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On the Limiting Behavior of Variations of Hodge Structures

by

Genival Francisco Fernandes da Silva Jr.

A dissertation presented to the  
Graduate School of Arts & Sciences  
of Washington University in  
partial fulfillment of the  
requirements for the degree  
of Doctor of Philosophy

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St. Louis, Missouri

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Genival Francisco Fernandes da Silva Jr.

*Washington University in St. Louis*

*May 2016*

To my parents Gonçala and Genival.

ABSTRACT OF THE DISSERTATION

On the Limiting Behavior of Variations of Hodge Structures

by

Genival Francisco Fernandes da Silva Jr.

Doctor of Philosophy in Mathematics

Washington University in St. Louis, 2016

Professor Matt Kerr, Chair

In this dissertation we will address three results concerning the limiting behavior of variations of Hodge structures. The first chapter introduces the main concepts involved and fixes some notation. In chapter two we discuss extension classes representing LMHS, compute them for a class of toric families and introduce an alternative method for the computation of VHS arising from middle convolution. The next chapter is concerned with the so called Apéry constants; we provide a method of computing such constants by using higher normal functions coming from geometry. Finally, in the last chapter we analyze a family of surfaces with geometric monodromy group  $G_2$ , and discuss the generic global Torelli theorem for such a family.



# Chapter 1

## Degenerating variations of Hodge structures

We recall the basic definitions and main concepts used throughout this dissertation, also we take this opportunity to present the notations and conventions used henceforward.

### 1.1 Hodge structures

Let  $V$  be a finitely generated abelian group. For  $k = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ ; we denote  $V_k := V \otimes k$ . A Hodge structure can be defined in the following equivalent ways:

**Definition 1.1.1.** A Hodge structure of weight  $n$  is a finitely generated abelian group  $V$  together with a decomposition:

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q} \quad (1.1.1)$$

such that  $\overline{V^{p,q}} = V^{q,p}$ .

**Definition 1.1.2.** A Hodge structure of weight  $n$  is a finitely generated abelian group  $V$  together with a decreasing filtration  $F^n \subset F^{n-1} \subset \dots \subset F^0 = V_{\mathbb{C}}$  such that:

$$F^p \oplus \overline{F^{n-p+1}} = V_{\mathbb{C}} \quad (1.1.2)$$

**Definition 1.1.3.** A Hodge structure of weight  $n$  is real representation  $\varphi : Res_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \rightarrow GL(V_{\mathbb{R}})$  such that  $\varphi(r) = r^n I_V$  for  $r \in \mathbb{G}_{m,\mathbb{R}}$ , where  $I_V$  is the identity on  $V_{\mathbb{R}}$ .

**Remark 1.1.4.** 1. Sometimes we prefer to use a rational vector space  $V$  instead of a finitely generated abelian group, the term used to reflect this change is of a **rational**

Hodge structure.

2. The  $\mathbb{Q}$ -Zariski closure of the image of  $\varphi$  in definition 1.1.3 is called the *Mumford-Tate group*.

The main example which motivated the definition of a Hodge structure is of the cohomology group  $H^k(X, \mathbb{Z})$  of a compact Kahler manifold  $X$ . Indeed, the Hodge decomposition theorem gives that  $H^k(X, \mathbb{Z})$  carries a Hodge structure of weight  $k$ .

**Example 1.1.5.** The Tate structure  $\mathbb{Q}(1)$  is defined to be  $(2\pi i)\mathbb{Q}$  with weight  $-2$ .

The usual operations of linear algebra can be easily extended to Hodge structures, in particular one has the direct sum of Hodge structures, the dual Hodge structure, the wedge product of Hodge structures, etc. Now let  $Q : V_{\mathbb{Q}} \times V_{\mathbb{Q}} \rightarrow \mathbb{Q}$  be a non-degenerate form with  $Q(x, y) = (-1)^n Q(y, x)$ .

**Definition 1.1.6.** A polarized Hodge structure of weight  $n$  is given by a Hodge structure  $(V, F^{\bullet})$  together with  $Q$  such that:

$$\begin{aligned} Q(F^p, F^{n-p+1}) &= 0 \\ i^{p-q} Q(x, \bar{x}) &> 0, 0 \neq x \in V^{p,q} \end{aligned} \tag{1.1.3}$$

A typical example of a polarized Hodge structure is the primitive cohomology group  $H_{pr}^k(X, \mathbb{Z})$  of a smooth projective variety  $X$ , together with the intersection product  $Q$ .

## 1.2 Variation of Hodge structures and the period mapping

The idea of varying Hodge structures in a family started with the work of Phillip Griffiths [19],[20],[21]. The motivation for such concept comes when one has a family of smooth

projective varieties  $\varphi : \mathcal{X} \rightarrow B$  and wonders what happens with the natural Hodge structure of the cohomology of the fibers  $H^n(X_t, \mathbb{Z})$  as we change  $t \in B$ . If  $B$  is contractible and  $\mathcal{X}$  trivial over it, we fix a point  $b_0 \in B$ , then we have diffeomorphisms  $g_t : X_{b_0} \rightarrow X_t$ , which induces isomorphisms on cohomology level:

$$g_t^* : H^k(X_t, \mathbb{C}) \rightarrow H^k(X_{b_0}, \mathbb{C}) \tag{1.2.1}$$

Therefore, we have a representation:

$$\rho : \pi_1(B \setminus \{b_0\}) \rightarrow \text{Aut}(H^k(X_{b_0}, \mathbb{C}))$$

We call  $\rho$  the *monodromy representation* and its image  $\Gamma$ , the *monodromy group*. Sometimes it is more convenient to work with the identity component of the monodromy group, so we define the *geometric monodromy group*  $\Pi$  as the identity component of  $\mathbb{Q}$ -Zariski closure of  $\Gamma$ .

Now set<sup>1</sup>  $f_p := \sum_{a \geq p} \dim H^{p,q}$ , then we can define the map:

$$\begin{aligned} \mathcal{P}^p : B &\rightarrow \text{Grass}(f_p, H^k(X_{b_0}, \mathbb{C})) \\ \mathcal{P}^p(t) &= g_t^*(F^p(X_t)) \end{aligned} \tag{1.2.2}$$

Griffiths proved that the above map is holomorphic:

**Theorem 1.2.1.** [21] The map  $\mathcal{P}^p(t)$  is holomorphic.

The  $H^{p,q}(X_t)$  can be glued together to form a holomorphic vector bundle  $\mathbb{H}$  with holomorphic sub-bundles  $\mathbb{F}^p \subset \mathbb{H}$  given by the  $F^p(X_t)$ . Associated to this data, we have a flat connection  $\nabla$ , the *Gauss-Manin connection*. As a corollary of theorem 1.2.1 we have:

---

<sup>1</sup>We are using the fact that the Hodge numbers are constant for  $b \in B$

**Corollary 1.2.2** (Griffiths Transversality). Let  $\sigma$  be a section of  $\mathbb{F}^p$  then:

$$\nabla(\sigma) \in \mathbb{F}^{p-1} \tag{1.2.3}$$

When  $t = b_0$ , the  $g_t^*$  above lies in  $Aut(H^k(X_{b_0}, \mathbb{C}))$  leading to a representation:

$$\rho : \pi_1(B) \rightarrow Aut(H^k(X_{b_0}, \mathbb{C})) \tag{1.2.4}$$

This is called the *Monodromy representation*. If we denote the image of  $\rho$  by  $\Gamma$ , then the period mapping can be seen as:

$$\mathcal{P} : B \rightarrow \Gamma/D \tag{1.2.5}$$

Where  $D$  is the set of all weight  $k$  Hodge structures and Hodge numbers  $f_p$  satisfying 1.1.3,  $D$  is called the *period domain*. Note that we have to quotient  $D$  by  $\Gamma$  in order for the map to be well-defined.

The following example illustrate the period map  $\mathcal{P}$  when the fibers  $X_t$  are elliptic curves.

**Example 1.2.3** (The Legendre family). Consider the following family of elliptic curves on  $\mathbb{P}^1 \setminus \{0, 1, \infty\}$

$$X_t := \{y^2 = x(x-1)(x-t)\} \subset \mathbb{P}^2 \tag{1.2.6}$$

On each  $X_t$  we have the holomorphic form:

$$\omega_t := \frac{dx}{y} = \frac{dx}{\sqrt{x(x-1)(x-t)}} \tag{1.2.7}$$

Recall that since an elliptic curve has genus 1, the space of holomorphic forms is 1-dimensional, hence  $\omega_t$  generates the whole space. Also, since  $X_t$  is a curve, we only care about  $H^1(X_t, \mathbb{C}) = H^{1,0} \oplus H^{0,1}$  to describe the period mapping.

Denote by  $\alpha, \beta$  the homology basis pictured in Figure 1.1, with  $\alpha \cdot \beta = 1$ . Now, recall

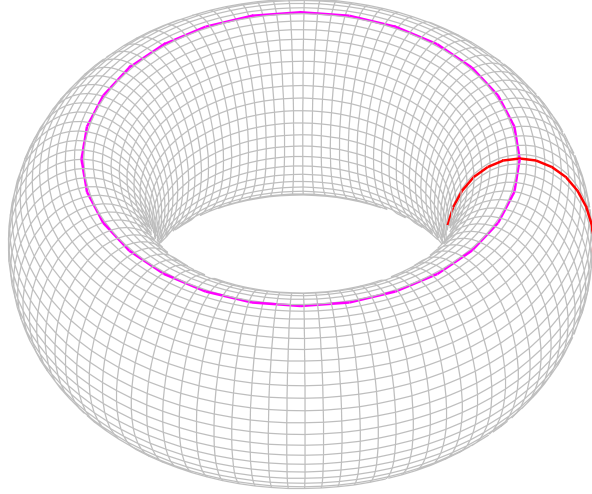


Figure 1.1: Homology basis

that the periods of  $X_t$  are by definition:

$$\begin{aligned} A(t) &:= \int_{\alpha} \omega_t \\ B(t) &:= \int_{\beta} \omega_t \end{aligned} \tag{1.2.8}$$

The period mapping in this case can be seen as:

$$\mathcal{P}(t) = \frac{B(t)}{A(t)} \tag{1.2.9}$$

All this geometric discussion motivates the definition of a variation of Hodge structure:

**Definition 1.2.4.** A variation of Hodge structure of weight  $k$  over a connected complex manifold  $B$  consists of a local system  $\mathbb{V}$  of Abelian groups together with a filtration of the vector bundle  $V := \mathbb{V} \otimes \mathcal{O}$ , by holomorphic sub-bundles:

$$F^p \subset F^{p-1} \subset \dots \subset F^0 = V \tag{1.2.10}$$

with sheaf of sections  $\mathcal{F}^p$  and satisfying the following conditions:

1.  $V = F^p \oplus \overline{F^{k-p+1}}$
2.  $\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1}$ , where nabla is the flat connection on  $V$  associated with the local system  $\mathbb{V}$ .

### 1.3 Limit mixed Hodge structures

The motivation in defining mixed Hodge structures comes from the idea of generalizing Hodge structures to non-compact and singular varieties, where the idea of pure Hodge structure might not exist, instead we might have a structure with different weights attached.

**Definition 1.3.1.** A *mixed Hodge structure* consists of a finitely generated abelian group  $V_{\mathbb{Z}}$ , a increasing filtration  $W_{\bullet}$  on  $V_{\mathbb{Q}}$  and a decreasing filtration  $F^{\bullet}$  on  $V_{\mathbb{C}}$  such that  $F^{\bullet}$  induces a weight  $k$  Hodge structure on  $Gr_k^W V_{\mathbb{Q}}$ .

**Example 1.3.2.** Let  $X$  be a smooth projective curve of genus 2, and consider  $\hat{X}$ , the resulting singular curve after we “pinch” one of the generators for the homology  $H^1(X)$ . Let  $\pi : \hat{X}_n \rightarrow \hat{X}$  be the normalization map, then we have the following exact sequence:

$$0 \rightarrow K \rightarrow H^1(\hat{X}) \xrightarrow{\pi^*} H^1(\hat{X}_n) \rightarrow 0 \quad (1.3.1)$$

where  $K$  denotes the kernel of  $\pi^*$ .

Let  $F^1$  be the subspace of  $H^1(\hat{X}, \mathbb{C})$  generated by any non-zero holomorphic form on  $\hat{X}$ , then  $(H^1(\hat{X}), F^1, K)$  defines a mixed Hodge structure, moreover  $H^1(\hat{X})/K \cong H^1(\hat{X}_n)$  is a weight one Hodge structure and  $K$  has a weight zero Hodge structure, therefore  $H^1(\hat{X})$  has nontrivial weight-graded pieces of weight one and zero.

Now consider the weight  $k$  polarized variation of Hodge structure  $\mathcal{V} = (\mathbb{V}, Q, \mathcal{F}^\bullet)$  over a complex manifold  $B$ . Assume that  $B$  has a smooth compactification  $\bar{B}$ , with a holomorphic disk embedding:

$$\begin{array}{ccccc} B & \subset & \bar{B} & \ni & x \\ \cup & & \cup & & \uparrow \\ \Delta^* & \subset & \Delta & \ni & 0 \end{array} \quad (1.3.2)$$

Let  $s$  be a choice of local coordinate on  $\Delta$ . Restricting  $\mathcal{V}$  to  $\Delta^*$ , and letting  $T$  denote the local monodromy, we have:

**Theorem 1.3.3** (Monodromy theorem).  $T$  is quasi-unipotent.

Therefore, after taking the unipotent part in the Jordan decomposition if necessary, we may assume  $T$  is unipotent. Then it makes sense to define  $\log(T) := \sum_i \frac{(-1)^{i-1}}{i} (T - I)^i$ . Set  $N = \log(T)$ , there exists a unique filtration  $W_\bullet = W(N)_\bullet$  on  $V$  such that:

$$\left\{ \begin{array}{l} N(W_k V) \subset W_{k-2} V \\ N^\ell : Gr_{n+k}^W V \xrightarrow{\cong} Gr_{n-k}^W V \end{array} \right\} \quad (1.3.3)$$

For example, if the weight is 1, the filtration is given by:

$$\{0\} \subset \text{Im}(N) \subset \text{Ker}(N) \subset V \quad (1.3.4)$$

and if the weight is 2 we have:

$$\{0\} \subset \text{Im}(N^2) \subset \text{Im}(N) \cap \text{Ker}(N) \subset \text{Im}(N) + \text{Ker}(N) \subset \text{Ker}(N^2) \subset V \quad (1.3.5)$$

If we set  $\ell(s) := \frac{\log(s)}{2\pi i}$ , then the local system  $\tilde{\mathbb{V}} := j_*(e^{-\ell(s)N}\mathbb{V})$  extends to  $\Delta$ . By the Nilpotent Orbit Theorem [35], the Hodge sheaves  $\mathcal{F}^p \subset \mathcal{V}$  extend to locally free subsheaves  $\mathcal{F}_e^p \subset \mathcal{V}_e := \tilde{\mathbb{V}} \otimes \mathcal{O}_\Delta$  on  $\Delta$ , and the  $SL_2$ -orbit Theorem[35] implies:

**Proposition 1.3.4.**  $(\tilde{V}, W_\bullet, \mathcal{F}_e^\bullet)|_x$  is a mixed Hodge structure polarized by  $N$ , called the limiting mixed Hodge structure (LMHS).

**Example 1.3.5.** Recall the Legendre family example 1.2.3, we compute the LMHS at 0 for this family. The monodromy matrix around 0 is  $T_0 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ . Therefore,  $N = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ , and both the kernel and image of  $N$  are 1-dimensional. In this case,  $V/\text{Ker}(N)$  is a 1-dimensional Hodge structure of weight 2, hence of type  $(1, 1)$  and  $\text{Im}(N)$  is a Hodge structure of weight  $(0, 0)$ . This is a typical example of LMHS of *Hodge-Tate* type (When there is no odd weight graded piece and graded pieces of weight  $2p$  are of type  $(p, p)$ )



## Chapter 2

# Mirror symmetry and Calabi-Yau Variation of Hodge structures

In this chapter we shall briefly describe how a recent result of Iritani [25] allows one to systematically compute LMHS of variations arising from families of anticanonical toric complete intersections. We shall carry this out for the 1-parameter,  $h^{2,1} = 1$  hypergeometric families of complete intersection C-Y threefolds classified in [15]. Each family yields a semistable degeneration over  $\mathbb{Q}$  with  $X_0$  the (suitably blown-up) “large complex structure limit” fiber. We also give a more explicit computation of the LMHS for another type of variations, by using Katz’s theory of the middle convolution [27, 13].

### 2.1 The $\mathbb{Z}$ -local system

Until recently, toric mirror symmetry (e.g., as described in [9] or [34]) only identified *complex* variations of Hodge structure arising from the A-model and B-model, because the Dubrovin connection on quantum cohomology merely provides a  $\mathbb{C}$ -local system on the A-model side. Iritani’s mirror theorem says that the integral structure on this local system provided by the  $\hat{\Gamma}$ -class (in the sense described below) completes the A-model  $\mathbb{C}$ -VHS to a  $\mathbb{Z}$ -VHS matching the one arising from  $H^3$  of fibers on the B-model side. The upshot is that to compute  $\Omega_{lim}$  (at 0) for a 1-parameter family of toric complete intersection Calabi-Yau 3-folds  $X_t \subset \mathbb{P}_\Delta$  over  $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ , we may use what boils down to characteristic class data from the mirror  $X_t^\circ \subset \mathbb{P}_{\Delta^\circ}$ .

In each case,  $V := H^{even}(X^\circ, \mathbb{C}) = \bigoplus_{j=0}^3 H^{j,j}(X^\circ)$  is a vector space of rank 4,  $\mathbb{P} := \mathbb{P}_{\Delta^\circ} = \mathbb{WP}(\delta_0, \dots, \delta_{3+r})$  is a weighted projective space<sup>1</sup> (with  $\delta_0 = \delta_1 = 1$ ), and  $X^\circ \subset \mathbb{P}$  is smooth<sup>2</sup> of multidegree  $(d_k)_{k=1}^r$  with  $\sum d_k = \sum \delta_i =: m$ . Let  $H$  denote the intersection with  $X^\circ$  of the vanishing locus of the weight 1 homogeneous coordinate  $X_0$ ; write  $\tau[H] \in H^{1,1}(X^\circ)$  for the Kähler class and  $q = e^{2\pi i \tau}$  for the Kähler parameter. We shall give a general recipe (following [14, sec. 1]) for constructing a polarized  $\mathbb{Z}$ -VHS, over  $\Delta^* : 0 < |q| < \epsilon$ , on  $\mathcal{V} := V \otimes \mathcal{O}_{\Delta^*}$ .

The easy parts are the Hodge filtration and polarization. Indeed, we simply put  $F^p := \bigoplus_{j \leq 3-p} H^{j,j} \subset V$  and  $\mathcal{F}_e^p := F^p \otimes \mathcal{O}_{\Delta} \subset V \otimes \mathcal{O}_{\Delta} =: \mathcal{V}_e$ . Similarly,  $Q$  on  $\mathcal{V}_e$  is induced from the form on  $V$  given by the direct sum of pairings  $Q_j : H^{j,j} \times H^{3-j,3-j} \rightarrow \mathbb{C}$  defined by  $Q_j(\alpha, \beta) := (-1)^j \int_{X^\circ} \alpha \cup \beta$ . A Hodge basis  $e = \{e_i\}_{i=0}^3$  of  $H^{even}$ , with  $e_i \in H^{3-i,3-i}(X^\circ)$  and  $[Q]_e$  of the form (2.3.2), is given by  $e_3 = [X^\circ]$ ,  $e_2 = [H]$ ,  $e_1 = -[L]$ , and  $e_0 = [p]$ . Here  $L$  is a copy of  $\mathbb{P}^1$  (parametrized by  $[X_0 : X_1]$ ) in  $X^\circ$  with  $L \cdot H = p$ , and  $[H] \cdot [H] = m[L]$ . The  $\{e_i\}$  give a Hodge basis<sup>3</sup> for  $\mathcal{V}_e$ .

$$Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \quad (2.1.1)$$

For the local system, we consider the generating series<sup>4</sup>  $\Phi_h(q) := \frac{1}{(2\pi i)^3} \sum_{d \geq 1} N_d q^d$  of the genus-zero Gromov-Witten invariants of  $X^\circ$ , and define the small quantum product on  $V$

<sup>1</sup>Technically, there are three exceptions to this amongst the examples we consider, which are weighted projective spaces  $\mathbb{WP}(\delta_0, \dots, \delta_n)$  for which the convex hull of  $\{e_1, \dots, e_n, -\sum \delta_i e_i\}$  is not reflexive. As described in [15], taking  $\Delta$  to be the convex hull of this set together with  $-e_n$  yields a reflexive polytope, and  $\mathbb{P}_{\Delta^\circ}$  is the blow-up of the  $\mathbb{WP}$  at a point not meeting (hence not affecting) the complete intersections we consider. Hence we may take  $X^\circ \subset \mathbb{P} = \mathbb{WP}(\delta_1, \dots, \delta_n)$ .

<sup>2</sup>The codimension of the singular locus in  $\mathbb{P}$  is at least 4 in every case, so does not meet a sufficiently general  $X^\circ$ .

<sup>3</sup>Note: in all bases we shall run the indices backwards ( $e = \{e_3, e_2, e_1, e_0\}$ , etc.) for purposes of writing matrices.

<sup>4</sup>derivatives  $\Phi_h^{(k)}$  will be taken with respect to  $\tau (= \ell(q))$

by  $e_2 * e_2 := -(m + \Phi_h'''(q))e_1$  and  $e_i * e_j := e_i \cup e_j$  for  $(i, j) \neq (2, 2)$ . This gives rise to the Dubrovin connection

$$\nabla := \text{id}_V \otimes d + (e_2 *) \otimes d\tau,$$

which we view as a map from  $\mathcal{V} \cong V \otimes \mathcal{O}_{\Delta^*} \rightarrow V \otimes \Omega_{\Delta^*}^1 \cong \mathcal{V} \otimes \Omega_{\Delta^*}^1$ , and the  $\mathbb{C}$ -local system  $\mathbb{V}_{\mathbb{C}} := \ker(\nabla) \subset \mathcal{V}$ .

Now define a map  $\tilde{\sigma} : V \rightarrow V \otimes \mathcal{O}(\Delta)$  by

$$\tilde{\sigma}(e_0) := e_0, \quad \tilde{\sigma}(e_1) := e_1, \quad \tilde{\sigma}(e_2) := e_2 + \Phi_h'' e_1 + \Phi_h' e_0,$$

$$\tilde{\sigma}(e_3) := e_3 + \Phi_h' e_1 + 2\Phi_h e_0.$$

For any  $\alpha \in V$ , one easily checks that

$$\sigma(\alpha) := \tilde{\sigma}(e^{-\tau[H]} \cup \alpha) := \sum_{k \geq 0} \frac{(-1)^k \tau^k}{k!} \tilde{\sigma}([H]^k \cup \alpha)$$

satisfies  $\nabla \sigma(\alpha) = 0$ , hence yields an isomorphism  $\sigma : V \xrightarrow{\cong} \Gamma(\mathfrak{H}, \rho^* \mathbb{V}_{\mathbb{C}})$  (where  $\rho : \mathfrak{H} \rightarrow \Delta^*$  sends  $\tau \mapsto q$ ). Writing<sup>5</sup>

$$\hat{\Gamma}(X^\circ) := \exp \left( -\frac{1}{24} ch_2(X^\circ) - \frac{2\zeta(3)}{(2\pi i)^3} ch_3(X^\circ) \right) \in V,$$

the image of

$$\begin{aligned} \gamma : K_0^{num}(X^\circ) &\longrightarrow \Gamma(\mathfrak{H}, \rho^* \mathbb{V}_{\mathbb{C}}) \\ \xi &\longmapsto \sigma(\hat{\Gamma}(X^\circ) \cup ch(\xi)) \end{aligned}$$

defines Iritani's  $\mathbb{Z}$ -local system  $\mathbb{V}$  underlying  $\mathbb{V}_{\mathbb{C}}$ . The filtration  $W_\bullet := W(N)_\bullet$  associated to its monodromy  $T(\gamma(\xi)) = \gamma(\mathcal{O}(-H) \otimes \xi)$  satisfies  $W_k \mathcal{V}_e = (\oplus_{j \geq 3-k/2} H^{j,j}) \otimes \mathcal{O}_\Delta$ .

---

<sup>5</sup>cf. §1 of [14] for the more general definition of  $\hat{\Gamma}(X^\circ)$

## 2.2 The limiting period matrix

In order to compute the limiting period matrix of this  $\mathbb{Z}$ -VHS over  $\Delta^*$ , we shall require a (multivalued) basis  $\{\gamma_i\}_{i=0}^3$  of  $\mathbb{V}$  satisfying  $\gamma_i \in W_{2i} \cap \mathbb{V}$ ,  $\gamma_i \equiv e_i \pmod{W_{2i-2}}$ , and  $[Q]_\gamma = [Q]_e$ . The corresponding  $\mathbb{Q}$ -basis of  $\tilde{\mathbb{V}}|_{q=0} =: V_{lim}$  is given by  $\gamma_i^{lim} := \tilde{\gamma}_i(0)$  where  $\tilde{\gamma}_i := e^{-\tau N} \gamma_i \in \Gamma(\Delta, \tilde{\mathbb{V}})$ . Of course, the  $e_i$  are another basis of  $V_{lim, \mathbb{C}}$ , and  $\Omega_{lim} = \gamma^{lim}[\text{id}]_e$ . Note that since  $N_{lim} = -(2\pi i) \text{Res}_{q=0}(\nabla) = -(e_2^*)|_{q=0} = -(e_2 \cup)|_{q=0}$ , we have

$$[N_{lim}]_e = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

A basis of the form we require is obtained by considering the Mukai pairing

$$\langle \xi, \xi' \rangle := \int_{X^\circ} ch(\xi^\vee \otimes \xi') \cup Td(X^\circ)$$

on  $K_0^{num}(X^\circ)$ . Since  $\langle \xi, \xi' \rangle = Q(\gamma(\xi), \gamma(\xi'))$ , any Mukai-symplectic<sup>6</sup> basis of  $K_0^{num}(X^\circ)$  of the form

$$\begin{aligned} \xi_1 &= \mathcal{O} + A\mathcal{O}_H + B\mathcal{O}_L + C\mathcal{O}_p \\ \xi_2 &= \mathcal{O}_H + D\mathcal{O}_L + E\mathcal{O}_p \\ \xi_3 &= -\mathcal{O}_L + F\mathcal{O}_p \\ \xi_4 &= \mathcal{O}_p \end{aligned} \tag{2.2.1}$$

will produce  $\gamma_i := \gamma(\xi_i)$  satisfying the above hypotheses. In this case, taking

$$\sigma_\infty(\alpha) := \lim_{q \rightarrow 0} \tilde{\sigma}(\alpha), \quad \gamma_\infty(\xi) := \sigma_\infty \left( \hat{\Gamma}(X^\circ) \cup ch(\xi) \right),$$

---

<sup>6</sup>That is,  $\langle \xi_i, \xi_{3-j} \rangle = 0$  unless  $i = j$ , in which case it is  $+1$  for  $i = 0, 1$  and  $-1$  for  $i = 2, 3$ .

we have  $\gamma_i^{lim} = \gamma_\infty(\xi_i)$ .

We now run this computation. Let  $c(X^\circ) = 1 + a[L] + b[p]$  be the Chern class of  $X^\circ$ ; note that there is no  $[H]$  term due to the fact that  $X^\circ$  is Calabi-Yau. Since the Chern character is  $ch(X^\circ) = 3 - a[L] + \frac{b}{2}[p]$  and the Todd class is  $Td(X^\circ) = 1 + \frac{a}{12}[L]$ ,  $\hat{\Gamma}(X^\circ) = 1 + \frac{a}{24}[L] - \frac{b\zeta(3)}{(2\pi i)^3}[p]$ . This yields

$$\begin{aligned}\gamma_3^{lim} &= e_3 + Ae_2 + \left(-B + \frac{m}{2}A - \frac{a}{24}\right)e_1 + \left(C - B + \frac{4m+a}{24}A - b\frac{\zeta(3)}{(2\pi i)^3}\right)e_0 \\ \gamma_2^{lim} &= e_2 + \left(-D + \frac{m}{2}\right)e_1 + \left(E - D + \frac{4m+a}{24}\right)e_0 \\ \gamma_1^{lim} &= e_1 + (F+1)e_0 \\ \gamma_0^{lim} &= e_0\end{aligned}$$

Imposing the symplectic condition produces constraints  $1 + F + A = 0$  and  $\frac{a+2m}{12} - D + E - AD + B = 0$ . After normalizing<sup>7</sup>  $A = B = C = D = 0$  ( $\implies F = -1, E = -\frac{a+2m}{12}$ ) in (2.2.1), expressing each  $e_i$  in terms of  $\{\gamma_i^{lim}\}$  gives the columns of

$$\Omega_{lim} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{a}{24} & -\frac{m}{2} & 1 & 0 \\ \frac{b\zeta(3)}{(2\pi i)^3} & \frac{a}{24} & 0 & 1 \end{pmatrix}. \quad (2.2.2)$$

To compute  $N$  (with these normalizations), we apply  $\mathcal{O}(-H) \otimes$  to the  $\xi_i$  in  $K_0^{num}(X^\circ)$ ;

---

<sup>7</sup> $A = 0$  is the canonical normalization of the local coordinate; the remaining choices are made to simplify the end result.

then

$$[T]_\gamma = [\mathcal{O}(-H) \otimes]_\xi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & m & 1 & 0 \\ -\frac{a+2m}{12} & m & 1 & 1 \end{pmatrix},$$

whereupon taking log gives

$$[N_{lim}]_{\gamma^{lim}} = [N]_\gamma = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \frac{m}{2} & m & 0 & 0 \\ -\frac{a}{12} & \frac{m}{2} & 0 & 0 \end{pmatrix}.$$

The data required to compute  $N$  and  $\Omega_{lim}$  for the complete intersection Calabi-Yau (CICY) examples from [15] is displayed in the table 2.1. Here for example “ $\mathbb{P}^5[3,3]$ ” means that  $X^\circ$  is a complete intersection of bidegree  $(3,3)$  in  $\mathbb{P}^5$ . Since  $X^\circ$  is smooth, the Chern numbers may be calculated using

$$c(X^\circ) = \frac{c(\mathbb{P})|_{X^\circ}}{c(N_{X^\circ/\mathbb{P}})} = \frac{\prod_{i=0}^{3+r} (1 + \delta_i[H])}{\prod_{k=1}^r (1 + d_k[H])}.$$

**Remark 2.2.1.** An interesting case not included amongst the CICY examples is the so called “14th case VHS”, labeled “**I**” in [loc. cit.]. It is shown in [6] that this VHS arises from the  $Gr_3^W H^3$  of a subfamily contained in the singular locus of a larger family of hypersurfaces in weighted-projective space. The LMHS of this sort of example is probably inaccessible to the above approach. The technique of the next section provides a possible approach to such examples.

$X^\circ$	m	a	b
$\mathbb{P}^4[5]$	5	50	-200
$\mathbb{P}^5[2, 4]$	8	56	-176
$\mathbb{P}^5[3, 3]$	9	54	-144
$\mathbb{P}^6[2, 2, 3]$	12	60	-144
$\mathbb{P}^7[2, 2, 2, 2]$	8	64	-128
$\mathbb{W}\mathbb{P}_{1,1,1,2,5}^4[10]$	10	340	-2880
$\mathbb{W}\mathbb{P}_{1,1,1,1,4}^4[8]$	8	176	-1184
$\mathbb{W}\mathbb{P}_{1,1,2,2,3,3}^5[6, 6]$	36	792	-4320
$\mathbb{W}\mathbb{P}_{1,1,1,2,2,3}^5[4, 6]$	24	384	-1872
$\mathbb{W}\mathbb{P}_{1,1,1,1,2}^4[6]$	6	84	-408
$\mathbb{W}\mathbb{P}_{1,1,1,1,1,3}^5[2, 6]$	12	156	-768
$\mathbb{W}\mathbb{P}_{1,1,1,1,2,2}^5[4, 4]$	16	160	-576
$\mathbb{W}\mathbb{P}_{1,1,1,1,1,2}^5[3, 4]$	12	96	-312

Table 2.1: LMHS parameters

## 2.3 The middle convolution approach

In this section we will describe a different approach in computing LMHS. By using Katz's theory of middle convolution we will construct certain variations of Hodge structures and explore some of its properties including the behavior on the limit and the verification of a conjecture presented in [18, Conjecture III.B.5].

For full details and computations we refer the reader to [11, sec. 5].

### 2.3.1 The construction of the variations

If  $\{\underline{a}\}$  and  $\{\underline{b}\}$  are finite sets of points in  $\mathbb{A}^1$  we define  $\{\underline{c}\} = \{\underline{a}\} * \{\underline{b}\}$  to be the set obtained by taking all sums of pairs  $a_j + b_k$  from  $\{\underline{a}\}$  and  $\{\underline{b}\}$ . Let  $U_1 = \mathbb{A}_x^1 \setminus \{a_1, \dots, a_m\}$ ,  $U_2 = \mathbb{A}_z^1 \setminus \{b_1, \dots, b_n\}$ , and  $U_3 = \mathbb{A}_y^1 \setminus \{c_1, \dots, c_p\}$ . Let  $U \subset \mathbb{A}_{(x,y)}^2$  be the Zariski open where  $\prod_j (x - a_j) \prod_k ((y - x) - b_k) \prod_l (y - c_l)$  does not vanish.

Given local systems  $\mathbb{V}_i \rightarrow U_i$  ( $i = 1, 2$ ), and projections  $\pi_1(x, y) := x$ ,  $\pi_2(x, y) := y - x$ ,

and  $\pi_3(x, y) = y$ . The *middle convolution* is the local system on  $U_3$  defined by:

$$\mathbb{V}_1 * \mathbb{V}_2 := R^1(\bar{\pi}_3)_*(j_*(\pi_1^*\mathbb{V}_1 \otimes \pi_2^*\mathbb{V}_2)) \quad (2.3.1)$$

By using middle convolution and quadratic twist [13, sec. 2.3-4], we are able to obtain a weight  $d$ , rank  $d + 1$  variation of Hodge structures  $\mathcal{V}_d$  over  $\mathbb{P}^1 \setminus \{0, 1, \infty\}$  with  $h^{d,0} = 1$ . The family  $\{X_d(t)\}$  produced (which are singular for  $d \geq 2$ ) all take the form  $w^2 = f_d(x_1, \dots, x_d, t)$ , for example:

- $d = 1$ :  $w^2 = (1 - tx)x(x - 1)$

(Legendre elliptic curve)

- $d = 3$ :  $w^2 = (1 - tx_3)x_3(x_2 - x_3)(x_2 - 1)(x_1 - x_2)(x_1 - 1)x_1$

(CY 3-fold family)

- $d = 6$ :  $w^2 = (1 - tx_6)(1 - x_6)(x_5 - x_6)x_5(x_4 - x_5)(1 - x_4) \times$   
 $(x_3 - x_4)x_3x_2 - x_3)(1 - x_2)(x_1 - x_2)x_1(1 - x_1).$

Also, it follows from [13, theorem 1.3.1] that for  $1 \leq d \leq 6$ ,  $\mathcal{V}_d$  has Hodge numbers all equal to 1. The monodromies of  $\mathcal{V}_d$  for some values of  $d$ , and the generic Mumford-Tate group are described in table 2.2 below.

	at 0	at 1	at $\infty$	MT-group
$d = 1$	$U(2)$	$U(2)$	$-U(2)$	$SL_2$
$d = 3$	$U(4)$	$-U(2) \oplus \mathbf{1}^{\oplus 2}$	$(-U(2))^{\oplus 2}$	$Sp_4$
$d = 6$	$U(7)$	$U(2)^{\oplus 2} \oplus U(3)$	$(-\mathbf{1})^{\oplus 4} \oplus \mathbf{1}^{\oplus 3}$	$G_2$

Table 2.2: Monodromies for  $\mathcal{V}_d$

In the  $Sp_4$  case, we may assume given a symplectic basis, so that the polarization takes



the form

$$Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \quad (2.3.2)$$

After conjugating by  $Sp_4(\mathbb{Q})$  to have

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ e & b & 0 & 0 \\ f & e & -a & 0 \end{pmatrix} \quad (2.3.3)$$

and canonically normalizing the local coordinate at 0, one knows (cf. [18]) that the limiting period matrix takes the form

$$\Omega_{lim} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{f}{2a} & \frac{e}{a} & 1 & 0 \\ \xi & \frac{f}{2a} & 0 & 1 \end{pmatrix} \quad (\xi \in \mathbb{C}).$$

The entries other than  $\xi$  are rational and correspond to torsion extension classes. The conjecture in [18, Conjecture III.B.5] basically says that the LMHS is  $\mathbb{Q}$ -motivated if and only if  $\xi = q \frac{\zeta(3)}{(2\pi i)^3}$  ( $q \in \mathbb{Q}$ ).

For  $G_2$ , again after appropriate normalizations, one has

$$\Omega_{lim} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ * & * & 1 & 0 & 0 & 0 & 0 \\ \textcircled{*} & * & * & 1 & 0 & 0 & 0 \\ * & \textcircled{*} & * & * & 1 & 0 & 0 \\ \xi & * & \textcircled{*} & * & * & 1 & 0 \\ * & \xi & * & \textcircled{*} & * & 0 & 1 \end{pmatrix}$$

where  $*$  denotes rational numbers. In this scenario, [18, Conjecture III.B.5] claims that  $\xi = q \frac{\zeta(5)}{(2\pi i)^5}$  ( $q \in \mathbb{Q}$ ).

### 2.3.2 Computing the limiting matrix

Henceforward we will restrict ourselves to the cases  $d = 1, 3, 6$ , and analyze the LMHS at  $t = 0$ .

Consider the form  $\omega_t = \frac{2^{d-1}}{(2\pi i)^d} \frac{dx_1 \wedge \dots \wedge dx_d}{w} \in \Omega^d(X_d(t))$ , which is holomorphic on a desingularization of  $X_d(t)$ . Then there is a basis  $\{\gamma_j\}_{j=0}^d$  of  $\mathbb{V}_d$  such that  $\int_{\gamma_0} \omega_t \rightarrow 1$  as  $t \rightarrow 0$ , and the monodromy matrix about  $t = 0$  is:

$$[T]_\gamma = \begin{pmatrix} 1 & 0 & \dots & 0 \\ a & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \dots & \pm a & & 1 \end{pmatrix} \tag{2.3.4}$$

We may assume after normalizing that  $(\gamma_0 \cdot \gamma_d) = 1$ . Let  $\ell(t) = \frac{\log(t)}{2\pi i}$ , then we define:

$$\Pi_d(t) := \int_{\gamma_d} \omega_t = (-1)^d \sum_{j=0}^d \ell^j(t) \sum_{k \geq 0} a_{jk} t^k \quad (2.3.5)$$

We have:

$$\frac{1}{2\pi i} \oint_{|t|=\epsilon} \frac{dt}{t} \int_{\gamma_d} \omega_t = \underbrace{(-1)^d \sum_{j=0}^d a_{j0} \ell^j(\epsilon)}_{=: \Pi_d^{nilp}(\epsilon)} + \underbrace{\mathcal{O}(\epsilon \log^d \epsilon)}_{\substack{\rightarrow 0 \\ \text{with } \epsilon}} \quad (2.3.6)$$

where  $\Pi_d^{nilp}(\epsilon)$  is the period of the “limiting” form  $\omega_\epsilon^{nilp}$  against  $\gamma_d$ . Its full period vector is:

$$[\omega_\epsilon^{nilp}]_\gamma = \begin{pmatrix} 1 \\ a_{10}^{(d-1)} \ell(\epsilon) + a_{00}^{(d-1)} \\ \vdots \\ \sum_{j=0}^{d-1} a_{j0}^{(1)} \ell^j(\epsilon) \\ \sum_{j=0}^d a_{j0} \ell^j(\epsilon) \end{pmatrix} \quad (2.3.7)$$

After scaling the parameter  $t = \alpha s$ , where  $\ell(\alpha) = -\frac{a_{00}^{(d-1)}}{a_{10}^{(d-1)}} = \frac{a_{00}^{(d-1)}}{a}$ , and applying  $e^{\ell(\alpha)N}$ , the period vector 2.3.7 becomes:

$${}^t \left( 1, 0, \dots, \mp \frac{\tilde{a}_{10}}{a}, \tilde{a}_{00} \right) \quad (2.3.8)$$

Therefore, the extension classes that characterizes the LMHS[18] are  $\tilde{a}_{00}(d = 3)$  and  $-\tilde{a}_{10}/a(d = 6)$ .

### 2.3.3 Computing the extension classes

We now address the computation of the extension classes. For  $d = 1$ , we have  $\tilde{a}_{00} = 0$ , yet  $a_{00}$  is non zero, as we shall verify.

Recall that we are after 2.3.6, which in this case translates to:

$$\frac{2}{(2\pi i)^2} \oint_{|t|=\epsilon} \frac{dt}{t} \int_1^{\frac{1}{t}} \frac{dx}{\sqrt{x(x-1)(1-tx)}} \quad (2.3.9)$$

After some work, the result is the following:

$$\frac{1}{\pi i} \int_{\eta}^1 \frac{du}{u\sqrt{1-u}} + \frac{1}{\pi^2 i} \sum_{m \geq 0} \frac{\left| \binom{-\frac{1}{2}}{m} \right| \epsilon^{-(m+\frac{1}{2})}}{(m+\frac{1}{2})} \int_0^{\eta} \frac{u^{m-\frac{1}{2}}}{(1-u)^{m+1}} du \quad (2.3.10)$$

Wich modulo  $\mathcal{O}(\epsilon \log \epsilon)$ , becomes:

$$\begin{aligned} &\equiv \frac{1}{\pi i} \left\{ \int_{\eta}^1 \frac{du}{u} + \sum_{k \geq 1} \left| \binom{-\frac{1}{2}}{k} \right| \int_{\eta}^1 u^{k-1} du \right\} + \frac{1}{\pi^2 i} \sum_{m \geq 0} \frac{\left| \binom{-\frac{1}{2}}{m} \right| \epsilon^{-(m+\frac{1}{2})}}{(m+\frac{1}{2})} \int_0^{\eta} u^{m-\frac{1}{2}} du \\ &\equiv -2\ell(\epsilon) + \underbrace{\frac{1}{\pi i} \left\{ \sum_{k \geq 1} \frac{\left| \binom{-\frac{1}{2}}{k} \right|}{k} + \frac{1}{\pi} \sum_{m \geq 0} \frac{\left| \binom{-\frac{1}{2}}{m} \right|}{(m+\frac{1}{2})^2} \right\}}_{-a_{00}} \end{aligned} \quad (2.3.11)$$

After some simplifications, we get that  $a_{00} = \frac{-2}{2\pi i} \{\log 4 + \log 4\} = \ell(\frac{1}{4})$ , moreover  $\frac{1}{4} \in \mathbb{Q}^*$ , which confirms the conjecture from [18, Conjecture III.B.5]

Now if  $d = 3$ , equation 2.3.6 reads:

$$\frac{2^3}{(2\pi i)^4} \oint_{|t|=\epsilon} \frac{dt}{t} \int_1^{\frac{1}{t}} \int_1^{x_3} \int_1^{x_2} \frac{1}{\sqrt{f_3(x_1, x_2, x_3, t)}} dx_1 dx_2 dx_3 \quad (2.3.12)$$

After we do a base change, we get  $\frac{2^3}{(2\pi i)^3}$  times:

$$\iiint_{[0,1]^{\times 3}} \left( \oint_{|t|=\epsilon} \frac{\sqrt{1-t}}{\sqrt{F_3(X_1, X_2, X_3, t)}} \frac{dt}{2\pi i t} \right) dX_1 dX_2 dX_3 \quad (2.3.13)$$

where  $F(X_1, X_2, X_3, t)$  is equal to:

$$\{(1 - X_3)t + X_3\} \{(1 - X_1X_2X_3)t + X_1X_2X_3\} X_1X_2 \prod_{i=1}^3 (1 - X_i). \quad (2.3.14)$$

After some computations we arrive conclude that the normalized  $\tilde{\Pi}_d^{nilp}(s)$  in this case is given by:

$$-\frac{4}{3}\ell^3(s) + 16\frac{\zeta(2)}{(2\pi i)^2}\ell(s) - \underbrace{48\frac{\zeta(3)}{(2\pi i)^3}}_{\tilde{a}_{00}} \quad (2.3.15)$$

And once again we confirm that  $\tilde{a}_{00}$  satisfies [18, Conjecture III.B.5].

The case  $d = 6$  is more subtle, and we could not prove [18, Conjecture III.B.5] in this case, but we do hope that the conjecture is still true, more precisely, we should have:

**Conjecture 2.3.1.** The canonically normalized  $\tilde{\Pi}_6^{nilp}(s)$  is given by

$$\frac{4}{45}\ell^6(s) + \frac{5}{9}\ell^4(s) + q_2\ell^2(s) + q_1\frac{\zeta(5)}{(2\pi i)^5}\ell(s) + q_0, \quad (2.3.16)$$

where  $q_0, q_1, q_2 \in \mathbb{Q}$ .

# Chapter 3

## The Arithmetic of the Landau-Ginzburg model of a certain class of threefolds

### 3.1 Apéry constants

The application of normal functions in areas peripheral to Hodge theory has emerged as a topic of research over the last decade [3],[4],[14],[24],[28],[33]; areas related to physics have accounted for much of this growth. The goal of this chapter is to use normal functions to give a ‘motivic’ meaning to constants arising in quantum differential equations associated to a certain class of Landau-Ginzburg models.

In [3], there is an explicit computation of a higher normal function associated with the Landau-Ginzburg mirror of a rank 4 Fano threefold, which turns out to be the value of a Feynman Integral. We want to present a similar approach, but instead of a Feynman integral, we will express some Apéry constants ([2],[23],[16],[17]) in terms of special values of the associated higher normal functions.

Landau-Ginzburg models are the natural object for ‘mirrors’ of Fano manifolds; more precisely, mirror symmetry relates a Fano variety with a dual object, which is a variety equipped with a non-constant complex valued function. For example, a LG model for  $\mathbb{P}^2$  is a family of elliptic curves and more generally, the LG model of a Fano  $n$ -fold is a family of Calabi-Yau  $(n - 1)$ -folds. In general, mirror symmetry relates symplectic properties of a Fano variety with algebraic ones of the mirror and vice versa.

In the following sections we will be mainly concerned with the Landau-Ginzburg models

for a special class of threefolds, namely the ones whose associated local system is of rank three, with a single nontrivial involution exchanging two maximally unipotent monodromy points. Looking at the classification in [7], one finds the short list  $V_{12}, V_{16}, V_{18}$  and “ $R_1$ ”, where the first three are rank 1 Fanos appearing in [23] and the latter is a rank 4 threefold with  $-K^3 = 24$  ( $K$  the canonical divisor). The involutions for these LG models have essentially been described in [23] and [3]. In the presence of an involution, it is possible to move the degeneracy locus of a higher cycle from the fiber over 0 to its involute, a property which we use for the construction of the desired normal function.

Let  $\mathbb{P}_{\Delta^\circ}$  be a toric degeneration of any of the varieties considered above; then each one of these will have a mirror Landau-Ginzburg model, which is a family of  $K3$  surfaces in  $\mathbb{P}_\Delta$ , that can be constructed as follows. Let  $\phi$  be a Minkowski polynomial for  $\Delta$ , then the family of  $K3$  is:

$$X_t := \overline{\{1 - t\phi(\mathbf{x}) = 0\}} \subset \mathbb{P}_\Delta \quad (3.1.1)$$

Let

$$\omega_t = \frac{1}{(2\pi i)^2} \text{Res}_{X_t} \left( \frac{\frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} \wedge \frac{dx_3}{x_3}}{1 - t\phi} \right) \quad (3.1.2)$$

and  $\gamma_t$  the invariant vanishing cycle about  $t = 0$ . We define the period of  $\phi$  by

$$\Pi_\phi(t) = \int_{\gamma_t} \omega_t = \sum a_n t^n \quad (3.1.3)$$

where  $a_n$  is the constant term of  $\phi^n$ . We say that  $a_n$  is the period sequence of  $\phi$ .

Consider a polynomial differential operator  $L = \sum F_k(t)P_k(D_t)$  where  $P_k(D_t)$  is a polynomial in  $D_t = t\frac{d}{dt}$ , then  $L \cdot \Pi_\phi(t) = 0$  is equivalent to a linear recursion relation. In practice, to compute  $L$  one uses knowledge of the first few terms of the period sequence and linear algebra to guess the recursion relation. The operator  $L$  is called a *Picard Fuchs operator*.

**Example 3.1.1.** The Picard-Fuchs operator for the threefold  $V_{12}$  is:

$$D^3 - t(1 + 2D)(17D^2 + 17D + 5) + t^2(D + 1)^3 \quad (3.1.4)$$

More generally, one also gets the same linear recursion on the power-series coefficients  $b_n$  of solutions of inhomogeneous equations  $L(\cdot) = G$ ,  $G$  a polynomial in  $t$ , for  $n \geq \deg(G)$ .

**Definition 3.1.2** ([23]). Given a linear homogeneous recurrence  $R$  and two solutions  $a_n, b_n \in \mathbb{Q}$  with  $a_0 = 1, b_0 = 0, b_1 = 1$ , if there is a  $L$ -function  $L(x)$  and  $c \in \mathbb{Q}^*$  such that:

$$\lim \frac{b_n}{a_n} = cL(x_0) \quad (3.1.5)$$

We say that 3.1.5 is the Apéry constant of  $R$ .

When we have a family of Calabi-Yau manifolds, a common way to look for Apéry constants is by considering the Picard-Fuchs equation. As described above, the coefficients of the power series expansion of the solutions of this equation satisfy a recurrence and in some cases the Apéry constant exists, see [2] for a wide class of examples. Beyond this “classical” case, we can also talk about quantum recurrences, which are recurrences arising from solutions of the Quantum differential equations satisfied by the quantum periods, which are defined using quantum products, see [22].

In [23], Golyshev uses quantum recurrences of the threefolds  $V_{10}, V_{12}, V_{14}, V_{16}, V_{18}$  to find Apéry constants; his method is basically to use a result of Beukers [23, Proposition 3.3] for the rational cases and apply a different approach for the non-rational ones. In the course of the proof of his results, he also describes the involution we mentioned above, but only for  $V_{12}, V_{16}$  and  $V_{18}$ . The main theorem of this chapter is:

**Theorem 3.1.3.** Let  $X$  be a Fano threefold, in the special class described above. Then there is a higher normal function  $\mathcal{N}$ , arising from a family of motivic cohomology classes on



the fibers of the LG model, such that the Apéry constant is equal to  $\mathcal{N}(0)$ .

As an immediate corollary of this result and Borel's theorem, the Apéry constant for these cases must be a  $\mathbb{Q}$ -linear combination of  $\zeta(3)$  and  $(2\pi i)^3$ , which provides a uniform conceptual explanation of this feature of the results in [23] and [3].

**Remark 3.1.4.** We note that throughout this chapter, the cycle groups are taken modulo torsion ( $\otimes \mathbb{Q}$ ).

## 3.2 Construction of the “toric” motivic classes

We assume the reader is familiar with the basic notions of Toric geometry, see [9] for a brief review or [10] for a more comprehensive treatment. Let

$$\phi = \sum a_{\mathbf{m}} \mathbf{x}^{\mathbf{m}} \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}] \quad (3.2.1)$$

be a Laurent polynomial with coefficients in  $\mathbb{C}$  and  $\Delta$  be the Newton polytope associated with  $\phi$ , which we will assume to be reflexive. (A list of all 3-dimensional reflexive polytopes is available at [7].) We briefly review the construction of the anti-canonical bundle and the facet divisors on the toric variety  $\mathbb{P}_{\Delta}$ . Let  $x, y, z$  be the toric coordinates on  $\mathbb{P}_{\Delta}$  and for each codimension 1 face  $\sigma \in \Delta(1)$ , choose a point  $\mathbf{o}_{\sigma}$  with integral coordinates, and write  $\mathbb{R}_{\sigma}$  for the 2-plane through  $\sigma$ . Then take a basis  $\mathbf{m}_1, \mathbf{m}_2$  for the translate  $(\mathbb{R}_{\sigma}^3 \cap \mathbb{Z}^3) - \mathbf{o}_{\sigma}$  and complete it to a basis  $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$  for  $\mathbb{Z}^3$  such that

$$\mathbb{R}_{\geq 0} \langle \pm \mathbf{m}_1, \pm \mathbf{m}_2, \mathbf{m}_3 \rangle \supset \Delta - \mathbf{o}_{\sigma} \quad (3.2.2)$$

Change coordinates, by setting  $x_j^{\sigma} = \mathbf{x}^{\mathbf{m}_j}$ ,  $j = 1, 2, 3$ . Consider the subset

$$\mathbb{D}_{\sigma}^* = \{x_1^{\sigma}, x_2^{\sigma} \in \mathbb{C}^*\} \cap \{x_3^{\sigma} = 0\} \quad (3.2.3)$$

of  $\mathbb{P}_\Delta$ ; let  $\mathbb{D}_\sigma$  be the Zariski closure of  $\mathbb{D}_\sigma^*$ , and set

$$\mathbb{D} := \sum_{\sigma \in \Delta(1)} [\mathbb{D}_\sigma] = \mathbb{P}_\Delta \setminus (\mathbb{C}^*)^3. \quad (3.2.4)$$

Henceforth we shall write  $x, y, z$  for  $x_1, x_2, x_3$ .

A standard result in toric geometry is that the sheaf  $\mathcal{O}(\mathbb{D})$  is ample and in case  $\Delta$  is reflexive; it is also the anti-canonical sheaf for  $\mathbb{P}_\Delta$ , and hence  $\mathbb{P}_\Delta$  is Fano in this case.

Given non vanishing holomorphic functions  $f_1, \dots, f_n$  on a quasi-projective variety  $Y$ , we denote the higher Chow cycle given by the graph of the  $f_j$  in  $Y \times (\mathbb{P}^1)^n$  by  $\langle f_1, \dots, f_n \rangle \in CH^n(Y, n)$ .

**Definition 3.2.1.** A 3 dimensional Laurent polynomial  $\phi$  is tempered if the symbol  $\langle x^\sigma, y^\sigma \rangle_{D_\sigma^*} \in CH^2(D_\sigma^*, 2)$  is trivial, for all facets  $\sigma$ , where  $D_\sigma^* \subset \mathbb{D}_\sigma^*$  is the zero locus of the facet polynomial  $\phi_\sigma$ .

**Remark 3.2.2.** The definition above can be restated as follows: For  $X_t$  a general  $K3$  surface of the family induced by  $\phi$ , let  $X_t^* = X_t \cap (\mathbb{C}^*)^3$ ; then  $\phi$  is tempered if the image of the higher Chow cycle  $\xi_t := \langle x, y, z \rangle_{X_t^*} \in CH^3(X_t^*, 3)$  under all residue maps vanishes. (Equivalently, viewed as an element of Milnor  $K$ -theory  $K_3^M(\mathbb{C}(X_t))$ ,  $\xi_t$  belongs to the kernel of the Tame symbol, cf. [29].)

In this chapter, we will focus on a special class of Laurent polynomials, namely Minkowski polynomials. See [1] for the basic definitions and properties of Minkowski polynomials.

**Example 3.2.3.** Consider the Minkowski polynomial  $\phi = x + y + z + (xyz)^{-1}$  with Newton polytope  $\Delta$  with vertices  $(1, 0, 0)$ ,  $(0, 1, 0)$ ,  $(0, 0, 1)$  and  $(-1, -1, -1)$ , see figure 3.1. Let  $\sigma$  be the facet with vertices  $(1, 0, 0)$ ,  $(0, 1, 0)$ ,  $(-1, -1, -1)$  and fix  $(-1, -1, -1)$  as the 'origin' of

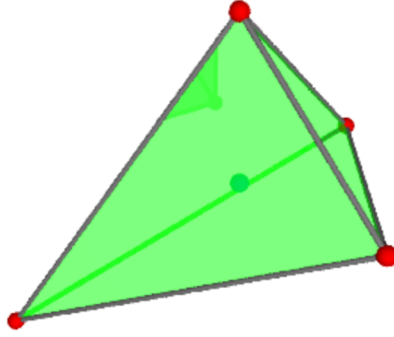


Figure 3.1: Newton polytope for the Laurent polynomial  $\phi = x + y + z + (xyz)^{-1}$ . Taken from [7]

the facet. Then clearly one possible choice of the new toric coordinates is:

$$\begin{aligned}
 x^\sigma &= x^2yz \\
 y^\sigma &= xy^2z \\
 z^\sigma &= x^{-1}
 \end{aligned}
 \tag{3.2.5}$$

Moreover  $\mathbb{D}_\sigma^* = \{z^\sigma = 0\}$ , so that  $D_\sigma^*$  is given by the zero locus of the facet polynomial  $\phi_\sigma = 1 + x^\sigma + y^\sigma$ . Therefore  $Res_{D_\sigma^*} \langle x, y, z \rangle_{X_t^*} = Res_{z^\sigma=0} \langle x^\sigma, y^\sigma, z^\sigma \rangle_{X_t^*} = \langle x^\sigma, y^\sigma \rangle_{D_\sigma^*} = \langle x^\sigma, -1 - x^\sigma \rangle = 0$ . Similarly, any other facet  $\sigma$  of this polytope has the property that  $\langle x^\sigma, y^\sigma \rangle_{D_\sigma^*} = 0$ .

The fact that the symbol  $\langle x^\sigma, y^\sigma \rangle_{D_\sigma^*}$  is trivial for all facets is not a coincidence; in fact, this is always the case for three-dimensional Minkowski polynomials. More precisely, we have:

**Proposition 3.2.4.** Every three-dimensional Minkowski polynomial is tempered.

*Proof.* In general, it is not true that every Laurent polynomial is tempered; one of the features of Minkowski polynomials is that they give rise to a decomposition in terms of rational

irreducible subvarieties, a fact that will be strongly used below. We use the equivalent definition of tempered as presented in remark 3.2.2.

Noting that  $D_\sigma := \mathbb{D}_\sigma \cap X_t$  and  $D = \mathbb{D} \cap X_t = \cup D_\sigma$  are independent of  $t \neq 0$ , and  $X_t^* = X_t \setminus D$ , let  $\iota : D \rightarrow X_t$  and  $j : X_t^* \rightarrow X_t$  be the natural inclusions. The localization exact sequence for higher Chow groups reads:

$$\cdots \rightarrow CH^2(D, 3) \xrightarrow{\iota_*} CH^3(X_t, 3) \xrightarrow{j^*} CH^3(X_t^*, 3) \xrightarrow{Res^D} CH^2(D, 2) \cdots \quad (3.2.6)$$

Now in general,  $D_\sigma$  is reducible, with components determined by the Minkowski decomposition of  $\sigma$ . Write  $D = \cup D_i$  as the resulting union of irreducible curves, and  $D_i^* = D_i \setminus \cup_j (D_i \cap D_j)$ . By the localization sequence (for  $D_i$ ), we have

$$CH^2(D_i, 2) = \ker \left\{ CH^2(D_i^*, 2) \xrightarrow{Res_{ij}} \oplus_j CH^1(D_i \cap D_j, 1) \right\}. \quad (3.2.7)$$

Since the edge polynomials of a Minkowski polynomial are cyclotomic,<sup>1</sup> for every  $i, j$  the composition

$$CH^3(X_t^*, 3) \xrightarrow{Res_i} CH^2(D_i^*, 2) \xrightarrow{Res_{ij}} \oplus_j CH^1(D_i \cap D_j, 1) \quad (3.2.8)$$

sends  $\xi_t$  to zero. By (3.2.7), we therefore have  $Res_i \xi \in CH^2(D_i, 2)$  for every  $i$ . Since in dimension 3 the irreducible pieces of a lattice Minkowski decomposition are either segments or triangles with no interior points, all the  $D_i$  are rational and smooth. Moreover, since both the Minkowski polynomial and the decomposition of the facet polynomials are defined over  $\bar{\mathbb{Q}}$ , the  $D_i$  are rational over  $\bar{\mathbb{Q}}$ . Now the  $Res_i \xi$  are clearly defined over  $\bar{\mathbb{Q}}$  (as the  $Res_\sigma \xi_t = \langle x^\sigma, y^\sigma \rangle$  are), and so belong to  $CH^2(\mathbb{P}^1, 2) \cong K_2(\bar{\mathbb{Q}}) = \{0\}$ ,

Therefore  $Res_i \xi_t$  is trivial, and  $\phi$  is tempered by Remark 3.2.2. □

**Remark 3.2.5.** The notion of Minkowski polynomial for dimension greater than 3 is not

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<sup>1</sup>in fact the roots are  $\pm 1$

yet well understood. However, if we assume the lattice polytopes in the Minkowski decompositions of facets have no interior points, then the proof above will extend to dimension 4, since we would still have rationality of the  $D_i$  (as above), and no significant problems appear in the local-global spectral sequence for higher Chow groups.

### 3.3 The Higher normal function $\mathcal{N}$

Recall that if  $S$  is a smooth projective variety, then

$$H_{\mathcal{M}}^n(S, \mathbb{Q}(n)) \cong CH^n(S, n) \cong Gr_{\gamma}^n K_n(S). \quad (3.3.1)$$

Not every member of our family  $X_t$  is smooth, but we can still have an element in the motivic cohomology. Such elements can be explicitly represented via higher Chow (double) complexes, so that we can still use standard formulas for Abel-Jacobi maps [32, §8]:

$$AJ^{m,n} : H_{\mathcal{M}}^n(X_t, \mathbb{Q}(n)) \rightarrow H^{n-1}(X_t, \mathbb{C}/\mathbb{Q}(n)). \quad (3.3.2)$$

The Landau-Ginzburg models for the threefolds  $V_{12}, V_{16}, V_{18}$ , and  $R_1$ , may be defined by (the Zariski closure of) the families  $\{1 - t\phi = 0\}$ , with  $\phi$  given by:

$$\begin{aligned} V_{12} : \phi &= \frac{(1+x+z)(1+x+y+z)(1+z)(y+z)}{xyz} \\ V_{16} : \phi &= \frac{(1+x+y+z)(1+z)(1+y)(1+x)}{xyz} \\ V_{18} : \phi &= \frac{(x+y+z)(x+y+z+xy+xz+yz+xyz)}{xyz} \\ R_1 : \phi &= \frac{(1+x+y+z)(xyz+xy+xz+yz)}{xyz} \end{aligned} \quad (3.3.3)$$

As these families of K3s all have Picard rank 19, their Picard-Fuchs operators take the form  $D_{PF} = \sum_{i=0}^3 F_i(t)(D_t)^k$ , with  $F_i(t)$  relatively prime polynomials. We call  $F_3(t) =: \sigma(D_{PF})$ ,

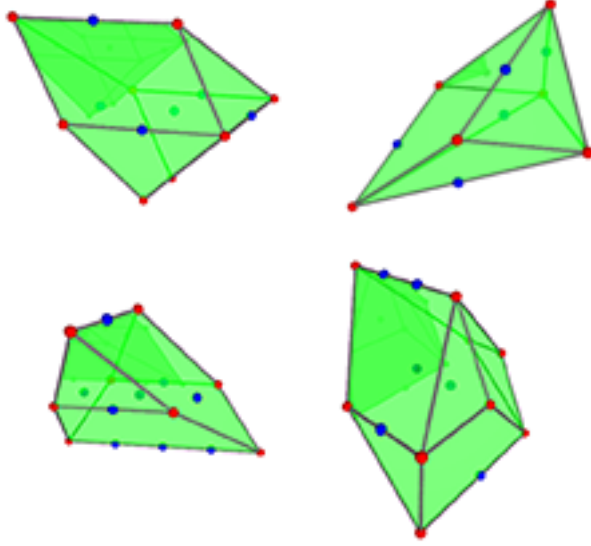


Figure 3.2: Newton polytopes for (top)  $V_{18}, R_1$  and (bottom)  $V_{12}, V_{16}$  respectively. Taken from [7]

which is taken to be monic, the *symbol* of  $D_{PF}$ . In the four cases the symbols are

$$1 - 34t + t^2, 1 - 24t + 16t^2, 1 - 18t - 27t^2, \text{ and } 1 - \frac{5}{16}t + \frac{1}{64}t^2, \quad (3.3.4)$$

respectively.

We shall adopt the notation  $\mathcal{X} \xrightarrow{\pi} \mathbb{P}^1$  for the total space of each family,  $\mathcal{X}^\circ = \mathcal{X} \setminus X_0 \xrightarrow{\pi^\circ} \mathbb{A}_{\frac{1}{t}}^1$  and  $\mathcal{X}_\circ = \mathcal{X} \setminus X_\infty \xrightarrow{\pi_\circ} \mathbb{A}_t^1$ , for restrictions. Henceforward,  $X$  will denote any threefold in the list  $V_{12}, V_{16}, V_{18}, R_1$ .

### Proof of theorem 3.1.3

Associated to  $X$  is a Newton polytope  $\Delta$ , and to the latter we associate a Minkowski polynomial  $\phi$ . Since by the proposition above,  $\phi$  is tempered, the family of higher Chow cycles lifts to a class  $[\Xi] \in CH^3(\mathcal{X}^\circ, 3)$  [[14],theorem 3.8], yielding by restriction a family of motivic cohomology classes  $[\Xi_t] \in H_{\mathcal{M}}^3(X_t, \mathbb{Q}(3))$  on the Landau-Ginzburg model. (On the smooth fibers these are just higher Chow cycles.)

The local system  $\mathbb{V} = R_{tr}^2 \pi_* \mathbb{Z}$  associated to the Landau-Ginzburg model of  $X$  has the following singular points:

- $V_{12} : t = 0, 17 \pm 12\sqrt{2}, \infty$
- $V_{16} : t = 0, 12 \pm 8\sqrt{2}, \infty$
- $V_{18} : t = 0, 9 \pm 6\sqrt{3}, \infty$
- $R_1 : t = 0, 4, 16, \infty$

(Besides 0 and  $\infty$ , these are just the roots of  $\sigma(D_{PF})$ .)

In each case, we have an involution  $\iota(t) = \frac{M}{t}$ , ( $M = 1, \frac{1}{16}, \frac{-1}{27}, 64$ ), exchanging say  $t_1$  and  $t_2$  with  $0 < |t_1| < |t_2| < \infty$ . The involution  $\iota$  gives then a correspondence  $I \in Z^2(\mathcal{X} \times \iota^* \mathcal{X})$  which gives a rational isomorphism between  $\mathbb{V}$  and  $\iota^* \mathbb{V}$ . Since  $I$  induces an isomorphism, the vanishing cycle  $\gamma_t$  at  $t = 0$  is sent to a rational multiple of the vanishing cycle  $\mu_t$  at  $t = \infty$ . Hence in a neighborhood of  $t = 0$ , we have:

$$\int_{\gamma_t} I^* \omega_{\iota(t)} = \int_{I_* \gamma_t} \omega_{\iota(t)} = n \int_{\mu_{\iota(t)}} \omega_{\iota(t)}, n \in \mathbb{Q}^* \quad (3.3.5)$$

Moreover, as a section of the Hodge bundle,  $\omega_t$  has a simple zero at  $t = \infty$  and no zero or poles anywhere else. So  $I^* \omega_{\iota(t)} = Ct \omega_t$ , for some  $C \in \mathbb{C}^*$ . If we set  $A(t) = \int_{\gamma_t} \omega_t$ , then  $A(0) = 1$ , and it follows that

$$\begin{aligned} C &= \lim_{t \rightarrow 0} \frac{n}{(2\pi i)^2 A(t)} \int_{\mu_{\iota(t)}} \text{Res}_{X_{\iota(t)}} \left( \frac{\frac{dx}{x} \wedge \frac{dy}{y} \wedge \frac{dz}{z}}{t - M\phi} \right) \\ &= -\frac{n}{M} \text{Res}_p^3 \left( \frac{dx \wedge dy \wedge dz}{xyz \cdot \phi(x, y, z)} \right), \end{aligned} \quad (3.3.6)$$

where  $p \in \text{sing}(X_\infty)$  is the point to which  $\mu_{\iota(t)}$  contracts. Hence  $C$  is rational and  $\tilde{\omega} := I^* \omega$  is a rational multiple of  $t\omega$ .

Now let  $\tilde{\Xi} := I^*\Xi \in H_{\mathcal{M}}^3(\mathcal{X}_o, \mathbb{Q}(3))$  be the pullback of the cycle, with fiberwise slices  $\tilde{\Xi}_t$ . If  $AJ$  is the Abel-Jacobi map<sup>2</sup> as above, then

$$AJ^{3,3}([\tilde{\Xi}_t]) \in H^2(X_t, \mathbb{C}/\mathbb{Q}(3)). \quad (3.3.7)$$

Taking  $\mathcal{R}_t$  to be any lift of this class to  $H^2(X_t, \mathbb{C})$ , we may define a normal function by:

$$\mathcal{N}(t) := \langle \mathcal{R}_t, \omega_t \rangle \quad (3.3.8)$$

By [14, Prop. 4.1],  $\mathcal{N}(t)$  has a power series of radius of convergence  $|t_2| > |t_1|$ . Moreover, by [14, p. 474], we have

$$D_{PF}(\mathcal{N}(t)) = \sigma(D_{PF})\mathcal{Y}(t), \quad (3.3.9)$$

where  $\mathcal{Y}(t) = (2\pi i)^2 \langle \tilde{\omega}_t, \nabla_{D_t}^2 \omega_t \rangle$  is the Yukawa coupling.

Applying [14, Rem. 4.4], the right-hand side of (3.3.9) takes the form  $kt$ , where (in view of (3.3.4))  $k = \lim_{t \rightarrow 0} \frac{\mathcal{Y}(t)}{t}$ . By writing  $\omega_t$  in terms of a basis of  $e^{\frac{\log(t)}{(2\pi i)}N} \mathbb{V}$  about  $t = 0$ , we find that  $k = C\kappa$  where  $\kappa$  is the (rational) nonzero entry of  $N^2$ . We conclude that

$$D_{PF}(\mathcal{N}(t)) = kt, k \in \mathbb{Q}^*. \quad (3.3.10)$$

Finally, if  $A(t) = \sum a_n t^n$  is the period sequence, then  $B(t) = \sum b_n t^n = -\mathcal{N}(t) + A(t)\mathcal{N}(0)$  is another solution for the Picard-Fuchs equation, so that

$$\mathcal{N}(t) = \sum (a_n \mathcal{N}(0) - b_n) t^n.$$

Since the radii of convergence for the generating series of  $a_n$  and  $b_n$  are both  $|t_1| < |t_2|$ , while that of  $a_n \mathcal{N}(0) - b_n$  is  $|t_2|$ , it follows that  $\frac{b_n}{a_n} \rightarrow \mathcal{N}(0)$ .  $\square$

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<sup>2</sup>In smooth fibers, AJ takes a rather simple form in terms of currents, see [31]



**Corollary 3.3.1.**  $\mathcal{N}(0)$  is (up to  $\mathbb{Q}(3)$ ) a multiple of  $\zeta(3)$ .

*Proof.* The proof is a direct consequence of the following commutative diagram (See [32]):

$$\begin{array}{ccc}
 H_{\mathcal{M}}^3(X_0, \mathbb{Q}(3)) & \xrightarrow{\cong} & K_5^{ind}(\mathbb{Q}) \\
 \downarrow_{AJ^{3,3}} & & \downarrow_{r_b} \\
 J^{3,3}(X_0) & \xrightarrow[\cong]{} & \frac{\mathbb{C}}{\mathbb{Q}(3)}
 \end{array} \tag{3.3.11}$$

Where the lower isomorphism is the pairing with  $\omega_0$  and  $r_b$  is the Borel regulator. The Abel-Jacobi map then reduces to the Borel regulator and by Borel's theorem it has to be multiple of  $\zeta(3)$ .  $\square$

**Remark 3.3.2.** An explicit computation of  $\mathcal{N}(0)$  for  $R_1$  has been written in [3]; the computation for  $V_{12}$  was done by M. Kerr and will be available in a forthcoming paper. Below we present the explicit computation of  $\mathcal{N}(0)$  in the case  $V_{16}$ :

**Example 3.3.3.** Consider  $V_{16}$  which has a Minkowski polynomial given by  $\phi = (x + 1)(y + 1)(z + 1)(1 + x + y + z)$ ; We change the coordinates to simplify the computations and use the same idea as [3]. The normal function  $\mathcal{N}$  at 0 takes the following form:

$$\mathcal{N}(0) = \int_{\nabla} R\{x, y, (1 - x - y)\} \tag{3.3.12}$$

Where  $\nabla$  is the “membrane”  $\nabla = \{(x,y) : -1 \leq y \leq 1, -y \leq x \leq 1\}$ . We have:

$$\begin{aligned}
\mathcal{N}(0) &= \int_{\nabla} \log(y) d\log(1-x-y) \wedge d\log(x) \\
&= \int_{-1}^1 \log(y) \left( \int_{-y}^1 \frac{dx}{x(1-x-y)} \right) dy \\
&= \int_{-1}^1 \log(y) \left( \int_{-y}^1 \frac{dx}{x(1-y)} + \int_{-y}^1 \frac{dx}{(1-y)(1-x-y)} \right) dy \\
&= 2 \int_{-1}^1 \log(y) \frac{\log(-y)}{(1-y)} dy \\
&\equiv 4 \int_{-1}^1 \log(1-y) \frac{\log(y)}{y} dy \quad \text{mod } \mathbb{Q}(3) \\
&\equiv -4 \sum_{k \geq 1} \frac{1}{k} \int_{-1}^1 \log(y) y^{k-1} dy \quad \text{mod } \mathbb{Q}(3) \\
&\equiv 8 \sum_{k \text{ odd}} \frac{1}{k^3} \quad \text{mod } \mathbb{Q}(3) \\
&\equiv 7\zeta(3) \quad \text{mod } \mathbb{Q}(3)
\end{aligned} \tag{3.3.13}$$

where the  $\mathbb{Q}(3)$  reflects the local ambiguity of  $\mathcal{N}$  by a  $\mathbb{Q}(3)$ -period of  $\tilde{\omega}$  (owing to the choice of lift  $\mathcal{R}$ ). Since the Apéry constant is a real number, we normalize  $\mathcal{N}$  locally by adding such a period to obtain  $\mathcal{N}(0) = 7\zeta(3)$ .

### 3.4 Maximally unipotent monodromy condition

The proof of Theorem 3.1.3 makes use of an involution of the family over  $t \mapsto \pm \frac{M}{t}$  to produce a cycle with no residues on the  $t = 0$  fiber, but with nontorsion associated normal function. That is, we use the involution to transport the residues of the cycle we *do* know how to construct (via temperedness) to over  $t = \infty$ .

What is absolutely certain is that without a second maximally unipotent monodromy fiber (at  $t = \infty$  in our four examples), such a normal function cannot exist. This follows

from injectivity of the topological invariant into

$$Hom_{\text{MHS}}(\mathbb{Q}(0), H^3(\mathcal{X}^*, \mathbb{Q}(3))) \subset \bigoplus_{\lambda \in \Sigma} Hom_{\text{MHS}}(\mathbb{Q}(0), H_2(X_\lambda, \mathbb{Q})),$$

where  $\Sigma \subset \mathbb{P}^1$  denotes the discriminant locus. As an immediate consequence, nothing like Theorem 3.1.3 can possibly hold for Golyshev's  $V_{10}$  and  $V_{14}$  examples.

While we could broaden the search to all local systems with more than one maximally unipotent monodromy point, those having an involution (or some other automorphism) represent our best chance for constructing cycles. Though it is required to apply a couple of the tools of [14] as written, the  $h_{tr}^2(X_t) = 3$  assumption is perhaps less essential; if we drop this, there are many other LG local systems with “potential involutivity”. Inspecting data from [7], we see that the period sequences 35, 49, 52, 53, 55, 59, 60, 62, 97 and 151 have monodromies that suggest the presence of an involution. This is something we will investigate in future works.

Finally, we omitted one case with  $h_{tr}^2(X_t) = 3$  and an involution, namely  $B_4$  (cf. [7]). This is because there is a second involution, namely  $t \mapsto -t$ , which probably rules out a meaningful Apéry constant (as  $|t_1| = |t_2|$ ).

# Chapter 4

## Elliptic surfaces with exceptional monodromy

There have been several constructions of family of varieties with exceptional monodromy group [13],[36]. In most cases, these constructions give Hodge structures with high weight(Hodge numbers spread out). Nicholas Katz was the first to obtain Hodge structures with low weight(Hodge numbers equal to  $(2, 3, 2)$ ) and geometric monodromy group  $G_2$ . In this chapter I will describe Katz's construction and prove explicitly that the geometric monodromy group of one of the family he constructed is  $G_2$ .

### 4.1 Katz family of elliptic surfaces

In [26], Nicholas Katz studies the appearance of  $G_2$  as the monodromy group of a family of elliptic surfaces. Katz describes 4 families, 3 of which have  $G_2$  as geometric monodromy group. For the sake of simplicity, I will work with one of the 3 families, but the exact same approach applies to the remaining two in which  $G_2$  occurs. We start off, by constructing the family of surfaces mentioned above.

Let  $\mathcal{E} \rightarrow \mathbb{P}^1 : y^2 = x(x-1)(x-z^2)$  be a rational elliptic surface with singular fibers at  $z = -1, 0, 1, \infty$ . For  $t \neq 0, \pm \frac{2}{3\sqrt{3}}, \infty$ , take a base change by:

$$E_t \rightarrow \mathbb{P}^1 : w^2 = tz(z-1)(z+1) + t^2 \tag{4.1.1}$$

The result is a family of elliptic surfaces  $X_t \rightarrow E_t$  with 7 singular fibers on each surface, as

described below:

$$\begin{array}{ccc}
\mathcal{X} & \leftrightarrow & X_t \\
\downarrow \pi & & \downarrow \pi_t \\
\mathcal{E} & \leftrightarrow & E_t \\
\downarrow & & \downarrow \\
\mathbb{P}^1 & \leftrightarrow & \{t\}
\end{array} \tag{4.1.2}$$

**Proposition 4.1.1.** For each  $X_t$  we have  $\dim(H_{tr}^2(X_t)) \leq 7$ .

*Proof.* Set  $X := X_t, E := E_t, \pi := X \rightarrow E$ . We have that  $X$  has 4 singular fibers of type  $I_2$ , 2 of type  $I_4$  and one of type  $I_8$ . Each fiber  $I_n$  contributes  $\frac{n}{12}$  to the degree of the Hodge bundle. Therefore,  $\pi_*\Omega_{X/E}^1$  has degree 2, i.e  $\pi_*\Omega_{X/E}^1 = \mathcal{O}(p+q)$ . Now:

$$\Omega_X^2 \cong \pi^*\pi_*\Omega_{X/E}^1 \otimes \pi^*\Omega_E^1 \cong \pi^*\Omega_E^1(p+q)$$

Which gives  $h_X^{2,0} = 2$ , and a similar analysis gives  $h_X^{1,0} = 1$ . Finally, Noether's formula gives:

$$h_X^{1,1} = 10 - 8h_X^{1,0} + 10h_X^{2,0} - K_X^2 = 22$$

We have 19 algebraic classes coming from the singular components of the singular fibers  $I_n$ , hence the transcendental part is at most  $26-19 = 7$ .  $\square$

**Remark 4.1.2.** In fact,  $\dim(H_{tr}^2(X_t)) = 7$  as we shall see.

We now describe a particular choice of 7-dimensional basis of 2-cycles that we will use henceforward. First, consider the 1-cycles  $\alpha, \beta, \gamma_{-1}, \gamma_0, \gamma_1$  over each  $E_t$ , as described in figure 4.1. Denote by  $\delta_1, \delta_2$  the basis for the local system over each point of  $E_t$ , with  $\delta_1 \cdot \delta_2 = 1$ . Let's see how they behave under the action of the monodromy, but first we analyze the situation over  $\mathbb{P}^1$  (with  $z^2$ -coordinate) before the double cover, i.e on  $y^2 = x(x-1)(x-z)$ , so that we can predict how the cycles change after the double cover.

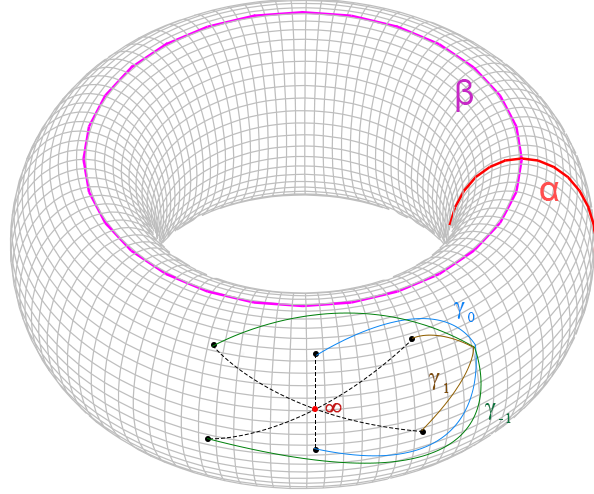


Figure 4.1: 1-cycles over the Base  $E_t$

The degeneration in this case is a nodal degeneration on  $0, 1$ , the monodromy matrices are then given by the Picard-Lefschetz formula:

$$\begin{aligned} T_0 &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\ T_1 &= \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \end{aligned} \tag{4.1.3}$$

Now consider the  $\mathbb{P}^1$  which has  $z$  coordinate. In order to make it simply connected, we draw some cuts over it. If we go through paths around  $-1, 0, 1$ , as described in figure 4.2, we can look at the image of those cycles under the double cover and see what the monodromy is. For example, as go around  $-1$  on the  $z$ -plane, the image goes to once around zero, then once around one and one more time around zero again on the  $z^2$ -plane, hence we can deduce that the local monodromy around  $-1$  is  $T_0 * T_1 * T_0^{-1}$ . Applying the same reasoning to  $0$  and  $1$ , we get the resulting monodromies:

$$\begin{aligned} \tilde{T}_{-1} &= \begin{pmatrix} -3 & 8 \\ -2 & 5 \end{pmatrix} \\ \tilde{T}_0 &= \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \\ \tilde{T}_1 &= \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \end{aligned} \tag{4.1.4}$$

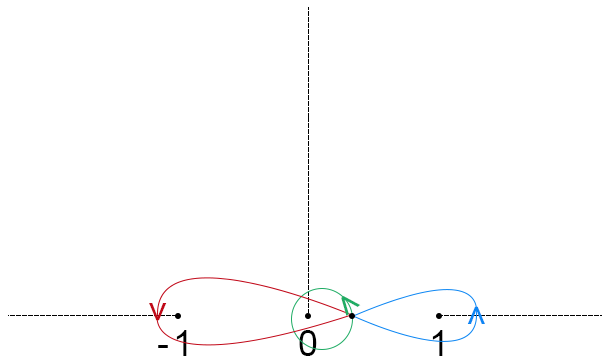


Figure 4.2: Cycles enclosing  $-1, 0$  and  $1$  in  $\mathbb{P}^1$  minus the cuts.

The vanishing cycle at each singular point is then:

- $2\delta_1 + \delta_2$  at  $-1$
- $\delta_1$  at  $0$
- $\delta_2$  at  $1$

Set  $\eta_1 = \delta_2$  and  $\eta_2 = 2\delta_1 + \delta_2$ , so  $\eta_1 \cdot \eta_2 = -2$  and the vanishing cycle at  $0$  is precisely  $\frac{1}{2}(\eta_1 + \eta_2)$ . We use henceforward the notation  $a \times b$  to denote the 2-cycle on  $X_t$  obtained by taking the 1-cycle  $a$  on a fiber of  $\pi_t$  and continuing it along the 1-cycle  $b$  on  $E_t$ .

## 4.2 Construction of the 2-cycles

Now that our notation is established we proceed with the definition of a 7-dimensional subspace of  $H_{tr}^2(X_t)$ :

$$\begin{aligned}
 A_1 &= \eta_1 \times \alpha & C_{-1} &= \eta_2 \times \gamma_{-1} \\
 A_2 &= \eta_2 \times \alpha & C_0 &= \frac{1}{2}(\eta_1 + \eta_2) \times \gamma_0 \\
 B_1 &= \eta_1 \times \beta & C_1 &= \eta_1 \times \gamma_1 \\
 B_2 &= \eta_2 \times \beta
 \end{aligned} \tag{4.2.1}$$

Note that,  $A_1, A_2, B_1, B_2$  are trivially transcendental, the same is not true for the  $C_i$ . The reason is that the  $C_i$  may—in fact they do—contain algebraic cycles resulting from classes of singular fibers. To overcome this, we have to “add” enough cycles in order to make all  $C_i$  transcendental.

Let’s take a closer look at the  $C_{-1}$ , for example. As we can see from figure 4.3, we can pick a cycle equivalent to  $C_{-1}$  but with minimal intersection, in other words:

$$\begin{aligned}
 C_{-1} \cdot D_- &= -1 \\
 C_{-1} \cdot D_+ &= 1 \\
 C_{-1} \cdot E_- &= -1 \\
 C_{-1} \cdot E_+ &= 1
 \end{aligned} \tag{4.2.2}$$

Now, let’s try to eliminate the intersections of  $C_{-1}$  with algebraic classes. Start by setting:

$$\tilde{C}_{-1} := C_{-1} + aD_- + bD_+ + cE_- + dE_+ \tag{4.2.3}$$

If  $\sigma$  is the class of the zero section, then the transcendental condition reduces to the following



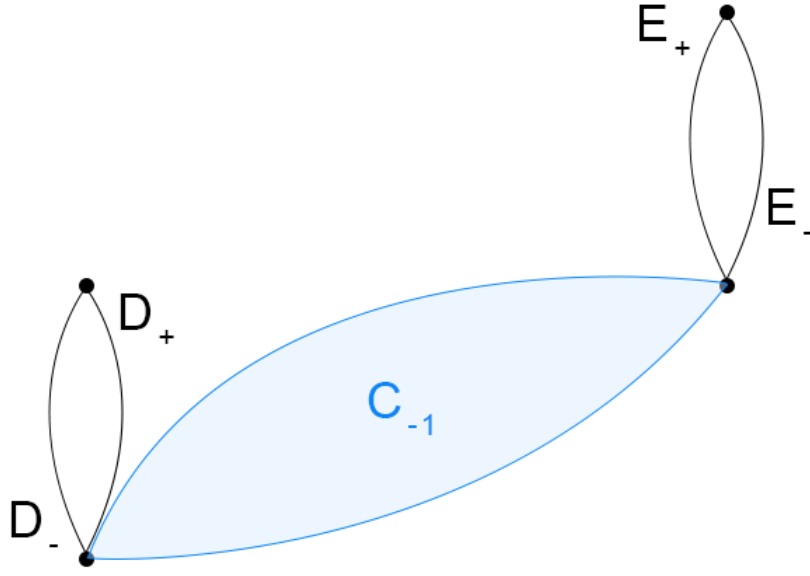


Figure 4.3: The 2-cycle  $C_{-1}$

system of linear equations:

$$\begin{aligned}
 \tilde{C}_{-1} \cdot D_{-} &= 0 \\
 \tilde{C}_{-1} \cdot D_{+} &= 0 \\
 \tilde{C}_{-1} \cdot \sigma &= 0 \\
 \tilde{C}_{-1} \cdot E_{-} &= 0 \\
 \tilde{C}_{-1} \cdot E_{+} &= 0
 \end{aligned} \tag{4.2.4}$$

Without loss of generality we may assume  $a = c = 0$ . Solving the system we get that:

$$\tilde{C}_{-1} = C_{-1} + \frac{1}{2}D_{-} + -\frac{1}{2}E_{-} \tag{4.2.5}$$

By following the exact same reasoning, we deduce that:

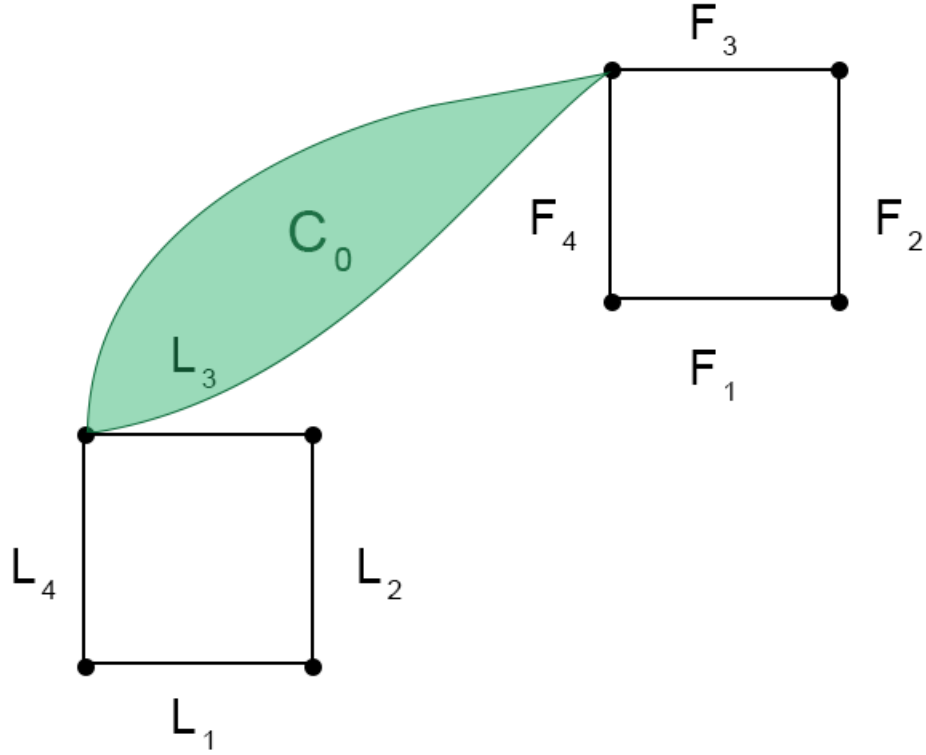


Figure 4.4: The 2-cycle  $C_0$

$$\tilde{C}_1 = C_1 + \frac{1}{2}G_- + -\frac{1}{2}H_- \quad (4.2.6)$$

where  $G_-$  and  $H_-$  are the components of the singular fibers of the endpoints.

Now we address  $C_0$ , consider the figure 4.4. Following the idea above, we set:

$$\tilde{C}_0 = C_0 + aL_1 + bL_2 + cL_3 - dF_1 - eF_2 - fF_3 \quad (4.2.7)$$

We again solve the system of equations required for transcendency:

$$\begin{aligned}
\tilde{C}_0 \cdot L_1 &= 0 \\
\tilde{C}_0 \cdot L_2 &= 0 \\
\tilde{C}_0 \cdot L_3 &= 0 \\
\tilde{C}_0 \cdot F_1 &= 0 \\
\tilde{C}_0 \cdot F_2 &= 0 \\
\tilde{C}_0 \cdot F_3 &= 0 \\
\tilde{C}_0 \cdot \sigma &= 0
\end{aligned} \tag{4.2.8}$$

The resulting cycle is:

$$\tilde{C}_0 = C_0 + \frac{3}{4}L_1 + \frac{1}{2}L_2 + \frac{1}{4}L_3 - \frac{3}{4}F_1 - \frac{1}{2}F_2 - \frac{1}{4}F_3 \tag{4.2.9}$$

### 4.3 Computation of the monodromies

Denote by  $V$  the space generated by the transcendental cycles  $(A_1, A_2, B_1, B_2, \widetilde{C}_{-1}, \widetilde{C}_0, \widetilde{C}_1)$ .

The intersection matrix is:

$$Q = \begin{bmatrix}
0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 2 \\
0 & 0 & 0 & 0 & 1 & -1/2 & -1 \\
0 & 0 & 0 & 0 & -2 & -1 & -1
\end{bmatrix}$$

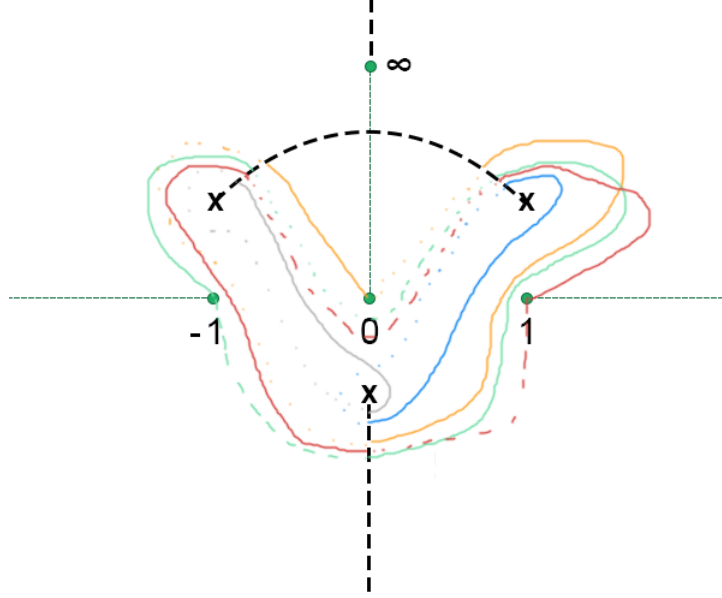


Figure 4.5: The 1-cycles  $\alpha$ ,  $\beta$ ,  $\gamma_{-1}$ ,  $\gamma_0$  and  $\gamma_1$  over the Elliptic curve  $E_t$

Notice that since  $\det(Q) \neq 0$ , we have  $\dim(V) = 7$ . Since  $V \subset H_{tr}^2(X_t)$ , Proposition 4.1.1 implies that  $\dim H_{tr}^2(X_t) = 7$ .

With our transcendental basis  $(A_1, A_2, B_1, B_2, \widetilde{C}_{-1}, \widetilde{C}_0, \widetilde{C}_1)$  defined, we now compute the monodromies matrices at the singular points  $t = \frac{-2}{3\sqrt{3}}, 0, \frac{2}{3\sqrt{3}}, \infty$ . For computational purposes, we will work with figure 4.5 instead of figure 4.1. When  $t \rightarrow \pm \frac{2}{3\sqrt{3}}$ , we have a nodal degeneration on the base curve  $E_t$ . In figure 4.5, such degeneration can be described as when the “x” of one cut merges itself with an “x” of the other cut.

It’s straightforward to conclude that in this case, the  $\widetilde{C}_i$  remain unchanged, while in the other cases the cycles over the vanishing cycles remain unchanged.

Finally, the cycles that do change, do it so according to the Picard-Lefschetz formula since the degeneration is nodal. In conclusion, we have the following monodromies for  $\frac{2}{3\sqrt{3}}, \frac{-2}{3\sqrt{3}}$

respectively:

$$\begin{aligned}
 M_+ &= \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
 M_- &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
 \end{aligned} \tag{4.3.1}$$

The situation when  $t \rightarrow 0$  is much more subtle. If one looks at figure 4.5, the endpoints of the cuts behave roughly as  $-1 - \frac{t}{2}$ ,  $t$  and  $1 - \frac{t}{2}$ , therefore when  $t$  go through a path around 0, the endpoints will certainly move, but this time not in a nice way as they did in the case above, they will instead make the  $\gamma_i$  cycles cross each other and also  $\alpha$  and  $\beta$ . This is the crucial point which results in  $G_2$  monodromy, as we shall verify.

Let's start off by analyzing the resulting cycle  $\tilde{\alpha}$  of the monodromy action on  $\alpha$ . If we look at figure 4.6, we see not only  $\alpha$  is no longer a vanishing cycle, but also that it crosses the cuts trivializing the local system. What that means basically is that  $\eta_1$  and  $\eta_2$  might change after monodromy; this is in fact the case, as we shall see.

Now, consider  $\tilde{\alpha} - \alpha$ , as depicted in figure 4.7. Note that the vanishing cycle at 1 is

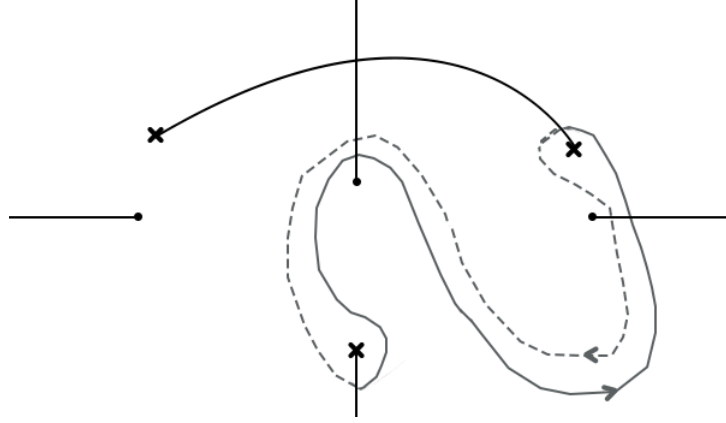


Figure 4.6:  $\tilde{\alpha}$ , the resulting cycle after monodromy

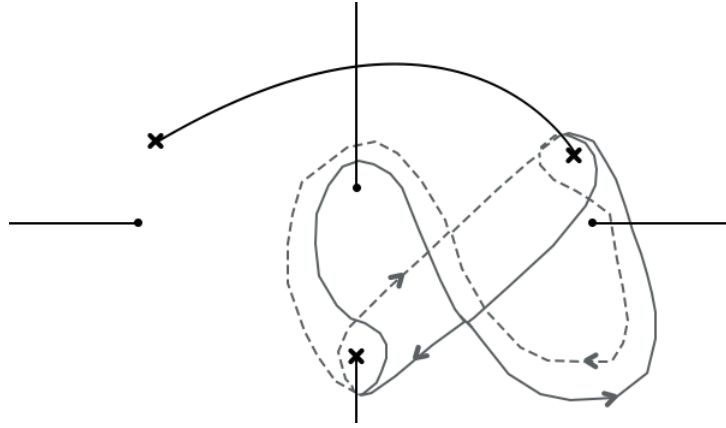


Figure 4.7:  $\tilde{\alpha} + \alpha$

$\eta_2$ , hence any cycle which is the continuation of  $\eta_2$  won't have monodromy around 1, so we can simplify  $\tilde{\alpha} - \alpha$  to encircle only 0, and vice-versa. Using the expression for the local monodromies 4.1.4, we can compute the resulting 2-cycles for the ones that are over  $\alpha$ , i.e  $A_1, A_2$ . Denote by  $M_0$  the monodromy at 0, then:

$$\begin{aligned}
 M_0(A_1) &= A_1 - 2A_2 + 2B_1 - 2B_2 - 4\tilde{C}_0 \\
 M_0(A_2) &= 2A_1 - 3A_2 + 6B_1 - 2B_2 - 4\tilde{C}_0 - 4\tilde{C}_1
 \end{aligned}
 \tag{4.3.2}$$

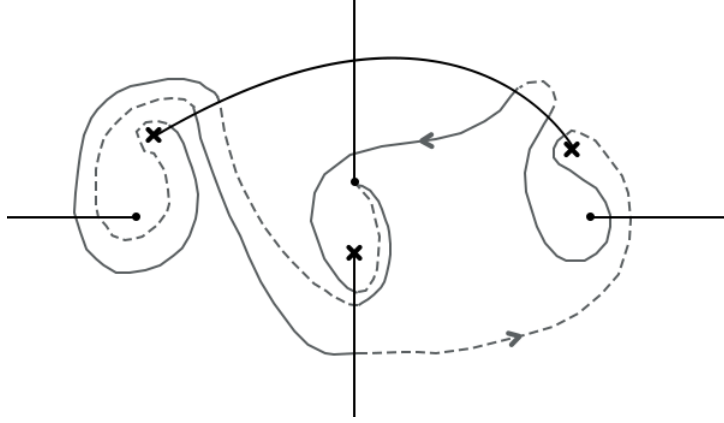


Figure 4.8:  $\tilde{\gamma}_0$ , the resulting cycle after monodromy

Similarly, we can follow exact the same procedure for  $\beta$ . We get:

$$\begin{aligned} M_0(B_1) &= -2A_1 + 6A_2 - 3B_1 + 2B_2 - 4\widetilde{C}_{-1} + 4\widetilde{C}_0 \\ M_0(B_2) &= -2A_1 + 2A_2 - 2B_1 + B_2 + 4\widetilde{C}_0 \end{aligned} \quad (4.3.3)$$

Now, as figure 4.8 suggest, the case for each  $\gamma_i$  is more subtle. Contrary to the  $\alpha, \beta$  cases, the 2-cycle  $\widetilde{C}_0$ , for example, is formed by continuing a 1-cycle that involves both  $\eta_1, \eta_2$ , therefore we can't ignore any of the points  $-1, 0, 1$  in computing the monodromy. The result is the following:

$$M_0(\widetilde{C}_0) = -A_1 + 3A_2 - 3B_1 + B_2 - 2\widetilde{C}_{-1} + \widetilde{C}_0 + 2\widetilde{C}_1 \quad (4.3.4)$$

Following the same procedure again for the remaining cycles, we get:

$$\begin{aligned} M_0(\widetilde{C}_{-1}) &= 2A_1 - 4A_2 + 6B_1 - 2B_2 + \widetilde{C}_{-1} - 4\widetilde{C}_0 - 4\widetilde{C}_1 \\ M_0(\widetilde{C}_1) &= -2A_1 + 6A_2 - 4B_1 + 2B_2 - 4\widetilde{C}_{-1} + 4\widetilde{C}_0 + \widetilde{C}_1 \end{aligned} \quad (4.3.5)$$

Now we can write our full monodromy  $M_0$ :

$$M_0 = \begin{bmatrix} 1 & 2 & -2 & -2 & 2 & -1 & -2 \\ -2 & -3 & 6 & 2 & -4 & 3 & 6 \\ 2 & 6 & -3 & -2 & 6 & -3 & -4 \\ -2 & -2 & 2 & 1 & -2 & 1 & 2 \\ 0 & 0 & -4 & 0 & 1 & -2 & -4 \\ -4 & -4 & 4 & 4 & -4 & 1 & 4 \\ 0 & -4 & 0 & 0 & -4 & 2 & 1 \end{bmatrix} \quad (4.3.6)$$

Since we can rearrange the loops around  $-1, 0, 1, \infty$  so that their product is the identity, we naturally get the expression for  $M_\infty$  as the inverse of the product  $M_- \cdot M_0 \cdot M_+$ , leading to:

$$M_\infty = \begin{bmatrix} 0 & -4 & 1 & 0 & -4 & 2 & 2 \\ 4 & 0 & 4 & 1 & -2 & 2 & 4 \\ -1 & 4 & -3 & -2 & 6 & -3 & -4 \\ 0 & -1 & 2 & 1 & -2 & 1 & 2 \\ -4 & 0 & -4 & 0 & 1 & -2 & -4 \\ 0 & 0 & 4 & 4 & -4 & 1 & 4 \\ 0 & -4 & 0 & 0 & -4 & 2 & 1 \end{bmatrix} \quad (4.3.7)$$

## 4.4 The period mapping for the Katz family

Recall that by the Monodromy theorem 1.3.3, all the monodromies are quasi-unipotent. Hence, all of them have a well-defined logarithm, which we will denote by  $N_i := \log(M_i)$ , see chapter 1 for a brief review of this topic.



A quick computation shows that  $M_0$  is in fact semi-simple, so the unipotent part  $(M_0)_{un}$  is the identity and hence  $N_0 = 0$ . The remaining monodromies do have non trivial logarithms:  $M_+, M_-$  are actually unipotent and  $M_\infty$  is the only non-unipotent. We can easily check that  $M_\infty^3$  is unipotent though.

If  $M_\infty = M_s \cdot M_u$  is the Jordan-Chevalley decomposition and  $I$  is the 7x7 identity matrix, then:

$$\begin{aligned}
N_+ &= M_+ - I \\
N_- &= M_- - I \\
N_\infty &:= \log(M_u) = \frac{1}{3} \log(M_\infty^3)
\end{aligned} \tag{4.4.1}$$

We have the following result concerning the monodromy group of the family  $X_t$ :

**Theorem 4.4.1.** The log-monodromies  $N_+, N_-, N_\infty$  generate  $\mathfrak{g}_2$ .

*Proof.* Consider the elements:

$$\begin{aligned}
Y_1 &= [N_-, N_+] & Y_8 &= [Y_5, Y_6] \\
Y_2 &= [N_-, N_\infty] & Y_9 &= [N_\infty, Y_5] \\
Y_3 &= [N_+, N_\infty] & Y_{10} &= [N_\infty, Y_9] \\
Y_4 &= [Y_1, Y_2] & Y_{11} &= [N_\infty, Y_{10}] \\
Y_5 &= [Y_1, Y_3] & Y_{12} &= [N_+, Y_{11}] \\
Y_6 &= [Y_2, Y_3] & Y_{13} &= [N_\infty, Y_{12}] \\
Y_7 &= [Y_2, Y_6] & Y_{14} &= [N_-, Y_{13}]
\end{aligned} \tag{4.4.2}$$

A quick computation leads us to:

**Lemma 4.4.2.** The elements  $N_-, N_+, Y_1, Y_4, Y_5, Y_6, Y_7, Y_8, Y_9, Y_{10}, Y_{11}, Y_{12}, Y_{13}, Y_{14}$  are linearly independent over  $\mathbb{Q}$ .

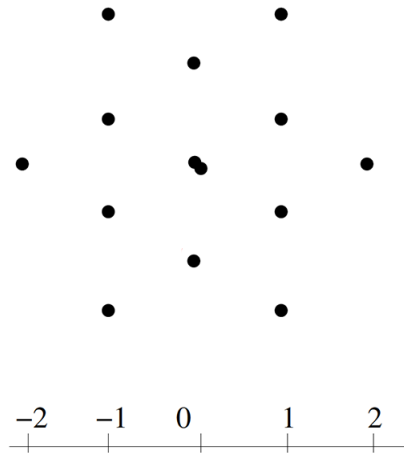


Figure 4.9: Roots of  $\mathfrak{g}_2$

Now define  $t_1 := Y_1$  and  $t_2 := [Y_4, Y_5]$ , a direct computation gives us that  $[t_1, t_2] = 0$ , moreover they both are diagonalizable. Let  $ad(\cdot)$  denotes the adjoint representation, if we act through  $ad(t_i), i = 1, 2$ , on  $\mathfrak{g}$ , we get 14 linearly independent (in both cases) eigenvectors with 1-dimensional eigenspaces, moreover we have:

- 1 with eigenvalue -2
- 4 with eigenvalue -1
- 4 with eigenvalue 0
- 4 with eigenvalue 1
- 1 with eigenvalue 2

Which are in 1-1 correspondence with the roots of  $\mathfrak{g}_2$ (see figure 4.9), therefore  $\mathfrak{h} := \langle t_1, t_2 \rangle$  is a Cartan subalgebra and  $\mathfrak{g} = \mathfrak{g}_2$ . □

This gives us the immediate corollary:

**Corollary 4.4.3.** The geometric monodromy group for the Katz family is  $G_2$ .

Henceforward, we denote by  $\Gamma \subset G_2$  the discrete subgroup generated by the monodromies  $M_-, M_0, M_+, M_\infty$ .

#### 4.4.1 On the generic global Torelli theorem

The generic Torelli theorem for the family  $X_t$  follows easily if the VHS determined by the latter satisfies the following proposition:

**Proposition 4.4.4** ([30]). Let  $\mathcal{V} \rightarrow B \setminus \{p_1, \dots, p_n\}$  be a variation of Hodge structures over a complete curve  $B$ , with associated period map  $\phi : B \rightarrow \Gamma \setminus D$ . If there is a point  $p_i$  such that the monodromy  $M_{p_i}$  is of infinite order and satisfies:

1.  $M_{p_i}$  is not a power of an element in  $\Gamma$ .
2. The limiting mixed Hodge structure at  $p_i$  is not of the same type from the one in  $p_k, k \neq i$ .

Then the map  $\phi$  is injective off a finite set.

*Proof.* Consider the variation of Hodge structure  $\mathcal{V} \boxtimes \mathcal{V}^* \rightarrow B \times B$ , given by the exterior tensor product of  $\mathcal{V}$  with  $\mathcal{V}^*$ . It has fiber  $Hom_{\mathbb{Z}}(V_{\mathbb{Z},b}, V_{\mathbb{Z},b'})$  over the point  $(b, b')$ . Let  $\Delta_B$  denote the diagonal of  $B \times B$ , by definition ( see chapter 1) we have that the Hodge locus  $B(id_{\mathcal{V}}) \supset \Delta_B$ .

Now suppose  $\phi$  is not injective off a finite set. Then there is a sequence  $s_n = (a_n, b_n) \in B(id_{\mathcal{V}}) \setminus \Delta_B$ , with distinct  $a_i, b_i$ . Since  $B(id_{\mathcal{V}})$  is algebraic [5], we either have  $B(id_{\mathcal{V}}) = B \times B$  or  $B(id_{\mathcal{V}})$  contains a 1 dimensional component  $C$  distinct from  $\Delta_B$ .

If the former holds, then  $\mathcal{V}$  is isotrivial and the limit mixed Hodge structures are the same (up to  $\Gamma$ -action), which contradicts our hypothesis of different LMHS.

If the latter holds, then  $\overline{B(id_V)}$  contains a point of the form  $(p_i, q)$ . Also:

$$\gamma(0_{id_V}) \in F_{lim,(p_i,q)}^0 \cap W_0^{(p_i,q)} \cap End(V_{\mathbb{Z}}) = Hom_{MHS}((V, F_{lim,p_i}^{\bullet}, W_{\bullet}^{p_i}), (V, F_{lim,q}^{\bullet}, W_{\bullet}^q))$$

Therefore,  $\gamma$  gives an isomorphism between the LMHS at  $p_i$  and  $q$ . By hypothesis, we can't have  $q = p_j$  for some  $j \neq i$ ; moreover if  $q \in C$ , then  $W_{\bullet}^q$  is trivial, but since  $M_{p_i}$  is of infinite order,  $W_{\bullet}^{p_i} = W(log(M_{p_i}))$  is not, a contradiction.

Lastly, if  $q = p_i$  and  $U$  is a neighborhood of  $p_i$ , we have that  $\phi$  is of degree  $d > 1$  on  $U \setminus p_i$ , hence is locally of the form:

$$\begin{array}{ccc} \Delta^* \cong U \setminus p_i & \rightarrow & \Delta^* \xrightarrow{\iota} \Gamma \setminus D \\ z & \longrightarrow & z^d \end{array} \quad (4.4.3)$$

set  $S := \iota_*(\text{generator of } \pi_1(\Delta^*))$ , then  $M_i = S^d$ , a contradiction.  $\square$

**Remark 4.4.5.** The argument in the proof above is still valid when we have LMHS of same type but not isomorphic.

Now in order to prove the generic global Torelli for the family  $X_t$ , all we have to do is to prove that  $M_-, M_0, M_+, M_{\infty}$  satisfy the hypothesis in the proposition above. The relevant LMHS are described in figure 4.10.

As we can see from figure 4.10, the LMHS at  $t = \infty$  can not be isomorphic to the ones at  $t_-, t_+$ . Moreover,  $M_{\infty}$  is of infinity order, hence the Torelli theorem in this case boils down to:

**Conjecture 4.4.6.** The monodromies  $M_-, M_+, M_0$  and  $M_{\infty}$  lie inside a copy of  $G_2(\mathbb{Z})$ . Moreover, if we denote by  $\Gamma \subset G_2(\mathbb{Z})$  the subgroup generated by them, then there is no  $S \in \Gamma$  such that  $M_{\infty} = S^k$  for some  $k \geq 2$ .

However, if we denote by  $M_{ss}$ , the semisimple part of  $M_{\infty}$ , then we have that  $M_{\infty} =$

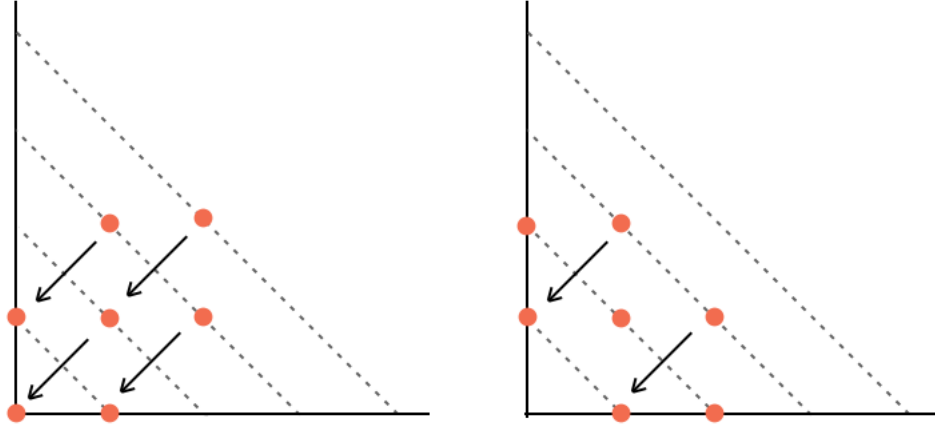


Figure 4.10: LMHS at  $\infty$  and  $t_-, t_+$  respectively. A bullet represent the dimension of the  $(p, q)$ -component of the Hodge structure on the Graded pieces.

$(e^{\frac{1}{2}N} M_{ss}^{-1})^2$ , which disproves the conjecture if  $e^{\frac{1}{2}N} M_{ss}^{-1} \in \Gamma$ . We do expect the following result to be true:

**Conjecture 4.4.7.** There is an automorphism  $\phi$  of the Katz family  $X_t$  such that  $\phi$  satisfies  $\pi \circ \phi = \iota \circ \pi$ , where  $\iota$  is the involution  $\iota : t \rightarrow -t$ . Moreover, the generic global Torelli holds for the family  $\tilde{X}_u$ , where  $u = t^2$ , and  $\tilde{X}_u$  obtained by the quotient of  $X_t$  by  $\phi$ .

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