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# Connectivity Bounds and S-Partitions for Triangulated Manifolds

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Connectivity Bounds and S-Partitions for Triangulated Manifolds

by

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A dissertation presented to  
The Graduate School  
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Dedicated to My Parents.



ABSTRACT OF THE DISSERTATION

Connectivity Bounds and S-Partitions for Triangulated Manifolds

by

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Two of the fundamental results in the theory of convex polytopes are Balinski's Theorem on connectivity and Bruggesser and Mani's theorem on shellability. Here we present results that attempt to generalize both results to triangulated manifolds. We obtain new connectivity bounds for complexes with certain missing faces and introduce a way to measure how far a manifold is from being shellable using S-partitions and the Stanley-Reisner Ring.

# Chapter 1

## Preliminaries

### 1.1 Introduction and Motivation:

This thesis has two main chapters that are somewhat separate but follow a similar narrative. The idea is to look at certain fundamental results in the field of convex polytopes and see how they generalize to triangulated manifolds. In particular we will be dealing with two well-known results:

**Theorem 1.1.** (*Balinski [3]*): *The graph of a  $d$ -dimensional convex polytope is  $d$ -connected.*

**Theorem 1.2.** (*Bruggesser, Mani [7]*): *Convex polytopes are shellable.*

In Chapter II we will provide a way to generalize Balinski's theorem to certain special classes of triangulated manifolds. The result that the theorem generalizes to triangulated manifolds is a well known theorem of Barnette([3]). More recently, stronger results were proved for special kinds of triangulations like flag complexes (see [2]). We introduce a simple approach rooted in algebraic topology that interpolates between the two results, and proves tighter bounds for complexes with missing faces of certain dimensions.

While Balinski's Theorem generalizes to manifolds, Theorem 1.2 fails for any manifolds that are not spheres or balls. The second part of this thesis (Chapter III) is dedicated to building a framework that can measure how badly manifolds fail to be shellable. It turns out a good approach here is to look at shellability both combinatorially and algebraically. We show that certain coarse S-partitions are good generalizations of shellability and explore

how these S-partitions give desirable properties of the topological and algebraic structure of the triangulated manifolds.

Why try to generalize to triangulated manifolds? There are many reasons: first of all, generalization is a natural mathematical inclination that will hopefully shed new light on the topic. Many of the most important questions in the field such as the g-conjecture are known to hold for convex polytopes but are still open for simplicial spheres and manifolds. Furthermore a lot of the proofs for convex polytopes rely on deep properties of convex geometry. However, in many cases (such as the Lower Bound Theorem [19]), it has been shown that the results extend to more complicated topological objects. Thus it seems worthwhile to explore what kind of results can be obtained solely using combinatorial and topological methods and seeing how this interplays with both geometric and algebraic techniques.

## 1.2 Simplicial Complexes

Throughout this thesis we will be dealing with simplicial complexes. A great resource for more in depth background on simplicial complexes and current problems related to their f-vectors is [19]. An (abstract) **simplicial complex**  $\Delta$  on a finite vertex set  $V(\Delta)$  is a collection of subsets of  $V(\Delta)$  that is closed under inclusion: if  $F \in \Delta$  and  $G \subseteq F$ , then  $G \in \Delta$ . The elements  $F \in \Delta$  are called **faces** and the maximal faces of  $\Delta$  under inclusion are called **facets**. The **dimension** of a face is  $\dim(F) = |F| - 1$  and the dimension of  $\Delta = \max\{\dim(F) | F \in \Delta\}$ . A simplicial complex is called **pure** if all facets have the same dimension.

Let  $\Delta$  be a  $(d-1)$  - dimensional simplicial complex. We define the **f-vector** to be the integer vector that counts the number of faces of  $\Delta$  in each dimension:  $f(\Delta) = (f_{-1}, f_0, f_1, \dots, f_{d-1})$  where  $f_i = |\{F \in \Delta | \dim(F) = i\}|$ . The entry  $f_{-1}$  is always 1 and corresponds to the empty set, the entry  $f_0$  counts all the vertices in the complex,  $f_1$  counts all the edges in the complex

and so on. We can also think of  $f$  as a polynomial instead of a vector:  $f(t, \Delta) = \sum_i f_{i-1} t^i$ . For various algebraic reasons discussed later it is more convenient to work with a certain integer transformation of the  $f$  vector called the  $h$ -vector. The **h-vector** =  $\{h_0, h_1, \dots, h_d\}$  is defined by the following polynomial relation:

$$\sum_{j=0}^d h_j(\Delta) t^{d-j} = \sum_{j=0}^d f_{j-1}(\Delta) (t-1)^{d-j}$$

From the relation we can express the  $h$ -vector as:

$$h_j(\Delta) = \sum_{i=0}^j (-1)^{j-i} \binom{d-i}{d-j} f_{i-1}(\Delta)$$

and the  $f$ -vector as:

$$f_{i-1}(\Delta) = \sum_{j=0}^i \binom{d-j}{d-i} h_j(\Delta)$$

The **join** of two simplicial complexes  $\Delta_1$  and  $\Delta_2$  on disjoint vertex sets, is the complex  $\Delta_1 \star \Delta_2 = \{F \cup G | F \in \Delta_1, G \in \Delta_2\}$ . It is easily seen that  $f(t, \Delta_1 \star \Delta_2) = f(t, \Delta_1) * f(t, \Delta_2)$ . If  $\Delta$  is a simplicial complex and  $F$  is a face of  $\Delta$  we define the **link** of  $F$  in  $\Delta$ ,  $lk(F)$  to be the following subcomplex:  $lk_{\Delta}(F) = \{G \in \Delta | G \cap F = \emptyset \text{ and } G \cup F \in \Delta\}$ . The **star** of a face  $F$  is the subcomplex  $st_{\Delta}(F) = \{G \in \Delta | G \cup F \in \Delta\}$ . Given a face  $F \in \Delta$  we define  $\overline{F}$  to be the simplicial complex that is the power set of  $F$  - that is  $\overline{F} = 2^F$ . Note that  $\overline{F}$  is the smallest simplicial complex containing  $F$ . Given a set of vertices  $W \in V(\Delta)$  we define the **induced subcomplex**  $\Delta_W$  to be  $\{F \in \Delta | F \subset W\}$ .

Let  $G := G(\Delta)$  be the graph of a simplicial complex i.e. the 1-dimensional simplicial complex formed by the edges and vertices in the complex. The **facet graph** of a pure complex  $\Delta$  is defined as the graph having as vertices the facets of  $\Delta$  with the facets  $F_1, F_2$  connected by an edge if their intersection has codimension 1:  $|F_1 \cap F_2| = |F_1| - 1 = |F_2| - 1$ . A **clique**  $F$  in  $G$  is a complete subgraph in  $G$ . Define the clique complex of  $G$  to be simplicial

complex  $X(G) = \{F \mid F \text{ is a clique in } G\}$ . A simplicial complex  $\Delta$  is called **flag** if it is equal to the clique complex of its 1-skeleton  $G(\Delta)$  i.e. if  $\Delta = X(G(\Delta))$ . There is an equivalent formulation of flag complexes in terms of missing faces that will come in handy later. Define  $F$  to be a **missing face** in  $\Delta$  if  $F \notin \Delta$  but  $F \setminus \{j\} \in \Delta$  for any  $j \in F$ . It is easy to prove that  $\Delta$  is flag if and only if it has no missing faces of dimension higher than 1.

Let's also introduce the notion of **banner complexes** ([8]) - these are lesser known but they are a good way to interpolate between flag and general complexes. Let  $\Delta$  be a  $(d - 1)$ -dimensional simplicial complex on the vertex set  $V(\Delta)$ . A subset  $W$  of  $V(\Delta)$  is called complete if every two vertices of  $W$  form an edge of  $\Delta$  (note that the notions of clique and complete set are the same, we just use complete sets here to follow the approach in [1]). A complete set  $W \subset V(\Delta)$  is critical if  $W \setminus \{v\}$  is a face of  $\Delta$  for some  $v \in W$ . We say that  $\Delta$  is **banner** if every critical complete set  $W$  of size at least  $d$  is a face of  $\Delta$ . We define the banner number of  $\Delta$  to be  $b(\Delta) = \min\{b \mid \text{lk}_\sigma \Delta \text{ is banner or the boundary of the } 2\text{-simplex for all faces } \sigma \in \Delta \text{ of cardinality } b\}$ . Note that a flag complex will always be banner so one can think of banner complexes as generalizations of flag complexes.

Although we will mostly be dealing with abstract simplicial complexes as defined above it is important to note that any abstract simplicial complex can be realized geometrically. A **convex polytope** is the convex hull of a finite set of points in  $\mathbb{R}^d$ . A **geometric  $k$ -simplex**  $\Delta^k$  is the convex hull of  $k + 1$  affinely independent points  $\{p_1, \dots, p_{k+1}\}$  in  $\mathbb{R}^n$  for some  $n > k$ . We can show that any abstract simplicial complex can be realized geometrically in Euclidean space: If  $n$  is the number of vertices of an abstract simplicial complex  $\Delta$ , then we can realize  $\Delta$  in  $\mathbb{R}^n$  by taking the basis elements of  $\mathbb{R}^n$  as the vertices and filling in all the simplices corresponding to faces in  $\Delta$ . We will denote this geometric realization by  $|\Delta|$ . For more details on geometric realizations and convex polytopes we point the reader towards the excellent book [24] by Ziegler.

## 1.3 Shellability

**Definition 1.3.** Let  $\Delta$  be a pure simplicial complex. A **shelling** of  $\Delta$  is an ordering of its facets:  $F_1, F_2, \dots, F_k$  such that  $\overline{F_i} \cap (\overline{F_1} \cup \dots \cup \overline{F_{i-1}})$  is pure and  $(\dim F_i - 1)$ -dimensional. A pure complex is called **shellable** if such a shelling exists.

Intuitively a shelling allows one to build a simplicial complex facet by facet, always gluing a new facet in a well-behaved way. It can be shown that if  $F_1, \dots, F_k$  is a shelling of  $\Delta$ , then for each  $i$ , there is some face  $R_i = r(F_i)$  of  $F_i$  such that  $\overline{F_i} \setminus \bigcup_{j=1}^{i-1} \overline{F_j}$  consists of all faces of  $F_i$  containing  $R_i$ . We write  $[R_i, F_i]$  for the interval of all such faces, and call  $R_i$  the **restriction face** of  $F_i$  - see [10] for more details. This construction partitions the simplicial complex into Boolean intervals - one for each shelling step:

$$\Delta = \cup_{i=1}^k [r(F_i), F_i]$$

Furthermore in the case of a shelling and corresponding partition the  $h$ -vector has a very simple, combinatorial interpretation:  $h_i = |\{r(F_j) \mid |r(F_j)| = i\}|$ . In other words,  $h_i$  just counts the restriction faces of size  $i$ . Chapter 2 of the thesis will focus on trying to generalize shellability and its consequences to manifolds that are not spheres or balls.

## 1.4 Simplicial Homology

Let's define homology for a simplicial complex - a great book for more info on this is [16]. Let  $k$  be a field. Let  $\Delta$  be a simplicial complex on vertex set  $\{1, 2, \dots, n\}$ . For  $i \in \mathbb{Z}$  let  $F_i(\Delta)$  be the set of  $i$ -dimensional faces of  $\Delta$  and for each  $\sigma \in F_i(\Delta)$ , let  $e_\sigma$  denote the corresponding basis vector in the  $k$ -vector space,  $k^{F_i(\Delta)}$ . The (augmented) **chain complex over  $k$**  is the complex:

$$0 \rightarrow k^{F_{n-1}(\Delta)} \xrightarrow{\partial_{n-1}} k^{F_{n-2}(\Delta)} \dots \xrightarrow{\partial_0} k^{F_0(\Delta)} \rightarrow 0$$

where, for all  $i = 0, 1, \dots, n - 1$ , and  $\sigma \in F_i(\Delta)$ :

$$\partial_i(e_\sigma) = \sum_{j \in \sigma} \text{sign}(j, \sigma) e_{\sigma-j}$$

We take  $\text{sign}(j, \sigma) = (-1)^{i-1}$  if  $j$  is the  $i$ -th element of  $\sigma$  when the elements of  $\sigma$  are listed in increasing order. The reader can make the routine check that  $\partial_i \circ \partial_{i+1} = 0$ .

For  $i \in \mathbb{Z}$ , the  $i$ -th **reduced homology** of  $\Delta$  is the  $k$ -vector space:

$$\bar{H}_i(\Delta, k) = \text{kernel}(\partial_i) / \text{image}(\partial_{i+1})$$

In particular, elements of  $\ker(\partial_i)$  are called  $i$ -**cycles** and elements of  $\text{Im}(\partial_{i+1})$  are called  $i$ -**boundaries**. Also we will denote the dimension of the homology groups as  $\beta_i(\Delta, k) = \dim H_i(\Delta, k)$  and call  $(\beta_0, \dots, \beta_d)$  the **Betti vector** of  $\Delta$  (with respect to some field  $k$ ).

Let's introduce some special cases of simplicial complexes that are all to some extent discrete versions of the notion of a manifold: A  $d$ -dimensional simplicial complex  $\Delta$  is a **weak pseudomanifold** if it is pure and every  $d - 1$  face is contained in exactly two  $d$  faces. If in addition the link of each face is connected we call  $\Delta$  a **normal pseudomanifold**. A **pseudomanifold** is a weak pseudomanifold in which the facet graph is connected. It's not hard to show that every normal pseudomanifold is indeed a pseudomanifold.

Let's state some basic results on pseudomanifolds - we will need these later and it serves as a good warm-up for the main theorems in Chapter 2:

**Lemma 1.4.** [16] Let  $\Delta$  be a weak  $d$ -pseudomanifold. Then the following are equivalent:

- a)  $\Delta$  is a pseudomanifold
- b)  $H_d(\Delta, \mathbb{Z}_2) = \mathbb{Z}_2$

**Lemma 1.5.** Let  $\Delta$  be a shellable weak  $d$ -pseudomanifold. Then  $|\Delta|$  is homeomorphic to a  $d$ -ball or a  $d$ -sphere.

*Proof.* This is a slightly weaker form of Theorem 11.4 in [9] since shellability implies constructibility. □

However pseudomanifolds in higher dimensions can be relatively badly behaved and have singularities so stronger homological conditions need to be imposed to have better behaved complexes. One route here is to define triangulated manifolds: A simplicial complex  $\Delta$  is a **triangulated manifold** if  $|\Delta|$  is homeomorphic to a topological manifold. However in practice working with the geometric realization is not very convenient so relaxing the conditions and making a definition in terms of simplicial homology is preferred: A  **$k$ -homology sphere** is a simplicial complex  $\Delta$  such that  $H_*(lk(F), k) = H_*(S^{d-|F|-1}, k)$  for all faces  $F \in \Delta$ , including  $\emptyset$ . Similarly a  **$k$ -homology manifold** is a complex  $\Delta$  such that for any non-empty face  $F$ , the link of  $F$  in  $\Delta$ ,  $lk(F)$  is  $k$ -homology  $(d - |F| - 1)$ -sphere. It is not hard to show that indeed a triangulated manifold will be a  $k$ -homology manifold over any field (see [16]).

For a simplicial sphere, the  $h$ -vector satisfies a nice symmetry giving a complete set of linear relations between the numbers of faces of different dimensions:

$$h_i = h_{d-i}, 0 \leq i \leq d$$

these are known as the **Dehn–Sommerville equations** [24].

## 1.5 The Face Ring

The **Stanley–Reisner ring** associated to a complex  $\Delta$  on vertex set  $\{1, 2, \dots, n\}$  is defined as the quotient ring

$$k[\Delta] = k[x_1, \dots, x_n]/I_\Delta$$



where  $I_\Delta$  is the ideal generated by the square-free monomials corresponding to the non-faces of  $\Delta$ . An **l.s.o.p** is a collection of linear forms  $\{\theta_1, \dots, \theta_d\}$  in  $k[x_1, \dots, x_n]$ , such that  $k[\Delta]/(\theta_1, \dots, \theta_d)$  is a finite dimensional  $k$ -vector space. We will denote  $k[\Delta]/(\theta_1, \dots, \theta_d)$  by  $k(\Delta)$  and call it the **reduced Stanley-Resiner ring** of  $\Delta$  for the specific l.s.o.p we have chosen. Note that  $I_\Delta$  is a homogeneous ideal, i.e. it is generated by homogeneous elements. Therefore  $I_\Delta$  is also both  $\mathbb{N}$ -graded and  $\mathbb{N}^n$ -graded by degree - so we get an induced  $\mathbb{N}$  and  $\mathbb{N}^n$  grading on the quotient  $k[\Delta]$ . Let  $R$  be an  $\mathbb{N}$   $k$ -algebra. We define the **Hilbert series** of  $R$  by  $Hilb(R) = \sum_i (\dim_k R_i) t^i$ . We will dive more into details on the face ring in Chapter 3 but let's state two fundamental theorems that relate the face ring to homological properties of  $\Delta$ . Let's also mention that a great reference on the Stanley-Reisner ring is [22].

**Theorem 1.6.** [22] *Let  $\Delta$  be a  $k$ -homology manifold sphere or ball of dimension  $d - 1$  and let  $\{\theta_1, \dots, \theta_d\}$  be an l.s.o.p for  $k[\Delta]$  then:*

$$\dim_k k(\Delta)_j = h_j(\Delta)$$

Note that one can state the theorem above more compactly as:  $Hilb(k(\Delta)) = h(\Delta, t)$ . Also note that this relates the graded dimensions of the reduced Stanley Reisner ring to the h-vector which is a purely combinatorial invariant. One can extend the theorem above to homology manifolds but since these are topologically more complicated (in terms of the Betti vector) we need to take the homology into account. This was done by Schenzel [20] in the following theorem:

**Theorem 1.7.** [20] *Let  $\Delta$  be a  $k$ -homology manifold (with or without boundary) of dimension  $d - 1$  and let  $\{\theta_1, \dots, \theta_d\}$  be an l.s.o.p for  $k[\Delta]$  then:*

$$\dim_k k(\Delta)_j = h_j(\Delta) + \binom{d}{j} \sum_{i=1}^{j-1} (-1)^{(j-i-1)} \beta_{i-1}(\Delta, k)$$

We are going to be using Theorem 1.7 quite extensively in Chapter III.

## 1.6 Discrete Morse Theory And Collapsibility

Chapter III focuses on  $S$ -partitions and these are closely related to discrete Morse functions so it is worthwhile to briefly sketch the main ideas behind Discrete Morse Theory. Great resources for learning more about Morse Theory are [14] and [5] - most of the material here is inspired from the aforementioned resources. Let  $\Delta$  be a simplicial complex. A discrete Morse function on  $\Delta$  is a function which, roughly speaking, assigns higher numbers to higher dimensional simplices, with at most one exception, locally, at each simplex. More precisely:

**Definition 1.8.** A map  $f : \Delta \setminus \{\emptyset\} \rightarrow \mathbb{R}$  is a **discrete Morse function** if for each face  $\sigma \in \Delta$ :

- (i) there is at most one boundary facet  $\rho$  of  $\sigma$  such that  $f(\rho) \geq f(\sigma)$  and
- (ii) there is at most one face  $\tau$  having  $\sigma$  as boundary facet such that  $f(\tau) \leq f(\sigma)$ .

**Definition 1.9.** A simplex  $\sigma$  is called **critical** if:

- i) there is no boundary facet  $\rho$  of  $\sigma$  such that  $f(\rho) \geq f(\sigma)$  and
- ii) there is no face  $\tau$  having  $\sigma$  as a boundary facet such that  $f(\tau) \leq f(\sigma)$

A simplicial complex is **collapsible** if it admits a discrete Morse function with only one critical cell. Roughly speaking, collapsible simplicial complexes can be progressively retracted to a single vertex via some sequence of elementary combinatorial moves. Each of these moves reduces the size of the complex by deleting exactly two faces. The only requirements are that these two faces should be of consecutive dimension, and the larger of the two should be the unique face properly containing the smaller one (which is usually called a “free face”).

One of the main results of Discrete Morse Theory is the following: Suppose  $\Delta$  is a simplicial complex with a discrete Morse function. Then  $\Delta$  is homotopy equivalent to a CW

complex with exactly one cell of dimension  $p$  for each critical simplex of dimension  $p$ . This result implies the following **Morse Inequalities**:

**Lemma 1.10.** [14] Let  $f$  be a discrete Morse function on  $\Delta$  with  $m_i$  critical faces of dimension  $i$ . Then

$$m_i \geq \beta_i(\Delta, k)$$

**Definition 1.11.** A discrete Morse function  $f$  on a  $(d - 1)$ -complex  $\Delta$  is called  **$k$ -perfect** if  $m_i = \beta_i(\Delta, k)$  for  $0 \leq i \leq d - 1$ .

# Chapter 2

## Connectivity of 1-skeletons of Pseudomanifolds

### 2.1 Preliminaries and History

Given a polytope or a triangulated manifold it is natural to try to understand the structure of its 1-dimensional skeleton. This was first done by Steinitz in 1922 [23] where he solved the problem in the 3-dimensional case: the graphs of 3-polytopes are exactly the 3-connected planar graphs. Balinski [3] extended these results to any dimension by proving that the graphs of boundaries of  $d$ -polytopes - which are  $(d - 1)$ -spheres - are  $d$ -connected. This was generalized further by Barnette [4] who showed that the graph of every  $(d - 1)$ -dimensional triangulated manifold is  $d$ -connected.

More recently these results have been sharpened in cases where more is known about the structure of the simplicial complex: Athanasiadis [2] proved better connectivity bounds for flag pseudomanifolds and Bjorner and Vorwerk [8] and Adiprasito, Goodarzi and Verbaro [1] extended the results for banner complexes.

Here we present a straightforward approach rooted in combinatorial topology that extends and simplifies previous approaches. Here is the basic idea: Let  $\Delta$  be a triangulation of a (pseudo)manifold from which we remove a subset of vertices  $W$  together with the induced subcomplex  $\Delta_W$ . We will analyze how the number of connected components of the 1-skeleton  $G(\Delta)$  on  $V \setminus W$  vertices relates to  $H_{d-1}(\Delta_W)$ . In some cases one completely determines the other, in other cases the way  $H_{d-1}(\Delta_W)$  "sits" inside  $H_{d-1}(\Delta)$  matters. For pseudomanifolds we obtain that if  $H_{d-1}(\Delta_W)$  is trivial removing  $W$  does not disconnect the graph and use this to prove lower bounds on connectivity of different classes of complexes.

Let's introduce some new notions that will be used shortly: Let  $\Delta$  be a simplicial complex. The underlying graph (or 1-skeleton)  $G(\Delta)$  of  $\Delta$  is the graph obtained by restricting  $\Delta$  to faces of cardinality at most two. A graph  $G$  is **k-connected** if it has at least  $k$ -vertices and removing any  $k - 1$  vertices does not disconnect  $G$ . The **connectivity** of a graph  $G$  denoted by  $k(G)$  is the size of smallest set of vertices that, when removed, renders  $G$  disconnected. All simplicial complexes we consider will be pure. We will work with homology over  $\mathbb{Z}_2$ .

## 2.2 Main Connectivity Bound

We will introduce the notion of strong connected components and use it to prove the main theorem on the connectivity of manifolds. Let  $W$  be a subset of the vertex set of  $\Delta$ . We define the following relation on the facets of  $\Delta$  not contained in  $W$ :  $F_1 \sim F_n$  if there is a sequence of facets  $F_1, F_2, \dots, F_n$  such that  $F_i \cap F_{i+1}$  has co-dimension 1 in  $\Delta$  and is not contained in  $W$ . It's easy to see that this is an equivalence relation and we will call the equivalence classes thus obtained **strong components of  $\Delta/W$** . We will denote the number of such classes by  $S(\Delta/W)$ .

**Theorem 2.1.** *Let  $\Delta$  be a  $d$ -pseudomanifold and  $W$  a subset of vertices. If  $H_{d-1}(\Delta_W) = 0$  then removing  $W$  does not disconnect  $G(\Delta)$ .*

*Proof.* We will in fact prove the stronger statement:

$$\dim H_0(\Delta_{V-W}) \leq S(\Delta/W) \leq \dim H_{d-1}(\Delta_W) + 1 \quad (2.2.1)$$

Let  $K$  be a strong component in  $\Delta/W$  and let  $V(K) = \cup F_i$  taken over all facets  $F_i$  contained in  $K$ . Let  $v_1, v_2$  be two vertices in  $V(K)$  with  $v_i \in F_i$  for  $i = 1, 2$  where  $F_1, F_2$  are facets in  $K$ . We can then build an edge-path between  $v_1, v_2$  in  $\Delta_{V-W}$  by following the sequence of facets connecting  $F_1$  and  $F_2$  and at each step choosing a point in  $F_i \cap F_{i+1}$  which

is not in  $W$ . This proves the first inequality.

For the second inequality let  $K_1, K_2, \dots, K_n$  be the strong components of  $\Delta/W$ . Given a strong component  $K_i$  we can construct a corresponding chain in  $C_d(\Delta)$  as follows:

$$K_i = \sum_{F_i \in K_i} F_i$$

The  $K_i$ 's are elements in  $C_d(\Delta)$  and  $\partial_d(K_i) \in C_{d-1}(\Delta_W)$  since otherwise one could extend the strong component  $K_i$  over a  $d - 2$  face not contained in  $W$ . Since  $\partial(\partial(K_i) = 0)$  the cycles  $[\partial(K_i)]$  will be elements in  $H_{d-1}(\Delta_W)$ .

Now assume that the boundaries of a subset  $S \subset \Delta/W$  of the strong components satisfy a linear relation in  $H_{d-1}(\Delta_W)$ :

$$\sum_{K \in S} \partial(K) = \partial(\sigma), \sigma \in C_d(\Delta_W)$$

This implies that:

$$\partial\left(\sum_{K \in S} K - \sigma\right) = 0$$

However since  $\Delta$  is a pseudomanifold, the top homology of  $\Delta$  must be supported on the entire complex by Lemma 1.4 and this can only happen if all the  $K_i$  are in the sum. Thus we get that any  $n - 1$  strong components are linearly independent so  $S(\Delta/W) = n \leq \dim H_{d-1}(\Delta_W) + 1$  □

## 2.3 Stronger Results for Normal Pseudomanifolds

One would expect better estimates on the connectivity of  $G(\Delta)$  if we restrict ourselves to spaces without singularities, say triangulated manifolds. In fact a much more lax condition

is necessary, namely requiring that the links be connected, to force the first inequality in equation 2.2.1 to become an equality.

**Lemma 2.2.** If  $\Delta$  is a normal pseudomanifold then

$$\dim H_0(\Delta_{V-W}) = S(\Delta/W) = \dim(H_d(\Delta, \Delta_W))$$

*Proof.* For the first equality: Let  $v, w$  be in the same connected component of  $G(\Delta_{V-W})$ . We now need to show that any two facets  $F$  and  $G$  with the first containing  $v$  and the second  $w$  are in the same strong component of  $S(\Delta/W)$ . The way we will build the facet sequence is by following the path between  $v$  and  $w$  and taking advantage of the facet connectivity of the links. Say  $v_i, v_{i+1}$  are two consecutive vertices in the path from  $v$  to  $w$  and  $F_i, F_{i+1}$  facets with  $v_i \in F_i$  and  $v_{i+1} \in F_{i+1}$ . Now the link of  $v_i$  is easily checked to also be a normal pseudomanifold and thus facet connected. Let  $F_{i,i+1}$  be a face in the link of  $\{v_i, v_{i+1}\}$ . We can find a sequence of facets in  $\Delta$  going from  $F_i$  to  $F_{i,i+1}$  using the facet connectivity of  $\text{lk}v_i$ . Furthermore any two such facets have  $v_i$  in common so this is a strong sequence in  $S(\Delta/W)$ . Analogously we can find a sequence from  $F_{i,i+1}$  to  $F_{i+1}$  so  $F_i$  and  $F_{i+1}$  are in the same strong component of  $S(\Delta/W)$ . Following the path between  $v$  and  $w$  and applying the procedure above one gets that  $F$  and  $G$  are in the same strong component of  $S(\Delta/W)$ .

For the second equality: Let  $K_1, \dots, K_n$  be the strong components of  $\Delta/W$ . We can think of these as nonzero chains in  $C_d(\Delta, \Delta_W)$ . Furthermore  $\partial_d(K_i) \subset C_{d-1}(\Delta_W)$  since otherwise one could extend the strong component so  $K_1, \dots, K_n$  are elements in  $(H_d(\Delta, \Delta_W))$ . Since every facet not contained in  $W$  is in exactly one strong component,  $K_1, \dots, K_n$  will be linearly independent. So all we now need to prove is that the  $K_i$ 's span  $(H_d(\Delta, \Delta_W))$ .

Let  $\sigma$  be a relative cycle in  $\ker \partial_d$ ,  $F$  be a facet of  $\Delta$  contained in  $\sigma$  and  $K_F$  the corresponding strong component. Assume there is a facet  $F'$  in  $K_F$  that is not in  $\sigma$ . There will

thus exist  $F_i, F_{i+1}$  in the facet sequence connecting  $F$  and  $F'$  in such that  $F_i \cap F_{i+1} = G \not\subset W$  with  $F_i \in \sigma$  and  $F_{i+1} \notin \sigma$ . But since  $G$  is contained in exactly two facets we get that  $G$  is in  $\partial_d(\sigma) \subset W$  - a contradiction. So every relative cycle is a sum of the strong components. It follows that the strong components of  $\Delta/W$  are a basis for the kernel and the result follows.  $\square$

**Remark 2.3.** The normality condition is required for the first equality in Lemma 2.3.1. Take for example any triangulation of the pinched torus like  $T_1$  below in Figure 2.1. and look at a cycle  $K$  that transverses the complex. In this case removing  $C$  gives us a connected complex that has 2 strong connected components even though  $H_1(C) = \mathbb{Z}_2$ .

**Remark 2.4.** One might also be interested whether one can generalize connectivity results to weak-pseudomanifolds. The result here is negative. Take  $T_2$  to be the doubly pinched torus on the right in Figure 2.1.  $T_2$  is not a pseudomanifold since  $\dim(H_2(\Delta)) = 2$  and one can see that removing the two "pinch" points disconnects the graph. One can generalize this construction to get weak-pseudomanifolds of any dimension whose graphs have connectivity 2 - simply glue along two  $d$ -spheres at 2 vertices.

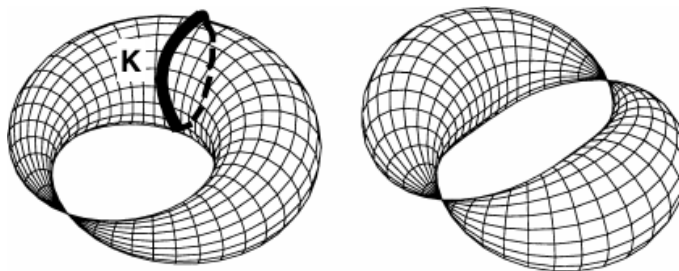


Figure 2.1: A Pinched Torus  $T_1$  and A Doubly Pinched Torus  $T_2$

The next theorem gives a more precise result in terms of not only a connectivity bound but how many connected components remain once some vertices are removed:



**Theorem 2.5.** Let  $\Delta$  be a normal pseudomanifold and let  $i$  be the inclusion map  $i : H_{d-1}(\Delta_W) \rightarrow H_{d-1}(\Delta)$ . Then  $\dim H_0(\Delta_{V-W}) = \dim(\ker i) + 1$ .

*Proof.* We will be using the fact that  $H_d(\Delta, \Delta_W)$  fits into the long exact sequence:

$$0 \longrightarrow H_d(\Delta) \longrightarrow H_d(\Delta, \Delta_W) \longrightarrow H_{d-1}(\Delta_W) \longrightarrow H_{d-1}(\Delta) \longrightarrow \dots \quad (2.3.1)$$

By using the exact sequence above we get  $S(\Delta/W) = \dim(H_d(\Delta, \Delta_W) = \dim(\ker i) + 1$  and combined with Lemma 2.2 the result follows.  $\square$

**Remark 2.6.** For normal pseudomanifolds we get that  $H_0(\Delta_{V-W}) \cong H_d(\Delta, \Delta_W)$  which can be interpreted as a weak form of Poincare-Lefschetz Duality for normal pseudomanifolds.

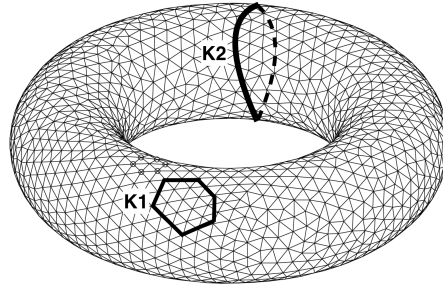


Figure 2.2: A Torus  $\Delta_T$  with two different cycles

**Example 2.7.** Theorem 2.5 tells us that the top homology of  $\Delta_W$  does not completely determine the connectivity of  $G(\Delta_{V-W})$ . In fact, we also need to know how the homology group  $H_{d-1}(\Delta_W)$  is embedded into  $H_d(\Delta)$ . As an illustrative example let's look at some triangulation of a torus  $\Delta_T$  as in Figure 2.2. Let  $K_2$  be a cycle cutting through the torus vertically and  $K_1$  be a cycle as in Figure 2.2.  $K_1$  and  $K_2$  both have isomorphic homology groups however removing  $K_1$  disconnects the graph  $G(\Delta_T)$  while removing  $K_2$  does not. Theorem 2.5 tells us why: in  $H_1(\Delta_T)$ ,  $K_2$  is a nontrivial cycle while  $K_1$  is trivial.

**Corollary 2.8.** If  $\Delta$  is a normal  $d$ -pseudomanifold with  $H_{d-1}(\Delta) = 0$  then  $\dim(H_0(\Delta_{V-W})) = \dim H_{d-1}(\Delta_W) + 1$  and  $k(G(\Delta)) = \min\{|W| : H_{d-1}(\Delta_W) \neq 0\}$

*Proof.* If  $H_{d-1}(\Delta) = 0$  the inclusion map  $i : H_{d-1}(\Delta_W) \rightarrow H_{d-1}(\Delta)$  is the zero map. So by Theorem 3.2 we get  $\dim(H_0(\Delta_{V-W})) = \dim H_{d-1}(\Delta_W) + 1$ .  $\square$

Note that the last corollary implies that  $\widetilde{H}_0(\Delta_{V-W}) \approx \widetilde{H}_{d-1}(\Delta_W)$  which is a weaker but more general form of the Alexander duality found in Stanley's Green Book [22]. The corollary tells us that normality and zero codimension one homology are enough to guarantee sphere-like homological behavior at the 1-skeleton level. It also tells us that in this case the connectivity of the 1-skeleton is completely determined by the  $d - 1$  homology of all induced sub-complexes.

## 2.4 Applications

We will now use Theorem 2.1 to to prove results on a class of complexes that interpolate between general simplicial complexes and flag complexes. Let  $C(i, d)$  the class of simplicial complexes with  $H_d(\Delta) \neq 0$  and no missing faces of dimension greater than  $i$ . These complexes were introduced in [17] by Nevo. For given  $i, d$  there exist unique integers  $0 \leq q$  and  $1 \leq r \leq i$  such that  $d + 1 = qi + r$ . Define

$$S(i, d) = \partial\sigma^i \star \dots \star \partial\sigma^i \star \partial\sigma^r$$

where  $\partial\sigma^i$ , the boundary of the  $i$ -simplex, appears  $q$ -times in the join. Nevo proved the following lower bounds on the number of vertices for complexes in  $C(i, d)$ .

**Lemma 2.9.** (Nevo [17]) If  $\Delta$  is in  $C(i, d)$  then  $f_0(\Delta) \geq f_0(S(i, d)) = q(i+1) + (r+1) = d + 1 + q + 1$

**Theorem 2.10.** *Let  $\Delta$  be a  $d$  - pseudomanifold with no missing faces of dimension higher than  $i$  then  $G(\Delta)$  is  $d + q + 1$  - connected.*

*Proof.* Assume removing the subset of vertices  $W$  disconnects  $G(\Delta)$ . By Theorem 1 we get that  $H_{d-1}(\Delta_W) \neq 0$  and since the complex  $\Delta_W$  is induced it cannot have any missing faces of dimension higher than  $i$ . It follows that  $\Delta_W \in C(i, d - 1)$  so  $f_0(\Delta_W) \geq f_0(S(i, d - 1)) \geq d + q + 1$ .

□

Note that the two previous results of Barnette [4](in the simplicial case) and Athanasiadis[2] follow from Theorem 2.10:

**Corollary 2.11.** Let  $\Delta$  be a pseudomanifold then  $G(\Delta)$  is  $d + 1$  connected. Furthermore if  $\Delta$  is a flag complex then  $G(\Delta)$  is  $2d$  connected.

**Example 2.12.** To make matters more concrete let's look at the case of 3-spheres. In this case we have three classes:  $C(1, 3), C(2, 3), C(3, 3)$ . Using theorem 2.10 we get the connectivity bounds in the table below. Further the theorem's proof gives us a good intuition as to why the bounds have these specific values - it has to do with the minimum number of vertices for polyhedra in the class  $S(i, 2)$  - see Figure 2.3 to visualize these polyhedra.

$\Delta$ in	Connectivity	$S(i, 2)$
$C(3, 3)$	4-connected	$\partial$ (tetrahedron)
$C(2, 3)$	5-connected	$\partial$ (triangular bypiramid)
$C(1, 3)$	6-connected	$\partial$ (octahedron)

Table 2.1: The connectivity bounds for 3-manifolds

**Remark 2.13.** : Results similar to Theorem 2.10 have been published in [1] using methods from algebraic topology similar to ours but also commutative algebra. The results there are

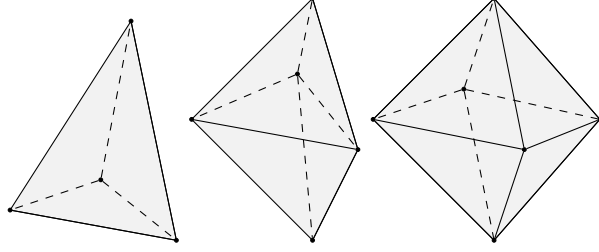


Figure 2.3: The polyhedra giving the lower connectivity bounds for  $C(i, 3)$

for a class of complexes called banner complexes which are a generalization of flag complexes. We will show however that the connectivity bounds we get are arbitrarily tighter for an infinite set of complexes. We will also show however that for certain complexes the bound obtained in [1] is better.

To construct our examples we will introduce a special complex as follows: let  $T_n = (C_3)^{n-1} \star C_4$ , where  $C_n$  is the empty cycle on  $n$  edges.  $T_n$  is homeomorphic to a  $2n - 1$ -sphere. Let's summarize the properties of  $T_n$  we are interested in the following lemma:

- Lemma 2.14.** : Let  $T_n$  as defined above. Then: a)  $T_n \in C(2, 2n - 1)$   
b)  $b(T_n) = 2(n - 1)$   
c)  $K(G(T_n)) = 3n - 1$

*Proof.* a)  $T_n$  is not flag since it contains empty triangles. We now need to show that it contains no empty simplices of dimension greater than 1. Assume  $S = \{x_1, \dots, x_n\} \subset V(T_n)$  such that  $\partial(2^S) \in T_n$ . Then  $S$  will be partitioned among the cycles in the join in  $T_n$  and no one part will have any empty faces since all  $n - 1$  subsets of  $S$  are contained in  $T_n$ . This implies that  $S \in T_n$  so  $T_n \in C(2, 2n - 1)$ .

b) It is easily seen that if  $\Delta$  is not a banner complex then the suspension of  $\Delta$  and  $\Delta \star \{x_1, x_2\}$  - the join with an edge, will not a banner complex either - simply join the critical face in the  $\Delta$  with a vertex or the new edge respectively to get a critical face in the new complex. For any  $k < 2(n - 1)$  we can find a set  $F$  in  $V(T_n)$  such that  $lk_{T_n}(F)$  is the

join of one 3-cycle, edges, and isolated vertices. Removing any vertex in the unique 3-cycle in the link gives us a critical face that is not contained in the link and this implies that  $lk_{T_n}(F)$  is not banner. In the case where  $k = 2(n-1)$  the links of all faces of cardinality  $k$  will be either 3 or 4 cycles and this implies  $b(T_n) = 2(n - 1)$ .

c) By the join construction one has to remove all the three cycles  $C_3$  in  $T_n$  to disconnect the graph  $G(T_n)$ . This leaves the 4-cycle  $C_4$  which can be disconnected by removing any two vertices. The total number of vertices removed is  $3(n - 1) + 2 = 3n - 1$  and the result follows. □

Now we can use the above lemma to compare the connectivity bounds: Using theorem 2.10 above together with Lemma 2.4.3 part b) we get that  $G(T_n)$  is  $3n - 1$  connected which as we have seen is in fact the best bound since it equals  $k(G(T_n))$ . However using Theorem 12 in [1] the bound is  $2(\dim(T_n) + 1) - b(T_n) - 2 = 2n$  which is a weaker bound that follows directly from Balinski's Theorem.

Conversely there are instances where Theorem 12 in [1] gives better bounds than the Theorem 2.10 above. This is in the case of triangulated spheres that are banner but not flag. Such an examples appears in [8] - see Example 3.5. In that case Bjorner and Vorwerk construct a banner 3-sphere, let's call it  $\Theta$ , that has a unique empty triangle. By the banner condition Theorem 12 in [1] we get the connectivity bound to be 6, better than 5, which is the best one could get using Theorem 2.10.

## 2.5 Manifolds with Boundary and f-vector approaches:

So far all of our results have been limited to manifolds without boundary. It seems natural to see if we can obtain similar results on the connectivity of the 1-skeleta of manifolds with boundary. Here the result is as follows:

**Lemma 2.15.** Let  $\Delta$  be a  $d$ -pseudomanifold with boundary. Then  $G(\Delta)$  is  $d$ -connected.

*Proof.* We will take advantage of the fact that  $\Delta$  is strongly connected i.e. that the facet graph of  $\Delta$  is connected. Assume  $\Delta$  has original vertex set  $V$  and that we remove  $d-1$  vertices  $W$ . We want to show  $G(\Delta_{V-W})$  remains connected. Let  $v_1, v_2$  be vertices in  $V \setminus W$ . We know that the facet graph of  $\Delta$  is connected. Pick two facets  $F, F'$  with  $v_1 \in F$  and  $v_2 \in F'$ . There will exist a sequence of facets connecting the facets  $F$  and  $F'$ :  $F = F_1, F_2, \dots, F_k = F'$ . such that  $|F_i \cap F_{i+1}| = d$ . Now since we only removed  $d - 1$  vertices we can always find a vertex in every intersection to build a path from  $v_1$  to  $v_2$ . This show that the graph remains connected when removing any  $d - 1$  collection of vertices so we are done.  $\square$

**Remark 2.16.** We actually haven't used the boundary properties in the proof above - the only requirement is that the facet graph of  $\Delta$  is strongly connected. In particular this the lemma above also holds for manifolds with boundary but we obtained stronger results using Lemma 2.11

**Remark 2.17.** Let's show that the connectivity bound in Lemma 2.15 is tight. For a 2-dimensional example look at the complex with facets  $\{123, 234\}$ . The graph of this complex is clearly 2-connected but not 3-connected. More generally one can glue two  $d$ -manifolds with boundary along a boundary  $d-1$  face of both - let's call this face  $F$ . This gives a new complex whose graph is  $d$ -connected and that can be disconnected by removing the vertices in the facet  $F$  which has  $d$ -vertices.

**Remark 2.18.** For manifolds with boundary we cannot obtain tighter results for flag complexes. In particular the complex with facets  $\{123, 234\}$  is a flag 2-ball but still has connectivity 2.

Now we will tackle a slightly different problem - an Upper Bound Theorem. While the following lemma does not use graph connectivity directly it fits very well in the overall theme of trying to generalize results from spheres to manifolds. One of the most well known

problems in the theory of face vector enumeration is the upper bound theorem. This states that the  $f$ -vector for a simplicial sphere is bounded above by the  $f$ -vector of a simplicial polytope called the cyclic polytope. The **cyclic d-polytope** on  $n$ -vertices  $C_n^d$  is defined as the convex hull of  $n$ -points on the moment curve  $(t, t^2, \dots, t^d)$ . Cyclic polytopes play a very important role in the theory of upper bound theorems because as we shall see in the next Lemma they maximize the number of faces for simplicial spheres. Before stating the lemma we also need to introduce the concept of neighborliness: a **k-neighborly polytope** is a convex polytope in which every set of  $k$  or fewer vertices forms a face. Not that we can also define neighborliness for simplicial complexes:  $\Delta$  is  $k$ -neighborly if every set of  $k$  vertices forms a face in  $\Delta$ . It can be shown (see [24]) that  $C_n^d$  is  $\lfloor d/2 \rfloor$  neighborly which is, in fact, the maximal level of neighborliness possible for simplicial polytopes if we exclude simplices.

**Definition 2.19.** A  $d - 1$  simplicial complex  $\Delta$  on  $n$  vertices satisfies the **f-UBC** if  $f_i(\Delta) \leq f_i(C_n^d)$

The following well-known result by Stanley states that the f-UBC holds for simplicial spheres:

**Lemma 2.20.** [22] Let  $\Delta$  be a  $d - 1$   $k$ -homology simplicial sphere. Then  $\Delta$  satisfies the f-UBC.

Here we present a lemma that allows one to extend the f-UBC by looking at the links of vertices:

**Lemma 2.21.** Let  $\Delta$  be a  $d - 1$  dimensional simplicial complex, with  $d - 1$  odd, such that the f-UBC holds for all the links of vertices. Then the f-UBC holds for  $\Delta$ .

*Proof.* Let  $n$  be the number of vertices and let  $C_n^d$  be the cyclic  $d$ -polytope on  $n$  vertices. We have the following string of polynomial equalities and inequalities. Note that all the inequalities are coefficient-wise:

$$f'(\Delta, t) = \sum_{v \in V(\Delta)} f(\text{lk}_{\Delta} v, t)$$

Which follows from counting vertex face incidences in 2 different ways.

$$\sum_v f(\text{lk}_{\Delta} v, t) \leq n f(C_{n-1}^{d-1}, t)$$

This follows since the link of a vertex will have at most  $n - 1$  vertices.

$$n f(C_{n-1}^{d-1}, t) = \sum_{v \in C_n^d} f(C_{n-1}^{d-1}, t) = \sum_v f(\text{lk}_{C_n^d} v, t)$$

Here we are using the fact that  $f(C_{n-1}^{d-1}, t) = f(\text{lk}_{C_n^d} v, t)$  for  $d - 1$ -odd. This equality is true since both  $C_{n-1}^{d-1}$  and  $C_n^d$  are  $\lfloor (d - 1)/2 \rfloor$  neighborly and the Dehn-Sommerville ([24]) relations determine the other half of the  $f$ -polynomial.

$$\sum_v f(\text{lk}_{C_n^d} v, t) = f'(C_n^d, t)$$

and the last equality is proved again by double counting. Thus we get that  $f'(\Delta, t) \leq f'(C_n^d, t)$  - element-wise so  $f(\Delta, t) \leq f(C_n^d, t)$  and the assertion is proved.

□

**Remark 2.22.** The argument breaks down for even dimensional complexes ( $d$ -odd) since the link in the cyclic  $d$  polytope will not be as neighborly as the  $d - 1$  cyclic polytope.

**Remark 2.23.** It would be interesting to see if we get something similar for the  $h$ -vector.

Setting

$$f(t) = (1 + t)^d h\left(\frac{t}{1 + t}\right)$$



in  $f'(\Delta, t) = \sum_v f(lk_{\Delta}v, t)$ , we get, after some computations, that:

$$dh(\Delta, t) + (1 - t)h'(\Delta, t) = \sum_v h(lk_{\Delta}v, t)$$

but it seems like assuming the links satisfy  $h$ -UBC won't get us anywhere.

Lemma 2.21 above allows us to easily generalize Upper Bounds on the  $f$ -vectors from spheres to manifolds in odd-dimensions. Note the  $f$ -UBC for odd-dimensional manifolds was first proved by Novik in [18] using commutative algebra techniques.

**Lemma 2.24.** [18] The  $f$ -UBC holds for odd-dimensional manifolds

*Proof.* This follows directly from lemma 2.21 and lemma 2.20 since the links of vertices in homology manifolds are homology spheres. □

# Chapter 3

## S-Shellings for Triangulated Manifolds

### 3.1 Preliminaries and History

Shellability is a fundamental concept in combinatorial topology and polytope theory. A big breakthrough in the theory of shellability came when Brugesser and Mani proved that convex polytopes are shellable [7]. The more general question of whether all triangulated spheres are shellable was answered in the negative for dimensions larger than 3 by Lickorish [15]. His and subsequent results rely on certain knot constructions. Furthermore shellability puts heavy topological restrictions on  $\Delta$ :

**Lemma 3.1.** Let  $\Delta$  be homology manifold with Betti vector  $(\beta_0, \dots, \beta_{d-1})$  over some field  $k$ . If  $\beta_i \neq 0$  for any  $i$  in  $\{1, \dots, d-2\}$  then  $\Delta$  is not shellable.

In particular this implies that any non-sphere manifold like a torus will not be shellable. So how can one try to extend the notion of shellability to more complicated objects like triangulated tori? One approach is to instead look not at the definition of shellability but rather at its implications, both at the poset level and algebraic level. Let's recall two well-known results that exhibit the usefulness of shellings. The first result can be found in the preliminaries, restated here for emphasis:

**Lemma 3.2.** [10] Any  $F_1, \dots, F_k$  shelling for  $\Delta$  partitions the simplicial complex into posets - one for each shelling step:

$$\Delta = \cup_{i=1}^k [r(F_i), F_i]$$

**Lemma 3.3.** (Klee-Kleinschmidt) [22] Let  $F_1, \dots, F_k$  be a shelling for  $\Delta$  and  $\theta$  be an l.s.o.p for  $k[\Delta]$ . Then  $\{x^{r(F_i)}\}$  - the set of restrictions of the facets is a  $k$ -basis for  $k(\Delta) = k[\Delta]/(\theta)$

A generalization of shellability should have similar implications as the two previous lemmas. A good place to start is the theory of  $S$ -partitions developed by Chari [11].  $S$ -partitions allow one to build *any* simplicial complex sequentially by adding intervals one at a time:

**Definition 3.4.** Let  $\Delta$  be a  $d-1$  dimensional simplicial complex. We define an **S-partition** to be an ordered list of (not necessarily maximal) faces of  $\Delta$ :  $F_1, F_2, \dots, F_k$  such that the following three conditions hold:

- (i)  $\overline{F_i} \cap (\overline{F_1} \cup \dots \cup \overline{F_{i-1}})$  is pure and  $(\dim F_i - 1)$ -dimensional
- (ii) all the facets of  $\Delta$  are included in the ordering
- (iii)  $F_1$  is  $d - 1$  dimensional

If all the faces in the ordering are *facets* we recover the Bjorner-Wachs [10] definition for a non-pure shelling. Note that the definition above is slightly different than the one introduced by Chari in [11]. The only difference is point (iii) - we want to start every  $S$ -partition with a face of dimension  $d-1$ . This will ensure some of the algebraic proofs we do later flow more nicely.

Similar to a shelling, an  $S$  - partition gives a partition of (the poset)  $\Delta$  into intervals  $[G_i, F_i]$  such that  $\cup_{i=1}^k [G_i, F_i]$  is a simplicial complex for any  $k$ . We will call the singleton intervals of the form  $[F, F]$  **critical** faces. We will call the intervals  $[G_i, F_i]$  **parts**. Also let's denote by  $\Delta_j := \cup_{i=1}^j [r(F_i), F_i]$  - the simplicial complex at step  $j$  of the  $S$ -partition.

**Lemma 3.5.** (Chari [12]) Given an  $S$ -partition for  $\Delta$  one can construct discrete Morse functions on  $\Delta$  whose critical faces are exactly the critical faces in the  $S$ -partition.

Note that any simplicial complex admits many  $S$ -partitions. In particular we have the trivial  $S$ -partition into singleton intervals (note that this is an  $S$ -partition in Chari's definition

only since it breaks condition (iii) above but one could also start with a facet and add only singleton intervals after). Clearly this is not a very useful  $S$ -partition. So in order to extend the definition of shellability to manifolds we need something more restrictive than just an  $S$ -partition. It turns out that the "coarser" the  $S$ -partition is the more information it will give us about our simplicial complex  $\Delta$  both at an algebraic and homological level. So we could try to minimize the number of faces in an  $S$ -partition - however finding a minimum number given a simplicial complex is not easy. It turns out that one way to enforce minimality is to look at the Stanley-Reisner ring of  $\Delta$  modulo an l.s.o.p. In order to make this more precise we will first need to develop some algebraic machinery and then we can finally define the new extended version of shellability to basically be an  $S$ -partitions that satisfy Lemma 3.3.

## 3.2 Algebraic Machinery

Let's start with defining some combinatorial invariants of  $S$ -partitions. Let's also define the  $h^S$  **triangle** as follows: Let  $h_{s,i}^S$  to be the number of intervals in  $\mathcal{S}$  of the form  $[r(F), F]$  such that  $|F| = s$  and  $|r(F)| = i$ . Notice that

$$h_i = h_{i,i} + h_{i+1,i} + \dots + h_{d,i}$$

and  $c_i := h_{i,i}$  is the number of critical  $i$ -cells in  $\mathcal{S}$  as well as in the corresponding Morse function  $f_{\mathcal{S}}$ . Denote by  $c^{\mathcal{S}}(t) = \sum c_i t^i$ .

Note that the  $h^S$  triangle determines the  $f$  vector by the following relation:

$$f(t) = \sum_{i,j} h_{i,j} t^j (1+t)^{i-j}$$

We can also express the  $h$ -vector in terms of the  $h^S$  triangle as follows. Using the definition

of the  $f$  polynomial in terms of the  $h$ -polynomial we have

$$(1+t)^d h\left(\frac{t}{1+t}\right) = \sum_{i,j} h_{i,j} t^j (1+t)^{i-j}$$

and by doing a change of variable  $\lambda = \frac{t}{1+t}$  we get

$$h(\lambda) = \sum_{i,j} h_{i,j} \lambda^j (1-\lambda)^{d-i}$$

Let's quickly introduce some algebraic notation - a good reference is [19]. Recall that the **Stanley-Reisner ring** associated to a complex  $\Delta$  on vertex set  $\{1, 2, \dots, n\}$  is defined as the quotient ring

$$k[\Delta] = k[x_1, \dots, x_n]/I_\Delta$$

where  $I_\Delta$  is the ideal generated by the square-free monomials corresponding to the non-faces of  $\Delta$ . An **l.s.o.p** is a collection of linear forms  $\{\theta_1, \dots, \theta_d\}$  in  $k[x_1, \dots, x_n]$ , such that  $k[\Delta]/(\theta_1, \dots, \theta_d)$  is a finite dimensional  $k$ -vector space. We will denote  $k[\Delta]/(\theta_1, \dots, \theta_d)$  by  $k(\Delta)$  and call it the **reduced Stanley-Reisner ring** of  $\Delta$  for the specific l.s.o.p we have chosen. Given a face  $F \in \Delta$  we will also define  $x^F$  to be the monomial in  $k[\Delta]$  whose support is  $F$ :  $x^F = \prod_{i \in F} x_i$ .

We will also need the following technical definitions and result due to Stanley that characterizes l.s.o.p's in terms of a choice function. We are following the presentation in [10], section 12. For a set of linear forms  $\{\theta_1, \theta_2, \dots, \theta_d\}$  in  $k[\Delta]$  let  $M = (m_{i,j})$  be the  $d \times n$  matrix defined by  $\theta_i = \sum_{j=1}^n m_{i,j} x_j$ . Let  $F_1, \dots, F_t$  be the *facets* of  $\Delta$  and call a function  $C : [t] \rightarrow 2^{[d]}$  a **nonsingular choice function** if  $|C(j)| = |F_j|$  and the square submatrix with rows in  $C(j)$  and columns in  $F_j$  is nonsingular, for all facets  $F_j$ .

**Lemma 3.6.** (Stanley in [21], page 150) Let  $\{\theta_1, \theta_2, \dots, \theta_d\}$  be a set of linear forms in  $k[\Delta]$ . Then  $\{\theta_1, \theta_2, \dots, \theta_d\}$  is an l.s.o.p if and only if there exists a non-singular choice function.

By Chari's results in [12] one can use the fundamental theorem of discrete Morse theory to show that the critical faces in an  $\mathcal{S}$ -partitions determine a spanning set for the homology  $H_*(\Delta)$ . We will show next that a similar results holds true at the algebraic level in  $k(\Delta)$  in the following Lemma. One can interpret as a weak version of the Klee-Kleinschmidt Lemma for shellable complexes.

**Lemma 3.7.** Let  $\mathcal{S}$  be an  $\mathcal{S}$ -partition for  $\Delta$  with  $\Delta = \cup_{i=0}^n [r(F_i), F_i]$ . The monomials  $\{x^{r(F)} : |r(F)| = i\}$  span  $k(\Delta)_i$ .

*Proof.* We will prove this by induction on the number of faces in  $\mathcal{S}$ . If the partition has one element, the restriction will be the empty set and  $k(\Delta) = k$  as a  $k$ -vector space so the lemma is true in this case.

Now assume we have added faces  $F_1, \dots, F_{k-1}$  and now we are adding the interval  $[r(F_k), F_k]$ . Since  $r(F_k)$  is the unique minimal non-face added we get that

$$k[\Delta_k]/(x^{r(F_k)}) = k[\Delta_{k-1}]$$

as rings.

Now let  $\theta$  be an l.s.o.p of  $\Delta_k$ . By Lemma 3.6 above this will also be an l.s.o.p for  $\Delta_{k-1}$  so we get that:

$$k(\Delta_k)/(x^{r(F_k)}) = k(\Delta_{k-1})$$

.

Now by the induction hypothesis we have that  $\{x^{r(F_1)}, \dots, x^{r(F_{k-1})}\}$  span  $k(\Delta)$  so it suffices to show that  $x^{r(F_k)}x_i = 0$  in  $k(\Delta)$  for any  $x_i$ .

If  $i \notin F_k$  then  $\{i\} \cup r(F_k)$  is not a face of  $\Delta_k$  so  $x^{r(F_k)}x_i = 0$  in  $k[\Delta]$  and thus in  $k(\Delta)$  as well.

Now let's assume  $i \in F_k$  and  $F_k$  has cardinality  $l \leq d$ . By Lemma 3.6 there exists a

nonsingular choice function  $C$ . Now let's select the  $C(k)$  rows in the matrix  $M = (m_{i,j})$  defined as above by  $\theta_i = \sum_{j=1}^n m_{i,j}x_j$ . This gives us a  $l \times n$  matrix. Since the  $l \times l$  restriction associated with the facet  $F_k$  is non-singular we can now use Gaussian elimination to express  $x_i$  in terms of the  $\theta$ 's and monomials not in  $F_k$ :

$$x_i = \sum_{j=1}^l \alpha_j \theta_{C(j)} + \sum_{j \notin F_k} \beta_j x_j$$

with the  $\alpha$ 's and  $\beta$ 's in  $k$ . When we multiply by  $x^{r(F_k)}$  we get both sums on the right to be zero in  $k(\Delta)$ . Thus  $x_i x^{r(F_k)} = 0$  in  $k(\Delta)$  and we are done. □

Based on the previous lemma and Chari's result on Morse functions we get the following:

**Corollary 3.8.**

$$c^{\mathcal{S}}(t) \geq \text{Hilb}(H_*(\Delta), k)(t)$$

$$h^{\mathcal{S}}(t) \geq \text{Hilb}(k(\Delta), k)(t)$$

Where  $\text{Hilb}(H_*(\Delta), k)(t)$  is the Betti polynomial for  $\Delta$  over  $k$  counting homology ranks and  $\text{Hilb}(k(\Delta), k)(t)$  is the  $\mathbb{Z}$ -graded Hilbert series for the reduced Stanley Reisner ring. These inequalities lead naturally to the following important definitions:

### 3.3 S-Shellings

Let  $\Delta$  be a simplicial complex - we are now ready to define what an  $S$ -shelling is.

**Definition 3.9.** Let  $k$  be a field and  $\theta$  be an l.s.o.p for  $\Delta$  - a simplicial complex. Furthermore let  $\mathcal{S}$  be an S-partition for  $\Delta$  with  $\Delta = \cup_{i=0}^n [r(F_i), F_i]$ . We call  $\mathcal{S}$  an  $(\mathcal{S}, \theta)$  - **shelling** if the restriction monomials  $\{x^{r(F_i)}\}$  are a basis for  $k(\Delta) = k[\Delta]/(\theta)$ .

**Remark 3.10.** Note that an  $S$ -shelling depends on both the field  $k$  and the l.s.o.p  $\theta$  so the complete notation would be  $(S, \theta, k)$  shelling. Whenever we use the notation  $S$ -shelling we mean that for a fixed  $k$  and fixed l.s.o.p  $\theta$ . Furthermore we will prove that for triangulated manifolds the choice of l.s.o.p is irrelevant - see Theorem 3.16.

**Definition 3.11.** An  $S$ -partition is  **$k$ -perfect** if  $c^S(t) = \text{Hilb}(H_*(\Delta), k)(t)$ . This is equivalent to saying that the Morse function associated to the  $S$ -partition is  $k$ -perfect.

**Definition 3.12.** An  $S$ -partition is **minimal** if it contains the smallest number of intervals possible.

Now that we gave a working definition of a working notion of  $S$ -shellability we will try to see how we can make it more intuitive. First let's show that  $S$ -shellings are indeed minimal:

**Lemma 3.13.** An  $S$ -shelling is a minimal  $S$ -partition.

*Proof.* This follows since any  $S$ -partition will give a spanning set of  $k(\Delta)$  by Lemma 3.7.  $\square$

The original motivation for the introduction of  $S$ -shellings was to generalize shellings - let's check that that is indeed the case:

**Lemma 3.14.** Let  $\Delta$  be a simplicial complex. Any (regular) shelling is an  $S$ -shelling for any  $k$  and any l.s.o.p  $\theta$ .

*Proof.* This follows directly from Lemma 3.3 which implies that the restriction monomials will always be a basis for  $k(\Delta)$ .  $\square$

**Lemma 3.15.** An  $S$ -partition is an  $(S, \theta)$ -shelling if and only if the following equality holds:

$$h^S(t) = \text{Hilb}(k(\Delta), k)(t)$$



*Proof.* let  $\mathcal{S}$  be an  $S$ -partition for  $\Delta$  with  $\Delta = \cup_{i=0}^n [r(F_i), F_i]$ . If the equality holds then the restriction monomials  $\{x^{r(F_i)}\}$  are a spanning set that has the same dimension as  $k(\Delta)$  so  $\mathcal{S}$  is an  $S$ -shelling. Conversely the result follows since the degree of the restriction monomials equals to the size of the corresponding face.  $\square$

As it stands the definition of an  $S$ -shelling, is dependent on the system of parameters we choose and defined algebraically. These are not very desirable features - we'd like a more combinatorial interpretation of when an  $S$ -partition is an  $S$ -shelling. It turns out that one can get such a characterization for all triangulated manifolds. This is because Schenzel's formula gives the graded dimensions of  $k(\Delta)$  in terms of the  $h$  vector and homology of  $\Delta$ . This allows us a clean characterization of an  $S$ -shelling without having to use the l.s.o.p directly:

**Theorem 3.16.** *Let  $\Delta$  be a triangulated  $d - 1$  manifold (with or without boundary) and  $\mathcal{S}$  an  $S$ -partition for  $\Delta$ . Then  $\mathcal{S}$  is an  $S$ -shelling if and only if it has length*

$$f_{d-1} + \sum_{i=1}^{d-1} \beta_{i-1}(\Delta) \binom{d-1}{i} \quad (3.3.1)$$

*Proof.* By Schenzel's Formula [19](Theorem 29) we can compute the graded dimensions of  $k(\Delta)$  as follows:

$$\dim_k(k(\Delta)_j) = h_j(\Delta) + \binom{d}{j} \sum_{i=1}^{j-1} (-1)^{j-1-i} \beta_{i-1}(\Delta, k)$$

Now adding all the graded parts we get that  $k(\Delta)$  has dimension:

$$\sum_{j=0}^d h_j(\Delta) + \sum_{j=0}^d \binom{d}{j} \sum_{i=1}^{j-1} (-1)^{j-1-i} \beta_{i-1}(\Delta, k)$$

The first sum adds to  $f_{d-1}$  and the second sum is equal to

$$\sum_{i=1}^{d-1} \beta_{i-1}(\Delta) \sum_{j=i+1}^d \binom{d}{j} (-1)^{j-i-1} = \sum_{i=1}^{d-1} \beta_{i-1}(\Delta) \binom{d-1}{i}$$

and the result follows.  $\square$

To show the usefulness of  $S$ -shellings let's use the previous lemma to prove something about how small  $S$ -partitions can be for manifolds. To obtain such a bound we could use Discrete Morse Theory to obtain the following bound:

$$|\mathcal{S}| \geq f_{d-1} + \sum_{i=1}^{d-1} \beta_{i-1}(\Delta) \tag{3.3.2}$$

This follows since given an  $S$ -partition once can construct a Morse function with the same critical faces - thus the inequality above follows from the Morse inequalities. However this bound is relatively weak - in fact we can get much better bounds using  $S$ -partitions:

**Lemma 3.17.** Let  $\Delta$  be a  $d - 1$  triangulated manifold and  $\mathcal{S}$  an  $S$ -partition for  $\Delta$ . Then the size of  $\mathcal{S}$  has the following lower bound:

$$|\mathcal{S}| \geq f_{d-1} + \sum_{i=1}^{d-1} \beta_{i-1}(\Delta) \binom{d-1}{i} \tag{3.3.3}$$

*Proof.* Let  $\mathcal{S}$  be an  $S$ -partition for  $\Delta$  with  $\Delta = \cup_{i=0}^n [r(F_i), F_i]$ . By Lemma 3.7 we get that the monomials  $\{x^{r(F_i)}\}$  span  $k(\Delta)$ . But by Schenzel's formula and the computation in the previous lemma we know that the dimension of  $k(\Delta)$  is exactly  $f_{d-1} + \sum_{i=1}^{d-1} \beta_{i-1}(\Delta) \binom{d-1}{i}$ . Since a spanning set must have higher cardinality than a basis in a vector space the result follows.  $\square$

**Remark 3.18.** For a 2-manifold Lemma 3.16 tells us that an  $S$ -partition is an  $S$ -shelling if and only if it has exactly  $f_2 + \beta_1(\Delta, k)$  parts. This means that our  $S$ -shelling will correspond

to adding the facets just as in a shelling plus critical edges, one for each basis cycle in  $H_1(\Delta, k)$ .

**Question 3.19.** A natural question to ask at this point is the following: What is the relationship between a  $k$ -perfect  $S$ -partition, minimal  $S$ -partition, and an  $S$ -shelling?

**Lemma 3.20.** Let  $\Delta$  be a 2-manifold. Then an  $S$ -shelling is both minimal and  $k$ -perfect.

*Proof.* An  $S$ -shelling will have exactly  $f_2 + \beta_1(\Delta, k)$  parts by the previous remark and will be minimal by lemma 3.3.1 so the result follows since every critical edge will index a basis cycle in  $H_1(\Delta, k)$ . □

Note that since the restriction monomials span  $k(\Delta)$ , an  $S$ -shelling will always be minimal. However a  $k$ -perfect  $S$ -partition need not be minimal. Any collapse of a collapsible complex will give a perfect  $S$ -partition but this will usually not be minimal since it contains only intervals of size two. One could try at this point to "consolidate" the 2-partitions to create a coarser  $S$ -partitions. However as we shall see there are complexes that admit  $k$ -perfect  $S$ -partitions but are not  $S$ -shellable.

**Lemma 3.21.** There exist triangulated manifolds that admit  $k$ -perfect  $S$ -partitions but do not admit  $S$ -shellings.

*Proof.* Notice that if we restrict ourselves to spheres an  $S$ -partition will be an  $S$ -shelling if and only if it is a pure shelling. This follows since a triangulated sphere only has non-trivial homology in the top dimension. Also  $\Delta$  will admit a perfect  $S$ -partition if and only if it admits a perfect Morse function. This follows from Chari's result.

So now in order to prove our lemma we have to come up with a triangulated sphere that is perfect but not shellable. Coming up with such examples is not very easy, however in [6][Section 5.5] Benedetti and Lutz give an example of a triangulated 3-sphere with a knotted trefoil knot on 3-edges that admits a minimal Morse vector of  $(1, 0, 0, 1)$ . However such a sphere cannot be shellable because of the presence of the knot. □

**Question 3.22.** : Will a  $S$ -shelling always be  $k$ -perfect for every  $k$ ? The answer is yes for 2-manifolds.

### 3.4 Examples

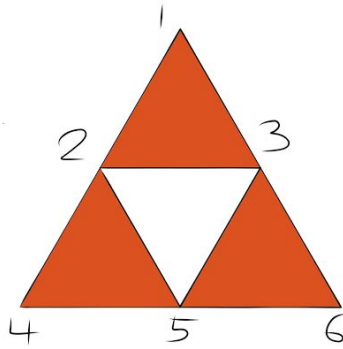


Figure 3.1: A Nonshellable 2-complex

**Example 3.23.** Let  $\Delta$  be the complex pictured in Figure 3.1 with facets  $\{1, 2, 3\}$ ,  $\{2, 4, 5\}$  and  $\{3, 5, 6\}$ . Note that  $I_\Delta = (x_1x_5, x_2x_6, x_3x_4, x_1x_4, x_4x_6, x_6x_1, x_2x_3x_5)$

An  $S$ -partition of  $\Delta$  is

$$\mathcal{S} = [\emptyset, 123], [5, 35], [6, 356], [25, 25], [4, 245]$$

$\mathcal{S}$  is a minimal  $S$ -partition as can be seen by brute-force trying all the possible facet orderings.

By Lemma 3.7 we get that the elements in  $\{1, x_5, x_6, x_2x_5, x_4\}$  span  $k(\Delta)$  however we shall see that they are not always linearly independent over  $k$ .

Let  $k = \mathbb{Q}$ . It's easy to check that  $\theta = (x_1 + x_5, x_2 + x_6, x_3 + x_4)$  is an l.s.o.p for  $k[\Delta]$ .

We then get the following relations in  $k(\Delta)$  :

$$x_4^2 = x_5^2 = x_6^2 = x_4x_5 = x_4x_6 = x_6x_5 = 0$$

so  $x_2x_5 = x_6x_5 = 0$  and  $\{1, x_5, x_6, x_4\}$  is a  $k$ -basis for  $k[\Delta]/(\theta)$ . We get that  $Hilb(k(\Delta)) = 1 + 3t \neq h_{\mathcal{S}}(t)$  for any  $S$ -partition  $\mathcal{S}$ . What this implies is that this complex does not have an  $S$ -shelling over  $\mathbb{Q}$  given the current l.s.o.p  $\theta$ , since any  $S$ -partition will have at least 5 parts.

In terms of simplicial homology the  $S$ -partition does give a basis for the homology in terms of the critical faces: When adding the interval  $[25, 25]$ ,  $5 - 2$  is a trivial cycle in  $\Delta_3$  since it is the boundary of, say,  $23+35$ . Thus  $\sigma_{25} = 23 + 35 - 52$  is a basis for  $H_1(\Delta)$ . In other words this  $S$ -partition is  $\mathbb{Z}_2$ -perfect.

**Example 3.24.** Minimal triangulation of  $RP^2$ :

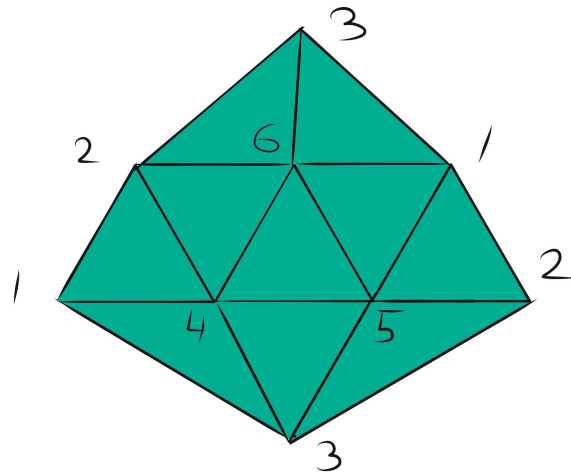


Figure 3.2: A minimal triangulation of  $RP^2$

Let  $\Delta$  be the complex pictured in Figure 3.2 - note the gluing along the boundaries.  $\Delta$  is a well known minimal triangulation of  $RP^2$ . We have the following minimal  $S$ -partition  $\mathcal{S}$ :

$[\emptyset, 124], [3, 143], [5, 345], [6, 456], [26, 264]$

$[36, 36], [23, 263], [16, 163], [15, 165], [25, 265], [235, 235]$

with

$$h_S(t) = 1 + 3t + 6t^2 + t^3$$

and spanning set

$$Q = \{1, x_3, x_5, x_6, x_2x_6, x_3x_6, x_2x_3, x_1x_6, x_1, x_5, x_2x_5, x_2x_3x_5\}$$

The regular  $h$ -vector of  $\Delta$  is  $(1,3,6,0)$  so Schenzel's formula gives us:

$$Hilb(k(\Delta)) = 1 + 3t + 6t^2 + \beta_1 t^3$$

So we see that  $Hilb(k(\Delta))(t) = h_S(t)$  iff  $\beta_1(\Delta, k) = 1$  iff  $k = \mathbb{Z}_2$ . So the  $S$ -partition above is an  $S$ -shelling only for  $k = \mathbb{Z}_2$ . Thus the restriction set  $Q$  is in fact a  $k$ -basis for  $k(\Delta)$  for any l.s.o.p when  $k = \mathbb{Z}_2$  and the theory of  $S$ -shellings allowed us to find it combinatorially.

In terms of future work related to  $S$ -shellings it would be interesting to see if there are any canonical ways of constructing  $S$ -shellings similar to line-shellings for polytopes [7]. The difficulty here is that there is no convex structure associated to an (abstract) simplicial triangulation. However even brute-force methods for finding  $S$ -shellings could be worthwhile exploring since these give  $k$ -bases for the reduced Stanley Reisner ring and would allow for problems in combinatorial commutative algebra to be approached from a purely combinatorial perspective.

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