)/\bar{G}\left(\frac{p_2^{n*}}{k+\alpha}\right) \geq g\left(\frac{p_2^{n*}}{1+\alpha}\right)/\bar{G}\left(\frac{p_2^{n*}}{1+\alpha}\right)$. Thus,

$$\left(\frac{p_2^{n*}-c_2+r_2}{k+\alpha}\right)\frac{g\left(\frac{p_2^{n*}}{k+\alpha}\right)}{\bar{G}\left(\frac{p_2^{n*}}{k+\alpha}\right)} > \left(\frac{p_2^{n*}-c_2}{1+\alpha}\right)\frac{g\left(\frac{p_2^{n*}}{1+\alpha}\right)}{\bar{G}\left(\frac{p_2^{n*}}{1+\alpha}\right)} = 1,$$

and, hence, $\partial_{p_2^r} v_2^r(p_2^{n*}) < 0$. Since $v_2^r(\cdot)$ is quasiconcave, $p_2^{r*} < p_2^{n*}$. On the other hand, if $k=1$ and $r_2=0$, $v_2^n(\cdot) \equiv v_2^r(\cdot)$ and thus $p_2^{n*} = p_2^{r*}$. This completes the proof of **Part (a)**.

Part (b). Because all new customers with willingness-to-pay $(1+\alpha)V$ greater than $p_2^n(X_2^n, X_2^r) \equiv p_2^{n*}$ would make a purchase. Hence,

$$Q_2^n(X_2^n, X_2^r) = \mathbb{E}[X_2^n 1_{\{(1+\alpha)V \geq p_2^{n*}\}}|X_2^n] = \bar{G}\left(\frac{p_2^{n*}}{1+\alpha}\right)X_2^n.$$

Analogously, all repeat customers with willingness-to-pay $(k+\alpha)V$ greater than $p_2^r(X_2^n, X_2^r) \equiv p_2^{r*}$ would make a purchase. Hence,

$$Q_2^r(X_2^n, X_2^r) = \mathbb{E}[X_2^r 1_{\{(k+\alpha)V \geq p_2^{r*}\}}|X_2^r] = \bar{G}\left(\frac{p_2^{r*}}{k+\alpha}\right)X_2^r.$$

This proves **Part (b)**.

Part (c). Since $\pi_2(X_2^n, X_2^r) := \max\{\Pi_2(p_2^n, p_2^r|X_2^n, X_2^r) : 0 \leq p_2^r \leq p_2^n\}$, it follows immediately that

$$\pi_2(X_2^n, X_2^r) = [\max v_2^n(p_2^n)]X_2^n + [\max v_2^r(p_2^r)]X_2^r.$$

To complete the proof, it remains to show that $\beta_n^* = [\max v_2^n(p_2^n)] > 0$ and $\beta_r^* = [\max v_2^r(p_2^r)] > 0$. By equations (C.2) and (C.3), we have $p_2^{n*} - c_2 > 0$, $\bar{G}\left(\frac{p_2^{n*}}{1+\alpha}\right) > 0$, $p_2^{r*} - c_2 + r_2 > 0$, and $\bar{G}\left(\frac{p_2^{r*}}{k+\alpha}\right) > 0$. Hence, $\beta_n^* = (p_2^{n*} - c_2)\bar{G}\left(\frac{p_2^{n*}}{1+\alpha}\right) > 0$ and $\beta_r^* = (p_2^{r*} - c_2 + r_2)\bar{G}\left(\frac{p_2^{r*}}{k+\alpha}\right) > 0$. This completes the proof of **Part (c)**. *Q.E.D.*

Proof of Theorem 4.3.1: Part (a). Since ξ_r^* satisfies that $\mathcal{U}_p = \mathcal{U}_w$, we have

$$\mathfrak{a}^*(\mathbb{E}[V] + \delta\mathbb{E}[(k+\alpha)V - \mathfrak{p}_2^{r*}]^+ - \xi_r^*) + (1-\mathfrak{a}^*)\delta\mathbb{E}[(1+\alpha)V - \mathfrak{p}_2^{n*}]^+ = \delta\mathbb{E}[(1+\alpha)V - \mathfrak{p}_2^{n*}]^+.$$

Direct algebraic manipulation yields that $\xi_r^* = \mu + \delta\mathbb{E}[(k+\alpha)V - \mathfrak{p}_2^{r*}]^+ - \delta\mathbb{E}[(1+\alpha)V - \mathfrak{p}_2^{n*}]^+$. Hence, by Definition C.1.1 and Lemma 7(a),

$$\begin{aligned} p_1^* = \mathfrak{r}_1^* = \xi_r^* &= \mu + \delta\mathbb{E}[(k+\alpha)V - \mathfrak{p}_2^{r*}]^+ - \delta\mathbb{E}[(1+\alpha)V - \mathfrak{p}_2^{n*}]^+ \\ &= \mu + \delta\mathbb{E}[(k+\alpha)V - p_2^{r*}]^+ - \delta\mathbb{E}[(1+\alpha)V - p_2^{n*}]^+. \end{aligned}$$

Hence,

$$\begin{aligned} \Pi_f(Q_1) &= p_1^*\mathbb{E}(X \wedge Q_1) - c_1 Q_1 + r_1\mathbb{E}(Q_1 - X)^+ + \delta\mathbb{E}\{\pi_2(X - (X \wedge Q_1), X \wedge Q_1)\} \\ &= (p_1^* - r_1)\mathbb{E}(X \wedge Q_1) - (c_1 - r_1)Q_1 + \delta\mathbb{E}[\beta_n^*(X - (X \wedge Q_1)) + \beta_r^*(X \wedge Q_1)] \\ &= (p_1^* + \delta(\beta_r^* - \beta_n^*) - r_1)\mathbb{E}(X \wedge Q_1) - (c_1 - r_1)Q_1 + \delta\beta_n^*\mathbb{E}(X) \\ &= (m_1^* - r_1)\mathbb{E}(X \wedge Q_1) - (c_1 - r_1)Q_1 + \delta\beta_n^*\mathbb{E}(X), \end{aligned}$$

where the second equality follows from $(Q_1 - X)^+ = Q_1 - (X \wedge Q_1)$, and the last from the identity $m_1^* = \mu + \delta\mathbb{E}[(k+\alpha)V - \mathfrak{p}_2^{r*}]^+ - \delta\mathbb{E}[(1+\alpha)V - \mathfrak{p}_2^{n*}]^+ + \delta(\beta_r^* - \beta_n^*)$. Therefore, Q_1^* is the solution to a newsvendor problem with marginal revenue $m_1^* - r_1$, marginal cost $c_1 - r_1$, and demand distribution $F(\cdot)$. Hence, $Q_1^* = \bar{F}^{-1}(\frac{c_1 - r_1}{m_1^* - r_1})$ and $\Pi_f^* = \Pi_f(Q_1^*) = (m_1^* - r_1)\mathbb{E}(X \wedge Q_1^*) - (c_1 - r_1)Q_1^* + \delta\beta_n^*\mathbb{E}(X)$. This proves **Part (a)**.

Part (b). Since myopic customers will make a purchase if and only if $p_1 \leq \mu$, $\tilde{p}_1^* = \tilde{\xi}_1^* = \mu$. Hence,

$$\begin{aligned}
\tilde{\Pi}_f(Q_1) &= \tilde{p}_1^*\mathbb{E}(X \wedge Q_1) - c_1Q_1 + r_1\mathbb{E}(Q_1 - X)^+ + \delta\mathbb{E}\{\pi_2(X - (X \wedge Q_1), X \wedge Q_1)\} \\
&= (\tilde{p}_1^* - r_1)\mathbb{E}(X \wedge Q_1) - (c_1 - r_1)Q_1 + \delta\mathbb{E}[\beta_n^*(X - (X \wedge Q_1)) + \beta_r^*(X \wedge Q_1)] \\
&= (\tilde{p}_1^* + \delta(\beta_r^* - \beta_n^*) - r_1)\mathbb{E}(X \wedge Q_1) - (c_1 - r_1)Q_1 + \delta\beta_n^*\mathbb{E}(X) \\
&= (\tilde{m}_1^* - r_1)\mathbb{E}(X \wedge Q_1) - (c_1 - r_1)Q_1 + \delta\beta_n^*\mathbb{E}(X),
\end{aligned}$$

where the second equality follows from $(Q_1 - X)^+ = Q_1 - (X \wedge Q_1)$, and the last from the identity $\tilde{m}_1^* = \mu + \delta(\beta_r^* - \beta_n^*)$. Therefore, $\tilde{Q}_1^*$ is the solution to a newsvendor problem with marginal revenue $\tilde{m}_1^* - r_1$, marginal cost $c_1 - r_1$, and demand distribution $F(\cdot)$. Hence, $\tilde{Q}_1^* = \bar{F}^{-1}(\frac{c_1 - r_1}{m_1^* - r_1})$ and $\tilde{\Pi}_f^* = \tilde{\Pi}_f(\tilde{Q}_1^*) = (m_1^* - r_1)\mathbb{E}(X \wedge \tilde{Q}_1^*) - (c_1 - r_1)\tilde{Q}_1^* + \delta\beta_n^*\mathbb{E}(X)$. This proves **Part (b)**. *Q.E.D.*

Proof of Lemma 8: The expressions for $\Pi_f(\cdot)$ and $\tilde{\Pi}_f(\cdot)$ have already been given in the proof of Theorem 4.3.1(a) and Theorem 4.3.1(b), respectively. We now compute $\Pi_f^u(Q_1)$. Following the same argument as the proof of Theorem 4.3.1(a), given the first-period production quantity Q_1, the first-period equilibrium price is

$$\begin{aligned}
p_1^u(Q_1) =& \mathbb{E}[V] + \delta[\mathbb{E}((k+\alpha)V - \mathfrak{p}_2^{u*})^+ - \mathbb{E}((1+\alpha)V - \mathfrak{p}_2^{u*})^+] \\
=& \mu + \delta[\mathbb{E}((k+\alpha)V - p_2^u(X_2^n, X_2^r))^+ - \mathbb{E}((1+\alpha)V - p_2^u(X_2^n, X_2^r))^+],
\end{aligned}$$

where $X_2^n = (X - Q_1)^+$ and $X_2^r = X \wedge Q_1$. Let $\pi_2^u(X_2^n, X_2^r) := \max_{p_2^u} \Pi_2^u(p_2^u | X_2^n, X_2^r)$. Hence,

$$\begin{aligned}
\pi_2^u(X_2^n, X_2^r) &= \max_{p_2^u \geq 0}\{X_2^n(p_2^u - c_2)\bar{G}\left(\frac{p_2^u}{1+\alpha}\right) + X_2^r(p_2^u - c_2)\bar{G}(\frac{p_2^u}{k+\alpha})\} \\
&= X_2^n(p_2^u(X_2^n, X_2^r) - c_2)\bar{G}\left(\frac{p_2^u(X_2^n, X_2^r)}{1+\alpha}\right) + X_2^r(p_2^u(X_2^n, X_2^r) - c_2)\bar{G}\left(\frac{p_2^u(X_2^n, X_2^r)}{k+\alpha}\right),
\end{aligned}$$

where $X_2^n = (X - Q_1)^+$ and $X_2^r = X \wedge Q_1$. Therefore,

$$\begin{aligned}
\Pi_f^u(Q_1) &= p_1^u(Q_1)\mathbb{E}(X \wedge Q_1) - c_1Q_1 + r_1\mathbb{E}(X - Q_1)^+ + \delta\mathbb{E}[\pi_2^u(X_2^n, X_2^r)] \\
&= (p_1^u(Q_1) - r_1)\mathbb{E}(X \wedge Q_1) - (c_1 - r_1)Q_1 + \delta\mathbb{E}[(X - Q_1)^+(p_2^u(X_2^n, X_2^r) - c_2)\bar{G}\left(\frac{p_2^u(X_2^n, X_2^r)}{1+\alpha}\right) \\
&\quad +(X \wedge Q_1)(p_2^u(X_2^n, X_2^r) - c_2)\bar{G}\left(\frac{p_2^u(X_2^n, X_2^r)}{k+\alpha}\right)] \\
&= (p_1^u(Q_1) + \mathbb{E}[(p_2^u(X_2^n, X_2^r) - c_2)\bar{G}\left(\frac{p_2^u(X_2^n, X_2^r)}{k+\alpha}\right) - (p_2^u(X_2^n, X_2^r) - c_2)\bar{G}\left(\frac{p_2^u(X_2^n, X_2^r)}{1+\alpha}\right)] \\
&\quad -r_1)\mathbb{E}(X \wedge Q_1) - (c_1 - r_1)Q_1 + \delta\mathbb{E}[(p_2^u(X_2^n, X_2^r) - c_2)\bar{G}\left(\frac{p_2^u(X_2^n, X_2^r)}{1+\alpha}\right)X] \\
&= (m_1^u(Q_1) - r_1)\mathbb{E}(X \wedge Q_1) - (c_1 - r_1)Q_1 + \delta\mathbb{E}[(p_2^u(X_2^n, X_2^r) - c_2)\bar{G}\left(\frac{p_2^u(X_2^n, X_2^r)}{1+\alpha}\right)X],
\end{aligned}$$

where

$$m_1^u(Q_1): \quad = \quad \mu + \delta\{\mathbb{E}[(p_2^u(X_2^n, X_2^r) - c_2)\bar{G}\left(\frac{p_2^u(X_2^n, X_2^r)}{k+\alpha}\right)] + \mathbb{E}((k+\alpha)V - p_2^u(X_2^n, X_2^r))^+$$
$$-\mathbb{E}[(p_2^u(X_2^n, X_2^r) - c_2)\bar{G}\left(\frac{p_2^u(X_2^n, X_2^r)}{1+\alpha}\right)] - \mathbb{E}((1+\alpha)V - p_2^u(X_2^n, X_2^r))^+\},$$

with $X_2^n = (X - Q_1)^+$ and $X_2^r = X \wedge Q_1$.

Analogously, since $\tilde{p}_1^{u*} = \mathbb{E}[V] = \mu$,

$$\begin{aligned}
\tilde{\Pi}_f^u(Q_1) &= \tilde{p}_1^{u*}\mathbb{E}(X \wedge Q_1) - c_1 Q_1 + r_1 \mathbb{E}(X - Q_1)^+ + \delta \mathbb{E}[\pi_2^u(X_2^n, X_2^r)] \\
&= (\mu - r_1)\mathbb{E}(X \wedge Q_1) - (c_1 - r_1)Q_1 + \delta\mathbb{E}[(X - Q_1)^+(p_2^u(X_2^n, X_2^r) - c_2)\bar{G}\left(\frac{p_2^u(X_2^n, X_2^r)}{1+\alpha}\right) \\
&\quad +(X \wedge Q_1)(p_2^u(X_2^n, X_2^r) - c_2)\bar{G}\left(\frac{p_2^u(X_2^n, X_2^r)}{k+\alpha}\right)] \\
&= (\mu + \mathbb{E}[(p_2^u(X_2^n, X_2^r) - c_2)\bar{G}\left(\frac{p_2^u(X_2^n, X_2^r)}{k+\alpha}\right) - (p_2^u(X_2^n, X_2^r) - c_2)\bar{G}\left(\frac{p_2^u(X_2^n, X_2^r)}{1+\alpha}\right)] \\
&\quad -r_1)\mathbb{E}(X \wedge Q_1) - (c_1 - r_1)Q_1 + \delta\mathbb{E}[(p_2^u(X_2^n, X_2^r) - c_2)\bar{G}\left(\frac{p_2^u(X_2^n, X_2^r)}{1+\alpha}\right) X] \\
&= (\tilde{m}_1^u(Q_1) - r_1)\mathbb{E}(X \wedge Q_1) - (c_1 - r_1)Q_1 + \delta\mathbb{E}[(p_2^u(X_2^n, X_2^r) - c_2)\bar{G}\left(\frac{p_2^u(X_2^n, X_2^r)}{1+\alpha}\right) X],
\end{aligned}$$

where

$$\tilde{m}_1^u(Q_1): \quad = \quad \mu + \delta\{\mathbb{E}[(p_2^u(X_2^n, X_2^r) - c_2)\bar{G}\left(\frac{p_2^u(X_2^n, X_2^r)}{k+\alpha}\right)] - \mathbb{E}[(p_2^u(X_2^n, X_2^r) - c_2)\bar{G}\left(\frac{p_2^u(X_2^n, X_2^r)}{1+\alpha}\right)]\},$$

with $X_2^n = (X - Q_1)^+$ and $X_2^r = X \wedge Q_1$. *Q.E.D.*

Before giving the proof of Theorem 4.4.1, we first prove Theorem 4.4.2.

Proof of Theorem 4.4.2: Part (a). If the firm charges a single price p_2^u in period 2, all new (repeat) customers with willingness-to-pay $(1+\alpha)V$ $((k+\alpha)V)$ greater than p_2^u will make a purchase (join the trade-in program). Hence, the second-period profit function of the firm $\Pi_2^u(p_2^u|X_2^n, X_2^r)$ is given by

$$\begin{aligned}
\Pi_2^u(p_2^u|X_2^n, X_2^r) &= X_2^n(p_2^u - c_2)\bar{G}\left(\frac{p_2^u}{1+\alpha}\right) + X_2^r(p_2^u - c_2)\bar{G}\left(\frac{p_2^u}{k+\alpha}\right) \\
&= X_2^n v_2^n(p_2^u) + X_2^r \hat{v}_2^r(p_2^u),
\end{aligned}$$

where $\hat{v}_2^r(p_2) := (p_2 - c_2)\bar{G}\left(\frac{p_2}{k+\alpha}\right)$. Clearly, $\hat{v}_2^r(\cdot)$ has a unique maximizer $\hat{p}_2^{r*}$, where $\hat{p}_2^{r*} \geq p_2^{r*}$ with the inequality being strict if $r_2 > 0$. Moreover, $\Pi_2^u(p_2^u|X_2^n, X_2^r) = \hat{\Pi}_2(p_2^u, p_2^u|X_2^n, X_2^r)$, where, by the proof of Lemma 7(a), $\hat{\Pi}_2(p_2^n, p_2^r|X_2^n, X_2^r) := X_2^n v_2^n(p_2^n) + X_2^r \hat{v}_2^r(p_2^r)$ is quasiconcave function of (p_2^n, p_2^r). Thus, the equilibrium second-period pricing strategy $p_2^u(X_2^n, X_2^r)$ is the maximizer of the second-period profit function, i.e., $p_2^u(X_2^n, X_2^r) = \text{argmax}_{p_2^u \geq 0}\Pi_2^u(p_2^u|X_2^n, X_2^r)$. Note that since $\hat{\Pi}_2(\cdot,\cdot|X_2^n, X_2^r)$ is quasiconcave in (p_2^n, p_2^r), $\Pi_2^u(p_2^u|X_2^n, X_2^r) = \hat{\Pi}_2(p_2^u, p_2^u|X_2^n, X_2^r)$ is also quasiconcave in p_2^u.

Observe that $\partial_{p_2^u}\Pi_2^u(p_2^u|X_2^n, X_2^r) \;=\; X_2^n\left[\bar{G}\left(\frac{p_2^u}{1+\alpha}\right) - \left(\frac{p_2^u - c_2}{1+\alpha}\right) g\left(\frac{p_2^u}{1+\alpha}\right)\right] + X_2^r\left[\bar{G}\left(\frac{p_2^u}{k+\alpha}\right) - \left(\frac{p_2^u - c_2}{k+\alpha}\right) g\left(\frac{p_2^u}{k+\alpha}\right)\right]$. Since $g(v)/\bar{G}(v)$ is increasing in v, $\partial_{p_2^u}\Pi_2^u(p_2^u|X_2^n, X_2^r) < 0$ if $p_2^u > p_2^{n*}$, and $\partial_{p_2^u}\Pi_2^u(p_2^u|X_2^n, X_2^r) > 0$ if $p_2^u < \hat{p}_2^{r*}$. Thus,

$$p_2^u(X_2^n, X_2^r) \in [\hat{p}_2^{r*}, p_2^{n*}] \subset [p_2^{r*}, p_2^{n*}] = [p_2^r(X_2^n, X_2^r), p_2^n(X_2^n, X_2^r)].$$

If $k < 1$, by the proof of Lemma 7(a), $\hat{p}_2^{r*} < p_2^{n*}$. Since $X_2^n, X_2^r > 0$, $\partial_{p_2^u}\Pi_2^u(\hat{p}_2^{r*}|X_2^n, X_2^r) = X_2^n\left[\bar{G}\left(\frac{\hat{p}_2^{r*}}{1+\alpha}\right) - \left(\frac{\hat{p}_2^{r*}-c_2}{1+\alpha}\right)g\left(\frac{\hat{p}_2^{r*}}{1+\alpha}\right)\right] > 0$ and $\partial_{p_2^u}\Pi_2^u(p_2^{n*}|X_2^n, X_2^r) = X_2^r\left[\bar{G}\left(\frac{p_2^{n*}}{k+\alpha}\right) - \left(\frac{p_2^{n*}-c_2}{k+\alpha}\right)g\left(\frac{p_2^{n*}}{k+\alpha}\right)\right] < 0$. Therefore, $p_2^{r*} \le \hat{p}_2^{r*} < p_2^u(X_2^n, X_2^r) < p_2^{n*}$ for all $X_2^n, X_2^r > 0$.

When $p_2^u \in [\hat{p}_2^{r*}, p_2^{n*}]$, $\bar{G}(\frac{p_2^u}{1+\alpha}) - (\frac{p_2^u-c_2}{1+\alpha})g(\frac{p_2^u}{1+\alpha}) \ge 0$ and $\bar{G}(\frac{p_2^u}{k+\alpha}) - (\frac{p_2^u-c_2}{k+\alpha})g(\frac{p_2^u}{k+\alpha}) \le 0$. Thus, $\Pi_2^u(p_2^u|X_2^n, X_2^r)$ is increasing in X_2^n and decreasing in X_2^r if $p_2^u \in [\hat{p}_2^{r*}, p_2^{n*}]$, i.e., $\Pi_2^u(p_2^u|X_2^n, X_2^r)$ is supermodular in (p_2^u, X_2^n) on the lattice $[\hat{p}_2^{r*}, p_2^{n*}] \times [0, +\infty)$, and submodular in (p_2^u, X_2^r) on the lattice $[\hat{p}_2^{r*}, p_2^{n*}] \times [0, +\infty)$. Therefore, $p_2^u(X_2^n, X_2^r)$ is continuously increasing in X_2^n and continuously decreasing in X_2^r. This proves **Part (a)**.

Part (b). Note that $p_1^{u*} = \mu + \delta[\mathbb{E}((k+\alpha)V - \mathfrak{p}_2^{u*})^+ - \mathbb{E}((1+\alpha)V - \mathfrak{p}_2^{u*})^+]$, where $\mathfrak{p}_2^{u*} \overset{d}{=} p_2^u((X - Q_1^{u*})^+, X \wedge Q_1^{u*}) \in [p_2^{r*}, p_2^{n*}]$. Therefore, $\delta[\mathbb{E}((k+\alpha)V - \mathfrak{p}_2^{u*})^+ - \mathbb{E}((k+\alpha)V - p_2^{r*})^+] \le 0$, $\delta[\mathbb{E}((1+\alpha)V - \mathfrak{p}_2^{u*})^+ - \mathbb{E}((1+\alpha)V - p_2^{n*})^+] \ge 0$, and thus

$$p_1^{u*} - p_1^* = \delta[\mathbb{E}((k+\alpha)V - \mathfrak{p}_2^{u*})^+ - \mathbb{E}((k+\alpha)V - p_2^{r*})^+] - \delta[\mathbb{E}((1+\alpha)V - \mathfrak{p}_2^{u*})^+ - \mathbb{E}((1+\alpha)V - p_2^{n*})^+] \le 0.$$

If $p_2^{r*} < p_2^{n*}$, at least one of the following two inequalities are strict: $\delta[\mathbb{E}((k+\alpha)V - \mathfrak{p}_2^{u*})^+ - \mathbb{E}((k+\alpha)V - p_2^{r*})^+] \le 0$ and $\delta[\mathbb{E}((1+\alpha)V - \mathfrak{p}_2^{u*})^+ - \mathbb{E}((1+\alpha)V - p_2^{n*})^+] \ge 0$. Hence, $p_1^{u*} < p_1^*$ if $p_2^{r*} < p_2^{n*}$.

It's straightforward to compute that, for any $Q_1 \ge 0$,

$$\begin{aligned}\Pi_f(Q_1) - \Pi_f^u(Q_1) =& (p_1^* - p_1^u(Q_1))\mathbb{E}(X \wedge Q_1) + \delta\mathbb{E}[(\beta_n^* - v_2^n(p_2^u((X-Q_1)^+, X \wedge Q_1)))(X - Q_1)^+ \\ &+ (\beta_r^* - \hat{v}_2^n(p_2^u((X-Q_1)^+, X \wedge Q_1)))(X \wedge Q_1)],\end{aligned}$$

where $p_1^u(Q_1) = \mu + \delta[\mathbb{E}((k+\alpha)V - \mathfrak{p}_2^{u*})^+ - \mathbb{E}((1+\alpha)V - \mathfrak{p}_2^{u*})^+] \le p_1^{u*}$ with $\mathfrak{p}_2^{u*} \overset{d}{=} p_2^u((X-Q_1)^+, X \wedge Q_1)$. Since $\beta_n^* \ge v_2^n(p_2)$ and $\beta_r^* \ge v_2^r(p_2) \ge \hat{v}_2^r(p_2)$ for any $p_2 \ge 0$, $\Pi_f(Q_1) \ge \Pi_f^u(Q_1)$ for all $Q_1 \ge 0$, and thus $\Pi_f^* = \max_{Q_1}\Pi_f(Q_1) \ge \max_{Q_1}\Pi_f^u(Q_1) = \Pi_f^{u*}$. If $p_2^{r*} < p_2^{n*}$, by the proof of part (a), $p_1^* > p_1^u(Q_1)$ and, hence, $\Pi_f(Q_1) > \Pi_f^u(Q_1)$ for all $Q_1 > 0$. Therefore, $\Pi_f^* = \Pi_f(Q_1^*) \ge \Pi_f(Q_1^{u*}) > \Pi_f^u(Q_1^{u*}) = \Pi_f^{u*}$. This proves **part (b)**.

Part (c). $\tilde{p}_1^{u*} = \tilde{p}_1^* = \mu$ follows immediately from that μ is the willingness-to-pay of myopic customers. Moreover, direct computation yields that, for any $Q_1 \ge 0$,

$$\begin{aligned}\tilde{\Pi}_f(Q_1) - \tilde{\Pi}_f^u(Q_1) =& \delta\mathbb{E}[(\beta_n^* - v_2^n(p_2^u((X-Q_1)^+, X \wedge Q_1)))(X - Q_1)^+ \\ &+ (\beta_r^* - \hat{v}_2^n(p_2^u((X-Q_1)^+, X \wedge Q_1)))(X \wedge Q_1)] \\ \ge& 0,\end{aligned}$$

where the inequality follows from the proof of part (b). If $p_2^{r*} < p_2^{n*}$, at least one of $\mathbb{E}[(\beta_n^* - v_2^n(p_2^u((X - Q_1)^+, X \wedge Q_1)))(X - Q_1)^+]$ and $\mathbb{E}[(\beta_r^* - \hat{v}_2^n(p_2^u((X-Q_1)^+, X \wedge Q_1)))(X \wedge Q_1)]$ is positive for $Q_1 > 0$. Hence, the same argument as the proof of part (b) yields that $\tilde{\Pi}_f^* > \tilde{\Pi}_f^{u*}$ if $\tilde{Q}_1^{u*} > 0$. This proves **part (c)**. *Q.E.D.*

Proof of Theorem 4.4.1: Part (a). Since $p_1^* - \tilde{p}_1^* = m_1^* - \tilde{m}_1^* = e^*$, it follows immediately that $p_1^* > \tilde{p}_1^*$ and $Q_1^* = \bar{F}^{-1}(\frac{c_1-r_1}{m_1^*-r_1}) > \bar{F}^{-1}(\frac{c_1-r_1}{\tilde{m}_1^*-r_1}) = \tilde{Q}_1^*$ if and only if $e^* > 0$. Moreover, for any Q_1, $\Pi_f(Q_1) - \tilde{\Pi}_f(Q_1) = e^*\mathbb{E}(X \wedge Q_1) > 0$ if and only if $e^* > 0$. Therefore, $\Pi_f^* = \max\Pi_f(Q_1) > \max\tilde{\Pi}_f(Q_1) = \tilde{\Pi}_f^*$ if and only if $e^* > 0$ and $Q_1^* > 0$.

Next, we show that $e^* > 0$ if and only if $r_2 > \bar{r}$. Observe that $v_2^r(p_2^r)$ is submodular in (p_2^r, r_2), so p_2^{r*} is decreasing in r_2. Moreover, e^* is decreasing in p_2^{r*}. Hence, e^* is increasing in r_2 and $e^* > 0$ if and only if $r_2 > \bar{r}$ for some $\bar{r}$. We now show that $\bar{r} \geq \frac{1-k}{1+\alpha}c_2$. It suffices to show that if $r_2 = \frac{1-k}{1+\alpha}c_2$, $e^* \leq 0$. If $r_2 = \frac{1-k}{1+\alpha}c_2$, $v_2^r(p_2^r) = (p_2^r - \frac{k+\alpha}{1+\alpha}c_2)\bar{G}(\frac{p_2^r}{k+\alpha})$. It's straightforward to check that $p_2^{r*} = \frac{k+\alpha}{1+\alpha}p_2^{n*}$. Hence,

$$\begin{aligned} e^* =& \mathbb{E}[(k+\alpha)V - p_2^{r*}]^+ - \mathbb{E}[(1+\alpha)V - p_2^{n*}]^+ \\ =& \mathbb{E}[(k+\alpha)V - \frac{k+\alpha}{1+\alpha}p_2^{n*}]^+ - \mathbb{E}[(1+\alpha)V - p_2^{n*}]^+ \\ =& -\frac{1-k}{1+\alpha}\mathbb{E}[(1+\alpha)V - p_2^{n*}]^+ \leq 0. \end{aligned}$$

This proves **Part (a)**.

Part (b). Observe that,

$$p_1^{u*} - \tilde{p}_1^{u*} = \delta[\mathbb{E}((k+\alpha)V - \mathfrak{p}_2^{u*})^+ - \mathbb{E}((1+\alpha)V - \mathfrak{p}_2^{u*})^+],$$

where $\mathfrak{p}_2^{u*} \overset{d}{=} p_2^u((X - Q_1^{u*})^+, X \wedge Q_1^{u*})$. Since $k \leq 1$, $p_1^{u*} \leq \tilde{p}_1^{u*}$ and the inequality is strict if $k < 1$. This establishes **part (b-i)**.

We now show part (b-ii). Direct computation yields that

$$\tilde{m}_1^u(Q_1) - m_1^u(Q_1) = \mathbb{E}((1+\alpha)V - p_2^u(X_2^n, X_2^r))^+ - \mathbb{E}((k+\alpha)V - p_2^u(X_2^n, X_2^r))^+,$$

where $X_2^n = (X - Q_1)^+$ and $X_2^r = X \wedge Q_1$. Since $k < 1$, we have $\mathbb{E}((1+\alpha)V - p_2^u(X_2^n, X_2^r))^+ - \mathbb{E}((k+\alpha)V - p_2^u(X_2^n, X_2^r))^+ > 0$.

Let $\Pi(Q_1, 1) = \tilde{\Pi}_f^u(Q_1)$ and $\Pi(Q_1, 0) = \Pi_f^u(Q_1)$,

$$\begin{aligned} \Pi(Q_1,1) - \Pi(Q_1,0) &= \tilde{\Pi}_f^u(Q_1) - \Pi_f^u(Q_1) = (\mu - p_1^u(Q_1))\mathbb{E}(X \wedge Q_1) \\ &= [\mathbb{E}((1+\alpha)V - p_2^u(X_2^n, X_2^r))^+ - \mathbb{E}((k+\alpha)V - p_2^u(X_2^n, X_2^r))^+]\mathbb{E}(X \wedge Q_1). \end{aligned}$$

Since $[\mathbb{E}((1+\alpha)V - p)^+ - \mathbb{E}((k+\alpha)V - p)^+]' = -\mathbb{P}(\frac{p}{1+\alpha} \leq V \leq \frac{p}{k+\alpha}) \leq 0$ and $p_2^u(X_2^n, X_2^r)$ is decreasing in Q_1, $\Pi(Q_1,1) - \Pi(Q_1,0) = (\mu - p_1^u(Q_1))(X \wedge Q_1)$ is increasing in Q_1, and, hence, $\Pi(\cdot,\cdot)$ is a supermodular function on the lattice $[0, +\infty) \times \{0,1\}$. Thus, $\tilde{Q}_1^{u*} = \text{argmax}_{Q_1 \geq 0}\Pi(Q_1, 1) \geq \text{argmax}_{Q_1 \geq 0}\Pi(Q_1, 0) = Q_1^{u*}$. This proves **part (b-ii)**.

Finally, since $\tilde{\Pi}_f^u(Q_1) - \Pi_f^u(Q_1) = (\mu - p_1^u(Q_1))(X \wedge Q_1) \geq 0$ where the inequality is strict if $p_2^{r*} < p_2^{n*}$. Hence, $\tilde{\Pi}_f^{u*} = \max_{Q_1 \geq 0} \tilde{\Pi}_f^u(Q_1) \geq \max_{Q_1 \geq 0} \Pi_f^u(Q_1) = \Pi_f^{u*}$. Moreover, the same argument as the proof of Theorem 4.4.2 (b-ii) implies that $\tilde{\Pi}_f^{u*} > \Pi_f^{u*}$ if $p_2^{r*} < p_2^{n*}$. This establishes **part (b)**. *Q.E.D.*

Proof of Theorem 4.4.3: Part (a). We first show that $m_1^u(Q_1)$ is decreasing in Q_1. Observe that $m_1^u(Q_1) = \mu + \delta[U_r(Q_1) - U_n(Q_1)]$, where

$$U_r(Q_1) := \mathbb{E}[(p_2^u(X_2^n, X_2^r) - c_2)\bar{G}\left(\frac{p_2^u(X_2^n, X_2^r)}{k+\alpha}\right)] + \mathbb{E}((k+\alpha)V - p_2^u(X_2^n, X_2^r))^+,$$

and

$$U_n(Q_1) := \mathbb{E}[(p_2^u(X_2^n, X_2^r) - c_2)\bar{G}\left(\frac{p_2^u(X_2^n, X_2^r)}{1+\alpha}\right)] + \mathbb{E}((1+\alpha)V - p_2^u(X_2^n, X_2^r))^+.$$

Let $u_r(p) := (p - c_2)\bar{G}(\frac{p}{k+\alpha}) + \mathbb{E}((k+\alpha)V - p)^+ = \mathbb{E}[(k+\alpha)V - c_2]1_{\{(k+\alpha)V \geq p\}}$ and $u_n(p) := (p - c_2)\bar{G}(\frac{p}{1+\alpha}) + \mathbb{E}((1+\alpha)V - p)^+ = \mathbb{E}[(1+\alpha)V - c_2]1_{\{(1+\alpha)V \geq p\}}$. It's clear that $u_r(\cdot)$ and $u_p(\cdot)$ are continuously decreasing in p. Moreover, $U_r(Q_1) = \mathbb{E}[u_r(p_2^u(X_2^n, X_2^r))]$ and $U_n(Q_1) = \mathbb{E}[u_n(p_2^u(X_2^n, X_2^r))]$, where $X_2^n = (X - Q_1)^+$ and $X_2^r = X \wedge Q_1$. Since $p_2^u(X_2^n, X_2^r)$ is increasing in X_2^n and decreasing in X_2^r, it is stochastically decreasing in Q_1. Hence, it suffices to show that $u_r(p) - u_n(p)$ is increasing in p. Observe that

$$\begin{aligned} u_r(p) - u_n(p) &= -[\int_{p/(1+\alpha)}^{p/(k+\alpha)} ((1+\alpha)V - p)g(V)\,\mathrm{d}V + \int_{p/(k+\alpha)}^{\bar{v}} (1-k)Vg(V)\,\mathrm{d}V] \\ &= -[\int_{p/(1+\alpha)}^{\bar{v}} ((1+\alpha)V - \max(p, (k+\alpha)V))g(V)\,\mathrm{d}V], \end{aligned}$$

which is continuously increasing in p. This establishes **part (a-i)**.

We now show that $m_1^u(Q_1) < m_1^*$ for all Q_1. Observe that

$$m_1^u(Q_1) - m_1^* = \mathbb{E}[u_r(p_2^u(X_2^n, X_2^r)) - u_r(p_2^{r*})] - \mathbb{E}[u_n(p_2^u(X_2^n, X_2^r)) - u_n(p_2^{n*})].$$

Because $p_2^{r*} \le p_2^u(X_2^n, X_2^r) \le p_2^{n*}$, $\mathbb{E}[u_r(p_2^u(X_2^n, X_2^r)) - u_r(p_2^{r*})] \le 0$ and $\mathbb{E}[u_n(p_2^u(X_2^n, X_2^r)) - u_n(p_2^{n*})] \ge 0$. Hence, $m_1^u(Q_1) \le m_1^*$. If $k < 1$, $p_2^{r*} < p_2^{n*}$, one of the inequalities $\mathbb{E}[u_r(p_2^u(X_2^n, X_2^r)) - u_r(p_2^{r*})] \le 0$ and $\mathbb{E}[u_n(p_2^u(X_2^n, X_2^r)) - u_n(p_2^{n*})] \ge 0$ must be strict. Therefore, $m_1^u(Q_1) < m_1^*$ for all $Q_1 \ge 0$. This proves **part (a-ii)**.

Next, we show that $Q_1^{u*} \le Q_1^*$. Observe that

$$\Pi_f^u(Q_1) - \Pi_f(Q_1) = (m_1^u(Q_1) - m_1^*)(X \wedge Q_1) + \delta\mathbb{E}[(p_2^u(X_2^n, X_2^r) - c_2)\bar{G}\left(\frac{p_2^u(X_2^n, X_2^r)}{1+\alpha}\right)] - \delta\beta_n^*\mathbb{E}(X).$$

Let $\Pi(Q_1, 1) = \Pi_f(Q_1)$ and $\Pi(Q_1, 0) = \Pi_f^u(Q_1)$. Then,

$$\Pi(Q_1, 1) - \Pi(Q_1, 0) = (m_1^* - m_1^u(Q_1))(X \wedge Q_1) + \delta\mathbb{E}X[\beta_n^* - (p_2^u(X_2^n, X_2^r) - c_2)\bar{G}\left(\frac{p_2^u(X_2^n, X_2^r)}{1+\alpha}\right)]$$

Note that for any realization of X, $p_2^u(X_2^n, X_2^r)$ and thus $(p_2^u(X_2^n, X_2^r) - c_2)\bar{G}(\frac{p_2^u(X_2^n, X_2^r)}{1+\alpha})$ is decreasing in Q_1. Therefore, by part (a-ii), $\Pi(Q_1, 1) - \Pi(Q_1, 0)$ is increasing in Q_1. Hence, $\Pi(\cdot, \cdot)$ is supermodular on the lattice $[0, +\infty) \times \{0, 1\}$. Hence, $Q_1^{u*} = \mathrm{argmax}_{Q_1 \ge 0}\Pi_f^u(Q_1) \le \mathrm{argmax}_{Q_1 \ge 0}\Pi_f(Q_1) = Q_1^*$. If $Q_1^{u*} > 0$, since $m_1^* > m_1^u(Q_1^{u*})$ $\Pi_f'(Q_1^{u*}) > \partial_{Q_1}\Pi_f^u(Q_1^{u*}) = 0$. Since $\Pi_f(\cdot)$ is concave in Q_1, $Q_1^* > Q_1^{u*}$. This proves **part (a-iii)**.

Part (b). We first show that $\tilde{m}_1^u(Q_1)$ is increasing in Q_1. Note that $\tilde{m}_1^u(Q_1) = \mu + \delta\mathbb{E}[\hat{v}_2^r(p_2^u(X_2^n, X_2^r)) - v_2^n(p_2^u(X_2^n, X_2^r))]$, where $X_2^n = (X - Q_1)^+$ and $X_2^r = X \wedge Q_1$. Because $\hat{p}_2^{r*} \le p_2^u(X_2^n, X_2^r) \le p_2^{n*}$ and $p_2^u(X_2^n, X_2^r)$ is increasing in X_2^n and decreasing in X_2^r. Thus, $\hat{v}_2^r(p_2^u(X_2^n, X_2^r))$ is stochastically increasing in Q_1 and $v_2^n(p_2^u(X_2^n, X_2^r))$ is stochastically decreasing in Q_1. Therefore, $\tilde{m}_1^u(Q_1) = \mu + \delta\mathbb{E}[\hat{v}_2^r(p_2^u(X_2^n, X_2^r)) - v_2^n(p_2^u(X_2^n, X_2^r))]$ is increasing in Q_1. This proves **part (b-i)**.

We now show **part (b-ii)**. Let $\hat{\beta}_r^* = \max_{p \ge 0}\hat{v}_2^r(p)$. It's clear that $\beta_r^* - \hat{\beta}_r^*$ is increasing in r_2, with $\beta_r^* = \hat{\beta}_r^*$ if $r_2 = 0$. Moreover, since $k < 1$, $\hat{\beta}_n^* := v_2^n(\hat{p}_2^{r*}) < \beta_n^*$. Therefore, let $\bar{r}_2 > 0$ be the threshold such that $\beta_r^* - \hat{\beta}_r^* = \beta_n^* - \hat{\beta}_n^*$. Hence, for all $r_2 < \bar{r}_2$, $\beta_r^* - \hat{\beta}_r^* < \beta_n^* - \hat{\beta}_n^*$. Moreover, by the monotone convergence theorem,

$$\lim_{Q_1 \to +\infty} \tilde{m}_1^u(Q_1) = \mu + \delta[v_2^r(\hat{p}_2^{r*}) - v_2^n(\hat{p}_2^{r*})] = \mu + \delta[\hat{\beta}_r^* - \hat{\beta}_n^*] > \mu + \delta[\beta_r^* - \beta_n^*] = \tilde{m}_1^*.$$

Part (b-i) shows that $\tilde{m}_1^u(Q_1)$ is increasing in Q_1. Hence, there exists a threshold $\bar{Q}(r_2)$ such that $\tilde{m}_1^u(Q_1) \ge \tilde{m}_1^*$ if and only if $Q_1 \ge \bar{Q}(r_2)$. To show that $\bar{Q}(r_2)$ is increasing in r_2, we observe that $\tilde{m}_1^*$ is increasing in r_2. Hence, $\bar{Q}(r_2) := \min\{Q_1 : \tilde{m}_1^u(Q_1) \ge \tilde{m}_1^*\}$ is increasing in r_2. This proves **part (b-ii)**.

Part (b-iii). Without loss of generality, assume that $Q_1^{u*} > 0$. Otherwise, the result holds trivially. It's clear that $Q_1^{u*} \uparrow \bar{X}$ and $Q_1^* \uparrow \bar{X}$ as $c_1 \downarrow 0$, where $\bar{X}$ is the upper bound of the support of X. Hence, there exists a threshold $\tilde{c}(r_2) > 0$, dependent on r_2, such that if $c_1 < \tilde{c}(r_2)$, $Q_1^{u*}, Q_1^* > \bar{Q}(r_2)$. Let $\hat{\pi}_2(Q_1) := \delta\mathbb{E}[v_2^n(p_2^u(X_2^n, X_2^r))X]$, where $X_2^n = (X - Q_1)^+$ and $X_2^r = X \wedge Q_1$. It's clear that $\hat{\pi}_2(\cdot)$ is differentiable and, by the chain rule

$$\hat{\pi}_2'(Q_1) = \delta\mathbb{E}[\partial_p v_2^n(p_2^u(X_2^n, X_2^r))(\partial_{X_2^n} p_2^u(X_2^n, X_2^r) + \partial_{X_2^r} p_2^u(X_2^n, X_2^r))1_{\{X \geq Q_1\}}X].$$

As $Q_1 \to \bar{X}$, for any realization of $X \leq \bar{X}$, $\partial_{X_2^n} p_2^u(X_2^n, X_2^r)$ and $\partial_{X_2^r} p_2^u(X_2^n, X_2^r)$ converges to 0. Hence, by the dominated convergence theorem, there exits a threshold $\hat{Q} \in [\bar{Q}(r_2), \bar{X})$, such that $\hat{\pi}_2'(Q_1) \in [-\epsilon\mathbb{P}(X \geq Q_1), 0]$ for all $Q_1 \geq \hat{Q}$, where $\epsilon := (\tilde{m}_1^u(\hat{Q}) - \tilde{m}_1^*)/2 > 0$. Let $\bar{c}_1(r_2) \in (0, \tilde{c}(r_2)]$ be the threshold such that, if $c_1 < \bar{c}_1(r_1)$, we have $Q_1^{u*}, Q_1^* > \hat{Q} \geq \bar{Q}(r_2)$. Therefore,

$$\begin{aligned}
\tilde{\Pi}_f'(Q_1^{u*}) &= (\tilde{m}_1^* - r_1)\mathbb{P}(X \geq Q_1^{u*}) - (c_1 - r_1) \\
&< (\tilde{m}_1^u(Q_1^{u*}) - r_1)\mathbb{P}(X \geq Q_1^{u*}) - \epsilon\mathbb{P}(X \geq Q_1^{u*}) - (c_1 - r_1) \\
&\leq (\tilde{m}_1^u(Q_1^{u*}) - r_1)\mathbb{P}(X \geq Q_1^{u*}) + \hat{\pi}_2'(Q_1^{u*}) - (c_1 - r_1) \\
&\leq \partial_{Q_1}\tilde{\Pi}_f^u(Q_1^{u*}) \\
&= 0,
\end{aligned}$$

where the first inequality follows from $\tilde{m}_1^u(Q_1^{u*}) - \tilde{m}_1^* \geq (\tilde{m}_1^u(\hat{Q}) - \tilde{m}_1^*) = 2\epsilon > \epsilon$, the second from $\hat{\pi}_2'(Q_1^{u*}) \in [-\epsilon\mathbb{P}(X \geq Q_1^{u*}), 0]$, and the last from the monotonicity that $\tilde{m}_1^u(\cdot)$ is increasing in Q_1. Because $\tilde{\Pi}_f(\cdot)$ is concave in Q_1, $\tilde{Q}_1^* = \text{argmax}_{Q_1}\tilde{\Pi}_f(Q_1) < \tilde{Q}_1^{u*}$ follows immediately. This establishes **part (b-iii)** and thus **Theorem 4.4.3**. *Q.E.D.*

Before presenting the proof Theorem 4.4.4, we give the following lemma that computes the equilibrium environmental impacts I_e^* and $\tilde{I}_e^*$.

Lemma 27 *(a) With strategic customers, the total expected environmental impact of the RE equilibrium is $I_e^* = I_e(Q_1^*)$, where*

$$I_e(Q_1) := (\kappa_1 - \iota_1)Q_1 + (\iota_1 + \delta(\kappa_2 - \iota_2)\bar{G}\left(\frac{p_2^{r*}}{k + \alpha}\right))\mathbb{E}(Q_1 \wedge X) + \delta\kappa_2\bar{G}\left(\frac{p_2^{n*}}{1 + \alpha}\right)\mathbb{E}(X - Q_1)^+.$$

(b) With myopic customers, the total expected environmental impact of the RE equilibrium is $\tilde{I}_e^* = I_e(\tilde{Q}_1^*)$.

(c) The function $I_e(\cdot)$ is strictly increasing in Q_1. Hence, $I_e^* \geq \tilde{I}_e^*$ if and only if $Q_1^* \geq \tilde{Q}_1^*$.

Proof of Lemma 27: Parts (a) and (b). Direct computation yields that

$$\begin{aligned}
I_e^* =&\mathbb{E}\{\kappa_1 Q_1^* + \delta\kappa_2(Q_2^n(X_2^{n*}, X_2^{r*}) + Q_2^r(X_2^{n*}, X_2^{r*})) - \iota_1(Q_1^* - X)^+ \\
&- \delta\iota_2 Q_2^r(X_2^{n*}, X_2^{r*})\} \\
=&\mathbb{E}\{\kappa_1 Q_1^* + \delta\kappa_2((X - Q_1^*)^+\bar{G}\left(\frac{p_2^{n*}}{1+\alpha}\right) + (X \wedge Q_1^*)\bar{G}\left(\frac{p_2^{r*}}{k+\alpha}\right)) - \iota_1(Q_1^* - X)^+ \\
&- \delta\iota_2(X \wedge Q_1^*)\bar{G}\left(\frac{p_2^{r*}}{k+\alpha}\right)\} \\
=&(\kappa_1 - \iota_1)Q_1^* + (\iota_1 + \delta(\kappa_2 - \iota_2)\bar{G}\left(\frac{p_2^{r*}}{k+\alpha}\right))\mathbb{E}(X \wedge Q_1^*) + \delta\kappa_2\bar{G}\left(\frac{p_2^{n*}}{1+\alpha}\right)\mathbb{E}(X - Q_1^*)^+ \\
=&I_e(Q_1^*),
\end{aligned}$$

where the second inequality follows from $X_2^{n*} = (X - Q_1^*)^+$ and $X_2^{r*} = X \wedge Q_1^*$, the third from $(Q_1^* - X)^+ = Q_1^* - (X \wedge Q_1^*)$, and the last from the definition of the function $I_e(\cdot)$. Analogously,

$$\begin{aligned}
\tilde{I}_e^* =&\mathbb{E}\{\kappa_1 \tilde{Q}_1^* + \delta\kappa_2(Q_2^n(\tilde{X}_2^{n*}, \tilde{X}_2^{r*}) + Q_2^r(\tilde{X}_2^{n*}, \tilde{X}_2^{r*})) - \iota_1(\tilde{Q}_1^* - X)^+ - \delta\iota_2 Q_2^r(\tilde{X}_2^{n*}, \tilde{X}_2^{r*})\} \\
=&\mathbb{E}\{\kappa_1 \tilde{Q}_1^* + \delta\kappa_2((X - \tilde{Q}_1^*)^+\bar{G}\left(\frac{p_2^{n*}}{1+\alpha}\right) + (X \wedge \tilde{Q}_1^*)\bar{G}\left(\frac{p_2^{r*}}{k+\alpha}\right)) - \iota_1(\tilde{Q}_1^* - X)^+ \\
&- \delta\iota_2(X \wedge \tilde{Q}_1^*)\bar{G}\left(\frac{p_2^{r*}}{k+\alpha}\right)\} \\
=&(\kappa_1 - \iota_1)\tilde{Q}_1^* + (\iota_1 + \delta(\kappa_2 - \iota_2)\bar{G}\left(\frac{p_2^{r*}}{k+\alpha}\right))\mathbb{E}(X \wedge \tilde{Q}_1^*) + \delta\kappa_2\bar{G}\left(\frac{p_2^{n*}}{1+\alpha}\right)\mathbb{E}(X - \tilde{Q}_1^*)^+ \\
=&I_e(\tilde{Q}_1^*).
\end{aligned}$$

This completes the proof of **Parts (a) and (b)**.

Part (c). To establish the monotonicity of $I_e(\cdot)$, observe that $I_e'(Q_1) = \kappa_1 - \iota_1 + (\iota_1 + \delta(\kappa_2 - \iota_2)\bar{G}\left(\frac{p_2^{r*}}{k+\alpha}\right))\mathbb{P}(X > Q_1) - \delta\kappa_2\bar{G}\left(\frac{p_2^{n*}}{1+\alpha}\right)\mathbb{P}(X > Q_1) > \kappa_1 - \iota_1 - \delta\kappa_2 > 0$, where the first inequality follows from $\bar{G}\left(\frac{p_2^{n*}}{1+\alpha}\right) \leq 1$ and $\mathbb{P}(X > Q_1) \leq 1$, and the second from the assumption that $\kappa_1 > \iota_1 + \kappa_2$. Hence, $I_e(\cdot)$ is strictly increasing in Q_1. This proves **Part (c)**. *Q.E.D.*

Proof of Theorem 4.4.4: Part (a). First, we compute I_e^{u*}. Given the market sizes (X_2^n, X_2^r), the equilibrium total second-period production quantity, $Q_2^u(X_2^n, X_2^r)$, is given by

$$Q_2^u(X_2^n, X_2^r) = X_2^n\bar{G}\left(\frac{p_2^u(X_2^n, X_2^r)}{1+\alpha}\right) + X_2^r\bar{G}\left(\frac{p_2^u(X_2^n, X_2^r)}{k+\alpha}\right).$$

Therefore,

$$\begin{aligned}
I_e^{u*} &= \mathbb{E}\{\kappa_1 Q_1^{u*} - \iota_1(Q_1^{u*} - X)^+ + \delta\kappa_2 Q_2^u(X_2^{n*}, X_2^{r*})\} \\
&= (\kappa_1 - \iota_1)Q_1^{u*} + \mathbb{E}\{[\iota_1 + \delta\kappa_2\bar{G}\left(\frac{p_2^u(X_2^{n*}, X_2^{r*})}{k+\alpha}\right)](Q_1^{u*} \wedge X)\} \\
&\quad + \delta\kappa_2\mathbb{E}[\bar{G}\left(\frac{p_2^u(X_2^{n*}, X_2^{r*})}{1+\alpha}\right)(X - Q_1^{u*})^+],
\end{aligned}$$

where $X_2^{n*} = (X - Q_1^{u*})^+$ and $X_2^{r*} = X \wedge Q_1^{u*}$. If $Q_1^* = 0$, $Q_1^{u*} = 0$ as well by Theorem 4.4.3(a). Hence, $I_e^* = I_e^{u*}$ regardless of the value of ι_2. In this case, part (a) trivially holds. On the other hand, if $Q_1^* > 0$,

I_e^* is strictly linearly decreasing in ι_2. Thus, let $\bar{\iota}_2^u := \max\{\iota_2 : I_e^* \geq I_e^{u*}\}$. We have $I_e^* \geq I_2^{u*}$ if and only if $\iota_2 \leq \bar{\iota}_2^u$. In particular, if $\iota_2 = 0$, $Q_1^* > Q_1^{u*}$ implies that

$$(\kappa_1 - \iota_1)Q_1^* + \delta\kappa_2\bar{G}\left(\frac{p_2^{n*}}{1+\alpha}\right)\mathbb{E}(X - Q_1^*)^+ > (\kappa_1 - \iota_1)Q_1^{u*} + \delta\kappa_2\mathbb{E}[\bar{G}\left(\frac{p_2^u(X_2^{n*}, X_2^{r*})}{1+\alpha}\right)(X - Q_1^{u*})^+],$$

and

$$(\iota_1 + \delta(\kappa_2 - \iota_2)\bar{G}\left(\frac{p_2^{r*}}{k+\alpha}\right))\mathbb{E}(X \wedge Q_1^*) \geq \mathbb{E}[(\iota_1 + \delta\kappa_2\bar{G}\left(\frac{p_2^u(X_2^{n*}, X_2^{r*})}{k+\alpha}\right))(X - Q_1^{u*})].$$

Thus, $\bar{\iota}_2^u > 0$. This establishes **part (a)**.

Part (b). As in the proof of part (a), we first compute $\tilde{I}_e^{u*}$:

$$\begin{aligned}
\tilde{I}_e^{u*} &= \mathbb{E}\{\kappa_1\tilde{Q}_1^{u*} - \iota_1(\tilde{Q}_1^{u*} - X)^+ + \delta\kappa_2 Q_2^u(\tilde{X}_2^{n*}, \tilde{X}_2^{r*})\} \\
&= (\kappa_1 - \iota_1)\tilde{Q}_1^{u*} + \mathbb{E}\{[\iota_1 + \delta\kappa_2\bar{G}\left(\frac{p_2^u(\tilde{X}_2^{n*}, \tilde{X}_2^{r*})}{k+\alpha}\right)](\tilde{Q}_1^{u*} \wedge X)\} \\
&\quad + \delta\kappa_2\mathbb{E}[\bar{G}\left(\frac{p_2^u(\tilde{X}_2^{n*}, \tilde{X}_2^{r*})}{1+\alpha}\right)(X - \tilde{Q}_1^{u*})^+],
\end{aligned}$$

where $\tilde{X}_2^{n*} = (X - \tilde{Q}_1^{u*})^+$ and $\tilde{X}_2^{r*} = X \wedge \tilde{Q}_1^{u*}$. By Theorem 4.4.4(b), $\tilde{Q}_1^{u*} \geq \tilde{Q}_1^*$. Hence,

$$(\kappa_1 - \iota_1)\tilde{Q}_1^* + \delta\kappa_2\bar{G}\left(\frac{p_2^{n*}}{1+\alpha}\right)\mathbb{E}(X - \tilde{Q}_1^*)^+ \leq (\kappa_1 - \iota_1)\tilde{Q}_1^{u*} + \delta\kappa_2\mathbb{E}[\bar{G}\left(\frac{p_2^u(\tilde{X}_2^{n*}, \tilde{X}_2^{r*})}{1+\alpha}\right)(X - \tilde{Q}_1^{u*})^+].$$

Let $\tilde{\iota}_2^u := (\bar{G}(\frac{p_2^{r*}}{k+\alpha}) - \bar{G}(\frac{p_2^{n*}}{k+\alpha}))\kappa_2/\bar{G}(\frac{p_2^{r*}}{k+\alpha}) < \kappa_2$. If $\iota_2 \geq \tilde{\iota}_2^u$, since $\tilde{Q}_1^{u*} \geq \tilde{Q}_1^*$,

$$\begin{aligned}
\mathbb{E}\{[\iota_1 + \delta\kappa_2\bar{G}\left(\frac{p_2^u(\tilde{X}_2^{n*}, \tilde{X}_2^{r*})}{k+\alpha}\right)](\tilde{Q}_1^{u*} \wedge X)\} &\geq \mathbb{E}\{[\iota_1 + \delta(\kappa_2 - \iota_2)\bar{G}\left(\frac{p_2^{n*}}{k+\alpha}\right)](\tilde{Q}_1^{u*} \wedge X)\} \\
&\geq \mathbb{E}\{[\iota_1 + \delta(\kappa_2 - \iota_2)\bar{G}\left(\frac{p_2^{n*}}{k+\alpha}\right)](Q_1^{u*} \wedge X)\}.
\end{aligned}$$

Therefore, if $\iota_2 \geq \tilde{\iota}_2^u$,

$$\begin{aligned}
\tilde{I}_e^{u*} &= (\kappa_1 - \iota_1)\tilde{Q}_1^{u*} + \mathbb{E}\{[\iota_1 + \delta\kappa_2\bar{G}\left(\frac{p_2^u(\tilde{X}_2^{n*}, \tilde{X}_2^{r*})}{k+\alpha}\right)](\tilde{Q}_1^{u*} \wedge X)\} \\
&\quad + \delta\kappa_2\mathbb{E}[\bar{G}\left(\frac{p_2^u(\tilde{X}_2^{n*}, \tilde{X}_2^{r*})}{1+\alpha}\right)(X - \tilde{Q}_1^{u*})^+] \\
&\geq (\kappa_1 - \iota_1)\tilde{Q}_1^* + \delta\kappa_2\bar{G}\left(\frac{p_2^{n*}}{1+\alpha}\right)\mathbb{E}(X - \tilde{Q}_1^*)^+ + \mathbb{E}\{[\iota_1 + \delta(\kappa_2 - \iota_2)\bar{G}\left(\frac{p_2^{n*}}{k+\alpha}\right)](Q_1^{u*} \wedge X)\} \\
&= \tilde{I}_e^*,
\end{aligned}$$

which proves **part (b)**. *Q.E.D.*

Proof of Theorem 4.4.5: Part (a). We first compute the equilibrium total customer surplus in the scenario of strategic customers, S_c^*. If a customer is a new customer in period 2, her expected total surplus is $\delta\mathbb{E}((1+\alpha)V - p_2^{n*})^+$ (since, by Lemma 7, $p_2^n(X_2^n, X_2^r) = p_2^{n*}$). Hence, the expected surplus of a strategic customer in the base model is given by:

$$\begin{aligned}
&\mathfrak{a}^*(\mu - p_1^* + \delta\mathbb{E}((k+\alpha)V - p_2^{r*})^+) + (1 - \mathfrak{a}^*)\delta\mathbb{E}((1+\alpha)V - p_2^{n*})^+ \\
=&\mathfrak{a}^*(\mu - \mu + \delta\mathbb{E}((1+\alpha)V - p_2^{n*})^+ - \delta\mathbb{E}((k+\alpha)V - p_2^{r*})^+ + \delta\mathbb{E}((k+\alpha)V - p_2^{r*})^+) \\
&+ (1 - \mathfrak{a}^*)\delta\mathbb{E}((1+\alpha)V - p_2^{n*})^+ \\
=&\delta\mathbb{E}((1+\alpha)V - p_2^{n*})^+.
\end{aligned}$$

Therefore, the equilibrium total customer surplus is given by $S_c^* = \mathbb{E}[\delta\mathbb{E}((1+\alpha)V - p_2^{n*})^+ X] = \delta\mathbb{E}[((1+\alpha)V - p_2^{n*})^+ X]$.

We now compute the equilibrium total customer surplus in the scenario of myopic customers, $\tilde{S}_c^*$. Since the customers are myopic, they get zero utility in period 1. Hence, in period 2, the expected surplus of a new customer is $\delta\mathbb{E}((1+\alpha)V - p_2^{n*})^+$, whereas that of a repeat customer is $\delta\mathbb{E}((k+\alpha)V - p_2^{r*})^+$. Therefore, the total customer surplus is given by

$$\begin{aligned}\tilde{S}_c^* =& \mathbb{E}[\delta\mathbb{E}((1+\alpha)V - p_2^{n*})^+ (X - \tilde{Q}_1^*)^+] + \mathbb{E}[\delta\mathbb{E}((k+\alpha)V - p_2^{r*})^+ (X \wedge \tilde{Q}_1^*)] \\ =& \delta\mathbb{E}[((1+\alpha)V - p_2^{n*})^+ (X - \tilde{Q}_1^*)^+] + \delta\mathbb{E}[((k+\alpha)V - p_2^{r*})^+ (X \wedge \tilde{Q}_1^*)].\end{aligned}$$

This proves **part (a)**.

Part (b). We first compute S_c^{u*}. If a customer is a new customer in period 2, her expected total surplus is $\delta\mathbb{E}((1+\alpha)V - \mathfrak{p}_2^{u*})^+$. Hence, the expected surplus of a strategic customer in the NTR model is given by:

$$\begin{aligned}&\mathfrak{a}^{u*}(\mu - p_1^{u*} + \delta\mathbb{E}((k+\alpha)V - \mathfrak{p}_2^{u*})^+) + (1-\mathfrak{a}^{u*})\delta\mathbb{E}((1+\alpha)V - \mathfrak{p}_2^{u*})^+ \\ =& \mathfrak{a}^{u*}(\mu - \mu + \delta\mathbb{E}((1+\alpha)V - \mathfrak{p}_2^{u*})^+ - \delta\mathbb{E}((k+\alpha)V - \mathfrak{p}_2^{u*})^+ + \delta\mathbb{E}((k+\alpha)V - \mathfrak{p}_2^{u*})^+) \\ &+ (1-\mathfrak{a}^{u*})\delta\mathbb{E}((1+\alpha)V - \mathfrak{p}_2^{u*})^+ \\ =& \delta\mathbb{E}((1+\alpha)V - \mathfrak{p}_2^{u*})^+.\end{aligned}$$

Therefore, the equilibrium total customer surplus is given by $S_c^{u*} = \mathbb{E}[\delta\mathbb{E}((1+\alpha)V - \mathfrak{p}_2^{u*})^+ X] = \delta\mathbb{E}[((1+\alpha)V - \mathfrak{p}_2^{u*})^+ X]$.

We now compute $\tilde{S}_c^{u*}$. Since the customers are myopic, they get zero utility in period 1. Hence, in period 2, the expected surplus of a new customer is $\delta\mathbb{E}((1+\alpha)V - \mathfrak{p}_2^{u*})^+$, whereas that of a repeat customer is $\delta\mathbb{E}((k+\alpha)V - \mathfrak{p}_2^{u*})^+$. Therefore, the total customer surplus is given by

$$\begin{aligned}\tilde{S}_c^{u*} =& \mathbb{E}[\delta\mathbb{E}((1+\alpha)V - \mathfrak{p}_2^{u*})^+ (X - \tilde{Q}_1^{u*})^+] + \mathbb{E}[\delta\mathbb{E}((k+\alpha)V - \mathfrak{p}_2^{u*})^+ (X \wedge \tilde{Q}_1^{u*})] \\ =& \delta\mathbb{E}[((1+\alpha)V - \mathfrak{p}_2^{u*})^+ (X - \tilde{Q}_1^{u*})^+] + \delta\mathbb{E}[((k+\alpha)V - \mathfrak{p}_2^{u*})^+ (X \wedge \tilde{Q}_1^{u*})].\end{aligned}$$

This proves **part (b)**.

Part (c). Note that, by Theorem 4.4.2(a), $p_2^{r*} \le \mathfrak{p}_2^{u*} \le p_2^{n*}$ with probability 1. It follows immediately that $S_c^{u*} = \delta\mathbb{E}[((1+\alpha)V - \mathfrak{p}_2^{u*})^+ X] \ge \delta\mathbb{E}[((1+\alpha)V - p_2^{n*})^+ X] = S_c^*$. In particular, if $k < 1$ and $Q_1^{u*} > 0$, $\mathfrak{p}_2^{u*} < p_2^{n*}$ with probability 1 and thus $S_c^{u*} > S_c^*$. This proves **part (c)**. *Q.E.D.*

Proof of Lemma 9: Part (a). Let $W_2(p_2^n, p_2^r | X_2^n, X_2^r)$ be the expected social welfare in period 2 when the price for new customers is p_2^n, and that for repeat customers is p_2^r. Since all new (repeat) customers with valuation $(1+\alpha)V \ge p_2^n$ ($(k+\alpha)V \ge p_2^r$) will make a purchase (trade the used products in), the firm profit equals $(p_2^n - c_2)\bar{G}(\frac{p_2^n}{1+\alpha})X_2^n + (p_2^r - c_2 + r_2)\bar{G}(\frac{p_2^r}{k+\alpha})X_2^r$, the expected customer surplus equals $X_2^n\mathbb{E}((1+\alpha)V - p_2^n)^+ + X_2^r\mathbb{E}((k+\alpha)V - p_2^r)^+$, and the environmental impact equals $\kappa_2 X_2^n \bar{G}(\frac{p_2^n}{1+\alpha}) + (\kappa_2 - \iota_2)X_2^r\bar{G}(\frac{p_2^r}{k+\alpha})$. Therefore, $W_2(p_2^n, p_2^r | X_2^n, X_2^r) = X_2^n w_n(p_2^n) + X_2^r w_r(p_2^r)$, where

$$w_n(p_2^n) := (p_2^n - c_2 - \kappa_2)\bar{G}(\frac{p_2^n}{1+\alpha}) + \mathbb{E}((1+\alpha)V - p_2^n)^+ = \mathbb{E}((1+\alpha)V - c_2 - \kappa_2)1_{\{(1+\alpha)V \ge p_2\}},$$

and

$$w_r(p_2^r) := (p_2^r - c_2 + r_2 - \kappa_2 + \iota_2)\bar{G}(\frac{p_2^r}{k+\alpha}) + \mathbb{E}((k+\alpha)V - p_2^r)^+ = \mathbb{E}((k+\alpha)V - c_2 + r_2 - \kappa_2 + \iota_2)1_{\{(k+\alpha)V \ge p_2^r\}}.$$

Thus, $w_n'(p_2^n) = \frac{p_2^n - c_2 - \kappa_2}{1+\alpha} g(\frac{p_2^n}{1+\alpha})$ and $w_r'(p_2^r) = \frac{p_2^r - c_2 + r_2 - \kappa_2 + \iota_2}{k+\alpha} g(\frac{p_2^r}{k+\alpha})$. Thus, $w_n'(p_2^n) > 0$ if $p_2^n < c_2 + \kappa_2$ and $w_n'(p_2^n) < 0$ if $p_2^n > c_2 + \kappa_2$. Analogously, $w_r'(p_2^r) > 0$ if $p_2^r < c_2 + \kappa_2 - r_2 - \iota_2$ and $w_r'(p_2^r) < 0$ if $p_2^r > c_2 + \kappa_2 - r_2 - \iota_2$. Hence, the unique maximizer of $w_n(\cdot)$ is $c_2 + \kappa_2$, and the unique maximizer of $w_r(\cdot)$ is $c_2 + \kappa_2 - r_2 - \iota_2$. Finally, it is straightforward to check that $c_2 + \kappa_2 - r_2 - \iota_2 \le c_2 + \kappa_2$, with the inequality being strict if and only if $r_2 > 0$ or $\iota_2 > 0$. Therefore, $p_{s,2}^n(X_2^n, X_2^r) \equiv p_{s,2}^{n*} = c_2 + \kappa_2$ and $p_{s,2}^r(X_2^n, X_2^r) \equiv p_{s,2}^{r*} = c_2 + \kappa_2 - r_2 - \iota_2$ for any realized (X_2^n, X_2^r). This proves **part (a)**.

Part (b). Under the equilibrium prices $(p_{s,2}^{n*}, p_{s,2}^{r*})$, a new customer will make a purchase if and only if her valuation $(1+\alpha)V \ge p_{s,2}^{n*}$, whereas a repeat customer will make a purchase (and join the trade-in program) if and only if her valuation $(k+\alpha)V \ge p_{s,2}^{r*}$. Therefore,

$$Q_{s,2}^n(X_2^n, X_2^r) = \mathbb{E}[X_2^n 1_{\{(1+\alpha)V \ge p_{s,2}^{n*}\}} | X_2^n] = X_2^n \bar{G}\left(\frac{p_{s,2}^{n*}}{1+\alpha}\right),$$

and

$$Q_{s,2}^r(X_2^n, X_2^r) = \mathbb{E}[X_2^r 1_{\{(k+\alpha)V \ge p_{s,2}^{r*}\}} | X_2^r] = X_2^r \bar{G}\left(\frac{p_{s,2}^{r*}}{k+\alpha}\right),$$

which proves **part (b)**.

Part (c). Plugging $p_{s,2}^{n*}$ and $p_{s,2}^{r*}$ into $w_2^n(\cdot)$ and $w_2^r(\cdot)$, respectively, we have $w_2^n(p_{s,2}^{n*}) = \mathbb{E}[(1+\alpha)V - p_{s,2}^{n*}]^+$ and $w_2^r(p_{s,2}^{r*}) = \mathbb{E}[(1+\alpha)V - p_{s,2}^{r*}]^+$. Therefore, $w_2(X_2^n, X_2^r) = X_2^n \mathbb{E}[(1+\alpha)V - p_{s,2}^{n*}]^+ + X_2^r \mathbb{E}[(1+\alpha)V - p_{s,2}^{r*}]^+$. This completes the proof of **part (c)**. *Q.E.D.*

Proof of Lemma 10: Part (a). Let $W_s(Q_1)$ be the expected social welfare with first-period production quantity Q_1 under strategic customer behavior. Following the same argument as the proof of Theorem 4.3.1(a), we have

$$\begin{aligned} p_{s,1}^* &= \mu + \delta\mathbb{E}[(k+\alpha)V - \mathfrak{p}_{s,2}^{r*}]^+ - \delta\mathbb{E}[(1+\alpha)V - \mathfrak{p}_{s,2}^{n*}]^+ \\ &= \mu + \delta\mathbb{E}[(k+\alpha)V - p_{s,2}^{r*}]^+ - \delta\mathbb{E}[(1+\alpha)V - p_{s,2}^{n*}]^+ \\ &= \mu + \delta(\beta_{s,r}^* - \beta_{s,n}^*) \\ &= m_{s,1}^*, \end{aligned}$$

which proves **part (a-i)**.

We now compute $W_s(Q_1)$. By Lemma 9(c), $w_2(X_2^n, X_2^r) = \beta_{s,n}^* X_2^n + \beta_{s,r}^* X_2^r$, so

$$\begin{aligned} W_s(Q_1) &= p_{s,1}^*\mathbb{E}(X \wedge Q_1) + (\mu - p_{s,1}^*)(X \wedge Q_1) - (c_1 + \kappa_1)Q_1 + (r_1 + \iota_1)\mathbb{E}(Q_1 - X)^+ \\ &\quad + \delta\mathbb{E}\{w_2(X - (X \wedge Q_1), X \wedge Q_1)\} \\ &= (\mu - r_1 - \iota_1)\mathbb{E}(X \wedge Q_1) - (c_1 - r_1 + \kappa_1 - \iota_1)Q_1 + \delta\mathbb{E}\{\beta_{s,n}^*(X - (X \wedge Q_1)) + \beta_{s,r}^*(X \wedge Q_1)\} \\ &= (m_{s,1}^* - r_1 - \iota_1)\mathbb{E}(X \wedge Q_1) - (c_1 - r_1 + \kappa_1 - \iota_1)Q_1 + \delta\beta_{s,n}^*\mathbb{E}(X). \end{aligned}$$

Therefore, $Q^*_{s,1}$ is the solution to a newsvendor problem with marginal revenue $m^*_{s,1} - r_1 - \iota_1$, marginal cost $c_1 + \kappa_1 - r_1 - \iota_1$, and demand distribution $F(\cdot)$. Hence, $Q^*_{s,1} = \bar{F}^{-1}(\frac{c_1+\kappa_1-r_1-\iota_1}{m^*_{s,1}-r_1-\iota_1})$, and the equilibrium social welfare is

$$W^*_s = W_s(Q^*_{s,1}) = (m^*_{s,1} - r_1 - \iota_1)\mathbb{E}(X \wedge Q^*_{s,1}) - (c_1 - r_1 + \kappa_1 - \iota_1)Q_1 + \delta\beta^*_{s,n}\mathbb{E}(X).$$

This proves **part (a-ii,iii)**.

Part (b). Let $\tilde{W}_s(Q_1)$ be the expected social welfare with myopic customers, if the first-period production quantity is Q_1. The willingness-to-pay of myopic customers is their expected valuation of the first-generation product μ. Thus, $\tilde{p}^*_{s,1} = \mu$. This proves **part (b-i)**.

We now compute $\tilde{W}_s(Q_1)$. By Lemma 9(c), $w_2(X^n_2, X^r_2) = \beta^*_{s,n}X^n_2 + \beta^*_{s,r}X^r_2$, so

$$\begin{aligned}
\tilde{W}_s(Q_1) &= \tilde{p}^*_{s,1}\mathbb{E}(X \wedge Q_1) + (\mu - \tilde{p}^*_{s,1})(X \wedge Q_1) - (c_1 + \kappa_1)Q_1 + (r_1 + \iota_1)\mathbb{E}(Q_1 - X)^+ \\
&\quad + \delta\mathbb{E}\{w_2(X - (X \wedge Q_1), X \wedge Q_1)\} \\
&= (\mu - r_1 - \iota_1)\mathbb{E}(X \wedge Q_1) - (c_1 - r_1 + \kappa_1 - \iota_1)Q_1 + \delta\mathbb{E}\{\beta^*_{s,n}(X - (X \wedge Q_1)) + \beta^*_{s,r}(X \wedge Q_1)\} \\
&= (\tilde{m}^*_{s,1} - r_1 - \iota_1)\mathbb{E}(X \wedge Q_1) - (c - r_1 + \kappa_1 - \iota_1)Q_1 + \delta\beta^*_{s,n}\mathbb{E}(X).
\end{aligned}$$

Therefore, $\tilde{Q}^*_{s,1}$ is the solution to a newsvendor problem with marginal revenue $\tilde{m}^*_{s,1} - r_1 - \iota_1$, marginal cost $c_1 + \kappa_1 - r_1 - \iota_1$, and demand distribution $F(\cdot)$. Hence, $\tilde{Q}^*_{s,1} = \bar{F}^{-1}(\frac{c_1+\kappa_1-r_1-\iota_1}{\tilde{m}^*_{s,1}-r_1-\iota_1})$, and the equilibrium social welfare is

$$\tilde{W}^*_s = \tilde{W}_s(\tilde{Q}^*_{s,1}) = (\tilde{m}^*_{s,1} - r_1 - \iota_1)\mathbb{E}(X \wedge \tilde{Q}^*_{s,1}) - (c - r_1 + \kappa_1 - \iota_1)\tilde{Q}_1 + \delta\beta^*_{s,n}\mathbb{E}(X).$$

This proves **part (b-ii,iii)**.

Part (c). Since $p^*_{s,1} - \tilde{p}^*_{s,1} = \beta^*_{s,r} - \beta^*_{s,n} = e^*_s$, $p^*_{s,1} \geq \tilde{p}^*_{s,1}$ if and only if $e^*_s \geq 0$. The equalities $Q^*_{s,1} = \tilde{Q}^*_{s,1}$ and $W^*_s = \tilde{W}^*_s$ follow from the fact that $m^*_{s,1} = \tilde{m}^*_{s,1}$. This establishes **part (c)**. *Q.E.D.*

Proof of Theorem 4.5.1: Part (a). With the unit subsidy rate s_r for remanufactured products, the expected per demand profit from repeat customers $v^r_2(p^r_2) = (p^r_2 + s_r + s_2 - c_2 + r_2)\bar{G}(\frac{p^r_2}{k+\alpha})$. Since $\partial_{p^r_2}\partial_{s_r}v^r_2(p^r_2) = -\frac{1}{1+\alpha}g(\frac{p^r_2}{1+\alpha}) \leq 0$, $v^r_2(p^r_2)$ is submodular in (p^r_2, s_r). Hence, $p^{r*}_2 = \text{argmax}_{p^r_2 \geq 0}v^r_2(p^r_2)$ is continuously decreasing in s_r. This completes the proof of **part (a-i)**. Because $Q^r_2(X^n_2, X^r_2) = X^r_2\bar{G}\left(\frac{p^{r*}_2}{k+\alpha}\right)$ and p^{r*}_2 is decreasing in s_r, $Q^r_2(X^n_2, X^r_2)$ is increasing in s_r, which proves **part (a-ii)**.

Part (b). By Theorem 4.3.1(a), $p^*_1 = \mu + \delta[\mathbb{E}((k+\alpha)V - p^{r*}_2)^+ - \mathbb{E}((1+\alpha)V - p^{n*}_2)^+]$, which is decreasing in p^{r*}_2. Since p^{r*}_2 is decreasing in s_r, p^*_1 is increasing in s_r. With the unit subsidy rate s_r for remanufactured product,

$$\Pi_f(Q_1) = (p^*_1 - r_1 - s_r)\mathbb{E}(X \wedge Q_1) - (c_1 - r_1 - s_r)Q_1 + \delta\beta^*_n\mathbb{E}(X),$$

Hence, $Q^*_1 = \bar{F}^{-1}\left(\frac{c_1-r_1-s_r}{p^*_1-r_1-s_r}\right)$. The critical fractile $\frac{c_1-r_1-s_r}{p^*_1-r_1-s_r}$ is decreasing in p^*_1 and s_r. Therefore, Q^*_1 is increasing in s_r. For each Q_1, $\Pi_f(Q_1)$ is increasing in s_r. Thus, $\Pi^*_f = \max_{Q_1 \geq 0}\Pi_f(Q_1)$ is increasing in s_r. By Lemma 27(a), $I^*_e = I_e(Q^*_1)$, which is increasing in Q^*_1. Thus, I^*_e is increasing in s_r as well. This establishes **part (b)**.

Part (c). By Theorem 4.3.1(b), $\tilde{p}_1^* = \mu$, which is independent of s_r. With the unit subsidy rate s_r for remanufactured product,

$$\tilde{\Pi}_f(Q_1) = (\tilde{p}_1^* - r_1 - s_r)\mathbb{E}(X \wedge Q_1) - (c_1 - r_1 - s_r)Q_1 + \delta\beta_n^*\mathbb{E}(X),$$

Hence, $\tilde{Q}_1^* = \bar{F}^{-1}\left(\frac{c_1-r_1-s_r}{\tilde{p}_1^*-r_1-s_r}\right)$. The critical fractile $\frac{c_1-r_1-s_r}{\tilde{p}_1^*-r_1-s_r}$ is decreasing in s_r. Therefore, $\tilde{Q}_1^*$ is increasing in s_r. For each Q_1, $\tilde{\Pi}_f(Q_1)$ is increasing in s_r. Thus, $\tilde{\Pi}_f^* = \max_{Q_1\geq 0}\tilde{\Pi}_f(Q_1)$ is increasing in s_r. By Lemma 27(b), $\tilde{I}_e^* = \tilde{I}_e(Q_1^*)$, which is increasing in $\tilde{Q}_1^*$. Thus, $\tilde{I}_e^*$ is increasing in s_r as well. This establishes **part (c)**. *Q.E.D.*

Proof of Theorem 4.5.2: Part (a). If s_2^* is the solution to $p_{s,2}^{n*} = \text{argmax}_{p_2^n\geq 0}(p_2^n + s_2 - c_2)\bar{G}\left(\frac{p_2^n}{1+\alpha}\right)$, it is clear that the subsidy/tax scheme with $s_2 = s_2^*$ can induce the equilibrium price for new customers $p_{s,2}^{n*}$. We now show that s_2^* exists. Since $v_2^n(p_2^n)$ is quasiconcave in p_2^n for any s_2, the first-order condition $\partial_{p_2^n}v_2^n(p_2^n) = 0$ guarantees the optimal price for new customers. Moreover,

$$\partial_{p_2^n}v_2^n(p_{s,2}^{n*}) = \bar{G}\left(\frac{p_{s,2}^{n*}}{1+\alpha}\right) - \frac{p_{s,2}^{n*} + s_2 - c_2}{1+\alpha}g\left(\frac{p_{s,2}^{n*}}{1+\alpha}\right),$$

which is strictly decreasing in s_2. Hence, there exists a unique s_2^*, such that $\partial_{p_2^n}v_2^n(p_{s,2}^{n*}) = 0$, thus inducing the socially optimal equilibrium price for new customers $p_{s,2}^{n*}$. This proves **part (a-i)**.

If s_r^* is the solution to $p_{s,2}^{r*} = \text{argmax}_{p_2^r\geq 0}(p_2^r + s_2^* + s_r - c_2 + r_2)\bar{G}\left(\frac{p_2^r}{k+\alpha}\right)$, the subsidy/tax scheme with $s_r = s_r^*$ can induce the equilibrium trade-in price for repeat customers $p_{s,2}^{r*}$. We now show that s_r^* exists. Since $v_2^r(p_2^r)$ is quasiconcave in p_2^r for any (s_2, s_r), the first-order condition $\partial_{p_2^r}v_2^r(p_2^r) = 0$ guarantees the optimal price for new customers. Moreover, if $s_2 = s_2^*$,

$$\partial_{p_2^r}v_2^r(p_{s,2}^{r*}) = \bar{G}\left(\frac{p_{s,2}^{r*}}{k+\alpha}\right) - \frac{p_{s,2}^{r*} + s_2^* + s_r - c_2 + r_2}{k+\alpha}g\left(\frac{p_{s,2}^{r*}}{k+\alpha}\right),$$

which is strictly decreasing in s_r. Hence, there exists a unique s_r^*, such that $\partial_{p_2^r}v_2^r(p_{s,2}^{r*}) = 0$ if $s_2 = s_2^*$, thus inducing the socially optimal equilibrium trade-in price for repeat customers $p_{s,2}^{r*}$. This proves **part (a-ii)**.

Given the subsidy/tax scheme (s_1, s_2^*, s_r^*), as shown above, the firm adopts the same second-period pricing strategy as the social welfare maximizing one: $(p_{s,2}^{n*}, p_{s,2}^{r*})$. Hence, the first-period price should also be the same as the one which is socially optimal and characterized by Lemma 10(a): $p_{s,1}^* = \mu + \delta[\beta_{s,r}^* - \beta_{s,n}^*]$. Thus, the expected profit of the firm in period 1 is

$$\begin{aligned}\Pi_f^s(Q_1) &= (p_{s,1}^* + s_1 - r_1)\mathbb{E}(X \wedge Q_1) - (c_1 - r_1)Q_1 + \delta\mathbb{E}[(X - X \wedge Q_1)(p_{s,2}^{n*} + s_2^* - c_2)\bar{G}\left(\frac{p_{s,2}^{n*}}{1+\alpha}\right) \\ &\quad +(X \wedge Q_1)(p_{s,2}^{r*} + s_2^* + s_r^* - c_2 + r_2)\bar{G}\left(\frac{p_{s,2}^{r*}}{k+\alpha}\right)] \\ &= (m_1^s(s_1) - r_1)\mathbb{E}(X \wedge Q_1) - (c_1 - r_1)Q_1 + \delta(p_{s,2}^{n*} + s_2^* - c_2)\bar{G}\left(\frac{p_{s,2}^{n*}}{1+\alpha}\right)\mathbb{E}(X),\end{aligned}$$

where $m_1^s(s_1) = s_1 + m_{s,1}^* + \delta[(\kappa_2 + s_2^* + s_r^* - \iota_2)\bar{G}(\frac{p_{s,2}^{r*}}{k+\alpha}) - (\kappa_2 + s_2^*)\bar{G}(\frac{p_{s,2}^{n*}}{1+\alpha})]$. Thus, $\Pi_f^s(Q_1)$ has a unique optimizer $\bar{F}^{-1}(\frac{c_1-r_1}{m_1^s(s_1)-r_1})$. Moreover, as shown in Lemma 10, $Q_{s,1}^* = \bar{F}^{-1}(\frac{c_1+\kappa_1-r_1-\iota_1-s_r^*}{m_{s,1}^*-r_1-s_r^*})$. Therefore, if s_1^* is the unique solution to $\frac{c_1-r_1}{m_1^s(s_1)-r_1} = \frac{c_1+\kappa_1-r_1-\iota_1-s_r^*}{m_{s,1}^*-r_1-s_r^*}$, the optimal production quantity with the linear subsidy/tax scheme $s_g^* = (s_1^*, s_2^*, s_r^*)$ is $Q_{s,1}^*$, which is the socially optimal first-period production quantity. This proves **part (a-iii)**.

We now show that s_2^* is increasing in κ_2. As shown in part (a-i), s_2^* satisfies $\bar{G}\left(\frac{p_{s,2}^{n*}}{1+\alpha}\right) - \frac{p_{s,2}^{n*}+s_2^*-c_2}{1+\alpha} g\left(\frac{p_{s,2}^{n*}}{1+\alpha}\right) = 0$, i.e.,

$$s_2^* = \frac{(1+\alpha)\bar{G}\left(\frac{p_{s,2}^{n*}}{1+\alpha}\right)}{g\left(\frac{p_{s,2}^{n*}}{1+\alpha}\right)} - p_{s,2}^{n*} + c_2 = \frac{(1+\alpha)\bar{G}\left(\frac{c_2+\kappa_2}{1+\alpha}\right)}{g\left(\frac{c_2+\kappa_2}{1+\alpha}\right)} - \kappa_2.$$

Because $g(v)/\bar{G}(v)$ is increasing in v, s_2^* is strictly decreasing in κ_2. Analogously, by part (a-ii), s_r^* satisfies $\bar{G}\left(\frac{p_{s,2}^{r*}}{k+\alpha}\right) - \frac{p_{s,2}^{r*}+s_2^*+s_r-c_2+r_2}{k+\alpha} g\left(\frac{p_{s,2}^{r*}}{k+\alpha}\right) = 0$, i.e.,

$$s_r^* = \frac{(k+\alpha)\bar{G}\left(\frac{p_{s,2}^{r*}}{k+\alpha}\right)}{g\left(\frac{p_{s,2}^{r*}}{k+\alpha}\right)} - p_{s,2}^{r*} - s_2^* + c_2 - r_2 = \frac{(k+\alpha)\bar{G}\left(\frac{c_2-r_2+\kappa_2-\iota_2}{k+\alpha}\right)}{g\left(\frac{c_2-r_2+\kappa_2-\iota_2}{k+\alpha}\right)} - s_2^* - \kappa_2 + \iota_2.$$

Because $g(v)/\bar{G}(v)$ is increasing in v, s_r^* is strictly increasing in ι_2.

By part (a-iii), s_1^* satisfies $\frac{c_1-r_1}{m_1^s(s_1^*)-r_1} = \frac{c_1+\kappa_1-r_1-\iota_1-s_r^*}{m_{s,1}^*-r_1-s_r^*}$, the left-hand-side of which is strictly decreasing in s_1^*, whereas the right-hand-side of which is strictly increasing in κ_1. Therefore, s_1^* is strictly decreasing in κ_1. This proves **part (a-iv)**.

Define $\bar{\kappa}_2^s$ as the solution to $\frac{(1+\alpha)\bar{G}\left(\frac{c_2+\kappa_2}{1+\alpha}\right)}{g\left(\frac{c_2+\kappa_2}{1+\alpha}\right)} = \kappa_2$, $\bar{\iota}_2^s$ as the solution to $\frac{(k+\alpha)\bar{G}\left(\frac{c_2-r_2+\kappa_2-\iota_2}{k+\alpha}\right)}{g\left(\frac{c_2-r_2+\kappa_2-\iota_2}{k+\alpha}\right)} - s_2^* - \kappa_2 + \iota_2 = 0$, and $\bar{\kappa}_1^s$ as the solution to $\frac{c_1-r_1}{m_1^s(0)-r_1} = \frac{c_1+\kappa_1-r_1-\iota_1-s_r^*}{m_{s,1}^*-r_1-s_r^*}$. Since $g(v)/\bar{G}(v)$ is increasing in v, $\bar{\kappa}_2^s$, $\bar{\iota}_2^s$, and $\bar{\kappa}_1^s$ are well-defined and unique. By the proof of part (a-iv), s_2^* is strictly decreasing in κ_2, s_r^* is strictly increasing in ι_2, and s_1^* is strictly decreasing in κ_1. Therefore, $s_1^* \geq 0$ if and only if $\kappa_1 \leq \bar{\kappa}_1^s$, $s_2^* \geq 0$ if and only if $\kappa_2 \leq \bar{\kappa}_2^s$, and $s_r^* \geq 0$ if and only if $\iota_2 \geq \bar{\iota}_2^s$. This proves **part (a-v)**.

Part (b). By Lemma 9 and Lemma 10 (c), the social-welfare-maximizing equilibrium outcome is the same with strategic customers and with myopic customers, except that $p_{s,1}^* = m_{s,1}^*$ and $\tilde{p}_{s,1}^* = \mu$. Therefore, exactly the same argument as the proof of part (a) proves **part (b)** as well. In particular, since the second-period decisions should be independent of whether the customers are strategic or myopic, $s_2^* = \tilde{s}_2^*$ and $s_r^* = \tilde{s}_r^*$.

Part (c). Since $m_{s,1}^* = \tilde{m}_{s,1}^*$, parts (a) and (b) imply that $\frac{c_1-r_1}{m_1^s(s_1^*)-r_1} = \frac{c_1-r_1}{\tilde{m}_1^s(\tilde{s}_1^*)-r_1}$. Thus, $m_1^s(s_1^*) = \tilde{m}_1^s(\tilde{s}_1^*)$ and, hence, $s_1^*+m_{s,1}^* = \tilde{s}_1^*+\mu$, i.e., $s_1^*-\tilde{s}_1^* = \mu-m_{s,1}^* = -e_s^*$. Therefore, $s_1^* \geq \tilde{s}_1^*$ if and only if $e_s^* \leq 0$. Moreover, since $\bar{\kappa}_1^s$ satisfies $\frac{c_1-r_1}{m_1^s(0)-r_1} = \frac{c_1+\bar{\kappa}_1^s-r_1-\iota_1-s_r^*}{m_{s,1}^*-r_1-s_r^*}$ and $\tilde{\bar{\kappa}}_1^s$ satisfies $\frac{c_1-r_1}{\tilde{m}_1^s(0)-r_1} = \frac{c_1+\tilde{\bar{\kappa}}_1^s-r_1-\iota_1-s_r^*}{\tilde{m}_{s,1}^*-r_1-s_r^*}$. Because $m_{s,1}^* = \tilde{m}_{s,1}^*$, $\bar{\kappa}_1^s \geq \tilde{\bar{\kappa}}_1^s$ if and only if $m_1^s(0) \leq \tilde{m}_1^s(0)$, i.e., $e_s^* \leq 0$. This proves **part (c)**. *Q.E.D.*

Proof of Theorem 4.5.3: Part (a). We first compute $C_g^* = C_g(s_g^*)$ and $\tilde{C}_g^* = \tilde{C}_g(\tilde{s}_g^*)$, observe that

$$\begin{aligned} C_g(s_g^*) =& \mathbb{E}\{s_1^*(X \wedge Q_{s,1}^*) + s_r^*(Q_{s,1}^* - X)^+ \\ &+ \delta[s_2^* Q_{s,2}^n((X - Q_{s,1}^*)^+, X \wedge Q_{s,1}^*) + (s_r^* + s_2^*)Q_{s,2}^r((X - Q_{s,1}^*)^+, X \wedge Q_{s,1}^*)]\}, \end{aligned}$$

and

$$\begin{aligned} \tilde{C}_g(\tilde{s}_g^*) =& \mathbb{E}\{\tilde{s}_1^*(X \wedge \tilde{Q}_{s,1}^*) + \tilde{s}_r^*(\tilde{Q}_{s,1}^* - X)^+ \\ &+ \delta[\tilde{s}_2^* \tilde{Q}_{s,2}^n((X - \tilde{Q}_{s,1}^*)^+, X \wedge \tilde{Q}_{s,1}^*) + (s_r^* + s_2^*)\tilde{Q}_{s,2}^r((X - \tilde{Q}_{s,1}^*)^+, X \wedge \tilde{Q}_{s,1}^*)]\}. \end{aligned}$$

By Lemma 10(c) and Theorem 4.5.2(c), $Q^*_{s,1} = \tilde{Q}^*_{s,1}$, $s^*_2 = \tilde{s}^*_2$, and $s^*_r = \tilde{s}^*_r$, it follows immediately that $C_g(s^*_g) - \tilde{C}_g(\tilde{s}^*_g) = (s^*_1 - \tilde{s}^*_1)\mathbb{E}(X \wedge Q^*_{s,1})$, which proves **part (a)**.

Part (b). By part (a), $C^*_g \geq \tilde{C}^*_g$ if and only if $s^*_1 \geq \tilde{s}^*_1$. By Theorem 4.5.2(c), $s^*_1 \geq \tilde{s}^*_1$ if and only if $e^*_s \leq 0$. Since $e^*_s = \mathbb{E}((k+\alpha)V - c_2 - \kappa_2 + r_2 + \iota_2)^+ - \mathbb{E}((1+\alpha)V - c_2 - \kappa_2)^+$ is strictly increasing in $r_2 + \iota_2$. Hence, let $\bar{\mathcal{V}} := \min\{r_2 + \iota_2 : e^*_s \geq 0\}$. It follows immediately that $e^*_s \leq 0$ if and only if $r_2 + \iota_2 \leq \bar{\mathcal{V}}_2$. We observe that $\mathbb{E}((k+\alpha)V - c_2 - \kappa_2)^+ - \mathbb{E}((1+\alpha)V - c_2 - \kappa_2)^+ < 0$. Thus, $\bar{\mathcal{V}}_2 > 0$. This establishes **part (b)**. *Q.E.D.*

D. Appendix for Chapter 5

D.1 Proofs of Statements

We use ∂ to denote the derivative operator of a single variable function, ∂_x to denote the partial derivative operator of a multi-variable function with respect to variable x, and $1_{\{\cdot\}}$ to denote the indicator function. The following lemma is used throughout our proof.

Lemma 28 *Let $F_i(z, Z)$ be a continuously differentiable and jointly concave function in (z, Z) for $i = 1, 2$, where $z \in [\underline{z}, \bar{z}]$ ($\underline{z}$ and $\bar{z}$ might be infinite) and $Z \in \mathbb{R}^n$. For $i = 1, 2$, let*

$$(z_i, Z_i) := \text{argmax}_{(z,Z)} F_i(z, Z),$$

be the optimizers of $F_i(\cdot, \cdot)$. If $z_1 < z_2$, we have:

$$\partial_z F_1(z_1, Z_1) \leq \partial_z F_2(z_2, Z_2).$$

Proof: $z_1 < z_2$, so $\underline{z} \leq z_1 < z_2 \leq \bar{z}$. Hence, $\partial_z F_1(z_1, Z_1) \begin{cases} = 0 & \text{if } z_1 > \underline{z}, \\ \leq 0 & \text{if } z_1 = \underline{z}; \end{cases}$

and $\partial_z F_2(z_2, Z_2) \begin{cases} = 0 & \text{if } z_2 < \bar{z}, \\ \geq 0 & \text{if } z_2 = \bar{z}. \end{cases}$ i.e., $\partial_z F_1(z_1, Z_1) \leq 0 \leq \partial_z F_2(z_2, Z_2)$. *Q.E.D.*

Proof of Lemma 11: Since $p(\cdot)$ and $\gamma(\cdot)$ are twice continuously differentiable, $R(\cdot, \cdot)$ is twice continuously differentiable, and jointly concave in (d_t, I_t^a) if and only if the Hessian of $R(d_t, I_t^a)$ is negative semi-definite, i.e., $\partial^2_{d_t} R(d_t, I_t^a) \leq 0$, and $\partial^2_{d_t} R(d_t, I_t^a) \partial^2_{I_t^a} R(d_t, I_t) \geq (\partial_{d_t} \partial_{I_t^a} R(d_t, I_t^a))^2$, where $\partial^2_{d_t} R(d_t, I_t^a) = p''(d_t)(d_t + \gamma(I_t^a)) + 2p'(d_t)$, $\partial_{d_t} \partial_{I_t^a} R(d_t, I_t^a) = p'(d_t)\gamma'(I_t^a)$, and $\partial^2_{I_t^a} R(d_t, I_t^a) = (p(d_t) - b - \alpha(c + r_d))\gamma''(I_t^a)$. It is easily verified that the Hessian of $R(d_t, I_t^a)$ is negative semi-definite if and only if $(p''(d_t)(d_t + \gamma(I_t^a)) + 2p'(d_t))(p(d_t) - b - \alpha(c + r_d))\gamma''(I_t^a) \geq (p'(d_t)\gamma'(I_t^a))^2$. *Q.E.D.*

Proof of Lemma 12: For part (a), if $\gamma''(I_t^a) = 0$, the left hand side of (5.3) equals to 0. Since the right hand side of (5.3) is greater than or equal to 0 and $(p'(d_t))^2 > 0$, the (5.3) holds only if $\gamma'(I_t^a) = 0$. For the second half of **part (a)**, it suffices to show that if $\gamma'(I^0) = 0$, $\gamma'(I_t^a) = 0$ for any $I_t^a \leq I^0$. Since $\gamma''(I_t^a) \leq 0$ for all $I_t^a \leq K_a$, $\gamma'(I_t^a) \geq \gamma'(I^0) = 0$ for any $I_t^a \leq I^0$. On the other hand, $\gamma'(I_t^a) \leq 0$ for all $I_t^a \leq K_a$, so $\gamma'(I_t^a) = 0$ and, thus, $\gamma''(I_t^a) = 0$ for all $I_t^a \leq I^0$.

Part (b): By **part (a)**, for any I_t^a such that $\gamma''(I_t^a) = 0$, $\gamma'(I_t^a) = 0$. $(\gamma'(I_t^a))^2 \leq -M\gamma''(I_t^a)$ for any $0 < M < +\infty$. Now we suppose $\gamma''(I_t^a) \neq 0$. Since $p(\cdot)$, $p'(\cdot)$ and $p''(\cdot)$ are continuous functions defined on a compact set $[\underline{d}, \bar{d}]$ with $p'(\cdot) < 0$ and $\gamma(K_a) \leq \gamma(I_t^a) \leq \gamma_0$, $(p''(d_t)(d_t + \gamma(I_t^a)) + 2p'(d_t))(p(d_t) - b - \alpha(c + r_d))/(p'(d_t))^2$ is uniformly bounded from below by a constant number, and we define this number to be $-M$. Hence, by (5.3), $(\gamma'(I_t^a))^2 \le -M\gamma''(I_t^a)$. *Q.E.D.*

Proof of Lemma 13:

Part (a). Observe that $\hat{p}'_\delta(\cdot) \equiv p'(\cdot)$ and $\hat{p}''_\delta(\cdot) \equiv p''(\cdot)$ for any $\delta > 0$. Thus, let

$$m := \max_{d_t\in[\underline{d},\bar{d}],I_t^a\le K_a}\{\frac{\hat{p}''_\delta(d_t)(d_t+\gamma(I_t^a))+2\hat{p}'_\delta(d_t)}{(\hat{p}'_\delta(d_t))^2}\} = \max_{d_t\in[\underline{d},\bar{d}],I_t^a\le K_a}\{\frac{p''(d_t)(d_t+\gamma(I_t^a))+2p'(d_t)}{(p'(d_t))^2}\} < 0,$$

$$k := \min_{d_t\in[\underline{d},\bar{d}]}\{p(d_t) - b - \alpha(c+r_d)\} \ge 0,$$

and

$$\delta^* := -\frac{M}{m} - k < +\infty.$$

Therefore, for any $\delta \ge \delta^*, d_t \in [\underline{d},\bar{d}], I_t^a \le K_a$,

$$\begin{aligned}
&\frac{(\hat{p}''_\delta(d_t)(d_t+\gamma(I_t^a))+2\hat{p}'_\delta(d_t))(\hat{p}_\delta(d_t)-b-\alpha(c+r_d))}{(\hat{p}'_\delta(d_t))^2}\gamma''(I_t^a)\\
=&\frac{p''(d_t)(d_t+\gamma(I_t^a))+2p'(d_t)}{(p'(d_t))^2}(p(d_t)+\delta-b-\alpha(c+r_d))\gamma''(I_t^a)\\
\ge&\frac{p''(d_t)(d_t+\gamma(I_t^a))+2p'(d_t)}{(p'(d_t))^2}(-\frac{M}{m}-k+p(d_t)-b-\alpha(c+r_d))\gamma''(I_t^a)\\
\ge&\frac{p''(d_t)(d_t+\gamma(I_t^a))+2p'(d_t)}{(p'(d_t))^2}\cdot(-\frac{M}{m})\gamma''(I_t^a)\\
\ge&-M\gamma''(I_t^a)\\
\ge&(\gamma'(I_t^a))^2,
\end{aligned}$$

where the first inequality follows from $\delta \ge \delta^*$, the second from $p(d_t) - b - \alpha(c + r_d) \ge k$, the third from the definition of m and the last from the assumption that $-M\gamma''(I_t^a) \ge (\gamma'(I_t^a))^2$ for any $I_t^a \le K_a$. Hence, by (5.3), for any $\delta \ge \delta^*$, $\hat{R}_\delta(\cdot,\cdot)$ is jointly concave on $d_t \in [\underline{d},\bar{d}], I_t^a \le K_a$.

Part (b). Observe that $\hat{\gamma}'_\varsigma(\cdot) \equiv \gamma'(\cdot)$ and $\hat{\gamma}''_\varsigma(\cdot) \equiv \gamma''(\cdot)$ for any $\varsigma > 0$. Since $p''(d_t) \ne 0$, let

$$n := \max_{d_t\in[\underline{d},\bar{d}]}\{\frac{(p(d_t)-b-\alpha(c+r_d))p''(d_t)}{(p'(d_t))^2}\} < 0,\ l := \min_{d_t\in[\underline{d},\bar{d}],I_t^a\le K_a}\{\gamma(I_t^a)+d_t+\frac{2p'(d_t)}{p''(d_t)}\} > 0,$$

and

$$\varsigma^* := -\frac{M}{n} - l < +\infty.$$

Therefore, for any $\varsigma \ge \varsigma^*, d_t \in [\underline{d},\bar{d}], I_t^a \le K_a$,

$$\begin{aligned}
&\frac{(p''(d_t)(d_t+\hat{\gamma}_\varsigma(I_t^a))+2p'(d_t))(p(d_t)-b-\alpha(c+r_d))}{(p'(d_t))^2}\hat{\gamma}''_\varsigma(I_t^a)\\
=&\frac{(p(d_t)-b-\alpha(c+r_d))(p''(d_t)(d_t+\gamma(I_t^a)+\varsigma)+2p'(d_t))}{(p'(d_t))^2}\gamma''(I_t^a)\\
=&\frac{(p(d_t)-b-\alpha(c+r_d))p''(d_t)}{(p'(d_t))^2}(\varsigma+\gamma(I_t^a)+d_t+\frac{2p'(d_t)}{p''(d_t)})\gamma''(I_t^a)\\
\ge&\frac{(p(d_t)-b-\alpha(c+r_d))p''(d_t)}{(p'(d_t))^2}(-\frac{M}{n}-l+\gamma(I_t^a)+d_t+\frac{2p'(d_t)}{p''(d_t)})\gamma''(I_t^a)\\
\ge&\frac{(p(d_t)-b-\alpha(c+r_d))p''(d_t)}{(p'(d_t))^2}(-\frac{M}{n})\gamma''(I_t^a)\\
\ge&-M\gamma''(I_t^a)\\
\ge&(\gamma'(I_t^a))^2 = (\hat{\gamma}'_\varsigma(I_t^a))^2,
\end{aligned}$$

where the first inequality follows from $\varsigma \geq \varsigma^*$, the second from $\gamma(I_t^a)+d_t+\frac{2p'(d_t)}{p''(d_t)} \geq l$, the third from the definition of n and the last from the assumption that $-M\gamma''(I_t^a) \geq (\gamma'(I_t^a))^2$ for any $I_t^a \leq K_a$. Hence, by (5.3), for any $\varsigma \geq \varsigma^*$, $\hat{R}_\varsigma(\cdot,\cdot)$ is jointly concave on $d_t \in [\underline{d}, \bar{d}], I_t^a \leq K_a$. *Q.E.D.*

Proof of Lemma 14: We prove parts (a) - (b) together by backward induction.

We first show, by backward induction, that if $V_{t-1}(I_{t-1}^a, I_{t-1}) - r_d I_{t-1}^a - cI_{t-1}$ is concavely decreasing in both I_{t-1}^a and I_{t-1}, both $g_t(x_t^a, x_t, d_t, I_t^a) := \mathbb{E}\{G_t(x_t^a - \delta(p(d_t), I_t^a, \epsilon_t), x_t - \delta(p(d_t), I_t^a, \epsilon_t))\}$ and $J_t(x_t^a, x_t, d_t, I_t^a, I_t)$ are jointly concave, $g_t(\cdot,\cdot,\cdot,I_t^a)$ and $J_t(\cdot,\cdot,\cdot,I_t^a,I_t)$ are strictly concave for any fixed I_t^a and I_t, and $V_t(I_t^a, I_t) - r_d I_t^a - cI_t$ is jointly concave and decreasing in I_t^a and I_t. It is clear that $V_0(I_0^a, I_0) - r_d I_0^a - cI_0 = -r_d I_0^a - cI_0$ is jointly concave, and decreasing in I_0^a and I_0. Hence, the initial condition holds.

Assume that $V_{t-1}(I_{t-1}^a, I_{t-1}) - r_d I_{t-1}^a - cI_{t-1}$ is concavely decreasing in both I_{t-1}^a and I_{t-1}. Therefore, $G_t(x, y)$ is jointly concave and decreasing in x and y. For every realization of $\epsilon_t = (\epsilon_t^a, \epsilon_t^m)$, we verify that $G_t(x_t^a - \delta(p(d_t), I_t^a, \epsilon_t), x_t - \delta(p(d_t), I_t^a, \epsilon_t))$ is jointly concave in (x_t^a, x_t, d_t, I_t^a)as follows: let $0 \leq \lambda \leq 1$, $x_*^a := \lambda x_1^a + (1-\lambda)x_2^a$, $x_* := \lambda x_1 + (1-\lambda)x_2$, $d_* := \lambda d_1 + (1-\lambda)d_2$ and $I_*^a := \lambda I_1^a + (1-\lambda)I_2^a$, we have:

$$\begin{aligned}
&\lambda G_t(x_1^a - (d_1 + \gamma(I_1))\epsilon_t^m - \epsilon_t^a, x_1 - (d_1 + \gamma(I_1))\epsilon_t^m - \epsilon_t^a) \\
&\quad + (1-\lambda)G_t(x_2^a - (d_2 + \gamma(I_2))\epsilon_t^m - \epsilon_t^a, x_2 - (d_2 + \gamma(I_2))\epsilon_t^m - \epsilon_t^a) \\
&\leq G_t(x_*^a - (d_* + \lambda\gamma(I_1) + (1-\lambda)\gamma(I_2))\epsilon_t^m - \epsilon_t^a, x_* - (d_* + \lambda\gamma(I_1) + (1-\lambda)\gamma(I_2))\epsilon_t^m - \epsilon_t^a) \\
&\leq G_t(x_*^a - (d_* + \gamma(I_*))\epsilon_t^m - \epsilon_t^a, x_* - (d_* + \gamma(I_*))\epsilon_t^m - \epsilon_t^a),
\end{aligned}$$

where the first inequality follows from the joint concavity of $G_t(\cdot,\cdot)$, the second from the concavity of $\gamma(\cdot)$, the monotonicity that $G_t(\cdot,\cdot)$ is decreasing in both of its arguments, and $\epsilon_t^m \geq 0$. Since concavity is preserved under expectation, $g_t(x_t^a, x_t, d_t, I_t^a) = \mathbb{E}\{G_t(x_t^a - \delta(p(d_t), I_t^a, \epsilon_t), x_t - \delta(p(d_t), I_t^a, \epsilon_t))\}$ is jointly concave in (x_t^a, x_t, d_t, I_t^a). Note that $R(d_t, I_t^a)$ is jointly concave in (d_t, I_t^a), $-\theta(x_t - I_t)^-$ is jointly concave in (x_t, I_t), and $-(r_d + r_w)(x_t^a - I_t^a)^-$ is jointly concave in (x_t^a, I_t^a). Therefore, $J_t(x_t^a, x_t, d_t, I_t^a, I_t)$ is jointly concave in $(x_t^a, x_t, d_t, I_t^a, I_t)$. The strict concavity of $g_t(\cdot,\cdot,\cdot,I_t^a)$ follows directly from the continuous distribution of D_t and that its support is an interval. Since $g_t(\cdot,\cdot,\cdot,I_t^a)$ is strictly concave and $R(\cdot, I_t^a)$ is concave for any fixed I_t^a, $J_t(\cdot,\cdot,\cdot,I_t^a,I_t)$ is strictly jointly concave for any fixed I_t^a and I_t.

Concavity is preserved under maximization (see, e.g., Section 3.2.5 of [32]), so the joint concavity of $V_t(I_t^a, I_t)$ follows immediately from the joint concavity of $J_t(\cdot,\cdot,\cdot,\cdot,\cdot)$. We now verify that $V_t(I_t^a, I_t)$ is decreasing in both I_t^a and I_t. Observe that $\gamma(I_t^a)$, $-(r_d + r_w)(x_t^a - I_t^a)^-$, and $G_t(x_t^a - \delta(p(d_t), I_t^a, \epsilon_t), x_t - \delta(p(d_t), I_t^a, \epsilon_t))$ are decreasing in I_t^a, and $-\theta(x_t - I_t)^-$ is decreasing in I_t. Hence, $J_t(x_t^a, x_t, d_t, I_t^a, I_t)$ is decreasing in I_t^a and I_t for any fixed (x_t^a, x_t, d_t). Assume $I_1^a > I_2^a$, we have $F(I_1^a) \subset F(I_2^a)$. Hence, for any I_t,

$$\begin{aligned}
V_t(I_1^a, I_t) - r_d I_1^a - cI_t &= \max_{(x_t^a, x_t, d_t) \in F(I_1^a)} J_t(x_t^a, x_t, d_t, I_1^a, I_t) \\
&\leq \max_{(x_t^a, x_t, d_t) \in F(I_2^a)} J_t(x_t^a, x_t, d_t, I_2^a, I_t) = V_t(I_2^a, I_t) - r_d I_2^a - cI_t,
\end{aligned}$$

where the inequality follows from the monotonicity that $J_t(x_t^a, x_t, d_t, I_t^a, I_t)$ is decreasing in I_t^a, and $F(I_1^a) \subset F(I_2^a)$, thus verifying $V_t(I_t^a, I_t)$ is decreasing in I_t^a. Analogously, if $I_1 > I_2$, for any I_t^a,

$$\begin{aligned} V_t(I_t^a, I_1) - r_d I_t^a - cI_1 &= \max_{(x_t^a, x_t, d_t) \in F(I_t^a)} J_t(x_t^a, x_t, d_t, I_t^a, I_1) \\ &\leq \max_{(x_t^a, x_t, d_t) \in F(I_t^a)} J_t(x_t^a, x_t, d_t, I_t^a, I_2) = V_t(I_t^a, I_2) - r_d I_t^a - cI_2, \end{aligned}$$

where the inequality follows from the monotonicity that $J_t(x_t^a, x_t, d_t, I_t^a, I_t)$ is decreasing in I_t.

Second, we show, again by backward induction, that if $V_{t-1}(\cdot, \cdot)$ is continuously differentiable, $g_t(\cdot, \cdot, \cdot, \cdot)$ and $V_t(\cdot, \cdot)$ are continuously differentiable on the interior of their domains. For $t = 0$, $V_t(I_t^a, I_t) = 0$ is clearly continuously differentiable. The initial condition holds.

Assume $V_{t-1}(I_{t-1}^a, I_{t-1})$ is continuously differentiable,

$$\begin{aligned} g_t(x_t^a, x_t, d_t, I_t) =& \mathbb{E}\{-(b + h_a)(x_t^a - (d_t + \gamma(I_t))\epsilon_t^m - \epsilon_t^a)^+ \\ & + \alpha[V_{t-1}(x_t^a - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a, x_t - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a) \\ & - r_d(x_t^a - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a) - c(x_t - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a)]\}. \end{aligned}$$

Since ϵ_t^a and ϵ_t^m are continuous, it is easy to compute the partial derivatives of $g_t(\cdot, \cdot, \cdot, \cdot)$ as follows:

$$\begin{aligned} \partial_{x_t^a} g_t(x_t^a, x_t, d_t, I_t^a) =& \mathbb{E}\{-(b + h_a)1_{\{x_t^a \geq (d_t + \gamma(I_t))\epsilon_t^m + \epsilon_t^a\}} \\ & + \alpha \partial_{I_{t-1}^a} V_{t-1}(x_t^a - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a, x_t - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a)\} - \alpha r_d, \\ \partial_{x_t} g_t(x_t^a, x_t, d_t, I_t^a) =& \mathbb{E}\{\alpha \partial_{I_{t-1}} V_{t-1}(x_t^a - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a, x_t - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a)\} - \alpha c, \\ \partial_{d_t} g_t(x_t^a, x_t, d_t, I_t^a) =& \mathbb{E}\{(b + h_a)\epsilon_t^m 1_{\{x_t^a \geq (d_t + \gamma(I_t^a))\epsilon_t^m + \epsilon_t^a\}} \\ & - \alpha \epsilon_t^m \partial_{I_{t-1}^a} V_{t-1}(x_t^a - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a, x_t - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a) \\ & - \alpha \epsilon_t^m \partial_{I_{t-1}} V_{t-1}(x_t^a - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a, x_t - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a)\} + \alpha(r_d + c), \\ \partial_{I_t^a} g_t(x_t^a, x_t, d_t, I_t^a) =& \mathbb{E}\{(b + h_a)\gamma'(I_t^a)\epsilon_t^m 1_{\{x_t^a \geq (d_t + \gamma(I_t^a))\epsilon_t^m + \epsilon_t^a\}} \\ & - \alpha \gamma'(I_t^a)\epsilon_t^m \partial_{I_{t-1}^a} V_{t-1}(x_t^a - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a, x_t - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a) \\ & - \alpha \gamma'(I_t^a)\epsilon_t^m \partial_{I_{t-1}} V_{t-1}(x_t^a - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a, x_t - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a)\} \\ & + \alpha(r_d + c)\gamma'(I_t^a), \end{aligned} \tag{D.0}$$

where the exchangeability of differentiation and expectation is easily justified using the canonical argument (see, for example, Theorem A.5.1 of [63], the condition of which can be easily checked observing the continuity of partial derivatives of $V_{t-1}(\cdot, \cdot)$, and that the distribution of D_t is continuous.). Since at least one of ϵ_t^a and ϵ_t^m follows a continuous distribution, $\partial_{x_t^a} g_t(x_t^a, x_t, d_t, I_t^a)$, $\partial_{x_t} g_t(x_t^a, x_t, d_t, I_t^a)$, $\partial_{d_t} g_t(x_t^a, x_t, d_t, I_t^a)$ and $\partial_{I_t^a} g_t(x_t^a, x_t, d_t, I_t^a)$ are continuous. Therefore, $g_t(\cdot, \cdot, \cdot, \cdot)$ is continuously differentiable.

Since $g_t(\cdot, \cdot, \cdot, I_t^a)$ is strictly concave and continuously differentiable, $J_t(\cdot, \cdot, \cdot, I_t^a, I_t)$ is strictly concave and continuously differentiable. Moreover, $J_t(\cdot, \cdot, \cdot, \cdot, \cdot)$ is continuously differentiable if $x_t^a \neq I_t^a$ and $x_t \neq I_t$, i.e., it is continuously differentiable almost everywhere. By envelope theorem, $V_t(\cdot, \cdot)$ is also differentiable on the interior of the feasible set $F(I_t^a)$ for $x_t^{a*}(I_t^a, I_t) \neq I_t^a$ and $x_t^*(I_t^a, I_t) \neq I_t$. For the case $x_t^{a*}(I_t^a, I_t) = I_t^a$ or $x_t^*(I_t^a, I_t) = I_t$, we show the continuous differentiability of $V_t(\cdot,\cdot)$ in the proof of Theorem 5.4.1. This completes the induction and, hence, the proof of Lemma 14. *Q.E.D.*

Proof of Theorem 5.4.1: Parts (a) - (d) and the differentiability of $V_t(I_t^a, I_t)$. We first show parts (a) - (d) and the continuous differentiability of $V_t(I_t^a, I_t)$.

Observe that if $x_t > I_t$ (i.e., the firm orders),

$$\partial_{x_t} J_t(x_t^a, x_t, d, I_t^a, I_t) = -\psi + \partial_{x_t} g_t(x_t^a, x_t, d_t, I_t^a) < 0.$$

Hence, if $x_t^*(I_t^a, I_t) > I_t$, $x_t^{a*}(I_t^a, I_t) = x_t^*(I_t^a, I_t) > I_t \geq I_t^a$ and the optimal policy is given by Equation (5.7). i.e., if $x_t^a(I_t^a) > I_t$, $(x_t^{a*}(I_t^a, I_t), x_t^*(I_t^a, I_t), d_t^*(I_t^a, I_t)) = (x_t^a(I_t^a), x_t^a(I_t^a), d_t(I_t^a))$. This completes the proof of **part (b)**.

If $x_t < I_t$ (i.e., the firm disposes), $-\theta(x_t - I_t)^- = \theta(x_t - I_t)$. Hence, the objective function

$$\begin{aligned} J_t(x_t^a, x_t, d_t, I_t^a, I_t) = & -\theta I_t + R(d_t, I_t^a) + (\theta-\psi)x_t - (r_d + r_w)(x_t^a - I_t^a)^- + \phi x_t^a \\ & + \mathbb{E}\{G_t(x_t^a - \delta(p(d_t), I_t^a, \epsilon_t), x_t - \delta(p(d_t), I_t^a, \epsilon_t)). \end{aligned}$$

Hence, if $x_t^*(I_t^a, I_t) < I_t$, the optimizer prescribed in Equation (5.9) is the optimal policy. i.e., if $\tilde{x}_t(I_t^a) < I_t$, $(x_t^{a*}(I_t^a, I_t), x_t^*(I_t^a, I_t), d_t^*(I_t^a, I_t)) = (\tilde{x}_t^a(I_t^a), \tilde{x}_t(I_t^a), \tilde{d}_t(I_t^a))$. **Part (c)** follows.

Next we show that $x_t^a(I_t^a) \leq \tilde{x}_t(I_t^a)$. If $x_t^a(I_t^a) > \tilde{x}_t(I_t^a)$, suppose $I_t \in (\tilde{x}_t(I_t^a), x_t^a(I_t^a))$. We have that:

$$\begin{cases} J_t(x_t^a(I_t^a), x_t^a(I_t^a), d_t(I_t^a), I_t^a, I_t) & > \sup_{x_t^a \leq I_t, d_t \in [\underline{d}, \bar{d}]}\{J_t(x_t^a, I_t, d_t, I_t^a, I_t)\}, \\ J_t(\tilde{x}_t^a(I_t^a), \tilde{x}_t(I_t^a), \tilde{d}_t(I_t^a), I_t^a, I_t) & > \sup_{x_t^a \leq I_t, d_t \in [\underline{d}, \bar{d}]}\{J_t(x_t^a, I_t, d_t, I_t^a, I_t)\}. \end{cases} \tag{D.1}$$

By the concavity of $J_t(\cdot,\cdot,\cdot, I_t^a, I_t)$,

$$\sup_{x_t^a \leq I_t, d_t \in [\underline{d}, \bar{d}]}\{J_t(x_t^a, I_t, d_t, I_t^a, I_t)\} \geq \lambda J_t(x_t^a(I_t^a), x_t^a(I_t^a), d_t(I_t^a), I_t^a, I_t) + (1-\lambda) J_t(\tilde{x}_t^a(I_t^a), \tilde{x}_t(I_t^a), \tilde{d}_t(I_t^a), I_t^a, I_t),$$

where $\lambda x_t^a(I_t^a) + (1-\lambda)\tilde{x}_t(I_t^a) = I_t$. The above inequality contradicts inequality (D.1). Hence, $x_t^a(I_t^a) \leq \tilde{x}_t(I_t^a)$. **Part (d)** thus follows from part (b), part (c), $x_t^a(I_t^a) \leq \tilde{x}_t(I_t^a)$, and the concavity of $J_t(\cdot,\cdot,\cdot, I_t^a, I_t)$. The second part of **part (a)** summarizes parts (b) - (d).

Since the proof of Lemma 14 already shows that $J_t(\cdot,\cdot,\cdot,\cdot,\cdot)$ is continuously differentiable, it suffices to show that $V_t(I_t^a, I_t)$ is continuously differentiable when $x_t^{a*}(I_t^a, I_t) = I_t^a$ or $x_t^*(I_t^a, I_t) = I_t$, given that $J_t(\cdot,\cdot,\cdot,\cdot,\cdot)$ is continuously differentiable. We only show that $\partial_{I_t} V_t(I_t^a, I_t)$ is continuous at the points where $x_t^*(I_t^a, I_t) = I_t$, because the continuity of $\partial_{I_t^a} V_t(I_t^a, I_t)$ at the points where $x_t^{a*}(I_t^a, I_t) = I_t^a$ follows from the same approach.

By the proof of Lemma 14, it suffices to check that the left and right partial derivatives, $\partial_{I_t} V_t(I_t^a, I_t-)$ and $\partial_{I_t} V_t(I_t^a, I_t+)$, are equal when $I_t = x_t^a(I_t^a)$ and $I_t = \tilde{x}_t(I_t^a)$. For $I_t = x_t^a(I_t^a)$, by the envelope theorem,

$$\begin{cases} \partial_{I_t} V_t(I_t^a, x_t^a(I_t^a)-) = & c \\ \partial_{I_t} V_t(I_t^a, x_t^a(I_t^a)+) = & c + \beta + \partial_{x_t^a} g(x_t^a(I_t^a), x_t^a(I_t^a), d_t(I_t^a), I_t^a) + \partial_{x_t} g(x_t^a(I_t^a), x_t^a(I_t^a), d_t(I_t^a), I_t^a). \end{cases}$$

The first order condition with respect to x_t^a and x_t implies that

$$\beta + \partial_{x_t^a} g(x_t^a(I_t^a), x_t^a(I_t^a), d_t(I_t^a), I_t^a) + \partial_{x_t} g(x_t^a(I_t^a), x_t^a(I_t^a), d_t(I_t^a), I_t^a) = 0.$$

Therefore, $\partial_{I_t} V_t(I_t^a, x_t^a(I_t^a)-) = \partial_{I_t} V_t(I_t^a, x_t^a(I_t^a)+)$. For $I_t = \tilde{x}_t(I_t^a)$, by the envelop theorem,

$$\begin{cases} \partial_{I_t} V_t(I_t^a, \tilde{x}_t(I_t^a)-) = & c - \theta \\ \partial_{I_t} V_t(I_t^a, \tilde{x}_t(I_t^a)+) = & c - \psi + \partial_{x_t} g(\tilde{x}_t^a(I_t^a), \tilde{x}_t(I_t^a), \tilde{d}_t(I_t^a), I_t^a). \end{cases}$$

The first order condition with respect to x_t at $I_t = \tilde{x}_t(I_t^a)$ implies that

$$\partial_{x_t} g(\tilde{x}_t^a(I_t^a), \tilde{x}_t(I_t^a), \tilde{d}_t(I_t^a), I_t^a) + \theta - \psi = 0.$$

Hence, $\partial_{I_t} V_t(I_t^a, \tilde{x}_t(I_t^a)-) = \partial_{I_t} V_t(I_t^a, \tilde{x}_t(I_t^a)+)$ and the partial derivative $\partial_{I_t} V_t(I_t^a, I_t)$ is continuous.

Part (e): Let

$$J_t^a(x_t^a, d_t, I_t^a) := R(d_t, I_t^a) + \beta x_t^a + g_t^a(x_t^a, d_t, I_t^a),$$

where $g_t^a(x_t^a, d_t, I_t^a) = \mathbb{E}[G_t^a(x_t^a - \delta(p(d_t), I_t^a, \epsilon_t))]$, with $G_t^a(x) = G_t^a(x, x)$.

We first show that $x_t^a(I_t^a)$ is decreasing in I_t^a. Let $\gamma_t := \gamma(I_t^a)$ and $y_t := d_t + \gamma_t$. Then, we have $J_t^a(x_t^a, d_t, I_t^a) = \hat{J}_t^a(x_t^a, y_t, \gamma_t)$, where

$$\hat{J}_t^a(x_t^a, y_t, \gamma_t) = R^*(y_t, \gamma_t) + \beta x_t^a + \mathbb{E}\{G_t^a(x_t^a - y_t \epsilon_t^m - \epsilon_t^a)\},$$

with $R^*(y_t, \gamma_t) := R(y_t - \gamma_t, I_t^a)$. We need the following lemma that establishes the supermodularity of $R^*(\cdot, \cdot)$ and $R(\cdot, \cdot)$:

Lemma 29 *(a) $R^*(y_t, \gamma_t)$ is strictly supermodular in (y_t, γ_t), where $y_t - \gamma_t = d_t \in [\underline{d}, \bar{d}]$ and $y_t \geq 0$. In addition, $R^*(y_t, \gamma_t)$ is strictly concave in y_t, for any fixed γ_t;*

(b) $R(d_t, I_t^a)$ is supermodular in (d_t, I_t^a), where $d_t \in [\underline{d}, \bar{d}]$ and $I_t^a \leq K_a$. In addition, $R(d_t, I_t^a)$ is strictly concave in d_t, for any fixed I_t^a.

Proof of Lemma 29: $R^*(y_t, \gamma_t) = (p(y_t - \gamma_t) - b - \alpha(c + r_d))y_t$ is twice continuously differentiable when $y_t - \gamma_t = d_t \in [\underline{d}, \bar{d}]$ and $y_t \geq 0$. To prove the supermodularity of $R^*(\cdot, \cdot)$, it suffices to show that $\partial_{y_t} \partial_{\gamma_t} R^*(y_t, \gamma_t) \geq 0$. Direct computation yields that: $\partial_{y_t} \partial_{\gamma_t} R^*(y_t, \gamma_t) = -(p''(y_t - \gamma_t)y_t + p'(y_t - \gamma_t))$. Since $p'(\cdot) < 0$ and $p''(\cdot) \leq 0$, $-(p''(y_t - \gamma_t)y_t + p'(y_t - \gamma_t)) > 0$. Hence, $R^*(y_t, \gamma_t)$ is strictly supermodular. Moreover, $\partial_{y_t}^2 R^*(y_t, \gamma_t) = p''(y_t - \gamma_t)y_t + 2p'(y_t - \gamma_t) < 0$, since $p''(\cdot) \leq 0$ and $p'(\cdot) < 0$. Hence, $R^*(y_t, \gamma_t)$ is strictly concave in y_t, for any fixed γ_t. This establishes part (a).

$R(\cdot, \cdot)$ is twice continuously differentiable, $\partial_{d_t} \partial_{I_t^a} R(d_t, I_t^a) = p'(d_t)\gamma'(I_t^a) \geq 0$. Hence, $R(\cdot, \cdot)$ is supermodular. In addition, $\partial_{d_t}^2 R(d_t, I_t^a) = p''(d_t)(d_t + \gamma(I_t^a)) + 2p'(d_t) < 0$, so $R(d_t, I_t^a)$ is strictly concave in d_t for any fixed I_t^a.*Q.E.D.*

As shown in the proof of Lemma 14, $G_t(\cdot, \cdot)$ and, thus, $G_t^a(\cdot)$, is concave. Note that $\epsilon_t^m \geq 0$, so, for any realization of $(\epsilon_t^a, \epsilon_t^m)$, it is easily verified that $G_t^a(x_t - y_t \epsilon_t^m - \epsilon_t^a)$ is supermodular in (x_t, y_t). Hence, $\mathbb{E}\{G_t^a(x_t - y_t \epsilon_t^m - \epsilon_t^a)\}$ is supermodular in (x_t, y_t), since supermodularity is preserved under expectation. By Lemma 29, $R^*(y_t, \gamma_t)$ is supermodular and, thus, $\hat{J}_t^a(x_t, y_t, \gamma_t)$ is supermodular in

(x_t, y_t, γ_t). Therefore, the optimal order-up-to level, $x_t^a(I_t^a)$, and optimal expected demand $y_t(I_t^a) := d_t(I_t^a) + \gamma_t$ are increasing in γ_t, and, since $\gamma(\cdot)$ is decreasing in I_t^a, decreasing in I_t^a.

We now proceed to show that the optimal expected price-induced demand $d_t(I_t^a)$ is increasing in I_t^a. Let $I_1^a > I_2^a$, $x_1^a := x_t^a(I_1^a)$, $x_2^a := x_t^a(I_2^a)$, $d_1 := d_t(I_1^a)$, $d_2 := d_t(I_2^a)$ $y_1 := d_1+\gamma(I_1^a)$, and $y_2 := d_2+\gamma(I_2^a)$. We prove that $d_1 \geq d_2$ by contradiction. Assume that $d_1 < d_2$. By Lemma 28, $d_1 < d_2$ implies that $\partial_{d_t} J_t^a(x_1^a, d_1, I_1^a) \leq \partial_{d_t} J_t^a(x_2^a, d_2, I_2^a)$.

$$\partial_{d_t} R(d_1, I_1^a) \geq \partial_{d_t} R(d_1, I_2^a) > \partial_{d_t} R(d_2, I_2^a),$$

where the first inequality follows from the supermodularity of $R(\cdot,\cdot)$ and the second inequality follows from the strict concavity of $R(\cdot, I_t^a)$. Hence,

$$\partial_{d_t} g_t^a(x_1^a, d_1, I_1^a) = \partial_{d_t} J_t^a(x_1^a, d_1, I_1^a) - \partial_{d_t} R(d_1, I_1^a) < \partial_{d_t} J_t^a(x_2^a, d_2, I_2^a) - \partial_{d_t} R(d_2, I_2^a) = \partial_{d_t} g_t^a(x_2^a, d_2, I_2^a). \tag{D.2}$$

Let

$$f(X) := -(b + h_a)1_{\{X \geq 0\}} + \alpha[\partial_{I_{t-1}^a} V_{t-1}(X, X) + \partial_{I_{t-1}} V_{t-1}^a(X, X) - r_d - c] \leq 0,$$

which is decreasing in X. We have:

$$\partial_{x_t^a} g_t^a(x_i^a, d_i, I_i^a) = \mathbb{E}\{f(x_i^a - y_i\epsilon_t^m - \epsilon_t^a)\} \text{ and } \partial_{d_t} g_t^a(x_i^a, d_i, I_i^a) = \mathbb{E}\{-\epsilon_t^m f(x_i^a - y_i\epsilon_t^m - \epsilon_t^a)\} \text{ for } i = 1, 2.$$

Recall that we have proved $x_2^a \geq x_1^a$ and $y_2^a \geq y_1^a$.

If $x_1^a = x_2^a$, $x_1^a - y_1^a\epsilon_t^m - \epsilon_t^a \geq x_2^a - y_2^a\epsilon_t^m - \epsilon_t^a$ for any realization of $(\epsilon_t^a, \epsilon_t^m)$. Hence,

$$\partial_{x_t^a} g_t^a(x_1^a, d_1, I_1^a) = \mathbb{E}\{f(x_1^a - y_1\epsilon_t^m - \epsilon_t^a)\} \leq \mathbb{E}\{f(x_2^a - y_2\epsilon_t^m - \epsilon_t^a)\} = \partial_{x_t^a} g_t^a(x_2^a, d_2, I_2^a),$$

where the inequality follows from that $f(\cdot)$ is decreasing.

If $x_2^a > x_1^a$, by Lemma 28, $\partial_{x_t^a} J_t^a(x_1^a, d_1, I_1^a) \leq \partial_{x_t^a} J_t^a(x_2^a, d_2, I_2^a)$ and, hence,

$$\partial_{x_t^a} g_t^a(x_1^a, d_1, I_1^a) = \partial_{x_t^a} J_t^a(x_1^a, d_1, I_1^a) - \beta \leq \partial_{x_t^a} J_t^a(x_2^a, d_2, I_2^a) - \beta = \partial_{x_t^a} g_t^a(x_2^a, d_2, I_2^a).$$

Note that there exists an ϵ_t^*, such that $x_1^a - y_1\epsilon_t^m \leq x_2^a - y_2\epsilon_t^m$ if $\epsilon_t^m \leq \epsilon_t^*$ and $x_1^a - y_1^a\epsilon_t^m > x_2^a - y_2^a\epsilon_t^m$ if $\epsilon_t^m > \epsilon_t^*$ (ϵ_t^* may equal $\underline{m}$ or $\overline{m}$.). Since $f(\cdot)$ is decreasing, $f(x_1^a - y_1\epsilon_t^m - \epsilon_t^a) - f(x_2^a - y_2\epsilon_t^m - \epsilon_t^a) \geq 0$ for any $\epsilon_t^m \in [\underline{m}, \epsilon_t^*]$ and any realization of ϵ_t^a. So

$$-\epsilon_t^m(f(x_1^a - y_1\epsilon_t^m - \epsilon_t^a) - f(x_2^a - y_2\epsilon_t^m - \epsilon_t^a)) \geq -\epsilon_t^*(f(x_1^a - y_1\epsilon_t^m - \epsilon_t^a) - f(x_2^a - y_2\epsilon_t^m - \epsilon_t^a)), \tag{D.3}$$

for any $\epsilon_t^m \in [\underline{m}, \epsilon_t^*]$ and any realization of ϵ_t^a. Analogously, for $\epsilon_t^m \in [\epsilon_t^*, \overline{m}]$, $f(x_1^a - y_1\epsilon_t^m - \epsilon_t^a) - f(x_2^a - y_2\epsilon_t^m - \epsilon_t^a) \leq 0$, and (D.3) holds for $\epsilon_t^m \in [\epsilon_t^*, \overline{m}]$ as well. Therefore, (D.3) holds for all $\epsilon_t^m \in [\underline{m}, \overline{m}]$ and any realization of ϵ_t^a.

Taking expectation, we have:

$$\begin{aligned}
\partial_{d_t} g_t^a(x_1^a, d_1, I_1^a) - \partial_{d_t} g_t^a(x_2^a, d_2, I_2^a) &= \mathbb{E}\{-\epsilon_t^m(f(x_1^a - y_1\epsilon_t^m - \epsilon_t^a) - f(x_2^a - y_2\epsilon_t^m - \epsilon_t^a))\} \\
&\geq \mathbb{E}\{-\epsilon_t^*(f(x_1^a - y_1\epsilon_t^m - \epsilon_t^a) - f(x_2^a - y_2\epsilon_t^m - \epsilon_t^a))\} \\
&= -\epsilon_t^*(\partial_{x_t^a} g_t^a(x_1^a, d_1, I_1^a) - \partial_{x_t^a} g_t^a(x_2^a, d_2, I_2^a)) \\
&\geq 0,
\end{aligned} \tag{D.4}$$

where the last inequality follows from $\partial_{x_t^a} g_t^a(x_1^a, d_1, I_1^a) \leq \partial_{x_t^a} g_t^a(x_2^a, d_2, I_2^a)$. (D.4) contradicts (D.2) and, hence, $d_1 \geq d_2$, i.e., $d_t(I_t^a)$ is increasing in I_t^a. The continuity of $x_t^a(I_t^a)$ and $d_t(I_t^a)$ follows directly from that the objective function $J_t^a(\cdot, \cdot, I_t^a)$ is strictly concave for any given I_t^a. The proof of **part (e)** follows. *Q.E.D.*

Remark D.1.1 *The supermodularity of $R^*(y_t, \gamma_t)$ implies that to better take advantage of the high demand induced by low inventory level, the firm should adjust its price to a level such that the expected demand will increase.*

Proof of Theorem 5.4.2: If $h_w \geq \alpha c - s$, $\theta - \psi = c - s - h_w - (1-\alpha)c = \alpha c - s - h_w \leq 0$. Since $g_t(x_t^a, x_t, d_t, I_t^a, I_t)$ is also decreasing in x_t, Equation (5.9) implies that $\tilde{x}_t(I_t^a) = \tilde{x}_t^a(I_t^a)$, for any t and I_t^a, which proves **part (a)**.

Observe that for any $(x_t^a, x_t, d_t, I_t^a, I_t)$,

$$\partial_{x_t} g_t(x_t^a, x_t, d_t, I_t^a, I_t) \geq -(\sum_{j=1}^{t} \alpha^j) h_w \geq -(\sum_{j=1}^{T} \alpha^j) h_w, \ t = T, T-1, \cdots 1,$$

where the first inequality holds as an equality if $x_j^*(I_j^a, I_j) = I_j$, for all $j \leq t-1$. Hence, $\partial_{x_t} g_t(x_t^a, x_t, d_t, I_t^a, I_t)$ is uniformly bounded from below by $-(\sum_{j=1}^{T} \alpha^j) h_w$, for any t. Thus, if $\theta - \psi = \alpha c - h_w - s \geq (\sum_{j=1}^{T} \alpha^j) h_w$, $\tilde{x}_t(I_t^a) = +\infty$ for any t and I_t^a. Hence, $s_* = \alpha c - (\sum_{j=0}^{T} \alpha^j) h_w$. This proves **part (b)**.

If $\inf_{I_t^a < K_a} \gamma'(I_t^a) \geq -M$, for any $(x_t^a, x_t, d_t, I_t^a, I_t)$,

$$\partial_{x_t^a} g_t(x_t^a, x_t, d_t, I_t^a, I_t) \geq -M(\sum_{j=1}^{t} \alpha^j)(\bar{p} + h_a) \geq -M(\sum_{j=1}^{T} \alpha^j)(\bar{p} + h_a), \ t = T, T-1, \cdots 1,$$

where $\bar{p}$ is the maximum marginal revenue and h_a is the maximum marginal holding cost. Hence, $\partial_{x_t^a} g_t(x_t^a, x_t, d_t, I_t^a, I_t)$ is bounded from below by $-M(\sum_{j=1}^{T} \alpha^j)(\bar{p} + h_a)$, for any t. Thus, if $r_d + r_w + \phi \geq M(\sum_{j=1}^{T} \alpha^j)(\bar{p} + h_a)$, $\tilde{x}_t^a(I_t^a) \geq I_t^a$, for any $I_t^a \leq K_a$.

If $\inf_{I_t^a < K_a} \gamma'(I_t^a) = -\infty$, $\lim_{I_t^a \to K_a} \gamma'(I_t^a) = -\infty$. Hence, for any x_t, d_t, and I_t,

$$\lim_{I_t^a \to K_a} \partial_{x_t^a} g_t(I_t^a, x_t, d_t, I_t^a, I_t) \leq \alpha(\underline{p} - b - (1-\alpha)(c + r_d)) \lim_{I_t^a \to K_a} \gamma'(I_t^a) = -\infty.$$

Hence, for any r_w, and any x_t, d_t and I_t,

$$\partial_{x_t^a} J_t(I_t^a -, x_t, d_t, I_t^a, I_t) = r_d + r_w + \phi + \partial_{x_t^a} g_t(I_t^a, x_t, d_t, I_t^a, I_t) \to -\infty, \text{ as } I_t^a \to K_a.$$

The above limit completes the proof of **Part (c)**.

For notational simplicity, we denote
$x^{a*} := x_t^{a*}(I_{t-1}^a, I_{t-1})$, $x^* := x_t^*(I_{t-1}^a, I_{t-1})$ and $d^* := d_t^*(I_{t-1}^a, I_{t-1})$. Observe that

$$\partial_{I_{t-1}^a} V_{t-1}(I_{t-1}^a, I_{t-1}) \leq (\underline{p} - b - \alpha(c + r_d))\gamma'(I_{t-1}^a) + \partial_{I_{t-1}^a} g_{t-1}(x^{a*}, x^*, d^*, I_{t-1}^a). \tag{D.5}$$

By Equation (D.0),

$$\begin{cases} \partial_{x_{t-1}^a} g_{t-1}(x^{a*}, x^*, d^*, I_{t-1}^a) = & \mathbb{E}\{f_1(\epsilon_{t-1}^m)\}, \\ \partial_{x_{t-1}} g_{t-1}(x^{a*}, x^*, d^*, I_{t-1}^a) = & \mathbb{E}\{f_2(\epsilon_{t-1}^m)\}, \\ \partial_{I_{t-1}^a} g_{t-1}(x^{a*}, x^*, d^*, I_{t-1}^a) = & -\gamma'(I_{t-1}^a)\mathbb{E}\{\epsilon_{t-1}^m[f_1(\epsilon_{t-1}^m) + f_2(\epsilon_{t-1}^m)]\}, \end{cases}$$

where

$$\begin{cases} f_1(\epsilon^m_{t-1}) = & \mathbb{E}_{\epsilon^a_{t-1}}\{-(b+h_a)1_{\{x^{a*}\geq(d_t+\gamma(I_t))\epsilon^m_{t-1}+\epsilon^a_{t-1}\}} \\ & +\alpha\partial_{I^a_{t-2}}V_{t-2}(x^{a*}-(d^*+\gamma(I^a_{t-1}))\epsilon^m_{t-1}-\epsilon^a_{t-1}, x^*-(d^*+\gamma(I^a_{t-1}))\epsilon^m_{t-1}-\epsilon^a_{t-1})\}-\alpha r_d \\ f_2(\epsilon^m_{t-1}) = & \mathbb{E}_{\epsilon^a_{t-1}}\{\alpha\partial_{I_{t-2}}V_{t-2}(x^{a*}-(d^*+\gamma(I^a_{t-1}))\epsilon^m_{t-1}-\epsilon^a_{t-1}, x^*-(d^*+\gamma(I^a_{t-1}))\epsilon^m_{t-1}-\epsilon^a_{t-1})\}-\alpha c. \end{cases}$$

The first order conditions with respect to x^a_{t-1} and x_{t-1} suggest that

$$\mathbb{E}\{f_1(\epsilon^m_{t-1})+f_2(\epsilon^m_{t-1})\}\leq-(\phi-\psi)=-\beta.$$

Since $f_1(\cdot)\leq 0$ and $f(\cdot)\leq 0$, we have:

$$\mathbb{E}\{\epsilon^m_{t-1}[f_1(\epsilon^m_{t-1})+f_2(\epsilon^m_{t-1})]\}\leq\mathbb{E}\{\underline{m}[f_1(\epsilon^m_{t-1})+f_2(\epsilon^m_{t-1})]\}=\underline{m}\mathbb{E}\{f_1(\epsilon^m_{t-1})+f_2(\epsilon^m_{t-1})\}\leq-\underline{m}\beta.$$

Therefore, by inequality (D.5),

$$\partial_{I^a_{t-1}}V_{t-1}(I^a_{t-1},I_t)\leq(\underline{p}-b-\alpha(c+r_d)+\underline{m}\beta)\gamma'(I^a_{t-1}). \tag{D.6}$$

So for any $d_t\in[\underline{d},\bar{d}]$ and any x_t,

$$\begin{aligned} \partial_{x^a_t}g_t(0,x_t,d_t,I^a_t)\leq&\alpha\mathbb{E}[\partial_{I^a_{t-1}}V_{t-1}(-(d_t+\gamma(I^a_t))\epsilon^m_t-\epsilon^a_t, x_t-(d_t+\gamma(I^a_t))\epsilon^m_t-\epsilon^a_t)] \\ \leq&\alpha\mathbb{E}[(\underline{p}-b-\alpha(c+r_d)+\underline{m}\beta)\gamma'(-(d_t+\gamma(I_t))\epsilon^m_t-\epsilon^a_t)] \\ \leq&\alpha(\underline{p}-b-\alpha(c+r_d)+\underline{m}\beta)(1-\iota)\gamma'(-\bar{D}) \\ \leq&-(r_d+r_w+\phi), \end{aligned} \tag{D.7}$$

where the first inequality follows from equation (D.0), the second from (D.6), and the last from the assumption that $\alpha(\underline{p}-b-\alpha(c+r_d)+\underline{m}\beta)(1-\iota)\gamma'(-\bar{D})+(r_d+r_w+\phi)\leq 0$. The third inequality of (D.7) follows from the following inequality:

$$\mathbb{E}[\gamma'(-D_t)]=\mathbb{E}_{D_t\geq\bar{D}}[\gamma'(-D_t)]+\mathbb{E}_{D_t\leq\bar{D}}[\gamma'(-D_t)]\leq 0+\mathbb{E}_{D_t\leq\bar{D}}[\gamma'(-\bar{D})]\leq(1-\iota)\gamma'(-\bar{D}),$$

where the first inequality follows from the concavity of $\gamma(\cdot)$ and the second inequality follows from the definition of $\bar{D}$. (D.7) implies that $x^{a*}_t(I^a_t,I_t)=0$ for all $I^a_t\leq K_a$ and all I_t, which completes the proof of **part (d)**. *Q.E.D.*

Before we proceed to prove the results in Section 5.5, we remark that $R^s_t(d_t,I^a_t)$ shares the same properties as $R(d_t,I^a_t)$. i.e., we have the following counterpart of Lemma 29 in the model without inventory withholding:

Lemma 30 *(a) $R^{s*}(y_t,\gamma_t)$ is strictly supermodular in (y_t,γ_t), where $R^{s*}(y_t,\gamma_t):=R^s(y_t-\gamma_t,I^a_t)$, $y_t-\gamma_t=d_t\in[\underline{d},\bar{d}]$ and $y_t\geq 0$. In addition, $R^{s*}(y_t,\gamma_t)$ is strictly concave in y_t, for any fixed γ_t;*

(b) $R^s(d_t,I^a_t)$ is supermodular in (d_t,I^a_t), where $d_t\in[\underline{d},\bar{d}]$ and $I^a_t\leq K_a$. In addition, $R^s(d_t,I^a_t)$ is strictly concave in d_t, for any fixed I^a_t.

Proof of Lemma 30: The proof is identical to that of Lemma 29, and hence omitted. *Q.E.D.*

Proof of Theorem 5.5.1: The proof is very similar to that of Lemma 14 and Theorem 5.4.1, so we only sketch it.

For **parts (a) - (c)**, the proof is exactly the same as that of Lemma 14, and hence omitted.

To show **parts (d) - (f)**, we define the following unconstrained optimizers:

$$(x_t^L(I_t^a), d_t^L(I_t^a)) := \mathrm{argmax}_{x_t^a \le K_a, d_t \in [\underline{d}, \bar{d}]}\{R^s(d_t, I_t^a) + \beta^s x_t^a + \mathbb{E}[G_t^s(x_t^a - \delta(p(d_t), I_t^a, \epsilon_t))]\},$$

and

$$(x_t^H(I_t^a), d_t^H(I_t^a)) := \mathrm{argmax}_{x_t^a \le K_a, d_t \in [\underline{d}, \bar{d}]}\{R^s(d_t, I_t^a) + (\beta^s + \theta) x_t^a + \mathbb{E}[G_t^s(x_t^a - \delta(p(d_t), I_t^a, \epsilon_t))]\}.$$

We need the following lemma:

Lemma 31 *Let* $\gamma_t := \gamma(I_t^a)$, $\Psi(x_t^a, y_t, \mu|\gamma_t) := R^{s*}(y_t, \gamma_t) + \mu x_t^a + \mathbb{E}\{G_t^s(x_t^a - y_t\epsilon_t^m - \epsilon_t^a)\}$ *is supermodular in* (x_t^a, y_t, μ) *for any given* γ_t.

Proof of Lemma 31: Since $G_t^s(\cdot)$ is concave and $\epsilon_t^m \ge 0$, $\mathbb{E}\{G_t^s(x_t^a - y_t\epsilon_t^m - \epsilon_t^a)\}$ is supermodular in (x_t^a, y_t). It's also clear that μx_t^a is strictly supermodular in (x_t^a, μ). Therefore, $\Psi(x_t^a, y_t, \mu|\gamma_t)$ is supermodular in (x_t^a, y_t, μ) for any given γ_t. *Q.E.D.*

Lemma 31 and its proof imply that $x_t^L(I_t^a) < x_t^H(I_t^a)$ since $\beta^s + \theta > \beta^s$. Exactly the same argument as in the proof of Theorem 5.4.1(e) implies that $x_t^L(I_t^a)$ and $x_t^H(I_t^a)$ are continuously decreasing in I_t^a and $d_t^L(I_t^a)$ and $d_t^H(I_t^a)$ are continuously increasing in I_t^a. $I_t^L := \sup\{I_t^a : I_t^a < x_t^L(I_t^a)\}$ and $I_t^H := \inf\{I_t^a : I_t^a > x_t^H(I_t^a)\}$. It's clear that I_t^L and I_t^H are the thresholds in **part (d)**. Therefore,

$$x_t^{s*}(I_t^a) = \begin{cases} x_t^L(I_t^a) & \text{if } I_t^a < I_t^L; \\ I_t^a & \text{if } I_t^L \le I_t^a \le I_t^H; \\ x_t^H(I_t^a) & \text{if } I_t^a > I_t^H. \end{cases}$$

It's clear that $x_t^{s*}(I_t^a)$ satisfies the statement in **part (e)**. Therefore, we have

$$d_t^{s*}(I_t^a) = \begin{cases} d_t^L(I_t^a) & \text{if } I_t^a < I_t^L; \\ \mathrm{argmax}_{d_t \in [\underline{d}, \bar{d}]} J_t^s(I_t^a, d_t, I_t^a) & \text{if } I_t^L \le I_t^a \le I_t^H; \\ d_t^H(I_t^a) & \text{otherwise.} \end{cases}$$

To prove **part (f)**, it remains to show that $d_t^{s*}(I_t^a)$ is increasing in I_t^a for $I_t^L \le I_t^a \le I_t^H$. Let $U_t^s(d_t, I_t^a) := J_t^s(I_t^a, d_t, I_t^a)$ and it is easily verified that $U_t^s(d_t, I_t^a)$ is supermodular in (d_t, I_t^a). Thus, $d_t^{s*}(I_t^a)$ is increasing in I_t^a, which completes the proof of Theorem 5.5.1. *Q.E.D.*

Proof of Theorem 5.5.2: We show both parts by backward induction.

For **part (a)**, we use backward induction to recursively show this result. For $t = 0$, $V_0^s(\cdot) = \hat{V}_0^s(\cdot) = 0$ and, hence, $\partial_{I_0^a} V_0^s(I_0^a) = \partial_{I_0^a} \hat{V}_0^s(I_0^a)$ for all I_0^a. We show that: if $\partial_{I_{t-1}^a} V_{t-1}^s(I_{t-1}^a) \le \partial_{I_{t-1}^a} \hat{V}_{t-1}^s(I_{t-1}^a)$ for all

$I^a_{t-1} \le K_a$, (a) $I^L_t \le \hat{I}^L_t$, (b) $I^H_t \le \hat{I}^H_t$, (c) $x^{s*}_t(I^a_t) \le \hat{x}^{s*}_t(I^a_t)$, (d) $d^{s*}_t(I^a_t) \ge \hat{d}^{s*}_t(I^a_t)$ and (e) $\partial_{I^a_t} V^s_t(I^a_t) \le \partial_{I^a_t} \hat{V}^s_t(I^a_t)$ for all $I^a_t \le K_a$. To prove these inequalities, we define $(\hat{x}^L_t(I^a_t), \hat{d}^L_t(I^a_t))$ and $(\hat{x}^H_t(I^a_t), \hat{d}^H_t(I^a_t))$ as the unconstrained optimizers in the model with demand $\hat{D}_t$, corresponding to $(x^L_t(I^a_t), d^L_t(I^a_t))$ and $(x^H_t(I^a_t), d^H_t(I^a_t))$, respectively. Let $y^L_t(I^a_t) := d^L_t(I^a_t) + \gamma(I^a_t)$, $\hat{y}^L_t(I^a_t) := \hat{d}^L_t(I^a_t) + \hat{\gamma}(I^a_t) = \hat{d}^L_t(I^a_t) + \gamma_0$, $\hat{R}^s(d_t, I^a_t) := R^s(d_t, -\infty)$, and $\hat{G}^s_t(y) := -(h_a + b)y^+ + \alpha[\hat{V}^s_{t-1}(y) - cy]$. We define the objective functions $J^L_t(x^a_t, d_t, I^a_t) := R^s(d_t, I^a_t) + \beta^s x^a_t + g^s_t(x^a_t, d_t, I^a_t)$, $\hat{J}^L_t(x^a_t, d_t, I^a_t) := \hat{R}^s(d_t, I^a_t) + \beta^s x^a_t + \hat{g}^s_t(x^a_t, d_t, I^a_t)$, where $\hat{g}^s_t(x^a_t, d_t, I^a_t) := \mathbb{E}\{\hat{G}^s_t(x^a_t - \hat{\delta}(p(d_t), I^a_t, \epsilon_t))\}$. Since $\epsilon^m_t = 1$ with probability 1, $g^s_t(x^a_t, d_t, I^a_t) = H^s_t(x^a_t - d_t - \gamma(I^a_t))$ and $\hat{g}^s_t(x^a_t, d_t, I^a_t) = \hat{H}^s_t(x^a_t - d_t - \gamma_0)$, where $H^s_t(X) := \mathbb{E}\{G^s_t(X - \epsilon^a_t)\}$ and $\hat{H}^s_t(X) := \mathbb{E}\{\hat{G}^s_t(X - \epsilon^a_t)\}$.

First, we show that, if $\partial_{I^a_{t-1}} V^s_{t-1}(I^a_{t-1}) \le \partial_{I^a_{t-1}} \hat{V}^s_{t-1}(I^a_{t-1})$ for all $I^a_{t-1} \le K_a$, $x^L_t(I^a_t) \le \hat{x}^L_t(I^a_t)$, $d^L_t(I^a_t) \ge \hat{d}^L_t(I^a_t)$, $x^H_t(I^a_t) \le \hat{x}^H_t(I^a_t)$, and $d^H_t(I^a_t) \ge \hat{d}^H_t(I^a_t)$. Since $\partial_{I^a_{t-1}} V^s_{t-1}(I^a_{t-1}) \le \partial_{I^a_{t-1}} \hat{V}^s_{t-1}(I^a_{t-1})$, $\partial_X H^s_t(X) \le \partial_X \hat{H}^s_t(X)$ for any X. We only show that $x^L_t(I^a_t) \le \hat{x}^L_t(I^a_t)$ and $d^L_t(I^a_t) \ge \hat{d}^L_t(I^a_t)$, while $x^H_t(I^a_t) \le \hat{x}^H_t(I^a_t)$ and $d^H_t(I^a_t) \ge \hat{d}^H_t(I^a_t)$ follow from the same argument.

We show by contradiction that $x^L_t(I^a_t) \le \hat{x}^L_t(I^a_t)$ and $d^L_t(I^a_t) \ge \hat{d}^L_t(I^a_t)$. Note that, for the model with inventory-independent demand (i.e., the firm faces $\hat{D}_t$), it is reduced to the classical joint pricing and inventory management problem with stochastic demand introduced in Federgruen and Heching (1999). Hence, $\hat{x}^L_t(I^a_t)$ and $\hat{d}^L_t(I^a_t)$ are constants independent of I^a_t.

Assume that $x^L_t(I^a_t) > \hat{x}^L_t(I^a_t)$. Lemma 28 yields that $\partial_{x^a_t} J^L_t(x^L_t(I^a_t), d^L_t(I^a_t), I^a_t) \ge \partial_{x^a_t} \hat{J}^L_t(\hat{x}^L_t(I^a_t), \hat{d}^L_t(I^a_t), I^a_t)$. Hence,

$$\begin{aligned}\partial_X H^s_t(x^L_t(I^a_t) - y^L_t(I^a_t)) =& \partial_{x^a_t} J^L_t(x^L_t(I^a_t), d^L_t(I^a_t), I^a_t) - \beta^s \\ \ge& \partial_{x^a_t} \hat{J}^L_t(\hat{x}^L_t(I^a_t), \hat{d}^L_t(I^a_t), I^a_t) - \beta^s \\ =& \partial_X \hat{H}^s_t(\hat{x}^L_t(I^a_t) - \hat{y}^L_t(I^a_t)).\end{aligned}$$

Since $\partial_X H^s_t(X) \le \partial_X \hat{H}^s_t(X)$ for any X and both of them are strictly decreasing, $y^L_t(I^a_t) > \hat{y}^L_t(I^a_t)$. Thus, $d^L_t(I^a_t) = y^L_t(I^a_t) - \gamma(I^a_t) > \hat{y}^L_t(I^a_t) - \gamma_0 = \hat{d}^L_t(I^a_t)$. Invoking Lemma 28, we have $\partial_{d_t} J^L_t(x^L_t(I^a_t), d^L_t(I^a_t), I^a_t) \ge \partial_{d_t} \hat{J}^L_t(\hat{x}^L_t(I^a_t), \hat{d}^L_t(I^a_t), I^a_t)$, and

$$\begin{aligned}\partial_{d_t} R^s(d^L_t(I^a_t), I^a_t) =& \partial_{d_t} J^L_t(x^L_t(I^a_t), d^L_t(I^a_t), I^a_t) + \partial_X H^s_t(x^L_t(I^a_t) - y^L_t(I^a_t)) \\ \ge& \partial_{d_t} \hat{J}^L_t(\hat{x}^L_t(I^a_t), \hat{d}^L_t(I^a_t), I^a_t) + \partial_X \hat{H}^s_t(\hat{x}^L_t(I^a_t) - \hat{y}^L_t(I^a_t)) \\ =& \partial_{d_t} \hat{R}^s(\hat{d}^L_t(I^a_t), I^a_t)\end{aligned}$$

Since $\partial_{d_t} R^s(d_t, I^a_t) = \partial_{y_t} R^{s*}(d_t + \gamma(I^a_t), \gamma(I^a_t))$, $\partial_{y_t} R^{s*}(y^L_t(I^a_t), \gamma(I^a_t)) \ge \partial_{y_t} R^{s*}(\hat{y}^L_t(I^a_t), \gamma_0)$. However, the strict concavity of $R^{s*}(\cdot, \gamma_t)$ and the supermodularity of $R^{s*}(\cdot, \cdot)$ yield that

$$\partial_{y_t} R^{s*}(y^L_t(I^a_t), \gamma(I^a_t)) < \partial_{y_t} R^{s*}(\hat{y}^L_t(I^a_t), \gamma(I^a_t)) \le \partial_{y_t} R^{s*}(\hat{y}^L_t(I^a_t), \gamma_0),$$

which leads to a contradiction. Therefore, we have $x^L_t(I^a_t) \le \hat{x}^L_t(I^a_t)$.

Assume that $d^L_t(I^a_t) < \hat{d}^L_t(I^a_t)$, so $y^L_t(I^a_t) = d^L_t(I^a_t) + \gamma(I^a_t) < \hat{d}^L_t(I^a_t) + \gamma_0 = \hat{y}^L_t(I^a_t)$. Lemma 28 yields that $\partial_{d_t} J^L_t(x^L_t(I^a_t), d^L_t(I^a_t), I^a_t) \le \partial_{d_t} \hat{J}^L_t(\hat{x}^L_t(I^a_t), \hat{d}^L_t(I^a_t), I^a_t)$. The strict concavity of $R^s(\cdot, I^a_t)$ and the supermodularity of $R^s(\cdot, \cdot)$ imply that

$$\partial_{d_t} R^s(d^L_t(I^a_t), I^a_t) > \partial_{d_t} R^s(\hat{d}^L_t(I^a_t), I^a_t) \ge \partial_{d_t} R^s(\hat{d}^L_t(I^a_t), -\infty) = \partial_{d_t} \hat{R}^s(\hat{d}^L_t(I^a_t), I^a_t).$$

Hence, we have:

$$\begin{aligned}\partial_X H_t^s(x_t^L(I_t^a) - y_t^L(I_t^a)) =& \partial_{d_t} R^s(d_t^L(I_t^a), I_t^a) - \partial_{d_t} J_t^L(x_t^L(I_t^a), d_t^L(I_t^a), I_t^a) \\ >& \partial_{d_t} \hat{R}^s(\hat{d}_t^L(I_t^a), I_t^a) - \partial_{d_t} \hat{J}_t^L(\hat{x}_t^L(I_t^a), \hat{d}_t^L(I_t^a), I_t^a) \\ =& \partial_X \hat{H}_t^s(\hat{x}_t^L(I_t^a) - \hat{y}_t^L(I_t^a)).\end{aligned}$$

The first order condition with respect to x_t^a implies that $\partial_X H_t^s(x_t^L(I_t^a) - y_t^L(I_t^a)) = \partial_X \hat{H}_t^s(\hat{x}_t^L(I_t^a) - \hat{y}_t^L(I_t^a)) = -\beta^s$, which leads to a contradiction. Hence, $d_t^L(I_t^a) \geq \hat{d}_t^L(I_t^a)$. We have thus proved that, if $\partial_{I_{t-1}^a} V_{t-1}^s(I_{t-1}^a) \leq \partial_{I_{t-1}^a} \hat{V}_{t-1}^s(I_{t-1}^a)$ for all $I_{t-1}^a \leq K_a$, $x_t^L(I_t^a) \leq \hat{x}_t^L(I_t^a)$, $d_t^L(I_t^a) \geq \hat{d}_t^L(I_t^a)$, $x_t^H(I_t^a) \leq \hat{x}_t^H(I_t^a)$, and $d_t^H(I_t^a) \geq \hat{d}_t^H(I_t^a)$. $I_t^L \leq \hat{I}_t^L$ and $I_t^H \leq \hat{I}_t^H$ follow immediately from $x_t^L(I_t^a) \leq \hat{x}_t^L(I_t^a)$ and $x_t^H(I_t^a) \leq \hat{x}_t^H(I_t^a)$.

Next, we show that $d_t^{s*}(I_t^a) \geq \hat{d}_t^{s*}(I_t^a)$, for all $I_t^a \leq K_a$. Since $d_t^L(I_t^a) \geq \hat{d}_t^L(I_t^a)$, $d_t^{s*}(I_t^a) = d_t^L(I_t^a) \geq \hat{d}_t^L(I_t^a)$, for all $I_t^a \leq I_t^L$. If $I_t^a \in [I_t^L, \hat{I}_t^L]$,

$$d_t^{s*}(I_t^a) \geq d_t^{s*}(I_t^L) = d_t^L(I_t^L) \geq \hat{d}_t^L(I_t^L) = \hat{d}_t^L(I_t^a),$$

where the first inequality follows from Theorem 5.5.1, the second from $d_t^L(I_t^a) \geq \hat{d}_t^L(I_t^a)$, and the last equality from Federgruen and Heching (1999) Theorem 1. If $I_t^a \in [\hat{I}_t^L, I_t^H]$ (it might be an empty set), $x_t^{s*}(I_t^a) = \hat{x}_t^{s*}(I_t^a) = I_t^a$. The supermodularity of $R^s(d_t, I_t^a)$ implies that

$$\partial_{d_t} R^s(\hat{d}_t^{s*}(I_t^a), I_t^a) \geq \partial_{d_t} R^s(\hat{d}_t^{s*}(I_t^a), -\infty) = \partial_{d_t} \hat{R}^s(\hat{d}_t^{s*}(I_t^a), I_t^a).$$

Since $\gamma_0 \geq \gamma(I_t^a)$, both $H_t^s(\cdot)$ and $\hat{H}_t^s(\cdot)$ are concave, and $\partial_X H_t^s(X) \leq \partial_X \hat{H}_t^s(X)$ for all X, so $-\partial_X H_t^s(I_t^a - \hat{d}_t^{s*}(I_t^a) - \gamma(I_t^a)) \geq -\partial_X \hat{H}_t^s(I_t^a - \hat{d}_t^{s*}(I_t^a) - \gamma_0)$. Hence,

$$\begin{aligned}\partial_{d_t} J_t^s(I_t^a, \hat{d}_t^{s*}(I_t^a), I_t^a) &= \partial_{d_t} R^s(\hat{d}_t^{s*}(I_t^a), I_t^a) - \partial_X H_t^s(I_t^a - \hat{d}_t^{s*}(I_t^a) - \gamma(I_t^a)) \\ &\geq \partial_{d_t} \hat{R}^s(\hat{d}_t^{s*}(I_t^a), I_t^a) - \partial_X \hat{H}_t^s(I_t^a - \hat{d}_t^{s*}(I_t^a) - \gamma_0) \\ &= \partial_{d_t} \hat{J}_t^s(I_t^a, \hat{d}_t^{s*}(I_t^a), I_t^a),\end{aligned}$$

i.e., $d_t^{s*}(I_t^a) \geq \hat{d}_t^{s*}(I_t^a)$. If $I_t^a \in [I_t^H, \hat{I}_t^H]$, $x_t^{s*}(I_t^a) \leq \hat{x}_t^{s*}(I_t^a) = I_t^a$. The first order condition with respect to x_t^a implies that $\partial_X H_t^s(x_t^{s*}(I_t^a) - d_t^{s*}(I_t) - \gamma(I_t^a)) = -(\beta^s + \theta) \leq \partial_X \hat{H}_t^s(I_t^a - \hat{d}_t^{s*}(I_t^a) - \gamma_0)$. If $d_t^{s*}(I_t^a) < \hat{d}_t^{s*}(I_t^a)$, Lemma 28 implies that $\partial_{d_t} J_t^s(x_t^{s*}(I_t^a), d_t^{s*}(I_t^a), I_t^a) \leq \partial_{d_t} \hat{J}_t^s(I_t^a, \hat{d}_t^{s*}(I_t^a), I_t^a)$. Hence,

$$\begin{aligned}\partial_{d_t} R^s(d_t^{s*}(I_t^a), I_t^a) =& \partial_{d_t} J_t^s(x_t^{s*}(I_t^a), d_t^{s*}(I_t^a), I_t^a) + \partial_X H_t^s(x_t^{s*}(I_t^a) - d_t^{s*}(I_t) - \gamma(I_t^a)) \\ \leq& \partial_{d_t} \hat{J}_t^s(I_t^a, \hat{d}_t^{s*}(I_t^a), I_t^a) + \partial_X \hat{H}_t^s(I_t^a - \hat{d}_t^{s*}(I_t^a) - \gamma_0) \\ =& \partial_{d_t} \hat{R}^s(\hat{d}_t^{s*}(I_t^a), I_t^a).\end{aligned} \tag{D.8}$$

The strict concavity of $R^s(\cdot, I_t^a)$ and the supermodularity of $R^s(\cdot, \cdot)$ imply that

$$\partial_{d_t} R^s(d_t^{s*}(I_t^a), I_t^a) > \partial_{d_t} R^s(\hat{d}_t^{s*}(I_t^a), I_t^a) \geq \partial_{d_t} R^s(\hat{d}_t^{s*}(I_t^a), -\infty) = \partial_{d_t} \hat{R}^s(\hat{d}_t^{s*}(I_t^a), I_t^a),$$

which contradicts inequality (D.8). Hence, $d_t^{s*}(I_t^a) \geq \hat{d}_t^{s*}(I_t^a)$. Finally, if $I_t^a \geq \hat{I}_t^H$, $d_t^{s*}(I_t^a) = d_t^H(I_t^a) \geq \hat{d}_t^H(I_t^a) = \hat{d}_t^{s*}(I_t^a)$. We have completed the proof of $d_t^{s*}(I_t^a) \geq \hat{d}_t^{s*}(I_t^a)$ for all $I_t^a \leq K_a$.

To complete the induction, it suffices to show that if $\partial_{I_{t-1}^a} V_{t-1}^s(I_{t-1}^a) \leq \partial_{I_{t-1}^a} \hat{V}_{t-1}^s(I_{t-1}^a)$ for all $I_{t-1}^a \leq K_a$, $\partial_{I_t^a} V_t^s(I_t^a) \leq \partial_{I_t^a} \hat{V}_t^s(I_t^a)$, for all $I_t^a \leq K_a$. Note that $\hat{x}_t^{d*}(I_t)$ and $\hat{d}_t^{s*}(I_t^a)$ are constant if $I_t^a \leq \hat{I}_t^L$ and

$I_t \geq \hat{I}_t^H$, by Theorem 1 in Federgruen and Heching (1999). Hence, $\partial_{I_t^a} V_t^s(I_t^a) \leq \partial_{I_t^a} \hat{V}_t^s(I_t^a)$ for all $I_t^a \leq \hat{I}_t^L$ and $I_t^a \geq \hat{I}_t^H$, since $\partial_{I_t^a} V_t^s(I_t^a) \leq \partial_{I_t^a} \hat{V}_t^s(I_t^a) = c$, if $I_t^a \leq \hat{I}_t^L$, and $\partial_{I_t^a} V_t^s(I_t^a) \leq \partial_{I_t^a} \hat{V}_t^s(I_t^a) = c - \theta = s$, if $I_t^a \geq \hat{I}_t^H$. If $\hat{I}_t^L \leq I_t \leq \hat{I}_t^H$, there are two possible cases: $\hat{I}_t^L \leq I_t^H \leq \hat{I}_t^H$ and $I_t^H \leq \hat{I}_t^L \leq \hat{I}_t^H$.

If $I_t^H \leq \hat{I}_t^L$, $\partial_{I_t^a} V_t^s(I_t^a) \leq \partial_{I_t^a} \hat{V}_t^s(I_t^a)$ for all $I_t^a \leq K_a$ follows immediately. Now assume that $I_t^H \in [\hat{I}_t^L, \hat{I}_t^H]$. If $I_t \in [\hat{I}_t^L, I_t^H]$, $x_t^{s*}(I_t^a) = \hat{x}_t^{s*}(I_t^a) = I_t^a$. Hence,

$$\begin{cases} \partial_{I_t^a} V_t^s(I_t^a) = & c + \beta^s + \partial_{I_t^a} R^s(d_t^{s*}(I_t^a), I_t^a) + \partial_X H_t^s(I_t^a - y_t^s(I_t^a)) - \gamma'(I_t^a)\partial_X H_t^s(I_t^a - y_t^s(I_t^a)), \\ \partial_{I_t^a} \hat{V}_t^s(I_t^a) = & c + \beta^s + \partial_X \hat{H}_t^s(I_t^a - \hat{y}_t^s(I_t^a)), \end{cases}$$

where $y_t^s(I_t^a) = d_t^{s*}(I_t^a) + \gamma(I_t^a)$ and $\hat{y}_t^s(I_t^a) = \hat{d}_t^{s*}(I_t^a) + \gamma_0$. It suffices to show that $\partial_X H_t^s(I_t^a - y_t^s(I_t^a)) \leq \partial_X \hat{H}_t^s(I_t^a - \hat{y}_t^s(I_t^a)$. We use the following lemma to prove this inequality:

Lemma 32 *Let* $y_1 = \text{argmax}_{y_t}\{R^{s*}(y_t, \gamma_0) + \hat{H}_t^s(I_t^a - y_t)\}$, $y_2 = \text{argmax}_{y_t}\{R^{s*}(y_t, \gamma_0) + H_t^s(I_t^a - y_t)\}$ *and* $y_3 = \text{argmax}_{y_t}\{R^{s*}(y_t, \gamma(I_t^a)) + H_t^s(I_t^a - y_t)\}$, *for* $I_t^a \in [\hat{I}_t^L, I_t^H]$. *We have* $\partial_X \hat{H}_t^s(I_t^a - y_1) \geq \partial_X H_t^s(I_t^a - y_2) \geq \partial_X H_t^s(I_t^a - y_3)$.

Proof of Lemma 32: Since $\partial_X \hat{H}_t^s(X) \geq \partial_X H_t^s(X)$,
$\partial_{y_t} R^{s*}(y_1, \gamma_0) - \partial_X H_t^s(I_t^a - y_1) \geq \partial_{y_t} R^{s*}(y_1, \gamma_0) - \partial_X \hat{H}_t^s(I_t^a - y_1)$, i.e., $y_1 \leq y_2$. If $y_1 = y_2$, $\partial_X \hat{H}_t^s(I_t^a - y_1) \geq \partial_X H_t^s(I_t^a - y_2)$ follows from $\partial_X \hat{H}_t^s(X) \geq \partial_X H_t^s(X)$ for any X. If $y_1 < y_2$, $\partial_{y_t} R^{s*}(y_1, \gamma_0) > \partial_{y_t} R^{s*}(y_2, \gamma_0)$ by the strict concavity of $R^{s*}(\cdot, \cdot)$, and $\partial_{y_t} R^{s*}(y_1, \gamma_0) - \partial_X \hat{H}_t^s(I_t^a - y_1) \leq \partial_{y_t} R^{s*}(y_2, \gamma_0) - \partial_X H_t^s(I_t^a - y_2)$ by Lemma 28. Hence, $\partial_X \hat{H}_t^s(I_t^a - y_1) > \partial_X H_t^s(I_t^a - y_2)$. For the second inequality, the supermodularity of $R^{s*}(\cdot, \cdot)$ yields that $y_2 \geq y_3$ and, thus, $\partial_X H_t^s(I_t^a - y_2) \geq \partial_X H_t^s(I_t^a - y_3)$. *Q.E.D.*

Invoking Lemma 32,

$$\partial_X H_t^s(I_t^a - y_t^s(I_t^a)) = \partial_X H_t^s(I_t^a - y_3) \leq \partial_X \hat{H}_t^s(I_t^a - y_1) = \partial_X \hat{H}_t^s(I_t^a - \hat{y}_t^s(I_t^a)).$$

Hence, $\partial_{I_t^a} V_t^s(I_t^a) \leq \partial_{I_t^a} \hat{V}_t^s(I_t^a)$ for all $I_t^a \in [\hat{I}_t^L, I_t^H]$. If $I_t^a \in [I_t^H, \hat{I}_t^H]$,

$$\partial_{I_t^a} V_t^s(I_t^a) \leq c - \theta = \partial_{I_t^a} \hat{V}_t^s(\hat{I}_t^H) \leq \partial_{I_t^a} \hat{V}_t^s(I_t^a),$$

where the first inequality follows from the first order condition with respect to x_t^a. This completes the induction and the proof of **part (a)**.

To prove **part (b)**, it suffices to show that if $\partial_{I_{t-1}^a} V_{t-1}^s(I_{t-1}^a) \leq \partial_{I_{t-1}^a} \hat{V}_{t-1}^s(I_{t-1}^a)$ for all $I_{t-1}^a \leq K_a$, (a) $x_t^L(I_t^a) \leq \hat{x}_t^L(I_t^a)$, (b) $x_t^H(I_t^a) \leq \hat{x}_t^H(I_t^a)$, and (c) $\partial_{I_t^a} V_t^s(I_t^a) \leq \partial_{I_t^a} \hat{V}_t^s(I_t^a)$, for all $I_t^a \leq K_a$. For $t = 0$, $\partial_{I_0^a} V_0^s(I_0^a) = \partial_{I_0^a} \hat{V}_0^s(I_0^a) = 0$ for $I_0^a \leq K_a$.

First, we show that $x_t^L(I_t^a) \leq \hat{x}_t^L(I_t^a)$, and the proof of $x_t^H(I_t^a) \leq \hat{x}_t^H(I_t^a)$ follows from the same argument. If $x_t^L(I_t^a) > \hat{x}_t^L(I_t^a)$, Lemma 28 yields that $\partial_{x_t^a} J_t^L(x_t^L(I_t^a), d_0, I_t^a) \geq \partial_{x_t^a} \hat{J}_t^L(\hat{x}_t^L(I_t^a), d_0, I_t^a)$. Hence,

$$\partial_X H_t^s(x_t^L(I_t^a) - y_t^L(I_t^a)) = \partial_{x_t^a} J_t^L(x_t^L(I_t^a), d_0, I_t^a) - \beta^s \geq \partial_{x_t^a} \hat{J}_t^L(\hat{x}_t^L(I_t^a), d_0, I_t^a) - \beta^s = \partial_X \hat{H}_t^s(\hat{x}_t^L(I_t^a) - \hat{y}_t^L(I_t^a)).$$

Since $\partial_X H_t^s(X) \leq \partial_X \hat{H}_t^s(X)$ for any X and both of them are strictly decreasing, $y_t^L(I_t^a) > \hat{y}_t^L(I_t^a)$. However, $y_t^L(I_t^a) = d_0 + \gamma(I_t^a) \leq d_0 + \hat{\gamma}(I_t^a) = \hat{y}_t^L(I_t^a)$. This contradiction shows that $x_t^L(I_t^a) \leq \hat{x}_t^L(I_t^a)$. $x_t^H(I_t^a) \leq \hat{x}_t^H(I_t^a)$ follows analogously.

To complete the proof, we need to show $\partial_{I_t^a} V_t^s(I_t^a) \leq \partial_{I_t^a} \hat{V}_t^s(I_t^a)$ for all $I_t^a \leq K_a$. For the case $I_t^a \in [\hat{I}_t^L, \hat{I}_t^H]$, the proof is identical to that of part (a), and, hence, omitted. If $I_t^a \leq I_t^L$,

$$\begin{cases} \partial_{I_t^a} V_t^s(I_t^a) = & c + (p_0 - b - \alpha c)\gamma'(I_t^a) - \gamma'(I_t^a)\partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)), \\ \partial_{I_t^a} \hat{V}_t^s(I_t^a) = & c + (p_0 - b - \alpha c)\hat{\gamma}'(I_t^a) - \hat{\gamma}'(I_t^a)\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)). \end{cases}$$

Since $x_t^{s*}(I_t^a) \leq \hat{x}_t^{s*}(I_t^a)$, there are two cases: (a) $x_t^{s*}(I_t^a) = \hat{x}_t^{s*}(I_t^a)$ and (b) $x_t^{s*}(I_t^a) < \hat{x}_t^{s*}(I_t^a)$. If $x_t^{s*}(I_t^a) = \hat{x}_t^{s*}(I_t^a)$, $x_t^{s*}(I_t^a) - y_t^s(I_t^a) \geq \hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)$ and, hence, $\partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)) \leq \partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a))$, since $\partial_X H_t^s(X) \leq \partial_X \hat{H}_t^s(X)$ for any X. If $x_t^{s*}(I_t^a) < \hat{x}_t^{s*}(I_t^a)$, Lemma 28 yields that $\partial_{x_t^a} J_t^s(x_t^{s*}(I_t^a), d_0, I_t^a) \leq \partial_{x_t^a} \hat{J}_t^s(\hat{x}_t^{s*}(I_t^a), d_0, I_t^a)$. Hence,

$$\partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)) = \partial_{x_t^a} J_t^s(x_t^{s*}(I_t^a), d_0, I_t^a) - \beta^s \leq \partial_{x_t^a} \hat{J}_t^s(\hat{x}_t^{s*}(I_t^a), d_0, I_t^a) - \beta^s = \partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)).$$

We have thus showed that $\partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)) \leq \partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a))$ in both cases. Therefore,

$$\begin{aligned} \partial_{I_t^a} V_t^s(I_t^a) =& c + \gamma'(I_t^a)(p_0 - b - \alpha c - \partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a))) \\ \leq& c + \hat{\gamma}'(I_t^a)(p_0 - b - \alpha c - \partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)) \\ =& \partial_{I_t^a} \hat{V}_t^s(I_t^a), \end{aligned}$$

where the inequality follows from $\gamma'(I_t^a) \leq \hat{\gamma}'(I_t^a) \leq 0$ and

$$p_0 - b - \alpha c - \partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)) \geq p_0 - b - \alpha c - \partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)) > 0.$$

The proof of the case $I_t^a \geq \hat{I}_t^H$ follows from the identical argument of the case $I_t^a \leq I_t^L$, and is, hence, omitted.

If $I_t^a \in [I_t^L, \hat{I}_t^L]$,

$$\begin{cases} \partial_{I_t^a} V_t^s(I_t^a) = & c + \beta^s + (p_0 - b - \alpha c)\gamma'(I_t^a) + \partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)) - \gamma'(I_t^a)\partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)), \\ \partial_{I_t^a} \hat{V}_t^s(I_t^a) = & c + (p_0 - b - \alpha c)\hat{\gamma}'(I_t^a) - \hat{\gamma}'(I_t^a)\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)). \end{cases}$$

Note that $\partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)) \leq -\beta^s = \partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a))$. Therefore,

$$\begin{aligned} \partial_{I_t^a} V_t^s(I_t^a) =& c + \beta^s + \partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)) + \gamma'(I_t^a)(p_0 - b - \alpha c - \partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a))) \\ \leq& c + \hat{\gamma}'(I_t^a)(p_0 - b - \alpha c - \partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a))) \\ =& \partial_{I_t^a} \hat{V}_t^s(I_t^a), \end{aligned} \tag{D.9}$$

where the inequality follows from $\gamma'(I_t^a) \leq \hat{\gamma}'(I_t^a) \leq 0$, $\partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)) \leq \partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a))$, and

$$p_0 - b - \alpha c - \partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)) \geq p_0 - b - \alpha c - \partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)) > 0.$$

We have thus showed $\partial_{I_t^a} V_t^s(I_t^a) \leq \partial_{I_t^a} \hat{V}_t^s(I_t^a)$ for all $I_t^a \leq K_a$, which completes the proof of **part (b)**. *Q.E.D.*

Proof of Theorem 5.5.3: We employ backward induction to prove **parts (a) - (d)** together. We define $H_t^s(X) := \mathbb{E}_{\epsilon_t^a}\{-(b+h_a)(X-\epsilon_t^a)^+ + \alpha(V_{t-1}^s(X) - cX)\}$ and $\hat{H}_t^s(X) := \mathbb{E}_{\epsilon_t^a}\{-(b+h_a)(X-\epsilon_t^a)^+ + \alpha(\hat{V}_{t-1}^s(X) - cX)\}$, so that $g_t^s(x_t^a, d_t, I_t^a) := H_t^s(x_t^a - d_t - \gamma(I_t^a))$ and $\hat{g}_t^s(x_t^a, d_t, I_t^a) := \hat{H}_t^s(x_t^a - d_t - \gamma(I_t^a))$. We define the objective functions $J_t^L(x_t^a, d_t, I_t^a) := R^s(d_t, I_t^a) + \beta^s x_t^a + g_t^s(x_t^a, d_t, I_t^a)$, $\hat{J}_t^L(x_t^a, d_t, I_t^a) := \hat{R}^s(d_t, I_t^a) + \beta^s x_t^a + \hat{g}_t^s(x_t^a, d_t, I_t^a)$, $J_t^H(x_t^a, d_t, I_t^a) := R^s(d_t, I_t^a) + (\beta^s+\theta)x_t^a + g_t^s(x_t^a, d_t, I_t^a)$, and $\hat{J}_t^H(x_t^a, d_t, I_t^a) := \hat{R}^s(d_t, I_t^a) + (\beta^s+\hat{\theta})x_t^a + \hat{g}_t^s(x_t^a, d_t, I_t^a)$, where $\hat{\theta} = c - \hat{s} \le c - s = \theta$. Let $\gamma_t := \gamma(I_t^a)$, $y_t^s(I_t^a) := d_t^{s*}(I_t^a) + \gamma_t$ and $\hat{y}_t^s(I_t^a) := \hat{d}_t^{s*}(I_t^a) + \gamma_t$.

It suffices to show that if $\partial_{I_{t-1}^a} \hat{V}_{t-1}^s(I_{t-1}^a) \ge \partial_{I_{t-1}^a} V_{t-1}^s(I_{t-1}^a)$ for all $I_{t-1}^a \le K_a$, (1) $\hat{I}_t^L \ge I_t^L$, (2) $\hat{x}_t^{s*}(I_t^a) \ge x_t^{s*}(I_t^a)$ for all $I_t^a \le \hat{I}_t^H$, (3) $\hat{d}_t^{s*}(I_t^a) \le d_t^{s*}(I_t^a)$, and (4) $\partial_{I_t^a} \hat{V}_t^s(I_t^a) \ge \partial_{I_t^a} V_t^s(I_t^a)$. Since $\partial_{I_{t-1}^a} \hat{V}_{t-1}^s(I_{t-1}^a) \ge \partial_{I_{t-1}^a} V_{t-1}^s(I_{t-1}^a)$, $\partial_X \hat{H}_t^s(X) \ge \partial_X H_t^s(X)$. For $t = 0$, $\partial_{I_0^a} \hat{V}_0^s(I_0^a) = \partial_{I_0^a} V_0^s(I_0^a) = 0$, so the initial condition is satisfied.

We first show that if $\partial_{I_{t-1}^a} \hat{V}_{t-1}^s(I_{t-1}^a) \ge \partial_{I_{t-1}^a} V_{t-1}^s(I_{t-1}^a)$, $\hat{x}_t^L(I_t^a) \ge x_t^L(I_t^a)$, $\hat{d}_t^L(I_t^a) \le d_t^L(I_t^a)$, and $\hat{d}_t^H(I_t^a) \le d_t^H(I_t^a)$. $\hat{x}_t^L(I_t^a) \ge x_t^L(I_t^a)$ and $\hat{d}_t^L(I_t^a) \le d_t^L(I_t^a)$ follows from the same argument as the proof of Theorem 5.5.2. We show by contradiction that $\hat{d}_t^H(I_t^a) \le d_t^H(I_t^a)$.

Assume that $d_t^H(I_t^a) < \hat{d}_t^H(I_t^a)$, so $y_t^H(I_t^a) = d_t^H(I_t^a) + \gamma_t < \hat{d}_t^H(I_t^a) + \gamma_t = \hat{y}_t^H(I_t^a)$. Lemma 28 yields that $\partial_{d_t} J_t^H(x_t^H(I_t^a), d_t^H(I_t^a), I_t^a) \le \partial_{d_t} \hat{J}_t^H(\hat{x}_t^H(I_t^a), \hat{d}_t^H(I_t^a), I_t^a)$. The strict concavity of $R^s(\cdot, I_t^a)$ imply that $\partial_{d_t} R^s(d_t^H(I_t^a), I_t^a) > \partial_{d_t} R^s(\hat{d}_t^H(I_t^a), I_t^a)$. Hence, we have:

$$\begin{aligned}
\partial_X H_t^s(x_t^H(I_t^a) - y_t^H(I_t^a)) =& \partial_{d_t} R^s(d_t^H(I_t^a), I_t^a) - \partial_{d_t} J_t^H(x_t^H(I_t^a), d_t^H(I_t^a), I_t^a) \\
>& \partial_{d_t} \hat{R}^s(\hat{d}_t^H(I_t^a), I_t^a) - \partial_{d_t} \hat{J}_t^H(\hat{x}_t^H(I_t^a), \hat{d}_t^H(I_t^a), I_t^a) \\
=& \partial_X \hat{H}_t^s(\hat{x}_t^H(I_t^a) - \hat{y}_t^H(I_t^a)).
\end{aligned}$$

The first order condition with respect to x_t^a implies that

$$\partial_X H_t^s(x_t^H(I_t^a) - y_t^H(I_t^a)) = -(\beta^s + \theta) < -(\beta^s + \hat{\theta}) = \partial_X \hat{H}_t^s(\hat{x}_t^H(I_t^a) - \hat{y}_t^H(I_t^a)),$$

which leads to a contradiction. Hence, $d_t^H(I_t^a) \ge \hat{d}_t^H(I_t^a)$. We have thus proved that, if $\partial_{I_{t-1}^a} \hat{V}_{t-1}^s(I_{t-1}^a) \ge \partial_{I_{t-1}^a} V_{t-1}^s(I_{t-1}^a)$, $\hat{x}_t^L(I_t^a) \ge x_t^L(I_t^a)$, $\hat{d}_t^L(I_t^a) \le d_t^L(I_t^a)$, and $\hat{d}_t^H(I_t^a) \le d_t^H(I_t^a)$.

Next, we show that $\hat{d}_t^{s*}(I_t^a) \le d_t^{s*}(I_t^a)$ for all $I_t^a \le K_a$. If $I_t^a \le I_t^L$ or $I_t^a \ge \max\{I_t^H, \hat{I}_t^H\}$, $\hat{d}_t^{s*}(I_t^a) \le d_t^{s*}(I_t^a)$ follows from $\hat{d}_t^L(I_t^a) \le d_t^L(I_t^a)$ and $\hat{d}_t^H(I_t^a) \le d_t^H(I_t^a)$. Now we assume that $I_t^a \in [I_t^L, \max\{I_t^H, \hat{I}_t^H\}]$. If $I_t^a \in [I_t^L, \hat{I}_t^L]$, $x_t^{s*}(I_t^a) = I_t^a \le \hat{x}_t^{s*}(I_t^a)$. If $\hat{d}_t^{s*}(I_t^a) > d_t^{s*}(I_t^a)$, by Lemma 28, $\partial_{d_t} J_t^s(x_t^{s*}(I_t^a), d_t^{s*}(I_t^a), I_t^a) \le \partial_{d_t} \hat{J}_t^s(\hat{x}_t^{s*}(I_t^a), \hat{d}_t^{s*}(I_t^a), I_t^a)$. The first order condition with respect to x_t^a implies that $\partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)) \le -\beta^s = \partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a))$. Therefore,

$$\begin{aligned}
\partial_{d_t} R^s(d_t^{s*}(I_t^a), I_t^a) =& \partial_{d_t} J_t^s(x_t^{s*}(I_t^a), d_t^{s*}(I_t^a), I_t^a) + \partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)) \\
\le& \partial_{d_t} \hat{J}_t^s(\hat{x}_t^{s*}(I_t^a), \hat{d}_t^{s*}(I_t^a), I_t^a) + \partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)) \\
=& \partial_{d_t} R^s(\hat{d}_t^{s*}(I_t^a), I_t^a).
\end{aligned}$$

However, $\hat{d}_t^{s*}(I_t^a) > d_t^{s*}(I_t^a)$ implies that $\partial_{d_t} R^s(d_t^{s*}(I_t^a), I_t^a) > \partial_{d_t} R^s(\hat{d}_t^{s*}(I_t^a), I_t^a)$. The contradiction shows that if $I_t^a \in [I_t^L, \hat{I}_t^L]$, $\hat{d}_t^{s*}(I_t^a) \le d_t^{s*}(I_t^a)$.

If $I_t^a \in [\hat{I}_t^L, I_t^H]$, $x_t^{s*}(I_t^a) = I_t^a \ge \hat{x}_t^{s*}(I_t^a)$. If $\hat{d}_t^{s*}(I_t^a) > d_t^{s*}(I_t^a)$, Lemma 28 implies that $\partial_{d_t} J_t^s(x_t^{s*}(I_t^a), d_t^{s*}(I_t^a), I_t^a) \le \partial_{d_t} \hat{J}_t^s(\hat{x}_t^{s*}(I_t^a), \hat{d}_t^{s*}(I_t^a), I_t^a)$. Since $\partial_X H_t^s(X) \le \partial_X \hat{H}_t^s(X)$ for any X and

$\hat{d}_t^{s*}(I_t^a) > d_t^{s*}(I_t^a)$, $\partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)) \leq \partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a))$. We apply the same argument as in the case $I_t^a \in [I_t^L, \hat{I}_t^L]$ and the contradiction shows that $\hat{d}_t^{s*}(I_t^a) \leq d_t^{s*}(I_t^a)$ for all $I_t^a \in [\hat{I}_t^L, I_t^H]$.

If $I_t^a \in [I_t^H, \hat{I}_t^H]$ (which might be an empty set), the first order condition with respect to x_t^a implies that

$$\partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^{s*}(I_t^a)) = -(\beta^s + \theta) < -(\beta^s + \hat{\theta}) \leq \partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^{s*}(I_t^a)). \tag{D.10}$$

If $\hat{d}_t^{s*}(I_t^a) > d_t^{s*}(I_t^a)$, Lemma 28 implies that $\partial_{d_t} J_t^s(x_t^{s*}(I_t^a), d_t^{s*}(I_t^a), I_t^a) \geq \partial_{d_t} \hat{J}_t^s(\hat{x}_t^{s*}(I_t^a), \hat{d}_t^{s*}(I_t^a), I_t^a)$. The same argument as in the case $I_t^a \in [\hat{I}_t^L, I_t^H]$ proves that $\hat{d}_t^{s*}(I_t^a) \leq d_t^{s*}(I_t^a)$ for all $I_t^a \in [I_t^H, \hat{I}_t^H]$. Hence, $\hat{d}_t^{s*}(I_t^a) \leq d_t^{s*}(I_t^a)$ for all $I_t^a \leq K_a$.

To complete the induction, we next show that if $\partial_{I_{t-1}^a} \hat{V}_{t-1}^s(I_{t-1}^a) \geq \partial_{I_{t-1}^a} V_{t-1}^s(I_{t-1}^a)$ for all $I_{t-1}^a \leq K_a$, $\partial_{I_t^a} \hat{V}_t^s(I_t^a) \geq \partial_{I_t^a} V_t^s(I_t^a)$ for all $I_t^a \leq K_a$.

If $I_t^a \leq I_t^L$, note that $\partial_{d_t} J_t^s(x_t^{s*}(I_t^a), d_t^{s*}(I_t^a), I_t^a) = \partial_{d_t} R^s(d_t^{s*}(I_t^a), I_t^a) - \partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a))$ and $\partial_{d_t} \hat{J}_t^d(\hat{x}_t^{s*}(I_t^a), \hat{d}_t^{s*}(I_t^a), I_t^a) = \partial_{d_t} R^s(\hat{d}_t^{s*}(I_t^a), I_t^a) - \partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a))$. By the first order condition with respect to x_t^a, $\partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)) = \partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)) = -\beta^s$. A simple contradiction argument leads to that $d_t^{s*}(I_t^a) = \hat{d}_t^{s*}(I_t^a)$, for $I_t^a \leq I_t^L$. Therefore:

$$\begin{cases} \partial_{I_t^a} V_t^s(I_t^a) & = c + (p(d_t^{s*}(I_t^a)) - b - \alpha c)\gamma'(I_t^a) - \gamma'(I_t^a)\partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)) \\ \partial_{I_t^a} \hat{V}_t^s(I_t^a) & = c + (p(\hat{d}_t^{s*}(I_t^a)) - b - \alpha c)\gamma'(I_t^a) - \gamma'(I_t^a)\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)). \end{cases}$$

Hence, $\partial_{I_t^a} \hat{V}_t^s(I_t^a) = \partial_{I_t^a} V_t^s(I_t^a)$, for $I_t^a \leq I_t^L$.

If $I_t^a \in [I_t^L, \hat{I}_t^L]$,

$$\begin{cases} \partial_{I_t^a} V_t^s(I_t^a) & = c + (p(d_t^{s*}(I_t^a)) - b - \alpha c)\gamma'(I_t^a) + \beta^s + (1 - \gamma'(I_t^a))\partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)) \\ \partial_{I_t^a} \hat{V}_t^s(I_t^a) & = c + (p(\hat{d}_t^{s*}(I_t^a)) - b - \alpha c)\gamma'(I_t^a) - \gamma'(I_t^a)\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)). \end{cases}$$

Note that the first order condition with respect to x_t^a implies that $\partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)) \leq -\beta^s = \partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a))$. If $d_t^{s*}(I_t^a) = \hat{d}_t^{s*}(I_t^a)$,

$$\begin{aligned} &\partial_{I_t^a} \hat{V}_t^s(I_t^a) - \partial_{I_t^a} V_t^s(I_t^a) \\ =& -\gamma'(I_t^a)(\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)) - \partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a))) + \beta^s + \partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)) \geq 0. \end{aligned}$$

If $d_t^{s*}(I_t^a) > \hat{d}_t^{s*}(I_t^a)$, Lemma 28 yields that $\partial_{d_t} J_t^s(x_t^{s*}(I_t^a), d_t^{s*}(I_t^a), I_t^a) \geq \partial_{d_t} \hat{J}_t^s(\hat{x}_t^{s*}(I_t^a), \hat{d}_t^{s*}(I_t^a), I_t^a)$, i.e.,

$$\partial_{d_t} R^s(d_t^{s*}(I_t^a), I_t^a) - \partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)) \geq \partial_{d_t} R^s(\hat{d}_t^{s*}(I_t^a), I_t^a) - \partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)). \tag{D.11}$$

We have:

$$\begin{aligned} \partial_{I_t^a} \hat{V}_t^s(I_t^a) - \partial_{I_t^a} V_t^s(I_t^a) =& [(p(\hat{d}_t^{s*}(I_t^a)) - p(d_t^{s*}(I_t^a))) - (\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)) \\ & - \partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)))]\gamma'(I_t^a) - (\beta^s + \partial_X H_t^s(x_t^{s*}(I^a) - y_t^s(I_t^a))) \\ \geq& [(p(\hat{d}_t^{s*}(I_t^a)) - p(d_t^{s*}(I_t^a))) - (\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)) \\ & - \partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)))]\gamma'(I_t^a) \\ \geq& [(p(\hat{d}_t^{s*}(I_t^a)) - p(d_t^{s*}(I_t^a))) - (\partial_{d_t} R^s(\hat{d}_t^{s*}(I_t^a), I_t^a) - \partial_{d_t} R^s(d_t^{s*}(I_t^a), I_t^a))]\gamma'(I_t^a) \\ =& [p'(d_t^{s*}(I_t^a))y_t^s(I_t^a) - p'(\hat{d}_t^{s*}(I_t^a))\hat{y}_t^s(I_t^a)]\gamma'(I_t^a) \\ \geq& 0, \end{aligned} \tag{D.12}$$

where the first inequality follows from $\partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)) + \beta^s \leq 0$, the second inequality from (D.11), and the last from the concavity of $p(\cdot)$ and $d_t^{s*}(I_t^a) > \hat{d}_t^{s*}(I_t^a)$.

If $I_t^a \in [\hat{I}_t^L, I_t^H]$, $x_t^{s*}(I_t^a) = I_t^a \geq \hat{x}_t^{s*}(I_t^a)$,

$$\begin{cases} \partial_{I_t^a} V_t^s(I_t^a) & = c + (p(d_t^{s*}(I_t^a)) - b - \alpha c)\gamma'(I_t^a) + \beta^s + (1-\gamma'(I_t^a))\partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)) \\ \partial_{I_t^a} \hat{V}_t^s(I_t^a) & = c + (p(\hat{d}_t^{s*}(I_t^a)) - b - \alpha c)\gamma'(I_t^a) + \beta^s + (1-\gamma'(I_t^a))\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)). \end{cases}$$

If $d_t^{s*}(I_t^a) = \hat{d}_t^{s*}(I_t^a)$, $\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)) \geq \partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a))$, and $\partial_{I_t^a}\hat{V}_t^s(I_t^a) \geq \partial_{I_t^a} V_t^s(I_t^a)$. If $d_t^{s*}(I_t^a) > \hat{d}_t^{s*}(I_t^a)$, as in (D.12),

$$\begin{aligned} \partial_{I_t^a}\hat{V}_t^s(I_t^a) - \partial_{I_t^a} V_t^s(I_t^a) \geq & [p'(d_t^{s*}(I_t^a))y_t^s(I_t^a) - p'(\hat{d}_t^{s*}(I_t^a))\hat{y}_t^s(I_t^a)]\gamma'(I_t^a) \\ & + (\partial_{d_t} R^s(\hat{d}_t^{s*}(I_t^a), I_t^a) - \partial_{d_t} R^s(d_t^{s*}(I_t^a), I_t^a)) \\ > & 0, \end{aligned} \tag{D.13}$$

where the second inequality follows from $d_t^{s*}(I_t^a) > \hat{d}_t^{s*}(I_t^a)$.

If $I_t^a \in [I_t^H, \hat{I}_t^H]$ (which might be an empty set),

$$\begin{cases} \partial_{I_t^a} V_t^s(I_t^a) = & c + (p(d_t^{s*}(I_t^a)) - b - \alpha c)\gamma'(I_t^a) - \gamma'(I_t^a)\partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)) - \theta \\ \partial_{I_t^a} \hat{V}_t^s(I_t^a) = & c + (p(\hat{d}_t^{s*}(I_t^a)) - b - \alpha c)\gamma'(I_t^a) + \beta^s + (1-\gamma'(I_t^a))\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)). \end{cases}$$

(D.10) implies that $\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)) > \partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a))$ and $\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)) + \beta^s + \theta \geq 0$. The same argument as in the case $I_t^a \in [I_t^L, \hat{I}_t^L]$ implies that $\partial_{I_t^a}\hat{V}_t^s(I_t^a) \geq \partial_{I_t^a} V_t^s(I_t^a)$ for $I_t^a \in [I_t^H, \hat{I}_t^H]$.

If $I_t^a \geq \max\{I_t^H, \hat{I}_t^H\}$,

$$\begin{cases} \partial_{I_t^a} V_t^s(I_t^a) = & c + (p(d_t^{s*}(I_t^a)) - b - \alpha c)\gamma'(I_t^a) - \gamma'(I_t^a)\partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)) - \theta \\ \partial_{I_t^a} \hat{V}_t^s(I_t^a) = & c + (p(\hat{d}_t^{s*}(I_t^a)) - b - \alpha c)\gamma'(I_t^a) - \gamma'(I_t^a)\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)) - \hat{\theta}. \end{cases}$$

Note that

$$\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)) = -(\beta^s + \hat{\theta}) \geq -(\beta^s + \theta) = \partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)).$$

If $d_t^{s*}(I_t^a) = \hat{d}_t^{s*}(I_t^a)$,

$$\hat{V}_t^s(I_t^a) - \partial_{I_t^a} V_t^s(I_t^a) = -\gamma'(I_t^a)(\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)) - \partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a))) + \theta - \hat{\theta} > 0.$$

If $d_t^{s*}(I_t^a) > \hat{d}_t^{s*}(I_t^a)$, the same argument as (D.12) implies that

$$\hat{V}_t^s(I_t^a) - \partial_{I_t^a} V_t^s(I_t^a) \geq [p'(d_t^{s*}(I_t^a))y_t^s(I_t^a) - p'(\hat{d}_t^{s*}(I_t^a))\hat{y}_t^s(I_t^a)]\gamma'(I_t^a) + \theta - \hat{\theta} > 0.$$

We have thus showed that, if $\partial_{I_{t-1}^a}\hat{V}_{t-1}^s(I_{t-1}^a) \geq \partial_{I_{t-1}^a} V_{t-1}^s(I_{t-1}^a)$ for all $I_{t-1}^a \leq K_a$, $\partial_{I_t^a}\hat{V}_t^s(I_t^a) \geq \partial_{I_t^a} V_t^s(I_t^a)$ for all $I_t^a \leq K_a$, which completes the proof of **Theorem 5.5.3**. *Q.E.D.*

Proof of Theorem 5.5.4: We first show **part (a)**. Observe that if $h_w \geq h_a$ and $\hat{\gamma}(I_t^a) \equiv \gamma_0$ for all $I_t^a \leq K_a$, withholding positive inventory is dominated by displaying this part of inventory to customers, because the holding cost at the customer-accessible storage is smaller than that at the warehouse, and there is no demand-suppressing effect of customer-accessible inventory. Therefore, the firm should not withhold any inventory if $h_w \geq h_a$ and $\hat{\gamma}(I_t^a) \equiv \gamma_0$ for all $I_t^a \leq K_a$.

Next, we show **part (b)** by backward induction. Since it is optimal for the firm not to withhold any inventory in the model with demand $\hat{D}_t$, this model is reduced to the one discussed in Section 5.5.1, i.e., the model without inventory withholding. Let $K_t(I_t^a) := V_t(I_t^a, I_t^a)$. It suffices to show that if $\partial_{I_{t-1}^a} \hat{V}_{t-1}^s(I_{t-1}^a) \geq \partial_{I_{t-1}^a} K_{t-1}(I_{t-1}^a)$, for all $I_{t-1}^a \leq K_a$, (a) $x_t^a(I_t^a) \leq \hat{x}_t^{s*}(I_t^a)$, (b) $d_t(I_t^a) \geq \hat{d}_t^{s*}(I_t^a)$, and (c) $\partial_{I_t^a} \hat{V}_t^s(I_t^a) \geq \partial_{I_t^a} K_t(I_t^a)$, for all $I_t^a \leq K_a$. For $t = 0$, $\hat{V}_0^s(I_0^a) = K_0(I_0^a) = 0$, so the initial condition is satisfied.

If $\partial_{I_{t-1}^a} \hat{V}_{t-1}^s(I_{t-1}^a) \geq \partial_{I_{t-1}^a} K_t(I_{t-1}^a)$ for $I_{t-1}^a \leq K_a$,

$$\partial_X \hat{H}_t^s(X) \geq \partial_X L_t(X, Y) + \partial_Y L_t(X, Y) \text{ for } X = Y,$$

where $\hat{H}_t^s(X)$ is defined in the proof of Theorem 5.5.2, and

$$L_t(X, Y) := \mathbb{E}_{\epsilon_t^a}\{-(h_a + b)(X - \epsilon_t^a)^+ + \alpha[V_{t-1}(X - \epsilon_t^a, Y - \epsilon_t^a) - cY]\}.$$

Therefore, the same argument as in the proof of Theorem 5.5.2(a) shows that $x_t^a(I_t^a) \leq \hat{x}_t^{s*}(I_t^a)$ and $d_t(I_t^a) \geq \hat{d}_t^{s*}(I_t^a)$.

To complete the induction, we show that if $\partial_{I_{t-1}^a} \hat{V}_{t-1}^s(I_{t-1}^a) \geq \partial_{I_{t-1}^a} K_{t-1}(I_{t-1}^a)$ for all $I_{t-1}^a \leq K_a$, $\partial_{I_t^a} \hat{V}_t^s(I_t^a) \geq \partial_{I_t^a} K_t(I_t^a)$, for $I_t^a \leq K_a$. Since $x_t^a(I_t^a) \leq \hat{x}_t^{s*}(I_t^a)$, $x_t^a(I_t^L) \leq x_t^{s*}(I_t^L) = I_t^L$. If $I_t^a \leq I_t^L$,

$$\partial_{I_t^a} K_t(I_t^a) \leq c + (\underline{p} - b - \alpha c)\gamma'(I_t^a) \leq c = \partial_{I_t^a} \hat{V}_t^s(I_t^a).$$

For the case $I_t^a \geq I_t^L$, the argument is very similar to that in the proof of Theorem 5.5.2, so we only sketch it. The key step is to show that

$$\partial_X \hat{H}_t^s(I_t^a - \hat{y}_t^s(I_t^a)) \geq \partial_X L_t(x_t^{a*}(I_t^a, I_t^a) - y_t(I_t^a), I_t^a - y_t(I_t^a)) + \partial_Y L_t(x_t^{a*}(I_t^a, I_t^a) - y_t(I_t^a), I_t^a - y_t(I_t^a)),$$

where $\hat{y}_t^s(I_t^a)$ is defined in the proof of Theorem 5.5.2 and $y_t(I_t^a) := d_t^{a*}(I_t^a, I_t^a) + \gamma(I_t^a)$. To show the above inequality, let $y_t^*(I_t^a)$ be the optimal expected demand in the system with demand $\hat{D}_t$ such that the firm is forced to display $x_t^{a*}(I_t^a, I_t^a)$ to customers and withhold $I_t^a - x_t^{a*}(I_t^a, I_t^a) > 0$ in the warehouse, when the current customer-accessible inventory level is $I_t^a > I_t^L$. Let

$$\hat{L}_t^s(X, Y) = \mathbb{E}_{\epsilon_t^a}\{-(h_a + b)(X - \epsilon_t^a)^+ + \alpha[\hat{V}_{t-1}^s(Y - \epsilon_t^a) - cY]\},$$

Following the same argument as the proof of Lemma 32, we have:

$$\begin{aligned}\partial_X \hat{H}_t^s(I_t^a - \hat{y}_t^s(I_t^a)) \geq& \partial_X \hat{L}_t^s(x_t^{a*}(I_t^a, I_t^a) - y_t^*(I_t^a), I_t^a - y_t^*(I_t^a)) + \partial_Y \hat{L}_t(x_t^{a*}(I_t^a, I_t^a) - y_t^*(I_t^a), I_t^a - y_t^*(I_t^a)) \\ \geq& \partial_X L_t(x_t^{a*}(I_t^a, I_t^a) - y_t(I_t^a), I_t^a - y_t(I_t^a)) + \partial_Y L_t(x_t^{a*}(I_t^a, I_t^a) - y_t(I_t^a), I_t^a - y_t(I_t^a)).\end{aligned} \tag{D.14}$$

Based on (D.14), the same argument as the proof of Theorem 5.5.2(a) yields that $\partial_{I_t^a} \hat{V}_t^s(I_t^a) \geq \partial_{I_t^a} K_t(I_t^a)$, for all $I_t^a \leq K_a$. This completes the induction and the proof of **Theorem 5.5.4(b)**. *Q.E.D.*

Proof of Theorem 5.5.5: We prove Theorem 5.5.5 by backward induction. Let $L_t(X, Y) := \mathbb{E}_{\epsilon_t^a}\{G_t(X - \epsilon_t^a, Y - \epsilon_t^a)\}$ and $H_t(X) := L_t(X, X)$, then $g_t^a(x_t^a, d_t, I_t^a) = H_t(x_t^a - d_t - \gamma(I_t^a))$. Let $K_t(I_t^a) = V_t(I_t^a, I_t^a)$.

It suffices to show that if $\partial_{I_{t-1}^a} K_{t-1}(I_{t-1}^a) \geq \partial_{I_{t-1}^a} V_{t-1}^s(I_{t-1}^a)$, for any $I_{t-1}^a \leq K_a$, (a) $x_t^a(I_t^a) \geq x_t^{s*}(I_t^a)$, (b) $d_t^*(I_t^a, I_t) \leq d_t^{s*}(I_t^a)$, and (c) $\partial_{I_t^a} K_t(I_t^a) \geq \partial_{I_t^a} V_t^s(I_t^a)$, for any $I_t^a \leq K_a$. For $t = 0$, $V_0^s(I_0^a) = K_0(I_0^a) = 0$, so the initial condition is satisfied. Because $\partial_{I_{t-1}^a} K_{t-1}(I_{t-1}^a) \geq \partial_{I_{t-1}^a} V_{t-1}^s(I_{t-1}^a)$, for any $I_{t-1}^a \leq K_a$, $\partial_X H_t(X) \geq \partial_X H_t^s(X)$ for any X.

Following the same argument as the proof of Theorem 5.5.3, we have that if $\partial_X H_t(X) \geq \partial_X H_t^s(X)$ for any X, $x_t^a(I_t^a) \geq x_t^L(I_t^a)$ and $d_t(I_t^a) \leq d_t^L(I_t^a)$. Hence, $I_t^L \leq I_t^* := \sup\{I_t^a : x_t^a(I_t^a) > I_t^a\}$. Therefore, we have that

$$d_t^*(I_t^a, I_t) = d_t(I_t^a) \leq d_t^L(I_t^a) \leq d_t^{s*}(I_t^a), \text{ if } I_t^a \leq I_t^*,$$

where the last inequality follows from the supermodularity of $J_t^s(x_t^a, d_t, I_t^a)$ in (x_t^a, d_t) for any fixed I_t^a.

If $I_t = I_t^a > I_t^*$, $x_t^{a*}(I_t^a, I_t) < x_t^*(I_t^a, I_t) = x_t^{s*}(I_t^a) = I_t^a = I_t$. Therefore,

$$\begin{aligned} d_t^{s*}(I_t^a) =& \text{argmax}_{d_t \in [\underline{d}, \bar{d}]} \{R(d_t, I_t^a) + H_t^s(I_t^a - d_t - \gamma(I_t^a))\} \geq \hat{d}_t(I_t^a) \\ :=& \text{argmax}_{d_t \in [\underline{d}, \bar{d}]} \{R(d_t, I_t^a) + H_t(I_t^a - d_t - \gamma(I_t^a))\}, \end{aligned}$$

since

$$\partial_{d_t} R(\hat{d}_t(I_t^a), I_t^a) - \partial_X H_t^s(I_t^a - \hat{d}_t(I_t^a) - \gamma(I_t^a)) \geq \partial_{d_t} R(\hat{d}_t(I_t^a), I_t^a) - \partial_X H_t(I_t^a - \hat{d}_t(I_t^a) - \gamma(I_t^a)),$$

where the inequality follows from $\partial_X H_t(X) \geq \partial_X H_t^s(X)$ for all X. Similar argument yields that:

$$\begin{aligned} d_t^*(I_t^a, I_t) =& \text{ argmax}_{d_t \in [\underline{d}, \bar{d}]} \{R(d_t, I_t^a) + L_t(x_t^{a*}(I_t^a, I_t) - d_t - \gamma(I_t^a), I_t^a - d_t - \gamma(I_t^a))\} \\ \leq& \hat{d}_t(I_t^a) = \text{argmax}_{d_t \in [\underline{d}, \bar{d}]} \{R(d_t, I_t^a) + L_t(I_t^a - d_t - \gamma(I_t^a), I_t^a - d_t - \gamma(I_t^a))\}, \end{aligned}$$

because $L_t(\cdot, Y)$ is concave for any fixed Y. Hence, $d_t^*(I_t^a, I_t) \leq \hat{d}_t(I_t^a) \leq d_t^{s*}(I_t^a)$ for any $I_t = I_t^a \geq I_t^*$.

To complete the induction, we need to show that if $\partial_{I_{t-1}^a} K_{t-1}(I_{t-1}^a) \geq \partial_{I_{t-1}^a} V_{t-1}^s(I_{t-1}^a)$, for any $I_{t-1}^a \leq K_a$, $\partial_{I_t^a} K_t(I_t^a) \geq \partial_{I_t^a} V_t^s(I_t^a)$, for any $I_t^a \leq K_a$. For $I_t^a \leq I_t^*$, $x_t^{a*}(I_t^a, I_t) = x_t^*(I_t^a, I_t)$. Same argument as in the proof of Theorem 5.5.3 implies that $\partial_{I_t^a} K_t(I_t^a) \geq \partial_{I_t^a} V_t^s(I_t^a)$, if $I_t^a \leq I_t^*$.

If $I_t^a > I_t^*$, the proof is based on the following lemma:

Lemma 33 *Assume that $I_t^a > I_t^*$. Let*

$$\hat{V}_t^s(I_t^a) = cI_t^a + \max_{d_t \in [\underline{d}, \bar{d}]} \{R(d_t, I_t^a) + \beta I_t^a + L_t(I_t^a - d_t - \gamma(I_t^a), I_t^a - d_t - \gamma(I_t^a))\}.$$

We have:

$$\partial_{I_t^a} V_t^s(I_t^a) \leq \partial_{I_t^a} \hat{V}_t^s(I_t^a) \leq \partial_{I_t^a} K_t(I_t^a). \tag{D.15}$$

Proof of Lemma 33: The first inequality follows from the same argument as the proof of Theorem 5.5.3. For the second inequality, observe that

$$\begin{aligned} \partial_{I_t^a} \hat{V}_t(I_t^a) =& c + \beta + (p(\hat{d}_t(I_t^a)) - b - \alpha c)\gamma'(I_t^a) \\ &+ (1 - \gamma'(I_t^a))\partial_X L_t(I_t^a - \hat{d}_t(I_t^a) - \gamma(I_t^a), I_t^a - \hat{d}_t(I_t^a) - \gamma(I_t^a)) \\ &+ (1 - \gamma'(I_t^a))\partial_Y L_t(I_t^a - \hat{d}_t(I_t^a) - \gamma(I_t^a), I_t^a - \hat{d}_t(I_t^a) - \gamma(I_t^a)), \\ \text{and } \partial_{I_t^a} K_t(I_t^a) =& c + \beta + (p(d_t^*(I_t^a, I_t)) - b - \alpha c)\gamma'(I_t^a) \\ &+ (1 - \gamma'(I_t^a))\partial_X L_t(x_t^{a*}(I_t^a, I_t) - d_t^*(I_t^a, I_t) - \gamma(I_t^a), I_t^a - d_t^*(I_t^a, I_t) - \gamma(I_t^a)) \\ &+ (1 - \gamma'(I_t^a))\partial_Y L_t(x_t^{a*}(I_t^a, I_t) - d_t^*(I_t^a, I_t) - \gamma(I_t^a), I_t^a - d_t^*(I_t^a, I_t) - \gamma(I_t^a))). \end{aligned}$$

Thus,

$$\begin{aligned}\partial_{I_t^a} K_t(I_t^a) - \partial_{I_t^a} \hat{V}_t(I_t^a) = & (p(d_t^*(I_t^a, I_t)) - p(\hat{d}_t(I_t^a)))\gamma'(I_t^a) \\ & - \gamma'(I_t^a)[\partial_X L_t(x_t^{a*}(I_t^a, I_t) - d_t^*(I_t^a, I_t) - \gamma(I_t^a), I_t^a - d_t^*(I_t^a, I_t) - \gamma(I_t^a)) \\ & - \partial_X L_t(I_t^a - \hat{d}_t(I_t^a) - \gamma(I_t^a), I_t^a - \hat{d}_t(I_t^a) - \gamma(I_t^a)) \\ & + \partial_Y L_t(x_t^{a*}(I_t^a, I_t) - d_t^*(I_t^a, I_t) - \gamma(I_t^a), I_t^a - d_t^*(I_t^a, I_t) - \gamma(I_t^a)) \\ & - \partial_Y L_t(I_t^a - \hat{d}_t(I_t^a) - \gamma(I_t^a), I_t^a - \hat{d}_t(I_t^a) - \gamma(I_t^a))] \\ & + \partial_X L_t(x_t^{a*}(I_t^a, I_t) - d_t^*(I_t^a, I_t) - \gamma(I_t^a), I_t^a - d_t^*(I_t^a, I_t) - \gamma(I_t^a)) \\ & - \partial_X L_t(I_t^a - \hat{d}_t(I_t^a) - \gamma(I_t^a), I_t^a - \hat{d}_t(I_t^a) - \gamma(I_t^a)) \\ & + \partial_Y L_t(x_t^{a*}(I_t^a, I_t) - d_t^*(I_t^a, I_t) - \gamma(I_t^a), I_t^a - d_t^*(I_t^a, I_t) - \gamma(I_t^a)) \\ & - \partial_Y L_t(I_t^a - \hat{d}_t(I_t^a) - \gamma(I_t^a), I_t^a - \hat{d}_t(I_t^a) - \gamma(I_t^a)).\end{aligned} \tag{D.16}$$

Based on the first order condition with respect to d_t and Lemma 28, the same argument as inequality (D.12) yields that $\partial_{I_t^a} K_t(I_t^a) - \partial_{I_t^a} \hat{V}_t(I_t^a) \geq 0$, and hence (D.15) holds. *Q.E.D.*

By Lemma 33, $\partial_{I_t^a} K_t(I_t^a) \geq \partial_{I_t^a} \hat{V}_t^s(I_t^a) \geq \partial_{I_t^a} V_t^s(I_t^a)$ for all $I_t^a \leq K_a$. This completes the induction in the proof of **Theorem 5.5.5**. *Q.E.D.*

Proof of Theorem 5.6.1: The proof, based on backward induction, is very similar to that of Lemma 14 and Theorem 5.4.1, so we only sketch it. In particular, the continuous differentiability of $V_t^r(I_t^a, I_t)$ follows from the same argument as in the proof of Lemma 14 and is, hence, omitted. Note that $V_0^r(I_0^r, I_0) - cI_0 - r_d I_0^a = -cI_0 - r_d I_0^a$ is jointly concave, continuously differentiable, and decreasing in both of its arguments.

If $V_{t-1}^r(I_{t-1}^a, I_{t-1}) - r_d I_{t-1}^a - cI_{t-1}$ is jointly concave and decreasing in I_{t-1}^a and I_{t-1}, $G_t^r(x, y)$ is decreasing in both x and y. Hence, the same argument as in the proof of Lemma 14 shows that, for any realization of $(\epsilon_t^a, \epsilon_t^m)$,

$$\begin{aligned}& -(r_d + r_w)(y_t^a - I_t^a)^- + \phi y_t^a + G_t^r(y_t^a - D_t, x_t - D_t) \\ = & -(r_d + r_w)(y_t^a - I_t^a)^- + \phi y_t^a + G_t^r(y_t^a - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a, x_t - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a)\end{aligned}$$

is jointly concave in (y_t^a, x_t, d_t, I_t^a). Concavity is preserved under maximization and expectation, so

$$\mathbb{E}_{D_t}\{\max_{\min\{D_t, I_t^a\} \leq y_t^a \leq \min\{K_a + D_t, x_t\}} \{-(r_d + r_w)(y_t^a - I_t^a)^- + \phi y_t^a + G_t^r(y_t^a - D_t, x_t - D_t)\}\}$$

is jointly concave in (x_t, d_t, I_t^a). Since $R(d_t, I_t^a) + r_d(d_t + \gamma(I_t^a))$ is jointly concave in (d_t, I_t^a), and $\theta(x_t - I_t)^-$ is jointly concave in (x_t, I_t),

$$\begin{aligned}& R(d_t, I_t^a) + r_d(d_t + \gamma(I_t^a)) - \theta(x_t - I_t)^- - \psi x_t \\ & + \mathbb{E}_{D_t}\{\max_{\min\{D_t, I_t^a\} \leq y_t^a \leq \{K_a + D_t, x_t\}} \{-(r_d + r_w)(y_t^a - I_t^a)^- + \phi y_t^a + G_t^r(y_t^a - D_t, x_t - D_t)\}\}\end{aligned}$$

is jointly concave in (x_t, d_t, I_t^a). Since concavity is preserved under maximization, $V_t^r(I_t^a, I_t)$ is jointly concave in (I_t^a, I_t).

Next, we show that $V_t^r(I_t^a, I_t) - r_d I_t^a - cI_t$ is decreasing in I_t^a and I_t. Since all of terms in $-(r_d+r_w)(y_t^a - I_t^a)^- + \phi y_t^a + G_t^r(y_t^a - (d_t+\gamma(I_t^a))\epsilon_t^m - \epsilon_t^a, x_t - (d_t+\gamma(I_t^a))\epsilon_t^m - \epsilon_t^a)$ is decreasing in I_t^a, it is decreasing in I_t^a itself, if the constraints $\min\{I_t^a, D_t\} \le y_t^a \le \min\{K_a + D_t, x_t\}$ is not binding.

If $y_t^a = I_t^a$,

$$\begin{aligned}&-(r_d+r_w)(y_t^a - I_t^a)^- + \phi y_t^a + G_t^r(y_t^a - (d_t+\gamma(I_t^a))\epsilon_t^m - \epsilon_t^a, x_t - (d_t+\gamma(I_t^a))\epsilon_t^m - \epsilon_t^a)\\ =&\phi I_t^a + G_t^r(I_t^a - (d_t+\gamma(I_t^a))\epsilon_t^m - \epsilon_t^a, x_t - (d_t+\gamma(I_t^a))\epsilon_t^m - \epsilon_t^a).\end{aligned}$$

If $\phi I_t^a + G_t^r(I_t^a - (d_t+\gamma(I_t^a))\epsilon_t^m - \epsilon_t^a, x_t - (d_t+\gamma(I_t^a))\epsilon_t^m - \epsilon_t^a)$ is strictly increasing in I_t^a, $-(r_d+r_w)(y_t^a - I_t^a)^- + \phi y_t^a + G_t^r(y_t^a - (d_t+\gamma(I_t^a))\epsilon_t^m - \epsilon_t^a, x_t - (d_t+\gamma(I_t^a))\epsilon_t^m - \epsilon_t^a)$ is strictly increasing in y_t^a in a small right-neighborhood of I_t^a: $[I_t^a, I_t^a + \xi)$, for a small enough $\xi > 0$. Under this condition, $y_t^a = I_t^a$ is not an optimizer. Hence,

$-(r_d+r_w)(y_t^a - I_t^a)^- + \phi y_t^a + G_t^r(y_t^a - (d_t+\gamma(I_t^a))\epsilon_t^m - \epsilon_t^a, x_t - (d_t+\gamma(I_t^a))\epsilon_t^m - \epsilon_t^a)$ is decreasing in I_t^a, if it is optimal to choose $y_t^a = I_t^a$.

If $y_t^a = D_t$,

$$\begin{aligned}&-(r_d+r_w)(y_t^a - I_t^a)^- + \phi y_t^a + G_t^r(y_t^a - (d_t+\gamma(I_t^a))\epsilon_t^m - \epsilon_t^a, x_t - (d_t+\gamma(I_t^a))\epsilon_t^m - \epsilon_t^a)\\ =&-(r_d+r_w)((d_t+\gamma(I_t^a))\epsilon_t^m + \epsilon_t^a - I_t^a)^- + \phi((d_t+\gamma(I_t^a))\epsilon_t^m + \epsilon_t^a) + G_t^r(0, x_t - (d_t+\gamma(I_t^a))\epsilon_t^m - \epsilon_t^a)\end{aligned}$$

is decreasing I_t^a.

Analogously, if $y_t^a = K_a + D_t$,

$$\begin{aligned}&-(r_d+r_w)(y_t^a - I_t^a)^- + \phi y_t^a + G_t^r(y_t^a - (d_t+\gamma(I_t^a))\epsilon_t^m - \epsilon_t^a, x_t - (d_t+\gamma(I_t^a))\epsilon_t^m - \epsilon_t^a)\\ =&-(r_d+r_w)(K_a + (d_t+\gamma(I_t^a))\epsilon_t^m + \epsilon_t^a - I_t^a)^- + \phi(K_a + (d_t+\gamma(I_t^a))\epsilon_t^m + \epsilon_t^a)\\ &+ G_t^r(K_a, x_t - (d_t+\gamma(I_t^a))\epsilon_t^m - \epsilon_t^a)\end{aligned}$$

is decreasing in I_t^a.

If $y_t^a = x_t$,

$$\begin{aligned}&-(r_d+r_w)(y_t^a - I_t^a)^- + \phi y_t^a + G_t^r(y_t^a - (d_t+\gamma(I_t^a))\epsilon_t^m - \epsilon_t^a, x_t - (d_t+\gamma(I_t^a))\epsilon_t^m - \epsilon_t^a)\\ =&-(r_d+r_w)(x_t - I_t^a)^- + \phi x_t + G_t^r(x_t - (d_t+\gamma(I_t^a))\epsilon_t^m - \epsilon_t^a, x_t - (d_t+\gamma(I_t^a))\epsilon_t^m - \epsilon_t^a)\end{aligned}$$

is decreasing in I_t^a.

Hence,

$$\begin{aligned}&\max_{\min\{D_t, I_t^a\} \le y_t^a \le \min\{x_t, D_t + K_a\}} \{-(r_d+r_w)(y_t^a - I_t^a)^- + \phi y_t^a\\ &+ G_t^r(y_t^a - (d_t+\gamma(I_t^a))\epsilon_t^m - \epsilon_t^a, x_t - (d_t+\gamma(I_t^a))\epsilon_t^m - \epsilon_t^a)\}\end{aligned}$$

is decreasing in I_t^a. Because, $-\theta(x_t - I_t)^-$ is decreasing in I_t and $F^r(I_1^a) \subset F^r(I_2^a)$ for any $I_1^a \ge I_2^a$,

$$\begin{aligned}V_t^r(I_t^a, I_t) - r_d I_t^a - cI_t = &\max_{(x_t,d_t)\in F^r(I_t^a)} \{R(d_t, I_t^a) + r_d(d_t + \gamma(I_t^a)) - \theta(x_t - I_t)^- - \psi x_t\\ &+ \mathbb{E}_{D_t}\{\max_{\min\{D_t, I_t^a\} \le y_t^a \le \min\{x_t, K_a + D_t\}} \{-(r_d+r_w)(y_t^a - I_t^a)^- + \phi y_t^a\\ &+ G_t^r(y_t^a - D_t, x_t - D_t)\}\}\}\end{aligned}$$

is decreasing in I_t^a and I_t. This concludes the proof of **part (a)**. **Part (b)** follows directly from the concavity of $V_{t-1}^r(\cdot, y)$ for any y and (5.12), while **part (c)** follows from the same argument as the proof of Theorem 5.4.1. *Q.E.D.*

E. Appendix for Chapter 6

E.1 Proofs of Statements

Proof of Lemma 17: We prove parts (a) - (c) together, using backward induction.

Since $V_0(\cdot|\theta_0) \equiv 0$ is concave and continuously differentiable in I_0 for any θ_0, it suffices to show that if $V_{t-1}(\cdot|\theta_{t-1})$ is concave and continuously differentiable in I_{t-1} for any θ_{t-1}, then, for any θ_t, (i) $\Psi_t(\cdot|\theta_t)$ is concave and continuously differentiable in z, (ii) $J_t(\cdot,\cdot,I_t|\theta_t)$ is strictly jointly concave and continuously differentiable in (d_t, q_t), and (iii) $V_t(\cdot|\theta_t)$ is concave and continuously differentiable in I_t.

Since $-H(\cdot)$ and $V_{t-1}(\cdot|\theta_{t-1})$ are concave and concavity is preserved under expectation, by Equation (6.14), $\Psi_t(z|\theta_t)$ is concave in z for any θ_t. Since ϵ_t follows a continuous distribution, $\Psi_t(z|\theta_t)$ is continuously differentiable in z.

By Assumption 6.4.1, $(\sum_{i\in\mathcal{N}} \Lambda_t^i R^i(d_t^i))$ is strictly jointly concave in d_t. The strict convexity of $C^j(\cdot|c_t^j)$ for each j, implies that $-\sum_{j\in\mathcal{M}} C^j(q_t^j|c_t^j)$ is strictly jointly concave in q_t. Moreover, by the concavity of $\Psi_t(\cdot|\theta_t)$, for any realization of ς_t, $\Psi_t(I_t + \sum_{j\in\mathcal{M}} q_t^j - (\sum_{i\in\mathcal{N}} \Lambda_t^i d_t^i)\varsigma_t|\theta_t)$ is jointly concave in (d_t, q_t, I_t). Therefore, by Equation (6.13),

$$J_t(d_t, q_t, I_t|\theta_t) = (\sum_{i\in\mathcal{N}} \Lambda_t^i R^i(d_t^i)) - \sum_{j\in\mathcal{M}} C^j(q_t^j|c_t^j) + \mathbb{E}_{\varsigma_t}\{\Psi_t(I_t + \sum_{j\in\mathcal{M}} q_t^j - (\sum_{i\in\mathcal{N}} \Lambda_t^i d_t^i)\varsigma_t|\theta_t)\}$$

is jointly concave in (d_t, q_t, I_t) and strictly jointly concave in (d_t, q_t). Since $R^i(\cdot)$ is continuously differentiable in d_t^i for any i, $C^j(\cdot|c_t^j)$ is continuously differentiable in q_t^j for any j and c_t^j, and $\Psi_t(\cdot|\theta_t)$ is continuously differentiable in z for any θ_t, $J_t(\cdot,\cdot,I_t|\theta_t)$ is continuously differentiable in (d_t, q_t) for any θ_t.

Since concavity is preserved under maximization, by Equation (6.11), $V_t(\cdot|\theta_t)$ is concave in I_t for any θ_t. The continuous differentiability of $V_t(\cdot|\theta_t)$ follows from the envelope theorem and its derivative is given by

$$\begin{aligned}\partial_{I_t} V_t(I_t|\theta_t) &= \partial_{I_t}\mathbb{E}_{\varsigma_t}\{\Psi_t(I_t + \sum_{j\in\mathcal{M}} q_t^{j*}(I_t,\theta_t) - (\sum_{i\in\mathcal{N}} \Lambda_t^i d_t^{i*}(I_t,\theta_t))\varsigma_t|\theta_t)\} \\ &= \mathbb{E}_{\varsigma_t}\{\partial_z\Psi_t(I_t + \sum_{j\in\mathcal{M}} q_t^{j*}(I_t,\theta_t) - (\sum_{i\in\mathcal{N}} \Lambda_t^i d_t^{i*}(I_t,\theta_t))\varsigma_t|\theta_t)\},\end{aligned} \quad \text{(E.1)}$$

where the first equality follows from the envelope theorem and the second from Theorem A.5.1 of [63] and the continuous differentiability of $\Psi_t(\cdot|\theta_t)$. *Q.E.D.*

Proof of Theorem 6.4.1: We prove part (b) first, part (c) second, and part (a) last.

Part (b). Let

$$\Phi_t(y|\theta_t) := \max_{d_t\in[0,d_{\max}]^n}\{(\sum_{i\in\mathcal{N}} \Lambda_t^i R^i(d_t^i)) + \mathbb{E}_{\varsigma_t}\{\Psi_t(y - (\sum_{i\in\mathcal{N}} \Lambda_t^i d_t^i)\varsigma_t|\theta_t)\}. \quad \text{(E.2)}$$

It's clear that $\Phi_t(\cdot|\theta_t)$ is concave and continuously differentiable in y, and

$$(q_t^{1*}(I_t,\theta_t), q_t^{2*}(I_t,\theta_t), \cdots, q_t^{m*}(I_t,\theta_t)) = \text{argmax}_{q_t\geq 0}\{-\sum_{j\in\mathcal{M}} C^j(q_t^j|c_t^j) + \Phi_t(I_t + \sum_{j\in\mathcal{M}} q_t^j|\theta_t)\}. \quad \text{(E.3)}$$

Invoke Lemma 15 with $p = m$, $q = 0$, $\gamma = -I_t$, $y_j = q_t^j$ $(1 \leq j \leq m)$, $\lambda_j = 1$ $(1 \leq j \leq m)$, $f_j(y_j|\gamma) = -C^j(q_t^j|c_t^j)$, $h(y_0|\gamma) = \Phi_t(I_t+y_0|\theta_t)$, and $\mathcal{Y}_j = [0,+\infty)$ for all $1 \leq j \leq m$. Since $\Phi_t(I_t+y_0|\theta_t)$ is supermodular in $(-I_t, y_0)$, $h(y_0|\gamma)$ is supermodular in (y_0, γ). Hence, Lemma 15 implies that $q_t^{j*}(I_t,\theta_t)$ is decreasing in I_t for any j and θ_t. The strict concavity of $J_t(\cdot,\cdot,I_t|\theta_t)$ yields that $q_t^{j*}(I_t,\theta_t)$ is continuous in I_t for any j and θ_t. Hence, $I_t^{q,j}(\theta_t) = \min\{I_t : q_t^{j*}(I_t,\theta_t) = 0\}$. If $j \in \mathcal{M}_t^*(\hat{I}_t,\theta_t)$, since $q_t^{j*}(I_t,\theta_t)$ is decreasing in I_t and $\hat{I}_t > I_t$, $q_t^{j*}(I_t,\theta_t) \geq q_t^{j*}(\hat{I}_t,\theta_t) > 0$. Thus, $j \in \mathcal{M}_t^*(I_t,\theta_t)$, and $\mathcal{M}_t^*(\hat{I}_t,\theta_t) \subset \mathcal{M}_t^*(I_t,\theta_t)$ follows immediately.

It remains to be shown that $I_t^{q,j}(\theta_t) < +\infty$. First observe that $V_t(I_t|\theta_t)$ is uniformly bounded from above by $\mathbb{E}[\sum_{s=1}^t \alpha^{t-s}(\sum_{j\in\mathcal{N}}\Lambda_s^j)|\theta_t]\bar{R} < +\infty$, where $\bar{R} := \max_{i\in\mathcal{N},d_t^i\in[0,d_{\max}]} R^i(d_t^i)$. Hence,

$$\lim_{z\to+\infty} \partial_z\Psi_t(z|\theta_t) \leq -\lim_{z\to+\infty} H'(z+) < 0.$$

By the envelope theorem, we have

$$\lim_{y\to+\infty} \partial_y\Phi_t(y|\theta_t) \leq -\lim_{y\to+\infty} H'(y+) < 0.$$

Thus, there exists a threshold $\bar{y}_t < +\infty$ such that $\partial_y\Phi_t(y|\theta_t) < 0$ for all $y \geq \bar{y}_t$. Therefore, for any $j \in \mathcal{M}$,

$$-\partial_{q_t^j} C^j(q_t^j|c_t^j) + \partial_y\Phi_t(I_t + \sum_{j\in\mathcal{M}} q_t^j|\theta_t) < 0 \text{ for all } I_t \geq \bar{y}_t \text{ and } q_t \geq 0.$$

Hence, $q_t^{j*}(I_t,\theta_t) = 0$ for all $I_t \geq \bar{y}_t$ and any $j \in \mathcal{M}$. Thus, $I_t^{q,j}(\theta_t) < +\infty$ for all θ_t and any $j \in \mathcal{M}$.

Part (c). The continuity of $x_t^*(I_t,\theta_t)$ follows from that of $q_t^{j*}(I_t,\theta_t)$ for each $j \in \mathcal{M}$. Assume, to the contrary, that $\hat{I}_t > I_t$ and $x_t^*(\hat{I}_t,\theta_t) < x_t^*(I_t,\theta_t)$. Hence, there exists a $j_0 \in \mathcal{M}$, such that $q_t^{j_0*}(\hat{I}_t,\theta_t) < q_t^{j_0*}(I_t,\theta_t)$. Without loss of generality, let $j_0 = 1$. The strict convexity of $C^1(\cdot|c_t^1)$ implies that $\partial_{q_t^1} C^1(q_t^{1*}(I_t,\theta_t)|c_t^1) > \partial_{q_t^1} C^1(q_t^{1*}(\hat{I}_t,\theta_t)|c_t^1)$. On the other hand, Lemma 16 yields that

$$-\partial_{q_t^1} C^1(q_t^{1*}(I_t,\theta_t)|c_t^1) + \partial_y\Phi_t(x_t^*(I_t,\theta_t)|\theta_t) \geq -\partial_{q_t^1} C^1(q_t^{1*}(\hat{I}_t,\theta_t)|c_t^1) + \partial_y\Phi_t(x_t^*(\hat{I}_t,\theta_t)|\theta_t).$$

Therefore, $\partial_y\Phi_t(x_t^*(\hat{I}_t,\theta_t)|\theta_t) < \partial_y\Phi_t(x_t^*(I_t,\theta_t)|\theta_t)$, which contradicts the concavity of $\Phi_t(\cdot|\theta_t)$. Hence, $x_t^*(I_t,\theta_t)$ is continuously increasing in I_t.

Part (a). The continuity of $d_t^{i*}(I_t,\theta_t)$ follows from the concavity of $J_t(\cdot,\cdot,I_t|\theta_t)$. For any given $\hat{I}_t$ and I_t $(\hat{I}_t > I_t)$, assume, to the contrary, that $d_t^{1*}(I_t,\theta_t) > d_t^{1*}(\hat{I}_t,\theta_t)$. Thus, $\partial_{d_t^1} R^1(d_t^{1*}(I_t,\theta_t)) < \partial_{d_t^1} R^1(d_t^{1*}(\hat{I}_t,\theta_t))$ by the strict concavity of $R^1(\cdot)$. On the other hand, Lemma 16 yields that
$\partial_{d_t^1} J_t(d_t^*(I_t,\theta_t), q_t^*(I_t,\theta_t), I_t|\theta_t) \geq \partial_{d_t^1} J_t(d_t^*(\hat{I}_t,\theta_t), q_t^*(\hat{I}_t,\theta_t), I_t|\theta_t)$. Thus,

$$\begin{aligned}\mathbb{E}_{\varsigma_t}\{\varsigma_t\partial_y\Psi_t(x_t^*(I_t,\theta_t) - (\sum_{i\in\mathcal{N}}\Lambda_t^i d_t^{i*}(I_t,\theta_t))\varsigma_t|\theta_t)\} =& \partial_{d_t^1} R^1(d_t^{1*}(I_t,\theta_t)) - \frac{1}{\Lambda_t^1}\partial_{d_t^1} J_t(d_t^*(I_t,\theta_t), q_t^*(I_t,\theta_t), I_t|\theta_t)\\ <& \partial_{d_t^1} R^1(d_t^{1*}(\hat{I}_t,\theta_t)) - \frac{1}{\Lambda_t^1}\partial_{d_t^1} J_t(d_t^*(\hat{I}_t,\theta_t), q_t^*(\hat{I}_t,\theta_t), I_t|\theta_t)\\ =& \mathbb{E}_{\varsigma_t}\{\varsigma_t\partial_y\Psi_t(x_t^*(\hat{I}_t,\theta_t) - (\sum_{i\in\mathcal{N}}\Lambda_t^i d_t^{i*}(\hat{I}_t,\theta_t))\varsigma_t|\theta_t)\}.\end{aligned} \quad \text{(E.4)}$$

For any $l = 2, 3, \cdots, n$, further assume that $d_t^{l*}(I_t, \theta_t) < d_t^{l*}(\hat{I}_t, \theta_t)$. Hence, $\partial_{d_t^l} R^l(d_t^{l*}(I_t, \theta_t)) > \partial_{d_t^l} R^l(d_t^{l*}(\hat{I}_t, \theta_t))$. On the other hand, Lemma 16 yields that
$\partial_{d_t^l} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t) \leq \partial_{d_t^l} J_t(d_t^*(\hat{I}_t, \theta_t), q_t^*(\hat{I}_t, \theta_t), I_t|\theta_t)$. Thus,

$$\begin{aligned}
\mathbb{E}_{\varsigma_t}\{\varsigma_t \partial_y \Psi_t(x_t^*(I_t, \theta_t) - (\sum_{i \in \mathcal{N}} \Lambda_t^i d_t^{i*}(I_t, \theta_t))\varsigma_t|\theta_t)\} =& \partial_{d_t^l} R^l(d_t^{l*}(I_t, \theta_t)) - \frac{1}{\Lambda_t^l} \partial_{d_t^l} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t) \\
>& \partial_{d_t^l} R^l(d_t^{l*}(\hat{I}_t, \theta_t)) - \frac{1}{\Lambda_t^l} \partial_{d_t^l} J_t(d_t^*(\hat{I}_t, \theta_t), q_t^*(\hat{I}_t, \theta_t), I_t|\theta_t) \\
=& \mathbb{E}_{\varsigma_t}\{\varsigma_t \partial_y \Psi_t(x_t^*(\hat{I}_t, \theta_t) - (\sum_{i \in \mathcal{N}} \Lambda_t^i d_t^{i*}(\hat{I}_t, \theta_t))\varsigma_t|\theta_t)\}.
\end{aligned} \tag{E.5}$$

Since (E.4) contradicts (E.5), $d_t^{l*}(I_t, \theta_t) \geq d_t^{l*}(\hat{I}_t, \theta_t)$ for all $l = 2, 3, \cdots, n$, if $d_t^{1*}(I_t, \theta_t) > d_t^{1*}(\hat{I}_t, \theta_t)$.

Hence, if $d_t^{1*}(I_t, \theta_t) > d_t^{1*}(\hat{I}_t, \theta_t)$, $d_t^{i*}(I_t, \theta_t) \geq d_t^{i*}(\hat{I}_t, \theta_t)$ for all $i \in \mathcal{N}$. By part (c), $x_t^*(\hat{I}_t, \theta_t) \geq x_t^*(I_t, \theta_t)$. Thus,

$$x_t^*(\hat{I}_t, \theta_t) - (\sum_{i \in \mathcal{N}} \Lambda_t^i d_t^{i*}(\hat{I}_t, \theta_t))\varsigma_t \geq x_t^*(I_t, \theta_t) - (\sum_{i \in \mathcal{N}} \Lambda_t^i d_t^{i*}(I_t, \theta_t))\varsigma_t$$

for any realization of ς_t. Thus, the concavity of $\Psi_t(\cdot|\theta_t)$ implies that

$$\varsigma_t \partial_y \Psi_t(x_t^*(\hat{I}_t, \theta_t) - (\sum_{i \in \mathcal{N}} \Lambda_t^i d_t^{i*}(\hat{I}_t, \theta_t))\varsigma_t|\theta_t) \leq \varsigma_t \partial_y \Psi_t(x_t^*(I_t, \theta_t) - (\sum_{i \in \mathcal{N}} \Lambda_t^i d_t^{i*}(I_t, \theta_t))\varsigma_t|\theta_t)$$

for any realization of ς_t. By taking expectation on both sides, we have

$$\mathbb{E}_{\varsigma_t}\{\varsigma_t \partial_y \Psi_t(x_t^*(\hat{I}_t, \theta_t) - (\sum_{i \in \mathcal{N}} \Lambda_t^i d_t^{i*}(\hat{I}_t, \theta_t))\varsigma_t|\theta_t)\} \leq \mathbb{E}_{\varsigma_t}\{\varsigma_t \partial_y \Psi_t(x_t^*(I_t, \theta_t) - (\sum_{i \in \mathcal{N}} \Lambda_t^i d_t^{i*}(I_t, \theta_t))\varsigma_t|\theta_t)\},$$

which contradicts (E.4). Therefore, $d_t^{1*}(I_t, \theta_t) \leq d_t^{1*}(\hat{I}_t, \theta_t)$. The same argument implies that $d_t^{i*}(I_t, \theta_t)$ is increasing in I_t for all $i \in \mathcal{N}$. Hence, $I_t^{d,i}(\theta_t) = \max\{I_t : d_t^{i*}(I_t, \theta_t) = 0\}$. If $i \in \mathcal{N}_t^*(I_t, \theta_t)$, since $d_t^{i*}(I_t, \theta_t)$ is increasing in I_t and $\hat{I}_t > I_t$, $d_t^{i*}(\hat{I}_t, \theta_t) \geq d_t^{i*}(I_t, \theta_t) > 0$. Thus, $i \in \mathcal{N}_t^*(\hat{I}_t, \theta_t)$, and $\mathcal{N}_t^*(I_t, \theta_t) \subset \mathcal{N}_t^*(\hat{I}_t, \theta_t)$ follows immediately.

To complete the proof, it remains to be shown that $I_t^{d,i}(\theta_t) < +\infty$ for all $i \in \mathcal{N}$. By the proof of part (b), $\lim_{z \to +\infty} \partial_z \Psi_t(z|\theta_t) < 0$. Moreover, observe that $x_t^*(I_t, \theta_t) \to +\infty$ as $I_t \to +\infty$. Thus, by the monotone convergence theorem,

$$\lim_{I_t \to +\infty} \mathbb{E}_{\varsigma_t}\{\varsigma_t \partial_y \Psi_t(x_t^*(I_t, \theta_t) - (\sum_{i \in \mathcal{N}} \Lambda_t^i d_t^i)\varsigma_t)\} < 0 \text{ for any } d_t \in [0, d_{\max}]^n.$$

Therefore,

$$\partial_{d_t^i} R^i(0) - \lim_{I_t \to +\infty} \mathbb{E}_{\varsigma_t}\{\varsigma_t \partial_y \Psi_t(x_t^*(I_t, \theta_t) - (\sum_{l \in \mathcal{N}, l \neq i} \Lambda_t^l d_t^l)\varsigma_t|\theta_t)\} > \partial_{d_t^i} R^i(0) = p_{\max}^i > 0$$

for any i and $d_t^{-i} \in [0, d_{\max}]^{n-1}$, where $d_t^{-i} := (d_t^1, d_t^2, \cdots, d_t^{i-1}, d_t^{i+1}, \cdots, d_t^n)$. Hence, for all $i \in \mathcal{N}$, $d_t^{i*}(I_t, \theta_t) > 0$ for sufficiently large I_t, i.e., $I_t^{d,i}(\theta_t) < +\infty$. *Q.E.D.*

Proof of Theorem 6.4.2:

Part (a). First, we show that if $\partial_{d_t^i} R^i(z) \geq \partial_{d_t^{\hat{i}}} R^{\hat{i}}(z)$ for all $z \in [0, d_{\max}]$, $d_t^{i*}(I_t, \theta_t) \geq d_t^{\hat{i}*}(I_t, \theta_t)$ for any (I_t, θ_t). Assume, to the contrary, that $d_t^{i*}(I_t, \theta_t) < d_t^{\hat{i}*}(I_t, \theta_t)$. By the inequality $\partial_{d_t^i} R^i(z) \geq \partial_{d_t^{\hat{i}}} R^{\hat{i}}(z)$

for all $z \in [0, d_{\max}]$ and the strict concavity of $R^i(\cdot)$, $\partial_{d_t^i} R^i(d_t^{i*}(I_t, \theta_t)) > \partial_{d_t^{\hat{i}}} R^{\hat{i}}(d_t^{\hat{i}*}(I_t, \theta_t))$. On the other hand, Lemma 16 yields that $\partial_{d_t^i} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t) \leq \partial_{d_t^{\hat{i}}} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t)$. Thus,

$$\begin{aligned}
\mathbb{E}_{\varsigma_t}\{\varsigma_t \partial_y \Psi_t(x_t^*(I_t, \theta_t) - (\sum_{l \in \mathcal{N}} \Lambda_t^l d_t^{l*}(I_t, \theta_t))\varsigma_t|\theta_t)\} =& \partial_{d_t^i} R^i(d_t^{i*}(I_t, \theta_t)) - \frac{1}{\Lambda_t^i} \partial_{d_t^i} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t) \\
>& \partial_{d_t^{\hat{i}}} R^{\hat{i}}(d_t^{\hat{i}*}(I_t, \theta_t)) - \frac{1}{\Lambda_t^{\hat{i}}} \partial_{d_t^{\hat{i}}} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t) \\
=& \mathbb{E}_{\varsigma_t}\{\varsigma_t \partial_y \Psi_t(x_t^*(I_t, \theta_t) - (\sum_{l \in \mathcal{N}} \Lambda_t^l d_t^{l*}(I_t, \theta_t))\varsigma_t|\theta_t)\},
\end{aligned}$$

which forms a contradiction. Thus, $d_t^{i*}(I_t, \theta_t) \geq d_t^{\hat{i}*}(I_t, \theta_t)$, and

$$I_t^{d,i}(\theta_t) = \max\{I_t : d_t^{i*}(I_t, \theta_t) = 0\} \leq \max\{I_t : d_t^{\hat{i}*}(I_t, \theta_t) = 0\} = I_t^{d,\hat{i}}(\theta_t).$$

For the second half of part (a), the inequality $I_t^{d,1}(\theta_t) \leq I_t^{d,2}(\theta_t) \leq \cdots \leq I_t^{d,n}(\theta_t)$ follows directly from the first half. It remains to be shown that $\mathcal{N}_t^*(I_t, \theta_t) = \{1, 2, \cdots, i^*\}$, where $i^* = \max\{i : I_t > I_t^{d,i}(\theta_t)\}$. Observe that $I_t > I_t^{d,i^*}(\theta_t) \geq I_t^{d,i^*-1}(\theta_t) \geq \cdots \geq I_t^{d,1}(\theta_t)$. Thus, by the definition of $I_t^{d,i}(\theta_t)$, $\{1, 2, \cdots, i^*\} \subset \mathcal{N}_t^*(I_t, \theta_t)$. Moreover, by the definition of i^*, $I_t \leq I_t^{d,i^*+1}(\theta_t) \leq I_t^{d,i^*+2}(\theta_t) \leq \cdots \leq I_t^{d,n}(\theta_t)$. Thus, $i \notin \mathcal{N}_t^*(I_t, \theta_t)$ for all $i \geq i^* + 1$ and, hence, $\mathcal{N}_t^*(I_t, \theta_t) = \{1, 2, \cdots, i^*\}$.

Part (b). We first show that if $\partial_{q_t^j} C^j(z|c_t^j) \geq \partial_{q_t^{\hat{j}}} C^{\hat{j}}(z|c_t^{\hat{j}})$ for any $z \geq 0$, $q_t^{j*}(I_t, \theta_t) \leq q_t^{\hat{j}*}(I_t, \theta_t)$ for any (I_t, θ_t). Assume, to the contrary, that $q_t^{j*}(I_t, \theta_t) > q_t^{\hat{j}*}(I_t, \theta_t)$. The inequality $\partial_{q_t^j} C^j(z|c_t^j) \geq \partial_{q_t^{\hat{j}}} C^{\hat{j}}(z|c_t^{\hat{j}})$ for any $z \geq 0$, together with the strict convexity of $C^j(\cdot|c_t^j)$, implies that $\partial_{q_t^j} C^j(q_t^{j*}(I_t, \theta_t)|c_t^j) > \partial_{q_t^{\hat{j}}} C^{\hat{j}}(q_t^{\hat{j}*}(I_t, \theta_t)|c_t^{\hat{j}})$. On the other hand, Lemma 16 implies that $\partial_{q_t^j} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t) \geq \partial_{q_t^{\hat{j}}} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t)$. Thus,

$$\begin{aligned}
\mathbb{E}_{\varsigma_t}\{\partial_y \Psi_t(x_t^*(I_t, \theta_t) - (\sum_{l \in \mathcal{N}} \Lambda_t^l d_t^{l*}(I_t, \theta_t))\varsigma_t|\theta_t)\} =& \partial_{q_t^j} C^j(q_t^{j*}(I_t, \theta_t)|c_t^j) + \partial_{q_t^j} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t) \\
>& \partial_{q_t^{\hat{j}}} C^{\hat{j}}(q_t^{\hat{j}*}(I_t, \theta_t)|c_t^{\hat{j}}) + \partial_{q_t^{\hat{j}}} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t) \\
=& \mathbb{E}_{\varsigma_t}\{\partial_y \Psi_t(x_t^*(I_t, \theta_t) - (\sum_{l \in \mathcal{N}} \Lambda_t^l d_t^{l*}(I_t, \theta_t))\varsigma_t|\theta_t)\},
\end{aligned}$$

which forms a contradiction. Thus, $q_t^{j*}(I_t, \theta_t) \leq q_t^{\hat{j}*}(I_t, \theta_t)$, and

$$I_t^{q,j}(\theta_t) = \min\{I_t : q_t^{j*}(I_t, \theta_t) = 0\} \leq \min\{I_t : q_t^{\hat{j}*}(I_t, \theta_t) = 0\} = I_t^{q,\hat{j}}(\theta_t).$$

For the second half of part (b), the inequality $I_t^{q,1}(\theta_t) \leq I_t^{q,2}(\theta_t) \leq \cdots \leq I_t^{q,m}(\theta_t)$ follows directly from the first half. It remains to be shown that $\mathcal{M}_t^*(I_t, \theta_t) = \{j^*, j^*+1, \cdots, m\}$, where $j^* = \min\{j : I_t < I_t^{q,j}(\theta_t)\}$. Observe that $I_t < I_t^{q,j^*}(\theta_t) \leq I_t^{q,j^*+1}(\theta_t) \leq \cdots \leq I_t^{q,m}(\theta_t)$. Thus, by the definition of $I_t^{q,j}(\theta_t)$, $\{j^*, j^*+1, \cdots, m\} \subset \mathcal{M}_t^*(I_t, \theta_t)$. Moreover, by the definition of j^*, $I_t \geq I_t^{q,j^*-1}(\theta_t) \geq I_t^{q,j^*-2}(\theta_t) \geq \cdots \geq I_t^{q,1}(\theta_t)$. Thus, $j \notin \mathcal{M}_t^*(I_t, \theta_t)$ for all $j \leq j^*-1$ and, hence, $\mathcal{M}_t^*(I_t, \theta_t) = \{j^*, j^*+1, \cdots, m\}$. *Q.E.D.*

Proof of Theorem 6.4.3: We show all parts together by backward induction. More specifically, we prove that if $\partial_{I_{t-1}} V_{t-1}(I_{t-1}|\hat{\theta}_{t-1}) \geq \partial_{I_{t-1}} V_{t-1}(I_{t-1}|\theta_{t-1})$ for all I_{t-1} and $\hat{\Lambda}_{t-1} > \Lambda_{t-1}$, then we have (i) $d_t^{i*}(I_t, \hat{\theta}_t) \leq d_t^{i*}(I_t, \theta_t)$ for all $i \in \mathcal{N}$, (ii) $q_t^{j*}(I_t, \hat{\theta}_t) \geq q_t^{j*}(I_t, \theta_t)$ for all $j \in \mathcal{M}$, (iii) $x_t^*(I_t, \hat{\theta}_t) \geq x_t^*(I_t, \theta_t)$, and (vi) $\partial_{I_t} V_t(I_t|\hat{\theta}_t) \geq \partial_{I_t} V_t(I_t|\theta_t)$ for all I_t and $\hat{\Lambda}_t > \Lambda_t$. Since $\partial_{I_0} V_0(I_0|\hat{\theta}_0) = \partial_{I_0} V_0(I_0|\theta_0) = 0$ for all

I_0 and $\hat{\Lambda}_0 > \Lambda_0$, the initial condition is satisfied. Since $\partial_{I_{t-1}} V_{t-1}(I_{t-1}|\hat{\theta}_{t-1}) \geq \partial_{I_{t-1}} V_{t-1}(I_{t-1}|\theta_{t-1})$ and $\xi_t^{\Lambda,i}(\hat{\Lambda}_t^i) \geq_{s.d.} \xi_t^{\Lambda,i}(\Lambda_t^i)$ for any $i \in \mathcal{N}$, $\partial_z \Psi_t(z|\hat{\theta}_t) \geq \partial_z \Psi_t(z|\theta_t)$ for any z.

First, we show that $d_t^{i*}(I_t, \hat{\theta}_t) \leq d_t^{i*}(I_t, \theta_t)$ for all $i \in \mathcal{N}$. Without loss of generality, we assume, to the contrary, that $d_t^{1*}(I_t, \hat{\theta}_t) > d_t^{1*}(I_t, \theta_t)$. The strict concavity of $R^1(\cdot)$ implies that $\partial_{d_t^1} R^1(d_t^{1*}(I_t, \hat{\theta}_t)) < \partial_{d_t^1} R^1(d_t^{1*}(I_t, \theta_t))$. On the other hand, Lemma 16 yields that $\partial_{d_t^1} J_t(d_t^*(I_t, \hat{\theta}_t), q_t^*(I_t, \hat{\theta}_t), I_t|\hat{\theta}_t) \geq 0 \geq \partial_{d_t^1} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t)$, i.e.,
$\partial_{d_t^1} J_t(d_t^*(I_t, \hat{\theta}_t), q_t^*(I_t, \hat{\theta}_t), I_t|\hat{\theta}_t)/\hat{\Lambda}_t^1 \geq 0 \geq \partial_{d_t^1} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t)/\Lambda_t^1$. Thus,

$$\begin{aligned}\partial_z \Psi_t(\Delta_t^*(I_t, \hat{\theta}_t)|\hat{\theta}_t) =& \partial_{d_t^1} R^1(d_t^{1*}(I_t, \hat{\theta}_t)) - \frac{1}{\hat{\Lambda}_t^1} \partial_{d_t^1} J_t(d_t^*(I_t, \hat{\theta}_t), q_t^*(I_t, \hat{\theta}_t), I_t|\hat{\theta}_t) \\ <& \partial_{d_t^1} R^1(d_t^{1*}(I_t, \theta_t)) - \frac{1}{\Lambda_t^1} \partial_{d_t^1} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t) \\ =& \partial_z \Psi_t(\Delta_t^*(I_t, \theta_t)|\theta_t).\end{aligned} \tag{E.6}$$

Since $\partial_z \Psi_t(z|\hat{\theta}_t) > \partial_z \Psi_t(z|\theta_t)$ for all z and $\Psi_t(\cdot|\theta_t)$ is concave in z, $\Delta_t^*(I_t, \hat{\theta}_t) > \Delta_t^*(I_t, \theta_t)$, i.e.,

$$I_t + \sum_{j \in \mathcal{M}} q_t^{j*}(I_t, \hat{\theta}_t) - (\sum_{i \in \mathcal{N}} \hat{\Lambda}_t^i d_t^{i*}(I_t, \hat{\theta}_t)) = \Delta_t^*(I_t, \hat{\theta}_t) > \Delta_t^*(I_t, \theta_t) = I_t + \sum_{j \in \mathcal{M}} q_t^{j*}(I_t, \theta_t) - (\sum_{i \in \mathcal{N}} \Lambda_t^i d_t^{i*}(I_t, \theta_t)).$$

Since $\hat{\Lambda}_t > \Lambda_t$ and $d_t^{1*}(I_t, \hat{\theta}_t) > d_t^{1*}(I_t, \theta_t)$, either (a) $d_t^{i*}(I_t, \hat{\theta}_t) < d_t^{i*}(I_t, \theta_t)$ for some $2 \leq i \leq n$, or (b) $q_t^{j*}(I_t, \hat{\theta}_t) > q_t^{j*}(I_t, \theta_t)$ for some $1 \leq j \leq m$.

In case (a), without loss of generality, we assume that $d_t^{2*}(I_t, \hat{\theta}_t) < d_t^{2*}(I_t, \theta_t)$. Lemma 16 yields that $\partial_{d_t^2} J_t(d_t^*(I_t, \hat{\theta}_t), q_t^*(I_t, \hat{\theta}_t), I_t|\hat{\theta}_t) \leq 0 \leq \partial_{d_t^2} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t)$, i.e.,
$\partial_{d_t^2} J_t(d_t^*(I_t, \hat{\theta}_t), q_t^*(I_t, \hat{\theta}_t), I_t|\hat{\theta}_t)/\hat{\Lambda}_t^2 \leq \partial_{d_t^2} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t)/\Lambda_t^2$. Thus, by (E.6),

$$\begin{aligned}\partial_{d_t^2} R^2(d_t^{2*}(I_t, \hat{\theta}_t)) =& \frac{1}{\hat{\Lambda}_t^2} \partial_{d_t^2} J_t(d_t^*(I_t, \hat{\theta}_t), q_t^*(I_t, \hat{\theta}_t), I_t|\hat{\theta}_t) + \partial_z \Psi_t(\Delta_t^*(I_t, \hat{\theta}_t)|\hat{\theta}_t) \\ <& \frac{1}{\Lambda_t^2} \partial_{d_t^2} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t) + \partial_z \Psi_t(\Delta_t^*(I_t, \theta_t)|\theta_t) \\ =& \partial_{d_t^2} R^2(d_t^{2*}(I_t, \theta_t)),\end{aligned}$$

which contradicts the strict concavity of $R^2(\cdot)$. Hence, $d_t^{2*}(I_t, \hat{\theta}_t) \geq d_t^{2*}(I_t, \theta_t)$ under the condition that $d_t^{1*}(I_t, \hat{\theta}_t) > d_t^{1*}(I_t, \theta_t)$. It follows from the same argument that $d_t^{i*}(I_t, \hat{\theta}_t) \geq d_t^{i*}(I_t, \theta_t)$ for all $i = 2, 3, \cdots, n$, under the condition that $d_t^{1*}(I_t, \hat{\theta}_t) > d_t^{1*}(I_t, \theta_t)$.

In case (b), without loss of generality, we assume that $q_t^{1*}(I_t, \hat{\theta}_t) > q_t^{1*}(I_t, \theta_t)$. Lemma 16 yields that $\partial_{q_t^1} J_t(d_t^*(I_t, \hat{\theta}_t), q_t^*(I_t, \hat{\theta}_t), I_t|\hat{\theta}_t) \geq \partial_{q_t^1} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t)$. Thus, by (E.6),

$$\begin{aligned}\partial_{q_t^1} C^1(q_t^{1*}(I_t, \hat{\theta}_t)|c_t^1) =& \partial_z \Psi_t(\Delta_t^*(I_t, \hat{\theta}_t)|\hat{\theta}_t) - \partial_{q_t^1} J_t(d_t^*(I_t, \hat{\theta}_t), q_t^*(I_t, \hat{\theta}_t), I_t|\hat{\theta}_t) \\ <& \partial_z \Psi_t(\Delta_t^*(I_t, \theta_t)|\theta_t) - \partial_{q_t^1} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t) \\ =& \partial_{q_t^1} C^1(q_t^{1*}(I_t, \theta_t)|c_t^1),\end{aligned}$$

which contradicts the strict convexity of $C^1(\cdot|c_t^1)$ in q_t^1. Hence, $q_t^{1*}(I_t, \hat{\theta}_t) \leq q_t^{1*}(I_t, \theta_t)$ under the condition that $d_t^{1*}(I_t, \hat{\theta}_t) > d_t^{1*}(I_t, \theta_t)$. It follows from the same argument that $q_t^{j*}(I_t, \hat{\theta}_t) \leq q_t^{j*}(I_t, \theta_t)$ for all $j = 1, 2, \cdots, m$, under the condition that $d_t^{1*}(I_t, \hat{\theta}_t) > d_t^{1*}(I_t, \theta_t)$. Combining cases (a) and (b), it follows that the initial assumption $d_t^{1*}(I_t, \hat{\theta}_t) > d_t^{1*}(I_t, \theta_t)$ is incorrect. Therefore, $d_t^{1*}(I_t, \hat{\theta}_t) \leq d_t^{1*}(I_t, \theta_t)$. The same argument yields that $d_t^{i*}(I_t, \hat{\theta}_t) \leq d_t^{i*}(I_t, \theta_t)$ for any $i = 1, 2, \cdots, n$. Hence, for each $i \in \mathcal{N}$,

$$I_t^{d,i}(\hat{\theta}_t) = \max\{I_t : d_t^{i*}(I_t, \hat{\theta}_t) = 0\} \geq \max\{I_t : d_t^{i*}(I_t, \theta_t) = 0\} = I_t^{d,i}(\theta_t).$$

For any $i \in \mathcal{N}_t^*(I_t, \hat{\theta}_t)$, $d_t^{i*}(I_t, \theta_t) \geq d_t^{i*}(I_t, \hat{\theta}_t) > 0$. Thus, $i \in \mathcal{N}_t^*(I_t, \theta_t)$, and $\mathcal{N}_t^*(I_t, \hat{\theta}_t) \subset \mathcal{N}_t^*(I_t, \theta_t)$ follows immediately.

Next, we show that $q_t^{j*}(I_t, \hat{\theta}_t) \geq q_t^{j*}(I_t, \theta_t)$ for all $j \in \mathcal{M}$. We assume, to the contrary, that $q_t^{1*}(I_t, \hat{\theta}_t) < q_t^{1*}(I_t, \theta_t)$. The strict convexity of $C^1(\cdot|c_t^1)$ in q_t^1 implies that $\partial_{q_t^1} C^1(q_t^{1*}(I_t, \hat{\theta}_t)|c_t^1) < \partial_{q_t^1} C^1(q_t^{1*}(I_t, \theta_t)|c_t^1)$. On the other hand, Lemma 16 yields that
$\partial_{q_t^1} J_t(d_t^*(I_t, \hat{\theta}_t), q_t^*(I_t, \hat{\theta}_t), I_t|\hat{\theta}_t) \leq \partial_{q_t^1} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t)$. Thus,

$$\begin{aligned}
\partial_z \Psi_t(\Delta_t^*(I_t, \hat{\theta}_t)|\hat{\theta}_t) =&\partial_{q_t^1} C^1(q_t^{1*}(I_t, \hat{\theta}_t)|c_t^1) + \partial_{q_t^1} J_t(d_t^*(I_t, \hat{\theta}_t), q_t^*(I_t, \hat{\theta}_t), I_t|\hat{\theta}_t) \\
<&\partial_{q_t^1} C^1(q_t^{1*}(I_t, \theta_t)|c_t^1) + \partial_{q_t^1} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t) \\
=&\partial_z \Psi_t(\Delta_t^*(I_t, \theta_t)|\theta_t).
\end{aligned} \tag{E.7}$$

Since $\partial_z \Psi_t(z|\hat{\theta}_t) > \partial_z \Psi_t(z|\theta_t)$ for all z and $\Psi_t(\cdot|\theta_t)$ is concave in z, $\Delta_t^*(I_t, \hat{\theta}_t) > \Delta_t^*(I_t, \theta_t)$, i.e.,

$$I_t + \sum_{j \in \mathcal{M}} q_t^{j*}(I_t, \hat{\theta}_t) - (\sum_{i \in \mathcal{N}} \hat{\Lambda}_t^i d_t^{i*}(I_t, \hat{\theta}_t)) = \Delta_t^*(I_t, \hat{\theta}_t) > \Delta_t^*(I_t, \theta_t) = I_t + \sum_{j \in \mathcal{M}} q_t^{j*}(I_t, \theta_t) - (\sum_{i \in \mathcal{N}} \Lambda_t^i d_t^{i*}(I_t, \theta_t)).$$

Since $\hat{\Lambda}_t > \Lambda_t$, and $q_t^{1*}(I_t, \hat{\theta}_t) < q_t^{1*}(I_t, \theta_t)$, either (a) $d_t^{i*}(I_t, \hat{\theta}_t) < d_t^{i*}(I_t, \theta_t)$ for some $1 \leq i \leq n$, or (b) $q_t^{j*}(I_t, \hat{\theta}_t) > q_t^{j*}(I_t, \theta_t)$ for some $2 \leq j \leq m$.

In case (a), without loss of generality, we assume that $d_t^{1*}(I_t, \hat{\theta}_t) < d_t^{1*}(I_t, \theta_t)$. Lemma 16 yields that
$\partial_{d_t^1} J_t(d_t^*(I_t, \hat{\theta}_t), q_t^*(I_t, \hat{\theta}_t), I_t|\hat{\theta}_t) \leq 0 \leq \partial_{d_t^1} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t)$, i.e.,
$\partial_{d_t^1} J_t(d_t^*(I_t, \hat{\theta}_t), q_t^*(I_t, \hat{\theta}_t), I_t|\hat{\theta}_t)/\hat{\Lambda}_t^1 \leq \partial_{d_t^1} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t)/\Lambda_t^1$. Thus, by (E.7),

$$\begin{aligned}
\partial_{d_t^1} R^1(d_t^{1*}(I_t, \hat{\theta}_t)) =&\frac{1}{\hat{\Lambda}_t^1} \partial_{d_t^1} J_t(d_t^*(I_t, \hat{\theta}_t), q_t^*(I_t, \hat{\theta}_t), I_t|\hat{\theta}_t) + \partial_z \Psi_t(\Delta_t^*(I_t, \hat{\theta}_t)|\hat{\theta}_t) \\
<&\frac{1}{\Lambda_t^1} \partial_{d_t^1} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t) + \partial_z \Psi_t(\Delta_t^*(I_t, \theta_t)|\theta_t) \\
=&\partial_{d_t^1} R^1(d_t^{1*}(I_t, \theta_t)),
\end{aligned}$$

which contradicts the strict concavity of $R^1(\cdot)$. Hence, $d_t^{1*}(I_t, \hat{\theta}_t) \geq d_t^{1*}(I_t, \theta_t)$ under the condition that $q_t^{1*}(I_t, \hat{\theta}_t) < q_t^{1*}(I_t, \theta_t)$. It follows from the same argument that $d_t^{i*}(I_t, \hat{\theta}_t) \geq d_t^{i*}(I_t, \theta_t)$ for all $i = 1, 2, \cdots, n$, under the condition that $q_t^{1*}(I_t, \hat{\theta}_t) < q_t^{1*}(I_t, \theta_t)$. Thus, under this condition, $d_t^{i*}(I_t, \hat{\theta}_t) = d_t^{i*}(I_t, \theta_t)$ for all $i = 1, 2, \cdots, n$.

In case (b), without loss of generality, we assume that $q_t^{2*}(I_t, \hat{\theta}_t) > q_t^{2*}(I_t, \theta_t)$. Lemma 16 yields that
$\partial_{q_t^2} J_t(d_t^*(I_t, \hat{\theta}_t), q_t^*(I_t, \hat{\theta}_t), I_t|\hat{\theta}_t) \geq \partial_{q_t^2} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t)$. Thus, by (E.7),

$$\begin{aligned}
\partial_{q_t^2} C^2(q_t^{2*}(I_t, \hat{\theta}_t)|c_t^2) =&\partial_z \Psi_t(\Delta_t^*(I_t, \hat{\theta}_t)|\hat{\theta}_t) - \partial_{q_t^2} J_t(d_t^*(I_t, \hat{\theta}_t), q_t^*(I_t, \hat{\theta}_t), I_t|\hat{\theta}_t) \\
<&\partial_z \Psi_t(\Delta_t^*(I_t, \theta_t)|\theta_t) - \partial_{q_t^2} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t) \\
=&\partial_{q_t^2} C^2(q_t^{2*}(I_t, \theta_t)|c_t^2),
\end{aligned}$$

which contradicts the strict convexity of $C^2(\cdot|c_t^2)$ in q_t^2. Hence, $q_t^{2*}(I_t, \hat{\theta}_t) \leq q_t^{2*}(I_t, \theta_t)$ under the condition that $q_t^{1*}(I_t, \hat{\theta}_t) < q_t^{1*}(I_t, \theta_t)$. It follows from the same argument that $q_t^{j*}(I_t, \hat{\theta}_t) \leq q_t^{j*}(I_t, \theta_t)$ for all $j = 2, \cdots, m$, under the condition that $q_t^{1*}(I_t, \hat{\theta}_t) < q_t^{1*}(I_t, \theta_t)$. Combining cases (a) and (b), it follows that the initial assumption $q_t^{1*}(I_t, \hat{\theta}_t) < q_t^{1*}(I_t, \theta_t)$ is incorrect. Therefore, $q_t^{1*}(I_t, \hat{\theta}_t) \geq q_t^{1*}(I_t, \theta_t)$. The same argument yields that $q_t^{j*}(I_t, \hat{\theta}_t) \geq q_t^{j*}(I_t, \theta_t)$ for any $j = 1, 2, \cdots, m$. Hence, for each $j \in \mathcal{M}$,

$$I_t^{q,j}(\hat{\theta}_t) = \min\{I_t : q_t^{j*}(I_t, \hat{\theta}_t) = 0\} \geq \min\{I_t : q_t^{j*}(I_t, \theta_t) = 0\} = I_t^{q,j}(\theta_t).$$

For any $j \in \mathcal{M}_t^*(I_t,\theta_t)$, $q_t^{j*}(I_t,\hat{\theta}_t) \geq q_t^{j*}(I_t,\theta_t) > 0$. Thus, $j \in \mathcal{M}_t^*(I_t,\hat{\theta}_t)$, and $\mathcal{M}_t^*(I_t,\theta_t) \subset \mathcal{M}_t^*(I_t,\hat{\theta}_t)$ follows immediately.

Finally, to complete the induction, we show that $\partial_{I_t} V_t(I_t|\hat{\theta}_t) \geq \partial_{I_t} V_t(I_t|\theta_t)$. Recall that $d_t^{i*}(I_t,\hat{\theta}_t) \leq d_t^{i*}(I_t,\theta_t)$ for any $i \in \mathcal{N}$. If $d_t^{1*}(I_t,\hat{\theta}_t) < d_t^{1*}(I_t,\theta_t)$, the strict concavity of $R^1(\cdot)$ implies that $\partial_{d_t^1} R^1(d_t^{1*}(I_t,\hat{\theta}_t)) > \partial_{d_t^1} R^1(d_t^{1*}(I_t,\theta_t))$. On the other hand, Lemma 16 yields that $\partial_{d_t^1} J_t(d_t^*(I_t,\hat{\theta}_t), q_t^*(I_t,\hat{\theta}_t), I_t|\hat{\theta}_t) \leq 0 \leq \partial_{d_t^1} J_t(d_t^*(I_t,\theta_t), q_t^*(I_t,\theta_t), I_t|\theta_t)$, i.e., $\partial_{d_t^1} J_t(d_t^*(I_t,\hat{\theta}_t), q_t^*(I_t,\hat{\theta}_t), I_t|\hat{\theta}_t)/\hat{\Lambda}_t^1 \leq \partial_{d_t^1} J_t(d_t^*(I_t,\theta_t), q_t^*(I_t,\theta_t), I_t|\theta_t)/\Lambda_t^1$. Thus,

$$\begin{aligned}
\partial_z \Psi_t(\Delta_t^*(I_t,\hat{\theta}_t)|\hat{\theta}_t) =& \partial_{d_t^1} R^1(d_t^{1*}(I_t,\hat{\theta}_t)) - \frac{1}{\hat{\Lambda}_t^1} \partial_{d_t^1} J_t(d_t^*(I_t,\hat{\theta}_t), q_t^*(I_t,\hat{\theta}_t), I_t|\hat{\theta}_t) \\
>& \partial_{d_t^1} R^1(d_t^{1*}(I_t,\theta_t)) - \frac{1}{\Lambda_t^1} \partial_{d_t^1} J_t(d_t^*(I_t,\theta_t), q_t^*(I_t,\theta_t), I_t|\theta_t) \\
=& \partial_z \Psi_t(\Delta_t^*(I_t,\theta_t)|\theta_t).
\end{aligned}$$

By Equation (E.1),

$$\partial_{I_t} V_t(I_t|\hat{\theta}_t) = \partial_z \Psi_t(\Delta_t^*(I_t,\hat{\theta}_t)|\hat{\theta}_t) > \partial_z \Psi_t(\Delta_t^*(I_t,\theta_t)|\theta_t) = \partial_{I_t} V_t(I_t|\theta_t).$$

The same argument implies that, if there exists an $i \in \mathcal{N}$, such that $d_t^{i*}(I_t,\hat{\theta}_t) < d_t^{i*}(I_t,\theta_t)$, we have $\partial_{I_t} V_t(I_t|\hat{\theta}_t) > \partial_{I_t} V_t(I_t|\theta_t)$.

Recall that $q_t^{j*}(I_t,\hat{\theta}_t) \geq q_t^{j*}(I_t,\theta_t)$ for any $j \in \mathcal{M}$. If $q_t^{1*}(I_t,\hat{\theta}_t) > q_t^{1*}(I_t,\theta_t)$, the strict convexity of $C^1(\cdot|c_t^1)$ in q_t^1 implies that $\partial_{q_t^1} C^1(q_t^{1*}(I_t,\hat{\theta}_t)|c_t^1) > \partial_{q_t^1} C^1(q_t^{1*}(I_t,\theta_t)|c_t^1)$. On the other hand, Lemma 16 yields that $\partial_{q_t^1} J_t(d_t^*(I_t,\hat{\theta}_t), q_t^*(I_t,\hat{\theta}_t), I_t|\hat{\theta}_t) \geq \partial_{q_t^1} J_t(d_t^*(I_t,\theta_t), q_t^*(I_t,\theta_t), I_t|\theta_t)$. Thus,

$$\begin{aligned}
\partial_z \Psi_t(\Delta_t^*(I_t,\hat{\theta}_t)|\hat{\theta}_t) =& \partial_{q_t^1} C^1(q_t^{1*}(I_t,\hat{\theta}_t)|c_t^1) + \partial_{q_t^1} J_t(d_t^*(I_t,\hat{\theta}_t), q_t^*(I_t,\hat{\theta}_t), I_t|\hat{\theta}_t) \\
>& \partial_{q_t^1} C^1(d_t^{1*}(I_t,\theta_t)|c_t^1) + \partial_{q_t^1} J_t(d_t^*(I_t,\theta_t), q_t^*(I_t,\theta_t), I_t|\theta_t) \\
=& \partial_z \Psi_t(\Delta_t^*(I_t,\theta_t)|\theta_t).
\end{aligned}$$

By Equation (E.1),

$$\partial_{I_t} V_t(I_t|\hat{\theta}_t) = \partial_z \Psi_t(\Delta_t^*(I_t,\hat{\theta}_t)|\hat{\theta}_t) > \partial_z \Psi_t(\Delta_t^*(I_t,\theta_t)|\theta_t) = \partial_{I_t} V_t(I_t|\theta_t).$$

The same argument implies that, if there exists a $j \in \mathcal{M}$, such that $q_t^{j*}(I_t,\hat{\theta}_t) < q_t^{j*}(I_t,\theta_t)$, we have $\partial_{I_t} V_t(I_t|\hat{\theta}_t) > \partial_{I_t} V_t(I_t|\theta_t)$.

Now we assume that for any $i \in \mathcal{N}$ and $j \in \mathcal{M}$, $d_t^{i*}(I_t,\hat{\theta}_t) = d_t^{i*}(I_t,\theta_t)$ and $q_t^{j*}(I_t,\hat{\theta}_t) = q_t^{j*}(I_t,\theta_t)$. Since $\hat{\Lambda}_t > \Lambda_t$,

$$\Delta_t^*(I_t,\hat{\theta}_t) = I_t + \sum_{j\in\mathcal{M}} q_t^{j*}(I_t,\hat{\theta}_t) - (\sum_{i\in\mathcal{N}} \hat{\Lambda}_t^i d_t^{i*}(I_t,\hat{\theta}_t)) \leq I_t + \sum_{j\in\mathcal{M}} q_t^{j*}(I_t,\theta_t) - (\sum_{i\in\mathcal{N}} \Lambda_t^i d_t^{i*}(I_t,\theta_t)) = \Delta_t^*(I_t,\theta_t).$$

Since $\partial_z \Psi_t(z|\hat{\theta}_t) \geq \partial_z \Psi_t(z|\theta_t)$ for any z, it follows that $\partial_z \Psi_t(\Delta_t^*(I_t,\hat{\theta}_t)|\hat{\theta}_t) \geq \partial_z \Psi_t(\Delta_t^*(I_t,\theta_t)|\theta_t)$ by the concavity of $\Psi_t(\cdot|\theta_t)$ in z. Thus, by Equation (E.1),

$$\partial_{I_t} V_t(I_t|\hat{\theta}_t) = \partial_z \Psi_t(\Delta_t^*(I_t,\hat{\theta}_t)|\hat{\theta}_t) \geq \partial_z \Psi_t(\Delta_t^*(I_t,\theta_t)|\theta_t) = \partial_{I_t} V_t(I_t|\theta_t).$$

This completes the induction and, thus, the proof of Theorem 6.4.3. *Q.E.D.*

Proof of Theorem 6.4.4: We show all parts together by backward induction. More specifically, we prove that if $\partial_{I_{t-1}} V_{t-1}(I_{t-1}|\hat{\theta}_{t-1}) \geq \partial_{I_{t-1}} V_{t-1}(I_{t-1}|\theta_{t-1})$ for all I_{t-1} and $\hat{c}_{t-1} > c_{t-1}$, then we have (i) $d_t^{i*}(I_t, \hat{\theta}_t) \leq d_t^{i*}(I_t, \theta_t)$ for all $i \in \mathcal{N}$, (ii) $q_t^{j*}(I_t, \hat{\theta}_t) \geq q_t^{j*}(I_t, \theta_t)$ for all $\{j \in \mathcal{M} : \hat{c}_t^j = c_t^j\}$, and (iii) $\partial_{I_t} V_t(I_t|\hat{\theta}_t) \geq \partial_{I_t} V_t(I_t|\theta_t)$ for all I_t and $\hat{c}_t > c_t$. Since $\partial_{I_0} V_0(I_0|\hat{\theta}_0) = \partial_{I_0} V_0(I_0|\theta_0) = 0$ for all I_0 and $\hat{c}_0 > c_0$, the initial condition is satisfied. Since $\partial_{I_{t-1}} V_{t-1}(I_{t-1}|\hat{\theta}_{t-1}) \geq \partial_{I_{t-1}} V_{t-1}(I_{t-1}|\theta_{t-1})$ and $\xi_t^{c,j}(\hat{c}_t^j) \geq_{s.d.} \xi_t^{c,j}(c_t^j)$ for any $j \in \mathcal{M}$, $\partial_z \Psi_t(z|\hat{\theta}_t) \geq \partial_z \Psi_t(z|\theta_t)$ for any z.

First, we show (i) and (ii). Without loss of generality, we assume that $\hat{c}_t^j > c_t^j$ for $j = 1, 2, \cdots, m_1$ ($1 \leq m_1 \leq m$) and $\hat{c}_t^j = c_t^j$ otherwise. Invoke Lemma 15 with $p = n + m - m_1$, $q = m_1$, $\Gamma = \{0, 1\}$, $y_i = -d_t^i$ ($1 \leq i \leq n$), $y_{n+j} = q_t^{m_1+j}$ ($1 \leq j \leq m - m_1$), $y_{n+m-m_1+j} = q_t^j$ ($1 \leq j \leq m_1$), $\lambda_i = \Lambda_t^i$ ($1 \leq i \leq n$), $\lambda_i = 1$ ($n + 1 \leq i \leq n + m$), $f_i(y_i) = \Lambda_t^i R^i(d_t^i)$ ($1 \leq i \leq n$), $f_{j+n}(y_{j+n}) = -C^{j+m_1}(q_t^{j+m_1}|c_t^{j+m_1})$ ($1 \leq j \leq m - m_1$), $g_{j+n+m-m_1}(y_{j+n+m-m_1}|\gamma) = \begin{cases} -C^j(q_t^j|c_t^j), & \text{if } \gamma = 0, \\ -C^j(q_t^j|\hat{c}_t^j), & \text{if } \gamma = 1, \end{cases}$ ($1 \leq j \leq m_1$), $h(y_0|\gamma) = \begin{cases} \Psi_t(I_t + y_0|\theta_t), & \text{if } \gamma = 0, \\ \Psi_t(I_t + y_0|\hat{\theta}_t), & \text{if } \gamma = 1, \end{cases}$ and $\mathcal{Y}_i = \begin{cases} [-d_{\max}, 0], & \text{if } 1 \leq i \leq n, \\ [0, +\infty), & \text{if } n + 1 \leq i \leq n + m. \end{cases}$ Since $C^j(q_t^j|c_t^j)$ is supermodular in (q_t^j, c_t^j) for any $1 \leq j \leq m$, and $\Psi_t(I_t + y_0|\theta_t)$ is supermodular in (y_0, c_t^j), $g_{j+p}(y_j|\gamma)$ ($1 \leq j \leq q$) is submodular in (y_j, γ), and $h(y_0|\gamma)$ is supermodular in (y_0, γ). Lemma 15 implies that $d_t^{i*}(I_t, \hat{\theta}_t) \leq d_t^{i*}(I_t, \theta_t)$ for all $i \in \mathcal{N}$, and $q_t^{j*}(I_t, \hat{\theta}_t) \geq q_t^{j*}(I_t, \theta_t)$ for all $\{j \in \mathcal{M} : \hat{c}_t^j = c_t^j\}$. For any $i \in \mathcal{N}_t^*(I_t, \hat{\theta}_t)$, $d_t^{i*}(I_t, \theta_t) \geq d_t^{i*}(I_t, \hat{\theta}_t) > 0$. Thus, $i \in \mathcal{N}_t^*(I_t, \theta_t)$, and $\mathcal{N}_t^*(I_t, \hat{\theta}_t) \subset \mathcal{N}_t^*(I_t, \theta_t)$ follows immediately.

To complete the induction, we show that $\partial_{I_t} V_t(I_t|\hat{\theta}_t) \geq \partial_{I_t} V_t(I_t|\theta_t)$. If $d_t^{1*}(I_t, \hat{\theta}_t) < d_t^{1*}(I_t, \theta_t)$, the strict concavity of $R^1(\cdot)$ implies that $\partial_{d_t^1} R^1(d_t^{1*}(I_t, \hat{\theta}_t)) > \partial_{d_t^1} R^1(d_t^{1*}(I_t, \theta_t))$. On the other hand, Lemma 16 yields that
$\partial_{d_t^1} J_t(d_t^*(I_t, \hat{\theta}_t), q_t^*(I_t, \hat{\theta}_t), I_t|\hat{\theta}_t) \leq \partial_{d_t^1} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t)$. Thus,

$$\begin{aligned}
\partial_z \Psi_t(\Delta_t^*(I_t, \hat{\theta}_t)|\hat{\theta}_t) =& \partial_{d_t^1} R^1(d_t^{1*}(I_t, \hat{\theta}_t)) - \frac{1}{\Lambda_t^1} \partial_{d_t^1} J_t(d_t^*(I_t, \hat{\theta}_t), q_t^*(I_t, \hat{\theta}_t), I_t|\hat{\theta}_t) \\
>& \partial_{d_t^1} R^1(d_t^{1*}(I_t, \theta_t)) - \frac{1}{\Lambda_t^1} \partial_{d_t^1} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t) \\
=& \partial_z \Psi_t(\Delta_t^*(I_t, \theta_t)|\theta_t).
\end{aligned}$$

By Equation (E.1),

$$\partial_{I_t} V_t(I_t|\hat{\theta}_t) = \partial_z \Psi_t(\Delta_t^*(I_t, \hat{\theta}_t)|\hat{\theta}_t) > \partial_z \Psi_t(\Delta_t^*(I_t, \theta_t)|\theta_t) = \partial_{I_t} V_t(I_t|\theta_t).$$

The same argument implies that, if there exists an $i \in \mathcal{N}$, such that $d_t^{i*}(I_t, \hat{\theta}_t) < d_t^{i*}(I_t, \theta_t)$, then we have $\partial_{I_t} V_t(I_t|\hat{\theta}_t) > \partial_{I_t} V_t(I_t|\theta_t)$. Now we assume that for all $i \in \mathcal{N}$, $d_t^{i*}(I_t, \hat{\theta}_t) = d_t^{i*}(I_t, \theta_t)$.

If $q_t^{j*}(I_t, \hat{\theta}_t) \leq q_t^{j*}(I_t, \theta_t)$ for all $j \in \mathcal{M}$,

$$\Delta_t^*(I_t, \hat{\theta}_t) = I_t + \sum_{j \in \mathcal{M}} q_t^{j*}(I_t, \hat{\theta}_t) - (\sum_{i \in \mathcal{N}} \Lambda_t^i d_t^{i*}(I_t, \hat{\theta}_t)) \leq I_t + \sum_{j \in \mathcal{M}} q_t^{j*}(I_t, \theta_t) - (\sum_{i \in \mathcal{N}} \Lambda_t^i d_t^{i*}(I_t, \theta_t)) = \Delta_t^*(I_t, \theta_t).$$

Since $\partial_z \Psi_t(z|\hat{\theta}_t) \geq \partial_z \Psi_t(z|\theta_t)$ for any z, the concavity of $\Psi_t(\cdot|\theta_t)$ in z implies that $\partial_z \Psi_t(\Delta_t^*(I_t, \hat{\theta}_t)|\hat{\theta}_t) \geq \partial_z \Psi_t(\Delta_t^*(I_t, \theta_t)|\theta_t)$. Thus,

$$\partial_{I_t} V_t(I_t|\hat{\theta}_t) = \partial_z \Psi_t(\Delta_t^*(I_t, \hat{\theta}_t)|\hat{\theta}_t) \geq \partial_z \Psi_t(\Delta_t^*(I_t, \theta_t)|\theta_t) = \partial_{I_t} V_t(I_t|\theta_t).$$

In the remaining case, $q_t^{j*}(I_t, \hat{\theta}_t) > q_t^{j*}(I_t, \theta_t)$ for some $1 \le j \le m$, Without loss of generality, assume that $q_t^{l*}(I_t, \hat{\theta}_t) > q_t^{l*}(I_t, \theta_t)$. In this case, the supermodularity of $C^l(\cdot|\cdot)$ in (q_t^l, c_t^l) and the strict convexity of $C^l(\cdot|c_t^l)$ in q_t^l imply that $\partial_{q_t^l} C^l(q_t^{l*}(I_t, \hat{\theta}_t)|\hat{c}_t^l) > \partial_{q_t^l} C^l(q_t^{l*}(I_t, \theta_t)|c_t^l)$. On the other hand, Lemma 16 implies that $\partial_{q_t^l} J_t(d_t^*(I_t, \hat{\theta}_t), q_t^*(I_t, \hat{\theta}_t), I_t|\hat{\theta}_t) \ge \partial_{q_t^l} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t)$. Thus,

$$\begin{aligned}\partial_z \Psi_t(\Delta_t^*(I_t, \hat{\theta}_t)|\hat{\theta}_t) =& \partial_{q_t^l} C^1(q_t^{l*}(I_t, \hat{\theta}_t)|\hat{c}_t^l) + \partial_{q_t^l} J_t(d_t^*(I_t, \hat{\theta}_t), q_t^*(I_t, \hat{\theta}_t), I_t|\hat{\theta}_t) \\ >& \partial_{q_t^l} C^l(q_t^{l*}(I_t, \theta_t)|c_t^l) + \partial_{q_t^l} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t) \\ =& \partial_z \Psi_t(\Delta_t^*(I_t, \theta_t)|\theta_t).\end{aligned}$$

Thus, by Equation (E.1),

$$\partial_{I_t} V_t(I_t|\hat{\theta}_t) = \partial_z \Psi_t(\Delta_t^*(I_t, \hat{\theta}_t)|\hat{\theta}_t) > \partial_z \Psi_t(\Delta_t^*(I_t, \theta_t)|\theta_t) = \partial_{I_t} V_t(I_t|\theta_t).$$

This completes the induction and, thus, the proof of Theorem 6.4.4. *Q.E.D.*

Proof of Theorem 6.4.5: We show all parts together by backward induction. More specifically, we prove that if $\partial_{I_{t-1}} \hat{V}_{t-1}(I_{t-1}|\theta_{t-1}) \ge \partial_{I_{t-1}} V_{t-1}(I_{t-1}|\theta_{t-1})$ for all I_{t-1}, then we have (i) $\hat{d}_t^{i*}(I_t, \theta_t) \le d_t^{i*}(I_t, \theta_t)$ for all $i \in \mathcal{N}$, (ii) $\hat{q}_t^{j*}(I_t, \theta_t) \ge q_t^{j*}(I_t, \theta_t)$ for all $j \in \mathcal{M}$, (iii) $\hat{x}_t^*(I_t, \theta_t) \ge x_t^*(I_t, \theta_t)$, (iv) $\hat{\Delta}_t^*(I_t, \theta_t) \ge \Delta_t^*(I_t, \theta_t)$, and (v) $\partial_{I_t} \hat{V}_t(I_t|\theta_t) \ge \partial_{I_t} V_t(I_t|\theta_t)$ for all I_t. Note that $\partial_{I_0} \hat{V}_0(I_0|\theta_0) = \partial_{I_0} V_0(I_0|\theta_0)$ for all I_0, so the initial condition is satisfied. Since $\partial_{I_{t-1}} \hat{V}_{t-1}(I_{t-1}|\theta_{t-1}) \ge \partial_{I_{t-1}} V_{t-1}(I_{t-1}|\theta_{t-1})$ and $\hat{\xi}_t^{\Lambda,i}(\Lambda_t^i) \ge_{s.d.} \xi_t^{\Lambda,i}(\Lambda_t^i)$ for any i and Λ_t, by Theorem 6.4.3(a), $\partial_z \hat{\Psi}_t(z|\theta_t) \ge \partial_z \Psi_t(z|\theta_t)$ for any z.

First, we show that $\hat{d}_t^{i*}(I_t, \theta_t) \le d_t^{i*}(I_t, \theta_t)$ for any $i \in \mathcal{N}$ and $\hat{q}_t^{j*}(I_t, \theta_t) \ge q_t^{j*}(I_t, \theta_t)$ for any $j \in \mathcal{M}$. We apply Lemma 15 to prove these results. Let $p = n + m$, $q = 0$, $\Gamma = \{0, 1\}$, $y_i = -d_t^i$ $(1 \le i \le n)$, $y_{j+n} = q_t^j$ $(1 \le j \le m)$, $\lambda_i = \Lambda_t^i$ $(1 \le i \le n)$, $\lambda_i = 1$ $(n + 1 \le i \le n + m)$, $f_i(y_i) = \Lambda_t^i R^i(d_t^i)$ $(1 \le i \le n)$, $f_{j+n}(y_{j+n}) = -C^j(q_t^j|c_t^j)$ $(1 \le j \le m)$, $h(y_0|0) = \Psi_t(I_t + y_0|\theta_t)$, $h(y_0|1) = \hat{\Psi}_t(I_t + y_0|\theta_t)$, and $\mathcal{Y}_i = \begin{cases} [-d_{\max}, 0], \ 1 \le i \le n, \\ [0, +\infty), \ n + 1 \le i \le n + m. \end{cases}$ Since $\partial_z \hat{\Psi}_t(z|\theta_t) \ge \partial_z \Psi_t(z|\theta_t)$ for any z, $h(y_0|\gamma)$ is supermodular in (y_0, γ). Lemma 15 implies that $\hat{d}_t^{i*}(I_t, \theta_t) \le d_t^{i*}(I_t, \theta_t)$ for any $i \in \mathcal{N}$ and $\hat{q}_t^{j*}(I_t, \theta_t) \ge q_t^{j*}(I_t, \theta_t)$ for any $j \in \mathcal{M}$. For any $i \in \hat{\mathcal{N}}_t^*(I_t, \theta_t)$, $d_t^{i*}(I_t, \theta_t) \ge \hat{d}_t^{i*}(I_t, \theta_t) > 0$. Thus, $i \in \mathcal{N}_t^*(I_t, \theta_t)$, and $\hat{\mathcal{N}}_t^*(I_t, \theta_t) \subset \mathcal{N}_t^*(I_t, \theta_t)$ follows immediately. For any $j \in \mathcal{M}_t^*(I_t, \theta_t)$, $\hat{q}_t^{j*}(I_t, \theta_t) \ge q_t^{j*}(I_t, \theta_t) > 0$. Thus, $j \in \hat{\mathcal{M}}_t^*(I_t, \theta_t)$, and $\mathcal{M}_t^*(I_t, \theta_t) \subset \hat{\mathcal{M}}_t^*(I_t, \theta_t)$ follows immediately.

Moreover,

$$\hat{x}_t^*(I_t, \theta_t) = I_t + \sum_{j \in \mathcal{M}} \hat{q}_t^{j*}(I_t, \theta_t) \ge I_t + \sum_{j \in \mathcal{M}} q_t^{j*}(I_t, \theta_t) = x_t^*(I_t, \theta_t),$$

and

$$\hat{\Delta}_t^*(I_t, \theta_t) = \hat{x}_t^*(I_t, \theta_t) - (\sum_{l \in \mathcal{N}} \Lambda_t^l \hat{d}_t^{l*}(I_t, \theta_t)) \ge x_t^*(I_t, \theta_t) - (\sum_{l \in \mathcal{N}} \Lambda_t^l d_t^{l*}(I_t, \theta_t)) = \Delta_t^*(I_t, \theta_t).$$

To complete the induction, we show that $\partial_{I_t} \hat{V}_t(I_t|\theta_t) \ge \partial_{I_t} V_t(I_t|\theta_t)$. Recall that $\hat{d}_t^{i*}(I_t, \theta_t) \le d_t^{i*}(I_t, \theta_t)$ for any $i \in \mathcal{N}$. If $\hat{d}_t^{1*}(I_t, \theta_t) < d_t^{1*}(I_t, \theta_t)$, the strict concavity of $R^1(\cdot)$ implies that

$\partial_{d_t^1} R^1(\hat{d}_t^{1*}(I_t,\theta_t)) > \partial_{d_t^1} R^1(d_t^{1*}(I_t,\theta_t))$. On the other hand, Lemma 16 yields that $\partial_{d_t^1} \hat{J}_t(\hat{d}_t^*(I_t,\theta_t), \hat{q}_t^*(I_t,\theta_t), I_t|\theta_t) \leq \partial_{d_t^1} J_t(d_t^*(I_t,\theta_t), q_t^*(I_t,\theta_t), I_t|\theta_t)$. Thus,

$$\begin{aligned}\partial_z \hat{\Psi}_t(\hat{\Delta}_t^*(I_t,\theta_t)|\theta_t) =& \partial_{d_t^1} R^1(\hat{d}_t^{1*}(I_t,\theta_t)) - \frac{1}{\Lambda_t^1}\partial_{d_t^1} \hat{J}_t(\hat{d}_t^*(I_t,\theta_t), \hat{q}_t^*(I_t,\theta_t), I_t|\theta_t)\\ >& \partial_{d_t^1} R^1(d_t^{1*}(I_t,\theta_t)) - \frac{1}{\Lambda_t^1}\partial_{d_t^1} J_t(d_t^*(I_t,\theta_t), q_t^*(I_t,\theta_t), I_t|\theta_t)\\ =& \partial_z \Psi_t(\Delta_t^*(I_t,\theta_t)|\theta_t).\end{aligned}$$

By Equation (E.1),

$$\partial_{I_t} \hat{V}_t(I_t|\theta_t) = \partial_z \hat{\Psi}_t(\Delta_t^*(I_t,\theta_t)|\theta_t) > \partial_z \Psi_t(\Delta_t^*(I_t,\theta_t)|\theta_t) = \partial_{I_t} V_t(I_t|\theta_t).$$

The same argument implies that, if there exists an $i \in \mathcal{N}$, such that $\hat{d}_t^{i*}(I_t,\theta_t) < d_t^{i*}(I_t,\theta_t)$, we have $\partial_{I_t} \hat{V}_t(I_t|\theta_t) > \partial_{I_t} V_t(I_t|\theta_t)$.

Recall that $\hat{q}_t^{j*}(I_t,\theta_t) \geq q_t^{j*}(I_t,\theta_t)$ for any $j \in \mathcal{M}$. If $\hat{q}_t^{1*}(I_t,\theta_t) > q_t^{1*}(I_t,\theta_t)$, the strict convexity of $C^1(\cdot|c_t^1)$ in q_t^1 implies that $\partial_{q_t^1} C^1(\hat{q}_t^{1*}(I_t,\theta_t)|c_t^1) > \partial_{q_t^1} C^1(q_t^{1*}(I_t,\theta_t)|c_t^1)$. On the other hand, Lemma 16 yields that $\partial_{q_t^1} \hat{J}_t(\hat{d}_t^*(I_t,\theta_t), \hat{q}_t^*(I_t,\theta_t), I_t|\theta_t) \geq \partial_{q_t^1} J_t(d_t^*(I_t,\theta_t), q_t^*(I_t,\theta_t), I_t|\theta_t)$. Thus,

$$\begin{aligned}\partial_z \hat{\Psi}_t(\hat{\Delta}_t^*(I_t,\theta_t)|\theta_t) =& \partial_{q_t^1} C^1(\hat{q}_t^{1*}(I_t,\theta_t)|c_t^1) + \partial_{q_t^1} \hat{J}_t(\hat{d}_t^*(I_t,\theta_t), \hat{q}_t^*(I_t,\theta_t), I_t|\theta_t)\\ >& \partial_{q_t^1} C^1(d_t^{1*}(I_t,\theta_t)|c_t^1) + \partial_{q_t^1} J_t(d_t^*(I_t,\theta_t), q_t^*(I_t,\theta_t), I_t|\theta_t)\\ =& \partial_z \Psi_t(\Delta_t^*(I_t,\theta_t)|\theta_t).\end{aligned}$$

By Equation (E.1),

$$\partial_{I_t} \hat{V}_t(I_t|\theta_t) = \partial_z \hat{\Psi}_t(\hat{\Delta}_t^*(I_t,\theta_t)|\theta_t) > \partial_z \Psi_t(\Delta_t^*(I_t,\theta_t)|\theta_t) = \partial_{I_t} V_t(I_t|\theta_t).$$

The same argument implies that, if there exists a $j \in \mathcal{M}$ such that $\hat{q}_t^{j*}(I_t,\theta_t) > q_t^{j*}(I_t,\theta_t)$, we have $\partial_{I_t} \hat{V}_t(I_t|\theta_t) > \partial_{I_t} V_t(I_t|\theta_t)$.

In the remaining case, $\hat{d}_t^{i*}(I_t,\theta_t) = d_t^{i*}(I_t,\theta_t)$ and $\hat{q}_t^{j*}(I_t,\theta_t) = q_t^{j*}(I_t,\theta_t)$ for any $i \in \mathcal{N}$ and $j \in \mathcal{M}$. We have

$$\hat{\Delta}_t^*(I_t,\theta_t) = I_t + \sum_{j\in\mathcal{M}} \hat{q}_t^{j*}(I_t,\theta_t) - (\sum_{i\in\mathcal{N}} \Lambda_t^i \hat{d}_t^{i*}(I_t,\theta_t)) = I_t + \sum_{j\in\mathcal{M}} q_t^{j*}(I_t,\theta_t) - (\sum_{i\in\mathcal{N}} \Lambda_t^i d_t^{i*}(I_t,\theta_t)) = \Delta_t^*(I_t,\theta_t).$$

Since $\partial_z \hat{\Psi}_t(z|\theta_t) \geq \partial_z \Psi_t(z|\theta_t)$ for any z, $\partial_z \hat{\Psi}_t(\hat{\Delta}_t^*(I_t,\theta_t)|\theta_t) \geq \partial_z \Psi_t(\Delta_t^*(I_t,\theta_t)|\theta_t)$. Thus, by Equation (E.1),

$$\partial_{I_t} \hat{V}_t(I_t|\theta_t) = \partial_z \hat{\Psi}_t(\hat{\Delta}_t^*(I_t,\theta_t)|\theta_t) \geq \partial_z \Psi_t(\Delta_t^*(I_t,\theta_t)|\theta_t) = \partial_{I_t} V_t(I_t|\theta_t).$$

This completes the induction and, thus, the proof of Theorem 6.4.5. *Q.E.D.*

Proof of Theorem 6.4.6: The proof of Theorem 6.4.6 follows from similar argument to that of Theorem 6.4.5, so we only sketch it.

We show all parts together by backward induction. More specifically, we prove that if $\partial_{I_{t-1}} \hat{V}_{t-1}(I_{t-1}|\theta_{t-1}) \geq \partial_{I_{t-1}} V_{t-1}(I_{t-1}|\theta_{t-1})$ for all I_{t-1}, then we have (i) $\hat{d}_t^{i*}(I_t,\theta_t) \leq d_t^{i*}(I_t,\theta_t)$ for all $i \in \mathcal{N}$, (ii) $\hat{q}_t^{j*}(I_t,\theta_t) \geq q_t^{j*}(I_t,\theta_t)$ for all $j \in \mathcal{M}$, (iii) $\hat{x}_t^*(I_t,\theta_t) \geq x_t^*(I_t,\theta_t)$, (iv) $\hat{\Delta}_t^*(I_t,\theta_t) \geq \Delta_t^*(I_t,\theta_t)$, and (v) $\partial_{I_t}\hat{V}_t(I_t|\theta_t) \geq \partial_{I_t}V_t(I_t|\theta_t)$ for all I_t. Note that $\partial_{I_0}\hat{V}_0(I_0|\theta_0) = \partial_{I_0}V_0(I_0|\theta_0)$ for all I_0, so the initial condition is satisfied. Since $\partial_{I_{t-1}}\hat{V}_{t-1}(I_{t-1}|\theta_{t-1}) \geq \partial_{I_{t-1}}V_{t-1}(I_{t-1}|\theta_{t-1})$ and $\hat{\xi}_t^{c,j}(c_t^j) \geq_{s.d.} \xi_t^{c,j}(c_t^j)$ for any j and c_t, by Theorem 6.4.4(a), $\partial_z\hat{\Psi}_t(z|\theta_t) \geq \partial_z\Psi_t(z|\theta_t)$ for any z.

First, we employ Lemma 15 to show that $\hat{d}_t^{i*}(I_t,\theta_t) \leq d_t^{i*}(I_t,\theta_t)$ for any $i \in \mathcal{N}$ and $\hat{q}_t^{j*}(I_t,\theta_t) \geq q_t^{j*}(I_t,\theta_t)$ for any $j \in \mathcal{M}$. Let $p = n+m$, $q = 0$, $\Gamma = \{0,1\}$, $y_i = -d_t^i$ $(1 \leq i \leq n)$, $y_{j+n} = q_t^j$ $(1 \leq j \leq m)$, $\lambda_i = \Lambda_t^i$ $(1 \leq i \leq n)$, $\lambda_i = 1$ $(n+1 \leq i \leq n+m)$, $f_i(y_i) = \Lambda_t^i R^i(d_t^i)$ $(1 \leq i \leq n)$, $f_{j+n}(y_{j+n}) = -C^j(q_t^j|c_t^j)$ $(1 \leq j \leq m)$, $h(y_0|0) = \Psi_t(I_t+y_0|\theta_t)$, $h(y_0|1) = \hat{\Psi}_t(I_t+y_0|\theta_t)$, and $\mathcal{Y}_i = \begin{cases} [-d_{\max},0], & 1 \leq i \leq n, \\ [0,+\infty), & n+1 \leq i \leq n+m. \end{cases}$ Invoking Lemma 15, we have that $\hat{d}_t^{i*}(I_t,\theta_t) \leq d_t^{i*}(I_t,\theta_t)$ for any $i \in \mathcal{N}$ and $\hat{q}_t^{j*}(I_t,\theta_t) \geq q_t^{j*}(I_t,\theta_t)$ for any $j \in \mathcal{M}$. $\hat{I}_t^{d,i}(\theta_t) \geq I_t^{d,i}(\theta_t)$ and $\hat{\mathcal{N}}_t^*(I_t,\theta_t) \subset \mathcal{N}_t^*(I_t,\theta_t)$ follow immediately from $\hat{d}_t^{i*}(I_t,\theta_t) \leq d_t^{i*}(I_t,\theta_t)$ for any $i \in \mathcal{N}$, whereas $\hat{I}_t^{q,j}(\theta_t) \geq I_t^{q,j}(\theta_t)$ and $\mathcal{M}_t^*(I_t,\theta_t) \subset \hat{\mathcal{M}}_t^*(I_t,\theta_t)$ follow immediately from $\hat{q}_t^{j*}(I_t,\theta_t) \geq q_t^{j*}(I_t,\theta_t)$ for any $j \in \mathcal{M}$. $\hat{x}_t^*(I_t,\theta_t) \geq x_t^*(I_t,\theta_t)$ and $\hat{\Delta}_t^*(I_t,\theta_t) \geq \Delta_t^*(I_t,\theta_t)$ also follow directly.

To complete the induction, we show that $\partial_{I_t}\hat{V}_t(I_t|\theta_t) \geq \partial_{I_t}V_t(I_t|\theta_t)$. Following the same argument as the proof of Theorem 6.4.5, we have that if there exists an $i \in \mathcal{N}$, such that $\hat{d}_t^{i*}(I_t,\theta_t) < d_t^{i*}(I_t,\theta_t)$, or there exists a $j \in \mathcal{M}$, such that $\hat{q}_t^{j*}(I_t,\theta_t) > q_t^{j*}(I_t,\theta_t)$, then $\partial_{I_t}\hat{V}_t(I_t|\theta_t) = \partial_z\hat{\Psi}_t(\hat{\Delta}_t^*(I_t,\theta_t)|\theta_t) > \partial_z\Psi_t(\Delta_t^*(I_t,\theta_t)|\theta_t) = \partial_{I_t}V_t(I_t|\theta_t)$.

In the remaining case, $\hat{d}_t^{i*}(I_t,\theta_t) = d_t^{i*}(I_t,\theta_t)$ and $\hat{q}_t^{j*}(I_t,\theta_t) = q_t^{j*}(I_t,\theta_t)$ for any $i \in \mathcal{N}$ and $j \in \mathcal{M}$. Thus, $\hat{\Delta}_t^*(I_t,\theta_t) = \Delta_t^*(I_t,\theta_t)$. Since $\partial_z\hat{\Psi}_t(z|\theta_t) \geq \partial_z\Psi_t(z|\theta_t)$ for any z, $\partial_z\hat{\Psi}_t(\hat{\Delta}_t^*(I_t,\theta_t)|\theta_t) \geq \partial_z\Psi_t(\Delta_t^*(I_t,\theta_t)|\theta_t)$. Thus, by Equation (E.1), $\partial_{I_t}\hat{V}_t(I_t|\theta_t) = \partial_z\hat{\Psi}_t(\hat{\Delta}_t^*(I_t,\theta_t)|\theta_t) \geq \partial_z\Psi_t(\Delta_t^*(I_t,\theta_t)|\theta_t) = \partial_{I_t}V_t(I_t|\theta_t)$. This completes the induction and, thus, the proof of Theorem 6.4.6. *Q.E.D.*

Proof of Theorem 6.4.7: We show all parts together by backward induction. More specifically, we prove that if $\partial_{I_{t-1}}\hat{V}_{t-1}(I_{t-1}|\theta_{t-1}) \geq \partial_{I_{t-1}}V_{t-1}(I_{t-1}|\theta_{t-1})$ for all I_{t-1}, then we have (i) $\hat{d}_t^{i*}(I_t,\theta_t) \leq d_t^{i*}(I_t,\theta_t)$ for all $i \in \mathcal{N}$, (ii) $\hat{q}_t^{j*}(I_t,\theta_t) \geq q_t^{j*}(I_t,\theta_t)$ for all $j \in \mathcal{M}$, (iii) $\hat{x}_t^*(I_t,\theta_t) \geq x_t^*(I_t,\theta_t)$, and (iv) $\partial_{I_t}\hat{V}_t(I_t|\theta_t) \geq \partial_{I_t}V_t(I_t|\theta_t)$ for all I_t. Since $\partial_{I_0}\hat{V}_0(I_0|\theta_0) = \partial_{I_0}V_0(I_0|\theta_0)$ for all I_0, the initial condition is satisfied. Since $\partial_{I_{t-1}}\hat{V}_{t-1}(I_{t-1}|\theta_{t-1}) \geq \partial_{I_{t-1}}V_{t-1}(I_{t-1}|\theta_{t-1})$, $\partial_z\hat{\Psi}_t(z|\theta_t) \geq \partial_z\Psi_t(z|\theta_t)$ for any z. Denote $\mathcal{N} = \{1,2,\cdots,n\}$ and $\hat{\mathcal{N}} = \{1,2,\cdots,\hat{n}\}$, where $\hat{n} > n$.

First, we show that $\hat{d}_t^{i*}(I_t,\theta_t) \leq d_t^{i*}(I_t,\theta_t)$ for all $i \in \mathcal{N}$. We assume, to the contrary, that $\hat{d}_t^{1*}(I_t,\theta_t) > d_t^{1*}(I_t,\theta_t)$. The strict concavity of $R^1(\cdot)$ implies that $\partial_{d_t^1}R^1(\hat{d}_t^{1*}(I_t,\theta_t)) < \partial_{d_t^1}R^1(d_t^{1*}(I_t,\theta_t))$. On the other hand, Lemma 16 yields that $\partial_{d_t^1}\hat{J}_t(\hat{d}_t^*(I_t,\theta_t),\hat{q}_t^*(I_t,\theta_t),I_t|\theta_t) \geq \partial_{d_t^1}J_t(d_t^*(I_t,\theta_t),q_t^*(I_t,\theta_t),I_t|\theta_t)$. Thus,

$$\begin{aligned}\partial_z\hat{\Psi}_t(\hat{\Delta}_t^*(I_t,\theta_t)|\theta_t) =& \partial_{d_t^1}R^1(\hat{d}_t^{1*}(I_t,\theta_t)) - \frac{1}{\Lambda_t^1}\partial_{d_t^1}\hat{J}_t(\hat{d}_t^*(I_t,\theta_t),\hat{q}_t^*(I_t,\theta_t),I_t|\theta_t) \\ <& \partial_{d_t^1}R^1(d_t^{1*}(I_t,\theta_t)) - \frac{1}{\Lambda_t^1}\partial_{d_t^1}J_t(d_t^*(I_t,\theta_t),q_t^*(I_t,\theta_t),I_t|\theta_t) \\ =& \partial_z\Psi_t(\Delta_t^*(I_t,\theta_t)|\theta_t).\end{aligned} \tag{E.8}$$

Since $\partial_z\hat{\Psi}_t(z|\theta_t) \geq \partial_z\Psi_t(z|\theta_t)$ for all z, $\hat{\Delta}_t^*(I_t,\theta_t) > \Delta_t^*(I_t,\theta_t)$, i.e.,

$$I_t + \sum_{j\in\mathcal{M}}\hat{q}_t^{j*}(I_t,\theta_t) - (\sum_{i\in\hat{\mathcal{N}}}\Lambda_t^i\hat{d}_t^{i*}(I_t,\theta_t)) = \hat{\Delta}_t^*(I_t,\theta_t) > \Delta_t^*(I_t,\theta_t) = I_t + \sum_{j\in\mathcal{M}}q_t^{j*}(I_t,\theta_t) - (\sum_{i\in\mathcal{N}}\Lambda_t^i d_t^{i*}(I_t,\theta_t)).$$

Since $\mathcal{N} \subset \hat{\mathcal{N}}$ and $\hat{d}_t^{1*}(I_t, \theta_t) > d_t^{1*}(I_t, \theta_t)$, either (a) $\hat{d}_t^{i*}(I_t, \theta_t) < d_t^{i*}(I_t, \theta_t)$ for some $i = 2, 3, \cdots, n$, or (b) $\hat{q}_t^{j*}(I_t, \theta_t) > q_t^{j*}(I_t, \theta_t)$ for some $j = 1, 2, \cdots, m$.

In case (a), without loss of generality, we assume that $\hat{d}_t^{2*}(I_t, \theta_t) < d_t^{2*}(I_t, \theta_t)$. By Lemma 16, we have $\partial_{d_t^2} \hat{J}_t(\hat{d}_t^*(I_t, \theta_t), \hat{q}_t^*(I_t, \theta_t), I_t|\theta_t) \leq \partial_{d_t^2} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t)$. Thus, by (E.8),

$$\begin{aligned}
\partial_{d_t^2} R^2(\hat{d}_t^{2*}(I_t, \theta_t)) =& \partial_z \hat{\Psi}_t(\hat{\Delta}_t^*(I_t, \theta_t)|\theta_t) + \frac{1}{\Lambda_t^2} \partial_{d_t^2} \hat{J}_t(\hat{d}_t^*(I_t, \theta_t), \hat{q}_t^*(I_t, \theta_t), I_t|\theta_t) \\
<& \partial_z \Psi_t(\Delta_t^*(I_t, \theta_t)|\theta_t) + \frac{1}{\Lambda_t^2} \partial_{d_t^2} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t) \\
=& \partial_{d_t^2} R^2(d_t^{2*}(I_t, \theta_t)),
\end{aligned}$$

which contradicts the strict concavity of $R^2(\cdot)$. Hence, $\hat{d}_t^{2*}(I_t, \theta_t) \geq d_t^{2*}(I_t, \theta_t)$ under the condition that $\hat{d}_t^{1*}(I_t, \theta_t) > d_t^{1*}(I_t, \theta_t)$. It follows from the same argument that $\hat{d}_t^{i*}(I_t, \theta_t) \geq d_t^{i*}(I_t, \theta_t)$ for all $i = 2, 3, \cdots, n$, under the condition that $\hat{d}_t^{1*}(I_t, \theta_t) > d_t^{1*}(I_t, \theta_t)$.

In case (b), without loss of generality, we assume that $\hat{q}_t^{1*}(I_t, \theta_t) > q_t^{1*}(I_t, \theta_t)$. By Lemma 16, we have $\partial_{q_t^1} \hat{J}_t(\hat{d}_t^*(I_t, \theta_t), \hat{q}_t^*(I_t, \theta_t), I_t|\theta_t) \geq \partial_{q_t^1} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t)$. Thus, by (E.8),

$$\begin{aligned}
\partial_{q_t^1} C^1(\hat{q}_t^{1*}(I_t, \theta_t)|c_t^1) =& \partial_z \hat{\Psi}_t(\hat{\Delta}_t^*(I_t, \theta_t)|\theta_t) - \partial_{q_t^1} \hat{J}_t(\hat{d}_t^*(I_t, \theta_t), \hat{q}_t^*(I_t, \theta_t), I_t|\theta_t) \\
<& \partial_z \Psi_t(\Delta_t^*(I_t, \theta_t)|\theta_t) - \partial_{q_t^1} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t) \\
=& \partial_{q_t^1} C^1(q_t^{1*}(I_t, \theta_t)|c_t^1),
\end{aligned}$$

which contradicts the strict convexity of $C^1(\cdot|c_t^1)$ in q_t^1. Hence, $\hat{q}_t^{1*}(I_t, \theta_t) \leq q_t^{1*}(I_t, \theta_t)$ under the condition that $\hat{d}_t^{1*}(I_t, \theta_t) > d_t^{1*}(I_t, \theta_t)$. It follows from the same argument that $\hat{q}_t^{j*}(I_t, \theta_t) \geq q_t^{j*}(I_t, \theta_t)$ for all $j = 1, 2, \cdots, m$, under the condition that $\hat{d}_t^{1*}(I_t, \theta_t) > d_t^{1*}(I_t, \theta_t)$. Combining cases (a) and (b), it follows that the initial assumption $\hat{d}_t^{1*}(I_t, \theta_t) > d_t^{1*}(I_t, \theta_t)$ is incorrect. Therefore, $\hat{d}_t^{1*}(I_t, \theta_t) \leq d_t^{1*}(I_t, \theta_t)$. The same argument yields that $\hat{d}_t^{i*}(I_t, \theta_t) \leq d_t^{i*}(I_t, \theta_t)$ for any $i = 1, 2 \cdots, n$. Hence, for each $i \in \mathcal{N}$,

$$\hat{I}_t^{d,i}(\theta_t) = \max\{I_t : \hat{d}_t^{i*}(I_t, \theta_t) = 0\} \geq \max\{I_t : d_t^{i*}(I_t, \theta_t) = 0\} = I_t^{d,i}(\theta_t).$$

If $i \in (\hat{\mathcal{N}}_t^*(I_t, \theta_t) \cap \mathcal{N})$, $d_t^{i*}(I_t, \theta_t) \geq \hat{d}_t^{i*}(I_t, \theta_t) > 0$. Thus, $i \in \mathcal{N}_t^*(I_t, \theta_t)$, and $(\hat{\mathcal{N}}_t^*(I_t, \theta_t) \cap \mathcal{N}) \subset \mathcal{N}_t^*(I_t, \theta_t)$ follows immediately.

Next, we show that $\hat{q}_t^{j*}(I_t, \theta_t) \geq q_t^{j*}(I_t, \theta_t)$ for all $j \in \mathcal{M}$. We assume, to the contrary, that $\hat{q}_t^{1*}(I_t, \theta_t) < q_t^{1*}(I_t, \theta_t)$. The strict convexity of $C^1(\cdot|c_t^1)$ in q_t^1 implies that $\partial_{q_t^1} C^1(\hat{q}_t^{1*}(I_t, \theta_t)|c_t^1) < \partial_{q_t^1} C^1(q_t^{1*}(I_t, \theta_t)|c_t^1)$. On the other hand, it follows from Lemma 16 that $\partial_{q_t^1} \hat{J}_t(\hat{d}_t^*(I_t, \theta_t), \hat{q}_t^*(I_t, \theta_t), I_t|\theta_t) \leq \partial_{q_t^1} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t)$. Thus,

$$\begin{aligned}
\partial_z \hat{\Psi}_t(\hat{\Delta}_t^*(I_t, \theta_t)|\theta_t) =& \partial_{q_t^1} C^1(\hat{q}_t^{1*}(I_t, \theta_t)|c_t^1) + \partial_{q_t^1} \hat{J}_t(\hat{d}_t^*(I_t, \theta_t), \hat{q}_t^*(I_t, \theta_t), I_t|\theta_t) \\
<& \partial_{q_t^1} C^1(q_t^{1*}(I_t, \theta_t)|c_t^1) + \partial_{q_t^1} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t) \qquad \text{(E.9)} \\
=& \partial_z \Psi_t(\Delta_t^*(I_t, \theta_t)|\theta_t).
\end{aligned}$$

Since $\partial_z \hat{\Psi}_t(z|\theta_t) \geq \partial_z \Psi_t(z|\theta_t)$ for each z and $\Psi_t(\cdot|\theta_t)$ is concave in z, $\hat{\Delta}_t^*(I_t, \theta_t) > \Delta_t^*(I_t, \theta_t)$, i.e.,

$$I_t + \sum_{j \in \mathcal{M}} \hat{q}_t^{j*}(I_t, \theta_t) - (\sum_{i \in \hat{\mathcal{N}}} \Lambda_t^i \hat{d}_t^{i*}(I_t, \theta_t)) = \hat{\Delta}_t^*(I_t, \theta_t) > \Delta_t^*(I_t, \theta_t) = I_t + \sum_{j \in \mathcal{M}} q_t^{j*}(I_t, \theta_t) - (\sum_{i \in \mathcal{N}} \Lambda_t^i d_t^{i*}(I_t, \theta_t)).$$

Since $\mathcal{N} \subset \hat{\mathcal{N}}$ and $\hat{q}_t^{1*}(I_t, \theta_t) < q_t^{1*}(I_t, \theta_t)$, either (a) $\hat{d}_t^{i*}(I_t, \theta_t) < d_t^{i*}(I_t, \theta_t)$ for some $i = 1, 2, \cdots, n$, or (b) $\hat{q}_t^{j*}(I_t, \theta_t) > q_t^{j*}(I_t, \theta_t)$ for some $j = 2, 3, \cdots, m$.

In case (a), without loss of generality, we assume that $\hat{d}_t^{1*}(I_t, \theta_t) < d_t^{1*}(I_t, \theta_t)$. By Lemma 16, we have $\partial_{d_t^1} \hat{J}_t(\hat{d}_t^*(I_t, \theta_t), \hat{q}_t^*(I_t, \theta_t), I_t|\theta_t) \le \partial_{d_t^1} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t)$. Thus, by (E.9),

$$\begin{aligned}
\partial_{d_t^1} R^1(\hat{d}_t^{1*}(I_t, \theta_t)) =& \partial_z \hat{\Psi}_t(\hat{\Delta}_t^*(I_t, \theta_t)|\theta_t) + \frac{1}{\Lambda_t^1} \partial_{d_t^1} \hat{J}_t(\hat{d}_t^*(I_t, \theta_t), \hat{q}_t^*(I_t, \theta_t), I_t|\theta_t) \\
<& \partial_z \Psi_t(\Delta_t^*(I_t, \theta_t)|\theta_t) + \frac{1}{\Lambda_t^1} \partial_{d_t^1} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t) \\
=& \partial_{d_t^1} R^1(d_t^{1*}(I_t, \theta_t)),
\end{aligned}$$

which contradicts the strict concavity of $R^1(\cdot)$. Hence, $\hat{d}_t^{1*}(I_t, \theta_t) \ge d_t^{1*}(I_t, \theta_t)$ under the condition that $\hat{q}_t^{1*}(I_t, \theta_t) < q_t^{1*}(I_t, \theta_t)$. It follows from the same argument that $\hat{d}_t^{i*}(I_t, \theta_t) \ge d_t^{i*}(I_t, \theta_t)$ for all $i = 1, 2, \cdots, n$, under the condition that $\hat{q}_t^{1*}(I_t, \theta_t) < q_t^{1*}(I_t, \theta_t)$. Thus, under this condition, $\hat{d}_t^{i*}(I_t, \theta_t) = d_t^{i*}(I_t, \theta_t)$ for all $i = 1, 2, \cdots, n$.

In case (b), without loss of generality, we assume that $\hat{q}_t^{2*}(I_t, \theta_t) > q_t^{2*}(I_t, \theta_t)$. By Lemma 16, we have $\partial_{q_t^2} \hat{J}_t(\hat{d}_t^*(I_t, \theta_t), \hat{q}_t^*(I_t, \theta_t), I_t|\theta_t) \ge \partial_{q_t^2} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t)$. Thus, by (E.9),

$$\begin{aligned}
\partial_{q_t^2} C^2(\hat{q}_t^{2*}(I_t, \theta_t)|c_t^2) =& \partial_z \hat{\Psi}_t(\hat{\Delta}_t^*(I_t, \theta_t)|\theta_t) - \partial_{q_t^2} \hat{J}_t(\hat{d}_t^*(I_t, \theta_t), \hat{q}_t^*(I_t, \theta_t), I_t|\theta_t) \\
<& \partial_z \Psi_t(\Delta_t^*(I_t, \theta_t)|\theta_t) - \partial_{q_t^2} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t) \\
=& \partial_{q_t^2} C^2(q_t^{2*}(I_t, \theta_t)|c_t^2),
\end{aligned}$$

which contradicts the strict convexity of $C^2(\cdot|c_t^2)$ in q_t^2. Hence, $\hat{q}_t^{2*}(I_t, \theta_t) \le q_t^{2*}(I_t, \theta_t)$ under the condition that $\hat{q}_t^{1*}(I_t, \theta_t) > q_t^{1*}(I_t, \theta_t)$. It follows from the same argument that $\hat{q}_t^{j*}(I_t, \theta_t) \ge q_t^{j*}(I_t, \theta_t)$ for all $j = 2, 3, \cdots, m$, under the condition that $\hat{q}_t^{1*}(I_t, \theta_t) < q_t^{1*}(I_t, \theta_t)$. Combining cases (a) and (b), it follows that the initial assumption $\hat{q}_t^{1*}(I_t, \theta_t) < q_t^{1*}(I_t, \theta_t)$ is incorrect. Therefore, $\hat{q}_t^{1*}(I_t, \theta_t) \ge q_t^{1*}(I_t, \theta_t)$. The same argument yields that $\hat{q}_t^{j*}(I_t, \theta_t) \ge q_t^{j*}(I_t, \theta_t)$ for any $j = 1, 2, \cdots, m$. Hence, for each $j \in \mathcal{M}$,

$$\hat{I}_t^{q,j}(\theta_t) = \min\{I_t : \hat{q}_t^{j*}(I_t, \theta_t) = 0\} \ge \min\{I_t : q_t^{j*}(I_t, \theta_t) = 0\} = I_t^{q,j}(\theta_t).$$

If $j \in \mathcal{M}_t^*(I_t, \theta_t)$, $\hat{q}_t^{j*}(I_t, \theta_t) \ge q_t^{j*}(I_t, \theta_t) > 0$. Thus, $j \in \hat{\mathcal{M}}_t^*(I_t, \theta_t)$, and $\mathcal{M}_t^*(I_t, \theta_t) \subset \hat{\mathcal{M}}_t^*(I_t, \theta_t)$ follows immediately. In addition,

$$\hat{x}_t^*(I_t, \theta_t) = I_t + \sum_{j \in \mathcal{M}} \hat{q}_t^{j*}(I_t, \theta_t) \ge I_t + \sum_{j \in \mathcal{M}} q_t^{j*}(I_t, \theta_t) = x_t^*(I_t, \theta_t).$$

Finally, to complete the induction, we show that $\partial_{I_t} \hat{V}_t(I_t|\theta_t) \ge \partial_{I_t} V_t(I_t|\theta_t)$. Recall that $\hat{d}_t^{i*}(I_t, \theta_t) \le d_t^{i*}(I_t, \theta_t)$ for any $i \in \mathcal{N}$. If $\hat{d}_t^{1*}(I_t, \theta_t) < d_t^{1*}(I_t, \theta_t)$, the strict concavity of $R^1(\cdot)$ implies that $\partial_{d_t^1} R^1(\hat{d}_t^{1*}(I_t, \theta_t)) > \partial_{d_t^1} R^1(d_t^{1*}(I_t, \theta_t))$. On the other hand, Lemma 16 implies that $\partial_{d_t^1} \hat{J}_t(\hat{d}_t^*(I_t, \theta_t), \hat{q}_t^*(I_t, \theta_t), I_t|\theta_t) \le \partial_{d_t^1} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t)$. Thus,

$$\begin{aligned}
\partial_z \hat{\Psi}_t(\hat{\Delta}_t^*(I_t, \theta_t)|\theta_t) =& \partial_{d_t^1} R^1(\hat{d}_t^{1*}(I_t, \theta_t)) - \frac{1}{\Lambda_t^1} \partial_{d_t^1} \hat{J}_t(\hat{d}_t^*(I_t, \theta_t), \hat{q}_t^*(I_t, \theta_t), I_t|\theta_t) \\
>& \partial_{d_t^1} R^1(d_t^{1*}(I_t, \theta_t)) - \frac{1}{\Lambda_t^1} \partial_{d_t^1} J_t(d_t^*(I_t, \theta_t), q_t^*(I_t, \theta_t), I_t|\theta_t) \\
=& \partial_z \Psi_t(\Delta_t^*(I_t, \theta_t)|\theta_t).
\end{aligned}$$

By Equation (E.1),

$$\partial_{I_t}\hat{V}_t(I_t|\theta_t) = \partial_z\hat{\Psi}_t(\hat{\Delta}^*_t(I_t,\theta_t)|\theta_t) > \partial_z\Psi_t(\Delta^*_t(I_t,\theta_t)|\theta_t) = \partial_{I_t}V_t(I_t|\theta_t).$$

The same argument implies that, if there exists an $i \in \mathcal{N}$, such that $\hat{d}^{i*}_t(I_t,\theta_t) < d^{i*}_t(I_t,\theta_t)$, we have $\partial_{I_t}\hat{V}_t(I_t|\theta_t) > \partial_{I_t}V_t(I_t|\theta_t)$.

Recall that $\hat{q}^{j*}_t(I_t,\theta_t) \geq q^{j*}_t(I_t,\theta_t)$ for any $j \in \mathcal{M}$. If $\hat{q}^{1*}_t(I_t,\theta_t) > q^{1*}_t(I_t,\theta_t)$, the strict convexity of $C^1(\cdot|c^1_t)$ in q^1_t implies that $\partial_{q^1_t}C^1(\hat{q}^{1*}_t(I_t,\theta_t)|c^1_t) > \partial_{q^1_t}C^1(q^{1*}_t(I_t,\theta_t)|c^1_t)$. On the other hand, it follows from Lemma 16 that $\partial_{q^1_t}\hat{J}_t(\hat{d}^*_t(I_t,\theta_t),\hat{q}^*_t(I_t,\theta_t),I_t|\theta_t) \geq \partial_{q^1_t}J_t(d^*_t(I_t,\theta_t),q^*_t(I_t,\theta_t),I_t|\theta_t)$. Thus,

$$\begin{aligned}\partial_z\hat{\Psi}_t(\hat{\Delta}^*_t(I_t,\theta_t)|\theta_t) =&\partial_{q^1_t}C^1(\hat{q}^{1*}_t(I_t,\theta_t)|c^1_t) + \partial_{q^1_t}\hat{J}_t(\hat{d}^*_t(I_t,\theta_t),\hat{q}^*_t(I_t,\theta_t),I_t|\theta_t)\\ >&\partial_{q^1_t}C^1(q^{1*}_t(I_t,\theta_t)|c^1_t) + \partial_{q^1_t}J_t(d^*_t(I_t,\theta_t),q^*_t(I_t,\theta_t),I_t|\theta_t)\\ =&\partial_z\Psi_t(\Delta^*_t(I_t,\theta_t)|\theta_t).\end{aligned}$$

By Equation (E.1),

$$\partial_{I_t}\hat{V}_t(I_t|\theta_t) = \partial_z\hat{\Psi}_t(\hat{\Delta}^*_t(I_t,\theta_t)|\theta_t) > \partial_z\Psi_t(\Delta^*_t(I_t,\theta_t)|\theta_t) = \partial_{I_t}V_t(I_t|\theta_t).$$

The same argument implies that, if there exists a $j \in \mathcal{M}$, such that $\hat{q}^{j*}_t(I_t,\theta_t) > q^{j*}_t(I_t,\theta_t)$, we have $\partial_{I_t}\hat{V}_t(I_t|\theta_t) > \partial_{I_t}V_t(I_t|\theta_t)$.

In the remaining case, $\hat{d}^{i*}_t(I_t,\theta_t) = d^{i*}_t(I_t,\theta_t)$ and $\hat{q}^{j*}_t(I_t,\theta_t) = q^{j*}_t(I_t,\theta_t)$ for any $i \in \mathcal{N}$ and $j \in \mathcal{M}$. Since $\hat{n} > n$ (or equivalently, $\mathcal{N} \subset \hat{\mathcal{N}}$),

$$\hat{\Delta}^*_t(I_t,\theta_t) = I_t + \sum_{j\in\mathcal{M}}\hat{q}^{j*}_t(I_t,\theta_t) - (\sum_{i\in\hat{\mathcal{N}}}\Lambda^i_t\hat{d}^{i*}_t(I_t,\theta_t)) \leq I_t + \sum_{j\in\mathcal{M}}q^{j*}_t(I_t,\theta_t) - (\sum_{i\in\mathcal{N}}\Lambda^i_t d^{i*}_t(I_t,\theta_t)) = \Delta^*_t(I_t,\theta_t).$$

Since $\partial_z\hat{\Psi}_t(z|\theta_t) \geq \partial_z\Psi_t(z|\theta_t)$ for each z, it follows that $\partial_z\hat{\Psi}_t(\hat{\Delta}^*_t(I_t,\theta_t)|\theta_t) \geq \partial_z\Psi_t(\Delta^*_t(I_t,\theta_t)|\theta_t)$ by the concavity of $\Psi_t(\cdot|\theta_t)$ in z. Thus, by Equation (E.1),

$$\partial_{I_t}\hat{V}_t(I_t|\theta_t) = \partial_z\hat{\Psi}_t(\hat{\Delta}^*_t(I_t,\theta_t)|\theta_t) \geq \partial_z\Psi_t(\Delta^*_t(I_t,\theta_t)|\theta_t) = \partial_{I_t}V_t(I_t|\theta_t).$$

This completes the induction and, thus, the proof of Theorem 6.4.7. *Q.E.D.*

Proof of Theorem 6.4.8: The proof of Theorem 6.4.8 follows from similar argument to that of Theorem 6.4.7, so we only sketch it.

We show all parts together by backward induction. More specifically, we prove that if $\partial_{I_{t-1}}\hat{V}_{t-1}(I_{t-1}|\theta_{t-1}) \leq \partial_{I_{t-1}}V_{t-1}(I_{t-1}|\theta_{t-1})$ for all I_{t-1}, then we have (i) $\hat{d}^{i*}_t(I_t,\theta_t) \geq d^{i*}_t(I_t,\theta_t)$ for all $i \in \mathcal{N}$, (ii) $\hat{q}^{j*}_t(I_t,\theta_t) \leq q^{j*}_t(I_t,\theta_t)$ for all $j \in \mathcal{M}$, and (iii) $\partial_{I_t}\hat{V}_t(I_t|\theta_t) \leq \partial_{I_t}V_t(I_t|\theta_t)$ for all I_t. Since $\partial_{I_0}\hat{V}_0(I_0|\theta_0) = \partial_{I_0}V_0(I_0|\theta_0) = 0$ for all I_0, the initial condition is satisfied. Since $\partial_{I_{t-1}}\hat{V}_{t-1}(I_{t-1}|\theta_{t-1}) \leq \partial_{I_{t-1}}V_{t-1}(I_{t-1}|\theta_{t-1})$, $\partial_z\hat{\Psi}_t(z|\theta_t) \leq \partial_z\Psi_t(z|\theta_t)$ for any z. Denote $\mathcal{M} = \{1, 2, \cdots, m\}$ and $\hat{\mathcal{M}} = \{1, 2, \cdots, \hat{m}\}$, where $\hat{m} > m$.

First, we show that $\hat{d}^{i*}_t(I_t,\theta_t) \geq d^{i*}_t(I_t,\theta_t)$ for all $i \in \mathcal{N}$. We assume, to the contrary, that $\hat{d}^{1*}_t(I_t,\theta_t) < d^{1*}_t(I_t,\theta_t)$. Following the same argument as that in the proof of Theorem 6.4.7, we have $\partial_z\hat{\Psi}_t(\hat{\Delta}^*_t(I_t,\theta_t)|\theta_t) > \partial_z\Psi_t(\Delta^*_t(I_t,\theta_t)|\theta_t)$, so $\hat{\Delta}^*_t(I_t,\theta_t) < \Delta^*_t(I_t,\theta_t)$, i.e.,

$$I_t + \sum_{j\in\hat{\mathcal{M}}}\hat{q}^{j*}_t(I_t,\theta_t) - (\sum_{i\in\mathcal{N}}\Lambda^i_t\hat{d}^{i*}_t(I_t,\theta_t)) = \hat{\Delta}^*_t(I_t,\theta_t) < \Delta^*_t(I_t,\theta_t) = I_t + \sum_{j\in\mathcal{M}}q^{j*}_t(I_t,\theta_t) - (\sum_{i\in\mathcal{N}}\Lambda^i_t d^{i*}_t(I_t,\theta_t)).$$

Since $\mathcal{M} \subset \hat{\mathcal{M}}$ and $\hat{d}_t^{1*}(I_t, \theta_t) < d_t^{1*}(I_t, \theta_t)$, either (a) $\hat{d}_t^{i*}(I_t, \theta_t) > d_t^{i*}(I_t, \theta_t)$ for some $i = 2, 3, \cdots, n$, or (b) $\hat{q}_t^{j*}(I_t, \theta_t) < q_t^{j*}(I_t, \theta_t)$ for some $j = 1, 2, \cdots, m$. We follow the same argument as that in the proof of Theorem 6.4.7 to reach a contradiction in either case (a) or case (b). Thus, $\hat{d}_t^{1*}(I_t, \theta_t) \geq d_t^{1*}(I_t, \theta_t)$. The same argument applies to show that $\hat{d}_t^{i*}(I_t, \theta_t) \geq d_t^{i*}(I_t, \theta_t)$ for any $i = 1, 2 \cdots, n$. $\hat{I}_t^{d,i}(\theta_t) \leq I_t^{d,i}(\theta_t)$ and $\mathcal{N}_t^*(I_t, \theta_t) \subset \hat{\mathcal{N}}_t^*(I_t, \theta_t)$ follows immediately.

Next, we show that $\hat{q}_t^{j*}(I_t, \theta_t) \leq q_t^{j*}(I_t, \theta_t)$ for all $j \in \mathcal{M}$. We assume, to the contrary, that $\hat{q}_t^{1*}(I_t, \theta_t) > q_t^{1*}(I_t, \theta_t)$. Following the same argument as that in the proof of Theorem 6.4.7, we have $\partial_z \hat{\Psi}_t(\hat{\Delta}_t^*(I_t, \theta_t)|\theta_t) > \partial_z \Psi_t(\Delta_t^*(I_t, \theta_t)|\theta_t)$, so $\hat{\Delta}_t^*(I_t, \theta_t) < \Delta_t^*(I_t, \theta_t)$, i.e.,

$$I_t + \sum_{j \in \hat{\mathcal{M}}} \hat{q}_t^{j*}(I_t, \theta_t) - (\sum_{i \in \mathcal{N}} \Lambda_t^i \hat{d}_t^{i*}(I_t, \theta_t)) = \hat{\Delta}_t^*(I_t, \theta_t) < \Delta_t^*(I_t, \theta_t) = I_t + \sum_{j \in \mathcal{M}} q_t^{j*}(I_t, \theta_t) - (\sum_{i \in \mathcal{N}} \Lambda_t^i d_t^{i*}(I_t, \theta_t)).$$

Since $\mathcal{M} \subset \hat{\mathcal{M}}$ and $\hat{q}_t^{1*}(I_t, \theta_t) > q_t^{1*}(I_t, \theta_t)$, either (a) $\hat{d}_t^{i*}(I_t, \theta_t) > d_t^{i*}(I_t, \theta_t)$ for some $i = 1, 2, \cdots, n$, or (b) $\hat{q}_t^{j*}(I_t, \theta_t) < q_t^{j*}(I_t, \theta_t)$ for some $j = 2, 3, \cdots, m$. We follow the same argument as that in the proof of Theorem 6.4.7 to reach a contradiction in either case (a) or case (b). Thus, $\hat{q}_t^{1*}(I_t, \theta_t) \leq q_t^{1*}(I_t, \theta_t)$. The same argument applies to show that $\hat{q}_t^{j*}(I_t, \theta_t) \leq q_t^{j*}(I_t, \theta_t)$ for any $j = 1, 2 \cdots, m$. Thus, $\hat{I}_t^{q,j}(\theta_t) \leq I_t^{q,j}(\theta_t)$ for any $j \in \mathcal{M}$. $(\hat{\mathcal{M}}_t^*(I_t, \theta_t) \cap \mathcal{M}) \subset \mathcal{M}_t^*(I_t, \theta_t)$ follows immediately.

To complete the induction, we show that $\partial_{I_t} \hat{V}_t(I_t|\theta_t) \leq \partial_{I_t} V_t(I_t|\theta_t)$. Following the same argument as that in the proof of Theorem 6.4.7, we have that if $\hat{d}_t^{i*}(I_t, \theta_t) > d_t^{i*}(I_t, \theta_t)$ for some $i \in \mathcal{N}$ or $\hat{q}_t^{j*}(I_t, \theta_t) < q_t^{j*}(I_t, \theta_t)$ for some $j \in \mathcal{M}$, $\partial_{I_t} \hat{V}_t(I_t|\theta_t) = \partial_z \hat{\Psi}_t(\hat{\Delta}_t^*(I_t, \theta_t)|\theta_t) < \partial_z \Psi_t(\Delta_t^*(I_t, \theta_t)|\theta_t) = \partial_{I_t} V_t(I_t|\theta_t)$.

In the remaining case, $\hat{d}_t^{i*}(I_t, \theta_t) = d_t^{i*}(I_t, \theta_t)$ and $\hat{q}_t^{j*}(I_t, \theta_t) = q_t^{j*}(I_t, \theta_t)$ for any $i \in \mathcal{N}$ and $j \in \mathcal{M}$. Since $\hat{m} > m$ (or equivalently, $\mathcal{M} \subset \hat{\mathcal{M}}$),

$$\hat{\Delta}_t^*(I_t, \theta_t) = I_t + \sum_{j \in \hat{\mathcal{M}}} \hat{q}_t^{j*}(I_t, \theta_t) - (\sum_{i \in \mathcal{N}} \Lambda_t^i \hat{d}_t^{i*}(I_t, \theta_t)) \geq I_t + \sum_{j \in \mathcal{M}} q_t^{j*}(I_t, \theta_t) - (\sum_{i \in \mathcal{N}} \Lambda_t^i d_t^{i*}(I_t, \theta_t)) = \Delta_t^*(I_t, \theta_t).$$

Since $\partial_z \hat{\Psi}_t(z|\theta_t) \leq \partial_z \Psi_t(z|\theta_t)$ for each z, it follows that $\partial_z \hat{\Psi}_t(\hat{\Delta}_t^*(I_t, \theta_t)|\theta_t) \leq \partial_z \Psi_t(\Delta_t^*(I_t, \theta_t)|\theta_t)$ by the concavity of $\Psi_t(\cdot|\theta_t)$ in z. Thus, by Equation (E.1), $\partial_{I_t} \hat{V}_t(I_t|\theta_t) = \partial_z \hat{\Psi}_t(\hat{\Delta}_t^*(I_t, \theta_t)|\theta_t) \leq \partial_z \Psi_t(\Delta_t^*(I_t, \theta_t)|\theta_t) = \partial_{I_t} V_t(I_t|\theta_t)$. This completes the induction and, thus, the proof of Theorem 6.4.8. *Q.E.D.*

Proof of Theorem 6.5.1:

Part (a). We consider an auxiliary game of N players $\hat{\mathcal{G}}$, in which the objective function of player i is $\hat{\Pi}_i(p|y, \theta) = (p_i - c_i)(\theta_i + f(Y) - b_i p_i + \sum_{j \neq i} \beta_{ij} p_j) - C_i(y_i)$, with decision variable $p_i \in [p_i^{\min}, p_i^{\max}]$. We first prove that $\hat{\mathcal{G}}$ has a unique equilibrium given by $A^{-1}(a(Y, \theta) + \kappa)$.

It is clear that, given any (y, θ), the objective function of player i in $\hat{\mathcal{G}}$, $\hat{\Pi}_i(p|y, \theta)$, is concave in p_i. Therefore, there exists an equilibrium in $\hat{\mathcal{G}}$.

We now show that the equilibrium in $\hat{\mathcal{G}}$, $\hat{p}^*(y, \theta)$, is an interior vector in the feasible set. Since $p_i^{\max}$ is sufficiently large so that it will not affect the equilibrium behavior, it remains to be shown that $\hat{p}_i^*(y, \theta) > p_i^{\min}$ for each i. Taking the first order derivative of the function $\hat{\Pi}_i(p|y, \theta)$ with respect to p_i, we have:

$$\partial_{p_i} \hat{\Pi}_i(p|y, \theta) = -b_i(p_i - c_i) + \theta_i + f(Y) - b_i p_i + \sum_{j \neq i} \beta_{ij} p_j.$$

Evaluating the above derivative at $p_i = c_i$, we have:

$$\partial_{p_i}\hat{\Pi}_i(p|y,\theta)|_{p_i=c_i} = [\theta_i + f(Y) - b_i p_i + \sum_{j\neq i}\beta_{ij}p_j]_{p_i=c_i}.$$

Following the assumption that $\lambda_i(p,y,\theta_i) > 0$ when $p_i = c_i$, we have $\partial_{p_i}\hat{\Pi}_i(p|y,\theta)|_{p_i=c_i} = \lambda_i(p,y,\theta_i)|_{p_i=c_i} > 0$ for any (p_{-i},y,θ), and, thus, $\hat{p}_i^*(y,\theta) > c_i = p_i^{\min}$. Hence, $\hat{p}^*(y,\theta)$ satisfies the first-order condition:

$$\partial_{p_i}\hat{\Pi}_i(\hat{p}^*(y,\theta)|y,\theta) = -b_i(\hat{p}_i^*(y,\theta) - c_i) + a_i(Y,\theta) - b_i\hat{p}_i^*(y,\theta) + \sum_{j\neq i}\beta_{ij}\hat{p}_j^*(y,\theta) = 0.$$

Equivalently,

$$A\hat{p}^*(y,\theta) = a(Y,\theta) + \kappa,$$

where the $(N\times N)$-matrix A and N-vectors $a(Y,\theta)$ and κ are defined in Section 6.5.1. Since A satisfies the diagonal-dominance condition, A^{-1} exists. Hence,

$$\hat{p}^*(y,\theta) = A^{-1}(a(Y,\theta) + \kappa) \tag{E.10}$$

is the unique equilibrium in $\hat{\mathcal{G}}$.

Note that $\hat{p}^*(y,\theta)$ continues to be an equilibrium in the original second-stage price competition, as long as it generates positive demand for each firm. If the firms select the price vector $\hat{p}^*(y,\theta)$ in the second-stage price competition, by (6.15), the associated demand for firm i is given by

$$\begin{aligned}
\hat{\lambda}_i^*(y,\theta) =& (a_i(Y,\theta) - (A\hat{p}^*(y,\theta))_i + b_i\hat{p}_i^*(y,\theta))^+ \\
=& b_i(\frac{a_i(Y,\theta)}{b_i} - \frac{(A\hat{p}^*(y,\theta))_i}{b_i} + \hat{p}_i^*(y,\theta))^+ \\
=& b_i(\frac{a_i(Y,\theta)}{b_i} - \frac{(AA^{-1}(a(Y,\theta)+\kappa))_i}{b_i} + \hat{p}_i^*(y,\theta))^+ \\
=& b_i(\hat{p}_i^*(y,\theta) - \frac{\kappa_i}{b_i})^+ \\
=& b_i(\hat{p}_i^*(y,\theta) - c_i) > 0,
\end{aligned}$$

where the third equality follows from (E.10), and the last from $\kappa_i = b_i c_i$. Hence, $\hat{p}^*(y,\theta)$ generates positive demand for each firm and, thus, forms an equilibrium in the second-stage price competition.

It remains to be shown that the original second-stage price competition does not have other equilibria. We assume, to the contrary, that there exists another equilibrium price vector $\bar{p}^*(y,\theta)$, with the associated equilibrium demand vector $\bar{\lambda}^*(y,\theta)$. Since

$$\partial_{p_i}\Pi_i([p_i,\bar{p}_{-i}^*(y,\theta)],y|\theta)|_{p_i=c_i} = \lambda_i([p_i,\bar{p}_{-i}^*(y,\theta)],y,\theta_i)|_{p_i=c_i} > 0,$$

$\bar{p}_i^*(y,\theta) > c_i$ for all i. If $\bar{\lambda}_i^*(y,\theta) > 0$ for all i, $\bar{p}^*(y,\theta)$ must satisfy the first-order condition given by (E.10), so $\bar{p}^*(y,\theta) = \hat{p}^*(y,\theta)$. In the remaining case, $\bar{\lambda}_i^*(y,\theta) = 0$ for some i. Without loss of generality, we assume that $\bar{\lambda}_1^*(y,\theta) = 0$. Since $\lambda_1([p_1^{\min},\bar{p}_{-1}^*(y,\theta)],y,\theta) > 0$, there exists a price $\bar{p}_1 > p_1^{\min} = c_1$ such that $\lambda_1([\bar{p}_1,\bar{p}_{-1}^*(y,\theta)],y,\theta) > 0$. Hence,

$$\Pi_1([\bar{p}_1,\bar{p}_{-1}^*(y,\theta)],y|\theta) = (\bar{p}_1 - c_1)\lambda_1([\bar{p}_1,\bar{p}_{-1}^*(y,\theta)],y,\theta) - C_1(y_1) > -C_1(y_1) = \Pi_1(\bar{p}^*(y,\theta),y|\theta),$$

which contradicts the assumption that $\bar{p}^*(y,\theta)$ is an equilibrium. Therefore, given any (y,θ), $p^*(y,\theta) = \hat{p}^*(y,\theta) = A^{-1}(a(Y,\theta)+\kappa)$ is the unique equilibrium in the second-stage price competition. Thus, for any i, $\lambda_i^*(y,\theta) = \hat{\lambda}_i^*(y,\theta) = b_i(p_i^*(y,\theta) - c_i) > 0$.

Part (b). By part (a), $p_i^*(y,\theta) = \sum_{l=1}^N (A^{-1})_{il}(\theta_l + f(Y) + \kappa_l)$. Therefore,

$$\partial_{y_j} p_i^*(y,\theta) = \partial_Y p_i^*(y,\theta) = \sum_{l=1}^N (A^{-1})_{il} f'(Y).$$

By Lemma 2 in [24], every entry of A^{-1} is nonnegative, so, together with the non-singularity of A^{-1} and the strict monotonicity of $f(\cdot)$, $\sum_{l=1}^N (A^{-1})_{il} f'(Y) > 0$. Thus, $\partial_{y_j} p_i^*(y,\theta) > 0$ for any i and j, and $p_i^*(y,\theta)$ is strictly increasing in Y for each i. Hence, $\lambda_i^*(y,\theta) = b_i(p_i^*(y,\theta) - c_i)$, which is strictly increasing in $p_i^*(y,\theta)$, is strictly increasing in Y and y_j for any i and j. *Q.E.D.*

Proof of Theorem 6.5.2:

Part (a). By (6.17), we have that $\pi_i(y|\theta) = b_i(\sum_{l=1}^N (A^{-1})_{il}(\theta_l + f(Y) + \kappa_l) - c_i)^2 - C_i(y_i)$. By Theorem 6.5.1(a), $p_i^*(Y,\theta) > c_i$ for any Y. Let $Y = 0$, we have $\psi_i := p_i^*(0,\theta) - c_i = \sum_{l=1}^N (A^{-1})_{il}(\theta_l + f_0 + \kappa_l) - c_i > 0$. Therefore,

$$\begin{aligned}\pi_i(y|\theta) =& b_i(\sum_{l=1}^N (A^{-1})_{il}(f(Y) - f_0) + \psi_i)^2 - C_i(y_i)\\ =& b_i(\sum_{l=1}^N (A^{-1})_{il})^2 (f(Y) - f_0)^2 + 2b_i\psi_i(\sum_{l=1}^N (A^{-1})_{il})(f(Y) - f_0) + b_i(\psi_i)^2 - C_i(y_i).\end{aligned}$$

Since $\psi_i > 0$, $f(\cdot)$ is concavely increasing in Y, and $C_i(\cdot)$ is convexly increasing in y_i, $\pi_i(y|\theta)$ is jointly concave in y under Assumption 6.5.1. Since $f(\cdot)$ is bounded from above by M, we have that $\lim_{y_i \to +\infty} \partial_{y_i} \pi_i(y|\theta) < 0$ for any θ. Hence, there exists an upper bound $y^{\max} < \infty$, such that the equilibrium of the first stage game is the same as that of a game with the same payoff functions, but the feasible set is constrained to $[0, y^{\max}]^N$. Since, for any given θ, $\pi_i(y|\theta)$ is concave in y_i for any i, and $[0, y^{\max}]^N$ is compact, the first-stage game has an equilibrium $y_{EF}^*(\theta) \in [0, y^{\max}]^N$. For any equilibrium $y_{EF}^*(\theta)$, we denote $Y_{EF}^*(\theta) := \sum_{i=1}^N y_{EF,i}^*(\theta)$.

Now, we show that $y_{EF}^*(\theta)$ is unique. Let $F_i(Y|\theta) := b_i(\sum_{l=1}^N (A^{-1})_{il}(\theta_l + f(Y) + \kappa_l) - c_i)^2$. Thus, $\pi_i(y|\theta) = F_i(Y|\theta) - C_i(y_i)$, where $Y := \sum_{i=1}^N y_i$. By our argument above, $F_i(\cdot|\theta)$ is concave and continuously differentiable in Y for any i and θ. Assume, to the contrary, that there exist two equilibria $\hat{y}_{EF}^*(\theta)$ and $y_{EF}^*(\theta)$ ($\hat{y}_{EF}^*(\theta) \neq y_{EF}^*(\theta)$). Without loss of generality, assume $\hat{Y}_{EF}^*(\theta) \geq Y_{EF}^*(\theta)$. Hence, there exists an i such that $\hat{y}_{EF,i}^*(\theta) > y_{EF,i}^*(\theta)$. Without loss of generality, we take $i = 1$. Lemma 16 yields that $\partial_{y_1}\pi_1(y_{EF}^*(\theta)|\theta) \leq \partial_{y_1}\pi_1(\hat{y}_{EF}^*(\theta)|\theta)$. Since $C_i(\cdot)$ is strictly convex, $C_1'(\hat{y}_{EF,1}^*(\theta)) > C_1'(y_{EF,1}^*(\theta))$. Thus,

$$\partial_Y F_1(\hat{Y}_{EF}^*(\theta)|\theta) = \partial_{y_1}\pi_1(\hat{y}_{EF}^*(\theta)|\theta) + C_1'(\hat{y}_{EF,1}^*(\theta)) > \partial_{y_1}\pi_1(y_{EF}^*(\theta)|\theta) + C_1'(y_{EF,1}^*(\theta)) = \partial_Y F_1(Y_{EF}^*(\theta)|\theta),$$

which contradicts the concavity of $F_i(\cdot|\theta)$. Hence, $\hat{y}_{EF,1}^*(\theta) \leq y_{EF,1}^*(\theta)$. The same argument shows that, for each i, $\hat{y}_{EF,i}^*(\theta) \leq y_{EF,i}^*(\theta)$. Hence, $\hat{Y}_{EF}^*(\theta) = \sum_i^N \hat{y}_{EF,i}^*(\theta) \leq \sum_i^N y_{EF,i}^*(\theta) = Y_{EF}^*(\theta)$, where the equality holds only when $\hat{y}_{EF,i}^*(\theta) = y_{EF,i}^*(\theta)$ for all i. Since $\hat{Y}_{EF}^*(\theta) \geq Y_{EF}^*(\theta)$ by assumption,

$\hat{Y}^*_{EF}(\theta) = Y^*_{EF}(\theta)$ and $\hat{y}^*_{EF}(\theta) = y^*_{EF}(\theta)$, which contradicts the initial assumption that $\hat{y}^*_{EF}(\theta) \neq y^*_{EF}(\theta)$. Therefore, the equilibrium in the first-stage effort level competition is unique.

Part (b). Part (b) follows immediately from part (a) and Theorem 6.5.1(a). *Q.E.D.*

Proof of Theorem 6.5.3:

Part (a). Since every entry of A^{-1} is nonnegative and A^{-1} is non-singular, $\partial_{\theta_j}\partial_Y F_i(Y|\theta) = 2b_i(A^{-1})_{ij}(\sum_{l=1}^N (A^{-1})_{il})f'(Y) \geq 0$. Hence $F_i(Y|\theta)$ is supermodular in (Y, θ_j) for any $1 \leq i, j \leq N$. Assume that $\hat{\theta} > \theta$ and $Y^*_{EF}(\theta) > Y^*_{EF}(\hat{\theta})$. Hence, the concavity and supermodularity of $F_i(\cdot|\theta)$ implies that, for each $1 \leq i \leq N$,

$$\partial_Y F_i(Y^*_{EF}(\theta)|\theta) \leq \partial_Y F_i(Y^*_{EF}(\hat{\theta})|\hat{\theta}). \tag{E.11}$$

If $y^*_{EF,1}(\theta) > y^*_{EF,1}(\hat{\theta})$, the strict convexity of $C_1(\cdot)$ yields that $C_1'(y^*_{EF,1}(\theta)) > C_1'(y^*_{EF,1}(\hat{\theta}))$. On the other hand, Lemma 16 implies that $\partial_{y_1}\pi_1(y^*_{EF}(\theta)|\theta) \geq \partial_{y_1}\pi_1(y^*_{EF}(\hat{\theta})|\hat{\theta})$. Thus,

$$\partial_Y F_1(Y^*_{EF}(\theta)|\theta) = \partial_{y_1}\pi_1(y^*_{EF}(\theta)|\theta) + C_1'(y^*_{EF,1}(\theta)) > \partial_{y_1}\pi_1(y^*_{EF}(\hat{\theta})|\hat{\theta}) + C_1'(y^*_{EF,1}(\hat{\theta})) = \partial_Y F_1(Y^*_{EF}(\hat{\theta})|\hat{\theta}),$$

which contradicts (E.11). Thus, under the condition $Y^*_{EF}(\theta) > Y^*_{EF}(\hat{\theta})$, $y^*_{EF,1}(\theta) \leq y^*_{EF,1}(\hat{\theta})$. Similar argument implies that, under the condition $Y^*_{EF}(\theta) > Y^*_{EF}(\hat{\theta})$, $y^*_{EF,i}(\theta) \leq y^*_{EF,i}(\hat{\theta})$ for any i. Therefore, if $Y^*_{EF}(\theta) > Y^*_{EF}(\hat{\theta})$, $Y^*_{EF}(\hat{\theta}) = \sum_i^N y^*_{EF,i}(\hat{\theta}) \geq \sum_i^N y^*_{EF,i}(\theta) = Y^*_{EF}(\theta)$, which forms a contradiction. Thus, $Y^*_{EF}(\theta)$ is increasing in θ_i for any i.

Part (b). Since every entry of A^{-1} is nonnegative, by part (a) and Theorem 6.5.1, $p^*_i(Y^*_{EF}(\theta), \theta) = \sum_{l=1}^N (A^{-1})_{il}(\theta_l + f(Y^*_{EF}(\theta)) + \kappa_l)$ is increasing in θ_j for any i and j. Thus, $\lambda^*_{EF,i}(\theta) = b_i(p^*_i(Y^*_{EF}(\theta), \theta) - c_i)$ is increasing in $p^*_i(Y^*_{EF}(\theta), \theta)$ and, hence, θ_j for any i and j. *Q.E.D.*

Proof of Theorem 6.5.4:

Part (a). We consider an auxiliary game of N players $\tilde{\mathcal{G}}$, in which the objective function of player i is $\tilde{\Pi}_i(p, y|\theta) = (p_i - c_i)(\theta_i + f(Y) - b_i p_i + \sum_{j\neq i}\beta_{ij}p_j) - C_i(y_i)$, with decision variable $p_i \in [p_i^{\min}, p_i^{\max}]$ and $y_i \geq 0$. We first prove that $\tilde{\mathcal{G}}$ has a unique equilibrium characterized by the unique solution of the system of equations (6.18) and (6.19). Note that (6.18) and (6.19) characterize the first-order condition: for any i, $\partial_{p_i}\tilde{\Pi}_i = 0$ and $\partial_{y_i}\tilde{\Pi}_i \begin{cases} = 0, & \text{if } y_i > 0, \\ \leq 0, & \text{otherwise.} \end{cases}$

We first show that (6.18) and (6.19) have a unique solution on $(p_1^{\min}, p_1^{\max}) \times (p_2^{\min}, p_2^{\max}) \times \cdots \times (p_N^{\min}, p_N^{\max}) \times [0, +\infty)^N$. Let $G_i(Y|\theta) := \frac{1}{2b_i\sum_{l=1}^N (A^{-1})_{il}} F_i(Y|\theta) = \frac{1}{2\sum_{l=1}^N (A^{-1})_{il}}(\sum_{l=1}^N (A^{-1})_{il}(\theta_l + f(Y) + \kappa_l) - c_i)^2$ for any i. By the proof of Theorem 6.5.2, $G_i(Y|\theta)$ is concave and continuously differentiable in Y for each i. Plugging (6.18) into (6.19), the left-hand-side of (6.19) becomes: $\partial_Y G_i(Y^*_{SC}(\theta)|\theta) - C_i'(y^*_{SC,i}(\theta))$, and (6.19) is reduced to:

$$\partial_Y G_i(Y^*_{SC}(\theta)|\theta) - C_i'(y^*_{SC,i}(\theta)) \begin{cases} = 0, & \text{if } y^*_{SC,i}(\theta) > 0, \\ \leq 0, & \text{otherwise,} \end{cases} \text{for all } i = 1, 2, \cdots, N, \tag{E.12}$$

where $Y^*_{SC}(\theta) = \sum_{i=1}^N y^*_{SC,i}(\theta)$. We define an auxiliary system of equations on $(Y_{SC}(\theta), y_{SC,1}(\theta), y_{SC,2}(\theta), \cdots, y_{SC,N}(\theta))$:

$$\partial_Y G_i(Y_{SC}(\theta)|\theta) - C_i'(y_{SC,i}(\theta)) \begin{cases} = 0, & \text{if } y_{SC,i}(\theta) > 0, \\ \leq 0, & \text{otherwise,} \end{cases} \text{for all } i = 1, 2, \cdots, N. \tag{E.13}$$

Note that the difference between (E.12) and (E.13) is that the identity $Y^*_{SC}(\theta) = \sum_{i=1}^N y^*_{SC,i}(\theta)$ [$Y_{SC}(\theta) = \sum_{i=1}^N y_{SC,i}(\theta)$] always holds [may not hold] in (E.12) [(E.13)]. Hence, for any solution of (E.13) $(Y_{SC}(\theta), y_{SC,1}(\theta), y_{SC,2}(\theta), \cdots, y_{SC,N}(\theta))$, if it also satisfies the identity $Y_{SC}(\theta) = \sum_{i=1}^N y_{SC,i}(\theta)$, it is also a solution to (E.12). Since $C_i(\cdot)$ is strictly convex in y_i for any i, there exists a unique vector $y_{SC}(\theta)$ that satisfies (E.13) for any fixed $Y_{SC}(\theta)$. Thus, we use $\mathcal{A} : \mathbb{R}^+ \to \mathbb{R}^N$ to denote the mapping from $Y_{SC}(\theta)$ to $y_{SC}(\theta)$, such that $y_{SC}(\theta) = \mathcal{A}(Y_{SC}(\theta))$ satisfies (E.13) for any given $Y_{SC}(\theta)$. Moreover, let $\mathcal{B} : \mathbb{R}^+ \to \mathbb{R}$ denote the following function: $\mathcal{B}(Y_{SC}(\theta)) = \sum_{i=1}^N y_{SC,i}(\theta)$, where $y_{SC}(\theta) = \mathcal{A}(Y_{SC}(\theta))$.

Now, we show that $\mathcal{B}(\cdot)$ has a unique fixed point on $\mathbb{R}^+$. It follows from the concavity of $G_i(\cdot|\theta)$ and the strict convexity of $C_i(\cdot)$ that $(\mathcal{A}(Y_{SC}(\theta)))_i$ is continuously decreasing in $Y_{SC}(\theta)$ for any i. Hence, $\mathcal{B}(Y_{SC}(\theta))$ is continuously decreasing in $Y_{SC}(\theta)$. By (E.13), $(\mathcal{A}(0))_i \geq 0$ for each i. Thus, $\mathcal{B}(0) \geq 0$. Let $\mathcal{C}(Y) := \mathcal{B}(Y) - Y$. Thus, $\mathcal{C}(\cdot)$ is strictly decreasing on $\mathbb{R}^+$ with $\mathcal{C}(0) \geq 0$ and $\lim_{Y\to+\infty} \mathcal{C}(Y) \leq \lim_{Y\to+\infty}(\mathcal{B}(0) - Y) = -\infty$. Therefore, $\mathcal{C}(\cdot)$ has a unique root on $\mathbb{R}^+$. Hence, $\mathcal{B}(\cdot)$ has a unique fixed point on $\mathbb{R}^+$ and, thus, (E.12) has a unique solution $y^*_{SC}(\theta)$. As shown by the proof of Theorem 6.5.1, given $y^*_{SC}(\theta)$, there exists a unique $p^*_{SC}(\theta)$ that satisfies (6.18), and $p^*_{SC}(\theta) \in (p_1^{\min}, p_1^{\max}) \times (p_2^{\min}, p_2^{\max}) \times \cdots \times (p_N^{\min}, p_N^{\max})$. Therefore, (6.18) and (6.19) has a unique solution $(p^*_{SC}(\theta), y^*_{SC}(\theta))$ on $(p_1^{\min}, p_1^{\max}) \times (p_2^{\min}, p_2^{\max}) \times \cdots \times (p_N^{\min}, p_N^{\max}) \times [0, +\infty)^N$.

We now show that the equilibrium in $\tilde{\mathcal{G}}$, $(\tilde{p}^*, \tilde{y}^*)$, if exists, must have an interior price vector $\tilde{p}^* \in (p_1^{\min}, p_1^{\max}) \times (p_2^{\min}, p_2^{\max}) \times \cdots \times (p_N^{\min}, p_N^{\max})$. Since $p_i^{\max}$ is sufficiently large for any i, it remains to be shown that $\tilde{p}_i^* > p_i^{\min} = c_i$ for all i. Assume, to the contrary, that $\tilde{p}_i^* = p_i^{\min} = c_i$ for some i. Without loss of generality, we take $i = 1$. Since $\lambda_1([p_1^{\min}, \tilde{p}^*_{-1}], \tilde{y}^*, \theta) > 0$, there exists a price $\tilde{p}_1 > p_1^{\min} = c_1$ such that $\lambda_1([\tilde{p}_1, \tilde{p}^*_{-1}], \tilde{y}^*, \theta) > 0$. Hence,

$$\Pi_1([\tilde{p}_1, \tilde{p}^*_{-1}], \tilde{y}^*, \theta) = (\tilde{p}_1 - c_1)\lambda_1([\tilde{p}_1, \tilde{p}^*_{-1}], \tilde{y}^*, \theta) - C_1(\tilde{y}_1^*) > -C_1(\tilde{y}_1^*) = \Pi_1(\tilde{p}^*, \tilde{y}^*, \theta),$$

which contradicts the assumption that $(\tilde{p}^*, \tilde{y}^*)$ is an equilibrium. Therefore, the equilibrium in $\tilde{\mathcal{G}}$, if exists, must have an interior price vector, and by the KKT necessary condition, must satisfy the first-order condition characterized by the system of equations (6.18) and (6.19).

It remains to be shown that the unique solution to (6.18) and (6.19), $(p^*_{SC}(\theta), Y^*_{SC}(\theta))$, is an equilibrium in $\tilde{\mathcal{G}}$. It suffices to prove that, for any i, given other firms' decisions $(p^*_{SC,-i}(\theta), y^*_{SC,-i}(\theta))$, $(p^*_{SC,i}(\theta), y^*_{SC,i}(\theta))$ maximizes

$$\tilde{\Pi}_i(p_i, y_i | p^*_{SC,-i}(\theta), y^*_{SC,-i}(\theta), \theta) := (p_i - c_i)(\theta_i + f(y_i + \sum_{j\neq i} y^*_{SC,j}(\theta)) - b_i p_i + \sum_{j\neq i} \beta_{ij} p^*_{SC,j}(\theta)) - C_i(y_i).$$

Following the same argument as that in the characterization of $(p^*_{SC}(\theta), y^*_{SC}(\theta))$, we have $(p^*_{SC,i}(\theta), y^*_{SC,i}(\theta))$ is the unique vector that satisfies the first-order condition:

$$\partial_{p_i}\tilde{\Pi}_i(p_i, y_i|p^*_{SC,-i}(\theta), y^*_{SC,-i}(\theta), \theta) \qquad = 0 \tag{E.14}$$

$$\partial_{y_i}\tilde{\Pi}_i(p_i, y_i|p^*_{SC,-i}(\theta), y^*_{SC,-i}(\theta), \theta) \qquad \begin{cases} = 0, & \text{if } y^*_{SC,i}(\theta) > 0, \\ \leq 0, & \text{otherwise.} \end{cases} \tag{E.15}$$

Since $\tilde{\Pi}_i(p_i, y_i|p^*_{SC,-i}(\theta), y^*_{SC,-i}(\theta), \theta)$ is a continuously differentiable function on a compact domain $[p_i^{\min}, p_i^{\max}] \times [0, y^{\max}]$, where $y^{\max}$ is defined in the proof of Theorem 6.5.2, it has a maximizer characterized by the first-order condition (E.14) and (E.15) for any given $(p^*_{SC,-i}(\theta), y^*_{SC,-i}(\theta))$. Therefore, given $(p^*_{SC,-i}(\theta), y^*_{SC,-i}(\theta))$, the unique solution to (E.14) and (E.15), $(p^*_{SC,i}(\theta), y^*_{SC,i}(\theta))$, is the unique maximizer of $\tilde{\Pi}_i(p_i, y_i|p^*_{SC,-i}(\theta), y^*_{SC,-i}(\theta), \theta)$ for any i. Therefore, the unique solution to (6.18) and (6.19), $(p^*_{SC}(\theta), y^*_{SC}(\theta))$, is the unique equilibrium in $\tilde{\mathcal{G}}$.

Note that $(p^*_{SC}(\theta), y^*_{SC}(\theta))$ continues to be an equilibrium in the original simultaneous competition, as long as it generates positive demand for each firm. If the firms select the price vector $p^*_{SC}(\theta)$ and the effort vector $y^*_{SC}(\theta)$ in the simultaneous competition, by (6.15) and (6.18), the associated demand for firm i is given by $\tilde{\lambda}^*_i(\theta) = (a_i(Y^*_{SC}(\theta), \theta) - (Ap^*_{SC}(\theta))_i + b_i p^*_{SC,i}(\theta))^+ = b_i(p^*_{SC,i}(\theta) - c_i) > 0$, where the inequality follows from $p^*_{SC,i}(\theta) > c_i$ for any i. Hence, $(p^*_{SC}(\theta), y^*_{SC}(\theta))$ generates positive demand for each firm and, thus, forms an equilibrium in the simultaneous competition.

It remains to be shown that the original simultaneous competition does not have other equilibria. We assume, to the contrary, that there exists another equilibrium $(\underline{p}^*(\theta), \underline{y}^*(\theta))$, with the associated equilibrium demand vector $\underline{\lambda}^*(\theta)$. Since

$$\partial_{p_i}\Pi_i([p_i, \underline{p}^*_{-i}(\theta)], \underline{y}^*(\theta)|\theta)|_{p_i=c_i} = \lambda_i([p_i, \underline{p}^*_{-i}(\theta)], \underline{y}^*(\theta), \theta_i)|_{p_i=c_i} > 0,$$

$\underline{p}^*_i > c_i$ for all i. If $\underline{\lambda}^*_i(\theta) > 0$ for all i, $(\underline{p}^*(\theta), \underline{y}^*(\theta))$ must satisfy the first-order condition (6.18) and (6.19), i.e., $(\underline{p}^*(\theta), \underline{y}^*(\theta)) = (p^*_{SC}(\theta), y^*_{SC}(\theta))$. In the remaining case, $\underline{\lambda}^*_i(\theta) = 0$ for some i. Without loss of generality, we assume that $\underline{\lambda}^*_1(\theta) = 0$. Since $\lambda_1([p_1^{\min}, \underline{p}^*_{-1}(\theta)], \underline{y}^*(\theta), \theta) > 0$, there exists a price $\underline{p}_1 > p_1^{\min} = c_1$ such that $\lambda_1([\underline{p}_1, \underline{p}^*_{-1}(\theta)], \underline{y}^*(\theta), \theta) > 0$. Hence,

$$\Pi_1([\underline{p}_1, \underline{p}^*_{-1}(\theta)], \underline{y}^*(\theta)|\theta) = (\underline{p}_1 - c_1)\lambda_1([\underline{p}_1, \underline{p}^*_{-1}(\theta)], \underline{y}^*(\theta), \theta) - C_1(\underline{y}^*_1(\theta)) > -C_1(\underline{y}^*_1(\theta)) = \Pi_1(\underline{p}^*(\theta), \underline{y}^*(\theta)|\theta),$$

which contradicts that $(\underline{p}^*(\theta), \underline{y}^*(\theta))$ is an equilibrium in the simultaneous competition. Therefore, given any θ, $(p^*_{SC}(\theta), y^*_{SC}(\theta))$, which is the unique solution of (6.18) and (6.19), is the unique equilibrium in the simultaneous competition. Thus, for any i, $\lambda^*_{SC,i}(\theta) = b_i(p^*_{SC,i}(\theta) - c_i) > 0$.

Part (b). By (E.12), $y^*_{SC}(\theta)$ is the unique equilibrium of an N-player game, in which the i^{th} player has the payoff function $\hat{\pi}_i(y|\theta) := G_i(Y|\theta) - C_i(y_i)$ and feasible set $\mathbb{R}^+$, where $Y = \sum_{i=1}^N y_i$. By the proof of Theorem 6.5.2(a) and Theorem 6.5.3(a), $F_i(Y|\theta)$, and hence $G_i(Y|\theta) = \frac{1}{2b_i \sum_{l=1}^N (A^{-1})_{il}} F_i(Y|\theta)$, are continuously differentiable and concave in Y and supermodular in (y_i, θ_j) for any (i, j). Assume $\hat{\theta} > \theta$ and $Y^*_{SC}(\theta) > Y^*_{SC}(\hat{\theta})$. Hence, the concavity and supermodularity of $G_i(\cdot|\theta)$ imply that, for each $1 \leq i \leq N$,

$$\partial_Y G_i(Y^*_{SC}(\theta)|\theta) \leq \partial_Y G_i(Y^*_{SC}(\hat{\theta})|\hat{\theta}). \tag{E.16}$$

If $y^*_{SC,1}(\theta) > y^*_{SC,1}(\hat{\theta})$, the strict convexity of $C_1(\cdot)$ yields that $C'_1(y^*_{SC,1}(\theta)) > C'_1(y^*_{SC,1}(\hat{\theta}))$. On the other hand, Lemma 16 implies that $\partial_{y_1}\hat{\pi}_1(y^*_{SC}(\theta)|\theta) \geq \partial_{y_1}\hat{\pi}_1(y^*_{SC}(\hat{\theta})|\hat{\theta})$. Thus,

$$\partial_Y G_1(Y^*_{SC}(\theta)|\theta) = \partial_{y_1}\hat{\pi}_1(y^*_{SC}(\theta)|\theta) + C'_1(y^*_{SC,1}(\theta)) > \partial_{y_1}\hat{\pi}_1(y^*_{SC}(\hat{\theta})|\hat{\theta}) + C'_1(y^*_{SC,1}(\hat{\theta})) = \partial_Y G_1(Y^*_{SC}(\hat{\theta})|\hat{\theta}),$$

which contradicts (E.16). Thus, under the condition that $Y^*_{SC}(\theta) > Y^*_{SC}(\hat{\theta})$, $y^*_{SC,1}(\theta) \leq y^*_{SC,1}(\hat{\theta})$. Similar argument implies that, under the condition that $Y^*_{SC}(\theta) > Y^*_{SC}(\hat{\theta})$, $y^*_{SC,i}(\theta) \leq y^*_{SC,i}(\hat{\theta})$ for any i. Therefore, if $Y^*_{SC}(\theta) > Y^*_{SC}(\hat{\theta})$, $Y^*_{SC}(\hat{\theta}) = \sum_i^N y^*_{SC,i}(\hat{\theta}) \geq \sum_i^N y^*_{SC,i}(\theta) = Y^*_{SC}(\theta)$, which forms a contradiction. Thus, $Y^*_{SC}(\theta)$ is increasing in θ_i for any i. Since every entry of A^{-1} is nonnegative and A^{-1} is non-singular, by (6.18), $p^*_{SC,i}(\theta) = \sum_{l=1}^N (A^{-1})_{il}(\theta_l + f(Y^*_{SC}(\theta)) + \kappa_l)$ is increasing in θ_j for any i and j. Thus, $\lambda^*_{SC,i}(\theta) = b_i(p^*_{SC,i}(\theta) - c_i)$ is increasing in $p^*_{SC,i}(\theta)$ and, hence, θ_j for any i and j. *Q.E.D.*

Proof of Theorem 6.5.5: We prove part (b) first, and parts (a) and (c) second.

Part (b). As shown in the proofs of Theorem 6.5.2 and Theorem 6.5.4, $y^*_{EF}(\theta)$ is the equilibrium of an N-player game with the concave objective function $\pi_i(y|\theta) = F_i(Y|\theta) - C_i(y_i)$ and feasible set $\mathbb{R}^+$ for player i, and $y^*_{CS}(\theta)$ is the equilibrium of an N-player game with the concave objective function $\hat{\pi}_i(y|\theta) = G_i(Y|\theta) - C_i(y_i)$ and feasible set $\mathbb{R}^+$ for player i. Note that $\partial_Y F_i(Y|\theta) = 2b_i(p^*_i(Y,\theta) - c_i)\partial_Y p^*_i(Y,\theta) > 0$, where the inequality follows from Theorem 6.5.1. Recall that $F_i(Y|\theta) = 2b_i \sum_{l=1}^n (A^{-1})_{il} G_i(Y|\theta)$. By Lemma 2 in [24], $2b_i \sum_{l=1}^n (A^{-1})_{il} \geq 2b_i(A^{-1})_{ii} \geq 1$. Thus, $\partial_Y F_i(Y|\theta) \geq \partial_Y G_i(Y|\theta) \geq 0$ for each i and θ. We assume, to the contrary, that $Y^*_{SC}(\theta) > Y^*_{EF}(\theta)$. Hence, for each i,

$$\partial_Y F_i(Y^*_{EF}(\theta)|\theta) \geq \partial_Y G_i(Y^*_{SC}(\theta)|\theta). \tag{E.17}$$

If $y^*_{SC,1}(\theta) > y^*_{EF,1}(\theta)$, the strict convexity of $C_1(\cdot)$ implies that $C'_1(y^*_{SC,1}(\theta)) > C'_1(y^*_{EF,1}(\theta))$. On the other hand, Lemma 16 yields that $\partial_{y_1}\hat{\pi}_1(y^*_{SC}(\theta)|\theta) \geq \partial_{y_1}\pi_1(y^*_{EF}(\theta)|\theta)$. Thus,

$$\partial_Y G_1(Y^*_{SC}(\theta)|\theta) = \partial_{y_1}\hat{\pi}_1(y^*_{SC}(\theta)|\theta) + C'_1(y^*_{SC,i}(\theta)) > \partial_{y_1}\pi_1(y^*_{EF}(\theta)|\theta) + C'_1(y^*_{EF,i}(\theta)) = \partial_Y F_1(Y^*_{EF}(\theta)|\theta),$$

which contradicts (E.17). Thus, under the condition that $Y^*_{SC}(\theta) > Y^*_{EF}(\theta)$, $y^*_{SC,1}(\theta) \leq y^*_{EF,1}(\theta)$. Similar argument implies that, under the condition that $Y^*_{SC}(\theta) > Y^*_{EF}(\theta)$, $y^*_{SC,i}(\theta) \leq y^*_{EF,i}(\theta)$ for any i. Therefore, if $Y^*_{SC}(\theta) > Y^*_{EF}(\theta)$, $Y^*_{EF}(\theta) = \sum_i^N y^*_{EF,i}(\theta) \geq \sum_i^N y^*_{SC,i}(\theta) = Y^*_{SC}(\theta)$, which forms a contradiction. Thus, $Y^*_{SC}(\theta) \leq Y^*_{EF}(\theta)$ for any θ.

Part (a). Since $Y^*_{SC}(\theta) \leq Y^*_{EF}(\theta)$ for any θ, by Theorems 6.5.1 and 6.5.4 and that every entry of A^{-1} is nonnegative,

$$p^*_{EF,i}(y^*_{EF}(\theta),\theta) = (\sum_{l=1}^N (A^{-1})_{il})(\theta_l + f(Y^*_{EF}(\theta)) + \kappa_l) \geq (\sum_{l=1}^N (A^{-1})_{il})(\theta_l + f(Y^*_{SC}(\theta)) + \kappa_l) = p^*_{SC,i}(\theta),$$

for any i and θ.

Part (c). Since $p^*_i(Y^*_{EF}(\theta),\theta) \geq p^*_{SC,i}(\theta)$, it follow immediately from Theorems 6.5.1 and 6.5.4 that

$$\lambda^*_i(Y^*_{EF}(\theta),\theta) = b_i(p^*_i(Y^*_{EF}(\theta),\theta) - c_i) \geq b_i(p^*_{SC,i}(\theta) - c_i) = \lambda^*_{SC,i}(\theta),$$

for any i and θ. *Q.E.D.*

E.2 Discussions on Stopping Condition (ii) of the Iterative Procedure

Lemma 15 is silent about how $y_i^*(\gamma)$ changes with γ for $p+1 \le i \le p+q$ in the convex program (6.1). In this section, we first present in detail how our comparative statics method stops in Step (d) without leading to a contradiction for $y_i^*(\gamma)$ $(p+1 \le i \le p+q)$. We then give an example to illustrate that, in the convex program (6.1), $y_i^*(\gamma)$ $(p+1 \le i \le p+q)$ may not be monotone in γ.

E.2.1 Comparative Statics Analysis of $y_i^*(\gamma)$ $(p+1 \le i \le p+q)$

We consider two (hypothetical and incorrect) scenarios: (a) $y_i^*(\gamma)$ is *increasing* in γ for all $\gamma \in \Gamma$ and some $p+1 \le i \le p+q$; and (b) $y_i^*(\gamma)$ is *decreasing* in γ for all $\gamma \in \Gamma$ and some $p+1 \le i \le p+q$.

For scenario (a), we assume, to the contrary, that $y_i^*(\hat{\gamma}) < y_i^*(\gamma)$ for some $\hat{\gamma} > \gamma$. By Lemma 16, $\partial_{y_i} F(y^*(\gamma)|\gamma) \ge \partial_{y_i} F(y^*(\hat{\gamma})|\hat{\gamma})$, i.e., $\partial_{y_i} g_i(y_i^*(\gamma)|\gamma) + \lambda_i \partial_{y_0} h(y_0^*(\gamma)|\gamma) \ge \partial_{y_i} g_i(y_i^*(\hat{\gamma})|\hat{\gamma}) + \lambda_i \partial_{y_0} h(y_0^*(\hat{\gamma})|\hat{\gamma})$. Since $g_i(\cdot|\cdot)$ is strictly concave in y_i and submodular in (y_i, γ), it may be possible that (i) $\partial_{y_i} g_i(y_i^*(\gamma)|\gamma) \ge \partial_{y_i} g_i(y_i^*(\hat{\gamma})|\hat{\gamma})$ or (ii) $\partial_{y_i} g_i(y_i^*(\gamma)|\gamma) < \partial_{y_i} g_i(y_i^*(\hat{\gamma})|\hat{\gamma})$. In case (i), the argument stops because we cannot obtain any monotone relationship between $\partial_{y_0} h(y_0^*(\gamma)|\gamma)$ and $\partial_{y_0} h(y_0^*(\hat{\gamma})|\hat{\gamma})$. Hence, no contradiction can be reached for this scenario.

For scenario (b), we assume, to the contrary, that $y_i^*(\hat{\gamma}) > y_i^*(\gamma)$ for some $\hat{\gamma} > \gamma$. By Lemma 16, $\partial_{y_i} F(y^*(\gamma)|\gamma) \le \partial_{y_i} F(y^*(\hat{\gamma})|\hat{\gamma})$, i.e., $\partial_{y_i} g_i(y_i^*(\gamma)|\gamma) + \lambda_i \partial_{y_0} h(y_0^*(\gamma)|\gamma) \le \partial_{y_i} g_i(y_i^*(\hat{\gamma})|\hat{\gamma}) + \lambda_i \partial_{y_0} h(y_0^*(\hat{\gamma})|\hat{\gamma})$. Since $g_i(\cdot|\cdot)$ is submodular in (y_i, γ) and strictly concave in y_i, $\partial_{y_i} g_i(y_i^*(\gamma)|\gamma) > \partial_{y_i} g_i(y_i^*(\hat{\gamma})|\hat{\gamma})$. Thus, we have that $\partial_{y_0} h(y_0^*(\gamma)|\gamma) < \partial_{y_0} h(y_0^*(\hat{\gamma})|\hat{\gamma})$. Since $h(\cdot|\cdot)$ is supermodular in (y_0, γ), we cannot obtain any monotone relationship between $y_0^*(\gamma)$ and $y_0^*(\hat{\gamma})$. Hence, the argument stops and no contradiction can be reached for this scenario.

Since the iterative procedure is stopped without reaching a contradiction, we suspect that, in the convex program (6.1), $y_i^*(\gamma)$ $(p+1 \le i \le p+q)$ may not be monotone in γ, and construct the following example.

Example E.2.1 *In the convex program* (6.1)*, let* $p = q = 1$, $\lambda_1 = \lambda_2 = 1$, $\Gamma = \mathbb{R}$, *and* $\mathcal{Y}_1 = \mathcal{Y}_2 = \mathbb{R}$. *Let*

$$f_1(y_1) = -(y_1)^2;\ g_2(y_2|\gamma) = \begin{cases} -(y_2)^2, & \text{if } \gamma \le 0, \\ -(y_2+\gamma)^2, & \text{otherwise;} \end{cases} \quad \text{and } h(y_0|\gamma) = \begin{cases} -(y_0-\gamma)^2, & \text{if } \gamma \le 0, \\ -(y_0)^2, & \text{otherwise.} \end{cases}$$

Clearly, $f_1(\cdot)$, $g_2(\cdot|\cdot)$, *and* $h(\cdot|\cdot)$ *satisfy the conditions of* (6.1). *It's easy to obtain that*

$$(y_1^*(\gamma), y_2^*(\gamma)) = \begin{cases} (\frac{\gamma}{3}, \frac{\gamma}{3}), & \text{if } \gamma \le 0, \\ (\frac{\gamma}{3}, -\frac{2\gamma}{3}), & \text{otherwise.} \end{cases}$$

Therefore, in this example, $y_2^*(\gamma)$ *is strictly increasing in* γ *for* $\gamma \le 0$*, and strictly decreasing in* γ *for* $\gamma > 0$.

Example E.2.1 implies that, in the convex optimization problem (6.1), $y_i^*(\gamma)$ $(p+1 \le i \le p+q)$ may not be monotone in γ for generally specified $\{f_i(\cdot), g_i(\cdot|\cdot), h(\cdot|\cdot)\}_{1 \le i \le p+q}$.