

## Washington University in St. Louis Washington University Open Scholarship

---

All Theses and Dissertations (ETDs)

---

January 2011

# Three Essays in Macroeconomics

Kyoung Jin Choi

*Washington University in St. Louis*

Follow this and additional works at: <https://openscholarship.wustl.edu/etd>

---

### Recommended Citation

Choi, Kyoung Jin, "Three Essays in Macroeconomics" (2011). *All Theses and Dissertations (ETDs)*. 64.  
<https://openscholarship.wustl.edu/etd/64>

This Dissertation is brought to you for free and open access by Washington University Open Scholarship. It has been accepted for inclusion in All Theses and Dissertations (ETDs) by an authorized administrator of Washington University Open Scholarship. For more information, please contact [digital@wumail.wustl.edu](mailto:digital@wumail.wustl.edu).

WASHINGTON UNIVERSITY

Department of Economics

Dissertation Examination Committee:

Rodolfo Manuelli, Chair

Costas Azariadis

Michele Boldrin

Hong Liu

Adrian Peralta-Alva

Yongseok Shin

THREE ESSAYS IN MACROECONOMICS

by

Kyoung Jin Choi

A dissertation presented to the  
Graduate School of Arts and Sciences  
of Washington University in  
partial fulfillment of the  
requirements for the degree  
of Doctor of Philosophy

May 2011

Saint Louis, Missouri

# Dissertation Abstract

THREE ESSAYS IN MACROECONOMICS

by  
Kyoung Jin Choi

Doctor of Philosophy in Economics

Washington University in St. Louis, 2011

Professor Rodolfo Manuelli, Chair

The scope of the dissertation is (broadly-defined) general macroeconomics. The first essay is on *optimal taxation and capital structure*, the second essay is on *firm dynamics*, and the third essay is on *financial crises*.

The first essay clarifies the role of the corporate income tax (as a form of double taxation) for achieving socially optimal allocations in the Mirrlees framework when the government cannot tax unrealized capital income at the individual level. Use of the corporate tax requires changes in the individual capital tax. The novelty of the paper is that the sophisticated tax system is designed to influence the individual agent's portfolio choice of debt and equity, which in turn endogenizes the leverage ratio. The optimum corporate tax is indeterminate, but a minimal level is necessary. An immediate question is what happens to capital structure if we increase or decrease the level of the corporate tax. Surprisingly, unlike in classical capital structure theories, in this optimal tax mechanism, the firm's leverage ratio is independent of the corporate tax rate.

The second essay examines firm dynamics to explain the following empirical facts: (i) The size of a firm and its growth rate are negatively correlated; (ii)

but, they are often independent for firms above a certain size. Existing theories of firm dynamics can explain the first fact, but cannot explain the second. This paper studies a dynamic moral hazard problem under an AK-technology. In a first best world, the expected growth rate is strictly decreasing with capital. However, with information asymmetry our theory is consistent with both empirical facts because the optimal contract dictates under-investment in low-level capital states and over-investment in high-level capital states. The reason is that the given convex production technology becomes nonconvex in equilibrium due to the information asymmetry and the degree of the nonconvexity differs by the level of capital. We also fully characterize the agent's incentives. The capital accumulation mechanism induces incentive schemes that are different from optimal contracts in the literature on principal-agent models.

Finally, in the third essay<sup>1</sup>, we propose a model of financial crises as transitions from an efficient and unstable state to an inefficient and stable state in a simple economy with sector-specific shocks. The main driving force of this transition is the unwinding of unsecured loans. Introducing public debt increases the volatility of stock prices. We also discuss possible policy interventions.

---

<sup>1</sup>This essay is a joint work with Costas Azariadis.

## Acknowledgements

I am delighted to acknowledge my thesis committee for their assistance. First of all, I would like to express my appreciation to the committee chair, Rodolfo Manuelli. He has provided invaluable supports and persistent intellectual stimulus since I met him in University of Wisconsin-Madison. It was the best decision to transfer to Washington University in St. Louis following him. I also would like to thank Costas Azariadis for his endless encouragement and constant advices. It has been my great privilege and pleasure to learn from him and to work with him. I wish to thank Yongseok Shin who has been a great mentor during my graduate school years. I also thank other committee members, Michele Boldrin, Hong Liu and Adrian Peralta-Alva.

My special thanks go to my son, Jake, for making my life joyful over the course of my studying. Finally, I would like to express my deepest appreciation to my wife, Mi Sun You, for her love, patience, and encouragement. This dissertation is dedicated to her.

# Contents

<b>Abstract</b>	<b>i</b>
<b>Acknowledgements</b>	<b>iv</b>
<b>1 Mirrlees Meets Modigliani-Miller: Optimal Taxation and Capital Structure</b>	<b>1</b>
1.1 Introduction . . . . .	1
1.2 A Simple Environment . . . . .	7
1.3 Constrained Planning Optimum . . . . .	9
1.4 Known Tax Schemes . . . . .	13
1.4.1 Ramsey Taxation Scheme . . . . .	14
1.4.2 Standard Dynamics Taxation Scheme . . . . .	15
1.4.3 Tax Timing Options . . . . .	19
1.5 The Third Best Taxation Scheme . . . . .	23
1.5.1 When No Debt, But Only Equity is Available . . . . .	23

1.5.2	When Both Debt and Equity are Available . . . . .	24
1.5.3	A Simple Example . . . . .	32
1.6	Aggregate Leverage and Some Comparative Statics Analysis . . . . .	35
1.6.1	Endogenous Leverage . . . . .	36
1.6.2	Comparative Statics: Labor Taxation . . . . .	38
1.7	More Than Two Types . . . . .	41
1.7.1	Basic Results: A Simple Extension . . . . .	42
1.7.2	Endogenous Leverage for More than Two Types . . . . .	45
1.7.3	Indeterminacy . . . . .	47
1.7.4	Comparative Statics: Corporate Taxation . . . . .	48
1.7.5	Non-zero Aggregate Capital Taxes . . . . .	50
1.8	Practical Discussion on the Tax Scheme . . . . .	51
1.8.1	On the Corporate Income Tax History in U.S. . . . .	51
1.8.2	On the Assumption . . . . .	52
1.8.3	On the Data . . . . .	53
1.9	Other Generalization . . . . .	54
1.9.1	More than Three Periods . . . . .	54

1.9.2	Aggregate Uncertainty: Production Shock . . . . .	55
1.10	Literature Review . . . . .	57
1.11	Conclusion . . . . .	59
1.12	References . . . . .	61
<b>Appendices</b>		<b>65</b>
1.A	Appendix for Section 1.3 . . . . .	65
1.B	Appendix for Section 1.4 . . . . .	66
1.C	Appendix for Section 1.5 . . . . .	71
1.D	Appendix for Section 1.6 . . . . .	76
1.E	Appendix for Section 1.7 . . . . .	79
<b>2</b>	<b>Optimal Contracts and Firm Dynamics with AK Technology</b>	<b>85</b>
2.1	Introduction . . . . .	85
2.2	The Model and The Problem . . . . .	93
2.3	The Optimal Contract . . . . .	96
2.3.1	The First Best Contract . . . . .	98
2.3.2	The Second Best Contract . . . . .	101
2.4	The Optimal Growth Rates . . . . .	103



2.4.1	The Optimal Effort . . . . .	104
2.4.2	The Investment Comparison . . . . .	106
2.4.3	The Expected Growth Rate . . . . .	109
2.5	The Optimal Payment Schedule . . . . .	112
2.6	The Capital Process near the Boundary: Implication to Poverty Traps . . . . .	116
2.7	Risk Sharing and Business Cycle Implication . . . . .	120
2.8	Other Production Functions: Constant Returns to Scales . . . . .	121
2.8.1	Multiplicity . . . . .	122
2.8.2	Cobb-Douglas Case . . . . .	122
2.9	Concluding Remark . . . . .	123
2.10	References . . . . .	125

**Appendices** **131**

2.A	Proofs of Theorem 2, Proposition 12, and Corollary 5 . . . . .	131
2.B	Proofs of Theorem 3, Proposition 13, and Corollary 6 . . . . .	133
2.C	Proof of Lemma 7 and 8 . . . . .	140
2.D	Proof of Proposition 14 and Proposition 15 . . . . .	142

2.E	Proof of Proposition 16 . . . . .	146
2.F	Proof of Proposition 17 and 18 . . . . .	147
<b>3</b>	<b>Notes on Reputational Lending and Financial Crises</b>	<b>150</b>
3.1	Introduction . . . . .	150
3.2	Set-up . . . . .	154
3.3	Without Public Debt . . . . .	156
3.3.1	Optimal Bubble States: Reputational Lending . . . . .	158
3.3.2	Suboptimal No-Bubble States: Internal Finance . . . . .	160
3.3.3	Comparison . . . . .	162
3.4	Public Debt . . . . .	163
3.4.1	Optimal Bubble States: Reputational Lending . . . . .	164
3.4.2	Suboptimal No-Bubble States: Short Sale Constraints . . . . .	167
3.5	Conclusion . . . . .	175
3.6	References . . . . .	176
	<b>Appendices</b>	<b>177</b>
3.A	Proofs of Chapter 3 . . . . .	177
3.B	Internal Finance Dynamics . . . . .	184

## List of Tables

3.1	The recession between 04/1960 and 02/1961 (11 months): This table shows that the analysis in section 3.4.2 is quite consistent with the data. In fact, the data for all recessions defined by NBER since 1960 show the same consistency. . . . .	167
3.2	The recession between 12/2007 and 06/2009 (18 months): This table shows that the analysis in section 3.4.2 is quite consistent with the data. In fact, the data for all the recessions defined by NBER since 1960 show the same consistency. . . . .	168
3.3	The initial price is $p(1) = 32.3464$ . $\pi = 0.99$ , $\beta = 0.9898$ . . . . .	192
3.4	The initial price is $p(1) = 32.3464$ . $\pi = 0.98$ , $\beta = 0.9898$ . . . . .	192
3.5	The initial price is $p(1) = 48.5196$ . $\pi = 0.9$ , $\beta = 0.9898$ . . . . .	192

## List of Figures

- 2.1 The optimal effort level: black-top (first best), blue-middle (second best), dotted-bottom (second best limit). The parameter values are given by  $h = 6, \delta = 0.1, \sigma = 0.3, \lambda = 2, a = 1$  . . . . . 104
- 2.2 The production technology frontiers: The left panels are the separate plots of  $hk$  and  $e(k)$  and the right panels are aggregation of them, i.e.,  $f(k, e(k))$ . The top panels are the first best cases and the bottom panels are the second best cases. . . . . 108
- 2.3 Expected growth rate of the first best (dotted blue curve) and the second best (solid green curve); parameter values,  $h = 2.5, \lambda = 0.6, \sigma = 0.3, \beta = 0.94, a = 1$  . . . . . 109
- 3.1 Internal Finance Equilibrium Dynamics:  $(\bar{k}, \bar{p}) = (\frac{2\beta}{1-\beta}, \frac{\beta^2}{1-\beta^2}) \approx \frac{1}{2}(k^*, p^*)$  if  $\beta \approx 1$ . . . . . 160

3.2	The curve at the top is the sum of two curves. All data are annual (from 1960 to 2009). The source for the market value of public debt (nominal) excluding debt held by government (e.g., the Federal Reserve and the Social Security Trust Fund) is obtained from CRSP. The nominal stock market capitalization is obtained from the Federal Reserve's Flow of Funds Accounts (Table L.213, line marked as market value of domestic corporations). . . . .	165
3.3	The recession between 04/1960 and 02/1961 (16 months): The blue line plots represents the percent change in stock market capitalization, i.e., $(P_{t+1} - P_t)/P_t$ and the red line represents the percent change in the government debt held by the public, i.e., $(B_{t+1} - B_t)/B_t$ . . .	166
3.4	The recession between 12/2007 and 06/2009 (18 months): The blue line represents the percent change in stock market capitalization, i.e., $(P_{t+1} - P_t)/P_t$ and the red line represents the percent change in government debt held by the public, i.e., $(B_{t+1} - B_t)/B_t$ . . . . .	166
3.5	Sample price dynamics and the corresponding risk premium when $\pi = 0.98$ and $\beta = 0.9897$ which is the quarterly value of 0.96. . . . .	191

# Mirrlees Meets Modigliani-Miller: Optimal Taxation and Capital Structure

## 1.1 Introduction

Corporate taxation has been widely criticized for several reasons. First, the corporate income tax is one type of capital income taxes. A standard result in Ramsey taxation models is that capital income taxes should be zero immediately or at least in the long-run (Judd (1985), Chamley (1986), Jones, Manuelli, and Rossi (1997), etc).<sup>1,2</sup> Thus, the corporate tax should be avoided as well in the Ramsey framework. Secondly, but more importantly, common investors consider corporate taxation as a source of inefficiency since it is double taxation: corporations are owned by individual investors who are already subject to individual capital income taxes.<sup>3</sup> Some economists probably do not pay much attention to literal words 'double taxation'.<sup>4</sup>

---

<sup>1</sup>There are a few exceptions: Conesa, Sagiri and Krueger (2009) argued that the optimal capital tax rate should be significantly positive in an overlapping generations model with idiosyncratic, uninsurable income shocks and borrowing constraints. Chen, Chen, and Wang (2010) also derived the similar conclusion in a human capital-based endogenous growth model with the frictional labor market. However, neither of them specified the role of corporate taxation.

<sup>2</sup>The similar result also holds in Mirrlees tax models. For example, the net (expected) capital income tax is zero in Kocherlakota (2005).

<sup>3</sup>Not all countries have the double tax system although many countries including U.S. hold it.

<sup>4</sup>Suppose that by a certain reason the optimal total capital income tax rate should be 40%. Then, what is the difference between (20%, 20%) and (30%, 10%) pairs of corporate and individual capital income taxes? If the answer is simply 'no', double taxation by itself has no problem and

More academically meaningful questions would be first, why we need to impose a *separate tax* on the firm's profits and second, whether it is possible to replace the corporate tax by a capital tax at the individual level and vice versa. In this paper, by using a simple model we investigate reasons and conditions where the corporate income tax is required. We also answer the above two questions.

With these motivations in mind, this paper studies a dynamic Mirrlees taxation model<sup>5</sup> with an additional but realistic constraint in the tax scheme that the government cannot impose tax on unrealized capital income at the individual level. The summary of the main results is as follows. Even under this restriction in the tax scheme, the socially optimal (second best) allocation still can be implemented, but in a fairly different tax system from the standard ones of Kocherlakota (2005) and Albanesi and Sleet (2006). Moreover, in this tax system, the corporate tax is crucial as a decentralization device. The introduction of the corporate tax requires proper adjustment in other individual capital taxes. This sophisticated tax system influences the individual agent's portfolio choice of debt and equity, which in turn endogenizes the firm's capital structure as well. The optimum corporate tax rate is generally indeterminate, but it must be greater than or equal to a positive minimal level. Thus, the tax authority can design the corporate tax rate flexibly by adjusting the other tax rates. Surprisingly, unlike in classical capital structure theories, this co-movement property makes the leverage ratio independent of the change in the corporate tax rate. Finally, we also investigate the impact of labor tax on the leverage ratio and find some new results. The rest of the introduction describes the

---

this paper should not be written.

<sup>5</sup>The standard assumption in the Mirrlees tax framework is that the skill of each agent is private information and stochastically move over time. See Section 1.2 for the detailed assumption. See Kocherlakota (2005, 2009), Albanesi and Sleet (2006), Golosov and Tsyvinski (2007, 2008), Fahri and Werning (2008 a, b) and the references therein.

intuition and the detailed reasons for these results.

We first start by showing how a standard dynamic public finance tax system fails to achieve a socially optimal allocation under the assumption of the tax scheme mentioned above. Notice that U. S. households pay personal property tax if they hold real estate, vehicles, intangible assets (e.g., copyrights and patents), durable goods, and other assets. However, capital gains tax is not paid until assets are sold. Since we are interested in the assets that are being traded every second in the market, i.e., debt (bond) and equity (stock), we abstract from those less frequently traded asset markets and take the extreme, but realistic assumption that no tax is imposed on unrealized capital income. In other words, agents never pay *individual* taxes just by holding assets.

This assumption creates a nontrivial value for the tax timing option of the low skill agent, which is the option of whether to cash in their investment gains. In other words, the agent can evade taxes by deferring the realization.<sup>6</sup> In order to understand the effect of a tax timing option, we should notice the regressive property of the capital taxation scheme in the dynamic public taxation models of Kocherlakota (2005, 2009) and Albanesi and Sleet (2006). Let us describe the idea using a simple example. Suppose that the economy has homogenous agents at time 0 and some of them become high skilled and the others become low skilled in the next period with some probability. In a standard dynamic Mirrlees tax system, a low skill agent pays the capital income taxes while a high skill agent receives the capital subsidy. Then, the low skill agent does not want to realize gains in

---

<sup>6</sup>This idea may go back to Stiglitz (1973). Interested readers can refer to literature on tax timing options or tax arbitrages, for example, Constantinides (1983). The important contribution in this paper is to endogenize the optimal taxation as well as the optimal capital structure.



capital income at this period if that will help evade taxes. This deviation, in turn, undermines the socially optimal allocation. In order to remove the value of this tax timing option, the government should set up an additional tax at the corporation level. In other words, they should tax the corporate profits<sup>7</sup>, which leads to double taxation.

Perhaps the most important contribution of this paper is that the capital structure of the corporation is endogenously determined together with the optimal individual/corporate capital tax system. Use of the corporate income tax by itself cannot achieve the social optimum. Suppose the corporate tax,  $\tau_c$ , is designed to get rid of tax timing options of low skill agents. Then, similar to a common argument in the trade-off theory of capital structure, one might suspect that every agent chooses to hold corporate debt rather than equity just to avoid double taxation.<sup>8</sup> This 100% debt financing also allows the consumption of agents to deviate from the socially optimal allocation. However, we carefully design the individual capital tax system in accordance with the corporate tax. Technically, this capital tax system matches the agent's Euler equations, state-by-state with respect to equity holding and in average with respect to debt holding. This mechanism makes firms indifferent to any capital structure. Each individual agent, however, faces a portfolio selection problem between debt and equity whose after tax returns are different for each type of agent. More precisely, ex-post high skill agents will prefer to hold debt while ex-post low skill agents will prefer to hold equity under the optimal capital tax code. Thus, ex-ante, each agent should optimally choose the ratio of portfolios of debt and equity one-period ahead, which in turn determines the aggregate leverage

---

<sup>7</sup>The definition of corporate profits in the paper is total output minus total wage and debt payments, which is what is left to equity holders.

<sup>8</sup>We do not consider bankruptcy. Hence, there is no default risk on debt.

ratio in the economy.

An important property of the corporate tax is its indeterminacy above a minimal level. If the corporate tax rate falls below the minimum level, then the value of the tax timing option becomes nontrivial. However, any corporate tax rates greater than the minimal level can achieve the constrained optimum allocation if individual taxes are properly adjusted. This minimal level requirement implies that the corporate income tax can never be replaced by any taxes at the individual level. Even when the current corporate tax is sufficiently high and the government decrease (or increase) the rate, the other individual capital tax rates are not adjusted one-to-one according to the change in the corporate tax rate.<sup>9</sup> In addition, due to the existence of corporate taxes, the aggregate capital tax is nonzero in this setting.<sup>10</sup> On the other hand, if corporate taxes are indeterminate, how can they influence the leverage ratio? This question is also important in a normative sense. Notice that the leverage ratio is positively correlated with the level of corporate tax in conventional capital structure theories. However, in our optimal tax system, changes in the corporate tax level need not influence the leverage ratio because adjustment of the individual capital income tax levels offset the effect of the change in the corporate tax level.<sup>11</sup>

Given this analysis, we may have two evaluations on the past U.S. tax reforms with respect to the corporate income tax. First, by the multiplicity of choosing cor-

---

<sup>9</sup>For example, suppose that the current corporate tax is 50%. Assume that the government decrease the rate by 10%. Then, some individual tax rates should increase, but *not* by 10% in the optimal tax code. In particular, capital income taxes on debt may not change at all.

<sup>10</sup>Notice that the aggregate capital tax (or the conditional expectation of the next period tax) is zero in Kocherlakota (2005).

<sup>11</sup>Since our theory is normative, it is not fair to compare our result with the result of positive theories. However, we need to mention the difference.

porate taxes, one cannot say without carefully examining individual capital income taxes that the U.S. tax system has been very inefficient due to the historically high corporate tax rates. Secondly, the past U.S. tax reforms may not be inconsistent with the two long-run time series data of the corporate income tax rate and the aggregate leverage ratio in U.S. (See Section 7 for more discussion).

Finally, we also investigate the impact of the labor tax on the leverage ratio,<sup>12</sup> an issue that is not treated in the literature on capital structure. In our tax mechanism, an agent chooses between debt and equity to insure against future skill shocks. Thus, how much subsidy (tax) an agent will receive (pay) for each future state should affect his/her portfolio choice. We show that if the tax system provides more (less) insurance against low skill shocks for the case of the balanced budget, then the leverage ratio increases (decreases) because ex-post low skill agents prefer equity to debt. More insurance against low skill shocks gives agents incentives to hold more debts. Similarly if the intertemporal resource transfer is allowed, the leverage ratio is positively correlated with the expected present value of labor subsidies conditional on being a low skill agent.

The rest of the paper is organized as follows. Section 2 introduces a simple environment. We first pin down the constrained optimum of the planner's problem in Section 3. In Section 4 we briefly review how to decentralize the constrained optimum using the capital/labor tax system using the known results. Then, we study how this result can be distorted if the government cannot tax unrealized capital income. Section 5 explains why we need to consider the corporate tax and

---

<sup>12</sup>Notice that not only the corporate tax but also the labor tax code are indeterminate. The indeterminacy of the labor tax is basically due to the Ricardian equivalence. See Bassetto and Kocherlakota (2004) and chapter 4 of Kocherlakota (2009).

we show how to endogenize the capital structure as well as the optimal tax system. We describe some comparative statics results in the leverage ratio with respect to labor taxes in Section 6. Section 7 extends the model for more than two types and explains the key properties of the corporate tax: (i) the optimal corporate tax rate is indeterminate and (ii) the leverage ratio is independent of the corporate tax. Section 8 provides practical discussion on the optimal tax code of this paper. We also provide a brief history of the U.S. tax system. Section 9 considers other generalizations: (i) with more than three periods and (ii) with (aggregate) uncertainty. Section 10 provide the related literature. Section 10 concludes. All proofs are in the appendix.

## 1.2 A Simple Environment

Here we first consider a simple model. Later, we also extend the model to a general case. The fundamental idea, however, is the same as the simple model introduced here. Suppose there are ex-ante identical unit measure of agents living for three periods with the following undiscounted utility function.<sup>13</sup> Then,

$$\sum_{t=0}^2 [u(c_t) - v(y_t)],$$

where  $c_t$  is consumption and  $y_t$  is labor provided by the agent in time  $t$ . In period 0, there is no uncertainty in types and all agents are homogeneous. In the beginning of each period, each agent privately learns his/her type. The agent has a high skill

---

<sup>13</sup>It is easy to generalize the model with many (possibly infinite) periods and discounting. But, there should be more than two periods since the tax timing option will not be created in the two period model. Without loss of generality we assume there are three periods.

with probability  $\pi$  and a low skill with probability  $1 - \pi$ . This distribution is i.i.d. over time and across agents.<sup>14</sup> If a high skill agent works, we get disutility  $v(y)$  from labor  $y$ . We assume that the low skill agents cannot provide labor, i.e.,  $y = 0$ . It is rather an extreme case: An agent is either able or completely disable at period 1 and 2. This is for simplicity, thus we only need to consider incentives for the high skill agents to work. Later we will extend the setup where there are more than two types and all types of agents can work in Section 1.7. The production technology is given by

$$F(K, Y) = rK + wY,$$

where  $K$  is aggregate capital and  $Y$  is aggregate labor.<sup>15</sup> Capital is depreciated at the rate  $\delta$  in each period and must be installed one-period ahead. Here without loss of generality we replace  $r + (1 - \delta)$  with  $r$ . The initial capital endowment is  $K_0$ . Every agent is assumed to have the same initial endowment  $k_0$ , so that  $k_0 = K_0$ .

We first investigate the constrained optimal allocation in Section 1.3. The main focus of this paper is on how to decentralize this social optimum by using a tax system. In more detail, the government's problem is to insure agents against skill risks and to provide incentives to work by using capital and labor income taxes. However, the government has the constraint in choosing a tax scheme since they cannot tax on unrealized capital income at the individual level. Assume that there is no government spending required.

---

<sup>14</sup>The i.i.d. assumption is for simplicity. All results are robust to the extension to a general stochastic environment beyond the i.i.d. case.

<sup>15</sup>The results are also preserved for a variety of constant returns to scale production functions.

### 1.3 Constrained Planning Optimum

The planner's problem is to choose  $(c_0, c_h, c_l, c_{hh}, c_{hl}, c_{lh}, c_{ll}, y_0, y_h, y_{hh}, y_{lh}, K_1, K_2)$ , each component of which is nonnegative to maximize an expected life time payoff

$$\begin{aligned} \max \quad & u(c_0) - v(y_0) + \pi(u(c_h) - v(y_h)) + (1 - \pi)u(c_l) \\ & + \pi^2(u(c_{hh}) - v(y_{hh})) + \pi(1 - \pi)u(c_{hl}) + \pi(1 - \pi)(u(c_{lh}) - v(y_{lh})) + (1 - \pi)^2u(c_{ll}) \end{aligned}$$

subject to the resource constraints

$$\begin{aligned} c_0 + K_1 &= rK_0 + wy_0, \\ \pi c_h + (1 - \pi)c_l + K_2 &= rK_1 + w\pi y_h, \\ \pi^2 c_{hh} + \pi(1 - \pi)c_{hl} + \pi(1 - \pi)c_{lh} + (1 - \pi)^2 c_{ll} \\ &= rK_2 + w(\pi^2 y_{hh} + \pi(1 - \pi)y_{lh}), \end{aligned}$$

and the incentive constraints

$$\begin{aligned} u(c_{hh}) - v(y_{hh}) &\geq u(c_{hl}), \\ u(c_{lh}) - v(y_{lh}) &\geq u(c_{ll}), \\ u(c_h) - v(y_h) + \pi(u(c_{hh}) - v(y_{hh})) + (1 - \pi)u(c_{hl}) \\ &\geq u(c_l) + \pi(u(c_{lh}) - v(y_{lh})) + (1 - \pi)u(c_{ll}), \\ u(c_h) - v(y_h) + \pi(u(c_{hh}) - v(y_{hh})) + (1 - \pi)u(c_{hl}) \\ &\geq u(c_h) - v(y_h) + \pi u(c_{hl}) + (1 - \pi)u(c_{hl}) \end{aligned}$$

$$\begin{aligned}
& u(c_h) - v(y_h) + \pi(u(c_{hh}) - v(y_{hh})) + (1 - \pi)u(c_{hl}) \\
& \geq u(c_l) + \pi u(c_{ll}) + (1 - \pi)u(c_{ll})
\end{aligned}$$

Let  $c := \{c_0, c_h, c_l, c_{hh}, c_{hl}, c_{lh}, c_{ll}\}$  is the consumption plan of an agent at time 0, working in period 1, non-working in period 1, working in both periods 1 and 2, working in period 1 and non-working in period 2, non-working in period 1 and working in period 2, and non-working in both periods 1 and 2, respectively.  $y := \{y_0, y_h, y_{hh}, y_{lh}\}$  is the amount of labor provided by corresponding agents. Note that the disables at each period never work, i.e.,  $y_l = y_{ll} = y_{hl} = 0$ . Notice the low type agents cannot work, so that they do not lie. Only high types can pretend to be low types. So, we have five incentive constraints that are specified above. However, in the finite horizon setting, the following temporal incentive constraints are sufficient to summarize all the truthful telling constraints:

$$u(c_{hh}) - v(y_{hh}) \geq u(c_{hl}), \quad (1.3.1)$$

$$u(c_{lh}) - v(y_{lh}) \geq u(c_{ll}), \quad (1.3.2)$$

$$u(c_h) - v(y_h) \geq u(c_l), \quad (1.3.3)$$

(1.3.1) and (1.3.2) are the truth-telling constraint for the high skill agents in period 2 who is high skilled in period 1 and is low skilled in period1, respectively. (1.3.3) is the instantaneous incentive constraint in period 1.

Let  $(c^*, y^*, K^*) := (\{c_0^*, c_h^*, c_l^*, c_{hh}^*, c_{hl}^*, c_{lh}^*, c_{ll}^*\}, \{y_0^*, y_h^*, y_{hh}^*, y_{lh}^*\}, \{K_1^*, K_2^*\})$  be the constrained optimum.<sup>16</sup> Then, it is easy to see from the first order necessary con-

---

<sup>16</sup>We have the following convention for notations. A small letter represents individual choice or allocation and a large letter represents an aggregate variable (a firm's choice if there is a single

ditions that the constrained optimum satisfies

$$\left\{ \begin{array}{l} u'(c_0^*) = \frac{r}{\frac{\pi}{u'(c_h^*)} + \frac{1-\pi}{u'(c_l^*)}}, \\ u'(c_h^*) = \frac{r}{\frac{\pi}{u'(c_{hh}^*)} + \frac{1-\pi}{u'(c_{hl}^*)}}, \\ u'(c_l^*) = \frac{r}{\frac{\pi}{u'(c_{lh}^*)} + \frac{1-\pi}{u'(c_{ll}^*)}}, \\ v'(y_0^*) = wu'(c_0^*), \quad v'(y_h^*) = wu'(c_h^*) \\ v'(y_{hh}^*) = wu'(c_{hh}^*), \quad v'(y_{lh}^*) = wu'(c_{lh}^*) \end{array} \right. , \quad (1.3.4)$$

$$\left\{ \begin{array}{l} c_0^* + K_1^* = rK_0 + wy_0^* \\ \pi c_h^* + (1-\pi)c_l^* + K_2^* = rK_1^* + w\pi y_h^* \\ \pi^2 c_{hh}^* + \pi(1-\pi)c_{hl}^* + \pi(1-\pi)c_{lh}^* + (1-\pi)^2 c_{ll}^* \\ = rK_2^* + w(\pi^2 y_{hh}^* + \pi(1-\pi)y_{lh}^*) \end{array} \right. , \quad (1.3.5)$$

and

$$\left\{ \begin{array}{l} u(c_h^*) - v(y_h^*) = u(c_l^*) \\ u(c_{hh}^*) - v(y_{hh}^*) = u(c_{hl}^*) \\ u(c_{lh}^*) - v(y_{lh}^*) = u(c_{ll}^*) \end{array} \right. \quad (1.3.6)$$

The above conditions are also sufficient since the solution is in the interior and unique. First notice that it is easy to show that all three incentive constraints (1.3.1), (1.3.2), and (1.3.3) are binding, which results in (1.3.6). For example, suppose  $u(c_h) - v(y_h) > u(c_l)$ . Then, by the concavity of  $u$ , the welfare goes up by increasing  $c_l$  a little bit and decreasing  $c_h$  a little bit without violating the resource constraint. The same argument applies to the second and the third equality.

---

firm). The superscript, \*, represents optimality, i.e., solutions to the planner's problem. For example,  $k_t$  is investment of an agent at  $t = 1, 2$  and  $K_t$  is the aggregate investment or capital raised by the representative firm.  $k_t^*$  and  $K_t^*$  are the optimal values of  $k_t$  and  $K_t$ , respectively.



The first three equations in (1.3.4) are so called the inverse Euler equations. Golosov, Kocherlakta, and Tsyvinski (2003) first pinned down the intertemporal wedge in a Pareto optimum between an individual's marginal benefit of investing in capital and his marginal cost of doing so, which suggests the positive tax on capital income. Since then and contemporaneously, several optimal taxation mechanisms have been developed. Among them, Kocherlakota (2005) first proposed how to implement a market economy that is closest to the classical workhorse dynamic general equilibrium models. He shows that the constrained optimum cannot be decentralized by simply imposing homogenous capital income equal to the (ex-ante) wedge. Instead he proposed capital income taxes equal to the *ex-post* wedge, which makes agents with different skills face different capital tax rates. The optimal capital income tax is zero in aggregate (or in the ex-ante expectation sense), but nonzero for individuals (in the ex-post sense). For example, people who are relatively low skilled in the next period pay a wealth tax; people who are relatively high skilled receive a wealth subsidy.

Before going further, we introduce the following lemma that will be used several times later to pin down size of optimal capital taxes.

**Lemma 1.** *The optimal allocation satisfies*

$$u'(c_0^*) < ru'(c_l^*).$$

*Proof.* See the Appendix. □

Lemma 1 still holds for a general case where there are many types of agents: When there are more than two types of agents,  $l$  should mean the lowest skill agents.

The following corollary of Lemma 1 is also used later.

**Corollary 1.** *The optimal allocation satisfies*

$$u'(c_0^*) > ru'(c_h^*).$$

*Proof.* See the Appendix. □

## 1.4 Known Tax Schemes

Two decentralization methods are examined in this section. In Section 1.4.1 we briefly introduce a kind of Ramsey taxation scheme and explain briefly why it does not work when the agent has private information on his/her skill, i.e, in the Mirrlees framework. Section 1.4.2 describes the standard dynamic Mirrless tax scheme as in Kocherlakota (2005) and Albanesi and Sleet (2006). Then, in Section 1.4.3 we explain why this standard dynamic taxation method also fails to decentralize the constrained optimal allocation. In particular, this section explicitly describes the assumption of this paper and presents the intuition of how to use the tax timing option.

In section 1.4.1 we define the Ramsey taxation scheme by the tax system including the capital income tax that matches the wedge in the (ex-ante) Euler equation. The next period capital income tax rate should be contingent on the information available at the current period. Next, in Section 1.4.2 we define the standard dynamics Mirrlees taxation scheme by the tax system including the capital income tax that matches the wedge in the ex-post Euler equation. The next period capital

income tax rates should be contingent on the full labor history including the next period (not even the current period).

Suppose there is a single firm that owns the technology. The firm rents capital and labor in each period to produce output. In period 0 and 1, the household decides how much to consume and work and how much capital to save (or accumulate). In period 2, agents decide how much to consume and work.

### 1.4.1 Ramsey Taxation Scheme

First consider a tax system  $\{\tau, \alpha_h, \alpha_l\}$  in period 1 where  $\tau$  is a capital tax rate and  $(\alpha_h, \alpha_l)$  are lump-sum taxes on the labor income of working/non-working agents. The key point here is that the capital tax rate imposed on all types of agents are the same. In particular, let us to set up  $\tau$  such that

$$u'(c_0^*) = E[r(1 - \tau)u(c_1^*)] = \pi r(1 - \tau)u'(c_h^*) + (1 - \pi)r(1 - \tau)u'(c_l^*). \quad (1.4.1)$$

This tax system works if there is no information asymmetry (in a Ramsey taxation world). With private information it fails to achieve the constrained optimum allocation. In particular, it fails to satisfy the incentive constraint of the high skill agent. The high skill agent will deviate by oversaving and pretending to be low skilled (See the two-period example in Kocherlakota (2005)).

## 1.4.2 Standard Dynamics Taxation Scheme

Secondly, we consider a tax system  $\{\tau_i, \alpha_i\}_{i=l,h}$  for period 1 and  $\{\tau_{ij}, \alpha_{ij}\}_{i,j=h,l}$  for period 2 proposed by Kocherlakota (2005). Note that  $l$  means that the agent does not work and  $h$  means that the agent works. For example,  $\tau_h$  is the (period 1) capital tax on the agent who works in period 1,  $\alpha_{lh}$  is the (period 2) labor income tax on the agent who does not work in period 1 and works in period 2. Notice that the tax mechanism has the full labor-history dependence up to the period when the corresponding capital tax is imposed. In essence, differentiating the tax rates on capital is required to achieve a constrained optimal allocation.

Given the tax plan  $\{\tau_i, \alpha_i\}_{i=l,h}$  and  $\{\tau_{ij}, \alpha_{ij}\}_{i,j=1,2}$ , an agent's problem is to choose consumption  $(c_0, c_h, c_l, c_{hh}, c_{hl}, c_{lh}, c_{ll})$ , labor  $(y_0, y_h, y_{hh}, y_{lh})$ , and investment  $(k_1, k_{2h}, k_{2l})$  to maximize

$$u(c_0) - v(y_0) + \pi(u(c_h) - v(y_h)) + (1 - \pi)u(c_l) \\ + \pi^2(u(c_{hh}) - v(y_{hh})) + \pi(1 - \pi)u(c_{hl}) + \pi(1 - \pi)(u(c_{lh}) - v(y_{lh})) + (1 - \pi)^2u(c_{ll})$$

subject to the following budget constraints. The constraint in  $t = 0$  is

$$c_0 = rk_0 - k_1 + wy_0,$$

the constraint in  $t = 1$  is

$$c_h = r(1 - \tau_h)k_1 - k_{2h} + wy_h + \alpha_h, \quad \text{if } y_h > 0 \\ c_l = r(1 - \tau_l)k_1 - k_{2l} + \alpha_l, \quad \text{otherwise,}$$

the constraint in  $t = 2$  when the agent works in  $t = 1$  is

$$\begin{aligned} c_{hh} &= r(1 - \tau_{hh})k_{2h} + wy_{hh} + \alpha_{hh}, & \text{if } y_{hh} > 0 \\ c_{hl} &= r(1 - \tau_{hl})k_{2h} + \alpha_{hl}, & \text{otherwise,} \end{aligned}$$

and finally the constraint in  $t = 2$  when the agent does not work in  $t = 1$  is

$$\begin{aligned} c_{lh} &= r(1 - \tau_{lh})k_{2l} + wy_{lh} + \alpha_{lh}, & \text{if } y_{lh} > 0 \\ c_{ll} &= r(1 - \tau_{ll})k_{2l} + \alpha_{ll}, & \text{otherwise.} \end{aligned}$$

Notice that positive  $\alpha$ 's represent subsidy and negative  $\alpha$ 's represent tax while positive  $\tau$ 's represent tax and negative  $\tau$ 's represent subsidy. The market clearing conditions are given by

$$\begin{aligned} (t = 0) \quad c_0 + k_1 &= rk_0 + wy_0, \\ (t = 1) \quad \pi c_h + (1 - \pi)c_l + \pi k_{2h} + (1 - \pi)k_{2l} &= rk_1 + w\pi y_h, \\ (t = 2) \quad \pi^2 c_{hh} + \pi(1 - \pi)c_{hl} + \pi(1 - \pi)c_{lh} + (1 - \pi)^2 c_{ll} \\ &= r[\pi k_{2h} + (1 - \pi)k_{2l}] + w(\pi^2 y_{hh} + \pi(1 - \pi)y_{lh}). \end{aligned}$$

Suppose the government does not period-by-period transfer resources, i.e., the government does not issue bonds. Then, the budget constraint of an agent and the market clearing condition imply the following government budget constraint in each period.

$$(t = 1) \quad [\pi\tau_h + (1 - \pi)\tau_l]rk_1 = \pi\alpha_h + (1 - \pi)\alpha_l, \tag{1.4.2}$$

$$\begin{aligned}
(t = 2) \quad & \pi[\pi\tau_{hh} + (1 - \pi)\tau_{hl}]k_{2h} + (1 - \pi)[\pi\tau_{lh} + (1 - \pi)\tau_{ll}]rk_{2l} \\
& = \pi\alpha_{hh} + \pi(1 - \pi)\alpha_{hl} + (1 - \pi)\pi\alpha_{lh} + (1 - \pi)^2\alpha_{ll}. \quad (1.4.3)
\end{aligned}$$

If we enable the government to finance their budget through government bonds, then the labor income tax should be indeterminate.<sup>17</sup> In this section we keep (1.4.2) and (1.4.3) for simplicity. However, from the next section on we will see the case where the government does issue bonds or does period-by-period transfer resources.

In order to achieve the constrained optimal competitive allocation, any tax system must be consistent with the ex-post Euler equation (not ex-ante Euler equation). Given the constrained optimum allocation  $(c_0^*, c_h^*, c_l^*, c_{hh}^*, c_{hl}^*, c_{lh}^*, c_{ll}^*)$ ,  $(y_0^*, y_h^*, y_{hh}^*, y_{lh}^*)$ , and  $(k_1^*, k_{2h}^*, k_{2l}^*)$ , we require the capital tax system  $\{\tau_h, \tau_l\}$  and  $\{\tau_{hh}, \tau_{hl}, \tau_{lh}, \tau_{ll}\}$  to be defined so that the ex-post Euler equation is satisfied with equality at each period and require the labor tax system  $\{\alpha_h, \alpha_l\}$  and  $\{\alpha_{hh}, \alpha_{lh}, \alpha_{hl}, \alpha_{ll}\}$  to satisfy the budget constraint as follows.

$$\begin{cases} r(1 - \tau_h)u'(c_h^*) = u'(c_0^*), & r(1 - \tau_l)u'(c_l^*) = u'(c_0^*) \\ \alpha_h = c_h^* + k_{2h}^* - r(1 - \tau_h)k_1^* - wy_h^* \\ \alpha_l = c_l^* + k_{2l}^* - r(1 - \tau_l)k_1^*. \end{cases} \quad (1.4.4)$$

with  $\pi k_{2h}^* + (1 - \pi)k_{2l}^* = K_2^*$ , and

---

<sup>17</sup>Interested readers can see the arguments in Section 4.4.3 in Kocherlakota (2009).

$$\left\{ \begin{array}{l} r(1 - \tau_{hh})u'(c_{hh}^*) = u'(c_h^*), \quad r(1 - \tau_{hl})u'(c_{hl}^*) = u'(c_h^*), \\ r(1 - \tau_{lh})u'(c_{lh}^*) = u'(c_l^*), \quad r(1 - \tau_{ul})u'(c_{ul}^*) = u'(c_l^*) \\ \alpha_{hh} = c_{hh}^* - r(1 - \tau_{hh})k_{2h}^* - wy_{hh}^* \\ \alpha_{hl} = c_{hl}^* - r(1 - \tau_{hl})k_{2h}^* \\ \alpha_{lh} = c_{lh}^* - r(1 - \tau_{lh})k_{2l}^* - wy_{lh}^* \\ \alpha_{ul} = c_{ul}^* - r(1 - \tau_{ul})k_{2l}^* \end{array} \right. \quad (1.4.5)$$

Then, it is not hard to see that the agent's optimal choice  $(c, y, k)$  is equal to the constrained optimum, i.e.,  $(c_0, c_h, c_l, c_{hh}, c_{hl}, c_{lh}, c_{ul}) = (c_0^*, c_h^*, c_l^*, c_{hh}^*, c_{hl}^*, c_{lh}^*, c_{ul}^*)$ ,  $(y_0, y_h, y_{hh}, y_{lh}) = (y_0^*, y_h^*, y_{hh}^*, y_{lh}^*)$ , and  $k_1 = K_1^*$ , conditional on  $(y_h, y_{hh}, y_{lh}) \gg 0$ . Then, we have the following proposition.

**Proposition 1** (Kocherlakota (2005)). *The competitive equilibrium is the constrained optimum allocation if the tax system satisfies (1.4.4) and (1.4.5).*

*Proof.* See the appendix □

Let us summarize the properties of the tax system of (1.4.4) and (1.4.5) in the following proposition.

**Proposition 2.** *The tax system defined in (1.4.4) and (1.4.5) also satisfies  $E_t[\tau_{t+1}] = 0$ , in other words,*

$$(a) \quad \pi\tau_h + (1 - \pi)\tau_l = 0 \text{ and } \tau_h < 0 < \tau_l.$$

$$(b) \quad \pi\tau_{hh} + (1 - \pi)\tau_{hl} = 0 = \pi\tau_{lh} + (1 - \pi)\tau_{ul}. \text{ Furthermore, } \tau_{hh} < 0 < \tau_{hl} \text{ and } \tau_{lh} < 0 < \tau_{ul}.$$

Proposition 2 implies that the expected ex-ante capital tax is zero although the ex-post capital tax is never zero. Notice that the expected labor income tax is not necessarily zero.<sup>18</sup> It is zero under the assumption that there is no intertemporal transfer of consumption by the planner, i.e., the government does not generate any debt repaid in the future. The working agents pay the labor income tax and the disable agents receive labor subsidy, which means the government insures the agents against the skill shocks. However, in order to give able agents incentives to work, the government should award the working agents the capital income (or wealth) subsidy instead of making them pay the labor income taxes for the disabled.

### 1.4.3 Tax Timing Options

The analysis in Section 1.4.2 is based on a standard framework of the dynamic Mirrless taxation models. In fact, the main contribution of this paper start from here. From this section on, we add real world features of the tax code into the model as in assumption 1 below. With this assumption, an agent is entitled with so called *a tax timing option* that is the option to realize capital incomes in each period. Then, as will be shown later, the decentralization method in the previous section fails to achieve the constrained optimum allocation. Notice that we are not criticizing dynamic Mirrless taxation models by citing practical problems. The idea of tax timing options can also be applied to break down any dynamic Ramsey models as well. What we want to focus on is how to correct this failure in the Mirrless framework, which eventually justifies corporate taxation. Now we introduce the

---

<sup>18</sup>If we assume that government never creates any bonds, we have  $\pi\alpha_h + (1 - \pi)\alpha_l = 0$  with  $\alpha_h < 0 < \alpha_l$  and  $\pi^2\alpha_{hh} + \pi(1 - \pi)\alpha_{hl} + (1 - \pi)\pi\alpha_{lh} + (1 - \pi)^2\alpha_{ll} = 0$ .



following simplifying assumptions.

**Assumption 1.** (i) *The government cannot impose tax on any unrealized capital returns of individual agents.*

(ii) *In period 1, an agent can resell equities to the corporation (or firms repurchase equities from the shareholders.) When they do so, they must pay the individual capital taxes. Otherwise, they do not pay the taxes just by holding equities.*

(iii) *There is no long-term debt, in other words, only one-period bonds are available in the market. Debt issued in period  $i$  must be paid in period  $i + 1$ ,  $i = 0, 1$ . Then, the individual taxes are imposed as well.*

The most important is Assumption 1-(i). As mentioned before in the introduction, in the real world, people annually need to pay taxes on some capital holding regardless of the capital gain realization, for example, real estates, vehicles, intangible assets (copyrights, patents, etc) and durable goods. However, these assets are not traded often and here we are interested in stocks and bonds that are being traded every second in the market. In addition, it is factual that the capital gain taxes are paid when stocks and bonds are sold.<sup>19</sup> Therefore, we take assumption 1-(i).

Assumption 1-(ii) implies that dividend distribution and share repurchase are identical. Practically, dividend payout usually has tax disadvantages relative to share repurchase. In particular, in the current U. S. tax code, the effective tax rates on dividends are slightly higher than those on share repurchases. Then, an

---

<sup>19</sup>The capital gain taxes are asymmetric. There are tax credits for capital loss. In this paper, we do not consider the tax credits. This is for simplicity. In fact, since the model has no uncertainty in production, we do not have to take capital loss into account.

immediate question is why firms are distributing dividends. However, those topics related to this dividend puzzle are beyond the scope of this paper (See Black (1976) and Miller (1986) for the dividend puzzle). Therefore, for simplicity, we assume that the share repurchasing is equivalent to dividends distribution. It also means that the agents realize capital gains by receiving cash in exchange for all or some fraction of the firm's outstanding equity that they hold or by selling all or some equity to any individual agent or the firm. In addition, we assume that there is no floatation cost and no friction in issuing equity and debt.

Assumption 1-(*iii*) identifies the difference between debt and equity. Basically, equity implies the ownership. Debt is the borrowing/lending contract between the firm and the investor, therefore it should be paid at the specified time. Notice that we do not consider bankruptcy of a firm. Technically, there are two differences. First, debt is corporate tax-free while equity is not. Second, debt has a maturity, so we assume for simplicity there is only one-period debt. However, equity can be realized (cashed) at any time upon an investor's request.

Now the intuition of the tax timing option is as follows. Although we have a three-period model, the model can be easily extended to a general case. Therefore, let us imagine that there are many periods and individual skills are arbitrarily evolving (potentially very persistently). Suppose the tax system is given by equations (1.4.4) and (1.4.5). If an agent sees that the capital income tax is high enough at the current period, then she can postpone realization of her capital income to the next period. In this case, the unrealized returns are left in the firm<sup>20</sup>, which is automatically transferred to reinvestment without taxes under assumption 1. In

---

<sup>20</sup>These unpaid retained earnings are sometimes called internal equities in the capital structure literature. Then, common stocks traded in the market are called external equities.

particular, the agent who has surprisingly low skill in the current period, therefore is facing positive capital taxes, will have the incentive to defer her capital income realization in order to evade the taxes. If she realizes her capital income at the time she becomes (surprisingly) high skilled in some periods later, she can receive even more subsidy proportional to the wealth accumulated without having paid taxes than what she would get if she realized her capital income earlier. In particular, currently low skill agents choose to exercise the tax timing option whereas the currently high type agents do not. Therefore, tax timing options provide typical arbitrage opportunities. Now we are ready to show the following proposition which is the starting point for the whole analysis in the remaining part of the paper.

**Proposition 3.** *Suppose Assumption 1 holds. Then, the the socially optimal allocation cannot be implemented by the tax system  $\{\tau_i, \alpha_i\}_{i=l,h}$  and  $\{\tau_{ij}, \alpha_{ij}\}_{i,j=h,l}$  in (1.4.4) and (1.4.5).*

*Proof.* See the Appendix. □

We have two remarks on Proposition 3. First, we focus only on the behavior of the low skill agents in period 1. The high skill agents already do not have incentives to deviate under the the second best world tax scheme. Second, although in the second best world we only investigated the case where there is no intertemporal transfer of resources, one should notice that, in general, the labor taxation is indeterminate. Therefore, the agent's investment (or saving) strategy depends on how much labor taxes will be assigned in period 1, in particular, how big  $(\alpha_h, \alpha_l)$  in (1.4.4) are. Proposition 3 is true for any labor tax system, in other words, it is valid regardless of whether the government period-by-period transfers resources.

Now, using the argument in Section 1.4.1 and the argument in proposition 3, we can establish the following corollary.

**Corollary 2.** *Suppose Assumption 1 holds. The constrained optimum cannot be decentralized by any tax systems using the capital income tax defined (i) to be equal to the ex-post wedge of the intertemporal Euler equation or (ii) to be equal to the ex-ante wedge in the intertemporal Euler equation.*

Corollary 2 gives a hint of how to design a optimal tax scheme in order to avoid the tax timing option. If the market would fail to achieve the optimal allocation by using only one of (i) and (ii) in the corollary, then one can think of a proper mixture of them as a solution. The next section shows an alternative way.

## 1.5 The Third Best Taxation Scheme

How does the government prevent agents from this deviation as in the proof of Proposition 3? For logical simplicity, we consider the following two cases step by step: (1) when firms do not issue debts and (2) when firms issue both equities and debts. In conclusion, the government should be required to tax unrealized returns or earnings in the firm level (as well as in the individual level), which is so called corporate taxation.

### 1.5.1 When No Debt, But Only Equity is Available

Assume firms are not allowed to issue debts. Then, corporate earnings in this case is equal to output minus labor shares. If the government sets any taxes in the

corporate level, then it makes all the agents pay capital income taxes although they do not realize their capital income. In particular, if this tax is set to be same as  $\tau_l$  in (1.4.4), then the low skill agents cannot defer to pay the capital income taxes to the next period, which means that they lose their tax timing options. More precisely, consider the following tax system  $(\tau_c^e, \tau_l^e, \tau_h^e)$  in period 1 where  $\tau_c^e$  is the corporate tax rate,  $\tau_l^e$  is the individual capital tax rate for non-working agents in period 1, and  $\tau_h^e$  is the individual capital tax rate for working agents in period 1 such that

$$\tau_c^e := \tau_l, \quad \tau_l^e := 0, \quad \tau_h^e := \tau_h - \tau_c^e. \quad (1.5.1)$$

where  $\tau_l$  and  $\tau_h$  are defined by (1.4.4). Notice that the low skill agents are now indifferent between realizing the return on capital investment and non-realizing. The high skill agents should pay the corporate tax  $\tau_c^e$ , but they can get back tax benefits  $\tau_h - \tau_l$  when they realized their capital income. Therefore, the net capital income is  $[(1 - \tau_h + \tau_l) - \tau_l]rk_1^* = (1 - \tau_h)rk_1^*$ , which is the same as that under the previous tax system (1.4.4) and (1.4.5).

### 1.5.2 When Both Debt and Equity are Available

Notice the tax system (1.5.1) is the optimal tax only if debt is unavailable. If debt is available and the individual capital taxes are given by  $(\tau_l^e, \tau_h^e)$ , then the agents in period 0 have no reason to buy equity since there is a positive corporate tax  $\tau_c > 0$ . Then, corporations raise 100% debt financing since we do not assume bankruptcy costs. Therefore, the optimal allocation cannot be obtained under (1.5.1).

Now suppose both debt and equity are available in the market. We need to

introduce more precise individual taxes as well as the corporate tax. Let us define  $\tau_c^*$  by the corporate tax rate and  $(\tau_l^B, \tau_l^E)$  and  $(\tau_h^B, \tau_h^E)$  by the individual capital income taxes of non-working ( $l$ ) and working ( $h$ ) agents, respectively. Superscript  $B$  represents debt and  $E$  represents equity. Then, we formalize the problem as follows: find the optimal tax system  $(\tau_c^*, \tau_l^B, \tau_l^E, \tau_h^B, \tau_h^E)$  such that given the agents tell the truth, the tax system guarantees that the agents choose the socially optimal allocations and given the agents optimally chooses their allocation, the agents choose to tell the truth.

Let us describe the idea of taxation as follows. Above all, unlike (1.5.1), we impose positive individual capital taxes on both equity and debt holdings of the low skill agents and, in particular, we set the tax rate on the debt holding of the low type agents greater than the corporate tax rate. It follows that the individual capital tax rates for the high skill agents should be adjusted to fit the Euler equation. Similarly to the above subsection, the tax rate on equity of the working agents should be negative. Then, the above idea is mathematically summarized as the following criterion.

$$0 < \tau_c^* < \tau_l^B \quad \text{and} \quad \tau_h^E < \tau_h < 0. \quad (1.5.2)$$

In fact, we need more constraints, but they are rather less important than (1.5.2). They will be specified in the below. This minor importance is due to the fact that if we set  $\tau_l^B = \tau_l^E$  and  $\tau_h^B = \tau_h^E$ , then the other criteria will trivially hold.

Let us consider in the ex-post sense who prefer debt and who prefer equity under (1.5.2). The high skill agents would be happier if they find themselves have more bonds. The low skill agents would be happier if they find themselves have more stocks. In other words, the high types prefer to be "debt holders" while the low

types prefer to be "equity holders" in ex-post. Therefore, in the ex-ante sense, in period 0, the risk-averse agents are facing a non-trivial portfolio selection problem between equities and bonds given the tax system.

Notice that no corporate tax is required in period 2 since all the firms are liquidated in period 2. Therefore,  $\{\tau_{ij}\}_{i,j=h,l}$  of (1.4.5) is still the optimal capital income tax in period 2. Define  $B_1$  and  $E_1$  by the amount of debt holdings and equity holdings, respectively. Then, given the tax system  $(\tau_c^*, \tau_l^B, \tau_l^E, \tau_h^B, \tau_h^E)$ , the agent's budget constraint in each period is as follows. In period 2, we have the same constraints as in the second-best case:

$$c_{hh} = r(1 - \tau_{hh})k_{2h} + wy_{hh} + \alpha_{hh}, \quad \text{if } y_{hh} > 0 \quad (1.5.3)$$

$$c_{hl} = r(1 - \tau_{hl})k_{2h} + \alpha_{hl}, \quad \text{otherwise} \quad (1.5.4)$$

and

$$c_{lh} = r(1 - \tau_{lh})k_{2l} + wy_{lh} + \alpha_{lh}, \quad \text{if } y_{lh} > 0 \quad (1.5.5)$$

$$c_{ll} = r(1 - \tau_{ll})k_{2l} + \alpha_{ll}, \quad \text{otherwise} \quad (1.5.6)$$

In period 1, however, we have

$$c_h = r(1 - \tau_h^B)B_1 + \max_{\{\text{realize, not}\}} ((1 - \tau_c^*)(1 - \tau_h^E), 1 - \tau_c^*)rE_1 - k_{2h} + wy_h + \alpha_h, \quad \text{if } y_h > 0 \quad (1.5.7)$$

$$c_l = r(1 - \tau_l^B)B_1 + \max_{\{\text{realize, not}\}} ((1 - \tau_c^*)(1 - \tau_l^E), 1 - \tau_c^*)rE_1 - k_{2l} + \alpha_l, \quad \text{otherwise} \quad (1.5.8)$$

Assume first the criteria in (1.5.2) is true. Moreover, suppose an agent enter period 1 with positive amount of both debt and equity. If in period 1 the agent finds him high skilled, then he would realize his return on equity since  $\tau_h^E < 0$ . Here, we need another criterion: He would be better with more debt if the net return on debt is greater than the net return on equity if

$$1 - \tau_h^B > (1 - \tau_c^*)(1 - \tau_h^E). \quad (1.5.9)$$

On the other hand, if in period 1 the agent finds him low skilled, then he would not realize his return on equity if we have

$$\tau_l^E > 0. \quad (1.5.10)$$

Then, he also would be better if he only holds equity since the net return on equity is greater than the net return on debt:

$$1 - \tau_c^* > 1 - \tau_l^B,$$

which is true by (1.5.2). Therefore, in period 0, if the tax system satisfies (1.5.2), (1.5.9), and (1.5.10), the agent faces a portfolio selection between  $(B_1, E_1)$  since he does not know which type he will be in period 1. The budget constraint in period 0 is as follows.

$$c_0 = rk_0 - (B_1 + E_1) + wy_0 \quad \text{with} \quad B_1 + E_1 = K_1^*. \quad (1.5.11)$$

We now introduce the optimal tax system in period 1 as follows (The period 2



capital taxes are the same as (1.4.5)). Define  $(\tau_c^*, \tau_l^B, \tau_l^E, \tau_h^B, \tau_h^E)$  and  $(\tau_{hh}, \tau_{hl}, \tau_{lh}, \tau_{ll})$  by

$$\left\{ \begin{array}{l} r(1 - \tau_c^*) = \frac{u'(c_0^*)}{u'(c_l^*)} \\ r(1 - \tau_c^*)(1 - \tau_h^E) = \frac{u'(c_0^*)}{u'(c_h^*)} \\ 1 - \tau_h^B > (1 - \tau_c^*)(1 - \tau_h^E) \\ \tau_l^E > 0 \\ \pi r(1 - \tau_h^B)u'(c_h^*) + (1 - \pi)r(1 - \tau_l^B)u'(c_l^*) = u'(c_0^*) \\ r(1 - \tau_{hh})u'(c_{hh}^*) = u'(c_h^*) \\ r(1 - \tau_{hl})u'(c_{hl}^*) = u'(c_h^*), \\ r(1 - \tau_{lh})u'(c_{lh}^*) = u'(c_l^*), \\ r(1 - \tau_{ll})u'(c_{ll}^*) = u'(c_l^*) \end{array} \right. \quad (1.5.12)$$

and define labor taxes  $(\alpha_h, \alpha_l)$  and  $(\alpha_{hh}, \alpha_{hl}, \alpha_{lh}, \alpha_{ll})$  such that their present values are matched:

$$\begin{aligned} & -\{\pi u'(c_h^*)\alpha_h + (1 - \pi)u'(c_l^*)\alpha_l + \pi u'(c_{hh}^*)\alpha_{hh} + (1 - \pi)u'(c_{ll}^*)\alpha_{ll}\} \\ & = u'(c_0^*)(rk_0 + wy_0^*) + \pi u'(c_h^*)wy_h^* + \pi u'(c_{hh}^*)wy_{hh}^* \\ & - \{u'(c_0^*)c_0^* + \pi u'(c_h^*)c_h^* + (1 - \pi)u'(c_l^*)c_l^* + \pi u'(c_{hh}^*)c_{hh}^* + (1 - \pi)u'(c_{ll}^*)c_{ll}^*\} \end{aligned} \quad (1.5.13)$$

Moreover, we have

$$\left\{ \begin{array}{l} u'(c_{hh}^*)\{c_{hh}^* - wy_{hh}^* - \alpha_{hh}\} = u'(c_{hl}^*)\{c_{hl}^* - \alpha_{hl}\} \\ u'(c_{lh}^*)\{c_{lh}^* - wy_{lh}^* - \alpha_{lh}\} = u'(c_{ll}^*)\{c_{ll}^* - \alpha_{ll}\} \end{array} \right. \quad (1.5.14)$$

Equation (1.5.13) results from adding the budget constraints (1.5.3), (1.5.4), (1.5.7),

(1.5.8), and (1.5.11), each of whom are multiplied by  $\pi u'(c_{hh}^*)$ ,  $(1-\pi)u'(c_{ll}^*)$ ,  $\pi u'(c_h^*)$ ,  $(1-\pi)u'(c_l^*)$ , and  $u'(c_0^*)$ , respectively and using the definitions of the capital income tax code (1.5.12). Equation (1.5.14) is also derived using the definition of the capital income tax code (1.5.12) such that

$$u'(c_{hh}^*)r(1 - \tau_{hh}^*) = u'(c_{hl}^*)r(1 - \tau_{hl}^*)$$

$$u'(c_{lh}^*)r(1 - \tau_{lh}^*) = u'(c_{ll}^*)r(1 - \tau_{ll}^*).$$

Technically  $\tau_c^*$  and  $\tau_h^E$  in (1.5.12) are first set up to be equal to the ex-post wedge between the MRT and the MRS that appear in the first order condition (or Euler equation) for the equity holding choice  $E_1$ . Then,  $(\tau_l^B, \tau_l^E)$  is determined in the first order condition for the debt holding choice  $B_1$ . There are two tax rates that can be flexibly chosen:  $(\tau_l^E, \tau_h^B)$ .  $\tau_l^E$  should be positive. Notice that in (1.5.12) we set

$$\tau_h^B < \tau_h^E + \tau_c - \tau_c \tau_h^E,$$

which is in fact from (1.5.9). By simple algebra we have, by Corollary 1,

$$\tau_h^E + \tau_c - \tau_c \tau_h^E = 1 - \frac{u'(c_0^*)}{r u'(c_h^*)} < 0,$$

which implies that  $\tau_h^B < 0$ . It is notable that either  $0 > \tau_h^B > \tau_h^E$  or  $0 > \tau_h^E > \tau_h^B$  is possible.

Notice that  $B_1 + E_1 = K_1^*$  should be satisfied since the agents are homogenous in period 0. On the other hand, it is not necessary that  $\pi k_{2h}^* + (1 - \pi)k_{2l}^* = K_2^*$  if we allow resource transfer between period 1 and 2. It is also easy to verify that the

tax system (1.5.12) satisfies the intuitive criteria given in (1.5.2). The capital tax system in period 1 of (1.5.12) can be rewritten as

$$\tau_c^* = 1 - \frac{u'(c_0^*)}{ru'(c_l^*)}, \quad (1.5.15)$$

$$\tau_h^E = 1 - \frac{u'(c_l^*)}{u'(c_h^*)}, \quad (1.5.16)$$

$$\tau_h^B < \tau_h^E + \tau_c - \tau_c \tau_h^E = 1 - \frac{u'(c_0^*)}{ru'(c_h^*)} \quad (1.5.17)$$

$$\tau_l^E > 0 \quad (1.5.18)$$

$$\tau_l^B = 1 - \frac{u'(c_0^*) - \pi r(1 - \tau_h^B)u'(c_h^*)}{(1 - \pi)ru'(c_l^*)} \quad (1.5.19)$$

Here, note again  $\tau_l^E$  is arbitrary. From equations (1.5.15), (1.5.16), (1.5.17), (1.5.18), and (1.5.19), we can directly confirm criteria (1.5.2), (1.5.9), and (1.5.10). We summarize this result as the following lemma that will be used later.

**Lemma 2.** *The tax system (1.5.12) satisfies*

$$0 < \tau_c^* < \tau_l^B \quad \text{and} \quad \tau_h^E < \tau_h < 0.$$

One may be interested in the case where  $\tau_h^B \approx \tau_h^E$ . The following lemma tells about this special case.

**Lemma 3.** *The tax system (1.5.12) is given. Then,  $\tau_h^B = \tau_h^E$  if and only if  $\tau_l^B = \frac{\tau_c^*}{1 - \pi}$ .*

*Proof.* This just results from (1.5.19). □

Now we are ready to state our main theorem.

**Theorem 1.** *Suppose the government can impose the corporate tax. Given the tax system (1.5.12), the consumption and labor allocation of the competitive equilibrium coincide with those of the constrained optimum allocation.*

*Proof.* See the Appendix. □

One may think that until now we have only considered the individual investors, so that the role of firms are ignored in debt and equity issuance. In fact, the effect of the corporate tax is offset by that of the individual capital taxes. Simple algebra shows that the expected tax rate on holding equity in  $t = 0$  is

$$\pi[1 - (1 - \tau_h^E)(1 - \tau_c^*)] + (1 - \pi)\tau_c^* = 0. \quad (1.5.20)$$

Therefore the tax system (1.5.12) makes firms indifferent to any capital structure as described in the proof of Theorem 1. In other words, the capital structure only results from the aggregate debt and equity portfolio choice of individual agents. Therefore, in the firm's point of view, the Modigliani-Miller theorem still holds. This idea is quite similar to that of Miller (1977).

**Corollary 3** (Modigliani-Miller Theorem Revisited). *The market value of any firm is independent of its capital structure.*

One important remark is that Corollary 3 is not automatically true for the case of more than two types. As will be explained in Section 1.7, if the number of types of agent is more than two (the number of assets in the market, debt and

equity), the expected tax rate on equity is not necessarily equal to zero since we have more degree of freedom to choose the tax rates. Therefore, the tax authority need to set the expected tax rate to be zero. Otherwise, the capital market does not clear. Therefore, for the case of more than two types of agents, Corollary 3 is not a property of the optimal tax system, but it should be a condition when setting up the optimal tax rates. This is the only one difference between the case where there are two types and the case where there are more than two types of agents.

### 1.5.3 A Simple Example

This section provides a very simple example. For the case where the utility function is logarithmic and the dis-utility function is linear, we describe some comparative statics results. In particular, the corporate tax rate increases in  $\pi$ , the probability of being a high skill agent. In this sense, we provide a simple regression result between the average schooling years and the corporate tax rates among OECD countries. although we need carefully interpret the result due to the indeterminacy property of the corporate tax when there are more than two types of agents (See Section 1.7).

Assume that the utility function is log and the disutility function is linear:

$$u(c) = \log(c), \quad v(y) = \kappa y. \quad (1.5.21)$$

Then, the the first order conditions (1.3.4) yields

$$c_0^* = c_h^* = c_{hh}^* = c_{lh}^* = \frac{w}{\kappa}. \quad (1.5.22)$$

Putting this into the inverse Euler equation in (1.3.4) to get

$$c_l^* = c_{hl}^* = \frac{(r - \pi)w}{(1 - \pi)\kappa}, \quad c_u^* = \frac{1}{1 - \pi} \left( \frac{r(r - \pi)}{1 - \pi} - \pi \right) \frac{w}{\kappa}. \quad (1.5.23)$$

Notice that we need the following assumption to get the well-defined solution.

$$\pi < r < 1 \quad \text{and} \quad r(r - \pi) > \pi(1 - \pi). \quad (1.5.24)$$

Recall the linear disutility function  $v(y)$  in (1.5.21). If (1.5.24) does not hold, then the agent will choose negative work (therefore negative disutility) in order to increase utility. Then, (1.3.6) gives

$$\begin{aligned} y_h^* &= \log(c_h^*) - \log(c_l^*) = \log\left(\frac{1 - \pi}{r - \pi}\right) > 0 \\ y_{hh}^* &= \log(c_{hh}^*) - \log(c_{hl}^*) = \log\left(\frac{1 - \pi}{r - \pi}\right) > 0 \\ y_{lh}^* &= \log(c_{lh}^*) - \log(c_{ll}^*) = \log\left(\frac{1 - \pi}{\frac{r(r - \pi)}{1 - \pi} - \pi}\right) = \log\left(\frac{1}{r(r - \pi) - \pi(1 - \pi)}\right) > 0, \end{aligned}$$

where all the equations are positive by (1.5.24). Then, finally we get  $y_0^*$ ,  $K_1^*$ , and  $K_2^*$  from (2.4) as follows:

$$\begin{aligned} rK_2^* &= \left[ \pi + \pi(1 - \pi) + \pi(1 - \pi) \left( \frac{r - \pi}{1 - \pi} \right) + (1 - \pi) \left( \frac{r(r - \pi)}{1 - \pi} - \pi \right) \right] \frac{w}{\kappa} \\ &\quad - w \left[ \pi^2 \log\left(\frac{1 - \pi}{r - \pi}\right) + \pi(1 - \pi) \log\left(\frac{1}{r(r - \pi) - \pi(1 - \pi)}\right) \right] \\ &= [\pi + \pi(r - \pi) + r(r - \pi)] \frac{w}{\kappa} - w\pi \log \left[ \left( \frac{1 - \pi}{r - \pi} \right)^\pi \left( \frac{1}{r(r - \pi) - \pi(1 - \pi)} \right)^{1 - \pi} \right]. \\ rK_1^* &= K_2^* + r \frac{w}{\kappa} - w\pi \log\left(\frac{1 - \pi}{r - \pi}\right) \end{aligned}$$

$$wy_0^* = K_1^* + \frac{w}{\kappa} - rK_0.$$

For the log utility case, we have the following optimal tax code:

$$\tau_c^* = 1 - \frac{c_l^*}{rc_0^*} \quad \text{and} \quad \tau_h^E = 1 - \frac{c_h^*}{c_l^*}. \quad (1.5.25)$$

Putting (1.5.22) and (1.5.23) into (1.5.25), we have

$$1 - \tau_c^* = \frac{u'(c_0^*)}{ru'(c_l^*)} = \frac{r - \pi}{r(1 - \pi)}.$$

Then, simple algebra gives the following proposition.

**Proposition 4.** *Suppose the agent has the log utility and the linear disutility functions of (1.5.21). Moreover, assume (1.5.24) is satisfied. Other things being equal, the following comparative statics analysis holds.*

- (i) *The corporate tax rate  $\tau_c^*$  increases in the measure (population) of high skill agents,  $\pi$ . In other words,  $\frac{d\tau_c^*}{d\pi} > 0$ .*
- (ii) *The corporate tax rate  $\tau_c^*$  decreases in the rate of return on investment,  $r$ . In other words,  $\frac{d\tau_c^*}{dr} < 0$ .*
- (ii) *The corporate tax rate  $\tau_c^*$  is independent of labor productivity,  $w$ . In other words,  $\frac{d\tau_c^*}{dw} = 0$ .*

Proposition 4 can be interpreted as follows. Assume there are two closed economies: (i) The corporate tax rate may be higher in the economy populated with more skilled workers. (ii) The corporate tax rate may be higher in the econ-

omy having higher return on investment. *(iii)* The corporate tax rate may be higher in the economy having higher labor productivity. Notice that statement *(i)* should be understood under the assumption of the law of large numbers. It is also generally true if the production technology is given by a constant returns to scale production function.

## 1.6 Aggregate Leverage and Some Comparative Statics Analysis

In this section we investigate how the taxes, in particular, the individual labor taxes affect the leverage ratio. We first identify the explicit solution for  $(B_1^*, E_1^*)$  in Section 1.6.1 and characterize its properties. It turns out that the debt and equity holding depends on the labor taxes. Therefore, the labor income taxes affect the leverage ratio. It implies that not only the capital income tax code (including the corporate tax) but also the labor income tax code are important, when we investigate the effect of a tax reform on the leverage ratio. However, the labor tax codes have been often ignored in capital structure theories.

In particular, we perform some comparative statics analysis on the aggregate leverage with respect to the change of labor tax codes. Recall that the labor taxes are indeterminate by the Ricardian equivalence. Section 1.6.2 deals with the effect of change in the labor tax. Suppose there is no period-by-period resource transfer. If the tax authority provides more (less) insurance against, then the leverage ratio increases (decreases). A similar result holds for the case when the intertemporal resource transfer is allowed.



### 1.6.1 Endogenous Leverage

Recall that we have two budget constraints of high and low skill agent in period 1 and the initial investment decision  $B_1 + E_1 = K_1^*$ . As described in Section 4.4.3 of Kocherlakota (2009), the timing and the amount of labor taxation is arbitrary as long as (1.5.13) and (1.5.14) is satisfied. Then, in fact, the individual optimal investment  $(B_1^*, E_1^*)$  in period 0 and  $(k_{2h}^*, k_{2l}^*)$  in period 1 depend on how the government, period-by-period, transfers labor taxes (or subsidies). The following proposition provide the analytic form of the debt and equity holding. In order for simpler exposition, we introduce some positive number  $\hat{k}_2$  which is equal to the period 1 aggregate investment,  $\pi k_{2h}^* + (1 - \pi)k_{2l}^* = \hat{k}_2$ .

**Proposition 5.** *Let  $\pi k_{2h}^* + (1 - \pi)k_{2l}^* = \hat{k}_2$ . Let  $(\tau_c^*, \tau_h^B, \tau_h^E, \tau_m^B, \tau_m^E, \tau_l^B, \tau_l^E)$  be an optimal capital tax system given by (1.5.12). Then, given the labor tax code,  $(\alpha_h, \alpha_l)$ , the optimal portfolio of debt and equity  $(B_1^*, E_1^*)$  is given by*

$$B_1^* = \frac{-X(\alpha_h, \alpha_l) - K_2^* + (\pi\tau_h^E + \tau_c^* - \pi\tau_h^E\tau_c^*)rK_1^*}{-r(\pi\tau_h^E\tau_c^* + \pi\tau_h^B - \pi\tau_h^E + \tau_c^* - (1 - \pi)\tau_l^B)} \quad (1.6.1)$$

$$E_1^* = \frac{X(\alpha_h, \alpha_l) + K_2^* - (\pi\tau_h^B + (1 - \pi)\tau_l^B)rK_1^*}{-r(\pi\tau_h^E\tau_c^* + \pi\tau_h^B - \pi\tau_h^E + \tau_c^* - (1 - \pi)\tau_l^B)} \quad (1.6.2)$$

where  $X(\alpha_h, \alpha_l) := (\pi\alpha_h + (1 - \pi)\alpha_l) - \hat{k}_2$ .

*Proof.* See the appendix. □

First notice that the denominator in (1.6.1) and (1.6.2) are positive, which is summarized in Lemma 5 in the appendix. Before describing the meaning of the

above proposition, we first narrow down the case where there is no period-by-period transfer by the government.

**Corollary 4.** *Let  $(\tau_c^*, \tau_h^B, \tau_h^E, \tau_m^B, \tau_m^E, \tau_l^B, \tau_l^E)$  be an optimal capital tax system given by (1.5.12). Suppose the period-by-period government budget is balanced. More precisely, suppose that we take some positive numbers  $\hat{k}_{1b}, \hat{k}_{1e}, \hat{k}_{2h}$ , and  $\hat{k}_{2l}$  to have  $(\alpha_h, \alpha_l)$  such that*

$$\begin{cases} \alpha_h = c_h^* - r(1 - \tau_h^B)\hat{k}_{1b} - r(1 - \tau_h^E)(1 - \tau_c^*)\hat{k}_{1e} + \hat{k}_{2h} - wy_h^* \\ \alpha_l = c_l^* - r(1 - \tau_l^B)\hat{k}_{1b} - r(1 - \tau_c^*)\hat{k}_{1e} + \hat{k}_{2l} \end{cases} \quad (1.6.3)$$

where

$$\hat{k}_{1b} + \hat{k}_{1e} = K_1^* \quad \text{and} \quad \pi\hat{k}_{2h} + (1 - \pi)\hat{k}_{2l} = K_2^*.$$

Then, we have  $B_1^* = \hat{k}_{1b}$ ,  $E_1^* = \hat{k}_{1e}$ ,  $k_{2h}^* = \hat{k}_{2h}$ , and  $k_{2l}^* = \hat{k}_{2l}$ .

*Proof.* See the appendix. □

Proposition 5 shows that the aggregate capital structure is determined by  $X(\alpha_h, \alpha_l)$  as well as the capital tax code. Therefore, we present comparative statics results with respect to change of the labor income tax code and change of the corporate income tax code in the next two subsections.

Notice  $B_1^* + E_1^* = K_1^*$  is fixed. Therefore, we only need to see the change of  $B_1^*$  in order to see the change of leverage ratio  $\frac{B_1^*}{B_1^* + E_1^*}$ .

## 1.6.2 Comparative Statics: Labor Taxation

For the comparative statics analysis on labor taxation, we should notice that the labor tax code must satisfy the Ricardian equivalence: (1.5.13) and (1.5.14). For example, if  $\alpha_l$  goes up, either or all of  $\alpha_l$ ,  $\alpha_{hh}$ , or  $\alpha_{hl}$  must go down as in (1.5.13). Although the tax authority cannot arbitrarily change the labor taxes, they have enough degree of freedom. Due to this indeterminacy property, we face too many cases. Hence, we focus on simple reasonable examples. Fixing the optimal allocation, we divide the analysis into two cases: (i) when only period 1 labor taxes ( $\alpha_h, \alpha_l$ ) is changed (without intertemporal resource transfer) and (ii) when the expected value of labor taxes will be changed (with the intertemporal resource transfer).

### Comparative Statics: Period 1 Labor Taxes ( $\alpha_h, \alpha_l$ )

Suppose, in this subsection, the labor taxes in the third period, ( $\alpha_{hh}, \alpha_{hl}, \alpha_{lh}, \alpha_{ll}$ ), is unchanged. (1.5.13) implies that  $\alpha_l$  is increased if and only if  $\alpha_h$  is decreased. This observation and proposition 5 give the following proposition.

**Proposition 6.** *Suppose that given the optimal allocation, the tax authority only changes the period 1 labor taxes whereas the period 2 labor taxes are fixed, i.e.,  $\alpha_{hh}, \alpha_{hl}, \alpha_{lh}$ , and  $\alpha_{ll}$  are fixed. Then, we have*

$$\frac{dB_1^*}{d\alpha_l} > 0 \quad \text{and} \quad \frac{dE_1^*}{d\alpha_l} < 0.$$

*In other words, if the tax system provide more (less) insurance against low skill*

shocks, then the leverage ratio goes up (down).

*Proof.* See the appendix. □

The intuition for Proposition 6 is as follows. Recall that in period 1 the ex-post low skill agents will prefer to hold more equities than debts. If the tax authority insures more against the low skill shocks, then the agent in period 0 generally wants to choose more debts. This effect pushes the leverage ratio up.

### Comparative Statics: Expected Labor Taxes

Even if the period-by-period resource transfer is allowed, the basic idea of proposition 6 still holds. The leverage ratio increases if the discounted expected subsidy on being a low skill agent onward increases. From the optimal tax code (1.5.12), we can have the relationship between the labor income tax and the optimal investment:

$$\begin{cases} k_{2h}^* = \frac{u'(c_{hh}^*)}{u'(c_h^*)}(c_{hh}^* - wy_{hh}^* - \alpha_{hh}) = \frac{u'(c_{hl}^*)}{u'(c_h^*)}(c_{hl}^* - \alpha_{hl}) \\ k_{2l}^* = \frac{u'(c_{lh}^*)}{u'(c_l^*)}(c_{lh}^* - wy_{lh}^* - \alpha_{lh}) = \frac{u'(c_{ll}^*)}{u'(c_l^*)}(c_{ll}^* - \alpha_{ll}) \end{cases} \quad (1.6.4)$$

Using (1.6.4), we can rewrite  $X(\alpha_h, \alpha_l)$  as

$$\begin{aligned} X(\alpha_h, \alpha_l) &= (\pi\alpha_h + (1 - \pi)\alpha_l) - (\pi k_{2h}^* + (1 - \pi)k_{2l}^*) \\ &= \frac{\pi}{u'(c_h^*)} [u'(c_h^*)\alpha_h + u'(c_{hh}^*)\alpha_{hh}] + \frac{1 - \pi}{u'(c_l^*)} [u'(c_l^*)\alpha_l + u'(c_{lh}^*)\alpha_{lh}] + C_1 \\ &= \frac{\pi}{u'(c_h^*)} [u'(c_h^*)\alpha_h + \pi u'(c_{hh}^*)\alpha_{hh} + (1 - \pi)u'(c_{hl}^*)\alpha_{hl}] \\ &\quad + \frac{1 - \pi}{u'(c_l^*)} [u'(c_l^*)\alpha_l + \pi u'(c_{lh}^*)\alpha_{lh} + (1 - \pi)u'(c_{ll}^*)\alpha_{ll}] + C_2, \end{aligned} \quad (1.6.5)$$

where  $C_1$  and  $C_2$  are some constants consisting of optimal values  $(c^*, y^*)$  independent of  $\alpha$ 's. Then, using the above expression (1.6.5) and the labor income budget constraints (1.5.13) and (1.5.14), we have the following proposition about how the change in labor taxes affects the debt and equity choice given the optimal allocation. To be more specific, we need to define the expected present value of labor taxes conditional on being a high skill agent:

$$A := u'(c_h^*)\alpha_h + \pi u(c_{hh}^*)\alpha_{hh} + (1 - \pi)u(c_{hl}^*)\alpha_{hl}.$$

**Proposition 7.** *Given the optimal allocation  $(c^*, k^*, y^*)$ , suppose the government changes the labor income tax codes  $(\alpha_h, \alpha_l)$  and  $(\alpha_{hh}, \alpha_{hl}, \alpha_{lh}, \alpha_{ll})$  that satisfies (1.5.13) and (1.5.14). Other things being equal, we have*

$$\frac{\partial B_1^*}{\partial A} < 0 \quad \text{and} \quad \frac{\partial E_1^*}{\partial A} > 0.$$

*Proof.* See the appendix. □

The intuition for Proposition 7 is quite similar to that of Proposition 6. Notice  $A$  is negative. Therefore,  $A$  goes up if and only if the expected present value of labor taxes conditional on being the high type goes down since the high skill agents in equilibrium should pay the labor taxes and the low skill agents receive the subsidy. In other words,  $A$  goes up if and only if the government provide less insurance against being low skilled. Thus, agents choose more amount of equity (therefore less amount of debt) for self-insuring her against the low skill shock. The ratio of debt holding is negatively correlated with the expected present value of labor taxes conditional on being the high type. On the other hands, the leverage goes up if the

expected discounted value of being low skill agent in period 1 and being whoever in period 2 increases, i.e., basically the tax authority provides more insurance against low skill shocks.

## 1.7 More Than Two Types

In this section, we extend the model of previous sections into the case for more than two types of agents. The fundamental idea is exactly same as before. We can explicitly derive the tax system and the optimal market portfolio of debt and equity that turn out to be easy extension of the previous results of the case for two types. However, there is one crucial difference, which is the reason why we write this section. The corporate tax rate when there are more than two types of agents is indeterminate while the uniqueness does hold when there are only two types.

We first summarize the tax code in Section 1.7.1 and the optimal portfolio of debt and equity in Section 1.7.2, which are analogues of previous results. Then, we continue to investigate the other properties. Section 1.7.3 shows the indeterminacy of the corporate tax level. In fact, it turns out to be that  $\tau_c$  suggested in Section 1.7.1 is the minimal level and the government can choose the corporate income tax rates greater than or equal to  $\tau_c$  by properly adjusting the other individual capital taxes according to the change of the corporate tax. This indeterminacy can provide a normative interpretation about the historically fairly high corporate income tax rates levied in many countries, in particular, during the last centuries in U. S..

The indeterminacy raises an immediate question: Given the current rate is high enough, what if we increase or decrease the corporate tax rate? Section 1.7.4 deals

with the effect of the change of the corporate tax on the firm's leverage ratio. Surprisingly, unlike the classical capital structure theories, the change of the corporate tax rate does not have impact on the leverage ratio. Finally, due to the existence of the corporate tax it is never surprising that the aggregate capital income tax is nonzero, which is different from the classical result of the Ramsey taxation (Section 1.7.5).

### 1.7.1 Basic Results: A Simple Extension

The previous analysis should also work for any finite number of agents. Since the basic intuition will be the same, here we show how to pin down the corporate tax and how to set up the individual capital taxes when there are three types of agents. It is straightforward to derive the general result for the case of  $n$  types of agents. Suppose that there are three skill types  $\{\theta_h, \theta_m, \theta_l\}$  with

$$\theta_h > \theta_m > \theta_l.$$

Let  $Pr(\theta = \theta_i) = \pi_i$  with  $i = h, m, l$ . So,  $\pi_h + \pi_m + \pi_l = 1$ .  $\theta_i$  is private information. Shocks are i.i.d. over time across agents as well. Everybody can work. Their utility functions are assumed to be the same as before:

$$\sum_{t=0}^2 u(c_t) - v(e_t)$$

with  $y_t = e_t \theta_t$ , where  $e_t$  is the effort level at time  $t$  and  $y_t$  is the labor provided by the agent.  $e_t$  is private information. The production function is the same as before:

$f(K, Y) = rK + wY$ . All the setup and the analysis are very similar as before.

It is tedious to write down the planner's problem again. Thus, we skip it. The first order conditions are similarly obtained. Assume that we have already characterized  $(c^*, y^*, k^*)$ , the constrained optimal allocation in this case. The most important key is the following inverse Euler equation in period 1:

$$u'(c_0^*) = \frac{r}{\frac{\pi_h}{u'(c_h^*)} + \frac{\pi_m}{u'(c_m^*)} + \frac{\pi_l}{u'(c_l^*)}}.$$

Each agent is indexed by subscripts  $h, m$ , and  $l$ , respectively. Then, the corporate tax rate  $\tau_c$  and the optimal individual capital tax code  $(\tau_h^B, \tau_h^E, \tau_m^B, \tau_m^E, \tau_l^B, \tau_l^E)$  in period 1 are give by

$$\left\{ \begin{array}{l} r(1 - \tau_c)u'(c_l^*) = u'(c_0^*) \\ r(1 - \tau_c)(1 - \tau_m^E)u'(c_m^*) = u'(c_0^*) \\ r(1 - \tau_c)(1 - \tau_h^E)u'(c_h^*) = u'(c_0^*) \\ \pi_h r(1 - \tau_h^B)u'(c_h^*) + \pi_m r(1 - \tau_m^B)u'(c_m^*) + \pi_l r(1 - \tau_l^B)u'(c_l^*) = u'(c_0^*), \\ (1 - \tau_h^B) > (1 - \tau_h^E)(1 - \tau_c) \\ (1 - \tau_m^B) > (1 - \tau_m^E)(1 - \tau_c) \\ (1 - \tau_l^B) < (1 - \tau_c) \\ \tau_l^E > 0 \end{array} \right. \quad (1.7.1)$$

and

$$\pi_l(1 - \tau_c) + \pi_m(1 - \tau_c)(1 - \tau_m^E) + \pi_m(1 - \tau_c)(1 - \tau_h^E) = 1. \quad (1.7.2)$$

The first three equations in (1.7.1) is derived by setting the capital tax rates equal to the ex-post wedges, each of which is the component of the Euler equation with



respect to  $E_1$ . The fourth equation is the Euler-equation derived from the first order condition with respect to  $B_1$ . The next four inequalities are the conditions where the high and middle skill agents will prefer debt while the lowest skill agents will prefer equity in the next period, which in turn remove the tax timing options of the lowest skill agents. Technically, we first pin down  $\tau_c, \tau_h^E$ , and  $\tau_l^E$ , and then choose  $\tau_h^B, \tau_m^B, \tau_l^B$  and  $\tau_l^E$  flexibly through the inequalities.

The crucial condition is (1.7.2). This condition is designed to make firms indifferent to choosing between debt and equity. (1.7.2) was not necessary for the case where there are two only types of agents. In that case, the last equation is automatically satisfied (See the proof of Theorem 1). However, for the case where there are more than two types of agents, we should impose this condition when setting up the capital tax rates. This is because the number of equity tax rates (equal to the number of types) to determine is more than the number of assets (debt and equity) in the market. If the last equation of (1.7.1) is not satisfied, then the firm will provide either 100 % debt or 100 % equity financing while every agent chooses both debt and equity with positive amount, which in turn fails to meet the market clearing condition.

This idea to set (1.7.1) is also easier to understand if we look at the following budget constraint of each type agent. In period 0,

$$c_0 = rk_0 - (B_1 + E_1) + wy_0 \quad \text{with} \quad B_1 + E_1 = K_1^*,$$

In period 1,

$$c_h = r(1 - \tau_h^B)B_1 + \max_{realize, not} ((1 - \tau_h^E)(1 - \tau_c), 1 - \tau_c)rE_1 - k_{2h} + wy_h + \alpha_h,$$

$$c_m = r(1 - \tau_m^B)B_1 + \max_{realize, not} ((1 - \tau_m^E)(1 - \tau_c), 1 - \tau_c)rE_1 - k_{2m} + wy_m + \alpha_m,$$

$$c_l = r(1 - \tau_l^B)B_1 + \max_{realize, not} ((1 - \tau_c)(1 - \tau_l^E), 1 - \tau_c)rE_1 - k_{2l} + wy_l + \alpha_l$$

Now, it is easy to show the following lemma which is an extension of Lemma 2.

**Lemma 4.** *The tax system (1.7.1) satisfies*

$$\tau_h^E < \tau_m^E < 0 < \tau_c < \tau_l^B.$$

*Proof.* See the appendix. □

Similarly to Lemma 2, Lemma 4 tells that this tax system makes the ex-post lowest skill agents prefer equity and all the other types prefer bonds. The only lowest skill agents need to pay individual capital income taxes in period 1. This is still true if we have more and more types. Only the lowest types of agents face a positive capital tax rates. However, in a model with more than 3 periods, it is no more true that the currently lowest type's capital tax rates is the highest. In Intuitively it would be usually true that the one who becomes very low skilled in the current period *relative* to the previous skill status pays the highest tax rates (See Section 1.9.1).

## 1.7.2 Endogenous Leverage for More than Two Types

The next proposition is analogous to Proposition 5. It provides the analytic form of the debt and equity holding. In order for simpler exposition, we introduce some

positive number  $\hat{k}_2$  which is equal to the period 1 aggregate investment,  $\pi_h k_{2h}^* + \pi_m k_{2m}^* + \pi_l k_{2l}^* = \hat{k}_2$ .

**Proposition 8.** *Let  $\pi_h k_{2h}^* + \pi_m k_{2m}^* + \pi_l k_{2l}^* = \hat{k}_2$ . Let  $(\tau_c, \tau_h^B, \tau_h^E, \tau_m^B, \tau_m^E, \tau_l^B, \tau_l^E)$  be the optimal tax system given in Proposition (1.7.1). Then, given the labor tax code,  $(\alpha_h, \alpha_m, \alpha_l)$ , the optimal portfolio of debt and equity  $(B_1^*, E_1^*)$  is given by*

$$B_1^* = \frac{-X(\alpha_h, \alpha_m, \alpha_l) - K_2^* + [\pi_h \tau_h^E + \pi_m \tau_m^E + \tau_c^* - (\pi_h \tau_h^E + \pi_m \tau_m^E) \tau_c] r K_1^*}{r D_3} \quad (1.7.3)$$

$$E_1^* = \frac{X(\alpha_h, \alpha_m, \alpha_l) + K_2^* - [\pi_h \tau_h^B + \pi_m \tau_m^B + \pi_l \tau_l^B] r K_1^*}{r D_3} \quad (1.7.4)$$

where  $X(\alpha_h, \alpha_m, \alpha_l) := (\pi_h \alpha_h + \pi_m \alpha_m + \pi_l \alpha_l) - \hat{k}_2$  and

$$D_3 = \pi_h [(1 - \tau_h^B) - (1 - \tau_h^E)(1 - \tau_c)] + \pi_m [(1 - \tau_m^B) - (1 - \tau_m^E)(1 - \tau_c)] + \pi_l [(1 - \tau_l^B) - (1 - \tau_c)].$$

*Proof.* See the appendix. □

First notice that the denominator in (1.7.3) and (1.7.4) are positive, which is summarized in Lemma 6 (easy extension of Lemma 5) in the appendix. One remark is that the comparative statics analysis with respect to the change in labor taxation is exactly same as shown in Proposition 6 and 7. The intuition is also the same, thus we skip this analysis.

### 1.7.3 Indeterminacy

The new result in this section is the indeterminacy of the capital income tax code. Notice that if the tax authority take the corporate tax level less than  $\tau_c$  in (1.7.1), then the low skill agents still have incentives to defer the realization of capital income. Then, what if the corporate tax level is higher than  $\tau_c$ ? The next proposition provide an answer to this question.

**Proposition 9.** *Let  $(\tau_c, \tau_h^B, \tau_h^E, \tau_m^B, \tau_m^E, \tau_l^B, \tau_l^E)$  be the optimal tax system given by (1.7.1). Let  $\tilde{\tau}_c = \tau_c + \epsilon$  for some  $\epsilon > 0$ . Then, there exist  $\delta_h > 0$  and  $\delta_m > 0$  such that  $(\tilde{\tau}_c, \tilde{\tau}_h^B, \tilde{\tau}_h^E, \tilde{\tau}_m^B, \tilde{\tau}_m^E, \tilde{\tau}_l^B, \tilde{\tau}_l^E)$  where*

$$\tilde{\tau}_c = \tau_c^* + \epsilon, \quad \tilde{\tau}_h^E = \tau_h^E - \delta_h, \quad \tilde{\tau}_m^E = \tau_m^E - \delta_m$$

*is also an optimal tax system. In addition, the other tax rates can be properly adjusted as long as the following inequalities are satisfied.*

$$(1 - \tilde{\tau}_h^B) > (1 - \tau_h^E + \delta_h)(1 - \tau_c - \epsilon)$$

$$(1 - \tilde{\tau}_m^B) > (1 - \tau_m^E + \delta_m)(1 - \tau_c - \epsilon)$$

$$(1 - \tilde{\tau}_l^B) < (1 - \tau_c - \epsilon)$$

$$\tilde{\tau}_l^E > 0$$

*Proof.* See the appendix. □

The proof of Proposition 9 is constructive, which means that we obtain  $\delta_h$  and  $\delta_m$  explicitly in the proof. Proposition 9 also tells that the corporate tax rate  $\tau_c$  in the

tax system (1.5.12) is the minimal level to support the socially optimal allocation. The tax authority can take  $\tilde{\tau}_c$  greater than this minimal value  $\tau_c$ . However, if the corporate tax rate increases by  $\epsilon$ , then the other individual capital taxes should be properly adjusted as well. In particular, the tax on equity of the higher skilled agents decreases by  $\delta_h$  and  $\delta_m$ , respectively. The other tax rates must satisfy the four inequalities and the Euler equation with respect to debt holding. In other words, these tax rates can be either increased or decreased.

Although the model has three periods, one can infer from this result that the corporate tax rates time series data of U.S. and many other OECD countries may be possible although we cannot say that it is optimal. In U.S. the effective corporate tax rates were over 50% during 1940-1950s and constantly decreased down to 25% in 2000s, which is around 50% change. The corporate tax rate might be initially too high. It is technically possible for the IRS to keep decreasing the rates during the last 60 years, in particular, in accordance with the constant requests of decreasing the rate from general investors. However, this story does not say that the IRS has been working optimally.

#### 1.7.4 Comparative Statics: Corporate Taxation

As shown in Proposition 9, the corporate tax is indeterminate as long as rate,  $\tilde{\tau}_c$  is greater than or equal to the minimal level  $\tau_c^*$  of (1.7.1). In other words, the tax authority is free to change the rates. Therefore, given the sufficiently high level of corporate tax rates, we can consider how the change in the rate affects the leverage ratio (or cross-country comparison). More precisely we rewrite (1.7.3) and (1.7.4)

using the tax code  $(\tilde{\tau}_c, \tilde{\tau}_h^B, \tilde{\tau}_h^E, \tilde{\tau}_m^B, \tilde{\tau}_m^E, \tilde{\tau}_l^B, \tilde{\tau}_l^E)$  suggested in Proposition 9. Thus, we introduce the following definition.

**Definition 1.** Let  $(\tilde{B}_1^*, \tilde{E}_1^*)$  be the debt and equity holding when the capital tax code is given by  $(\tilde{\tau}_c, \tilde{\tau}_h^B, \tilde{\tau}_h^E, \tilde{\tau}_m^B, \tilde{\tau}_m^E, \tilde{\tau}_l^B, \tilde{\tau}_l^E)$ .

Classical capital structure literature often predicts the positive correlation between the leverage ratio and the corporate tax rates, namely,

$$\frac{d\tilde{B}_1^*}{d\tilde{\tau}_c} > 0. \quad (1.7.5)$$

In particular, the leverage ratio decreases if the corporate tax rate decreases because the use of debt becomes less advantageous. Surprisingly, however, our paper predict that the leverage ratio is independent of the change in corporate tax rates. The change of the corporate tax need not affect the firm's leverage ratio in the optimal tax framework.

**Proposition 10.** Assume there is no period-by-period resource transfer and  $(\alpha_h, \alpha_m, \alpha_l)$  are fixed. Let the current tax system be given by  $(\tilde{\tau}_c, \tilde{\tau}_h^B, \tilde{\tau}_h^E, \tilde{\tau}_m^B, \tilde{\tau}_m^E, \tilde{\tau}_l^B, \tilde{\tau}_l^E)$  and  $\tilde{\tau}_c$  is sufficiently higher than the minimal level  $\tau_c$  defined by (1.7.1). Let the debt and equity holding be given by  $(\tilde{B}_1^*, \tilde{E}_1^*)$  corresponding to the current tax system. If there no change in  $(\tilde{\tau}_h^B, \tilde{\tau}_m^B, \tilde{\tau}_l^B, \tilde{\tau}_l^E)$ , then

$$\frac{d\tilde{B}_1^*}{d\tilde{\tau}_c} = \frac{d\tilde{E}_1^*}{d\tilde{\tau}_c} = 0.$$

*Proof.* See the appendix. □

Notice that from Proposition 9, if  $\tilde{\tau}_c$  changes, then  $\tilde{\tau}_h^E$  and  $\tilde{\tau}_h^B$  do change as well.

However, the other tax rates,  $(\tilde{\tau}_h^B, \tilde{\tau}_m^B, \tilde{\tau}_l^B, \tilde{\tau}_l^E)$ , do not necessarily change. If these tax rates are constant, then the leverage ratio is unchanged although the corporate tax rate is changing. Therefore, Proposition 10 tells that the changes in the other individual tax rates are much more important rather than that of the corporate tax rates when we examine the impact of tax reforms on the leverage ratio. Notice that the aggregate leverage ratio in U.S. is around 0.4, which has been quite stationary during the last 5-60 years (See Frank and Goyal (2010)). Notice that the results in this section is only a comparative static analysis and this theory is normative, not positive. Therefore, a right interpretation about Proposition 9 is that the past U.S. tax reforms might not be unreasonable in the long run in terms of corporate income taxes.

### 1.7.5 Non-zero Aggregate Capital Taxes

In the classical Ramsey models, the optimal capital tax rates should be zero if the agents have constant relative risk aversion utility function or should converge to zero as time goes by if they have general utility functions. It is still true in Kocherlakota (2005) that the aggregate capital income taxes are zero (therefore capital income taxes are purely redistributed) although individual capital taxes are never zero. In this paper, even the aggregate capital taxes are never zero since the corporate tax exists.

**Proposition 11.** *Suppose the capital income tax code is given by (1.5.12). In period 0, the aggregate (expected) optimal capital tax of period 1 is negative.*

*Proof.* See the appendix. □

In the proof of Proposition 11, the aggregate total capital income tax of period 1 is given by

$$r \underbrace{(\pi_h \tilde{\tau}_h^B + \pi_m \tilde{\tau}_m^B + \pi_l \tilde{\tau}_l^B)}_{(a)} B_1^* + r \underbrace{(1 - \{\pi_l(1 - \tilde{\tau}_c) + \pi_m(1 - \tilde{\tau}_c)(1 - \tilde{\tau}_m^E) + \pi_m(1 - \tilde{\tau}_c)(1 - \tilde{\tau}_h^E)\})}_{(b)} E_1^*. \quad (1.7.6)$$

Component (a) of equation (1.7.6) is negative and component (b) of equation (1.7.6) is 0. This means that the capital taxes from equity are purely redistributive while the capital taxes from debt are not.

## 1.8 Practical Discussion on the Tax Scheme

### 1.8.1 On the Corporate Income Tax History in U.S.

The modern form of the corporate income tax in U.S. was introduced by the Revenue Act 1909.<sup>21</sup> Since the individual income tax was revived in 1913, a separate corporate tax has remained until now. It is widely accepted that the first inception of the corporate income tax was mainly for increasing the tax revenue. However, the government and the IRS were certainly aware of individual incentives to avoid taxes. They have continuously amended the tax law in this dimension.

One of notable evidence is the Revenue Act 1936 which introduced a surtax on the undistributed profits of a firm. According to Lent (1948), this additional tax was

---

<sup>21</sup>The federal corporate income tax was first introduced in 1894 but found unconstitutional the following year.



designed to remove the inequality in corporate taxes on the shares of stockholders who could afford to escape high surtaxes by withholding distribution of earnings. The idea of withholding distribution of earnings is quite similar to the tax timing option in the paper. Although the act itself was repealed several years later, the notion of removing inequality due to withholding distribution was probably incorporated in the next tax reforms again and again. The Internal Revenue Report (2002) concretely stated that from almost the beginning of the corporate income tax, there have been restrictions or additional taxes on excessive accumulations of undistributed corporate profits and special rules and rates for individuals who incorporate to avoid taxes. Therefore, we believe that the tax scheme in this paper is not far away from the real world tax scheme in spirit.

### 1.8.2 On the Assumption

Whether the government can tax unrealized capital income depends on how well it can monitor asset transactions among shareholders. Corporate taxation, in fact, is never required if the Internal Revenue Service (IRS) can easily keep track of all shareholders of a corporation. The constrained optimum can be implemented simply by using an individual capital/labor income tax code (as in Kocherlakota (2005) or Albanesi and Sleet (2006)) without using the additional tax instrument such of the corporate tax. A real example is the existence of C corporations and S corporations in the US tax code: C corporations can have an unlimited number of shareholders, while S corporations are restricted to no more than 100 shareholders. C corporations can have non-US residents as shareholders, but S corporations cannot.<sup>22</sup> Because S

---

<sup>22</sup>Other differences are as follows: S corporations cannot be owned by C corporations, other S corporations, LLCs, partnerships, or many trusts. C corporations are not subject to these

corporations have simple ownership structures which can be easily accessed by the tax authority, they are not required to face taxes at the corporate level. On the other hand, the owners of a C corporation are changing every second in the stock market, including foreign investors who are out of the control of the IRS. Therefore, there is a role for corporate taxes on C corporations.

### 1.8.3 On the Data

It is notable that effective corporate tax rates in U.S. have decreased constantly and significantly from over 50% in the 1940-50s to around 25% in the 2000s (Friedman, 2004).<sup>23</sup> According to the standard capital structure theory, the leverage ratio should have significantly decreased as well. However, a stylized empirical fact on capital structure is that the aggregate market-based leverage ratio<sup>24</sup> is fairly stationary during the last century with surprisingly small fluctuations (See Frank and Goyal (2007)). Our theory is not inconsistent with two time series data. However, again, this theory is normative, so we do not want to compare between our result and the result from positive theories. We hope that this kind of a general equilibrium approach will shed lights on solving the anomaly between two time series data.

---

same restrictions. S corporations can have only one class of stock (disregarding voting rights). C corporations can have multiple classes of stock.

<sup>23</sup>The effective tax rate is the corporate tax receipts as a percent of corporate profits.

<sup>24</sup>The market-based leverage ratio is defined by  $\text{debt}/(\text{debt} + \text{market value of equity})$ .

## 1.9 Other Generalization

### 1.9.1 More than Three Periods

The model also can be extended to a multi-period model even incorporating many types of agents suggested in the previous section. Although the analysis might not be very tractable, the idea is simply preserved. The crucial thing is to how to take the corporate tax in each period.

Suppose we already characterize the constrained optimal allocation in a multi-period setting although we do not specify it here. Recall that the corporate tax is designed to remove the tax timing option of the lowest skill agents in the three period model. The lowest skill agent is the one who should pay the maximum capital income taxes in the standard Mirrlees model. Then, we should remove the tax timing option of the agent who faces the largest capital income tax in each period. That is, the corporate income tax,  $\tau_{t+1,c}^*$ , in period  $t+1$  (contingent on  $t+1$  history) is set to be

$$1 - \tau_{t+1,c}^* = \inf \frac{u'(c_t^*)}{\beta r u'(c_{t+1}^*)},$$

given  $c_t^*$  is the socially optimal allocation in period  $t$  and  $\beta$  is the discount factor.

Then, the other individual capital taxes should be adjusted according to  $\tau_{t+1,c}^*$ .

## 1.9.2 Aggregate Uncertainty: Production Shock

Suppose that the production function in period 2 is given by  $f(k, y) = \tilde{r}k + wy$ , where  $\tilde{r}$  is a random variable independent of  $\theta$ ,

$$\tilde{r} = \begin{cases} r_1, & \text{with probability } p \\ r_2, & \text{with probability } 1 - p \end{cases}$$

with  $r_1 < r < r_2$ . Note that  $\tilde{r} = r$  in period 0 and 2. Let  $c_i^*(\tilde{r})$ ,  $i = l, h$  denote the optimal consumption under the aggregate shock. Then, the optimal allocation should satisfy the inverse Euler equation with  $\lambda(r_i) > 0$ ,  $i = 1, 2$ :

$$\lambda(r_i)u'(c_0^*) = \frac{1}{E\left[\frac{1}{u'(c_1^*(\tilde{r}))} \mid \tilde{r} = r_i\right]} = \frac{1}{\frac{\pi}{u'(c_h^*(r_i))} + \frac{1-\pi}{u'(c_l^*(r_i))}} \quad i = 1, 2$$

$$p\lambda(r_1)r_1 + (1-p)\lambda(r_2)r_2 = 1.$$

Now the corporation raises funds by equities and debts. Let  $R_1$  and  $R(\tilde{r})$  be the period 1 return on one unit of debt and equity in period 0. Then, their relation is given by

$$R(\tilde{r}) = \frac{\tilde{r}(B_1 + E_1) - R_1 B_1}{E_1}. \quad (1.9.1)$$

Then, each period budget constraint is rewritten as follows. In period 0,

$$c_0 = k_0 - (B_1 + E_1) + wy_0 \quad \text{with} \quad B_1 + E_1 = k_1^* \quad (1.9.2)$$

In period 1,

$$c_h(\tilde{r}) = (1 - \tau_h(\tilde{r}))R_1B_1 + \max_{realize, not} \{(1 - \tau_h(\tilde{r}))(1 - \tau_c(\tilde{r})), 1 - \tau_c(\tilde{r})\}R(\tilde{r})E_1 - k_{2h}(\tilde{r}) + wy_h + \alpha_h(\tilde{r}), \quad (1.9.3)$$

$$c_l(\tilde{r}) = (1 - \tau_l(\tilde{r}))R_1B_1 + \max_{realize, not} \{(1 - \tau_c(\tilde{r}))(1 - \tau_l(\tilde{r})), 1 - \tau_c(\tilde{r})\}R(\tilde{r})E_1 - k_{2l}(\tilde{r}) + \alpha_l(\tilde{r}), \quad (1.9.4)$$

where each variable is contingent on  $\tilde{r}$ . The optimal tax system shows the state-contingency:  $\{\tau_c(\tilde{r}), \tau_h^B(\tilde{r}), \tau_h^E(\tilde{r}), \tau_l^B(\tilde{r}), \tau_l^E(\tilde{r})\}$  with  $\tilde{r} = r_1, r_2$  satisfying

$$\begin{cases} R(\tilde{r})(1 - \tau_c(\tilde{r}))u'(c_l^*(\tilde{r})) = \lambda(\tilde{r})\tilde{r}u'(c_0^*) \\ R(\tilde{r})(1 - \tau_c(\tilde{r}))(1 - \tau_h^E(\tilde{r}))u'(c_h^*(\tilde{r})) = \lambda(\tilde{r})\tilde{r}u'(c_0^*) \\ \pi R_1(1 - \tau_h^B(\tilde{r}))u'(c_h^*(\tilde{r})) + (1 - \pi)R_1(1 - \tau_l^B(\tilde{r}))u'(c_l^*(\tilde{r})) = \lambda(\tilde{r})\tilde{r}u'(c_0^*) \end{cases} \quad (1.9.5)$$

In sum, there are two equations from (1.9.1), 4 equations from (1.9.3) and (1.9.4), and the following three equations:

$$\pi k_{2h}(\tilde{r}) + (1 - \pi)k_{2l}(\tilde{r}) = K_2^*, \quad (\tilde{r} = r_1, r_2)$$

$$B_1 + E_1 = K_1^*$$

Then, we can get 9 unknowns:  $R_1, (B_1, E_1)$  and  $(k_{2h}(\tilde{r}), k_{2l}(\tilde{r}))_{\tilde{r}=r_1, r_2}, \{R(\tilde{r})\}_{\tilde{r}=r_1, r_2}$ . It is not hard to see that there is an interior solution of  $(B_1, E_1)$ . Therefore, the aggregate shock affects the capital structure in the quantitative sense.

## 1.10 Literature Review

**Capital Structure Theory** The literature on capital structure is too large to summarize. Roughly speaking there are two widely held views. One is the *trade-off theory* and the other is the *pecking order theory*. The main driving force determining the use of debt in the trade-off theory is the trade-off between tax benefits and bankruptcy costs. In the pecking order theory, information asymmetry provides a strict order of financing: due to adverse selection, internal funds are used first, debt is issued if internal funds are depleted, and equity is a last resort if it is not sensible to issue more debt. Each theory can explain many features of corporate financing. As mentioned before, however, neither of them are satisfactory in terms of the stylized empirical long run stability of the leverage ratio and the downward trend in the corporate tax rates.<sup>25</sup>

Notice that it is not an entirely new view to explain the capital structure in the general equilibrium context, in particular, using the difference between individual and corporate taxes. Miller (1977) first proposed the idea that the aggregate leverage ratio results from different individual tax rates among investors given the corporate taxes. DeAngelo and Masulis (1980) and Auerbach and King (1983) formalize more micro-founded models. They, in addition, find that individual short-

---

<sup>25</sup>The pecking order theory is empirically rejected since firms often issue equities in wrong times. The two most common critiques on the standard trade-off theory are that (i) measured bankruptcy costs are too small, and moreover (ii) firms use too little debt. Dynamic versions of the trade-off theory seem to successfully explain that the observed levels of debt are not surprising (See Fischer et al. (1989), Hennessy and Whited (2005), Goldstein et al (2001), etc). In this sense, the recent dynamic trade-off theory becomes more compelling although any judgement on the results is still tentative. However, the amount of bankruptcy costs is still questionable and the long-term stability of the leverage ratio is another concern. See Frank and Goyal (2007) for excellent empirical surveys.

sale constraints are necessary for the existence of the equilibrium.<sup>26</sup> The Miller equilibrium, however, should be quite sensitive to the relative ratio of the corporate to the highest individual tax rates.<sup>27</sup> The investors are separated into two groups: Those agents whose individual tax rate is greater than the corporate tax rate should be specialized in equities and the others in debts.<sup>28</sup> Therefore, a change in corporate taxes, *ceteris paribus*, should directly affect the leverage ratios. This is also counterfactual to the stability of the leverage ratio given the very large changes in corporate tax rates during the last century.<sup>29</sup> Furthermore, the agents are not completely separated in either equity or debt in this paper.

**New Dynamic Public Taxation** One notable progress in recent taxation theory is called *the new dynamic public finance*, which developed the optimal tax system by extending the seminal work of Mirrlees (1971) to a dynamic setting. The main assumption in this literature is that agents in the economy have private information about their skills, which evolve stochastically over time. They consider the capital income taxes as a key device to implement the second best allocation. Our paper follows this spirit and builds on Kocherlakota (2005).

Other papers closely related to this one are Golosov and Tsyvinski (2007) and Albanesi (2006). Golosov and Tsyvinski (2007) study *asset testing* mechanisms

---

<sup>26</sup>The short-shale constraints are not necessary in our model.

<sup>27</sup>On the other hand, Graham (2003) and McDonald (2006) point out that the Miller equilibrium in the 1970's was plausible, when the highest personal tax rates exceeded the highest corporate rates, but, in the 1980's, the relative tax rates for corporations increased, making the Miller equilibrium implausible.

<sup>28</sup>Miller (1977), DeAngelo and Masulis (1980), and Auerbach and King (1983) all predict that high income people (with high tax bracket) hold equity whereas low income people (with low tax bracket) hold debt.

<sup>29</sup>Even before these models appeared, Stiglitz (1973) stated "Empirical studies of the effects of taxation on corporate financial structure suggest that taxation has not had a very significant effect on corporate financial structure, let alone the dramatic change that one might have anticipated given the very large increases in the corporate tax rates in the last fifty years."

in the disability insurance system in which a disability transfer is paid only if an agent has assets below a specified threshold.<sup>30</sup> An asset test deters false claims by penalizing the strategy of oversaving and not working. This idea can be applied the mechanism where the high type agent should be prevented from oversaving in order to avoid work. However, in our model oversaving is not the essential problem. Whether the agent deviates does not directly hinge on the the amount of agent's current wealth, but on the fact that he has chance to be a high type worker in the future. Albanesi (2006) considers the dynamic moral hazard problem of entrepreneurs facing idiosyncratic capital risk. She investigates differential asset taxation to implement the optimal allocation. She also shows that the double taxation is optimal if entrepreneurs sell the ownership of their firms and buy the ownership of other firms. The corporate tax in Albanesi (2006) is levied only on outside investors, but not on the entrepreneur who also has the ownership. The corporate tax, however, is the tax imposed on the earnings of each firm. To our knowledge, our model is the closest one that explains the real world double taxation mechanism. More importantly, the capital structure and optimal tax system are endogenously determined in our paper.

## 1.11 Conclusion

We clarify the role the corporate tax in order to achieve the constrained optimal allocation under the Mirrleesian taxation framework with an additional realistic assumption. In addition, the existence of the corporate tax requires the individ-

---

<sup>30</sup>The disability shock in Golosov and Tsyvinski is an absorbing state; once the agent declares disability, he/she can never come back to work.



ual taxation properly adjusted. This sophisticated tax system affects an individual agent's portfolio holdings of debt and equity, in turn, it determines the aggregate leverage ratio. Along this line, this paper investigates the endogenous characteristics between the optimal tax system and the capital structure. The optimal tax mechanism in this paper is designed to prevent the agents from using tax timing options. Understanding the capital structure in optimal taxation framework may seem somewhat unusual because taxation is often regarded as a normative theory. However, we hope this approach can potentially shed on light in designing a workhorse model in understanding capital structure issues better.

## 1.12 References

1. Acemoglu, C., Golosov, M., and A. Tsyvinski, (2009) "Dynamic Mirrlees Taxation and Political Economy", forthcoming in Review of Economic Studies.
2. Albanesi, S. (2006), "Optimal Taxation of Entrepreneurial Capital with Private Information", Working paper.
3. Albanesi, S. and C. Sleet (2006), "Dynamic Optimal Taxation with Private Information", forthcoming, Review of Economic Studies.
4. Auerbach, A. and M. King (1983), "Taxation, Portfolio Choice, and Debt-Equity Ratios: A General Equilibrium Model", Quarterly Journal of Economics, 98(4), 587-610.
5. Bassetto, M. and N. Kocherlakota (2004), "On the irrelevance of government debt when taxes are distortionary", Journal of Monetary Economic 51, 299-304.
6. Black, F., (1976), "The Dividend Puzzle", Journal of Portfolio Management, 2, 5-8.
7. Chen, B, Chen, H. and Wang, P. (2010) "Taxing Capital is Not a Bad Idea Indeed: The Role of Human Capital and Labor-Market Frictions", Working paper.
8. Conesa, W. C., K. Sagiri and D. Krueger (2009) "Taxing Capital? Not a Bad Idea After All!", American Economic Review 99, 25-48.

9. Constantinides, G. (1984), "Optimal Stock Trading with Personal Taxes: Implications for Prices and the Abnormal January Returns", *Journal of Financial Economics*, 13, 65-89.
10. Farhi, E. and I. Werning (2008 a), "The Political Economy of Nonlinear Capital Taxation", Working paper.
11. Farhi, E. and I. Werning (2008 b), "Progressive Estate Taxation", Working paper.
12. Fischer, E., R. Heinkel, and J. Zechner (1989), "Dynamic Capital Structure Choice: Theory and Tests", *Journal of Finance* 44, 19-40.
13. Goldstein, R., N. Ju, and H. Leland, (2001), "An EBIT-Based Model of Dynamic Capital Structure", *Journal of Business* 74, 483-512.
14. DeAngelo, H. and R. Masulis (1980), "Optimal Capital Structure Under Corporate and Personal Taxation", *Journal of Finance*, 42(5), 1143-1166.
15. Frank, M. and G. Goyal (2007), "Trade-off and Pecking Order Theories of Debt", Working paper.
16. Friedman, J. (2003), *The Decline of Corporate Income Tax Revenues*, Center On Budget and Polity Priorities.
17. Golosov, M., Kocherlakota, N. and Tsyvinski, A., (2003), "Optimal Indirect and Capital Taxation", *Review of Economic Studies* 70, pp. 569-587.
18. Golosov, M and A Tsyvinski (2006), "Designing Optimal Disability Insurance: A Case for Asset Testing", *Journal of Political Economy* 114(2), pp. 257-269.

19. Golosov, M and A Tsyvinski (2007), "Optimal Taxation with Endogenous Insurance Markets", *Quarterly Journal of Economics*, 122(2), pp. 487-534.
20. Graham, J. (1999), "Do Personal Taxes Affect Corporate Financing Decisions?", *Journal of Public Economics*, 73, 147-185
21. Graham, J. (2000), "How Big Are the Tax Benefits of Debt?", *Journal of Finance*, 55(5), 1901-1941.
22. Graham, J. (2003), "Taxes and Corporate Finance: A Review", *Review of Financial Studies*, 16(4), 1075-1129.
23. Graham, J. and A. Tucker (2006), "Tax Shelters and Corporate Debt Policy", *Journal of Financial Economics* 81, 563-594.
24. Henneesy, C.A. and T. A. Whited (2005), "Debt Dynamics", *Journal of Finance* 60, 1129-1165
25. Ju, N., R. Parrino, A. M Poteshman, and M. S. Weisbach (2005), "Horses and Rabbits? Trade-Off Theory and Optimal Capital Structure", *Journal of Financial and Quantitative Analysis* 40, 259-281.
26. Kocherlakota, N., (2005), "Zero Expected Wealth Taxes: A Mirrlees Approach to Dynamic Optimal Taxation", *Econometrica* 73, pp. 1587-1621.
27. Kocherlakota, N., (2009), "The New Dynamic Public Finance", forthcoming, Princeton University Press.
28. G. E. Lent (1948), "The Impact of The Undistributed Profits Tax 1936-1937", New York: Columbia University Press.

29. McDonald, R. (2006), "Portfolio Choice and Corporate Financial Policy When There Are Tax-Intermediating Dealers", working paper.
30. Miller, M. (1977), "Debt and Taxes", *Journal of Finance*, 32(2), 261-275.
31. Miller, M. (1986), "Behavioral Rationality in Finance: The Case of Dividends", *Journal of Business* 59, S451-S468.
32. Stiglitz, J. (1969), "A Re-Examination of the Modigliani-Miller Theorem", *American Economic Review* 59, 784-793.
33. Stiglitz, J. (1973), "Taxation, Corporate Financial Policy and The Cost of Capital", *Journal of Public Economics*, 2, 1-34.
34. The Internal Revenue Service Report (2002), "Corporation Income Tax Brackets and Rates, 1909-2002."
35. Weichenrieder, A. (2005). "(Why) Do We Need Corporate Taxation?", Working paper.

## Appendices

### 1.A Appendix for Section 1.3

#### Proof of Lemma 1

*Proof.* Recall the inverse Euler equation.

$$\frac{r}{u'(c_0^*)} = \frac{\pi}{u'(c_h^*)} + \frac{1-\pi}{u'(c_l^*)}.$$

Then, by the Jensen inequality we have

$$u'(c_0^*) < r\pi u'(c_h^*) + r(1-\pi)u'(c_l^*) < r\pi u'(c_l^*) + r(1-\pi)u'(c_l^*) = ru'(c_l^*),$$

which completes the proof. □

#### Proof of Corollary 1

*Proof.* From the inverse Euler equation, we have

$$\frac{u'(c_0^*)}{ru'(c_h^*)} = \frac{1}{\pi} - \frac{(1-\pi)u'(c_0^*)}{\pi ru'(c_l^*)} > \frac{1}{\pi} - \frac{(1-\pi)}{\pi} = 1,$$

where the inequality follows by Lemma 1. □

## 1.B Appendix for Section 1.4

### Proof of Proposition 1

*Proof.* In fact, this proposition can be regarded as a special case of the general theorem shown in Kocherlakota (2005). Hence, Readers who are interested in the general set-up and its proof should refer Kocherlakota (2005). Under the tax system (1.4.4) and (1.4.5) we rewrite the agent's budget constraint as following:

$$\left\{ \begin{array}{l} c_l = c_l^* + r(1 - \tau_l)(k_1 - k_1^*) \\ c_h = c_h^* + r(1 - \tau_h)(k_1 - k_1^*) + w(y_h - y_h^*) \\ c_{hh} = c_{hh}^* + r(1 - \tau_{hh})(k_{2h} - k_{2h}^*) + w(y_{hh} - y_{hh}^*) \\ c_{hl} = c_{hl}^* + r(1 - \tau_{hl})(k_{2h} - k_{2h}^*) \\ c_{lh} = c_{lh}^* + r(1 - \tau_{lh})(k_{2l} - k_{2l}^*) + w(y_{lh} - y_{lh}^*) \\ c_{ll} = c_{ll}^* + r(1 - \tau_{ll})(k_{2l} - k_{2l}^*) \end{array} \right.$$

Then, the first order conditions are given by

$$\left\{ \begin{array}{l} u'(c_0) = \pi r(1 - \tau_h)u'(c_h) + (1 - \pi)r(1 - \tau_l)u'(c_l), \\ u'(c_h) = \pi r(1 - \tau_{hh})u'(c_{hh}) + (1 - \pi)r(1 - \tau_{hl})u'(c_{hl}), \\ u'(c_l) = \pi r(1 - \tau_{lh})u'(c_{lh}) + (1 - \pi)r(1 - \tau_{ll})u'(c_{ll}), \\ wu'(c_0) = v'(y_0), \quad wu'(c_h) = v'(y_h), \quad wu'(c_{hh}) = v'(y_{hh}), \quad wu'(c_{lh}) = v'(y_{lh}) \end{array} \right.$$

$$\left\{ \begin{array}{l} c_0 + k_1 = wy_1 + k_0 \\ \pi c_h + (1 - \pi)c_l + \pi k_{2h} + (1 - \pi)k_{2l} = rk_1 + w\pi y_h \\ \pi^2 c_{hh} + \pi(1 - \pi)c_{hl} + \pi(1 - \pi)c_{lh} + (1 - \pi)^2 c_{ll} \\ \quad = r(\pi k_{2h} + (1 - \pi)k_{2l}) + w(\pi^2 y_{hh} + \pi(1 - \pi)y_{lh}) \end{array} \right.$$

Then, it is not hard to see that the solution to the above system coincides with the constrained optimal solution. In fact, we need to check whether the individual agent will optimally choose the corresponding planner's allocation in each of following cases: (i)  $y_h > 0$ ,  $y_{hh} > 0$ , (ii)  $y_h = 0$ ,  $y_{hh} > 0$ , (iii)  $y_h > 0$ ,  $y_{hh} = 0$ , (iv)  $y_h = 0$ ,  $y_{hh} = 0$ . Since the agent's derived utility is strict concave with respect to  $(y, k)$ , each pair of allocation  $(c, k, y)$  corresponding to all cases from (i) to (iv) is the unique solution coinciding with the socially optimal allocation by using the above first order conditions. We omit the tedious algebra. □

## Proof of Proposition 2

*Proof.* Notice the following 3 equations for the first equality of (a):

$$r(1 - \tau_h)u'(c_h^*) = u'(c_0^*), \quad r(1 - \tau_l)u'(c_l^*) = u'(c_0^*), \quad u'(c_0^*) = \frac{r}{\frac{\pi}{u'(c_h^*)} + \frac{1-\pi}{u'(c_l^*)}}.$$

Then,

$$\pi\tau_h + (1 - \pi)\tau_l = \pi \left( 1 - \frac{u'(c_0^*)}{ru'(c_h^*)} \right) + (1 - \pi) \left( 1 - \frac{u'(c_0^*)}{ru'(c_l^*)} \right)$$



$$= 1 - \frac{u'(c_0^*)}{r} \left( \frac{\pi}{u'(c_h^*)} + \frac{1 - \pi}{u'(c_l^*)} \right) = 0.$$

Then, since  $c_h^* > c_l^*$ , we have the second property of (a). The proof for (b) is similar.

For the proof of footnote 18, if there is no intertemporal transfer of resources through the government, we have  $\pi\alpha_h + (1 - \pi)\alpha_l = r(\pi\tau_{kh} + (1 - \pi)\tau_{kl})k_1 = 0$ .

□

### Proof of Proposition 3

Before we start the proof of Proposition 3, there are two comments for easier understanding. First, the proof focuses only on the behavior of the low skill agents in period 1. The high skill agents already do not have incentives to deviate under the the second best world tax scheme. Second, although in the second best world we only investigated the case where there is no intertemporal transfer of resources, one should notice that, in general, the labor taxation is indeterminate. More precisely,  $(k_{2h}^*, k_{2l}^*)$  in the tax system (1.4.4) can be assigned arbitrarily as long as the sum of optimal capital accumulation of all the agents is equal to the capital investment of the constrained optimum, in other words, as long as  $\pi k_{2h}^* + (1 - \pi)k_{2l}^* = K_2^*$  is satisfied. Therefore, the agent's investment (or saving) strategy depends on how much labor taxes will be assigned in period 1, in particular, how big  $(\alpha_h, \alpha_l)$  in (1.4.4) are. Due to this indeterminacy the proof of proposition 3 is divided into 2 cases. Therefore, the proof is valid regardless of whether the government period-by-period transfers resources.

*Proof.* Consider an agent who *exclusively* owns a firm in period 0 become a *low skill* agent in period 1. If she gets the capital income  $rk_1^*$ , consume  $c_l^*$ , and invest  $k_{2l}^*$  as in Section 1.4.2, her remaining expected utility  $X$  at period 1 is

$$X := u(c_l^*) + \pi u(c_{lh}^*) - \pi v(y_{lh}^*) + (1 - \pi)u(c_u^*). \quad (1.B.1)$$

Now we investigate the two cases. In each case, we suggest a strategy to deviate from the socially optimal allocation and show that this allocation gives the low skill agent better off, which completes the proof.

First suppose

$$k_{2l}^* \geq r(1 - \tau_l)k_1^*,$$

which means that the low skill agent get enough labor subsidy. Consider the strategy that the firm does not distribute the capital rent  $rk_1^*$  and she additionally invest  $k'_1$  into her firm. In this case her consumption in period 1 is  $\alpha_l - k'_1$  since she does not pay the capital tax and gets the subsidy  $\alpha_l$ . Then, her remaining expected utility  $Y$  is now

$$\begin{aligned} Y &:= u(\alpha_l - k'_1) + \pi u[r(1 - \tau_{lh})(rk_1^* + k'_1) + wy_{lh} + \alpha_{lh}] - \pi v(y_{lh}^*) \\ &\quad + (1 - \pi)u[r(1 - \tau_u)(rk_1^* + k'_1) + \alpha_u] \\ &= u(c_l^* + k_{2l}^* - r(1 - \tau_l)k_1^* - k'_1) + \pi u\left(c_{lh}^* + \frac{c_{lh}^*}{c_l^*}(rk_1^* + k'_1 - k_{2l}^*)\right) - \pi v(y_{lh}^*) \\ &\quad + (1 - \pi)u\left(c_u^* + \frac{c_u^*}{c_l^*}(rk_1^* + k'_1 - k_{2l}^*)\right) \quad (1.B.2) \end{aligned}$$

In this case, we have  $X < Y$  as long as we can pick any  $k'_1$  satisfying

$$k_{2l}^* - r(1 - \tau_l)k_1^* \geq k'_1 \geq k_{2l}^* - rk_1^*.$$

This is possible since  $\tau_l > 0$  and  $k_{2l}^* \geq r(1 - \tau_l)k_1^*$ . Note that  $k'_1 = 0$  can be allowed.

Secondly, suppose

$$r(1 - \tau_l)k_1^* > k_{2l}^*,$$

which means that the labor subsidy is not enough, so the agent cannot afford to invest more. Consider the strategy that the firm distributes only  $r\tilde{k}_1 < rk_1^*$  amount of capital rent to the owner (the disable agent). In this case, she pays  $r\tau_l\tilde{k}_1$  as a capital income tax and has  $\alpha_l + r(1 - \tau_l)\tilde{k}_1$  as net consumption in period 1. The rest of capital rent ( $rk_1^* - r\tilde{k}_1$ ) is just remained (therefore reinvested) in the firm without being taxed. Then, her remaining expected utility  $Y$  is

$$\begin{aligned} Y &:= u(\alpha_l + r(1 - \tau_l)\tilde{k}_1) + \pi u[r(1 - \tau_{lh})(rk_1^* - r\tilde{k}_1) + wy_{lh} + \alpha_{lh}] - \pi v(y_{lh}^*) \\ &\quad + (1 - \pi)u[r(1 - \tau_u)(rk_1^* - r\tilde{k}_1) + \alpha_u] \\ &= u(c_l^* + k_{2l}^* - r(1 - \tau_l)(k_1^* - \tilde{k}_1)) + \pi u\left(c_{lh}^* + \frac{c_{lh}^*}{c_l^*}(rk_1^* - r\tilde{k}_1 - k_{2l}^*)\right) - \pi v(y_{lh}^*) \\ &\quad + (1 - \pi)u\left(c_u^* + \frac{c_u^*}{c_l^*}(rk_1^* - r\tilde{k}_1 - k_{2l}^*)\right) \quad (1.B.3) \end{aligned}$$

We compare (1.B.1) with (1.B.3). Notice that  $r(1 - \tau_l)k_1^* - k_{2l}^* < rk_1^* - k_{2l}^*$ . Then, if we take  $\tilde{k}_1 > 0$  such that

$$r(1 - \tau_l)\tilde{k}_1 \approx r(1 - \tau_l)k_1^* - k_{2l}^*,$$

then  $Y - X > 0$ . This completes the proof.  $\square$

## 1.C Appendix for Section 1.5

### Proof of Lemma 2

*Proof.* By simple algebra, showing  $0 < \tau_c^*$  and  $\tau_h^E < \tau_h$  is equivalent to showing

$$u'(c_0^*) < ru'(c_l^*),$$

which is result of Lemma 1. On the other hand, from (1.5.15) and (1.5.19),  $\tau_c^* < \tau_l^B$  is equivalent to  $\tau_h^B < 1 - \frac{u'(c_0^*)}{ru'(c_h^*)}$ , which is exactly (1.5.18).  $\square$

### Proof of Theorem 1

*Proof.* Only the period 1 tax codes are different between the second and the third best world. The period 2 tax codes are the same. The optimal choice of the agent between period 1 and 2 is same as the constrained optimal allocation, i.e., the agent's consumption in  $t = 2$  and investment in  $t = 1$  are the same as the constrained optimal allocation (This is simply the result of Proposition 1. Readers can refer Kocherlakota (2005) for more general proof). Therefore, we focus on the allocation between  $t = 0$  and  $t = 1$  given that

$$(k_{2h}, k_{2l}, c_{hh}, c_{hl}, c_{ll}, c_{lh}, y_{hh}, y_{lh}) = (k_{2h}^*, k_{2l}^*, c_{hh}^*, c_{hl}^*, c_{ll}^*, c_{lh}^*, y_{hh}^*, y_{lh}^*) \quad (1.C.1)$$

Without loss of generality we also assume that there is no period-by-period transfer of resources. The result can be easily generalized for the case of resource transfer although the individual investment  $\{k_1(= B_1 + E_1), k_{2h}, k_{2l}\}$  will be different from the constrained optimal allocation for this case.

First, consider the individual agent's problem. Notice that after choosing between realizing and not-realizing their capital income, the budget constraints of the agent are

$$c_0 = rk_0 + wy_0 - (B_1 + E_1)$$

$$c_h = r(1 - \tau_h^B)B_1 + (1 - \tau_c^*)(1 - \tau_h^E)rE_1 - k_{2h} + wy_h + \alpha_h, \quad \text{if } y_h > 0$$

$$c_h = r(1 - \tau_l^B)B_1 + (1 - \tau_c^*)rE_1 - k_{2l}\alpha_l, \quad \text{if } y_h = 0$$

$$c_l = r(1 - \tau_l^B)B_1 + (1 - \tau_c^*)rE_1 - k_{2l} + \alpha_l.$$

We only need to consider two strategies of a high skill agent since a low skill agent cannot tell a lie. Suppose the agent works if she becomes a high skill agent in period 1. Substituting  $(c_h, c_l)$  into the objective function, we get the first order conditions with respect to  $B_1$  and  $E_1$  as follows.

$$u'(c_0) = \pi r(1 - \tau_h^B)u'(c_h) + (1 - \pi)r(1 - \tau_l^B)u'(c_l)$$

$$u'(c_0) = \pi r(1 - \tau_h^E)(1 - \tau_c^*)u'(c_h) + (1 - \pi)r(1 - \tau_c^*)u'(c_l)$$

$$v'(y_0) = wu'(c_0), \quad v'(y_h) = wu'(c_h)$$

$$c_0 = rk_0 + wy_0 - (B_1 + E_1)$$

$$c_h = r(1 - \tau_h^B)B_1 + (1 - \tau_c^*)(1 - \tau_h^E)rE_1 - k_{2h} + wy_h + \alpha_h$$

$$c_l = r(1 - \tau_l^B)B_1 + (1 - \tau_c^*)rE_1 - k_{2l} + \alpha_l.$$

Notice the objective function is strictly concave. Given (1.C.1),  $(c_0, c_l, c_h, y_h) = (c_0^*, c_l^*, c_h^*, y_h^*)$  is satisfied since the above first order conditions are the same as those first order conditions for the constrained optimal allocation in (1.3.4), (1.3.5), and (1.3.6) in Section 1.4.2.

The similar argument also applies for  $y = 0$ . Suppose a high skill agent does not work in  $t = 1$ , i.e.  $y_h = 0$ . Then, under the given tax system, he will choose 100 % equity investment since  $\tau_c^* < \tau_l^B$ . The first order conditions in this case are

$$u'(c_0) = r(1 - \tau_c^*)u'(c_h) = r(1 - \tau_c^*)u'(c_l)$$

$$v'(y_0) = wu'(c_0),$$

$$c_0 = rk_0 + wy_0 - E_1$$

$$c_h = c_l = (1 - \tau_c^*)rE_1 - k_{2l} + \alpha_l.$$

Given (1.C.1), setting  $(c_0, c_h, c_l, y_0, B_1, E_1)$  equal to  $(c_0^*, c_l^*, c_l^*, y_0^*, 0, k_1^*)$  satisfies the above first-order conditions by comparing these with (1.3.4), (1.3.5), and (1.3.6). Hence, the agent is indifferent between working  $y_h > 0$  in period 1 (when becoming high skilled) and not working in period 1.

Second, we consider the firm's problem. Again we only focus on the firm's decision for period 0 capital structure to install capital and period 1 labor employment, assuming period 1 investment and period 2 labor employment optimally take place. In fact, in period 1, the market becomes the classical second best world, that is the Modigliani-Miller theorem world. Thus, we can, without loss of gener-

ality, assume that the firm only the spot market to rent capital in period 1 as in classical macroeconomic models. Define  $f$  by any general constant-returns-to-scale production function (Thus, this proof is for general CRS production functions).

Let  $(r_b, r_e)$  denotes by the return on equity and debt and  $w'$  denotes by the price of labor. Here we first show that  $r_b = r_e$  in equilibrium. Given the next period investment plan  $K_2$ , the firm's problem is to raise debt  $B_1$  and equity  $E_1$  to install capital  $K_1$  in period 0 and rent labor  $Y_1$  in period 1 to maximize

$$r_e E_1 := \max_{(K_1, B_1, Y_1)} (1 - \tau_c^*) E[f(K_1, Y_1) - w'Y_1 - r_b B_1]$$

subject to  $B_1 + E_1 \geq K_1$

Notice that  $K_2 = K_2^*$  and this does not affect the value of equity in period 0. Then, putting  $B_1 + E_1 = K_1$ , we write the expectation operator in detail as follows.

$$\begin{aligned} r_e E_1 &= \max_{B_1, Y_1} (1 - \tau_c^*) \{ \pi(1 - \tau_h^E) + (1 - \pi) \} [f(E_1 + B_1, Y_1) - w'Y_1 - r_b B_1] \\ &= \max_{B_1, Y_1} f(E_1 + B_1, Y_1) - w'Y_1 - r_b B_1. \end{aligned}$$

since the tax code satisfies

$$\pi(1 - \tau_h^E)(1 - \tau_c^*) + (1 - \pi)(1 - \tau_c^*) = \frac{\pi u'(c_0^*)}{r u'(c_h^*)} + \frac{(1 - \pi) u'(c_0^*)}{r u'(c_l^*)} = 1. \quad (1.C.2)$$

by the inverse Euler equation. Suppose there is an interior solution  $B_1 \in (0, K_1^*)$ .

First order conditions with market clearing provide

$$r_b = f_1(K_1^*, Y_1^*) \quad \text{and} \quad w' = f_2(K_1^*, Y_1^*)$$

Since  $f$  is CRS, we also obtain  $r_e = r_b = f_1(K_1^*, Y_1^*)$ . (This also justifies why we have used  $r_e = r_b = r$  in the main context without special comment when  $f(k, y) = rk + wk$ . It is also clear to have  $w' = w$  for this case.) On the other hand, no arbitrage argument also can be applied: If  $r_e > r_b$ , then an agent will buy a stock using a money from selling a bond with interest rate  $r' \in (r_b, r_e)$ , which gives arbitrage. If  $r_b > r_e$ , then one will establish his own firm with no debt financing to get  $r$  return, instead of investing into a firm with return  $r_e$ .

Now consider equation (1.C.2). This is the expected effective after tax net return on equity, which is one. Thus, in aggregation, the representative shareholder does not pay the corporate tax. Since there is no bankruptcy, the firm is indifferent to choosing between debt and equity. In addition, the firm value is indifferent to capital structure. More precisely, suppose that there is an general equilibrium that the firm has a particular value of debt and equity  $(B_1^c, E_1^c)$ . Then, we have

$$rE_1^c = f(K_1^*, Y_1^*) - w'Y_1^* - rB_1^c.$$

or

$$E_1^c + B_1^c = \frac{f(K_1^*, Y_1^*) - w'Y_1^*}{r}.$$

Thus, the firm value depends on the aggregate variable, which is determined by the market supply of capital and labor. The idea is quite similar to Stiglitz (1969). This completes the proof. □



## 1.D Appendix for Section 1.6

### Proof of Proposition 5

*Proof.* Given the tax system, we already know that the constrained optimal solution of consumption and labor vectors  $(c^*, y^*)$  coincide with the solution to the competitive equilibrium. Now,  $(k_{2h}^*, k_{2l}^*, B_1^*, E_1^*)$  are obtained by solving the following system of equations:

$$\begin{bmatrix} \pi & (1-\pi) & 0 & 0 \\ 0 & 1 & -r(1-\tau_l^B) & -r(1-\tau_c^*) \\ 1 & 0 & -r(1-\tau_h^B) & -r(1-\tau_h^E)(1-\tau_c^*) \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_{2h} \\ k_{2l} \\ B_1 \\ E_1 \end{bmatrix} = \begin{bmatrix} \hat{k}_2 \\ \alpha_l - c_l^* \\ \alpha_h - c_h^* + wy_h^* \\ k_1^* \end{bmatrix} \quad (1.D.1)$$

Solving the above matrix equation (1.D.1), we have (1.6.1) and (1.6.2).  $\square$

### Sign of Denominators of (1.6.1) and (1.6.1)

The following lemma is useful to figure out the sign of aggregate debt and equity holding in Proposition 5.

**Lemma 5.** *Let  $D_2 = -(\pi\tau_h^E\tau_c^* + \pi\tau_h^B - \pi\tau_h^E + \tau_c^* - (1-\pi)\tau_l^B)$ . Then, we have  $D_2 > 0$ .*

*Proof.*

$$\begin{aligned}
D_2 &= \pi[(1 - \pi_h^B) - (1 - \tau_h^E)(1 - \tau_c^*)] + (1 - \pi)[(1 - \tau_l^B) - (1 - \tau_c^*)] \\
&= \pi(1 - \tau_h^B) - \frac{\pi u'(c_0^*)}{ru'(c_h^*)} + \frac{u'(c_0^*)}{ru'(c_l^*)} - \pi(1 - \tau_h^B) \frac{u'(c_h^*)}{u'(c_l^*)} - \frac{(1 - \pi)u'(c_0^*)}{ru'(c_l^*)} \\
&= \pi(1 - \tau_h^B) \left(1 - \frac{u'(c_h^*)}{u'(c_l^*)}\right) - 1 + \frac{u'(c_0^*)}{ru'(c_l^*)}, \\
&> \frac{\pi u'(c_0^*)}{ru'(c_h^*)} \left(1 - \frac{u'(c_h^*)}{u'(c_l^*)}\right) + \frac{u'(c_0^*)}{ru'(c_l^*)} - 1 = \frac{\pi u'(c_0^*)}{ru'(c_h^*)} + \frac{(1 - \pi)u'(c_0^*)}{ru'(c_l^*)} - 1 = 0.
\end{aligned}$$

where the first inequality is a rewriting of  $D_2$ , the second equality is by using (1.5.12), the third and the last equality are by the inverse Euler equation, and the third inequality is by (1.5.17).  $\square$

## The Proof of Corollary 4

*Proof.* Given  $(\alpha_h, \alpha_l)$ , the aggregate transfer of labor income subsidy is given by

$$X(\alpha_h, \alpha_l) = \pi\alpha_h + (1 - \pi)\alpha_l = r(\pi\tau_h^* + (1 - \pi)\tau_l^*)\hat{k}_{1b} + r(\pi\tau_h^* + \tau_c^*)\hat{k}_{1e}.$$

Since there is no governmental transfer,  $\hat{k}_1 = k_1^*$ . Plugging the above equation and  $\hat{k}_1 = k_1^*$  into (1.6.1) and (1.6.2), we have the required result.  $\square$

## Proof of Proposition 6

*Proof.* If  $\alpha_l$  goes up by  $\epsilon$ , then  $\alpha_h$  should be decreased by  $\frac{(1-\pi)u'(c_l^*)}{\pi u'(c_h^*)}\epsilon$  from (1.5.13).

Therefore, the change in  $X(\alpha_h, \alpha_l)$  is

$$\begin{aligned}\Delta X(\alpha_h, \alpha_l) &= \pi \left( -\frac{(1-\pi)u'(c_l^*)}{\pi u'(c_h^*)}\epsilon \right) + (1-\pi)\epsilon \\ &= \epsilon(1-\pi)\frac{u'(c_h^*) - u'(c_l^*)}{u'(c_h^*)} < 0.\end{aligned}$$

In this case, (1.6.1) and (1.6.2) tell that change in debt will be positive and the change in equity is negative, which shows that the leverage ratio goes up. On the other hand, if  $\alpha_l$  goes down, then the opposite implication holds, which means the leverage ratio goes down. This completes the proof.  $\square$

## Proof of Proposition 7

*Proof.* Using (1.5.14), we can rewrite (1.5.13) as

$$\begin{aligned}\pi[u'(c_h^*)\alpha_h + \pi u(c_{hh}^*)\alpha_{hh} + (1-\pi)u(c_{hl}^*)\alpha_{hl}] \\ + (1-\pi)[u'(c_l^*)\alpha_h + \pi u(c_{lh}^*)\alpha_{lh} + (1-\pi)u(c_{ll}^*)\alpha_{ll}] = D_1,\end{aligned}$$

where  $D_1$  is some constant consisting of optimal values  $(c^*, y^*)$  independent of  $\alpha$ 's.

Then, plugging this into (1.6.5) and rearranging the equation to get

$$X(\alpha) = \pi \left( \frac{1}{u'(c_h^*)} - \frac{1}{u'(c_l^*)} \right) [u'(c_h^*)\alpha_h + \pi u(c_{hh}^*)\alpha_{hh} + (1-\pi)u(c_{hl}^*)\alpha_{hl}] + D_2,$$

for some constant  $D_2$  consisting of optimal values  $(c^*, y^*)$  independent of  $\alpha$ 's. Notice that  $c_h^* > c_l^*$ . Then,  $X(\alpha)$  has the same sign with the expected present value of labor subsidies conditional on being the high type,  $A$ ,

$$A := u'(c_h^*)\alpha_h + \pi u(c_{hh}^*)\alpha_{hh} + (1 - \pi)u(c_{hl}^*)\alpha_{hl}.$$

This shows that  $\frac{\partial k_{1b}^*}{\partial A} < 0$  and  $\frac{\partial k_{1e}^*}{\partial A} > 0$  since  $\pi > \frac{\tau_l^* - \tau_c^*}{\tau_l^*}$ . This completes the proof.  $\square$

## 1.E Appendix for Section 1.7

### Proof of Lemma 4

*Proof.* First two inequalities result from  $c_l^* < c_m^* < c_h^*$ . Showing the third inequality is equivalent to showing

$$u'(c_0^*) < ru'(c_l^*). \tag{1.E.1}$$

Recall the inverse Euler equation.

$$\frac{r}{u'(c_0^*)} = \frac{\pi_h}{u'(c_h^*)} + \frac{\pi_m}{u'(c_m^*)} + \frac{\pi_l}{u'(c_l^*)}.$$

Then, inequality (1.E.1) comes from the Jensen's inequality:

$$\begin{aligned} u'(c_0^*) &< r\pi_h u'(c_h^*) + r\pi u'(c_m^*) + \pi_l u'(c_l^*) \\ &< r\pi_l u'(c_l^*) + \pi_m u'(c_l^*) + r\pi_l u'(c_l^*) = ru'(c_l^*). \end{aligned}$$

This completes the proof.  $\square$

## Proof of Proposition 8

*Proof.* The proof is basically the extension of the proof of Proposition 5. Given the tax system, we already know that the constrained optimal solution of consumption and labor vectors  $(c^*, y^*)$  coincide with the solution to the competitive equilibrium. Now,  $(k_{2h}^*, k_{2l}^*, B_1^*, E_1^*)$  are obtained by solving the following system of equations:

$$\begin{bmatrix} \pi_h & \pi_m & \pi_l & 0 & 0 \\ 0 & 0 & 1 & -r(1 - \tau_l^B) & -r(1 - \tau_c) \\ 0 & 1 & 0 & -r(1 - \tau_m^B) & -r(1 - \tau_m^E)(1 - \tau_c) \\ 1 & 0 & 0 & -r(1 - \tau_h^B) & -r(1 - \tau_h^E)(1 - \tau_c) \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_{2h} \\ k_{2m} \\ k_{2l} \\ B_1 \\ E_1 \end{bmatrix} = \begin{bmatrix} \hat{k}_2 \\ \alpha_l - c_l^* + wy_l^* \\ \alpha_m - c_m^* + wy_m^* \\ \alpha_h - c_h^* + wy_h^* \\ K_1^* \end{bmatrix} \quad (1.E.2)$$

Solving the above matrix equation (1.E.2), we have (1.7.3) and (1.7.4).  $\square$

## Sign of Denominators of (1.7.3) and (1.7.4) in Proposition 8

The following lemma is useful to characterize the sign of aggregate debt and equity holding. This lemma is also used later.

**Lemma 6.** *We have  $D_3 > 0$ .*

*Proof.*

$$\begin{aligned} D_3 = & \pi_h[(1 - \tau_h^B) - (1 - \tau_h^E)(1 - \tau_c)] + \pi_m[(1 - \tau_m^B) - (1 - \tau_m^E)(1 - \tau_c)] \\ & + \pi_l[(1 - \tau_l^B) - (1 - \tau_c)] \end{aligned}$$

$$\begin{aligned}
&= \pi_h(1 - \tau_h^B) - \frac{\pi_h u'(c_0^*)}{r u'(c_h^*)} + \pi_m(1 - \tau_m^B) - \frac{\pi_m u'(c_0^*)}{r u'(c_m^*)} \\
&\quad + \frac{u'(c_0^*)}{r u'(c_l^*)} - \pi_h(1 - \tau_h^B) \frac{u'(c_h^*)}{u'(c_l^*)} - \pi_m(1 - \tau_m^B) \frac{u'(c_m^*)}{u'(c_l^*)} - \frac{\pi_l u'(c_0^*)}{r u'(c_l^*)} \\
&= \pi_h(1 - \tau_h^B) \left(1 - \frac{u'(c_h^*)}{u'(c_l^*)}\right) + \pi_m(1 - \tau_m^B) \left(1 - \frac{u'(c_m^*)}{u'(c_l^*)}\right) + \frac{u'(c_0^*)}{r u'(c_l^*)} - 1, \\
&> \frac{\pi_h u'(c_0^*)}{r u'(c_h^*)} \left(1 - \frac{u'(c_h^*)}{u'(c_l^*)}\right) + \frac{\pi_m u'(c_0^*)}{r u'(c_m^*)} \left(1 - \frac{u'(c_m^*)}{u'(c_l^*)}\right) + \frac{u'(c_0^*)}{r u'(c_l^*)} - 1 \\
&= \frac{\pi_h u'(c_0^*)}{r u'(c_h^*)} + \frac{\pi_m u'(c_0^*)}{r u'(c_m^*)} + \frac{\pi_l u'(c_0^*)}{r u'(c_l^*)} - 1 = 0.
\end{aligned}$$

where the second equality is by using (1.7.1), the third and the last equality are by the inverse Euler equation, and the third inequality is by (1.7.1).  $\square$

## Proof of Proposition 9

*Proof.* We will find  $(\delta_h, \delta_m, \delta_l)$  explicitly. The first order conditions in the individual agent problem under the tax system  $(\tilde{\tau}_c, \tilde{\tau}_h^B, \tilde{\tau}_h^E, \tilde{\tau}_m^B, \tilde{\tau}_m^E, \tilde{\tau}_l^B, \tilde{\tau}_l^E)$  are given by

$$\begin{aligned}
u'(c_0) &= \pi_l r [1 - (\tau_c + \epsilon)] u'(c_l) + \pi_m r [1 - (\tau_c + \epsilon)] [1 - (\tau_m^E - \delta_m)] u'(c_m) \\
&\quad + \pi_h r [1 - (\tau_c + \epsilon)] [1 - (\tau_h^E - \delta_h)] u'(c_h) \tag{1.E.3}
\end{aligned}$$

$$u'(c_0) = \pi_l r [1 - \tilde{\tau}_l^B] u'(c_l) + \pi_m r [1 - \tilde{\tau}_m^B] u'(c_m) + \pi_h r [1 - \tilde{\tau}_h^B] u'(c_h) \tag{1.E.4}$$

In order to make the firm indifferent to issuing between debt and equity, we have the following condition

$$\pi_l(1 - \tilde{\tau}_c) + \pi_m(1 - \tilde{\tau}_c)(1 - \tilde{\tau}_m^E) + \pi_m(1 - \tilde{\tau}_c)(1 - \tilde{\tau}_h^E) = 1. \tag{1.E.5}$$

for any optimal tax system. In this case,

$$\pi_l[1 - (\tau_c + \epsilon)] + \pi_m[1 - (\tau_c + \epsilon)][1 - (\tau_m^E - \delta_m)] + \pi_h[1 - (\tau_c + \epsilon)][1 - (\tau_h^E - \delta_m)] = 1.$$

Let us define (1.E.3\*) and (1.E.4\*) by resulting equations after putting the optimal solution  $(c_l^*, c_m^*, c_h^*)$  into (1.E.3) and (1.E.4). Solving (1.E.3\*) and (1.E.5), we have

$$\delta_m = \frac{(1 - \tau_m^E)\epsilon}{1 - \tau_c - \epsilon} + \frac{\pi_l\epsilon}{\pi_m(1 - \tau_c - \epsilon)} \left( \frac{u'(c_l^*) - u'(c_h^*)}{u'(c_m^*) - u'(c_h^*)} \right) \quad (1.E.6)$$

$$\delta_h = \frac{(1 - \tau_h^E)\epsilon}{1 - \tau_c - \epsilon} + \frac{\pi_l\epsilon}{\pi_h(1 - \tau_c - \epsilon)} \left( \frac{u'(c_m^*) - u'(c_l^*)}{u'(c_m^*) - u'(c_h^*)} \right) \quad (1.E.7)$$

Finally, the other tax rates,  $\tilde{\tau}_h^B$ ,  $\tilde{\tau}_m^B$ ,  $\tilde{\tau}_l^B$ , and  $\tilde{\tau}_l^E$  can be arbitrarily determined by (1.E.4) and the following four inequalities

$$(1 - \tilde{\tau}_h^B) > (1 - \tau_h^E + \delta_h)(1 - \tau_c - \epsilon)$$

$$(1 - \tilde{\tau}_m^B) > (1 - \tau_m^E + \delta_m)(1 - \tau_c - \epsilon)$$

$$(1 - \tilde{\tau}_l^B) < (1 - \tau_c - \epsilon)$$

$$\tilde{\tau}_l^E > 0$$

Now, finally if we take the tax system  $(\tilde{\tau}_c, \tilde{\tau}_h^B, \tilde{\tau}_h^E, \tilde{\tau}_m^B, \tilde{\tau}_m^E, \tilde{\tau}_l^B, \tilde{\tau}_l^E)$ , then it is easy to see that  $(c_0^*, c_h^*, c_m^*, c_l^*)$  is the solution to the agent's problem since  $(c_h^*, c_m^*, c_l^*)$  is the solution to the Euler equation (1.E.3) and (1.E.4) and the concavity is still preserved under this transform with  $(\delta_h, \delta_m, \delta_l)$ .  $\square$

## Proof of Proposition 10

*Proof.* Suppose  $\tilde{\tau}_c$  increases by  $\epsilon$ . Let operator  $\Delta$  denote by the change in any variable corresponding to  $\epsilon$  amount increase in  $\tilde{\tau}_c$ . For example,  $\Delta\tilde{\tau}_h^E = -\delta_h$  and  $\Delta\tilde{\tau}_m^E = -\delta_m$  by Proposition 9. We will show that  $\Delta D_3 = 0$ . Recall that in order to make the firm indifferent to issuing between debt and equity, for any optimal tax system  $(\tilde{\tau}_c, \tilde{\tau}_h^B, \tilde{\tau}_h^E, \tilde{\tau}_m^B, \tilde{\tau}_m^E, \tilde{\tau}_l^B, \tilde{\tau}_l^E)$ , the following equation should be satisfied.

$$\pi_l(1 - \tilde{\tau}_c) + \pi_m(1 - \tilde{\tau}_c)(1 - \tilde{\tau}_m^E) + \pi_h(1 - \tilde{\tau}_c)(1 - \tilde{\tau}_h^E) = 1.$$

Using the above equation, we can rewrite  $D_3$  as

$$D_3 = \pi_h(1 - \tilde{\tau}_h^B) + \pi_m(1 - \tilde{\tau}_h^m) + \pi_l(1 - \tilde{\tau}_l^B) - 1.$$

Since  $\Delta\tilde{\tau}_i^B = 0$  for all  $i = h, m, l$  by the condition of the Proposition, we have  $\Delta D_3 = 0$ .

Note that  $\Delta X(\alpha_h, \alpha_m, \alpha_l) = 0$  since  $(\alpha_h, \alpha_m, \alpha_l)$  is fixed. Then, By using the similar analysis, the numerators of  $\tilde{B}_1^*$  and  $\tilde{E}_1^*$  are unchanged. In sum, there is no change in the numerators and the denominators in  $\tilde{B}_1^*$  and  $\tilde{E}_1^*$ , which completes the proof.  $\square$



## proof of Proposition 11

*Proof.* The expected capital taxes (as the income of the government) are given as

$$r \underbrace{(\pi_h \tilde{\tau}_h^B + \pi_m \tilde{\tau}_m^B + \pi_l \tilde{\tau}_l^B)}_{:= (a)} B_1^* + r \underbrace{(1 - \{\pi_l(1 - \tilde{\tau}_c) + \pi_m(1 - \tilde{\tau}_c)(1 - \tilde{\tau}_m^E) + \pi_m(1 - \tilde{\tau}_c)(1 - \tilde{\tau}_h^E)\})}_{:= (b)} E_1^*.$$

Notice that (b) is zero (due to the condition that firms are indifferent between issuing debt and equity), i.e.,

$$\pi_l(1 - \tilde{\tau}_c) + \pi_m(1 - \tilde{\tau}_c)(1 - \tilde{\tau}_m^E) + \pi_m(1 - \tilde{\tau}_c)(1 - \tilde{\tau}_h^E) = 1$$

Now we will show that part (a) is negative, which completes the proof as follows.

$$\pi_h \tilde{\tau}_h^B + \pi_m \tilde{\tau}_m^B + \pi_l \tilde{\tau}_l^B = 1 - \{\pi_h(1 - \tilde{\tau}_h^B) + \pi_m(1 - \tilde{\tau}_m^B) + \pi_l(1 - \tilde{\tau}_l^B)\} < 0. \quad (1.E.8)$$

since we have

$$\pi_h(1 - \tilde{\tau}_h^B) + \pi_m(1 - \tilde{\tau}_m^B) + \pi_l(1 - \tilde{\tau}_l^B) = D_3 + (b) = D_3 > 0,$$

by Lemma 6. □

# Optimal Contracts and Firm Dynamics with AK Technology

## 2.1 Introduction

Gibrat's law states that the size of a firm and its growth rate are independent (Gibrat (1931) and Masfield (1962)). Since then, there have been many studies on the relationship between firm size and growth. The following empirical facts are generally accepted:<sup>1</sup>

- (i) There is a slightly negative correlation between the size of a firm and its growth rate in various industries;<sup>2</sup>
- (ii) The growth rate of a firm is often independent of its size and age for firms above a certain size level.

These empirical regularities hold not only for cross-sectional data, but also for the time-series data. The majority of firm dynamics papers have considered (i) as a

---

<sup>1</sup>For fact (i), see Kumar (1985), Dunne and Hughes (1994), Bottazzi and Secchi (2003), Gabe and Krybill (2002), etc. For (ii) see Bottazzi et al. (2005), Droucopoulos (1983), Hardwick and Adams (2002), Audretsch et al. (2004), Mansfield (1962), Mowery (1983), Hart and Oulton (1996), Lotti et al. (2003), Geroski and Gugler (2004), Caves (1998), etc. See Coad (2007) for a nice survey.

<sup>2</sup>In some industries, Gibrat's law cannot be rejected.

*stylized fact* and have successively built theories to explain it, for example, Albuquerque and Hopenhayn (2004), Clementi and Hopenhayn (2006), and Cooley and Quadrini (2001). The key assumptions are financial market frictions such as limited liability, limited enforcement, information asymmetry about profit realizations, external financing costs, and others. Notice that all of these papers assume a strictly concave production function. Without those financing constraints, capital should be immediately allocated at the efficient steady state level. Under those constraints, optimal lending contracts dictate a gradual growth of capital up to the efficient level at a decreasing rate. Once it reaches the steady state, the growth rate becomes zero. Thus, although these models can explain growth of small firms well, they are rather silent for large firms. In particular, there is no growth after the steady state is reached, which is not consistent with *(ii)*. Notice that fact *(ii)* is based on more recent findings than *(i)*. We also observe many large firms showing sustained growth at even faster rates. This paper is first motivated by the discrepancy between the existing theory and the empirical facts.

The simplest way to introduce endogenous growth is to adapt an Ak-technology.<sup>3</sup> Thus, we hold a linear production function with respect capital and examine what other factors or constraints can derive results similar to the empirical fact. First, under the Ak-technology the steady state is discarded and the above financial frictions becomes meaningless or do not play a significant role in designing the contract implying the gradual growth of capital. Thus, we abstract from such financing contracts and their impact on firm growth. Then, instead of shutting down the channel of lending or borrowing from the outside entities such as banking or insurance sec-

---

<sup>3</sup>There are many papers trying to understand firm growth in terms of heterogeneity of firms with a general equilibrium, e.g - Bartelsman, Haltiwanger, and Scarpetta (2008) and Luttmer (2008, 20010). Our paper focuses on the partial equilibrium aspects.

tors, we stick to the case where a firm grows through the internal stochastic capital accumulation mechanism. In turn, we also abstract from the debt contracts and capital structure decisions of a firm. Then, our first objective is to find the simplest economic environment (with Ak technology) consistent with the empirical regularity introduced above.

This paper studies a continuous time principal-agent model in which capital grows under the optimal contract. We adopt the classical moral hazard set-up with an Ak technology, which is the key difference from existing firm dynamics models.<sup>4</sup> A contract is signed at time 0 and is continuous-history dependent. The principal commits to a contract that is incentive compatible to the agent. At each time  $t$ , profits are determined by a simple constant returns to scale (linear) production function with two arguments: capital provided by the principal and effort from the agent which is unobservable to the principal. The contract specifies the instantaneous payment to the agent and the instantaneous dividend paid to the principal. Then the remaining output is invested for future production, which in turn specifies the size of capital and determines the growth rate of the firm.

The explicit solution can be obtained when both the principal and the agent have the constant absolute risk aversion utility functions. In the first best case, the expected growth rate is always strictly decreasing in firm size, more precisely, inversely related to the size of capital. However, thanks to the Ak technology the growth rate is asymptotically constant in capital. Gibrat's law cannot be rejected

---

<sup>4</sup>Clementi and Hopenhayn (2006) and DeMarzo et al. (2009) also study the moral hazard problem, but they do not consider the production contribution from the agent. What it matter in Clementi and Hopenhayn is to make entrepreneurs truthfully reveal their profits. In DeMarzo et al. (2009), the manager only chooses a binary effort choice,  $\{work, shirk\}$  under which working is basically equivalent to the truthful revelation of profits. In our model, the manager really contributes to production by choosing different levels of efforts, hence we put the wording "classical" moral hazard.

for those firms in the tail. Therefore, our model might be consistent with both empirical facts (i) and (ii) even without considering the moral hazard problem. In other word, it is sufficient to have a very simple model of the first best principal-agent problem with linear production with respect to capital from the principal and labor from the agent for explaining the firm dynamics with respect to firm size and growth.

One potential weakness of the first best solution is that the growth rate for low capital state seems too high. This requests further analysis for the second best case in which the optimal contract would generate a proper growth behavior of the firm in the *quantitative* sense. In the presence of moral hazard, in particular, the optimal investment decision deviates from the first best contract, which is the deriving force to have the reasonable result. We prove that in the second best case there are more incentives to under-invest when the level of capital is low and that there are more incentives to over-invest when the level of capital is high. As a consequence, the growth rates for small firms become relatively lower and the growth rate for large firms become relatively higher than those without information asymmetry, while we still have the negative relationship between firm size and growth. Thus, the moral hazard case is more consistent with both empirical facts (i) and (ii).<sup>5</sup>

There are two reasons why over/under-investment appears to depend on the size of capital. First, in the production side, the given linear (therefore convex) technology becomes nonconvex in the optimal contract. Furthermore, the degree of nonconvexity differs by the level of capital,  $k$ , due to the incentive compatibility condition for the manager's effort. Due to the nonconvexity, *equilibrium* marginal

---

<sup>5</sup>One remark about this model is that we do not try to fit the growth rate distributions. In particular, we do not consider entry and exit of firms.

production to capital, when  $k$  is small, is smaller than when  $k$  is big.<sup>6</sup> Therefore, it is advantageous to have more capital accumulation (or invest more) when  $k$  is relatively big. On the other hand, since the nonconvexity is more severe when  $k$  is small, fast growth through more investment is not optimal. In other words, the marginal cost of investment, which is the principal's marginal utility of dividend, is relatively higher for lower level of capital. This is the reason to under-invest for low capital states.

The second contribution of the paper is that we fully analyze the incentive scheme when both the principal and the agent are risk averse. In the purely theoretical sense, the model is a principal-agent problem with capital accumulation. The immediate question is how capital growth affects the incentive provision for the agent. The consideration about the firm's capital accumulation yields a quite different optimal contract from classical principal-agent models that only consider profit sharing. Notice that the volatility of the shock is assumed to be proportional to the square root of the size of capital,  $\sqrt{k}$ . By this capital size effect, we can show that given the level of capital more effort derives less risky payments, which is opposite to the result of classical principal-agent problems without growth. In the classical framework, if the agent puts more effort, then the compensation gets more volatile, which gives the agent an incentive to work harder. In our model, however, as the firm size increases the sensitivity of the payment gets larger according to the square root of the size regardless of the level of effort, thus it is not quite necessary for the agent to put more effort and it is even better for the agent to work less. Then how does the principal get the agent to exert effort? We find that there must be an adjustment term or a stick in the incentive provision that makes the payment

---

<sup>6</sup>The definition of equilibrium marginal production is in Section 2.4.2

drop instantaneously if the agent works less. This supports proper production at each time and guarantees sufficient growth.

The third contribution of this paper is that the model can partially explain a firm poverty trap, i.e., why some small firms with enough profits do not grow. We assume that the volatility of the growth rate is inversely related to the size of capital. From this assumption, we derive an interesting phenomenon: firms under certain conditions show little growth, which is optimal. The sufficient conditions are very reasonable: *(i)* the project is sufficiently risky, *(ii)* the subjective discount factor is high enough, or *(iii)* both the principal and the agent are fairly risk averse. Under one of these conditions, *ceteris paribus*, there is a positive probability that the capital process hits the zero boundary in finite time. Although the process reflects to the positive region as soon as it hits the boundary, the process may go back to the boundary again in finite time and reflect again, and so on. The under-investment in small firms, in turn, reinforces the slow growth. In consequence, it takes longer time for those firms to escape from the low capital states. We believe that this might be one potential reasons for firm level poverty traps.

There is also a large literature on over- or under-investment issues.<sup>7</sup> Dow, Gorton and Krishnamurthy (2005) (henceforth DGK) and Albuquerque and Wang (2008) predict overinvestment. DGK's model is based on Jensen's (1996, 1993) free cash flow theory. The over-investment in DGK results from the assumption that managers are empire-builders and so the shareholders should hire costly monitoring

---

<sup>7</sup>According to Stein (2003), there are broadly two categories of literature with respect to the investment issue. One is the models of agency conflicts and the other is the models of costly external finance. The former generally predicts over-investment and the latter generally predicts under-investment. Here instead of surveying all those papers, we only introduce the models having 'dynamic' features in order to narrow down the scope.

auditors to control the manager’s decision making on payout and investment. On the other hand, Albuguerque and Wang (2008) consider the agency conflict between the controlling shareholder and outside investors. In Albuguerque and Wang, the controlling shareholder does all the decision making for payout and investment although he has a relatively small ownership of the firm compared to the outside shareholders. Therefore, it is intuitively rather straightforward that firms overinvest in both DGK and Albuguerque and Wang. But, in our model how much is invested is specified by the optimal contract, not by an exogenous assumption about the agency power. We neither assume the existence of extra costs in auditing managers as in DGK nor assume exogenous costs when managers steal the outside shareholders’ profits as in Albuguerque and Wang. The inefficient investment in our model is generated by the moral hazard problem, not by the imperfect protection of the shareholders.

On the other hands, the usual dynamic contracting theory for small firm-level corporate finance such as Albuquerque and Hopenhayn (2004) and Clementi and Hopenhayn (2006), DeMarzo et al. (2009), etc, often predicts underinvestment in financing a small firm’s project.<sup>8</sup> Albuguerque and Wang (2008) point out that overinvestment is likely to be the dominant issue for larger firms around the world whereas the underinvestment implied by these contracting models is potentially more important for smaller firms. To our knowledge, our model is the first that has both under- or over-investment features *depending on the firm size*. This paper, in turn, possibly sheds light on the integration of two separate theories in corporate finance on investment decisions and the capital structure of small and large firms.

---

<sup>8</sup>The essential ingredient for overinvestment in our paper is that the effort choice set of the manager is larger than those of most other papers. See footnote 4 on this.



For technical side of the paper, we use the martingale method developed by Sannikov (2008).<sup>9</sup> Recently, several continuous time principal-agent models have been studied such as DeMarzo and Sannikov (2006), He (2008), Sannikov (2008), and Williams (2009). However, none of those papers' focuses are similar to ours. The example in Williams (2009) looks similar to ours, but there is a substantial difference as he did not consider capital accumulation. The output process can take arbitrary negative values and he only cares about profit-sharing under hidden action and hidden saving.

Finally, one also has concerns about a special assumption on the production function in our model although this assumption gives a simple and tractable solution. Therefore, we also examine other possible production technologies in Section 8. Notice that the focus in this paper is to keep the asymptotic constant returns to scale (CRS) property in terms of the size of a firm in order to have a positive constant expected growth rate in the long run. It turns out that those seemingly reasonable production functions, in equilibrium, violate the CRS property. Therefore, we believe that our assumption on the technology is not only simple enough to have main results, but also reasonable to be consistent with the empirical results.

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 shows the first and second best solutions. Section 4 explains the main result of this paper. Section 5 explains the optimal payment schedule. Section 6 describes the dynamics of the firm in low capital states and its implication to poverty traps. Section 7 provides brief explanation about risk sharing and business

---

<sup>9</sup>In the earlier version, we used the method from Williams (2009) and some literature on backward stochastic differential equations (Yong and Zhou (1999), Pardoux and Peng (1990), and El Karoui, Peng, and Quenez (1997, 2001)).

cycle implication about the model. Section 8 examines whether there are other production functions that can provide the main result of the paper. We show that common cases often violate the CRS property. Section 9 gives concluding remarks including some points for future research agenda. All proofs are in the appendix.

## 2.2 The Model and The Problem

We assume constant absolute risk aversion (CARA) preferences of a principal and an agent (or a manager). Let us denote by their utility functions  $u_p(d)$  and  $u_m(c, e)$ , respectively.

$$u_p(d) = -\exp(-\lambda d), \quad \text{and} \quad u_m(c, e) = -\exp\left(-\lambda\left(c - \frac{e^2}{2a}\right)\right),$$

where  $\lambda > 0$  is a risk aversion parameter and  $a$  is a positive number.  $c$  is the payment to the agent and  $e$  is the level of the agent's effort.  $d$  is the dividend to the principal. In particular, the agent's utility function is non-separable. This assumption discards the income effect and makes algebra simple. They are expected utility maximizers. The agent's utility is given by  $E_0 \left[ \int_0^\infty e^{-\beta t} u_m(c_t, e_t) dt \right]$  and the principal's utility is  $E_0 \left[ \int_0^\infty e^{-\beta t} u_p(d_t) dt \right]$ .

The principal is endowed with an unique production technology. But, in order to produce output she should hire an agent. Only if she can hire a worker, she can operate her firm. The production technology<sup>10</sup> without considering the uncertainty is

$$f(k, e) = hk + e,$$

---

<sup>10</sup>In section x, we introduce the case for other production technologies.

where  $k$  is capital provided by the principal and  $e \in \mathcal{E}$  is the effort level exerted by the agent where  $\mathcal{E}$  will be specified later. We will work with a Brownian motion  $W_t$  in a standard probability space  $(\Omega, \mathcal{F}, P)$  with a continuous time framework. The capital process  $k(t)$  follows the stochastic differential equation:

$$\begin{aligned} dk(t) &= (f(k(t), e(t)) - c(t) - d(t) - \delta k(t))dt + \sigma \sqrt{k(t)}dW_t, \\ &= g(k(t), c(t), d(t), e(t))dt + \sigma \sqrt{k(t)}dW_t, \end{aligned} \tag{2.2.1}$$

where  $\delta$  is the rate of capital depreciation and

$$g(k, c, d, e) := f(k, e) - c - d - \delta k.$$

Notice  $f(k(t), e(t))dt + \sigma \sqrt{k(t)}dW_t$  is instantaneous production,  $(c(t) + d(t))dt$  is the instantaneous payment, and therefore  $f(k(t), e(t))dt + \sigma \sqrt{k(t)}dW_t - (c(t) + d(t))dt$  is instantaneous investment. Here,  $\sigma \sqrt{k}dW_t$  can be regarded as an instantaneous production shock which depends on the size of capital being installed.<sup>11</sup> Specifically, we assume neither the principal and the agent has saving and borrowing technology. In particular, as mentioned in the introduction, we abstract from the outside financing opportunities. Otherwise, in the Ak-technology framework, this problem has no solution.

The discrete time analogue of (2.2.1) is

$$\begin{aligned} k_{t+1} &= (1 - \delta)k_t + i_t, \\ i_t &= f(k_t, e_t) - c_t - d_t + \sigma \sqrt{k_t} \epsilon_{t+1}, \end{aligned}$$

---

<sup>11</sup>The size of shocks can be regarded to depend on investment at each time. This is related to the idea of Greenwood, Hercowitz, and Krusell (1997).

where  $\epsilon_{t+1}$  are i.i.d. normal. The reason why we assume the square root process is two-fold: First, we do not allow negative values of  $k$ . The zero-capital level, here, is the minimum bound for capital. If we set  $k = k_0$  as the minimum in which the firm can start the project, then we can define the volatility structure as  $\sigma\sqrt{k - k_0}dW_t$ . Without loss of generality we have  $k_0 = 0$ . Second, more importantly, the volatility of the growth rate  $\frac{dk}{k}$  is assumed to be negatively related to the level of capital, that is,  $\frac{\sigma}{\sqrt{k}}$ . It is easy to see if we rewrite (2.2.1) as

$$\frac{dk(t)}{k(t)} = \left( \frac{g(k(t), c(t), d(t), e(t))}{k(t)} \right) dt + \frac{\sigma}{\sqrt{k(t)}} dW_t.$$

This is based on the empirical finding of Hymer and Pashingian (1962), Amaral et al. (1997), Bottazzi et al. (2001), etc. It is also consistent with some empirical literature on the relation between GDP growth and volatility (Ramey and Ramey (1995)).<sup>12</sup>

The triplet  $(c(t), d(t), e(t))$  is a contract between the principal and the manager.  $c(t)$  is the rate of payment schedule for the agent at each time  $t$  and  $d(t)$  is the rate of dividend delivered to the principal at each  $t$ .  $e(t) \in \mathcal{E}$  is the effort choice by the manager which is unobservable to the principal, where  $\mathcal{E}$  be the set of progressively measurable processes with respect to  $\mathcal{F}_t$  whose support is  $[0, M]$  with some large number  $M > a$ . Let  $\mathcal{S}$  be the set of feasible contracts  $(c(t), d(t), e(t))$  if  $e(t) \in \mathcal{E}$  and  $c$  and  $d$  are contingent on the all the previous history  $\mathcal{G}_t$ , the completion of the  $\sigma$ -algebra generated by all histories of capital,  $\{k_s\}_{s=0}^t$ .

The principal's problem is to offer an feasible contract  $(c(t), d(t), e(t)) \in \mathcal{S}$  to

---

<sup>12</sup>Section 3.2 of Jones and Manuelli (2005) provides the excellent survey for the empirical results between the mean growth rate and volatility.

maximize her utility, satisfying the individual rationality (IR) and the incentive compatibility (IC) for the manager. Mathematically,

$$\max_{(c,d,e) \in \mathcal{S}} E_0 \left[ \int_0^\infty e^{-\beta t} u_p(d(t)) dt \right]$$

subject to (2.2.1), (IR), and (IC):

$$\begin{aligned} \text{(IR)} \quad & E_0 \left[ \int_0^\infty e^{-\rho t} u_m(c_t, e_t) dt \right] \geq q_0 \quad \text{and} \\ \text{(IC)} \quad & e(t) \in \arg \max_{e(t) \in \mathcal{E}} E_0 \left[ \int_0^\infty e^{-\rho t} u_m(c_t, e_t) dt \right], \end{aligned}$$

where  $q_0$  is the reservation utility of the manager. Note that  $\{e(t)\}_{t=0}^\infty$  is incentive compatible if it maximizes the agent utility given  $\{c(t)\}_{t=0}^\infty$  and  $\{d(t)\}_{t=0}^\infty$ .

### 2.3 The Optimal Contract

In this section, we pin down the first best solution and heuristically derive the second best solution. The formal proofs are in Appendix A and B. We follow the method developed in Sannikov (2008). Taking for granted that the continuation value  $q_t$  and capital  $k_t$  serve as two state variables, we rewrite the problem as follows. The principal's problem is to offer an incentive compatible admissible  $(c(t), d(t), e(t)) \in \mathcal{S}$  to the manager that maximizes

$$E_0 \left[ \int_0^\infty e^{-\beta t} u_p(d(t)) dt \right]$$

subject to two underlying stochastic differential equations:

$$\begin{aligned} dk_t &= [(h - \delta)k_t + e_t - c_t - d_t]dt + \sigma \sqrt{k_t}dW_t, \\ dq_t &= [\beta q_t - u_m(c_t, e_t)]dt + \gamma(c_t, e_t)\sigma \sqrt{k(t)}\{dk_t - g(k_t, c_t, d_t, e_t)\} \end{aligned}$$

with the initial condition  $(k(0), q(0)) = (k_0, q_0)$  and  $\gamma(c, e)$  is given by

$$\gamma(c, e) := \min\{y \in [0, \infty) \mid e \in \arg \max_{e' \in \mathcal{E}} u_m(c, e') + ye'\}. \quad (2.3.1)$$

In this case, we have

$$\gamma(c, e) = -\frac{\partial u_m(c, e)/\partial e}{\partial g(k, c, d, e)/\partial e} = -\frac{\partial u_m}{\partial e}(c, e) = \frac{\lambda e}{a} \exp(-\lambda(c - e^2/2a)). \quad (2.3.2)$$

Notice that  $\gamma(t)$  in the volatility part of  $q(t)$  process plays an important role.  $\gamma(t)$  is often called sensitivity of the payment in the principal-agent literature.  $q(t)$  is derived by the Martingale Representation Theorem and  $\gamma(t)$  is derived through the incentive compatibility condition in the appendix. Using the underlying processes we can derive the Bellman equation. The value function  $J(k, q)$  for the principal satisfies the following Bellman equation

$$\begin{aligned} \beta J(k, q) &= \max_{c, d, e} u_p(d) + J_k[(h - \delta)k + e - c - d] + J_q(\beta q - u_m(c, e)) \\ &\quad + \frac{1}{2} (J_{kk} + 2J_{kq}\gamma(c, e) + J_{qq}\gamma(c, e)^2) \sigma^2 k. \end{aligned} \quad (2.3.3)$$

Before solving the problem we introduce the following assumption for the parameter values in order for the problem to be well-posed.

**Assumption 2.**

$$A_1 := \frac{h - \delta}{1 + \frac{\lambda\sigma^2}{4}} > \beta$$

Assumption 2 holds throughout the paper. This guarantees that the first best capital process does not fall below the zero-boundary. It is the condition that the given technology should satisfy. Otherwise the contract cannot be made. It, however, neither discard the case where the boundary is attracting nor is attainable. In other words, the under a certain condition the equilibrium capital process starting from the positive level can reach to the zero level even in finite expected time (See proposition 17 and 18).

Here note that the domain of the solution (or the capital process) is defined on  $[0, \infty)$ . The zero boundary should be interpreted as the minimum level of capital that can initiate the firm. Then, the value function  $V(0, q)$  is the endogenous genuine value of the project that the principal owns.

### 2.3.1 The First Best Contract

Suppose that the principal can perfectly observe the agent's action. Then, we do not have to consider (IC). The usual way to solve the first best solution (or Pareto optimal solution) is to solve

$$\max_{c,d,e} E_0 \left[ \int_0^\infty e^{-\beta t} u_p(d(t)) dt \right]$$

subject to (2.2.1) and (IR)

$$E_0 \left[ \int_0^\infty e^{-\beta t} u_m(c(t), e(t)) dt \right] \geq q_0.$$

On the other hand, since we will compare the first with the second best solutions later, for a simple exposition, it is convenient to reuse the Bellman equation (2.3.3) as follows.

$$\begin{aligned} \beta J(k, q) = \max_{c, d, e, \gamma} & u_p(d) + J_k[(h - \delta)k + e - c - d] + J_q(\beta q - u_m(c, e)) \\ & + \frac{1}{2} (J_{kk} + 2J_{kq}\gamma + J_{qq}\gamma^2) \sigma^2 k. \end{aligned} \quad (2.3.4)$$

Notice that  $\gamma$  is now an arbitrary choice variable of the principal instead of being derived through the incentive compatibility. Then, it can be easily verified that we obtain the optimal solution from the above Bellman equation.

**Proposition 12.** *The optimal solution  $(c^f, d^f, e^f)$  to (2.3.4) is given by*

$$\begin{aligned} c^f(k, q) &= \frac{a}{2} - \frac{1}{\lambda} \ln(-qA_1) \\ d^f(k, q) &= A_1 k + B_1 - \frac{1}{\lambda} \ln\left(\frac{A_1}{-q}\right) \\ e^f(k) &= a, \end{aligned}$$

where  $A_1 = \frac{h-\delta}{1+\frac{\lambda\sigma^2}{4}}$  and  $B_1 = \frac{1}{\lambda} \left( \frac{2(\beta-A_1)}{A_1} + 2 \log A_1 + \frac{\lambda a}{2} \right)$ .

*Proof.* See the appendix. □

**Theorem 2** (The First Best Solution). *The solution to the Bellman equation (2.3.4) gives the first-best value of the principal. In other words, if the initial capital is given*



by  $k_0$  and the reservation utility of the agent is given by  $q_0$ , then the value of the principal at time 0 is  $J(k_0, q_0)$ , where  $J$  is the solution to (2.3.4).

*Proof.* See the appendix. □

**Corollary 5.** *The first best optimal capital stock  $k$  and the agent's discounted utility  $q$  follows*

$$dk = \left[ (h - \delta - A_1)k + \frac{2(A_1 - \beta)}{\lambda A_1} \right] dt + \sigma \sqrt{k} dW_t \quad (2.3.5)$$

$$\frac{dq}{q} = (\beta - A_1)dt - \frac{\lambda \sigma A_1}{2} \sqrt{k} dW_t. \quad (2.3.6)$$

*Proof.* See the appendix. □

Since we have the explicit solution to (2.3.4), we can derive the first best optimal contract  $\{c_t^f, d_t^f, e_t^f\}_{t=0}^\infty$  using the functional forms  $(c^f(\cdot, \cdot), d^f(\cdot, \cdot), e^f(\cdot))$  described in Proposition 12 as follows:

$$(c_t^f, d_t^f, e_t^f) = (c^f(k_t, q_t), d^f(k_t, q_t), e^f(k_t, q_t)), \quad (2.3.7)$$

where  $(k_t, q_t)$  is given by (2.3.5) and (2.3.6) in Corollary 5.

Notice that the discrete time analogue of (2.3.5) is

$$k_{t+1} = (h - \delta - A_1 + 1)k_t + \frac{2(A_1 - \beta)}{\lambda A_1} + \sigma \sqrt{k_t} \epsilon_{t+1}, \quad \epsilon_{t+1} \sim \text{i.i.d.},$$

which is so called a CIR (Cox, Ingersoll, and Ross) process. It slightly differs from a AR(1) process in the sense that there is a square root term  $\sqrt{k_t}$  in the volatility

part, which makes the process nonnegative.

### 2.3.2 The Second Best Contract

The next proposition is analogous to Proposition 12. While we get the explicit solution for (2.3.4), we only have the semi-explicit solution form of (2.3.3). Thus, we characterize the second best solution by using a system of ordinary differential equations.

**Proposition 13.** *Let  $(e(k), \theta(k))$  satisfy*<sup>13</sup>

$$\theta(k) > 0, \quad \theta'(k) > 0, \quad \lambda\theta'(k)^2 > 2\theta''(k), \quad \forall k \in [0, \infty)$$

*and be the  $C^1$ -solution to the following system of the first order ordinary differential equations*

$$e'(k) = F(e(k), \theta(k), k)$$

$$\theta'(k) = H(e(k), k)$$

*with initial conditions*

$$e(0) = a \quad \text{and} \quad \theta(0) = \frac{1}{\lambda} \left[ \frac{2(\beta - \theta'(0))}{\theta'(0)} + 2 \ln \theta'(0) + \frac{\lambda a}{2} \right],$$

*where the functional forms of  $F$  and  $G$  and the derivation of the initial conditions*

---

<sup>13</sup>These are the sufficient conditions for the concavity of  $J$

are given in the Appendix. Let  $\psi$  be the function of  $e$  defined by

$$\psi(e) = \frac{a + \lambda e^2 - \lambda a e}{a} = 1 + \frac{\lambda e}{a}(e - a).$$

Then, the solution  $(c^s, d^s, e^s)$  to the Bellman equation (2.3.3) is given by

$$c^s(k, q) = \frac{e(k)^2}{2a} - \frac{1}{\lambda} \ln [(-q)\theta'(k)\psi(e(k))]$$

$$d^s(k, q) = \theta(k) - \frac{1}{\lambda} \ln \left( \frac{\theta'(k)}{-q} \right)$$

$$e^s(k) = e(k)$$

*Proof.* See the appendix. □

Now we are ready to describe the main theorem (verification theorem). In fact, the proof of the theorem uses the properties in several lemmas that will be appeared later which results from characterizing the system of ordinary differential equations in Proposition 13 (e.g., Lemma 7 and Lemma 9). Therefore, one can skip the proof of Theorem 2 in the first reading. We locate the theorem here for simple exposition. It is also convenient to compare Section and Section .

**Theorem 3** (The Second Best Solution). *The solution to the Bellman equation (2.3.3) gives the second-best value of the principal. In other words, if the initial capital is given by  $k_0$  and the reservation utility of the agent is given by  $q_0$ , then the value of the principal at time 0 is  $J(k_0, q_0)$ , where  $J$  is the solution to (2.3.3).*

*Proof.* See the appendix. □

**Corollary 6.** *The second best optimal capital  $k$  and the agent's utility process evolve as*

$$dk = \left[ (h - \delta)k - \theta(k) - \frac{e(k)^2}{2a} + e(k) + \frac{1}{\lambda} \ln (\theta'(k)^2 \psi(e(k))) \right] dt + \sigma \sqrt{k} dW_t \quad (2.3.8)$$

$$\frac{dq}{q} = [\beta - \theta'(k)\psi(e(k))] dt - \frac{\lambda\sigma}{a} \theta'(k)e(k)\psi(e(k))\sqrt{k} dW_t \quad (2.3.9)$$

*Proof.* See the appendix. □

Since we have characterized the solution to (2.3.3), we can derive the second best optimal contract  $\{c_t^s, d_t^s, e_t^s\}_{t=0}^\infty$  using the functional forms  $(c^s(\cdot, \cdot), d^s(\cdot, \cdot), e^s(\cdot))$  described in Proposition 13 as follows:

$$(c_t^s, d_t^s, e_t^s) = (c^s(k_t, q_t), d^s(k_t, q_t), e^s(k_t, q_t)), \quad (2.3.10)$$

where  $(k_t, q_t)$  is given by (2.3.8) and (2.3.9) in Corollary 6.

## 2.4 The Optimal Growth Rates

This section describes the main result of the paper. Recall the empirical facts that the growth rate of the firm has slightly negative dependence on its size, but, Gibrat's law holds for firms above a certain size level. In order to explain how the growth rate changes as a firm grows, we first focus on how the investment decision changes over the firm size. In particular, we need to compare the first with the second best investment. The first step is to pin down the optimal effort levels.

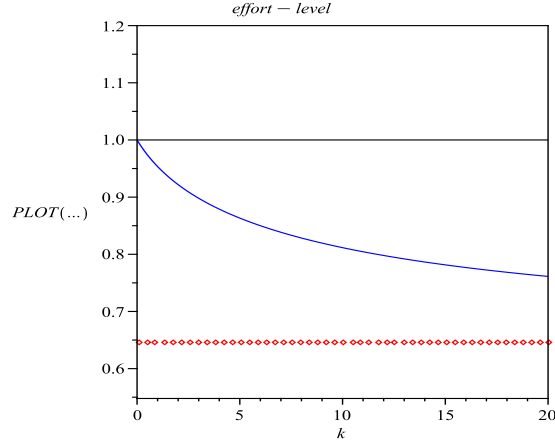


Figure 2.1: The optimal effort level: black-top (first best), blue-middle (second best), dotted-bottom (second best limit). The parameter values are given by  $h = 6$ ,  $\delta = 0.1$ ,  $\sigma = 0.3$ ,  $\lambda = 2$ ,  $a = 1$

### 2.4.1 The Optimal Effort

Notice that the effort level is independent of the agent's continuation value. This is because the agent's utility function is assumed to be nonseparable so that there is no income effect. Thus, the agent's effort is nicely pinned down as only a function of  $k$ . We show that starting from  $e(0) = a$  at  $k = 0$ , the second best effort level is converging to the constant level as the level of capital goes to infinity. This helps us a lot to analyze the properties of the solution.

**Lemma 7.** *The second best optimal effort level  $e = e(k)$  satisfies*

$$(1) \ e(0) = a \text{ and } 0 < e(k) < a, \quad \forall k > 0.$$

(2)  $\lim_{k \rightarrow \infty} e(k) = e^*$ , where  $e^*$  is the largest solution in  $(\frac{a}{2}, a)$  to the following

*cubic polynomial equation:*<sup>14</sup>

$$e\psi(e) = \frac{a}{2} \iff \lambda e^3 - \lambda a e^2 + a e - \frac{a^2}{2} = 0.$$

Moreover, we have  $e^* < e(k) < a$  for all  $k > 0$ .

*Proof.* See the appendix. □

Lemma 7 immediately results in the following corollary which gives the bound for  $\psi(e(k))$  in Theorem 2. This property is also important to characterize the properties of the solution in the following subsections.

**Corollary 7.**  $\psi(e)$  is decreasing in  $e$  and bounded with respect to  $k$  such that

$$1 + \frac{\lambda}{a}(e^* - a) < \psi(e(k)) < 1, \quad \forall k \in [0, \infty).$$

Figure 2.1 shows an example of effort level with respect to capital. The effort level is decreasing and asymptotically converging to the dotted line in the bottom. The top constant line is the first best effort. Numerical examples show that the second best effort  $e = e(k)$  is monotonically decreasing in  $k$  as seen in figure 2.1. We suspect that this is generally true. But, it is hardly proven. Whether this is true or not, all other asymptotic properties on consumptions and investment that are introduced next subsections still hold by using lemma 7.

---

<sup>14</sup>The existence of the solution is easily seen by the intermediate value theorem.

## 2.4.2 The Investment Comparison

In this subsection we compare the results of the first and the second best cases. We can define the first best investment  $I_f$  and the second best investment  $I_s$  at time  $t$  in a discrete time approximation as follows using the solutions in theorem 1 and 2:<sup>15</sup>

$$I_f(t) = (h - A_1)k_t + \frac{2(A_1 - \beta)}{\lambda A_1} + \sigma\sqrt{k_t}\epsilon_{t+1}$$

$$I_s(t) = hk_t - \theta(k_t) - \frac{e(k)^2}{2a} + e(k) + \frac{1}{\lambda} \log(\theta'(k)^2\psi(e(k))) + \sigma\sqrt{k_t}\epsilon'_{t+1},$$

where  $\epsilon_{t+1}$ 's and  $\epsilon'_{t+1}$ 's are i.i.d. normal  $N(0, 1)$ . Rather than comparing the time series of  $I_f(t)$  and  $I_s(t)$ , it is more convenient to compare their conditional expectations with respect to  $k_t$  that are defined by  $I^i(k) = E_t[I_i(t)|k_t = k]$  for  $i = f, s$ . Hence,

$$I^f(k) = (h - A_1)k + \frac{2(A_1 - \beta)}{\lambda A_1}$$

$$I^s(k) = hk - \theta(k) - \frac{e(k)^2}{2a} + e(k) + \frac{1}{\lambda} \log(\theta'(k)^2\psi(e(k)))$$

The the difference between  $I^f$  and  $I^s$  is

$$I^f(k) - I^s(k) = \theta(k) - A_1k + \frac{2(A_1 - \beta)}{\lambda A_1} + \frac{e(k)^2}{2a} - e(k) - \frac{1}{\lambda} \log(\theta'(k)\psi(e(k)))$$

---

<sup>15</sup>Notice that the continuous time version of the instantaneous investment at time  $t$  is heuristically written as

$$I_i(t) = f(k(t), e^i(t)) + \sigma\sqrt{k(t)}dW_t - c^i(t) - d^i(t), \quad i = f, s.$$

$$= \theta(k) - (A_1 k + B_1) + \frac{(e(k) - a)^2}{2a} - \frac{1}{\lambda} \log \left( \frac{\theta'(k)^2}{A_1^2} \psi(e(k)) \right). \quad (2.4.1)$$

**Proposition 14** (Investment When  $k$  is Small). *When the level of capital is sufficiently low, there is under-investment. More precisely,  $I^f(0) > I^s(0)$ .*

*Proof.* See the appendix. □

**Proposition 15** (Investment When  $k$  is Big). *There is over-investment as capital grows sufficiently large. More precisely,*

$$\lim_{k \rightarrow \infty} I^f(k) - I^s(k) = -\infty. \quad (2.4.2)$$

*Moreover, the investment-capital ratio and the investment-output ratio for the second best case converge those of the first best case. In other words,*

$$\lim_{k \rightarrow \infty} \frac{I^f(k) - I^s(k)}{k} = 0. \quad (2.4.3)$$

*Proof.* See the appendix. □

Proposition 14 and 15 tell that there are more incentives to under-invest when the level of capital is low and that there are more incentives to over-invest when the level of capital is high. However, notice that their investment-output ratios as in (2.4.3) are asymptotically same. This distinction might be empirically important.

Note the production function is  $f(k, e) = hk + e$  that is linear in  $k$  and  $e$ .  $h$  is (explicit) marginal product of capital. But, the effort level is a function of  $k$ ,  $e = e(k)$  in the optimal contract, so that we define the *equilibrium marginal*



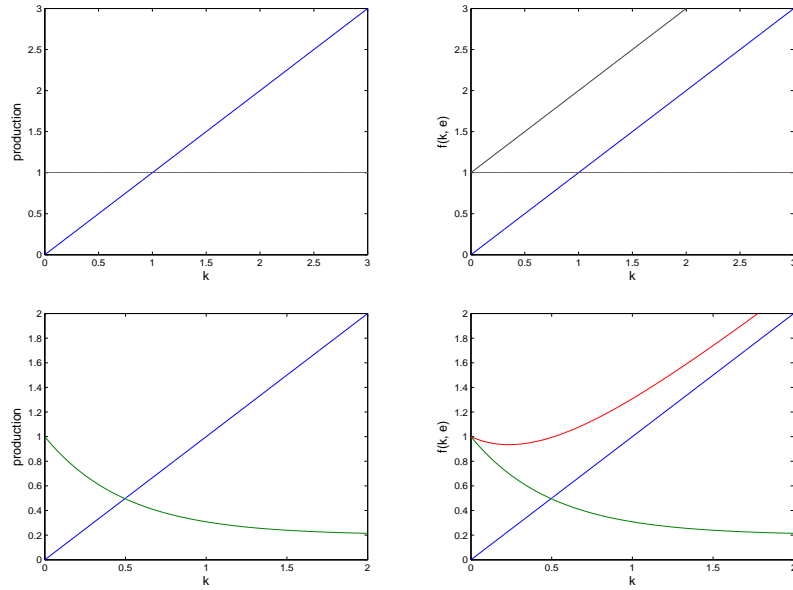


Figure 2.2: The production technology frontiers: The left panels are the separate plots of  $hk$  and  $e(k)$  and the right panels are aggregation of them, i.e.,  $f(k, e(k))$ . The top panels are the first best cases and the bottom panels are the second best cases.

*production to capital*(EMPK) by

$$\frac{d}{dk} f(k, e(k)) = h + e'(k).$$

Notice that EMPK is generally increasing in  $k$  since  $e(k)$  is generally decreasing as shown in Figure 2.1. Thus, the given convex technology becomes nonconvex in the optimal contract. Comparing the top and bottom panels in Figure 2.2, we can guess how the first and the second best are different in terms of technology.

EMPK, when  $k$  is small, is smaller than when  $k$  is high. Therefore, it is advantageous to have more capital accumulation especially when  $k$  is high. Notice this only tells that the investment-capital ratio or investment-production ratio, when  $k$  is big, is higher than when  $k$  is small, however, it does not give the sufficient

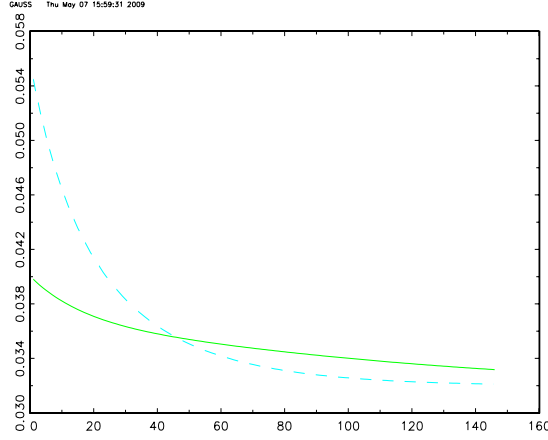


Figure 2.3: Expected growth rate of the first best (dotted blue curve) and the second best (solid green curve); parameter values,  $h = 2.5$ ,  $\lambda = 0.6$ ,  $\sigma = 0.3$ ,  $\beta = 0.94$ ,  $a = 1$

reason why there is even over-investment for high  $k$ . Notice that the second best is always less efficient than the first best for any  $k > 0$ . The only way to get over the current inefficiency is to grow faster so as to get more compensation from the high dividend stream in the future. This motivation derives over-investment for high  $k$  (for relatively high EMPK states). Notice that EMPK converges to  $h$ , the greatest lower bound, as  $k$  goes to infinity. On the other hand, since the marginal production is fairly small when  $k$  is small, the fast growth through more investment cannot compensate the principal. This is the reason to under-invest for low capital states.

### 2.4.3 The Expected Growth Rate

From the results of previous subsections, we know that  $e(k)$  and  $\psi(k)$  are bounded and  $\theta'(k)$  are all bounded for  $k$ . Therefore, the optimal investment is  $O(k)$  order and asymptotically linear in  $k$ . Then, the equilibrium capital processes are given

by

$$\begin{aligned} \text{(1st best)} \quad & \frac{dk}{k} = \left( h - \delta - A_1 + \frac{2(A_1 - \beta)}{\lambda A_1 k} \right) dt + \frac{\sigma}{\sqrt{k}} dW_t, \\ \text{(2nd best)} \quad & \frac{dk}{k} = \left( h - \delta - \frac{\theta(k)}{k} + \frac{R(k)}{k} \right) dt + \frac{\sigma}{\sqrt{k}} dW_t \end{aligned}$$

where  $R(k)$  is define by

$$R(k) = -\frac{e(k)^2}{2a} + e(k) + \frac{1}{\lambda} \log [\theta'(k)^2 \psi(e(k))].$$

Notice  $\lim_{k \rightarrow \infty} \frac{R(k)}{k} = 0$  since the numerator is bounded as  $k \rightarrow \infty$ . The second best drift term, the investment minus depreciation, still has the order of  $O(k)$  since  $\lim_{k \rightarrow \infty} \frac{\theta(k)}{k} = A_1$  from Lemma 9 in the appendix. When the level of capital is high enough, the process approximately looks like a stochastic process with affine drift and square root volatility terms.

The first best and second best expected growth rate  $g_f(k)$  and  $g_s(k)$  are defined by

$$\begin{aligned} g_f(k) &:= \frac{I^f(k) - \delta k}{k} = (h - \delta - A_1) + \frac{2(A_1 - \beta)}{\lambda A_1 k} \\ g_s(k) &:= \frac{I^s(k) - \delta k}{k} = (h - \delta) - \frac{\theta(k)}{k} + \frac{R(k)}{k}. \end{aligned}$$

Notice that  $g_f(k)$  is strictly decreasing with order  $O(1/k)$ , but  $g_s(k)$  is not obvious. Notice that the growth rate at the tail in the first best case is already constant. This asymptotical property is thanks to the size effect or the linear technology in capital. In fact, the first best solution already shows two empirical results, which means that we may not require the agency friction to have the main result. In other

words, we already found the simplest framework to show the empirical regularity.

On the other hand, notice the the growth rate  $g_f(k)$  around very low capital state is too high (See Figure 2.3). So, there is a problem of the first best solution in the quantitative sense. We know  $\lim_{k \rightarrow \infty} g_s(k) = h - \delta - A_1 = \lim_{k \rightarrow \infty} g_s(s)$ . Furthermore, proposition 14 and 15 tell that

$$g_f(k) > g_s(k) \quad \text{for small } k \quad \text{and} \quad g_f(k) < g_s(k) \quad \text{for large } k,$$

which means the investment decision due to incentive compatibility make the second best expected growth rate drop for small  $k$  and increases for large  $k$  in comparison with the first best expected growth rate. Figure 2.3 shows this mechanism.<sup>16</sup> It is still true that the expected growth rate is generally decreasing in  $k$ , showing the negative negative relationship between firm size and growth. This story is more consistent with the two empirical evidence.

Remark that in our model we do not consider the entry or exit of firms therefore the long run distribution for  $g_i$  is degenerate. We do not try to match the firm size or growth rate distribution in this model. This is the limitation of our model.

---

<sup>16</sup>For numerical simulations, we rewrite the first and second best capital processes as the following discrete time versions:

$$\begin{aligned} \text{(1st best)} \quad k_{t+1}^f &= [h - \delta + 1 - A_1]k_t^f + \frac{2(A_1 - \beta)}{\lambda A_1} + \sigma \sqrt{k_t^f} \epsilon_{t+1} \\ \text{(2nd best)} \quad k_{t+1}^s &= [h - \delta + 1]k_t^s - \theta(k_t^s) - R(k_t^s) + \sigma \sqrt{k_t^s} \epsilon_{t+1}. \end{aligned}$$

The expected growth rate is computed by  $g_i(k_t^i) \approx \frac{k_{t+1}^i - k_t^i}{k_t^i}$  for  $i = f, s$ .

## 2.5 The Optimal Payment Schedule

We can write down the optimal payment schedule,  $c^i(t)$ ,  $i = f, s$  with respect to capital by substituting the continuation value processes  $q_t$  of (2.3.6) and (2.3.9).

**Proposition 16.** *The optimal payment schedule is decomposed into following six components:*

$$\begin{aligned}
 c^f(t) = & \underbrace{-\frac{1}{\lambda} \log(-q_0)}_{(1f)} + \underbrace{\frac{a}{2}}_{(2f)} + \underbrace{\int_0^t \frac{\sigma A_1}{2} \sqrt{k_s} dW_s}_{(3f)} \\
 & + \underbrace{\frac{\lambda}{2} \int_0^t \frac{\sigma^2 A_1^2 k_s}{4} ds}_{(4f)} - \underbrace{\frac{1}{\lambda} \log(A_1)}_{(5f)} + \underbrace{\frac{1}{\lambda} \int_0^t (A_1 - \beta) ds}_{(6f)}.
 \end{aligned}$$

and

$$\begin{aligned}
 c^s(t) = & \underbrace{-\frac{1}{\lambda} \log(-q_0)}_{(1s)} + \underbrace{\frac{e(k_s)^2}{2a}}_{(2s)} + \underbrace{\int_0^t \frac{\sigma \theta'(k_s) e(k_s) \psi(e(k_s))}{a} \sqrt{k_s} dW_s}_{(3s)} \\
 & + \underbrace{\frac{\lambda}{2} \int_0^t \frac{\sigma^2 \theta'(k_s)^2 e(k_s)^2 \psi(e(k_s))^2 k_s}{a^2} ds}_{(4s)} \\
 & - \underbrace{\frac{1}{\lambda} \log(\theta'(k_t) \psi(e(k_t)))}_{(5s)} + \underbrace{\frac{1}{\lambda} \int_0^t (\theta'(k_s) \psi(e(k_s)) - \beta) ds}_{(6s)} \tag{2.5.1}
 \end{aligned}$$

*Proof.* See the appendix. □

First notice that we align similar terms at similar positions in the first and

second best payment schedule. They look similar at the first glance, however, it is easy to realize that the second best payment depends on the effort level in order for the agent to have proper incentives to work while the first best is not. We will pin down this effect in detail.

Notice that (2.5.1) can be interpreted as following:

$$\begin{aligned} c(t) = & (1s) \text{ reservation level of consumption} + (2s) \text{ the agent's actual cost} \\ & + (3s) \text{ compensation risk due to unobservability of the effort} \\ & + (4s) \text{ risk premium due to the compensation risk} \\ & + (5s) \text{ adjustment of compensation for future production} \\ & + (6s) \text{ general wage-backloading} \end{aligned}$$

The above decomposition fully characterizes the agent payment schedule. Before giving detail description of each term, let us recall the structural difference of ours from usual moral hazard models. There is capital accumulation, so that the agent's current effort level not only determines current production, but also affects future output since high effort today helps to accumulate the more capital so that it also contributes high output for the future.

Term (1s) and (2s) are easy to understand. Term (3s) is due to the fact that the contract is based not on the unobservable agent's effort levels, but on the observable production realizations. (4s) follows from (3s) since the agent is risk averse. Notice that it is proportional to the risk-aversion parameter of the agent. Terms from (1s) to (4s) are also found in Holmström and Milgrom (1987) and Shättler and Sung

(1993).<sup>17</sup> Not surprisingly those 4 terms appear in our setup as well.

Even if (3s) and (4s) are analogue to the result of Holmström and Milgrom (1987) and Shättler and Sung (1993), implications are quite different. The integrand of term (3s) can be rewritten as<sup>18</sup>

$$\theta'(e(k))e(k)\psi(e(k))\sqrt{k} = \frac{a}{\lambda\sigma^2\sqrt{k}} \cdot \frac{e(k)(a - e(k))}{(-a + e(k)\psi(e(k)))}.$$

First, the payment schedule is less volatile when  $k$  is higher. This is obvious because in our model capital becomes relatively more important in production than the agent's labor as  $k$  grows. Secondly, in order to understand the effect of effort let us fix  $k$ . Then, it is easy to see  $\frac{e(a-e)}{-a+e\psi(e)}$  is decreasing in  $e \in (e^*, a]$ . This tells that more effort derives less risky payment at  $t$ , given the level of capital. This is opposite to the result of general principal-agent models. For example, the payment schedule of Holmström and Milgrom (1987) in our terminology can be written as

$$L(T) = -\log(-q_0) + \int_0^T e(t)dt + \int_0^T C_e(e(t))dW_t + \frac{\lambda}{2} \int_0^T \sigma^2 C_e(e(t))^2 dt,$$

where  $L(T)$  is the lump-sum payment at  $T$  and  $C(e)$  is disutility incurred by effort  $e$  that is not dependent on  $k$ .<sup>19</sup> Notice  $C(e)$  is assumed to be convex, so that its derivative,  $C_e(e)$ , is increasing in  $e$ . The third term in the above equation tells that if the agent puts more effort, then the wage schedule becomes more volatile. This gives the agent incentives to work harder. In our model, however, as capital

---

<sup>17</sup>In their models, the payment occurs only at the end of the contract period  $T$ .

<sup>18</sup>This is done by substituting  $\theta'(k)$  by equation (2.B.7) in the proof of Theorem 2.

<sup>19</sup>Although it is not quite proper to compare, we can define the lump-sum proxy  $L(T)$  in our model by  $L(T) = \int_0^T e^{\beta t} c(t) dt$ . By the Fubini theorem, we can change the order of integration to get the similar form with (2.5.1). The only difference is the new one has a proper discount factor in the integrand.

increases size of the shock becomes larger according to  $\sqrt{k}$ , thus it is not quite necessary for the agent to put more effort and it is even better for the agent to work less to have enough compensation since the probability of getting high output goes higher as capital gets larger.

Then how does the principal make the agent to exert effort? In (5s), we rewrite the term inside the log as

$$\theta'(k)\psi(e(k)) = \frac{a}{\lambda\sigma^2k} \cdot \frac{a - e(k)}{(-a + \psi(e(k))e(k))}.$$

It is also easy to see that given fixed  $k$ ,  $\frac{a-e}{-a+\psi(e)e}$  is decreasing in  $e \in (e^*, a]$ . Note that the minus sign in front of term (5s). Thus, given fixed  $k$  the payment drops instantaneously as the agent works less. Therefore, term (5s) works as *adjustment or stick* through which the principal prevent the agent from decreasing too much effort. Hence, this supports proper production at each time  $t$  to guarantee sufficient growth.

Finally the payment schedule in our model is continuous so that the principal must consider tradeoff of how much she compensates the agent between today and future. Term (6s) shows this tradeoff. It comes from the drift of the agent's continuation process,  $q$ . In the first best case, (6f) shows that the wage is back-loaded if we ignore all the other terms since  $A_1 - \beta > 0$  by Assumption 2. In the 2nd best case, it is not quite obvious whether  $\beta$  is less than  $\theta'(k)\psi(e(k))$  for all  $k$ , but we expect that at least numerically for reasonable parameter values.<sup>20</sup>

---

<sup>20</sup>Note that  $\lim_{k \rightarrow \infty} \theta'(k)\psi(e(k)) = A_1\psi(e^*) > A_1[1 - \frac{\lambda}{a}(a - e^*)] > \beta$  if  $a$  is big enough from corollary 7.



The analogous result is appeared in Sannikov (2008). He shows that the wage can be frontloaded or backloaded depending on how patient the agent is under the possibility of firing the manager. In particular, the frontloading occurs if the continuation value is close enough to the threshold levels of firing the manager when the manager is sufficiently patient. In our model, there is no reason to replace the manager even if the capital level hits the zero boundary, while the manager should be fired in Sannikov (2008) due to the income effect when the continuation value is too high (meaning the firm is fairly profitable). Therefore, there is no reason for our model to have wage-backloading.

## 2.6 The Capital Process near the Boundary: Implication to Poverty Traps

Before we introduce results, we need some definitions in order to characterize the boundary behavior of the capital process in the next propositions. Let  $T_x$  be the first hitting time to  $x$  of  $k(t)$  and define a random time  $T_{l+} = \lim_{x \downarrow l} T_x$  for  $l \in \mathbb{R}$ . The boundary  $l$  is *attracting* if

$$\Pr(\{T_{l+} \leq T_x \mid k(0) = k_0\}) > 0, \quad \text{for all } l < k_0 < x$$

and is *attainable* if

$$\Pr(\{T_l < \infty \mid K(0) = k_0\}) > 0.$$

In other words,  $l$  is attracting if there is a positive probability for the process to hit  $l$  and is attainable if, in addition, it can hit  $l$  in finite time.

**Proposition 17.** *If  $\frac{4(A_1-\beta)}{\lambda A_1} \geq \sigma^2$ , then the first best capital process never reaches to the zero-boundary almost surely. If  $\frac{4(A_1-\beta)}{\lambda A_1} < \sigma^2$ , then the zero boundary is attracting and attainable.*

*Proof.* See the appendix. □

Proposition 17 shows that the capital process reaches to the zero-capital level in finite expected time with a positive probability if  $\sigma$  is big enough or the subjective discount factor is very close to  $A_1$  or the agent is fairly risk-averse, i.e.,  $\frac{4(A_1-\beta)}{\lambda A_1} < \sigma^2$ . It, however, reflects into the positive region as soon as it reaches to the boundary since

$$dk = \frac{2(A_1 - \beta)}{\lambda A_1} dt \quad \text{at } k = 0 \quad (2.6.1)$$

and  $A_1 > \beta$  by Assumption 2.

The following proposition is analogous to Proposition 17.

**Proposition 18.** *Suppose  $\beta$  is small enough such that  $\beta < \theta'(0)$ .<sup>21</sup> If  $\frac{4(\theta'(0)-\beta)}{\lambda \theta'(0)} \geq \sigma^2$ , then the second best capital process never reaches to the zero-boundary almost surely. If  $\frac{4(\theta'(0)-\beta)}{\lambda \theta'(0)} < \sigma^2$ , then the zero boundary is attracting and attainable.*

*Proof.* See the appendix. □

Proposition 18 implies that the zero-capital boundary can be obtained in finite time, other things being equal, if either (i)  $\sigma$  is high enough, or (ii) the subjective

---

<sup>21</sup>It seems at least numerically true with reasonable parameter values. Note that the strict concavity of the value function  $J$  requires the strict increasing property of  $\theta(k)$ , i.e.,  $\theta'(k) > 0$  for all  $k$ . In addition, here we require sufficient curvature of  $\theta(k)$  around 0. If this condition fails, then after the process reaches the boundary, it not only never reflects, but also, the problem is not well-defined. In other words, such contracts cannot satisfy the participation constraint at all.

discount factor is close enough to  $\theta'(0)$ , or (iii) both agents are fairly risk averse.<sup>22</sup> Although the basic intuition is similar to the first best case, we infer from the under-investment result of proposition 14 that the force driving the process to grow is much weaker in the 2nd best case.

An easy way to check this is to investigate the process near the boundary. Similarly to the first best case, the process reflects immediately into the positive region once it touches the zero-boundary.

$$dk = \frac{2(\theta'(0) - \beta)}{\lambda\theta'(0)} dt \quad \text{at } k = 0 \quad (2.6.2)$$

From lemma 8 in the below, equations (2.6.1) and (2.6.2) tell that the speed of escaping from the boundary is slower for the second best case than the first best.

**Lemma 8.**  $\theta'(0) < A_1$ .<sup>23</sup>

*Proof.* See the appendix. □

In sum, under any one of conditions (i) to (iii) in the above, we have a positive probability that the capital process hits the minimum boundary in finite time and this probability is strictly bigger for the second best case. Although the process reflects to the positive region as soon as it hits the boundary, it is still true that the process may go back to the boundary in finite time and reflect again, and so

---

<sup>22</sup>Notice that here the subjective discount factors  $\beta$ 's for the the manager and the principal are the same, so it is not a good idea to have the economic interpretation in terms of whether agents are impatient or impatient. On the other hand, fixing  $\beta$ , we focus on how  $h$ ,  $\lambda$ , and  $\sigma$  affect the firm dynamics.

<sup>23</sup>Recall Assumption 2 where  $\beta < A_1$ . Mathematically, however, it can be shown that if  $A_1 < \beta$ , then  $\theta'(0) > A_1$ . In fact proposition 17 also tells that if  $A_1 < \beta$ , the zero-boundary is attainable. Notice also that Lemma 8 does not say about the comparison between  $\beta$  and  $\theta'(0)$ .

on. Under-investment, in turn, reinforces slow growth in the second best case. Therefore, it takes much longer time for small firms to escape from the low capital states if they face moral hazard problems.

In the micro-finance level industrial organization theory, there is a question of why small firms do not grow. An easy answer might be that it is due to credit constraints. Many small firms seem to be unable to get external financing. If so, the next question is why they do not bootstrap by saving more portion of profits. Recent empirical studies suggest that small firms have enough profits and it is hardly believed that their production technologies are poor (for example, see McKenzie and Woodruff (2006) and Michael, Lee, and Robinson (2008)).

The main driving force in this model making small firms spend longer time in a low capital status is related to the assumption that the firm has capital-size-specific shocks (not productivity shocks), i.e., recall  $\sigma\sqrt{k}dW_t$  term. In the accordance with this assumption, condition (i) might be the most important condition for the firm-level poverty trap. Also, notice that low saving (or under-investment) results from moral hazard. Although our model does not directly aim for explaining the firm-level poverty trap, an answer is quite related to find a mechanism that hinders small firms from having more investment. The incentive compatibility under moral hazard might be one of the potential answers. We hope this view may shed light on the line of research.

## 2.7 Risk Sharing and Business Cycle Implication

One might be interested in how the payment, the dividend, and the investment change corresponding to the shocks. Suppose there is a positive shock  $\Delta W > 0$  when  $k_t = k$ . In this case, we have  $\sigma\sqrt{k}\Delta W$  amount of more production. Who gets this? (2.5.1) shows that the payment to the agent increases by

$$\frac{\sigma\theta'(k)e(k)\psi(e(k))}{a}\sqrt{k}\Delta W.$$

whereas the dividend to the principal decreases by the exactly same amount.<sup>24</sup> So, they are offset. If there is an negative shock, we have the opposite case: the payment decreases and the dividend increases by the same amount. This is because the principal and the agent have the same degree of risk aversion. Intuitively it would be the case that if the agent is more risk-averse, then he would get paid less at a good shock and have less payment decrease at a bad shock.

The important change occurs in the investment side. The whole  $\sigma\sqrt{k}\Delta W$  is added to investment. That is, the principal instantaneously wants the firm to grow faster rather than to get more dividend, so that she will get compensated in the future from higher production. However, notice that this result is not from the moral hazard side, but from the growth setup since we have the same result in the first best case. The information asymmetry only affects the sensitivity of the payment corresponding to the shocks.

---

<sup>24</sup>From the result of theorem 2, one can verify that  $c(t) + d(t)$  is independent of  $q$ . Given capital, the shock affects the change of the continuation value. So, the total amount of  $c(t) + d(t)$  is not changed according to the shock.

## 2.8 Other Production Functions: Constant Returns to Scales

One may be curious about whether the results of the model are robust to the assumption for the production function. One may argue that the given production function is too simple. In fact, if we take a different production function, then the result will be changed. However, it is easy to show that the asymptotic growth rate is never constant for other possibly reasonable production functions. The constant returns to scale property is violated. Therefore, this analysis justifies our assumption on the production function form. Basically, we investigate two cases. both case will violate the CRS property in equilibrium so that they are not good models for investigating firm dynamics. We also can infer that other similar production functions are not very appropriate for a firm dynamics model.

Here we only consider the first best case when the production function is given by  $f(k, e)$ . Without loss of generality we assume that capital is fully depreciated ( $\delta = 0$ ). Then, we have the following HJB equation.

$$\begin{aligned} \beta J(k, q) = \max_{c, d, e, \gamma} & u_p(d) + J_k[f(k, e) + e - c - d] + J_q(\beta q - u_m(c, e)) \\ & + \frac{1}{2} (J_{kk} + 2J_{kq}\gamma + J_{qq}\gamma^2) \sigma^2 k. \end{aligned} \quad (2.8.1)$$

Then, by using first order conditions, we have

$$e = a \frac{\partial f(k, e)}{\partial e} \quad (2.8.2)$$

### 2.8.1 Multiplicity

Suppose

$$f(k, e) = (h + e)k$$

where  $h$  is some constant. Then, (2.8.2) gives

$$e_t^* = ak_t.$$

Then, the equilibrium production is  $f(k, e^*) = ak^2 + hk$ . Therefore, the expected growth rate becomes linear in  $k$ , which does not make sense.

### 2.8.2 Cobb-Douglas Case

Suppose

$$f(k, e) = hk^\alpha e^{1-\alpha}$$

where  $h$  is some constant. Then, (2.8.2) gives

$$e = ((1 - \alpha)ah)^{\frac{1}{1+\alpha}} k^{\frac{\alpha}{1+\alpha}}.$$

Then, the equilibrium production is  $f(k, e^*) = Ck^{\frac{2\alpha}{1+\alpha}}$ , where for some constant  $C$ . Thus,  $\alpha = 1$  should be taken in order to have the CRS. This is the case where the manager has no impact on production.

In sum, by observing cases in Subsection 2.8.1 and Subsection 2.8.2, we can conclude that our assumption on the production function is not unreasonable in

order to keep the constant returns to scale.

## 2.9 Concluding Remark

We study the moral hazard problem affecting the firm's growth by distortion of the investment decision. We assume that technology is an Ak with small managerial contribution onto production. This slight modification makes huge difference on investment and growth of economies. These results seem more relevant to the world economic growth phenomena rather than to the individual firms. There is still ongoing debate on the Gibrat's law. We also do not insist that the moral hazard problem is the main driving force of the poverty trap.<sup>25</sup> Maybe it might be too hasty to consider things like poverty traps in a partial equilibrium context. However, this exercise is still meaningful in the sense that moral hazard can be a potential problem in growth. On the other hand, in order to consider how the moral hazard problem affect the growth rate distribution, one needs to allow possibility of entry and exit of firms or heterogeneity of firms, which we leave as a future research topic.

The usual asset pricing implication of the models with agency conflict is sometimes quite straightforward. Since the dividend delivered to the outside shareholders are less than that of the first best case, the stock price of the second best is smaller than that of the first best such as Albuquerque and Wang (2008) and Dow, Gorton, and Krishnamurthy (2005). But, this is not quite obvious in our model due to the nonlinearity between the amount of investment to the firm and the dividend paid to

---

<sup>25</sup>For literature on poverty traps, see Azariadis and Stachurski (2005).



the capital owner, not only in the current partial equilibrium setup, but furthermore in the aggregation of all contracts in the market. This makes the general equilibrium analysis fairly hard, so that the problem looks untractable. One notable exception is Sung and Wan (2008). They study a general equilibrium model of a moral-hazard economy and suggest several important results in terms of asset pricing. But, their model is static in the sense that investment is one time event at the beginning of the finite horizon and neither the continuous payment and dividend nor the capital accumulation through the continuous investment are considered. We expect that the dynamic general equilibrium approach might lead to quite different economic intuition.

## 2.10 References

1. Albuquerque, R., and H. Hopenhayn (2004) "Optimal Lending Contracts and Firm Dynamics", *Review of Economic Studies*, Vol. 71(2), 285-315.
2. Albuquerque, R., and N. Wang (2008) "Agency Conflicts, Investment, and Asset Pricing", *Journal of Finance*, Vol. 63(1), 1-40.
3. Amaral, L. A. N, S. V. Buldyrev, S. Havlin, M. A. Salinger, H. E. Stanley and M. H. R. Stanley (1997) "Scaling Behavior in Economics: the Problem of Quantifying Company Growth", *Physica A*, Vol 244, 1-24.
4. Audretsch, D. B., L. Klomp, E. Santarelli, and A. R. Thurik (2004) "Gibrat's law: Are the Services Different?" *Review of Industrial Organization*, Vol. 24, 301-324.
5. Azariadis, C. and J. Stachurski (2005) "Poverty Traps", *HANDBOOK OF ECONOMIC GROWTH*, Chapter 6.
6. Bartelsman, E., J. Haltiwanger, and S. Scarpetta (2008) "Cross Country Differences in Productivity: The Role of Allocative Efficiency", *working paper*
7. Bottazi, G., G. Dosi, M. Lippi, F. Pmmolli, and M. Riccaboni (2001) "Innovation and Corporate Growth in the Evolution of the Drug Industry" *International Journal of Industrial Organization*, Vol. 19, 1161-1187.
8. Bottazi, G. and A. Secchi (2003) "Common Properties and Sectoral Specificities in the Dynamics of U.S. Manufacturing Companies", *Review of Industrial Organization*, Vol. 23, 217-232.

9. Bottazzi, G. Coad, A., Jacoby, N., and A, Secchi (2005) "Corporate Growth and Industrial Dynamics: Evidence from French Manufacturing", *working paper*
10. Caves, R. E. (1998), "Industrial Organization and New Findings on the Turnover and Mobility of Firms", *Journal of Economic Literature*, Vol. 36, 1947-1982.
11. Clementi, G. AND H. Hopenhayn (2006) "A Theory of Financing Constraints and Firm Dynamics" *Quarterly Journal of Economics*, Vol. 121, 229-265
12. Coad, A. (2007) "Firm Growth: A Survey", *Working Paper*
13. Cooley, T. F. and V. Quadrini (2001) "Financial Markets and Firm Dynamics", *American Economic Review*, Vol. 91, 1286-1310.
14. Droucopoulos, V. (1983) "International Big Business Revisited: on the Size and Growth of the World's Largest Firms", *Managerial and Decision Economics*, Vol 4, 244-252.
15. DeMarzo, P., M. Fishman, Z. He, and N. Wang (2009) "Dynamic agency and the q theory of investment", *working paper*
16. DeMarzo, P., and Y. Sannikov (2006) "Optimal Security Design and Dynamic Capital Structure in a Continuous-Time Agency Model", *Journal of Finance*, vol 61, 2681-2724
17. Dow, J., Gorton, G., and Krishnamurthy, A. (2005) "Equilibrium Investment and Asset Prices under Imperfect Corporate Control", *American Economic Review*, Vol 95, 659-681.

18. Dunne, P. and A. Hughes (1994) "Age, Size, Growth and Survival: UK companies in the 1980's", *Journal of Industrial Economics*, Vol. 42, 115-140.
19. El Karoui, N., Peng, S. and Quenez, M. C. (1997) "Backward stochastic differential equations in finance", *Mathematical Finance* Vol. 7, 171.
20. El Karoui, N., Peng, S. and Quenez, M. C. (2001) "A Dynamic Maximum Principles for the Optimization of Recursive Utilities under Constraints", *The Annals of Applied Probability*, Vol. 11, 664-693.
21. He, Z. (2009) "Optimal Executive Compensation when Firm Size Follows Geometric Brownian Motion" *The Review of Financial Studies*, Vol. 22, 859-892
22. Gabe, T. M. and D. S. Kraybill (2002) "The Effect of State Economic Development Incentives on Employment Growth of Establishments", *Journal of Regional Science*, Vol. 42, 703-730.
23. Geroski, P. A., and K. Gugler (2004) "Corporate Growth Convergence in Europe", *Oxford Economic Papers*, Vol. 56, 597-620.
24. Greenwood, J., Z. Hercowitz, and P. Krusell (1997), "Long-Run Implications of Investment-Specific Technological Change", *American Economic Review*, Vol. 87, 342-362.
25. Hardwick, P. and M. Adams (2002) "Firm Size and Growth in the United Kingdom Life Insurance Industry", *Journal of Risk and Insurance*, Vol. 69, 577-593.

26. Hart, P. E. and N. Oulton (1996) "The Growth and Size of Firms", *Economic Journal*, Vol 106, 1242-1252.
27. Holmström, B. and P. Milgrom (1987) "Aggregation and Linearity in the Provision of Intertemporal Incentives", *Econometrica*, Vol. 55, 303-328.
28. Hymer, S. and P. Pashigian (1962) "Firm Size and Rate of Growth", *Journal of Political Economy*, Vol. 70, 556-569.
29. Jensen, M. (1986) "The Agency Costs of Free Cash Flow: Corporate Finance and Takeovers", *American Economic Review* 76, 323-330.
30. Jensen, M. (1993) "The Modern Industrial Revolution, Exit, and the Failure of Internal Control Systems", *Journal of Finance* 48, 831-880.
31. Jones, L. and R. Manuelli, (2005) " Neoclassical Models of Endogenous Growth: The Effects of Fiscal Policy, Innovation and Fluctuations", *HANDBOOK OF ECONOMIC GROWTH*, Chapter 1.
32. Kumar, M. S. (1985), "Growth, Acquisition Activity and Firm Size: Evidence from the United Kingdom", *Journal of Industrial Economics*, Vol. 33, 327-338
33. Lotti, F. E., E. Snatarelli, and M. Vivarelli (2003) 'Does Gibrat's Law hold among young, small firms?', *Journal of Evolutionary Economics*, Vol. 13, 213-235.
34. Mansfield, E. (1962) "Entry, Gibrat's Law, Innovation and the Growth of Firms", *American Economic Review*, Vol. 52, 1023-1051.

35. McKenzie, D. and C. Woodruff (2006) "Do entry costs provide an empirical basis for poverty traps? evidence from mexican microenterprises" *Economic De- velopment and Cultural Change*, Vol. 55, 3-32.
36. Michael, K., J. Lee, and J. Robinson (2008) "The return to capital for small retailers in kenya: Evidence from inventories", *working paper*
37. Mowery, D. C. (1983) "Industrial Research and Firm Size, Survival, and Growth in American Manufacturing, 1921-1946: An Assessment", *Journal of Economic History*, Vol. 43, 953-980.
38. Oyer, P. (2004) "Why Do Firms Use Incentives that have No Incentive Effects?", *Journal of Finance*, Vol. 59, 1619-1649
39. Pardoux, P. and Peng, S. (1990) "Adapted Solution of a Backward Stochastic Differential Equation", *Systems and Control Letters* vol. 14, 5561.
40. Phelan, C. and R. Townsend (1991) "Computing Multi-period, Information Constrained Optima.", *Review of Economic Studies*, Vol. 58(5), 853-881.
41. Sannikov, Y. (2008) "A Continuous-Time Version of the Principal-Agent Problem", *forthcoming in Review of Financial Studies*.
42. Schättler, H. and J. Sung (1993) "The First-Order Approach to the Continuous-Time Principal-Agent Problem with Exponential Utility", *Journal of Economic Theory*, Vol. 61, 331-371.
43. Stein, J. (2003) "Agency, Information and Corporate Investment", *Handbook of Economics of Finance*, Chapter 2.

44. Sung, J. and X. Wan (2008) "Equilibrium Equity Premium, Interest Rate and Cost of Capital in a Large-Firm Economy under Moral Hazard", working paper
45. Thomas, J. and T. Worrall (1988) "Self-Enforcing Wage Contracts" *Review of Economic Studies*, Vol. 51(2), 541-554.
46. Williams, N. (2009) "On Dynamic Principal-Agent Problems in Continuous Time" *working paper*
47. Yamada, T. and Watanabe, S. (1971) "On the Uniqueness of Solutions of Stochastic Differential Equations", *Journal of Mathematics of Kyoto University*, Vol 11, 155-167.
48. Yong, J. and Z. Y. Zhou (1999), "Stochastic Controls: Hamiltonian Systems and HJB Equations", Springer-Verlag, New York.

## Appendices

### 2.A Proofs of Theorem 2, Proposition 12, and Corollary 5

Since the first-best solution is easy to understand through the second-best solution, we just present the explicit solution of the first-best case. All the verification procedures are omitted.

*Proof.* The first order conditions are

$$[c]: \quad -J_k - J_q \lambda \exp\left(-\lambda\left(c - \frac{e^2}{2a}\right)\right) = 0$$

$$[d]: \quad \lambda \exp(-\lambda d) - J_k = 0$$

$$[e]: \quad J_k + J_q \frac{\lambda e}{a} \exp\left(-\lambda\left(c - \frac{e^2}{2a}\right)\right) = 0$$

$$[\gamma]: \quad \gamma = -\frac{J_{kq}}{J_{qq}}$$

Guess the value function  $J$  as

$$J(k, q) = \frac{1}{q} \exp(-\lambda \theta(k)),$$

where  $\theta : [0, \infty) \rightarrow \mathbb{R}$  is a  $C^2$  function. Then, we have

$$J_k = -\lambda \theta'(k) J, \quad J_q = -\frac{J}{q},$$

$$J_{kk} = (-\lambda \theta''(k) + \lambda^2 \theta'(k)^2) J, \quad J_{kq} = \frac{\lambda \theta'(k)}{q} J, \quad J_{qq} = \frac{2}{q^2} J.$$



From FOC  $[\gamma]$ , we first have

$$\gamma = -\frac{\lambda\theta'(k)q}{2}. \quad (2.A.1)$$

From FOC  $[c]$ ,

$$\lambda \exp\left(-\lambda\left(c - \frac{e^2}{2a}\right)\right) = -\frac{J_k}{J_q} = -\lambda\theta'(k)q \quad \text{or} \quad c = \frac{e^2}{2a} - \frac{1}{\lambda} \ln(-q\theta'(k)) \quad (2.A.2)$$

From FOC  $[d]$ ,

$$\lambda \exp(-\lambda d) = -\lambda\theta'(k)J \quad \text{or} \quad d = \theta(k) - \frac{1}{\lambda} \ln\left(\frac{\theta'(k)}{-q}\right) \quad (2.A.3)$$

FOC  $[e]$  and (2.A.2) give the optimal effort  $e^* = a$ , which is constant. To simplicity, using (2.A.2) and (2.A.3) together with this constant effort we define  $I$ , the drift term without considering depreciation  $\delta k$ , by

$$I(k) = hk - c - d + a = hk - \theta(k) + \frac{2}{\lambda} \ln(\theta'(k)) + \frac{a}{2}.$$

Now putting the above optimal policies into the HJB equation we derive the following ODE with respect to  $k$ .

$$\begin{aligned} \beta &= \theta'(k) - \lambda\theta'(k)(I(k) - \delta k) - \frac{1}{q}(\beta q - \theta'(k)q) \\ &\quad + \frac{\sigma^2 k}{2} \left[ (-\lambda\theta''(k) + \lambda^2\theta'(k)^2) + 2\frac{\lambda\theta'(k)}{q} \left( \frac{-\lambda\theta'(k)q}{2} \right) + \frac{2}{q^2} \left( \frac{\lambda^2\theta'(k)^2 q^2}{4} \right) \right] \\ \implies 2\beta &= \theta'(k)(2 - \lambda I(k) + \lambda\delta k) - \frac{\sigma^2 \lambda}{2} k\theta''(k) + \frac{\sigma^2 \lambda^2}{4} k\theta'(k)^2 \\ \text{or } 2\beta &= \theta'(k) \left( 2 - \lambda(h - \delta)k + \lambda\theta(k) - 2 \log(\theta'(k)) - \frac{\lambda a}{2} \right) \\ &\quad - \frac{\sigma^2 \lambda}{2} k\theta''(k) + \frac{\sigma^2 \lambda^2}{4} k\theta'(k)^2 \end{aligned}$$

Putting  $\theta(k) = A_1 k + B_1$  into the above ODE, we can get

$$A_1 = \frac{h - \delta}{1 + \frac{\lambda \sigma^2}{4}} \quad \text{and} \quad B_1 = \frac{1}{\lambda} \left( \frac{2(\beta - A_1)}{A_1} + 2 \log A_1 + \frac{\lambda a}{2} \right).$$

Finally, if we plug  $c^f$ ,  $d^f$ , and  $e^f$  into the capital  $k$  and continuation  $q$  processes, the proofs of proposition 12 and Corollary 5 are completed.  $\square$

## 2.B Proofs of Theorem 3, Proposition 13, and Corollary 6

We first characterize incentive compatibility by using a similar method to martingale techniques developed by Sannikov (2008). For notational simplicity, let  $\sigma(k) = \sigma\sqrt{k}$  and  $g(k, c, d, e) = (h - \delta)k - c - d + e$ . Recall the underlying process is given by

$$dk(t) = g(k(t), c(t), d(t), e(t))dt + \sigma(k(t))dW_t.$$

Attentive readers might be worried that  $\sigma(k) = \sigma\sqrt{k}$  does not satisfy the Lipschitz continuity. The existence and the uniqueness, in our model with the square root volatility, follows by classical results of Yamada and Watanabe (1971).

Given an arbitrary pair of consumption, dividend, and effort,  $(c, d, e) = (\{c_t\}, \{d_t\}, \{e_t\})$ , the agent's expected remaining utility  $q_t$  at time  $t$  is defined by

$$q_t(c, d, e) = E^e \left[ \int_t^\infty e^{-\beta(s-t)} u_m(c_s, e_s) dt \mid \mathcal{F}_t \right], \quad (2.B.1)$$

where  $E^e$  denotes the expectation under the probability measure  $Q^e$  induced by the agent's effort  $e$ . Then, we have the following stochastic representation of the

continuation value process,  $q = \{q_t\}$ .

**Proposition 19.** *There exists a progressively measurable process  $\gamma = \{\gamma_t\}$  such that*

$$dq_t(c, d, e) = (\beta q_t - u_m(c_t, e_t))dt + \gamma_t \sigma \sqrt{k(t)} \left( \frac{g(k_t, c_t, d_t, e_t)}{\sigma \sqrt{k(t)}} dt - dW_t \right). \quad (2.B.2)$$

with  $E^e[\int_0^t \gamma_s^2 ds] < \infty$  for all  $t \in [0, \infty)$ .

*Proof.* This can be easily shown by a standard application of the Martingale Representation Theorem (see theorem 1.3.13 of Karatzas and Shreve (1991)). One can also find the formal argument in the proof of proposition 1 of Sannikov (2008).  $\square$

**Proposition 20.** *Given the contract  $(c, d, e) = (\{c_t\}, \{d_t\}, \{e_t\})$ , suppose  $\gamma_t$  is the process from Proposition 19 representing  $q_t(c, d, e)$ . Then,  $e$  is incentive compatible if and only if*

$$u_m(c_t, e_t) - \gamma_t g(k_t, c_t, d_t, e_t) \geq u_m(c_t, e') - \gamma_t g(k_t, c_t, d_t, e'), \quad \forall e' \in \mathcal{E} \quad (2.B.3)$$

for all  $t \in [0, \infty)$  and  $Q^e$ -almost surely

*Proof.* The proof is basically the same as the proof of proposition 2 in Sannikov (2008). On the other hands, equivalently, one can refer the comparison theorem (Theorem 3.2 in El Karoui, Peng, and Quenez (2001)): The necessary and sufficient condition for  $q(0)$  being maximized is to minimize the drift term. More general proof (in the finite horizon) can be found in Proposition 5.1 in Williams (2009) or Theorem 4.2 in Schattler and Sung (1993).  $\square$

Notice that condition (2.B.3) is equivalent to condition (2.3.1) since  $d$  is irrelevant as seen in the equation. Therefore, we have  $\gamma_t = \gamma_t(c_t, e_t)$ . Then, the first order condition to (2.3.1) pins down  $\gamma = \gamma(c, e)$  as in (2.3.2), i.e.,

$$\gamma(c, e) = -\frac{\partial u_m(c, e)/\partial e}{\partial g(k, c, d, e)/\partial e} = -\frac{\partial u_m}{\partial e}(c, e) = \frac{\lambda e}{a} \exp(-\lambda(c - e^2/2a)).$$

The above  $q_t$  plays as a role of a state variable for the principal's utility maximization problem. Then,  $(k(t), q(t))$  provides the principal's problem a Markovian structure. Notice that from the direct calculation we have

$$\begin{aligned} \gamma_c(c, e) &= -\lambda\gamma(c, e), & \gamma_e(c, e) &= \frac{a + \lambda e^2}{ae}\gamma(c, e), \\ u_m(c, e) &= -\frac{a}{\lambda e}\gamma(c, e), & \frac{\partial u_m}{\partial c}(c, e) &= \frac{a\gamma(c, e)}{e}, & \frac{\partial u_m}{\partial e}(c, e) &= -\gamma(c, e) \end{aligned} \quad (2.B.4)$$

We will use (2.B.4) for notational convenience when we calculate the first order conditions. Then, the first order conditions to Bellman equation (2.3.3) are given by

$$[c] : \quad -J_k - J_q u_c(c, e) - [J_{kq}\gamma(c, e) + J_{qq}\gamma^2(c, e)]\lambda\sigma^2 k = 0.$$

$$[d] : \quad \lambda \exp(-\lambda d) - J_k = 0.$$

$$[e] : \quad J_k - J_q u_e(c, e) + [J_{kq}\gamma(c, e) + J_{qq}\gamma^2(c, e)]\frac{a + \lambda e^2}{ae}\sigma^2 k = 0.$$

## Proofs of Proposition 13 and Corollary 6

The following is the proof for Proposition 13 and Corollary 6 given that solution  $J(k, q)$  to the Bellman equation (2.3.3) is the value of the principal when  $k_0 = k$

and  $q_0 = q$ .

*Proof.* Similarly to the first best case we also guess the value function  $J$  as

$$J(k, q) = \frac{1}{q} \exp(-\lambda\theta(k)).$$

From FOC [c], FOC [e], and (2.B.4) we can get

$$\begin{aligned} & -J_k - J_q u_c(c, e) + (J_k - J_q u_e(c, e)) \frac{\lambda a e}{a + \lambda e^2} = 0 \\ \implies & \lambda\theta'(k) + \frac{1}{q} \left( \frac{a\gamma(c, e)}{e} \right) + \left( -\lambda\theta'(k) - \frac{\gamma(c, e)}{q} \right) \frac{\lambda a e}{a + \lambda e^2} = 0 \\ \implies & \lambda\theta'(k) \frac{a + \lambda e^2 - \lambda a e}{a + \lambda e^2} + \frac{\gamma(c, e)}{q} \frac{a^2}{e(a + \lambda e^2)} = 0 \\ \implies & \gamma(c, e) = \lambda(-q)\theta'(k) \frac{e(a + \lambda e^2 - \lambda a e)}{a^2} \equiv \frac{\lambda}{a}(-q)\theta'(k) e \psi(e), \end{aligned} \quad (2.B.5)$$

where

$$\psi(e(k)) = \frac{a + \lambda e(k)^2 - \lambda a e(k)}{a}.$$

(2.B.5) yields the optimal consumption for the agent

$$c = \frac{e(k)^2}{2a} - \frac{1}{\lambda} \ln((-q)\theta'(k)\psi(e(k))). \quad (2.B.6)$$

Putting (2.B.5) back into FOC [c],

$$\begin{aligned} & \lambda\theta'(k) + \frac{1}{q} \left( \frac{a\gamma(c, e)}{e} \right) - \lambda\sigma^2 k \left[ \frac{\lambda\theta'(k)}{q} \gamma(c, e) + \frac{2}{q^2} \gamma(c, e)^2 \right] = 0 \\ \implies & 1 - \psi(e) - \frac{\lambda^2 \sigma^2}{a^2} k (-a + 2\psi(e)e) \psi(e) e \theta'(k) = 0 \end{aligned}$$

$$\text{or } \theta'(k) = \frac{a(a - e(k))}{\lambda\sigma^2k\psi(e(k))(-a + 2\psi(e(k))e(k))} \quad (2.B.7)$$

From FOC [d],

$$\begin{aligned} \lambda \exp(-\lambda d) &= -\lambda\theta'(k)J \\ \implies d &= \theta(k) - \frac{1}{\lambda} \ln \left( \frac{\theta'(k)}{-q} \right) \end{aligned} \quad (2.B.8)$$

Similarly to the first best case, using (2.B.6) and (2.B.8), we define  $I(k)$ , the drift term of the capital process without considering the depreciation  $\delta k$  by

$$\begin{aligned} I(k; \theta(k), \theta'(k), e(k)) &= hk - c - d + e \\ &= hk - \theta(k) - \frac{e(k)^2}{2a} + e(k) + \frac{1}{\lambda} \ln (\psi(e(k))\theta'(k)^2) \end{aligned} \quad (2.B.9)$$

Putting the above optimal policies into the HJB equation, we get

$$\begin{aligned} \beta &= \theta'(k) - \lambda\theta'(k)(I(k) - \delta k) - \frac{1}{q} \left( \beta q + \frac{a}{\lambda e} \gamma(c, e) \right) \\ &+ \frac{\sigma^2 k}{2} \left[ (-\lambda\theta''(k) + \lambda^2\theta'(k)^2) + \frac{2\lambda\theta'(k)}{q} \gamma(c, e) + \frac{2}{q^2} \gamma(c, e)^2 \right] \quad \text{or} \\ 2\beta &= \theta'(k) \left[ 1 + \psi(e(k)) - \lambda(h - \delta)k + \lambda\theta(k) + \frac{\lambda e(k)^2}{2a} - \lambda e(k) - \ln (\psi(e(k))\theta'(k)^2) \right] \\ &- \frac{\lambda\sigma^2 k}{2} \theta''(k) + \frac{\lambda^2\sigma^2 k\theta'(k)^2}{2} \left[ 1 - \frac{2}{a} e(k)\psi(e(k)) + \frac{2}{a^2} e(k)^2\psi(e(k))^2 \right] \end{aligned} \quad (2.B.10)$$

Now the system of the ordinary differential equations is derived by manipulating (2.B.5) and (2.B.10). It is described at the end of this proof. Then, plugging  $c^s$ ,  $d^s$ , and  $e^s$  into  $k_t$  and  $q_t$  processes, we completes the proofs of Proposition 13 and Corollary 6.

The following is the way of deriving the system of the ordinary differential equations. First note that we have

$$k\theta''(k) = \frac{2}{\sigma^2\lambda}\theta'(k)[1 + \psi(e(k)) - \lambda I(k) + \lambda\delta k] \\ + \lambda k \left[ 1 - \frac{2}{a}e(k)\psi(e(k)) + \frac{2}{a^2}e(k)^2\psi(e(k))^2 \right] \theta'(k)^2 - \frac{4\beta}{\sigma^2\lambda}.$$

Notice that the numerical algorithm is in the next subsection.

1. Let us define functions  $H$  and  $G$  such that

$$H(e, k) = \frac{a(a - e)}{\lambda\sigma^2k\psi(e)(-a + 2\psi(e)e)} \\ G(e, \theta_0, \theta_1, k) = \frac{2}{\sigma^2\lambda}\theta_1[1 + \psi(e) - \lambda I(k; \theta_0, \theta_1, e) + \lambda\delta k] \\ + \lambda k \left[ 1 - \frac{2}{a}e\psi(e) + \frac{2}{a^2}e^2\psi(e)^2 \right] \theta_1^2 - \frac{4\beta}{\sigma^2\lambda}$$

2. Rewrite (2.B.7) and (2.B.10) as

$$\theta'(k) = H(e(k), k) \tag{2.B.11}$$

$$k\theta''(k) = G(e(k), \theta(k), \theta'(k), k). \tag{2.B.12}$$

3. Taking a derivative in (2.B.11) with respect to  $k$ ,

$$\theta''(k) = H_1(e(k), k)e'(k) + H_2(e(k), k). \tag{2.B.13}$$

4. From (2.B.12) and (2.B.13) we have the following system of 1st order ordinary

differential equations with two variables  $(e(k), \theta(k))$ :

$$e'(k) = \frac{G(e(k), \theta(k), H(e(k), k), k)}{kH_1(e(k), k)} - \frac{H_2(e(k), k)}{H_1(e(k), k)} := F(e(k), \theta(k), k)$$

$$\theta'(k) = H(e(k), k) \tag{2.B.14}$$

□

### Proof of Theorem 3 (Verification)

*Proof.* Suppose  $J(k, q)$  is a solution to the Bellman equation (2.3.3). For any incentive compatible contract  $\{c, d, e\}$ , define the principal's auxiliary gain process  $G$  by

$$G_t(c, d, e) = \int_0^t e^{-\beta s} u_m(d_s) ds + e^{-\beta t} J(k_t, q_t)$$

where  $k_t$  and  $q_t$  are capital and continuation value processes induced by  $\{c, d, e\}$  as in Proposition 19. We will show that  $G_t$  is super-martingale and indeed martingale when  $\{c, d, e\}$  is an optimal contract. By Ito's lemma, we have

$$dG_t = \beta e^{-\beta t} A_t dt + \beta e^{-\beta t} \sigma \sqrt{k_t} (J_k(k_t, q_t) - q J_q(k_t, q_t) \frac{\lambda}{a} \theta'(k_t) e(k_t) \psi(e(k_t))) dW_t$$

where the drift term  $A_t$  is

$$A_t = u_p(d_t) - \beta J(k_t, q_t) + J_k((h - \delta)k_t + e_t - c_t - d_t) + J_q(\beta q_t - u_m(c_t, e_t))$$

$$+ \frac{1}{2} (J_{kk} + 2J_{kq} \gamma(c_t, e_t) + J_{qq} \gamma(c_t, e_t)^2) \sigma^2 k_t.$$



Now it is easy to see that  $A_t$  is nonpositive by (2.3.3) and zero at optimum  $(c, d, e)$  as in Proposition 13. Therefore, we are left to show that the diffusion term of  $G_t$  at optimal  $\{c, d, e\}$  is bounded. The drift term can be rewritten as

$$\sigma \exp(-\theta(t)) \left( -\lambda \theta'(k_t) + \frac{\lambda}{a} \theta'(k_t) e(k_t) \psi(e(k_t)) \right) e^{-\beta t} \frac{\sqrt{k_t}}{q_t}.$$

Note that  $\theta'(k)$ ,  $e(k)$ , and  $\psi(e(k))$  are all bounded by Lemma 7 and Lemma 9.  $\exp(-\theta(t))$  is also bounded. Thus, we need to show  $\frac{e^{-\beta t} \sqrt{k_t}}{q_t}$  is a square integrable, i.e., for any  $T < \infty$ ,

$$E \left[ \int_0^T \frac{e^{-\beta s} \sqrt{k_s}}{q_s} ds \right] < \infty. \quad (2.B.15)$$

Notice that  $q_t$  should be bounded away from 0 for any time  $T$ . Suppose  $q_t$  reaches 0 in some finite time  $S$ . This implies the payment  $c_t$  takes infinite value at  $t = S$ , which is never feasible. Therefore, (2.B.15) should be satisfied. This completes the proof.  $\square$

## 2.C Proof of Lemma 7 and 8

### Proof of Lemma 7

*Proof.* (1) Recall from the proof of Theorem 2

$$\theta'(k) = \frac{a(a - e(k))}{\lambda \sigma^2 k \psi(e(k))(-a + 2\psi(e(k))e(k))} > 0. \quad (2.C.1)$$

It is easy to see  $e(0) = a$ , otherwise  $\theta'(k) \rightarrow \infty$  as  $k \rightarrow 0$ . Note that  $e(k)$  is  $C^1$  and  $e(0) = a$ . Suppose  $k$  is small. Then, it is easy to see that

$$\psi(e(k))(-a + 2\psi(e(k))e(k)) > 0 \quad \text{when} \quad e(k) \in (a - \epsilon, a + \epsilon)$$

for some  $\epsilon > 0$  by the continuity of  $e(k)$ . Thus, it must be that case  $e(k) < a$  for such  $k$ . Once  $e(k)$  enters the region below  $a$ , it cannot cross  $a$  as  $k$  grows, otherwise it violates  $\theta'(k) > 0$ .

(2) Notice that  $\psi(e) > 0$  for  $e \in [e^*, a]$ . Suppose not, i.e.,  $\lim_{k \rightarrow \infty} e(k) < e^*$ . This implies  $\psi(e(k))(-a + 2\psi(e(k))e(k))$  is bounded away from 0. Then,  $k\theta'(k)$  is bounded by (2.C.1). It follows that

$$\lim_{k \rightarrow \infty} \theta'(k) = 0, \quad \lim_{k \rightarrow \infty} \theta''(k) = 0, \quad \lim_{k \rightarrow \infty} k\theta'(k)^2 = 0, \quad \lim_{k \rightarrow \infty} k\theta''(k) = 0 \quad (2.C.2)$$

Consider ODE (2.B.10) derived from the bellman equation in the proof of Theorem 2.

$$\begin{aligned} 2\beta &= \theta'(k)[1 + \psi(e(k)) - I(k; (\theta(k), \theta'(k), e(k) + \lambda\delta k) - \frac{\lambda\sigma^2 k}{2}\theta''(k) \\ &\quad + \frac{\lambda^2\sigma^2 k\theta'(k)^2}{2} \left[ 1 - \frac{2}{a}e(k)\psi(e(k)) + \frac{2}{a^2}e(k)^2\psi(e(k))^2 \right]. \end{aligned}$$

Taking  $k \rightarrow \infty$  on both sides, we get

$$2\beta = -\lambda(h - \delta) \lim_{k \rightarrow \infty} k\theta'(k)$$

using (2.C.2), which is a contradiction since the right hand side is non-positive.  $\square$

## Proof of Lemma 8

*Proof.* Let us define a function  $f : (0, \infty) \rightarrow R$  such that

$$f(x) = \frac{1}{\lambda} \left[ \frac{2\beta}{x} - 2 + 2 \log x + \frac{\lambda a}{2} \right].$$

Then,  $B_1 = f(A_1)$  and  $\theta(0) = f(\theta'(0))$ . First notice that  $\theta(0) < B_1$  since the first best solution is always bigger than the second best solution. Otherwise it would be that the first best effort and the second best effort should be equal. But, it is not true.  $f(x)$  has the global minimum value at  $x = \beta$ .  $\lim_{x \rightarrow 0} f(x) = \infty$  and  $\lim_{x \rightarrow \infty} f(x) = \infty$ .  $f(x)$  is strictly decreasing in  $(0, \beta)$  and strictly increasing in  $(\beta, \infty)$ . Notice again  $A_1 > \beta$ . Then, it is easy to see that  $\theta'(0) < A_1$  since  $f(\theta'(0)) < f(A_1)$ . Sketching the graph of  $f(x)$  confirms the proof.  $\square$

## 2.D Proof of Proposition 14 and Proposition 15

### Lemma 9

First, we need following lemma that is useful to show the limiting behavior of the investment level.

**Lemma 9.** (1)  $\lim_{k \rightarrow \infty} \frac{\theta(k)}{k} = \lim_{k \rightarrow \infty} \theta'(k) = A_1$ .

$$(2) \lim_{k \rightarrow \infty} k(-a + 2\psi(e(k))e(k)) = \frac{a(a-e^*)}{\lambda\sigma^2 A_1 \psi(e^*)}$$

$$(3) \lim_{k \rightarrow \infty} k^2 e'(k) = -\frac{a(a-e^*)}{2\lambda\sigma^2 A_1 \psi(e^*) (\psi'(e^*)e^* + \psi(e^*))}$$

$$(4) \lim_{k \rightarrow \infty} (A_1 - \theta'(k))k = -\infty.$$

*Proof.* (1) Since  $\frac{\theta(k)}{k} < A_1 + \frac{B_1}{k}$ , it is straightforward  $\lim_{k \rightarrow \infty} \frac{\theta(k)}{k} \leq A_1$ . Note  $\theta(k) \in C^2$  so that the limit must exist and let  $\lim_{k \rightarrow \infty} \frac{\theta(k)}{k} = \lim_{k \rightarrow \infty} \theta'(k) = x$ . Then, dividing ODE (2.B.10) in the proof of Theorem 2 by  $k$  and taking the limit, we get

$$x = -\frac{\lambda\sigma^2}{4}x + (h - \delta) \quad \text{or} \quad x = \frac{h - \delta}{1 + \frac{\lambda\sigma^2}{4}} = A_1$$

since  $e(k)$  and  $\psi(e(k))$  are bounded and  $\psi(e(k))e(k) \rightarrow \frac{a}{2}$  and  $\theta''(k) \rightarrow 0$  as  $k \rightarrow \infty$ .

(2) Recall  $\theta'(k) = \frac{a(a-e(k))}{\lambda\sigma^2 k \psi(e(k))(-a+\psi(e(k))e(k))}$ . Then, taking limit on the both sides to get the required result.

(3) Applying L'Hopital's rule to (2), we can easily get the required result.

(4) Again by L'Hopital's rule it is equivalent to show  $\lim_{k \rightarrow \infty} k^2 \theta''(k) = -\infty$ . Rearranging ODE (2.B.10), we have

$$\frac{\lambda\sigma^2}{2} k^2 \theta''(k) = \theta'(k) k \Phi(k),$$

where

$$\begin{aligned} \Phi(k) := & 1 + \psi(e(k)) + \frac{\lambda e(k)^2}{2a} - \lambda e(k) - \log(\psi(e(k))\theta'(k)^2) - \frac{2\beta}{\theta'(k)} \\ & - \lambda(h - \delta)k + \lambda\theta(k) + \frac{\lambda\sigma^2\theta'(k)}{2}k \left[ 1 - \frac{2}{a}\psi(e(k))e(k) + \frac{2}{a^2}\psi(e(k))^2 e(k)^2 \right] \end{aligned}$$

It is enough to show that there are positive numbers  $\epsilon$  and  $K$  such that  $\Phi(k) < -\epsilon$  for all  $k > K$ . First, we can write a Lorentz series

$$1 - \frac{2}{a}\psi(e(k))e(k) + \frac{2}{a^2}\psi(e(k))^2 e(k)^2 \approx \frac{1}{2} + \frac{b_2}{k^2} + \frac{b_3}{k^3} + \dots \quad (2.D.1)$$

since from (2) we can express  $\psi(e(k))e(k) \approx \frac{a}{2} + \frac{a_1}{k} + \frac{a_2}{k} + \dots$  for some constants  $a_i$ 's and  $b_i$ 's. Note that there is no  $\frac{1}{k}$  term in series (2.D.1). Second, we know  $\theta(k) < A_1k + B_2$ . Using these two facts together with (1), we have there exists some number  $K$  such that if  $k > K$ ,

$$\begin{aligned} \Phi(k) &< 1 + \psi(e^*) + \frac{\lambda e^{*2}}{2a} - \lambda e^* - \log(\psi(e^*)A_1^2) - \frac{2\beta}{A_1} \\ &\quad - \lambda(h - \delta)k + \lambda A_1 k + \lambda B_1 + \frac{\lambda \sigma^2 A_1}{2} \left[ \frac{k}{2} + \frac{b_1}{k} \right] + \delta_1 \\ &< 1 + \psi(e^*) + \frac{\lambda e^{*2}}{2a} - \lambda e^* - \log(\psi(e^*)A_1^2) - \frac{2\beta}{A_1} + \lambda B_1 + \delta_2 \end{aligned}$$

for some small numbers  $\delta_1, \delta_2 > 0$  since  $A_1 = \frac{h-\delta}{1+\frac{\lambda\sigma^2}{4}}$ . Now we are left to show that

$$\begin{aligned} &1 + \psi(e^*) + \frac{\lambda e^{*2}}{2a} - \lambda e^* - \log(\psi(e^*)\theta'(k)^2) - \frac{2\beta}{A_1} + \lambda B_1 < 0 \\ \iff &\frac{3\lambda e^{*2} - 4\lambda a e^* + \lambda a^2}{2a} - \log(\psi(e^*)) - 2 \log\left(\frac{A_1}{\theta'(0)}\right) < 0 \end{aligned} \quad (2.D.2)$$

Now define  $G(e) = \frac{3\lambda e^{*2} - 4\lambda a e^* + \lambda a^2}{2a} - \log\left(\frac{a + \lambda e^2 - \lambda a e}{a}\right)$  for  $e \in [a/2, a]$ . It is easy to show that

$$G'(e) < 0, \quad \text{for } e \in (a/2, a), \quad \text{and} \quad G'(a) = 0$$

if  $a \geq 1$ . Hence  $G(e)$  attains the maximum at  $e = a$  and it is 0. On the other hand  $\theta'(0) < A_1$  by Lemma 8. This shows (2.D.2). So, the proof is completed.  $\square$

## Proof of Proposition 14

*Proof.* Recall from Theorem 1 and 2

$$B_1 = \frac{1}{\lambda} \left[ \frac{2(\beta - A_1)}{A_1} + 2 \log(A_1) + \frac{\lambda a}{2} \right],$$

$$\theta(0) = \frac{1}{\lambda} \left[ \frac{2(\beta - \theta'(0))}{\theta'(0)} + 2 \log(\theta'(0)) + \frac{\lambda a}{2} \right].$$

Using the above equations, it is straightforward from (2.4.1) that

$$I^f(0) - I^s(0) = \frac{2\beta}{\lambda} \left[ \frac{1}{\theta'(0)} - \frac{1}{A_1} \right]$$

since  $A(k)$  is bounded and  $e(0) = a$ . Hence,  $I^f(0) > I^s(0)$  from Lemma 8.  $\square$

## Proof of Proposition 15

*Proof.* We first rearrange ODE (2.B.14) in the proof of Theorem 2 that is derived from the HJB equation as following. This is an analogue to the 1st best solution ( $A_1 k + B_1$ ):

$$\theta(k) = A(k)k + B(k), \tag{2.D.3}$$

where

$$A(k) := (h - \delta) + \frac{\sigma^2 \theta''(k)}{2\theta'(k)} - \frac{\lambda \sigma^2 \theta'(k)}{2} \left[ 1 - \frac{2}{a} e(k) \psi(e(k)) + \frac{2}{a^2} e(k)^2 \psi(e(k))^2 \right]$$

$$B(k) := \frac{2\beta}{\lambda \theta'(k)} - \frac{2}{\lambda} - \frac{3e(k)^2}{2a} + 2e(k) + \frac{1}{\lambda} \log(\psi(e(k)) \theta'(k)^2)$$

Replacing  $\theta(k)$  of (2.4.1) with (2.D.3), we have

$$I^f(k) - I^s(k) = (A(k) - A_1)k + \frac{2\beta}{\lambda} \left[ \frac{1}{\theta'(k)} - \frac{1}{A_1} \right] - \frac{e(k)^2}{a} + e(k). \quad (2.D.4)$$

From (1) and (3) in Lemma 9, (2.D.3), and (2.D.4), to show (2.4.2) is equivalent to show that  $\lim_{k \rightarrow \infty} (A(k) - A_1)k = -\infty$ . We have

$$A(k) - A_1 = \frac{(h - \delta) \frac{\lambda \sigma^2}{4}}{1 + \frac{4\lambda \sigma^2}{4}} + \frac{\sigma^2 \theta''(k)}{2\theta'(k)} - \frac{\lambda \sigma^2 \theta'(k)}{2} \left[ 1 - \frac{2}{a} e(k) \psi(e(k)) + \frac{2}{a^2} e(k)^2 \psi(e(k))^2 \right].$$

Note that  $\lim_{k \rightarrow \infty} k \theta''(k) = 0$  and  $\lim_{k \rightarrow \infty} \psi(e(k))e(k) = \frac{a}{2}$ . Thus,

$$\lim_{k \rightarrow \infty} (A(k) - A_1)k = \frac{\lambda \sigma^2}{4} \lim_{k \rightarrow \infty} (A_1 - \theta'(k))k = -\infty$$

by (4) of Lemma 9, which shows (2.4.2). Now, it is straightforward to see (2.4.3) since

$$\frac{I^f(k) - I^s(k)}{k} = \frac{\theta(k)}{k} - A_1 + \frac{1}{k} \times [\text{some bounded term}]$$

from (2.4.1). □

## 2.E Proof of Proposition 16

*Proof.* Here we only show the 2nd best solution since the 1st best solution is obtained in the same way. We solve (2.3.9) to get the explicit form of the agent's continuation value. By Ito's lemma, we have

$$\log(-q_t) = \log(-q_0) + \int_0^t (\beta - \theta'(k_s) \psi(e(k_s))) ds$$

$$\begin{aligned}
& - \int_0^t \frac{\lambda\sigma}{a} e(k_s) \theta'(k_s) \psi(e(k_s)) \sqrt{k_s} dW_s \\
& - \frac{1}{2} \int_0^t \frac{\sigma^2 \lambda^2}{a^2} e(k_s)^2 (\theta'(k_s))^2 \psi(e(k_s))^2 k(s) ds
\end{aligned} \tag{2.E.1}$$

Putting (2.E.1) into the optimal consumption in Theorem 2, we can rewrite the instantaneous payment schedule for the agent as the stochastic version (2.5.1) with a single state variable  $k_t$ .  $\square$

## 2.F Proof of Proposition 17 and 18

### Proof of Proposition 17

*Proof.* Now, for proposition 17 we use the results of Lemma 6.1 and 6.2 in Karlyn and Taylor (1981) (henceforce KT). We first define

$$s(\xi) = \exp\left(-\int_{\xi_0}^{\xi} \frac{2a_1 k + 2b_1}{\sigma^2 k} dk\right) \quad \text{and} \quad S(x) = \int_{x_0}^x s(\xi) d\xi,$$

where  $a_1 = h - \delta - A_1$  and  $b_1 = \frac{2(A_1 - \beta)}{\lambda A_1}$ . Note that  $a_1 k + b_1$  is the drift term of the 1st-best capital process. Here,  $\xi_0$  and  $x_0$  are some constants whose value are not important for the proof. Notice that  $a_1 > 0$  and  $b_1 > 0$  by assumption.

Let  $S(0, c_1] = S(c_1) - \lim_{x \downarrow 0} S(x)$  for some  $c_1 > 0$ . We have

$$S(x) = C(\xi_0) \int_{x_0}^x e^{-\frac{2a_1}{\sigma^2} \xi} \xi^{-\frac{2b_1}{\sigma^2}} d\xi$$

with some constant  $C(\xi_0)$  only depending on  $\xi_0$ . If  $2b_1 \geq \sigma^2$ , then  $S(0, c_1] = \infty$ ,



which shows that the capital process never reaches the zero-boundary almost surely by Lemma 6.1 of KT. On the other hands, if  $2b_1 < \sigma^2$ , then  $S(0, c_1]$  is finite, which means the zero-boundary is attracting.

To show the attainability, we define  $m(x)$  and  $\Sigma(l)$  by

$$m(x) = \frac{1}{\sigma^2 x s(x)} \quad \text{and} \quad \Sigma(l) = \int_l^{l_0} \left( \int_\eta^{l_0} m(\xi) d\xi \right) S(\eta) d\eta.$$

Note  $l_0$  is any positive number. Then, we have

$$\Sigma(0) = C(\xi_0, x_0) \int_0^{l_0} \left( \int_\eta^{l_0} e^{\frac{2a_1}{\sigma^2} \xi} \xi^{\frac{2b_1}{\sigma^2} - 1} d\xi \right) e^{\frac{-2a_1}{\sigma^2} \eta} \eta^{-\frac{2b_1}{\sigma^2}} d\eta$$

for some constants  $C(\xi_0, x_0)$  only depending on  $\xi_0$  and  $x_0$ . Since  $-1 < \frac{2b_1}{\sigma^2} - 1 < 0$ , we have  $\left( \int_\eta^{l_0} e^{\frac{2a_1}{\sigma^2} \xi} \xi^{\frac{2b_1}{\sigma^2} - 1} d\xi \right) < M$  bounded by some constant  $M$  as  $\eta \downarrow 0$  and thus

$$\Sigma(0) < MC(\xi_0, x_0) \int_0^{l_0} e^{\frac{-2a_1}{\sigma^2} \eta} \eta^{-\frac{2b_1}{\sigma^2}} d\eta < \infty,$$

which shows that the zero-boundary is attainable by Lemma 6.2 of KT. □

## Proof of Proposition 18

*Proof.* Now, the proof of proposition 18 is analogous to that of proposition 17. we also use the results of Lemma 6.1 and 6.2 in KT. We first define  $s(\xi)$  and  $S(x)$  by

$$s(\xi) = \exp \left( - \int_{\xi_0}^{\xi} \frac{\mu(k)}{\sigma^2 k} dk \right) \quad (\mu(k) \text{ is the drift of } k(t)\text{-process} )$$

$$= \exp \left( - \int_{\xi_0}^{\xi} \frac{2(h - \delta)k - 2\theta(k) - \frac{e(k)^2}{a} + 2e(k) + \frac{2}{\lambda} \log(\theta'(k)^2 \psi(e(k)))}{\sigma^2 k} dk \right)$$

and

$$S(x) = \int_{x_0}^x s(\xi) d\xi = C(\xi_0) \int_{x_0}^x \exp \left( - \frac{2(h - \delta)}{\sigma^2} \xi \right) L(\xi) d\xi,$$

where  $C(\xi_0)$  is a constant only dependent on  $\xi_0$  and

$$L(\xi) = \exp \left( - \int_{\xi_0}^{\xi} \frac{-2\theta(k) - \frac{e(k)^2}{a} + 2e(k) + \frac{2}{\lambda} \log(\theta'(k)^2 \psi(e(k)))}{\sigma^2 k} dk \right).$$

Here,  $\xi_0$  and  $x_0$  are any positive constants. We want to check whether  $S(0)$  is finite or infinite. If  $S(0)$  is infinite, we next need to check if  $\Sigma(0)$  is finite or infinite. (The definition of  $\Sigma(l)$  is in the proof of proposition 17.) By the continuity of  $\theta(k)$ ,  $\theta'(k)$  and  $e(k)$ , we have with some constant  $C_1(\xi_0)$

$$L(\xi) \approx C_1(\xi_0) \xi^{-\frac{-2\theta(0)+a+\frac{2}{\lambda} \log(\theta'(0)^2)}{\sigma^2}} \quad \text{for sufficiently small } \xi > 0.$$

Thus, we can apply the similar argument as in proposition 17 to show the assertion. The important criteria in proposition 17 is the size comparison between  $2b_1 = \frac{4(A_1 - \lambda_1)}{\lambda A_1}$  and  $\sigma^2$ . Similarly here  $-2\theta(0) + a + \frac{2}{\lambda} \log(\theta'(0)^2)$  plays the same role with  $2b_1$ . Finally, notice that  $-2\theta(0) + a + \frac{2}{\lambda} \log(\theta'(0)^2) = \frac{4(\theta'(0) - \beta)}{\lambda \theta'(0)}$ . This completes the proof.  $\square$

# Notes on Reputational Lending and Financial Crises

## 3.1 Introduction

We propose to study financial crises as transitions from "good" to "bad" regimes, or steady states, triggered by random shocks and facilitated by non-interventionist economic policies. By "good" states we mean bubble-like equilibria with strong lending and a mixture of desirable and undesirable properties. These equilibria feature high aggregate income, inflated asset prices, much unsecured borrowing and consumption smoothing, and complete re-allocation of capital. We view such regimes as welfare-desirable or "constrained efficient"; they provide as much welfare as could be furnished by a hypothetical central planner endowed with all the allocative powers of existing markets. On the minus side, bubble states are fragile or dynamically unstable in technical language.

"Bad" states or regimes are fundamental equilibria in which all borrowing is frozen.<sup>1</sup> These states have inferior welfare features among which are low values for asset prices, lending and aggregate income, limited consumption smoothing, and

---

<sup>1</sup>The borrowing is only secured by collateral if a collateral asset is introduced in the model. The same is the logic of the freezing reputational lending under the presence of the collateral asset.

poor allocation of capital. We regard such states as "constrained inefficient"; they could be bettered by a central planner. To make matters worse, fundamental states are dynamically stable and inherently difficult to change.

Our aim is to propose the simplest possible frame of reference that captures the key properties of those two states, as articulated in the previous two paragraphs, to describe the transition from the bubble regime to the fundamental one, and then to ask the obvious policy questions: Are there activist policies that can alter or reverse the stability properties of the two regimes? Is it possible to rule out convergence to the bad state? By which policies?

We consider a two sector production economy with sectoral productivity shocks. The financial market is complete and it includes a productive capital asset and contingent claims. For simplicity we ignore collateral lending, factor accumulation, aggregate total factor productivity (TFP) shocks, and labor markets. Instead we only focus on the sectoral allocation of fixed capital stock through the financial system which reallocates capital from the low marginal product of capital (MPK) sector to the high MPK sector. Rapid reallocation leads to good economic outcomes and slow reallocation due to a freeze of the capital market leads to bad outcomes. The price of capital reflects MPK (or dividends) plus the shadow value of constraints; Equilibrium leverage is endogenously determined so as to rule out default. In this model the crisis is an unwinding of a reputational bubble which destroys unsecured lending. Investors in each sector plan to buy capital for the event of high future MPK. This investment can be financed partly by internal funds (reduced current consumption) and partly by contingent loans. We assume that all loans are secured by reputation which conveys right to participate in future asset mar-

kets; default is punished by perpetual market exclusion.<sup>2</sup> Then, reputation can be considered as a bubble: It is high-valued if the expected leverage is high, but not otherwise. Dynamic complementarity connects future debt/equity limits to current ones. Shrinking debt limits reduce investment demand and capital asset prices, slow down capital reallocation from the low MPK to the high MPK sector, and eventually hurt GDP.

One of the key features of the financial crisis is that the movement of the stock price is more volatile than the fundament (macroeconomic) variable, i.e., output or GDP. Moreover, the magnitude of the price drop is also much higher than that of output. Suppose that the investors have log utility functions. In the baseline model (in section 2), the price of capital and output turn out perfectly correlated, so that the percentage change in the price is exactly the same as the percentage change in output. However, if we introduce public debt, we can generate a higher volatility in the stock price than in output, which results from the investors' optimal portfolio choice between government debt and stocks. We provide a reasonable calibration exercise about this as well.

Nevertheless, if the preference is log and public debt is available, the sum of the market capitalization and the total amount of public debt is still perfectly correlated with the output. This result suggests a surprisingly simple empirical prediction: Big fluctuation in stock prices and small fluctuations in output are equivalent to the following events: The ex-post return on stocks is lower (higher) than the ex-post return on public debt when the ex-post return on stocks is negative (positive). We verify that these phenomena have been accrued during all the recession periods in

---

<sup>2</sup>Default is punished by seizure of collateral in the case of collateral borrowing.

U.S. since 1960.

The closest paper to ours perhaps is Kocherlakota (2009) in the sense that the existence of bubbles in his model is related with production efficiency. He introduced the bubble bursting mechanism inspired by Kiyotaki and Moore (2008) which is based on collateral borrowing and lending. However, Kocherlakota (2009) does not consider unsecured reputational lending. In collateral-asset-based models like his, the bubble component of the price reflects collateral services (shadow value of debt constraints) with a given exogenous leverage. In particular, models with collateral constraints need to assume no dividends from collateral assets and no utility gain from the dividend. Otherwise, the bubble vanishes. In our paper, the unsecured reputational lending is critical to play a role of bubble. Geanakoplos (1997, 2002, 2009) considers an incomplete market collateral general equilibrium models with heterogenous-belief agents. A particular aspect of Geanakoplos (1997, 2002, 2009) is that the model allows for endogenous default. So his models can generate endogenous leverage, but unfortunately do not show the dynamic features of the financial crisis. Azariadis and Kaas (2009) consider the similar framework with this paper, but in their model there is no financial asset.

The rest of paper is as follows. Section 1 introduces the model set-up. Section 2 analyzes a baseline case and its properties. The model is extended for economies with public debt in section 3. Section 4 concludes. All the proofs are in appendix A. Appendix B describes the detailed dynamics of the internal finance case; for example, it covers the transition density function of the internal finance economy.

## 3.2 Set-up

We assume for simplicity that there are two sectors  $i = 1, 2$  in the economy. In this model, we abstract from total factor productivity shocks and only focus on sectoral technology shocks. This means that the technology frontier does not fluctuate, but instead the capital mis-allocation in each sector may play the same role as TFP fluctuations in a business cycle such as in Hsieh and Klenow (2009) and Azariadis and Kaas (2009).

There are two states  $s_t \in \{1, 2\}$ . Define history  $s^t = (s_0, s_1, \dots, s_t)$  and  $S^t$  by the set of all  $s^t$ 's. Let  $s^{t+1} = (s^t, s_{t+1})$ . The transition density  $\pi(s^t|s^\tau)$  for  $t > \tau \geq 0$  is given by

$$\pi(s'|s) = \Pr(s_{t+1} = s' | s_t = s).$$

Each sector is populated by continuum of entrepreneurs and workers with equal mass. Entrepreneurs with a unit measure at each sector have common utility preference:

$$\bar{u}_i = E \left[ \sum_{t=0}^{\infty} \beta^t u(c_i(s^t)) | s_0 \right] = \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t | s_0) u(c_i(s^t)),$$

where  $u(\cdot)$  is a general common utility function with strict concavity. Each entrepreneur has a proprietary constant returns to scale (CRS) production technology  $F(K, N)$ , where  $K$  is capital (or land holding) and  $N$  is labor from workers within the sector. Output  $Y(i, s^t)$  of sector  $i$  at state  $s^t$  is

$$Y(i, s^t) = z(i, s^t) F(k(i, s^t), N(s^t)),$$

where  $z(i, s^t) = 1$  if  $s_t = i$  and  $z_i(s^t) = \alpha \in [0, 1]$  if  $s_t \neq i$ . However, without loss of generality we set  $\alpha = 0$ .  $k(i, s^t)$  is capital of investors in sector  $i$  at history  $s^t$ .

Later we will see the general case. For simplicity, first we assume an  $Ak$ -technology and abstract from labor, i.e.,

$$F(K, N) = K.$$

Furthermore, we assume that capital (land) is non-depreciable and non-producible durable good with fixed supply:

$$k(1, s^t) + k(2, s^t) = 2, \quad \forall s^t \in S^t.$$

Investors produce, consume, and trade financial assets, which will be specified shortly. Let us define  $p(s^t)$  by the price of capital given history  $s^t$  in period  $t$ . The return (yield) on capital,  $Q(i, s^t)$ , is defined by

$$Q(i, s^t) = \frac{z(i, s^t) + p(s^t)}{p(s^{t-1})}. \quad (3.2.1)$$

There two kinds of contingent claims in the market for consumption and capital. Their prices are  $q(s^t, s_{t+1})$  and  $q_K(s^t, s_{t+1})$  with current history  $s^t$  and future state  $s_{t+1}$ . By no-arbitrage, the price of capital is given by

$$p(s^t) = \sum_{s_{t+1}} q_k(s^t, s_{t+1}),$$

The quantities of each claim purchased by investors in sector  $i$  are defined as  $b(i, s^t, s_{t+1})$  and  $k(i, s^t, s_{t+1})$  respectively. Then the budget constraint of agent  $i$



is given by

$$\begin{aligned} c(i, s^t) + \sum_{s_{t+1}} [q_k(s^t, s_{t+1})k(i, s^t, s_{t+1}) + q(s^t, s_{t+1})b(i, s^t, s_{t+1})] \\ = b(i, s^t) + Q(i, s^t)p(s^{t-1})k(i, s^t). \end{aligned} \quad (3.2.2)$$

The expected payoff of Investor  $i$  at history  $s^t$  is defined by

$$v(i, s^t) = E \left[ \sum_{j=0}^{\infty} \beta^j u[c(i, s^{t+j})] \mid s^t \right].$$

Investors have the following debt constraint

$$(1 + \lambda(i, s^t))b(i, s^t) + \lambda(i, s^t)p(s^t)k(i, s^t) \geq 0, \quad (3.2.3)$$

where  $\lambda(i, s^t)$  is the endogenous leverage ratio for agent  $i$  in history  $s^t$ , in other words,  $\lambda(i, s^t)$  is the largest allowable ratio of debt to net worth and endogenously set by market to deter default in all possible histories soon, which will be specified clearly in the definition of the equilibrium later.

### 3.3 Without Public Debt

This section characterizes the properties of the equilibria in the economy where there is no public debt. Let us start with the definition of competitive equilibrium.

**Definition 2.** *A quantity list  $(c(i, s^t), k(i, s^t), b(i, s^t))$ , a price list  $(p(s^t), q(s^t, s_{t+1}), q_k(s^t, s_{t+1}))$ , and leverage ratio  $\lambda(i, s^t)$  are a competitive equilibrium if, for each  $s^t$ , they satisfy*

the following requirements:

(i) HHs maximize payoff  $v(i, s^t)$  subject to debt and budget constraints in (3.2.2) and (3.2.3).

(ii) Markets clear, in other words,

$$\begin{aligned}
 (\text{Goods}) \quad & \sum_{i=1,2} c(i, s^t) = k(h, s^t), \text{ for } h = s_t \\
 (\text{Capital}) \quad & \sum_{i=1,2} k(i, s^t) = 2 \\
 (\text{Consumption}) \quad & \sum_{i=1,2} b(i, s^t) = 0.
 \end{aligned}$$

(iii) Leverage ratios are the largest values  $\lambda(i, s^t)$  consistent with no default in any history, i.e.,

$$E \left[ \sum_{s^{t+j} \succ s^t} \beta^j u[c(i, s^{t+j})] \mid s^t \right] \geq V_{IF}(k(i, s^t)), \quad (3.3.1)$$

where  $V_{IF}(k)$  is the payoff of autarky or internal finance for given capital  $k$ .

The LHS of (3.3.1) is the payoff from solvency given history  $s^t$ ; the RHS is the payoff from default (i.e, from financial autarky) at  $t$  with capital  $k(i, s^t)$ . In autarky, the agent can trade capital but cannot participate the loan market. More precisely, the agent can buy and sell claims on capital but cannot buy and sell claims on consumption.

### 3.3.1 Optimal Bubble States: Reputational Lending

Here we first characterize the optimal bubble states with reputational lending, which is the limited enforcement equilibrium with debt constraints. We first assume that

$$u(1) \geq \theta u(2) + (1 - \theta)u(0) \quad (3.3.2)$$

with  $\theta \equiv (1 - \beta\pi)/(1 + \beta - 2\beta\pi)$ .

**Theorem 4.** *Suppose (3.3.2) holds. Then the equal-treatment allocation is a competitive equilibrium with high unsecured lending and slack debt limits. More precisely, we have perfect consumption smoothing, perfect capital mobility, and maximal aggregate output*

$$\begin{aligned} c(i, s^t) &= 1, \forall (i, s^t) \\ k(i, s^t) &= \begin{cases} 2 & \text{if } i = s_t \\ 0 & \text{if } i \neq s_t \end{cases} \\ y(s^t) &= 2, \forall s^t \end{aligned}$$

and prices for capital, claims on consumption, and claims on capital are given by

$$\begin{aligned} p(s^t) &= \frac{\beta}{1 - \beta} \\ q(s^t, s') &= \beta\pi(s'|s^t) \\ q_k(s^t, s') &= \frac{q(s^t, s')}{1 - \beta}. \end{aligned}$$

The financial portfolio of investors is

$$b(i, s^t) = \begin{cases} \frac{-1}{1-\beta} & \text{if } i = s_t \\ \frac{1}{1-\beta} & \text{if } i \neq s_t \end{cases}$$

and the leverage ratio is

$$\lambda(i, s^t) \geq \frac{1}{2\beta - 1}.$$

*Proof.* See the appendix. □

The efficient equilibrium has perfect consumption smoothing and perfect capital mobility. This efficiency helps to produce the maximum amount of consumption goods so that the current price is the highest. Notice that the yield on equity in the efficient equilibrium is  $Q(i, s^t) = \frac{1}{\beta}$ . The price of capital is, in fact, rewritten as

$$p(s^t) = \frac{\beta}{2(1-\beta)} k(h, s^t), \quad \text{for } s = s_t \tag{3.3.3}$$

where  $k(s^t, s^t)$  is the amount of capital held by the productive investor (or the sector with a good technology shock). This relationship is also true not for the suboptimal case alone and but also for the intermediate transition from the efficient equilibrium to the inefficient equilibrium. This tells that the price of capital and output are perfectly correlated.

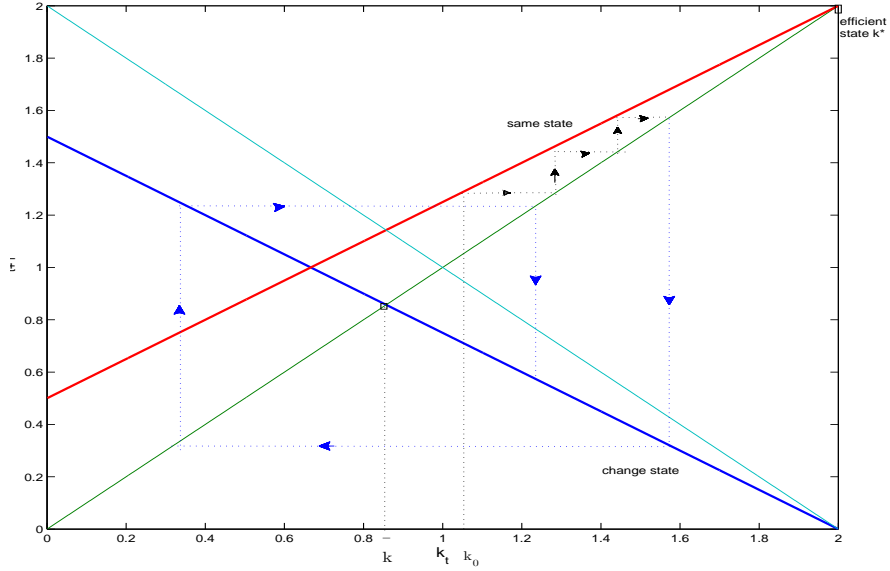


Figure 3.1: Internal Finance Equilibrium Dynamics:  $(\bar{k}, \bar{p}) = (\frac{2\beta}{1-\beta}, \frac{\beta^2}{1-\beta^2}) \approx \frac{1}{2}(k^*, p^*)$  if  $\beta \approx 1$ .

### 3.3.2 Suboptimal No-Bubble States: Internal Finance

Now, we analyze the suboptimal no-bubble state, which does not allow any borrowing and lending, so investors only rely on internal finance. This equilibrium is unintermediated and financially autarkic. No loan is available and no financial asset is traded, but spot market trades exchanging capital for consumption are open.

Decisions generally depend on the entire history of events. Now the budget constraint of investor  $i$  at current history  $s^t$  is

$$c(i, s^t) + \sum_{s_{t+1}} q_k(s^t, s_{t+1})k(i, s^t, s_{t+1}) = W(i, s^t) = \begin{cases} (p(s^t) + 1)k(i, s^t) & \text{if } i = s_t \\ p(s^t)k(i, s^t) & \text{if } i \neq s_t \end{cases}$$

where  $W(i, s^t)$  is the financial wealth of the investor. Investment decision simplifies when  $u(c) = \log(c)$  such as

$$c(i, s^t) = (1 - \beta)W(i, s^t)$$

$$\sum_{s_{t+1}} q_k(s^t, s_{t+1})k(i, s^t, s_{t+1}) = \beta W(i, s^t)$$

We characterize the dynamics of the internal finance equilibrium.

**Theorem 5.** *Let  $k_t$  be the amount of capital owned by a productive sector and  $p_t = p(s^t)$ . Then, we have*

$$p_t = \frac{\beta}{2(1 - \beta)}k_t. \quad (3.3.4)$$

*The aggregate output dynamics of  $(k_t, p_t)$  obeys*

$$k_{t+1} = \begin{cases} 2(1 - \beta) + \beta k_t, & \text{if } s_{t+1} = s_t \text{ (w.p. } \pi), \\ 2\beta - \beta k_t, & \text{if } s_{t+1} \neq s_t \text{ (w.p. } 1 - \pi), \end{cases} \quad (3.3.5)$$

and

$$p_{t+1} = \begin{cases} \beta + \beta p_t, & \text{if } s_{t+1} = s_t \text{ (w.p. } \pi) \\ \frac{\beta^2}{1 - \beta} - \beta p_t, & \text{if } s_{t+1} \neq s_t \text{ (w.p. } 1 - \pi) \end{cases} \quad (3.3.6)$$

*Proof.* See the appendix. □

The internal finance equilibrium is volatile with invariant set  $[0, 2]$  and unknown asymptotic distribution. (See the figure 3.1.) If the sectoral shocks have positive persistence ( $\pi \approx 1$ ), then economy spends much time on a path to the efficient state

$k^* = 2$  and  $p^* = \frac{\beta}{1-\beta}$ . If the sectoral shocks have negative persistence ( $\pi \approx 0$ ), then economy spends much time gyrating around  $\bar{k} = \frac{2\beta}{1+\beta}$  and  $\bar{p} = \frac{\beta^2}{1-\beta^2}$ .

### 3.3.3 Comparison

Here we compare the allocations at the efficient and the inefficient states. Let index *eff* represent the efficient equilibrium and index *in* represent the internal finance equilibrium. Then, for output per capita,  $Y_t$ , we have

$$Y^{eff} = 1 > Y^{in}(k_t), \quad \forall t$$

where  $k_t$  is capital held by the productive sector. For price of capital,

$$P^{eff} = \frac{\beta}{1-\beta} > P^{in}(k_t).$$

On the other hand, the baseline model shows that price and output are perfectly correlated for all periods regardless of the states of the economy.

**Proposition 21.** *The percent change in output is the same as the percent change in price, in other words,*

$$\frac{Y_t - Y_{t+1}}{Y_t} = \frac{P_t - P_{t+1}}{P_t}, \forall t.$$

This means that the percentage change in GDP is the same as the percentage change in prices, which is not consistent with the data. The above proposition comes from the fact that in all equilibria - in the efficient state or the internal

equilibrium or even during the transition from the efficient to the inefficient state - we have the following relationship

$$p(s^t) = \frac{\beta}{1 - \beta} Y(s^t)$$

as seen in (3.3.3) and (3.3.4).

### 3.4 Public Debt

If we introduce public debt in the basic model, then there is only one change in Definition 2 of the competitive equilibrium. Let  $2B_t$  is the total amount of debt issued by the government and held by the public. Now the market clearing condition for consumption becomes

$$\sum_i b(i, s^t) = 2B_t,$$

instead of  $\sum_i b(i, s^t) = 0$  in the definition of the equilibrium. Then, the government debt constraint is

$$q_t B_{t+1} = B_t,$$

where  $q_t$  is the price of the debt, which means the government cannot issue debt arbitrarily. In addition, there is no government spending in this framework.

The previous analysis is still valid under log utility assumption. The only difference is the following equation with respect to the price dynamics.

$$P_t + B_t = \frac{\beta}{2(1 - \beta)} y(k_t) \tag{3.4.1}$$



for  $y(\cdot)$  is the GDP and  $k_t$  is again capital held by the productive sector. For the Ak-model, we have  $y(k_t) = k_t$ , where  $k_t$  is the holding of capital in the high productivity sector with  $z(i, s^t) = 1$ . Before going any further, we plot Figure 3.2, market value of public debt relative to capital income at the bottom, stock market capitalization relative to capital income in the middle, and the sum of two curves at the top in using U.S. data from last 50 years. We can see that even in the low frequency data stock market value has fluctuated relatively strongly while the debt held by the public has not fluctuated much.

### 3.4.1 Optimal Bubble States: Reputational Lending

The efficient state in the presence of government debt is basically the same as the efficient state without government debt. In other words,  $B_t = 0$  in the efficient state. This is because government debt is redundant: investors can borrow and lend to finance their own projects without any frictions. Therefore, basically, we have the same results as in Theorem 4. Note that the dynamics of price and capital is

$$p(s^t) = \frac{\beta}{2(1-\beta)}k_t = \frac{\beta}{1-\beta}, \quad (3.4.2)$$

since  $k_t = 2$  for all  $t$ .

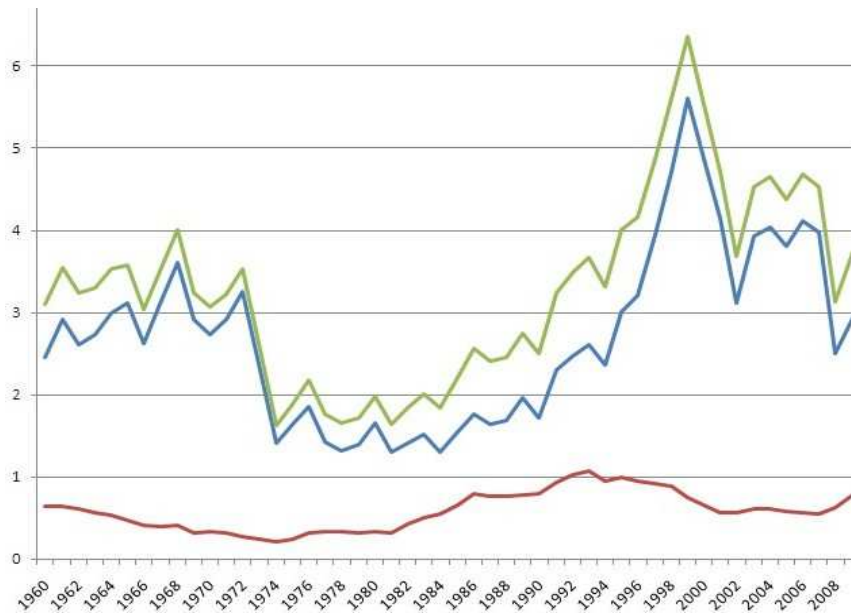


Figure 3.2: The curve at the top is the sum of two curves. All data are annual (from 1960 to 2009). The source for the market value of public debt (nominal) excluding debt held by government (e.g., the Federal Reserve and the Social Security Trust Fund) is obtained from CRSP. The nominal stock market capitalization is obtained from the Federal Reserve's Flow of Funds Accounts (Table L.213, line marked as market value of domestic corporations).

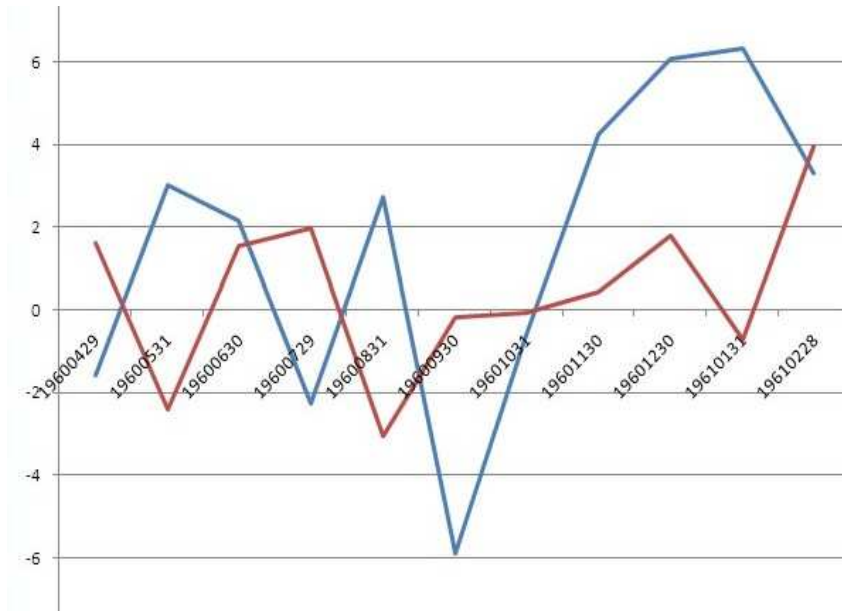


Figure 3.3: The recession between 04/1960 and 02/1961 (16 months): The blue line plots represents the percent change in stock market capitalization, i.e.,  $(P_{t+1} - P_t)/P_t$  and the red line represents the percent change in the government debt held by the public, i.e.,  $(B_{t+1} - B_t)/B_t$ .

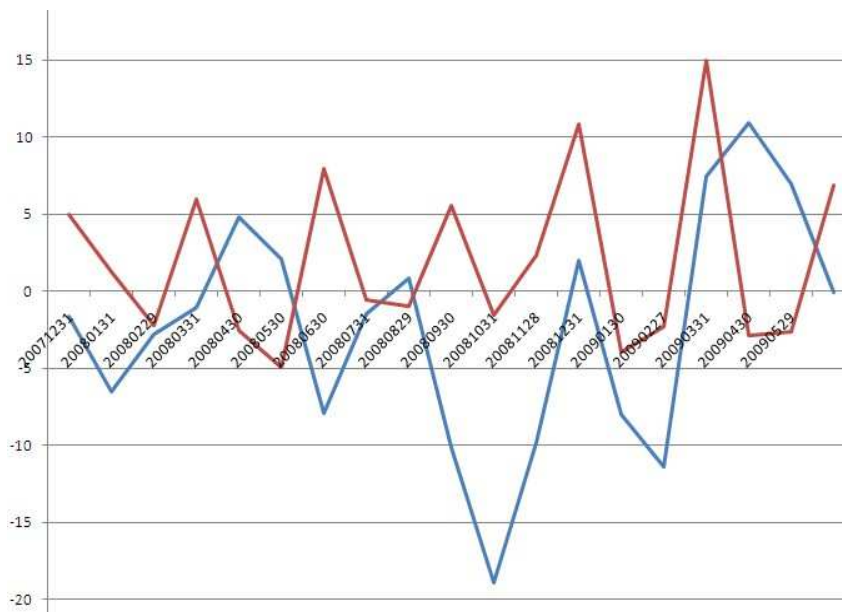


Figure 3.4: The recession between 12/2007 and 06/2009 (18 months): The blue line represents the percent change in stock market capitalization, i.e.,  $(P_{t+1} - P_t)/P_t$  and the red line represents the percent change in government debt held by the public, i.e.,  $(B_{t+1} - B_t)/B_t$ .

Recession Period	% change in $P_t$	% change in $B_t$	consistency
04/1960	-1.57	1.63	yes
05/1960	3.03	-2.40	yes
06/1960	2.14	1.53	yes
07/1960	-2.25	1.96	yes
08/1960	2.73	-3.04	yes
09/1960	-5.88	-0.20	yes
10/1960	-0.59	-0.08	yes
11/1960	4.25	0.44	yes
12/1960	6.06	1.81	yes
01/1961	6.34	-0.73	yes
02/1961	3.30	3.96	-

Table 3.1: The recession between 04/1960 and 02/1961 (11 months): This table shows that the analysis in section 3.4.2 is quite consistent with the data. In fact, the data for all recessions defined by NBER since 1960 show the same consistency.

### 3.4.2 Suboptimal No-Bubble States: Short Sale Constraints

In the inefficient state when no private borrowing is allowed, there is a role of public debt. Since public debt is available instead of private lending, the budget constraint of investors in sector  $i$  is

$$\begin{aligned}
c(i, s^t) + \sum_{s_{t+1}} [q_k(s^t, s_{t+1})k(i, s^t, s_{t+1}) + q(s^t, s_{t+1})b(i, s^t, s_{t+1})] \\
= b(i, s^t) + [p(s^t) + z(i, s^t)]k(i, s^t).
\end{aligned}$$

with

$$b(i, s^t, s_{t+1}) \geq 0, \quad \forall s^t, s_{t+1}, i \tag{3.4.3}$$

Recession Period	% change of $P_t$	% change of $B_t$	consistency
12/2007	-1.673057558	5.006265543	yes
01/2008	-6.509446333	1.261840481	yes
02/2008	-2.803570454	-2.227989569	yes
03/2008	-1.067556935	5.975709145	yes
04/2008	4.838793371	-2.504127775	yes
05/2008	2.088034702	-4.94043583	yes
06/2008	-7.902467504	7.92938367	yes
07/2008	-1.475947109	-0.592132354	yes
08/2008	0.887202423	-0.949081126	yes
09/2008	-10.17136462	5.55515366	yes
10/2008	-18.91537163	-1.552650689	yes
11/2008	-9.839817725	2.339153746	yes
12/2008	1.988322784	10.82486971	-
01/2009	-8.024960209	-3.905810843	yes
02/2009	-11.3864756	-2.306937452	yes
03/2009	7.451325519	14.98809852	-
04/2009	10.891628	-2.859539126	yes
05/2009	6.925002044	-2.630704213	yes
06/2009	-0.041403965	6.905923561	yes

Table 3.2: The recession between 12/2007 and 06/2009 (18 months): This table shows that the analysis in section 3.4.2 is quite consistent with the data. In fact, the data for all the recessions defined by NBER since 1960 show the same consistency.

The market clearing condition is given by

$$b(1, s^t) + b(2, s^t) = 2B_t, \quad \forall s^t,$$

which is, in fact, in equilibrium,

$$b(i, s^t) = 2B_t \quad \text{if } i \neq s_t.$$

since the investor cannot borrow against the high productive state for the next period. The government budget condition (or feasibility of public debt) is

$$2B_t = \sum_i \sum_{s_{t+1}} q(s^t, s_{t+1})b(i, s_t, s_{t+1}), \quad (3.4.4)$$

which is, in fact, in equilibrium, by the same reason

$$2B_t = q(s^t, s')b(i, s_t, s') + q(s^t, s)b(j, s_t, s), \text{ for } s_t = s, i = s, j = s'.$$

**Theorem 6.** *Let  $p_t = p(s^t)$ . Summing over  $i = 1, 2$ , we have*

$$p_t + B_t = \frac{\beta}{2(1-\beta)}k_t. \quad (3.4.5)$$

*The dynamics of the price is given by*

$$p_{t+1} = \begin{cases} \beta + \beta p_t & \text{if } s_{t+1} = s_t \\ \frac{\beta^2}{1-\beta} - \beta p_t & \text{if } s_{t+1} \neq s_t \end{cases} \quad (3.4.6)$$

*Proof.* See the appendix. □

The price dynamics with public debt as in (3.4.6) is exactly same as the price dynamics without public debt as in (3.3.6).

Although investors cannot borrow in order to finance the future good project, they can save through government debt. It is easy to see that consumption and investment when public debt is available are better than when public debt is unavailable (as in section 3.3.2) in terms of welfare.

However, the existence of the public debt makes the crucial difference in the price dynamics. We divide the price dynamics into two cases: (i) when  $P_t > P_{t+1}$  and (ii) when  $P_t < P_{t+1}$  in the ex-post sense.

First, when  $P_t > P_{t+1}$ , there is change in states and  $k_t > k_{t+1}$ , we have a GDP drop such that

$$\frac{\Delta y_t}{y_t} = \frac{y_t - y_{t+1}}{y_t} = \frac{(P_t - P_{t+1}) + (B_t - B_{t+1})}{P_t + B_t}. \quad (3.4.7)$$

Therefore, we want to have

$$\frac{P_t - P_{t+1}}{P_t} > \frac{(P_t - P_{t+1}) + (B_t - B_{t+1})}{P_t + B_t} = \frac{y_t - y_{t+1}}{y_t}. \quad (3.4.8)$$

The above inequality is equivalent to

$$\frac{B_{t+1}}{P_{t+1}} > \frac{B_t}{P_t}. \quad (3.4.9)$$

equivalently

$$\frac{B_{t+1} - B_t}{B_t} > \frac{P_{t+1} - P_t}{P_t}. \quad (3.4.10)$$

This means that the market capitalization drops much more than the market value of the debt held by the public during the crisis time when there is change in states. In this case, the important point is the ex-post return, i.e., the right hand side of (3.4.10) is negative. In other words, when the (ex-post) return on stocks is negative, the bond return is greater than the stock return.

Reversely, if the price goes up during recession (no change in states), i.e.,  $k_{t+1} >$

$k_t$ , all the inequalities above are reversed:

$$\frac{B_{t+1} - B_t}{B_t} < \frac{P_{t+1} - B_t}{P_t}, \quad (3.4.11)$$

which means that when the (ex-post) stock return is positive, the stock return is greater than the bond return. Notice that (3.4.10) and (3.4.11) are very consistent with the data, in particular, the data in recession periods. For example, see Table 3.1 and Table 3.2. The data in two tables show that our theory is reasonable. We just picked two recessions. However, not only these two recessions but also all other recessions show almost same results with (3.4.10) and (3.4.11)

For comparison, we analyze the simplest case with deterministic shocks, i.e.,  $\pi = 1$ .

$$(z(1, s^t), z(2, s^t)) \equiv (z(t, t), z(2, t)) = \begin{cases} (0, 1), & t = 0, 2, 4, \dots \\ (1, 0), & t = 1, 3, 5, \dots \end{cases}$$

In this case, investors budget constraints can be written as

$$c(i, t) + p(t)k(i, t + 1) + q(t)b(i, t + 1) = b(i, t) + (z(i, t) + p(t))k(i, t),$$

$$\text{with } b_{t+1}^i \geq 0, \quad \forall i, t. \quad (3.4.12)$$

Equation (3.4.12) means that investors cannot borrow in time  $t$  although they have a good investment opportunity in the next period ( $z(i, t + 1) = 1$ ). However, they can accumulate government debt for the consumption smoothing against the bad shock in the next period ( $z(i, t + 1) = 0$ ). Thus, the market clearing condition in



the deterministic case implies

$$b(H, t) = 0 \quad \text{and} \quad b(L, t) = 2B_t.$$

Notice that there is no stationary density in the general case, i.e.,  $\pi \neq 1$ . However, we have the periodic steady state in the deterministic case. See the following theorem.

**Theorem 7** (Suboptimal Equilibrium: Deterministic Case with Public Debt). *Suppose  $\pi = 1$  and the public debt is available.*

(i) *There exists an unique internal finance stationary state such that*

$$(c_t^i, k_t^i) = \begin{cases} (\hat{c}_H, \hat{k}), & \text{if } z(i, t) = 1 \\ (\hat{c}_L, 2 - \hat{k}), & \text{if } z(i, t) = 0 \end{cases}$$

(ii) *Given  $\hat{b}$ , the vector  $(\hat{c}^H, \hat{c}^L, \hat{k})$  is the solution to*

$$1 + \frac{1}{\hat{p}} = \frac{u'(\hat{c}_L)}{\beta u'(\hat{c}_H)},$$

$$\hat{c}_H = (1 + 2\hat{p})\hat{k} - 2\hat{p} - \hat{b} \quad (\text{or } \hat{c}_L = 2\hat{p} - 2\hat{p}\hat{k} + \hat{b}),$$

$$\hat{c}_H + \hat{c}_L = \hat{k},$$

*where the asset price,  $\hat{p}$ , and the public debt price,  $\hat{q}$ , are given by*

$$\hat{p} = \frac{\beta^2}{1 - \beta^2} \quad \text{and} \quad \hat{q} = 1.$$

(iii) If  $u(c) = \log c$ , then

$$\begin{aligned}\hat{c}_H &= \frac{2\beta}{(1+\beta)^2} + \frac{(1-\beta)}{\beta(1+\beta)}\hat{b} \\ \hat{c}_L &= \frac{2\beta^2}{(1+\beta)^2} + \frac{(1-\beta)}{1+\beta}\hat{b} \\ \hat{k} &= \frac{2\beta}{1+\beta} + \frac{(1-\beta)}{\beta}\hat{b}\end{aligned}$$

*Proof.* See the appendix. □

There are several comments on Theorem 7. Notice that  $\hat{b} = 2B_t$  by market clearing in equilibrium, meaning that the total market value of the public debt is just fixed as soon as the economy enters the internal finance stage while it has been growing during the transition from the efficient state to the current inefficient state. Its value basically depends on the amount of consumption saving of the investor who has the good technology. The government debt can be considered as the saving technology for the investors since the private borrowing and lending market is frozen. This implies that if the government issues more debt above this equilibrium amount, the price of debt drops so that the total market value is fixed, which is the similar phenomenon as the inflation occurs when the government prints more money. On the other hand, during the transition from the efficient equilibrium to the internal finance equilibrium, the gross interest rate  $1/q_t$  has been decreased in average (although it can fluctuate) and it finally reaches to 1.

Secondly, let us compare  $\Delta CI$  the percent change in capital income (or output) with  $\Delta P$  that in price from the efficient state to the inefficient state as in the

following proposition. We first define

$$\Delta CI := \frac{k^e - \hat{k}}{k^e} \quad \text{and} \quad \Delta P := \frac{p^e - \hat{p}}{p^e} \quad (3.4.13)$$

Note that  $k^e = 2$  and  $p^e = \frac{\beta}{1-\beta}$  for the efficient state.

**Proposition 22.** *The percent change in capital income (or equivalently output) from the efficient state to the inefficient state is lower than the percent change in price, in other words, we have*

$$\Delta CI < \Delta P \quad (3.4.14)$$

The proof follows from the simple algebra:

$$\Delta CI = \frac{1}{1+\beta} - \frac{1-\beta}{\beta} B \quad \text{and} \quad \Delta P = \frac{1}{1+\beta}.$$

Thus, the inequality in (3.4.14) becomes deeper if B is bigger. Let me apply a very simple calibration analysis. Let  $\beta = 0.9$ . Then,  $\Delta P = 55.56$ , which means that the price drops by about 56%. In this case, in order for having 27% drop of capital income we should obtain  $B = 2.3$ , which is the public debt per capita. Notice that the curve in the bottom of Figure 3.2 shows the market value of the U.S. public debt as a fraction of GDP since 1960. The average value during the last 50 years is 0.59, which is corresponds to 1.77 as a fraction of capital income. This value seems close enough to the calibrated value 2.3 although our model is simple enough.

### 3.5 Conclusion

A model of the financial crisis is studied. In particular, crisis is viewed as the transition from an efficient and unstable state to an inefficient and stable state by using a two sector economy with sector-specific shocks. The main driving force of this transition is the unsecured reputational lending. If we add collateral borrowing and/or a labor market in this model, we may generate quantitatively more reasonable results. Introducing public debt generates volatile stock prices during the crisis time. In particular, this theory tells that the stock price is more volatile than output during recession is equivalent to that the ex-posed stock return is lower (higher) than the return on public debt when the ex-posed stock return realized negative (positive). This empirical conjecture turns out to be fairly consistent with monthly data for all recession periods since 1960, which strongly suggests the importance of public debt.

### 3.6 References

- 1 Alvarez, F. and U. Jermann (2002), "Efficiency, Equilibrium, and Asset Pricing with Risk of Default", *Econometrica*, Vol. 68, pp. 775-797.
- 2 Azariadis, C. and L. Kaas (2006), "Credit Market Development, Growth and Volatility", *working paper*
- 3 Azariadis, C. and L. Kaas (2007), "Asset Price Fluctuations without Aggregate Shocks", *Journal Economic Theory*, Vol. 136, pp. 126-143.
- 4 Azariadis, C. (2008), "Financial Frictions in Dynamic Economics", Lecture Note, *Washington University in St. Louis*
- 5 Kocherlakota, N. (1992), "Bubbles and constraints on debt accumulation", *Journal of Economic Theory* 57, 245-256.
- 6 Kocherlakota, N. (2009), "Bursting Bubbles: Consequences and Cures", *working paper*
- 7 Kiyotaki, N., and J. Moore (2008), "Liquidity, business cycles, and monetary policy", *working paper*
- 8 Kehoe, P. and F. Perri (2002), "International Business Cycles with Endogenous Incomplete Markets", *Econometrica*, Vol. 70, pp. 907 - 928.
- 9 Kehoe, P. and F. Perri (2004), "Competitive equilibria with limited enforcement", *Econometrica*, Vol. 119, pp. 184 - 206.
- 10 Santos, M., and M. Woodford (1997), "Rational asset pricing bubbles", *Econometrica* 65, 19-57.

## Appendices

### 3.A Proofs of Chapter 3

#### A.1 Proof of Theorem 4

*Proof.* The proof of Theorem 4 is easy to understand by characterizing the perfect enforcement equilibrium without debt constraints. Ignoring (3.2.3) and assuming equal initial wealth for all agents leads to the unique stationary equilibrium as specified in the theorem. More precisely, without any borrowing friction, the investors can have perfect consumption, i.e.,  $c(i, s^t) = 1$  for all  $i$  and  $s^t$  and complete reallo-

cation of capital, i.e.,  $k(i, s^t) = \begin{cases} 2 & \text{if } i = s_t \\ 0 & \text{if } i \neq s_t \end{cases}$

. Using budget constraint (3.2.2) and FOC, we have

$$\frac{q(s^t, s')}{\pi(s'|s^t)} = \frac{q_k(s^t, s')}{\pi(s'|s^t)p(s^t)Q(s^t, s')} = \frac{\beta u'[c(i, s^t, s')]}{u'[c(i, s^t)]} = \beta.$$

Lastly, it is easy to see this allocation and the leverage ratio  $\lambda(i, s^t)$  under the condition (3.3.2) satisfy the solvency constraint, (3.3.1). This completes the proof.

□

## A.2 Proof of Theorem 5

*Proof.* First order conditions imply that

$$q_k(s^t, s_{t+1})u'(c(i, s^t)) = \beta(p(s^t, s_{t+1}) + z(i, s^t, s_{t+1}))u'(c(i, s^t, s_{t+1}))\pi(s_{t+1}|s^t), \quad \forall s^t, i \quad (3.A.1)$$

If the preference is log-utility, then

$$\sum_{s'} q_k(s^t, s')k(i, s^t, s') = \beta[p(s^t) + z(i, s^t)]k(i, s^t), \quad \forall s^t, i.$$

Note  $k_t$  be the amount of capital held by the productive sector at  $t$  and  $p_t = p(s^t)$ .

Summing the above equation over  $i = 1, 2$ ,

$$2p_t = 2 \sum_{s'} q_k(s^t, s') = \beta\{(p_t + 1)k_t + p_t(2 - k_t)\},$$

equivalently we have

$$p_t = \frac{\beta}{2(1 - \beta)}k_t.$$

Now notice that

$$\begin{aligned} c(i, s^t) &= (1 - \beta)[p_t + z(i, s^t)]k(i, s^t) \\ &= \begin{cases} (1 - \beta)(p_t + 1)k_t & \text{if } i = s_t \\ (1 - \beta)p_t(2 - k_t) & \text{if } i \neq s_t \end{cases} \end{aligned} \quad (3.A.2)$$

Putting (3.A.2) into the first order condition (3.A.1), we have

$$\frac{q_k(s^t, s_{t+1})}{(1-\beta)(p_t+1)k_t} = \frac{\beta(p_{t+1}+1)}{(1-\beta)(p_{t+1}+1)k_{t+1}} \pi(s_{t+1}|s^t) \quad \text{if } s_{t+1} = s_t \quad (3.A.3)$$

$$\frac{q_k(s^t, s_{t+1})}{(1-\beta)(p_t+1)k_t} = \frac{\beta p_{t+1}}{(1-\beta)p_{t+1}(2-k_{t+1})} \pi(s_{t+1}|s^t) \quad \text{if } s_{t+1} \neq s_t \quad (3.A.4)$$

$$\frac{q_k(s^t, s_{t+1})}{(1-\beta)p_t(2-k_t)} = \frac{\beta(p_{t+1}+1)}{(1-\beta)(p_{t+1}+1)k_{t+1}} \pi(s_{t+1}|s^t) \quad \text{if } s_{t+1} \neq s_t \quad (3.A.5)$$

$$\frac{q_k(s^t, s_{t+1})}{(1-\beta)p_t(2-k_t)} = \frac{\beta p_{t+1}}{(1-\beta)p_{t+1}(2-k_{t+1})} \pi(s_{t+1}|s^t) \quad \text{if } s_{t+1} = s_t \quad (3.A.6)$$

The price of arrow security in (3.A.3) should be the same as that in (3.A.6). The price in (3.A.4) should be the same as (3.A.5). Therefore,

$$\begin{aligned} \frac{(p_t+1)k_t}{k_{t+1}} &= \frac{p_t(2-k_t)}{2-k_{t+1}} \quad \text{if } s_{t+1} = s_t \quad \text{and} \\ \frac{(p_t+1)k_t}{2-k_{t+1}} &= \frac{p_t(2-k_t)}{k_{t+1}} \quad \text{if } s_{t+1} \neq s_t, \end{aligned}$$

Equivalently we have

$$k_{t+1} = \begin{cases} 2(1-\beta) + \beta k_t & \text{if } s_{t+1} = s_t \\ 2\beta - \beta k_t & \text{if } s_{t+1} \neq s_t \end{cases}$$

This completes the proof. □



### A.3 Proof of Theorem 6

*Proof.* Given  $k(i, s^t, s') > 0$ , we have the same first order conditions as (3.A.1) with respect to Arrow securities for capital. For all  $s^t, s_{t+1}$  and  $i = 1, 2$ ,

$$q_k(s^t, s_{t+1})u'(c(i, s^t)) = \beta(p(s^t, s_{t+1}) + z(i, s', s_{t+1}))u'(c(i, s^t, s_{t+1}))\pi(s'|s^t). \quad (3.A.7)$$

With respect to public debt holding notice that (3.4.3) is binding for  $s_{t+1} = s_t$ , in other words, the investor cannot borrow for the high productive state in the next period. Therefore, the first order conditions for  $b(i, s^t, s')$  are

$$q(s^t, s')u'(c(i, s^t)) = \beta u'(c(i, s^t, s'))\pi(s'|s^t), \quad \text{if } i \neq s' \text{ and } i = s_t, \quad (3.A.8)$$

$$q(s^t, s')u'(c(i, s^t)) = \beta u'(c(i, s^t, s'))\pi(s'|s^t), \quad \text{if } i = s' \text{ and } i \neq s_t, \quad (3.A.9)$$

putting  $s' = s_{t+1}$  for notational convenience.  $q(s^t, s')$  in (3.A.8) represents the price of Arrow security when there is no change in state between today and tomorrow.  $q(s^t, s')$  in (3.A.9) represents the price of Arrow security when there is change in state. The investor in (3.A.8) has high productivity today and low productive tomorrow and the investor in (3.A.9) has low productivity today and tomorrow.

The log utility assumption gives

$$\sum_{s'} [q_k(s^t, s')k(i, s^t, s') + q(s^t, s')b(i, s^t, s')] = \beta[p(s^t) + z(i, s^t)]k(i, s^t), \quad \forall s^t, i.$$

Let  $k_t$  be the amount of capital held by the productive sector at  $t$ . Let  $p_t = p(s^t)$ .

Summing over  $i = 1, 2$ , we have the first required result, i.e.,

$$p_t + B_t = \frac{\beta}{2(1 - \beta)} k_t.$$

Let  $b_t$  be the amount of debt held by the unproductive sector at the beginning of time  $t$ , i.e.,  $b_t = b(i, s^t)$  for  $i \neq s_t$ . Now notice that

$$\begin{aligned} c(i, s^t) &= (1 - \beta)(b(i, s^t) + p_t + z(i, s^t))k(i, s^t) \\ &= \begin{cases} (1 - \beta)(p_t + 1)k_t & \text{if } s_{t+1} = s_t \\ (1 - \beta)[b_t + p_t(2 - k_t)] & \text{if } s_{t+1} \neq s_t \end{cases} \end{aligned} \quad (3.A.10)$$

Then, putting (3.A.10) into (3.A.7), we get

$$\frac{q_k(s^t, s')}{(1 - \beta)(p_t + 1)k_t} = \frac{\beta(p_{t+1} + 1)}{(1 - \beta)(p_{t+1} + 1)k_{t+1}} \pi(s'|s^t) \quad \text{if } s' = s_t \quad (3.A.11)$$

$$\frac{q_k(s^t, s')}{(1 - \beta)(p_t + 1)k_t} = \frac{\beta p_{t+1}}{(1 - \beta)[b_{t+1} p_{t+1} (2 - k_{t+1})]} \pi(s'|s^t) \quad \text{if } s' \neq s_t \quad (3.A.12)$$

$$\frac{q_k(s^t, s')}{(1 - \beta)[b_t + p_t(2 - k_t)]} = \frac{\beta(p_{t+1} + 1)}{(1 - \beta)(p_{t+1} + 1)k_{t+1}} \pi(s'|s^t) \quad \text{if } s' \neq s_t \quad (3.A.13)$$

$$\frac{q_k(s^t, s')}{(1 - \beta)[b_t + p_t(2 - k_t)]} = \frac{\beta p_{t+1}}{(1 - \beta)[b_{t+1} + p_{t+1}(2 - k_{t+1})]} \pi(s'|s^t) \quad \text{if } s' = s_t \quad (3.A.14)$$

The price of arrow security in (3.A.11) should be the same as that in (3.A.14). The price in (3.A.12) should be the same as (3.A.13). Notice that if  $b_t = 0$  and  $b_{t+1} = 0$ , in other words, if public debt is unavailable, then (3.A.11) – (3.A.14) collapse to (3.A.3) – (3.A.6), respectively.

Therefore,

$$\begin{aligned} \frac{(p_t + 1)k_t}{k_{t+1}} &= \frac{p_{t+1}[b_t + p_t(2 - k_t)]}{b_{t+1} + p_{t+1}(2 - k_{t+1})} && \text{if } s' = s_t \quad \text{and} \\ \frac{p_{t+1}(p_t + 1)k_t}{b_{t+1} + p_{t+1}(2 - k_{t+1})} &= \frac{b_t + p_t(2 - k_t)}{k_{t+1}} && \text{if } s' \neq s_t. \end{aligned}$$

Thanks to (3.4.5), we can rewrite the above equations as

$$\begin{aligned} \frac{(p_t + 1)k_t}{k_{t+1}} &= \frac{p_{t+1} \left[ \frac{\beta}{1-\beta} - p_t \right] k_t}{\left[ \frac{\beta}{1-\beta} - p_{t+1} \right] k_{t+1}} && \text{if } i = s' \quad \text{and} \\ \frac{p_{t+1}(p_t + 1)k_t}{\left[ \frac{\beta}{1-\beta} - p_{t+1} \right] k_{t+1}} &= \frac{\left[ \frac{\beta}{1-\beta} - p_t \right] k_t}{k_{t+1}} && \text{if } i \neq s'. \end{aligned}$$

Equivalently we have the dynamics of the price

$$p_{t+1} = \begin{cases} \beta + \beta p_t & \text{if } i = s' \\ \frac{\beta^2}{1-\beta} - \beta p_t & \text{if } i \neq s' \end{cases}$$

□

## A.4 Proof of Theorem 7

*Proof.* First we have the following first order conditions:

$$\begin{aligned} u'(c_t^H)p_t &= \beta u'(c_{t+1}^L)p_{t+1} \\ u'(c_t^H)q_t &= \beta u'(c_{t+1}^L) \end{aligned}$$

$$u'(c_t^L)p_t = \beta u'(c_{t+1}^H)(1 + p_{t+1}),$$

where  $c_t^H$  and  $c_t^L$  represent the consumption of the investor who has a good productivity shock and a bad shock at time  $t$ , respectively. Then at the steady state  $(c_t^H, c_t^L) = (\hat{c}^H, \hat{c}^L)$ , we have

$$q_t = \hat{q} = 1 \quad \text{and} \quad p_t = \hat{p} = \frac{\beta^2}{1 - \beta^2}.$$

and

$$c^L = \beta c^H. \tag{3.A.15}$$

Notice that the investor who will have a good shock tomorrow is rationed today, in other words,  $b_{t+1}^H = 0$ . Thus the budget constraint in the stationary periodic equilibrium is

$$\hat{c}^H + \hat{p}(2 - \hat{k}) + \hat{b} = (1 + \hat{p})\hat{k} \tag{3.A.16}$$

$$\text{and} \quad \hat{c}^L + \hat{p}\hat{k} = \hat{b} + (0 + \hat{p})\hat{k} \tag{3.A.17}$$

The market clearing implies

$$\hat{c}^H + \hat{c}^L = \hat{k} \quad \text{and} \quad \hat{b} = 2B_t.$$

For the log-utility case, the total investment at each period is always the  $\beta$  fraction of the financial wealth of the investor, i.e.,

$$\hat{p}(2 - \hat{k}) + \hat{b} = \beta(1 + \hat{p})\hat{k} \quad \text{and} \quad \hat{p}\hat{k} = \beta(\hat{b} + \hat{p}(2 - \hat{k})).$$

Equivalently we have

$$\hat{k} = \frac{2\beta}{1+\beta} + \frac{1-\beta}{\beta}\hat{b}. \quad (3.A.18)$$

Putting (3.A.18) into (3.A.16) and (3.A.17), we have the required solutions for (iii).  $\square$

### 3.B Internal Finance Dynamics

In this appendix we characterize the dynamics and the probability density function of the price and capital in the inefficient state. We only focus on the case where there is no public debt. Recall the process  $s_t \in \{1, 2\}$  obeys a simple symmetric Markov process with conditional prob:

$$\pi(s'|s) = \Pr(s_{t+1} = s' | s_t = s), \quad \pi = \pi(1|1) = \pi(2|2)$$

Let us write state history  $s^t = (s_0, \dots, s_t)$ .

Let  $\beta \in (0, 1)$ . Let  $k_0 \in [0, 2]$ . Under this transition probability, recall the dynamics of  $(k_t, p_t)$  is given by

$$k_{t+1} = \begin{cases} 2(1-\beta) + \beta k_t, & \text{if } s_{t+1} = s_t \text{ (w.p. } \pi), \\ 2\beta - \beta k_t, & \text{if } s_{t+1} \neq s_t \text{ (w.p. } 1-\pi). \end{cases} \quad (3.B.1)$$

and  $p_t = \frac{\beta}{2(1-\beta)}k_t$ , i.e.,

$$p_{t+1} = \begin{cases} g^1(p_t) := \beta + \beta p_t, & \text{if } s_{t+1} = s_t \text{ (w.p. } \pi), \\ g^2(p_t) := \frac{\beta^2}{1-\beta} - \beta p_t, & \text{if } s_{t+1} \neq s_t \text{ (w.p. } 1 - \pi). \end{cases} \quad (3.B.2)$$

The recursive relation between the cumulative density of capital,  $F_t^k$  at time  $t$  and  $F_{t+1}^k$  at time  $t + 1$  can be derived in the following recursive.

$$\begin{aligned} F_{t+1}^k(x) &= \Pr(k_{t+1} \leq x) \\ &= \Pr(k_{t+1} \leq x | s_{t+1} = s_t) \Pr(s_{t+1} = s_t) + \Pr(k_{t+1} \leq x | s_{t+1} \neq s_t) \Pr(s_{t+1} \neq s_t) \\ &= \pi \Pr(2(1-\beta) + \beta k_t \leq x | s_{t+1} = s_t) + (1-\pi) \Pr(2\beta - \beta k_t \leq x | s_{t+1} \neq s_t) \\ &= \pi \Pr\left(k_t \leq \frac{x-2+2\beta}{\beta}\right) + (1-\pi) \Pr\left(k_t \geq 2 - \frac{x}{\beta}\right) \\ &= \pi F_t^k\left(\frac{x-2+2\beta}{\beta}\right) + (1-\pi) \left[1 - F_t^k\left(2 - \frac{x}{\beta}\right)\right]. \end{aligned} \quad (3.B.3)$$

The limiting density  $F^k = \lim_{t \rightarrow \infty} F_t^k$  should satisfy the following relation.

$$F^k(x) = \frac{\pi}{\beta} F^k\left(\frac{x-2+2\beta}{\beta}\right) + (1-\pi) \left[1 - F^k\left(2 - \frac{x}{\beta}\right)\right]. \quad (3.B.4)$$

Notice that  $k_t$  and  $p_t$  are discrete processes, so they have point density functions, i.e., probability mass functions. Therefore, we cannot get pmf by differentiating  $F_t$  or  $F$ .

Similarly, we can derive the recursive relation for the cumulative density function

of the price,  $F_t^p$  and  $F_{t+1}^p$  and its limiting density  $F^p$ .

$$F_{t+1}^p(x) = \pi F_t^p\left(\frac{x-\beta}{\beta}\right) + (1-\pi) \left[1 - F_t^p\left(\frac{\beta}{1-\beta} - \frac{x}{\beta}\right)\right]. \quad (3.B.5)$$

and

$$F^p(x) = \pi F^p\left(\frac{x-\beta}{\beta}\right) + (1-\pi) \left[1 - F^p\left(\frac{\beta}{1-\beta} - \frac{x}{\beta}\right)\right]. \quad (3.B.6)$$

It is hard to use representations (3.B.3) and (3.B.5). Therefore, we directly draw the density function by using somewhat complicated notations. Since we will use the probability density of the price in the next section, we only derive the density for  $p_t$ . Let  $k_0$  be given. Then,  $p_0 = \frac{\beta}{2(1-\beta)}k_0$  is also given. Now recall (3.B.2). First, it is easy to see that given  $p_0$ ,  $p_2$  has the following probability mass function.

$$p_2 = \begin{cases} (g^1 \circ g^1)(p_0) & \text{if } s_2 = s_1 = s_0 \text{ (w.p. } \pi^2), \\ (g^2 \circ g^1)(p_0) & \text{if } s_2 \neq s_1 = s_0 \text{ (w.p. } \pi(1-\pi)), \\ (g^1 \circ g^2)(p_0) & \text{if } s_2 = s_1 \neq s_0 \text{ (w.p. } (1-\pi)\pi), \\ (g^2 \circ g^2)(p_0) & \text{if } s_2 \neq s_1 \neq s_0 \text{ (w.p. } (1-\pi)^2), \end{cases}$$

where  $g^i \circ g^j$  is the function composition such that  $(g^i \circ g^j)(p) = g^i(g^j(p))$  for all  $p > 0$ . Similarly,  $p_3$  can also be represented by using a total of 8 cases, each of which with probability  $\pi^3, \pi^2(1-\pi), \pi(1-\pi)^2, \dots, (1-\pi)^3$ , respectively. Likewise, there are  $2^t$  possible values for  $p_t$ . Let

$$A_t = \{(i_1, i_2, \dots, i_t) \mid i_n \in \{1, 2\} \text{ for } n = 1, 2, \dots, t\}.$$

Then, set  $A_t$  has  $2^t$  number of pairs of  $(i_1, i_2, \dots, i_t)$ . The probability mass function

for  $p_t$  is given as follows. For each pair of  $(i_1, i_2, \dots, i_t) \in A_t$ ,

$$p_t = (g^{i_t} \circ g^{i_{t-1}} \circ \dots \circ g^{i_2} \circ g^{i_1})(p_0), \quad \text{with prob. } \pi^a(1 - \pi)^b, \quad (3.B.7)$$

where

$a :=$  the number of 1's among  $\{i_1, i_2, \dots, i_t\}$  and

$b :=$  the number of 2's among  $\{i_1, i_2, \dots, i_t\}$ .

Notice that since it is not true that  $g^i = g^j \neq g^j \circ g^i$  (Operation  $\circ$  does not commute), the general representation is not simplified.

## B.1 Risk Premium

To compute returns on safe and risky assets in efficient and inefficient states. We first define prices for un-traded contingent claims from Household FOC's in the internal finance state: Generally, we have

$$q(s^t, s') = \beta\pi(s'|s^t) \max_i \left\{ \frac{u'[c(i, s^t, s')]}{u'[c(i, s^t)]} \right\}.$$

$$q_k(s^t, s') = \beta\pi(s'|s^t) \max_i \left\{ \frac{Q(i, s^t, s')u'[c(i, s^t, s')]}{u'[c(i, s^t)]} \right\}.$$

More precisely, for log utility,

$$\frac{q(s^t, s')}{\beta\pi(s'|s^t)} = \max_i \frac{c(i, s^t)}{c(i, s^t, s')} = \max_i \frac{(p_t + z_t^i)k(i, s^t)}{\left(\frac{p_{t+1} + z_{t+1}^i}{p_t}\right) p_t k(i, s^t, s')}$$



$$= \max_i \frac{(p_t + z_t^i)k(i, s^t)}{\left(\frac{p_{t+1} + z_{t+1}^i}{p_t}\right) \beta (p_t + z_t^i)k(i, s^t)} = \frac{1}{\beta} \max_i \left( \frac{p_t}{p_{t+1} + z_{t+1}^i} \right) = \frac{p_t}{\beta p_{t+1}}.$$

Hence, we have

$$q(s^t, s') = \frac{p_t \pi(s'|s^t)}{p_{t+1}}. \quad (3.B.8)$$

Then, from (3.B.2)

$$\sum_{s'} q(s^t, s') = \frac{\pi p_t}{\beta(1 + p_t)} + \frac{(1 - \pi)p_t}{\frac{\beta^2}{1 - \beta^2} - \beta p_t}.$$

The risk-free rate  $R_{t+1}^F$  is defined by

$$R_{t+1}^F = \frac{1}{\sum_{s'} q(s^t, s)} = \frac{\beta(1 + p_t)}{p_t \left[ 2\pi - 1 + \frac{1 - \pi}{\beta - (1 - \beta)p_t} \right]}. \quad (3.B.9)$$

It is easy to see that  $R_t^F \rightarrow 0$  as  $p_t \rightarrow p^* = \frac{\beta}{1 - \beta}$  and  $R_{t+1}^F \rightarrow \infty$  as  $p_t \rightarrow 0$ . Notice that time  $t + 1$  expected return of high technology sector at  $t$  is

$$R_{t+1}^H = \pi \left( \frac{1 + \beta + \beta p_t}{p_t} \right) + (1 - \pi) \left[ \frac{\beta^2}{(1 - \beta)p_t} - \beta \right].$$

Time  $t + 1$  expected return of low technology sector at  $t$  is

$$R_{t+1}^L = \pi \left( \frac{\beta + \beta p_t}{p_t} \right) + (1 - \pi) \left[ \frac{1 + \frac{\beta^2}{1 - \beta^2}}{p_t} - \beta \right].$$

Therefore, economy-wide (average) return on investment is

$$\bar{R}_{t+1} = \frac{R_{t+1}^H k_t + R_{t+1}^L (2 - k_t)}{2}$$

$$= \frac{\pi\beta + (1 - \pi) \left(1 + \frac{\beta^2}{1 - \beta^2}\right)}{p_t} + (2\pi - 1) \left(\beta - 1 + \frac{1}{\beta}\right). \quad (3.B.10)$$

since  $p_t = \frac{\beta}{2(1-\beta)}k_t$ . Then, the risk-premium is

$$\begin{aligned} R_{t+1}^e(p_t) &:= \bar{R}_{t+1} - R_{t+1}^F \\ &= \frac{\pi\beta + (1 - \pi) \left(1 + \frac{\beta^2}{1 - \beta^2}\right)}{p_t} - \frac{\beta(1 + p_t)}{p_t \left[2\pi - 1 + \frac{1 - \pi}{\beta - (1 - \beta)p_t}\right]} + (2\pi - 1) \left(\beta - 1 + \frac{1}{\beta}\right) \end{aligned} \quad (3.B.11)$$

## B.2 Persistent Shocks: $\pi \approx 1$

Notice that the risk-premium when the shocks are highly persistent ( $\pi \approx 1$ ) is

$$\bar{R}_{t+1} - R_{t+1}^F \approx \frac{1 - \beta}{\beta}.$$

Historically 6% risk-premium implies  $\beta \approx 0.9434$ .

## B.3 General Case: $0 \ll \pi \ll 1$

The risk-premium defined  $R_{t+1}^e$  by (3.B.11) is the function of  $p_t$ . In this case, we can have the expected value of  $R_{t+1}^e$  given  $p_0$  or  $k_0$  by using the probability mass function (3.B.7).

$$E[R_{t+1}^e | p_0] = \sum_{(i_1, \dots, i_t) \in A_t} R_{t+1}^e(p_t) \Pr(p_t = (g^{i_t} \circ g^{i_{t-1}} \circ \dots \circ g^{i_1})(p_0)) \quad (3.B.12)$$

Equation (3.B.12) is analytically hard to simplify, but it is not hard to numerically calculate (3.B.12). Take  $t \rightarrow \infty$  and we can also get the long-run average value. Here, we may consider the variance of  $R_{t+1}^e$  as well. Then, matching the mean and variance of  $R_{t+1}^e$  for a long run calibrates  $\beta$  and  $\pi$ . It can be performed numerically.

The direct computation using equation (3.B.12) is problematic. In particular, the number of states grows exponentially (by  $2^t$ ) as time goes by. Then, we face memory problems. It is fairly hard to run a program more than 20 periods ( $t \geq 20$ ), which technically implies that only short term equity premium less than 5 years (20/4 quarters) is available by direct calculation. This limitation lead us to try numerical simulations such as Monte Carlo simulations.

The figure shows one simulation generating a dynamics for a price and its corresponding dynamics for the risk-premium ( $t = 1000$  periods). The price and the risk-premium are non-stationary. Notice that neither  $\frac{\sum_{t=0}^T p(t)}{T}$  nor  $\frac{\sum_{t=0}^T R_{t+1}}{T}$  converges as  $T \rightarrow \infty$ . One thing interesting is that when the price is increasing with decreasing rates, the premium is going up (not going down; not in the other direction). At the time of regime switching, both the price and the risk-premium go down together. This is typical in autarky dynamics.

The simulation results are summarized in Table 3.3, Table 3.4 and Table 3.5. We generate three sample paths (in each table) of 4000 periods. In order to check whether the sample mean and standard variation are convergent, we also observe subperiods (1000-periods and 2000-periods) sample means and and variances. If we increase the number of periods (e.g.  $t = 40,000$ ), we still have similar results. The results shows that they are not convergent, which means the stationary density does not exist.

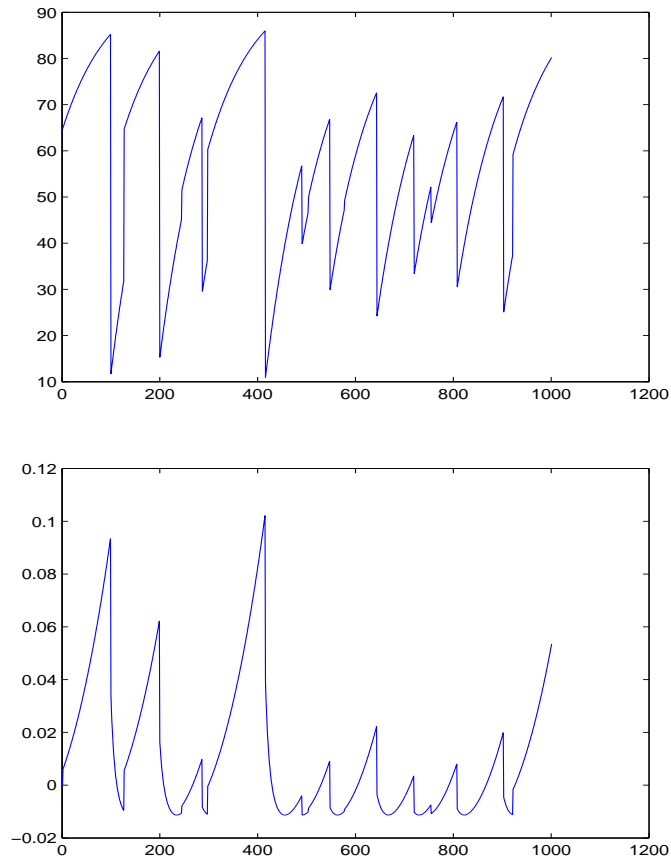


Figure 3.5: Sample price dynamics and the corresponding risk premium when  $\pi = 0.98$  and  $\beta = 0.9897$  which is the quarterly value of 0.96.

t	sample mean (price)	sample s.d. (price)	sample mean (return)	sample s.d. (return)
1000	63.3697	20.8728	0.0303	0.0453
2000	65.2133	22.4009	0.0508	0.0901
4000	64.0350	22.6531	0.0436	0.0748

Table 3.3: The initial price is  $p(1) = 32.3464$ .  $\pi = 0.99$ ,  $\beta = 0.9898$ .

t	sample mean (price)	sample s.d. price)	sample mean (return)	sample s.d. (return)
1000	66.2186	22.4743	0.0422	0.0603
2000	74.4292	23.3482	0.0989	0.1245
4000	69.6540	24.1760	0.1054	0.1773

Table 3.4: The initial price is  $p(1) = 32.3464$ .  $\pi = 0.98$ ,  $\beta = 0.9898$ .

t	sample mean (price)	sample s.d. (price)	sample mean (return)	sample s.d. (return)
1000	63.6494	21.3277	0.0294	0.0424
2000	63.2871	22.5166	0.0376	0.0619
4000	63.4216	22.2575	0.0341	0.0527

Table 3.5: The initial price is  $p(1) = 48.5196$ .  $\pi = 0.9$ ,  $\beta = 0.9898$ .