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The Kumon Approach to Learning Mathematics: An Educator's Perspective

Thomas H. Fuller Jr.

Largely unnoticed amid the cry for better mathematics teaching, Kumon is quietly helping a million mathematics students (from infants to adults; 75% are elementary children). Though conservative in diction and device (including 5000+ worksheets to be solved in "standard" times), it is surprisingly student-centered in practice. The author's investigation during the past year reflects his background in both education and computer science. The paper considers the demands, theories, methods and record of Kumon mathematics from the standpoint of educational theory, cognitive science, and language processing. It considers syntactic and semantic learning of mathematics, arguing that their proper relative positioning... **Read complete abstract on page 2.**

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An Educator's Perspective**

Thomas H. Fuller, Jr.

WUCS-91-49

December 1991

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Abstract

Largely unnoticed amid the cry for better mathematics teaching, Kumon is quietly helping a million mathematics students (from infants to adults; 75% are elementary children). Though conservative in diction and device (including 5000+ worksheets to be solved in "standard" times), it is surprisingly student-centered in practice. The author's investigation during the past year reflects his background in both education and computer science. The paper considers the demands, theories, methods, and record of Kumon mathematics from the standpoint of educational theory, cognitive science, and language processing. It considers *syntactic* and *semantic* learning of mathematics, arguing that their proper relative positioning helps lead the student to higher-order thinking. Future research issues are suggested. The Kumon Machine is briefly introduced; it incorporates the Kumon method into "silicon paper" — a notebook-sized pen-based computer.

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Matt was slipping in his seventh grade math class; he lacked both interest and self-confidence. In his words, he just didn't *do* math well. Peter was unusually bright and outgoing — one of nine students chosen from 250 sixth graders for a select math group. But his quick answers hid flaws that would sabotage his success in later math courses. His sister Allison had a debilitating combination of attention deficit disorder and receptive and expressive language disorder, limiting even normal motor control; math was a towering wall over which she was never expected to see, much less climb. Although in the fifth year of her special school, she could not count reliably even in single digits. Third grader Jacob was bright, but, according to his mother, "bored out of his mind in school" (Walters 1991).

All four children enrolled in the Kumon math program, devoting from fifteen to thirty minutes per day (sixty minutes for Allison) to worksheets, and attending a Kumon class once or twice a week. By high school, Matt's new confidence and enthusiasm for mathematics led him to honors math courses; he received 770 on the Math SAT. After six months, Peter is now working nearly back at grade level, but with sound algorithms that are preparing him for success in advanced math courses. (He is thinking about an engineering or medical career). Allison has advanced past counting to single digit addition (well beyond expectations) after six months in the program. Jacob now multiplies two and three digit numbers with unswerving accuracy. The third grader's mother notes, "Jacob now grabs the bill at restaurants and figures the tip before anyone else can." (Walters 1991). The former three cases were shared with the author by the parents and the Kumon instructor. They are not exceptional. Children in the Kumon program often move a year or more ahead of their classmates. Thousands have advanced to calculus problems before leaving junior high school. Two of the six students on the Japanese national mathematics team are Kumon students.

On the other side of the educational coin, Kumon has found its way into three hundred regular schools — public and private. The Sumiton School was languishing at 19th place among the 21 schools in Walker County, Alabama on the Stanford Achievement Test. After a year of Kumon in mathematics classes, they moved to 9th. Patricia Hollingsworth, Director and Teacher at the University of Tulsa School for Gifted Children told the Ninth World Conference of Gifted Children last year of her school's inclusion of Kumon activities (1991):

For the nine years of its existence, [our school] has used math manipulatives, calculators, and higher-level thinking and problem solving to teach math. And we

plan to continue to do so. Students here understand math concepts, enjoy math, and do well on standardized tests. Yet, until about a year ago, we had one small problem. Students had not committed math facts to memory, and no amount of jingles, jumping to numbers, or classroom games had brought them to the point of owning those basic math facts....My concern was for their futures...Too many students who love math in elementary school drop by the way side in middle school or high school, particularly girls....Math is the most sequential of all academic subjects....Unlike in other subjects, skills in math must be *overlearned* in order to be successful in subsequent math. Kumon is so sequential that, for the most part, it is totally self-teaching....At our school, in fact, we had 95 students working on Kumon at 95 different stages.

The cry for improved mathematics teaching in the US is swelling to crescendo (for example, Husen 1967, Dossey et al. 1988, NCTM 1989 and 1991, Mathematical Sciences Education Board 1990, Mullis et al. 1991, Strait 1991). Two of the six national education goals adopted last year by President Bush and the 50 US governors refer specifically to mathematics. Unnoticed amid the roar, Kumon is quietly helping a million students in mathematics (from infants to adults, but 75% elementary children). Though conservative in diction and device, it is radically student-centered in practice. Students advance at their own pace with miniscule input from instructors; some pass their teachers and parents. Moreover, if parents are able and willing, Kumon can actively involve them in their children's learning.

At first approach, Kumon is an intimidating mountain of 5000+ worksheets to be solved in breathtakingly short "standard" times. Its emphasis on precise technique and repetition to the point of mastery is reminiscent of Suzuki musical methods. (In fact, Suzuki and Kumon have been close friends for years; Suzuki now shares office space in the Kumon headquarters.) It seems at odds with the socioconstructivism increasingly prominent in mathematics education academe. When a respected colleague (also a math teacher) heard of my interest in Kumon, she shuddered at the spectre of the draconian math mills of yesteryear. What could interest a modern educator like me (who, after all, uses mathy manipulatives even in the college classroom) in such pedagogical atavism? Like a haiku poem though, the simple appearance of Kumon conceals secrets undetected at first reading.

Overview of this paper

This paper seeks to stir a discussion about the instructional implications and challenges raised by the method and record of Kumon — to prime the pump in a possibly fruitful, if un-faddish, area of investigation. It also indexes Kumon against the measuring reeds of educational theorists and cognitive scientists such as Skinner, Gagne, Bruner, Piaget, Papert, and Davis. It doesn't offer new experimental evidence for or against Kumon methods, although it may bring some overlooked evidence to the fore. Its approach is not clinical, but more in the form of an extended analysis by a newcomer to Kumon — a newcomer with a background in education and computer science who has spent much of the last year investigating this system.

First, Kumon philosophy and methods are introduced. Then, the present US political and philosophical demands on mathematics education are briefly reviewed. Relevant educational

theory is juxtaposed with the recent contributions from cognitive science. The theory and practice of Kumon bear a message to contemporary mathematics instruction, and vice versa; both are explored. A thematic metaphor — borrowed from computer science — emerges from this comparison: *both* syntactic facility in skills *and* semantic grasp of concepts are needed for children to be successful in mathematics. Furthermore, the relative positioning of facility and concept is often skewed to the detriment of both in mathematics teaching. Finally, future research issues are suggested, including the Kumon Machine — the incorporation of the Kumon method into "silicon paper" (a notebook-sized "pencil"-based computer).

In writing at the intersection of mathematics education and computer science, I realize that I speak to a mixed audience — educators interested in Kumon and in what computer science research can contribute to learning models, and computer scientists interested in the relation of computer science paradigms to mathematics education... Such a mix risks boring the reader on one hand with explanations of what is already known, or frustrating the reader on the other hand by language and concepts not adequately explained. I ask the collegial indulgence of the reader of either discipline as I expand upon one or paraphrase the other.

1 What is the Kumon approach?

An "unreview" of the literature on Kumon

Literature about Kumon is sparse. It is either written in the popular press (such as Finn 1989, Murr 1989, Reingold 1990, Nash 1991, and Walters 1991), or written by Kumon instructors and employees, or published only in Japanese — all equally unsatisfying for the (English-speaking) investigator. All 21 of founder Toru Kumon's books are available only in Japanese. The International Encyclopedia of Education Research and Studies (Husen and Postlethwaite 1985) omits the word *Kumon* in its otherwise dauntingly thorough ten volumes. A search of ERIC (Educational Resources Information Center, which indexes more than 750 educational periodicals) reveals only one article between 1966 and 1991 (Shiba 1986), written by a professor of industrial management at the University of Tsukuba, Japan. Such institutional oversight (in Japan and the US) is reminiscent of the dearth of research into the remarkable successes of Jaime Escalante (the real-life hero of East Los Angeles portrayed in the movie *Stand and Deliver*). Both seem uninteresting to academe — possibly because they emphasize performance rather than articulate supporting theory.

What is Kumon?

Briefly, it's the world's largest home study mathematics program. It does not replace regular schools; in this country and Japan, it is only aimed at about 35 to 40% of the curriculum — arguably (at least for the earlier grades) the easier portion to teach: basic operations, skills, and some problem-solving. In later grades, it extends through calculus, physics, probability, statistics, and more.

In 1954, Toru Kumon had taught secondary math for twenty years. When his second grade son came home one afternoon with a poor grade on a math test, Mrs. Kumon persuaded him to tutor the boy. He examined the textbook and found much that he considered superfluous to gaining a strong command of math. He shopped around for supplemental drill texts, but found them unsatisfactory as well. In his words (Kumon Institute of Education 1987(a), p. 5):

...I found he was taking too much time on things that were not particularly important. Naturally, this was no fun for him. Thinking it would not do for my son to waste his time, I finally decided to design the teaching materials myself.... Eliminating what I considered was a waste of time and expanding what was important, I devised an order of presentation that proceeded step by step from the most elementary stage. This was the prototype of the Kumon materials with their array of calculation problems.... He continued to study like this for 30 minutes every day, and just after he entered the sixth grade, he had completed the differential and integral calculus of the high school curriculum. I tried giving him a few college entrance examination problems and he was able to solve most of them. I remember how relieved I felt in the realization that, as he had reached this level, he would have no worries over future entrance examinations.

The current 29 volumes of worksheets encompass a curriculum from songs for newborns (Worksheet Series 8A) to line-drawing (Worksheet Series 5A) to integral calculus, permutations, probability, and college physics and mathematics (Worksheet V). The guiding philosophy is deceptively simple. In Kumon's words, "I feel that any child presented study materials that precisely match his or her scholastic ability will enjoy learning and come to know the joy of developing his or her abilities. When children are unable to find joy in learning, it is because the content of what they are studying is not at their own level."

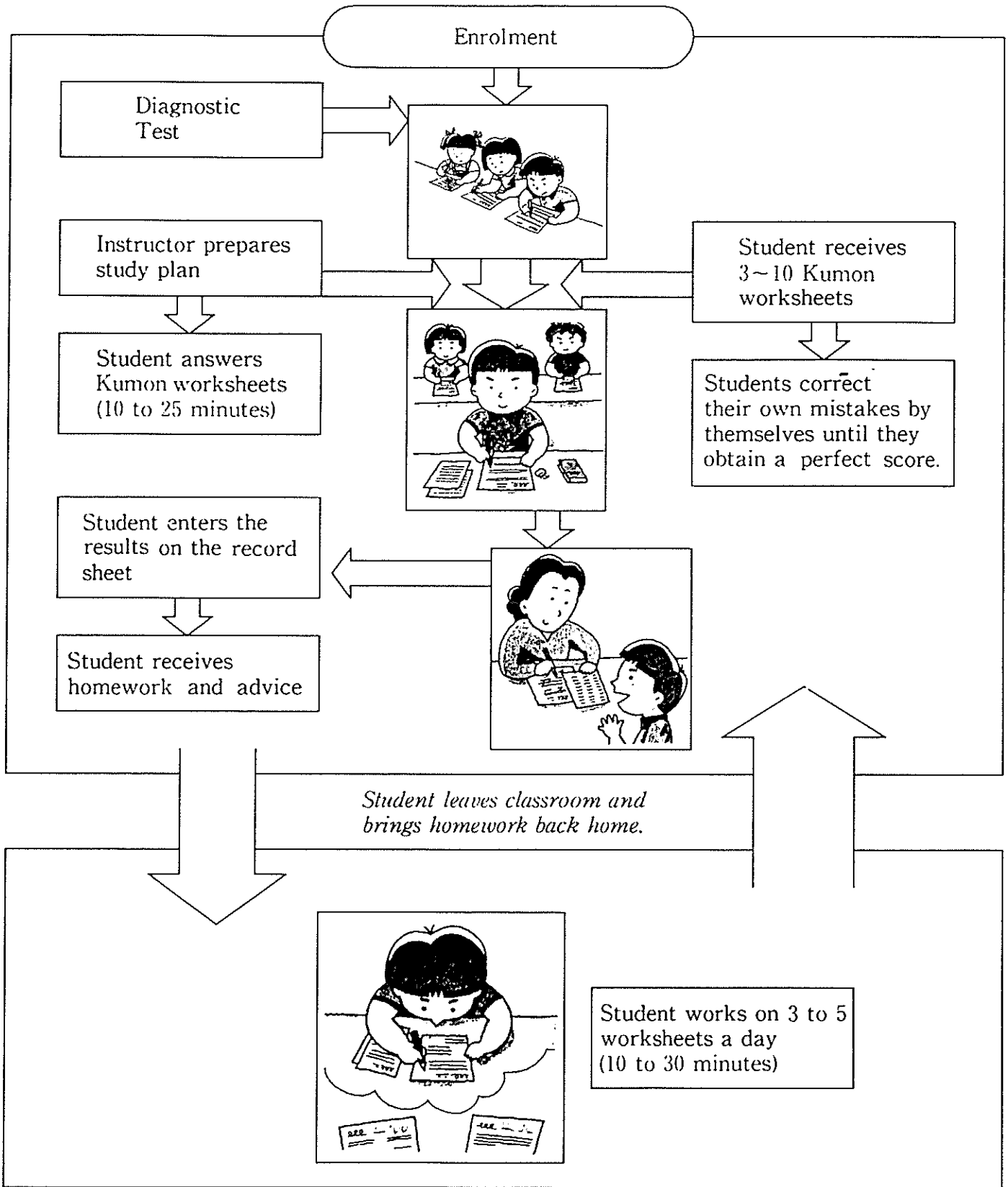
This basic philosophy is expressed in six organizing principles:

- (1) Articulated learning (fine-grained instruction) leading directly to calculus.
- (2) Entry at exactly the right place for the student.
- (3) Natural and universal ability to calculate quickly and accurately.
- (4) Study without strain or pressure.
- (5) Home study (including parental support, oversight, and often correction).
- (6) Independent learning.

Kumon includes a fixed curriculum that is administered by standardized worksheets, diagnostic tests, achievement tests, games, and puzzles. It is rooted in the philosophy that *any* child can learn mathematics through calculus if it is presented in small, understandable segments, and if *mastery* is assured at each level before proceeding to the next.

The tightly-structured curriculum is softened by active parental participation, by completely individualized assignments, and by the coaching, correction and planning of the Kumon instructors. (See the flowchart on the following page.) The instructors in turn teach parents to be more effective coaches. There are monthly publications, case studies, conferences, instructional guidance, and puzzles. There are also some competitions (the highest achievers are publicized worldwide), but the main competition is with the standards and with oneself. The worksheets receive no letter grade. The student's progress is reported and graphed regularly.

The fine-grained instruction is reminiscent of both Suzuki music and the programmed learning of the 1960's (see Gagne 1984, and the discussion below; cf. with the discussion of Suppes in Solomon 1986), but the differences are significant. The process begins with a diagnostic test to determine the entry point. Both time and accuracy are measured. Errors may be assessed. Older children often start a grade or several grades below their current (regular school) level. Watching a high school student start with fractions is deeply disconcerting to parents, but the intent is to have the child start at a level where she can work comfortably, confidently, and above all, independently. (This left one engineer starting with whole number multiplication, Worksheet Series C!) This entry position sets the stage for the student to *teach herself* at every step of the process. Usually the student will reach materials at her grade level in several months, and often will be somewhat past her grade level within two years.



Study in the classroom (twice a week, 10 to 20 or 30 to 40 minutes each time, according to age)

Study at home (10 to 30 minutes a day)

The typical Kumon classroom (attended twice per week) has students of all ages side by side quietly working on worksheets. Sixth graders may be proving trigonometric identities; high schoolers may be reducing fractions. The overarching goal is to cultivate each student's ability and desire to learn independently. The results of each sheet are marked against answer sheets (often by the instructors, sometimes by the parents or the student themselves), and the student corrects the marked problems, resubmits them, corrects and resubmits until the worksheet is perfect. The only acceptable grade is 100%. Worksheets are typically redone (from scratch) two or three times before reaching this level. The habit of correcting one's own mistakes before moving on is intended to build mastery, alertness, and confidence. Most of the work is done at home, averaging 10 to 20 minutes a day for elementary students, and 30-40 for secondary. Results are entered on progress sheets.

Kumon instructors

The instructor intervenes very little. When needed, instruction is always administered one-to-one. The Kumon *Instruction Manual* expressly discourages grouping of students for instruction. The instructor reviews the worksheets, shares hints designed to help the student discover the answer herself, and assigns more worksheets. Some Kumon instructors are teachers from public and private schools operating Kumon centers nights and weekends. During their three days of training, they are admonished that the student's self-confidence is built by placing her at the appropriate starting point, and then insisting on mastery of each level before advancing. Besides the long weekend of initial training at regional offices, Kumon instructors attend monthly meetings, visit each other's classes, and are tested about twice per year in their math proficiency (both accuracy and speed).

Instructors are taught that speed is significant up to a point, as long as the repetition involved in acquiring it does not become discouraging. Although the Standard Completion Times (SCT) are usually enforced, too much practice in any one area is, in the words of the *Instruction Manual* "obviously counterproductive." The SCT's calculated for each worksheet include the time required to correct any errors. The goal for every worksheet is perfect work in standard time.

Instructors are urged to remain sensitive to the attitude of the student in order to assess and maintain the best balance of mastery and challenge. Similarly, the instructor must wisely judge the student's performance on diagnostic and achievement tests, and develop appropriate individual goals (usually extending six months, a year, and two years). Such goals involve the commitment of students and the parents to the long-term effort necessary to meet them.

There is no long-term *financial* commitment however. Parents pay (currently in the US) \$65 per month, one month at a time. The only monetary penalty for dropping out of the program is the \$30 (US) fee to rejoin.

After decades of success in teaching mathematics, Kumon has now expanded to teach Japanese, English, French, and German to Japanese students, Japanese to foreign students, English to American and Australian students, penmanship, and the game Go (very popular in Japan). Besides the US, the company has exported Kumon schools to Taiwan, Australia, Brazil,

Germany, and about a dozen other countries. The materials are translated, but not further customized to local environments; Kumon students in all countries work the same problems.

Facility before concept

Children naturally learn facilities before they learn concepts. For example, they copy letter shapes before understanding words. They mimic tunes before understanding music. They race through scales before composing songs. Seymour Papert worked with Jean Piaget in investigating children's learning, and is the author of the popular computer-learning program LOGO. He appropriates a phrase from one of his LOGO students to emphasize that learning advances in "mind-sized bites." Papert of course was including "bites" of both concepts and capabilities, but the users of LOGO gain many new skills (besides mathematics) while mastering its clever language. They become adept, often more quickly than adults, in expressing the desired graphics from the viewpoint of the "turtle" — the on-screen pointer of LOGO. Thus a *facility* for maneuvering in the mindscape of geometry is born even before the *conceptual* contours of that mindscape are consciously grasped.

Kumon assumes that facility *necessarily* precedes concepts. Moreover, the Kumon materials are targeted precisely at cultivating the *facility* of mathematics and letting the concepts follow. One instructor calls it, only half-humorously, "math in muscle memory." This empowers (in one sense) the learner to form her own (even contradictory) mishmash of theories and tricks as long as are they are *consistent* and *reliable*. Hence the emphasis on the careful crafting of worksheets, the demand for writing intermediate steps, the constant self-correction, and the review by the instructor; these are all designed to prevent the processes from wandering too far afield. This facility learning is held to represent about 35-40% of the elementary curriculum. The social and some of the conceptual aspects of mathematics learning are consigned to the student's regular school. However, the growth of the facilities is usually accompanied by some natural conceptual learning. In fact, Kumon instructors hold that once the facilities are thoroughly mastered, the underlying concepts are easily explained. For example, they find that much less time is required to explain the concept of adding fractions if the conceptual explanation follows thousands of successful operations with fractions.

Syntax before semantics

Kumon may force a new look at learning theory, borrowing language from computer science in the process. This, at least, is posited by Dan Kimura, Kumon instructor and Professor of Computer Science at Washington University in St. Louis. He suggests that the facility learning of Kumon positions *syntax* chronologically much further in front of *semantics* than popular methods (discussed later) which de-emphasize the mechanics of symbol manipulation in favor of conceptual content.

Syntax (from the Greek *syn-tassein* to arrange together) denotes "a connected or orderly system dealing with the formal [i.e. form-defined, form-expressed] properties of languages or calculi, the harmonious arrangement of parts or elements, especially the way in which words are put together to form phrases, clauses, or sentences, and the part of grammar dealing with this" (Merriam-

Webster 1988). Syntax, to the computer scientist (as to the linguist), refers purely to form and positional relationship of symbols. *Semantics* denotes their meaning. Children often first learn music (and language to a degree) syntactically; they mimic and frolic in the forms before they grasp the content (cf. Bruner et al. 1966, Chapter 2, Kieran 1980, Johnson-Laird in Hirst 1988).

Kimura claims that much more of mathematics than heretofore realized is learned syntactically before being understood semantically. Moreover, learning methodology based on this model may be as powerful and natural in mathematics as it is in music and language. Much more will be said about this following the elaboration of the relevant educational theory.

Instruction examples

The transition from addition to multiplication illustrates this approach. By the time multiplication is introduced in Kumon, the six workbooks 5A through B have guided the child from connecting dots all the way up to addition and subtraction. The first 10 worksheets of Volume C are ostensibly just a review of addition (up to three digits). But mixed in are problem sets like these from Worksheet C5a:

$$\begin{aligned}2 + 2 &= \\2 + 2 + 2 &= \\2 + 2 + 2 + 2 &= \\2 + 2 + 2 + 2 + 2 &= \\2 + 2 + 2 + 2 + 2 + 2 &= \\&\dots\end{aligned}$$

The answers form a obvious pattern 4, 6, 8, ... allowing the children to move quickly through the sheet. By Worksheet C7a problems like the following are given:

$$\begin{aligned}5 + 5 + 5 &= \\5 + 5 + 5 + 5 &= \\5 + 5 + 5 + 5 + 5 &= \\5 + 5 + 5 + 5 + 5 + 5 &= \\&\dots \\6 + 6 + 6 &= \\6 + 6 + 6 + 6 &= \\6 + 6 + 6 + 6 + 6 &= \\6 + 6 + 6 + 6 + 6 + 6 &= \\&\dots\end{aligned}$$

The next two sheets present problems like the two above mixed with the two and three digit column addition. Then Worksheet C10a includes:

$$\begin{aligned}2 + 2 + 2 + 2 &= \\3 + 3 + 3 + 3 + 3 &= \\4 + 4 + 4 + 4 &= \\&\dots\end{aligned}$$



Time	:	to	:	Date
Name				

§ 2. Multiplication up to 4 (5 pts. each)

Table

$2 \times 2 = 4$
$3 \times 2 = 6$
$4 \times 2 = 8$
$5 \times 2 = 10$
$6 \times 2 = 12$
$7 \times 2 = 14$
$8 \times 2 = 16$
$9 \times 2 = 18$

1. Fill in the blanks.

(1) $\boxed{2} \times \boxed{4} = \boxed{6} \times \boxed{8} = \boxed{}$

(2) $\boxed{4} \times \boxed{6} = \boxed{8} \times \boxed{10} = \boxed{}$

(3) $\boxed{6} \times \boxed{8} = \boxed{10} \times \boxed{12} = \boxed{}$

2. Fill in the blanks.

(1) $\boxed{2} \times \boxed{4} = \boxed{6} \times \boxed{8} = \boxed{}$

(2) $\boxed{4} \times \boxed{6} = \boxed{8} \times \boxed{10} = \boxed{}$

(3) $\boxed{6} \times \boxed{8} = \boxed{10} \times \boxed{12} = \boxed{}$

3.

(1) $2 \times 2 =$ (9) $3 \times 2 =$

(2) $3 \times 2 =$ (10) $5 \times 2 =$

(3) $4 \times 2 =$ (11) $7 \times 2 =$

(4) $5 \times 2 =$ (12) $9 \times 2 =$

(5) $6 \times 2 =$ (13) $2 \times 2 =$

(6) $7 \times 2 =$ (14) $4 \times 2 =$

(7) $8 \times 2 =$ (15) $6 \times 2 =$

(8) $9 \times 2 =$ (16) $8 \times 2 =$

The next sheet, C11a (previous page), introduces a new *syntactic* element, the multiplication symbol as a table of examples, and a progression that mimics the addition examples that preceded it.

C12 and C13 continue the process of multiplying single digits times 2. C14 introduces the table for 3 and C15 and C16 give practice in single digits times 3. By Worksheet C40, all the single digits have been introduced. C41 begins the process of two-digit multiplication (again by analogy with addition), and the process continues.

Typically students may have to repeat these worksheets a time or two to complete them without mistakes in the Standard Completion Times. The high degree of familiarity with each operation before tackling the next reinforces visual patterns. Following this paradigm, division is introduced in Worksheet C111 (following page) as a visual analogy with multiplication. Again, a new symbol is learned by analogy until the analogy is withdrawn (Worksheet C113) and division gains its own syntactic life.

In a similar fashion, Volume F (nominally, sixth grade) introduces algebra as a syntactic extension of arithmetic. The preceding worksheets drill the learner on the order of operations with fractions (including decimal), parentheses, negative numbers, and the four operations — often mixing all of these in a single expression. Thus the first algebraic equations are not solved by applying the field properties of real numbers (as so often explained by many mathematics texts). They are solved by undoing (reversing) operations in the normal order. Subtracting 7 undoes adding 7, etc. Thus equations are *syntactically* processed. After further gradual development, the learners will have moved up to the following examples from Volume H (eighth grade, linear algebra and analytic geometry) to Volume L (tenth grade, advanced functions).

$$\begin{aligned} \text{H98b:5} \quad & 2(x+y-1)-5 = x-2y \\ & 3x-2 = 1+6y \end{aligned}$$

$$\begin{aligned} \text{H135a:3} \quad & x+y+z+w = 2 \\ & x-y+z+2w = 5 \\ & 3x+2y-z+w = -3 \\ & 2x+y+z-3w = -2 \end{aligned}$$

H158b:6 There are 500 grams of an 8% salt solution. How many grams of water must be evaporated from this solution to obtain a 10% solution?

H159b:5 At what time between 3 and 4 o'clock are both hands of a clock pointing in exactly opposite directions?

H180a:2 Two cargo trains are linked together forming a long one with 22 cars. The cars of the first train carry 15 tons apiece and those of the second 20 tons. If the total cargo weight is between 380 and 400 tons, then what is the range of the number of 20 ton cars?



CIIIa

Time	:	to	:	Date	
Name					

CIII

§ 12. Introduction to Division (2 pts. each)

(1) $2 \times \square = 4$

(11) $3 \times \square = 18$

(2) $2 \times \square = 6$

(12) $3 \times \square = 24$

(3) $2 \times \square = 10$

(13) $3 \times \square = 15$

(4) $2 \times \square = 12$

(14) $3 \times \square = 21$

(5) $2 \times \square = 16$

(15) $3 \times \square = 27$

(6) $2 \times \square = 18$

(16) $4 \times \square = 16$

(7) $2 \times \square = 8$

(17) $4 \times \square = 32$

(8) $2 \times \square = 0$

(18) $4 \times \square = 20$

(9) $2 \times \square = 14$

(19) $4 \times \square = 36$

(10) $2 \times \square = 2$

(20) $4 \times \square = 24$

I179a:2 Find the coordinates of the points of intersection of the two quadratic functions $y = x^2 + (2a + b)x + ab$ and $y = -x^2$.

I190a:3 The quadrilateral ABCD is a parallelogram with given vertices A (2,4), B (-2,1) and D (1,-2). Find the coordinates of vertex C and the length of the diagonal AC.

J164a:1 Solve for x and y:

$$x^2 + 9xy + y^2 = 23$$

$$xy + x + y = 5$$

K50b:9 Find the range of a so that $x^2 + 10x + a > 0$ is true for all real values of x.

K110b:1 Solve *graphically*: $\sqrt{x+3} = \frac{2}{x}$

K178b:3 Prove the following equalities:

$$(3) \cos^2 A \sin^2 B - \sin^2 A \cos^2 B = \cos^2 A - \cos^2 B.$$

L15a:1 Graph $y = 2 \sin \frac{1}{2}(x + \frac{\pi}{2})$.

L35a1 Prove the following equalities:

$$(1) (\sin \beta + \cos \beta)^2 = 1 + \sin 2\beta$$

L73a Complete the proof of Heron's formula for the area of a triangle.

L139b:4 The circle $x^2 + y^2 - 2a(x + y - 1) = 1$ passes through two fixed points regardless of the value of a. Find the coordinates of the two fixed points.

L140b:3 Prove that the circle $x^2 + y^2 - 2ay = 1$ passes through two fixed points regardless of the value of a.

Note how the proof required in L140b:3 builds upon the practice gained in solving problems similar to L139b:4.

L160a:2 Find the locus of point P to make the lengths of the tangent lines drawn to the following two circles equal.

$$x^2 + y^2 = 3 \qquad (x + 2)^2 + (y - 2)^2 = 2$$

L160b:4 When β takes any real value, find the locus of point P(2 + cos β , sin (β - 3)).

Kumon students may reach these worksheets *before* they have studied the corresponding concepts in their regular school. More than 500 *preschoolers* have reached H20 — linear algebra — including one precocious toddler of 3 years and 2 months. Moreover, they often gain these substantial new concepts without significant intervention by adults. They (hypothetically, at least) appear to absorb them as syntactic processes — a type of eye-hand processing perhaps kindred to the video wizardry where children so "handily" beat their elders.

Founder Kumon's reasoning for the rejection of typical drill and practice materials for his son is significant in light of his fundamental reliance on worksheets. In fact, at least two California companies have tried to duplicate the Kumon method using computer-generated sequences of drills. Drill by itself is not enough, nor are examples. Learning demands fine granularity of concept in a balance of repetition, novelty, encouragement, and "mind-size bites" that make the construction of learning materials still more an art than a science.

World's largest mathematics learning lab

What kind of experimentation would be possible if a million learners and twenty thousand instructors in twenty countries focused on a single set of materials and a shared methodology? One answer is apparent from the fine-grained modifications to Kumon materials. For example, the critical transition from counting to addition is based on the relation between the successor function and primitive notions of addition (cf. Suppes and Groen 1967, Groen and Resnick 1977, Widaman et al. 1989, and Geary and Brown 1991). For many years, a series of worksheets at this juncture asked the student to perform a number of successor operations of the form:

2 → __	[The answer is 3]
3 → __	[4, ...]
4 → __	
...	

Since counting has been mastered by now, the child easily fills in the successor of each. Next the students see combined examples of the form

2 → __	2 + 1 = __
3 → __	3 + 1 = __
4 → __	4 + 1 = __
...	

As is done for virtually all new material, the earliest examples are shown with answers and then with lightly shaded answers. Finally, student fill in answers independently. After several such worksheets, the successor operations are withdrawn, and addition problems are presented by themselves. In effect the new syntax of $\langle \text{number} \rangle + 1$ has replaced the familiar syntax of $\langle \text{number} \rangle \rightarrow$.

At first, problems are given in numerical order (as shown above). Later, problems are presented with very slight exceptions to the order (2+1, 3+1, 5+1, 4+1, etc.). Again the student is weaned from dependence on earlier learning (ordinality) to become familiar with the new syntax.

Recently, this method was changed slightly to a somewhat more direct introduction of addition — one that did not make explicit use of the successor function as shown above. The new approach was found less successful by Kumon instructors. In response, Kumon is reprinting the whole series of worksheets to incorporate the earlier method. This process of fine tuning, testing, listening and re-tuning characterizes Kumon's *corporate* learning. It draws upon the outcomes of its worldwide learning laboratory ("staffed" by twenty thousand independent instructors) to improve the materials. The process is not perfect, (as the aforesaid switch and switchback indicate), and ponderously slow sometimes, but it encourages a spirit of experimentation and revision that is healthy for any corporate endeavor.

Kumon and parents

Virtually all educators agree on the central importance of parents in the success of children's education. The National Council of Teachers of Mathematics *Standards* (NCTM 1989, 1991), the NAEP reports, and many others (Committee on the Mathematical Sciences 1990, Ashlock 1990, Goldstein 1990, Tregaskis 1991, Mathews 1988, Bennett 1988) are unanimous in placing the parent at the center of improved achievement by US students. "In a joint statement of NCTM, the Parent-Teacher Association (PTA), and the Mathematical Association of America (MAA) on parental involvement, the parent is recognized as the child's first teacher and *the most important continuing teacher* [emphasis mine] that a student has. Parental attitudes toward mathematics are critical factors in making all students mathematically powerful" (Frye 1990). Generally though, despite many efforts to the contrary, (such as the NCTM's Family Math programs) the average parent feels at a disadvantage in trying to support mathematics learning.

In fairness, at least part of the problem rests with parents' attitudes toward mathematics. Especially pernicious is the belief held by many US parents that mathematical achievement depends on natural ability; Asian parents believe (with Escalante) that mathematical achievement is the result of hard work (L. F. Cavazos, *The Journal of NIH Research*, April, 1990, qtd. in Moore 1990). Hence, the popularity of the *juku* and Kumon in Japan. Some suggest that the classrooms, curriculum, and learning materials are not deterministically different in Japan, Germany, Israel, and East Los Angeles, nor are the children significantly more able. The dominant variable is parent's insistence on achievement in mathematics (cf. Committee on the Mathematical Sciences 1990).

The pressures of single parenting, dual career couples, and generally harried schedules prevent many parents from contributing as they would. When home life permits, parents across cultures can help by supporting teachers, showing personal interest in the learning, valuing what children are learning, and working problems with them. "They can help children see number and shape in the world around them. They can help children understand the importance and usefulness of mathematics" (Ashlock 1990, p. 46).

Parents of Kumon students are encouraged to provide strong support for home learning, sometimes to correct papers (at some centers, their *corrections* are corrected!), to stay close to the progress of students, and to grow in their skills as coaches. Kumon instructors keep in touch with parents through special meetings, letters, phone calls, and newsletters. The emphasis on parenting may also be observed in the collection of games and early learning materials. Folksy guidelines are offered for enriching the child's environment — beginning immediately after birth — through singing, reading and observing pictures/symbols. In a spirit more akin to Japanese quality control circles than Dr. Spock, Kumon mothers boast of how quickly their children memorize the 200 songs (of each family's own choosing) considered minimal for early reading readiness.

Though amusing at first, such practices are organically related to founder Kumon's philosophy that "first of all, the objective of the Kumon Method is not to teach children arithmetic or math skill as an end in itself. Rather, it is an educational method that uses arithmetic and math as a means to develop the fullest potential of each individual child." It is felt that the lack of individualized learning in regular schools leads to the failures and frustrations in math for so many children. "This convinces children that they are incapable, gives them a feeling of inferiority, and makes them dislike math even more. It is not the confused children who are in the wrong; it is the methods and materials being used by the teacher." Hence the emphasis on confidence, modest successes (even at levels below grade), and concentration skills. These prepare the student for the process of self-teaching. Of those starting below grade, about 90% are said by Kumon to catch up to their school curriculum within a year.

The student, instructor, and parent meet a couple of times a year for systematic planning to set goals for the next six months and the next year or two years. These goals require corresponding commitment to consistent effort — 10 to 20 minutes each day for elementary students, up to 40 minutes per day for high school students and twice weekly meetings with the Kumon instructor.

For students with deeper disabilities, several special recommendations are made to the instructors. If reading skills are insufficient, the reading intensive word problems of worksheets F151-180 may be omitted. Interestingly, these are the *only* worksheets recommended for omission. [Note: not all instructors follow the guidelines in this or other matters. Newer ones usually do, but after a couple of years, the instructors individually and through meetings with other instructors tend to develop variations on the guidelines.] In special needs teaching, the student's ability to learn must be thoroughly and honestly developed. This may lead the instructor to work out some problems together, furnish extra examples, carefully assess the intermediate written [never omitted] steps of the process, and use supportive manipulatives. In the guidance for special-needs children, even more emphasis is placed on the Standard Completion Times. These times are usually given as a range (say 3 to 5 minutes for a worksheet

of fraction problems). It is recommended that these children repeat a worksheet until the *shortest* SCT's are reached in order to assure mastery before moving ahead to more difficult work. There is also a special Kumon magazine *Handicapped Children*.

The emphasis on intermediate steps is important for all children. These steps are critical to assessing processes and debugging mistakes. As one instructor notes: "The answer is secondary; the *process* is primary. If the process is correct, trustworthy answers will follow." Peter, mentioned in the opening, was performing all operations mentally — even long division — through a remarkable jungle of tangled processes. He was rapidly approaching the point where such mental gymnastics would fail him. His mother has taught for 25 years; his dad is a physician, and Peter has had excellent math teaching (remember that he was one of the nine out of 250 in the advanced math class). But, as his dad says, the bright students who get right answers often get little attention. Only after Peter was forced to systematically write out the processes was it possible to begin correcting them. Also, the techniques of Kumon are designed to promote the most efficient synergistic use of paper and mind in the solution. Surprisingly, once consistent processes are assimilated, children can work faster when writing down a few well-chosen steps than when omitting them.

Kumon in regular schools

For most of its thirty-three years, Kumon math was only offered as a home study supplement to the regular (public or private) school curriculum. In Japan, nearly 10% of the elementary school population now takes Kumon outside of the public school program. Over the past few years, it has been adopted by a growing number of public and private schools in the US. Some schools "steal" time from the usual 45 to 60 minutes allotted to math. Some open early enough to accommodate "Kumon time." Surprisingly, the children seem to like it. Third grade teacher Betty Whitten from Sumiton School in Alabama tells of students who asked to stay after the class Easter party to do an extra set of worksheets. "We'd already had a lesson that morning. But we did another lesson after the party. There's just that much motivation." Many of the schools who use Kumon to supplement their usual classes report significant gains in standardized testing of basic skills, and (lagging a year or two) higher order mathematical thinking as well.

Parents seem to be the main conduit of enthusiasm for Kumon — virtually its only advertising. But enthusiasm can also be found among teachers and administrators who have tried Kumon in their schools. The Sumiton School increased the school's placement in the Stanford Achievement Test from 19th of the 21 schools in Walker County to 9th after a year of Kumon in mathematics classes. Interestingly the seventh and eighth grades which did not participate in Kumon remained as low as before. Principal Ilene Black emphasizes that this was an overall ranking on *all* subjects, not just mathematics; she attributes the general improvement to the students' increased level of concentration. "The children simply stay on task more." A first grade teacher came to Black asking how many A's in math were allowable. After being assured that there is no limit, she gave 24 A's in math to a class of 29.

C. E. Craft, (Curriculum Director of the McComb, Mississippi School District) reported that a test group of students had increases averaging 8 percentile points (above a control group) after less than 3 months of Kumon instruction (Craft 1990). Woodrow Goins (principal of Cleora

School) reported an overall gain of 9 percentile points over grades 1-8 after a year of Kumon. Kreole Elementary School in Moss Point, Mississippi reported a rise in the second grade Class Average Percentile Rank from 44 to 88, and 33 to 49 for the fourth grade. Houston Preparatory School noted very significant early gains in SRA computation scores, but a reduction in problem-solving scores (See Table 1 below). This dip in conceptual learning was also mentioned by some other administrators. Longer-term results (beyond a year) generally show conceptual areas catching up and passing earlier levels. In all these cases, *overall* math achievement rose.

Table 1
Houston Preparatory School
(scores before and after a year of Kumon)

SRA Test	Grade/Year		Change in percentile rank
	National percentile rank		
	Second, 1989	Third, 1990	
Problem Solving	62	77	+15
Computation	74	88	+14
Total Math	75	87	+12
	Third, 1989	Fourth, 1990	
Problem Solving	70	63	-7
Computation	68	86	+18
Concepts (no test in Third Grade)		75	
Total Math	71	77	+6
	Fourth, 1989	Fifth, 1990	
Problem Solving	59	53	-6
Computation	45	78	+33
Concepts	49	63	+14
Total Math	45	66	+21
	Fifth, 1989	Sixth, 1990	
Problem Solving	48	36	-12
Computation	39	52	+13
Concepts	48	62	+14
Total Math	39	51	+12
	Sixth, 1989	Seventh, 1990	
Problem Solving	59	60	+1
Computation	47	73	+26
Concepts	69	75	+6
Total Math	56	72	+16

Austin Elementary School in Harlingen, Texas reported that their third grade TEAMS scores improved from 72% to 83%. In a letter to parents last October, the school said, "We feel strongly that Kumon was in large part responsible for this gain because students worked much faster and with a greater sense of success than they had in the past." Principal Juana Simmons noted that the average students in the second through fifth grades gained 2.5 grade levels during less than a year with Kumon, "a very significant gain." They also investigated the Kumon levels and concluded that an advance of one Kumon level is basically equivalent to an advance of one grade level (cf. Lewis and Guidry 1990).

In a unique corporate twist, one company has started using Kumon to remediate the math skills of 27 of its employees. They spend 30 minutes per day (paid time) on the worksheets. A company clerk does the grading of the worksheets, and a vice president administers the program.

Students' feelings about mathematics

Feelings about mathematics have been shown to deeply affect achievement in mathematics (e.g. Shields 1991, Tocci and Engelhard 1991, and Mullis 1991). Matt's mother (mentioned in the opening paragraph) credits much of his improvement to the cultivation of a sturdy confidence in his ability to "do" mathematics. As Shields notes, more difficult problem-solving requires a confidence that is not shaken when answers are protracted. Shields, Tocci and Engelhard all indite the absence of such confidence as a significant factor in the mathematics gender gap. Bruner observes that "Learners do not learn much in an anxious environment." (Bruner 1966, p. 52) Also, "... cognitive or intellectual mastery is rewarding. It is particularly so when the learner recognizes the cumulative power of learning, that learning one thing permits him to go on to something that before was out of reach..." (ibid, p. 30)

The following letter to the Kumon office in Houston affords an anecdotal illustration of the longer-term benefits of this restoration of confidence:

Dear Kumon Mathematex:

Since coming to Kumon, I have learned a great deal. I dare say that I have learned more in the nine months I have been here than in all my high school and secondary school [sic] experience.

I was never a good math student in high school and just barely got by. In college I failed the same remedial algebra class 4 times in a row and had to sit out a year due to academic probation because of my poor math grades.

After taking two-and-a-half years off from school, I decided to do something about it. I enrolled in Kumon and began "relearning" everything I ever knew. The program made learning math so easy and made it enjoyable to someone who literally despised working numbers.

I am still in the [Kumon] program and back in school pursuing an engineering career. My grades in math are near the top of my classes at college.

Thank you, Kumon.

Mark

The Kumon office notes that Mark advanced from Level D (whole number multiplication and division) to Level H (linear algebra) during those nine months.

Role shifts: teacher, parent, "coach"

Although progress and instruction in Kumon *are* radically individual, its reliance on standard worksheets and standard times feels out of sync with Western notions of individuality. But our worship of "individuality" cuts both ways. On its altar we have tragically sacrificed a majority of our students with the pseudo-individual assertion that math just isn't for everyone. The NCTM *Standards* insist that some math *is* for everyone. Kumon lets students move in genuinely individual paths. (Recall the very different tracks of Peter and his sister Allison mentioned in the opening). Not surprisingly, the NCTM feels that "the teacher is the key to effective learning in the classroom" (NCTM President Frye 1990). Kumon places less emphasis on the instructor — a role better described as "coach" than teacher. Although the materials themselves are tightly structured from materials for babies who can't hold a pencil to college calculus, most of the individualization is found in the coaching.

The Kumon center, despite its emphasis on focused individual learning, may hint at the mathematics learning community in some unexpected ways. It includes children of diverse ages and need; no grades are given; competition is primarily with oneself (as a swimmer races against her personal best); and the coach/instructor is typically constant over several years. All these are qualities that characterize successful learning communities. On the other hand, the learning is not carried out through discourse within the community, but the support of parents, friends, and instructor can be felt.

The value of the longer-term continuity (which extends through summers as well) is not to be overlooked as a contributor to the success of Kumon students. Jaime Escalante, the teacher portrayed in *Stand and Deliver*, insists that he must teach his students *three years* if they are to succeed in AP calculus. He has helped more than 500 students through the Calculus AP examinations. When he started at Garfield High School in 1978, only six classes in first year algebra were offered. Twelve years later, twenty-five algebra classes are filled. He feels that long-term commitment alone cultivates the mutual care and trust that form the foundation of a real learning community (Davis, et al. 1991, p. 191, Escalante and Dirmann 1990, Mathews 1988).

2 Kumon and Education Theory

The hue and cry for better mathematics teaching

Two of the six national education goals adopted by President Bush and the 50 United States Governors refer to mathematics:

By the year 2000:

...

Goal 3. American students will leave grades four, eight, and twelve having demonstrated competency in challenging subject matter, including English, mathematics, science, history, and geography; and every school will ensure that all students learn to use their minds well, so they may be prepared for responsible citizenship, further learning, and productive employment.

Goal 4. US students will be first in the world in science and mathematics achievement.

...

To reach the fourth goal, we must reverse a quarter century of poor US showing in international comparisons. In a massive study from 1959 to 1966, involving 132,775 students in twelve industrialized nations (Husen 1967), US thirteen-year-olds barely edged ahead of their Swedish peers for next to last in math skills. Japanese students were highest; Germany was not far behind. For students in the last year of secondary school (about seventeen years old), US students were last by a considerable margin. To add insult to injury, the study further reported the total educational expenditures per inhabitant were *six times higher* in the US than in Japan, and expenditures per pupil were almost *seven times higher* in the US.

Since then, the National Assessment of Educational Progress (NAEP, conducted by ETS under contract to the US Department of Education) has assessed mathematics achievement in the school years 1972-3, 1977-8, 1981-2, 1985-6, and 1989-90. In 1989-90, 26,000 students in 1300 schools participated nationally. Another 2500 students (in about 100 public schools) in *each* of 40 states participated in trial state testing (Mullis et al. 1991).

Their 1986 report (Dossey et al. 1988) opened with the gloomy news that the *average* Japanese students exhibit higher levels of achievement than the *top 5%* of American students enrolled in college preparatory courses. It states that US math achievement is up slightly this decade, but primarily in "lower-order skills." Much more improvement is needed in the higher-level skills and concepts. Only 5% of the high-school seniors tested (1990 report, down a percent from 1986) were proficient at the level of multi-step problem solving, geometry, and algebra. Fewer than half (46% in the 1990 report) of high school seniors demonstrated a consistent grasp of decimals, percents, fractions, and simple algebra. David Kearns, CEO of Xerox, laments that a

new Japanese semiconductor plant in the southeastern US had to hire *graduate-school students* to perform statistical quality control functions performed by *high-school graduates* in Japan (1986 report) . American businesses currently spend \$25 billion every year to teach their employees to read, write, and count (1986 report) .

This year's SAT tests continued a gradual decline in both math and verbal scores; since 1969, average math scores have dropped 19 points on the 200-800 scale (Germani 1991). Although SAT results have the eye of politician and public, they are not really a reliable measure of national achievement in mathematics for several reasons: the American College Test predominates in the 28 states not on the seaboard (though the ACT does show a similar decline), and many students take neither test, presumably self-selecting out of the college track (*St. Louis Post-Dispatch* 1991, Owen 1985, e.g. p. 271).

Stigler et al. (1990, cf. Mayer 1991) conducted a carefully designed and administered study comparing both computational skills and higher-level conceptual understanding of students in the US, China, and Japan. The study, sponsored by the NCTM, was carefully balanced across social and economic strata, classrooms, and schools. It included group testing of computational skills (the four basic operations with whole numbers, fractions, and decimals) and a battery of nine individually-administered tests in the following areas:

- word problems — from first to seventh grade level
- conceptual knowledge — such as place value, positive and negative numbers
- mathematical operations — related to real world situations
- graphing — extracting information from tabular and graphical data
- estimation
- visualization and mental folding
- transformation of spatial relations
- mental calculation

Altogether, 5,524 first and fifth grade children in Sendai (Japan), Taipei (Taiwan), and Chicago were tested in three sessions. Additionally the first graders were given a test of reasoning, and the fifth graders were tested in geometric concepts. The nine individual tests were administered to representative subsets of the children who took the group computational tests.

For the group testing of computational skills, the team reported "highly significant differences among the scores for the three cities at both grades, [$p < .001$], but no significant differences by sex." In light of Husen's earlier findings, this is hardly surprising.

The results of the individual tests ("higher-order thinking" topics) were more disconcerting:

The American students received significantly lower scores than the Japanese and Chinese students at both grade levels, but the difference is much greater at fifth grade than at first grade.... [T]he relatively low performance of the American children relative to the Asian children is pervasive, occurring in all topics tested.... Between first and fifth grades, the relative status of the American children shows a striking decline and the performance of Chinese students shows remarkable

improvement. The conclusion is clear. Asian children's high level of performance in mathematics is not restricted to a narrow range of well-rehearsed, automatic computational skills but is manifest across a wide range of tasks and problems. (ibid., p. 13)

Again the significance was strong ($p < .001$) and few differences by sex were found. The team notes that Taiwan teachers emphasize very rapid retrieval of answers. Although Japanese teachers de-emphasize speed in favor of thoughtful, careful reflection, Japanese children are also significantly quicker than American children in mental addition and multiplication.

In the general discussion, the investigators comment, "The deficiencies displayed by the American children were more serious than we expected.... With problems as diverse as estimating the distance between a tree and a hidden treasure on a map, deciding who won a race on the basis of data in a graph, trying to explain subtraction to visiting Martians, or calculating the sum of 19 and 45, significantly fewer American than Chinese or Japanese children solved the problems correctly" (ibid., p. 25). Geary, Fan, and Bow-Thomas (1991) found that Chinese first graders exhibited more developmentally mature mixes of addition strategies as well as more reliable retrieval from memory than their US peers. Such findings stir a consideration of the subtle relations between the conceptual grasp of mathematics and operational skills. Is the relation one of co-dependency, orthogonality, or inverse dependency.

Philosophies of mathematics education

As the recently-developed NCTM *Standards* imply, much of the debate over mathematics education involves *what* mathematics is to be taught. What are the purposes of education? Is mathematics education primarily to serve the society's needs, as suggested by NAEP above? In support of this utilitarian view, note that the college graduates of 2006 are now first graders. The National Science Foundation (NSF) estimates that the US will face a shortfall of 400,000 scientists and 275,000 engineers by 2006, "compromising its economic strength, quality of life and national security." (C. Holden, *Science*, June 30, 1989, qtd. in Moore 1990).

None would deny these needs, but hasn't mathematics *another* mission beyond economic needs, namely to liberate the mind of the individual thinker (Dewey 1938, Paul Hirst 1965)? Hirst includes mathematics as one of his seven *forms* or intellectual frameworks of human thinking. (Other *forms* are physical science, language and fine arts, social science, and so on.) He holds that the human mind requires a thorough "initiation" into each of these forms in order to be freed from the bonds of ignorant or rigid patterns of thought.

To others, education should be the vehicle of cultural transmission; and if so, then *whose* culture (Adler 1982 and 1984, Friere 1968, Martin 1981)? Are these three broad goals — societal, individual, cultural — at odds or coincident? (Cf. Kamens and Benavot 1991.) Who should decide the goals and how can they be assessed? Some suggest that the real decision has already succumbed to political forces, and has swung well into the conservative camp in recent years (Shor 1986). Such issues can be resolved only by the jostling interplay of political, economic, cultural, and educational forces.

The NCTM embraces both societal and individual goals in its labors over the last decade to build a national consensus among many professional and academic organizations on this issue. As then NCTM President Frye said last year, "Just as the teacher is the key to effective learning in the classroom, the National Council of Teachers of Mathematics is the key to effective leadership in mathematics education. NCTM provides nationwide and international direction for curriculum, instruction, and preservice education for the mathematics to be taught to students at all levels." Many influential organizations have swung behind the standards including American Mathematical Society, Mathematical Association of America, Mathematical Sciences Education Board, National Association of Biology Teachers, National Association of Principals, State Boards of Education, National Congress of Parents and Teachers, and many more. US Secretary of Education Lamar Alexander points to these standards as a model for other disciplines (Bencivenga 1991).

[Two documents were actually issued: *Curriculum and Evaluation Standards for School Mathematics* (1989), and *Professional Standards for Teaching Mathematics* (1991). Unless the context requires the distinction, these closely complementary volumes are referred to collectively as the *Standards*.]

The *Standards* address many dimensions of mathematics teaching from preschool to college and from curriculum developers to boards to teachers to supervisors. The main new directions are summarized:

Woven into the fabric of the *Professional Standards for Teaching Mathematics* are five major shifts in the environment of mathematics classrooms that are needed to move from current practice to mathematics teaching for the empowerment of students. We need to shift —

toward classrooms as mathematical communities — away from classrooms as simply a collection of individuals;

toward logic and mathematical evidence as verification — away from the teacher as the sole authority for right answers;

toward mathematical reasoning — away from merely memorizing procedures;

toward conjecturing, inventing, and problem solving — away from an emphasis on mechanistic answer-finding;

toward connecting mathematics, its ideas, and its applications — away from treating mathematics as a body of isolated concepts and procedures.

The *Standards* also state that the question of "what math?" is answered by: trigonometry, analytic geometry, logic, statistics, and so forth, which emphatically *are* "for everyone." The spirit of the manifesto is deeply *inclusive*, to not block any student's potential achievement or access to opportunities. Significantly, it is recognized that this may require adopting a

curriculum that does *not depend upon success with preceding mathematical material*. The idea of such an adoption seems especially problematic to those who work with the students later in life, such as college teachers and employers. If calculus, for example, can be taught such that it requires no success with algebra or fractions, it is certainly available to a wider audience, and it may help students feel better about themselves and about mathematics. But tough questions remain. How much calculus will such students really gain? And will the improved self-esteem stand up to lost educational and vocational opportunities?

Ironically, American students already lead the world in *their own* (and their parents') estimate of their mathematical knowledge. This attitude may be based less on world-class studies, and more on US-normed tests, which like "grades on the curve," are subject to glacial drift with the ability of both teachers and students. Exemplifying this tendency, some (such as Lester and Kroll 1991) propose grading schemes that give 75 to 80% of full credit for well-explained approaches that merely fail to get correct answers. Although some circumstances may justify this practice, it does risk further slippage in the international arena where correct answers are undeniably the currency of this world's success.

A curriculum that does not depend upon success with preceding mathematical material may well be less threatening to underprepared students (or teachers), but will it achieve the national education goals for mathematics education? Perhaps, say some (cf. Mayer 1991); but this means that it will have to leapfrog European and Asian nations as they rest by their rigid (often nationally-mandated) curricula. Such a hope remains unproven. Perhaps a readiness-constrained curriculum might plausibly be advanced as an *interim* step as we work earnestly to better prepare the students rising through the system. Then, as these students reach the upper-level courses, we can teach such courses in ways that do depend on success in prerequisites without loss of inclusiveness.

Methods of understanding mathematics education

How do we measure mathematics education? The most widely-known assessments are multiple choice standardized tests. Robert Davis, long a leader among mathematical reformers, points to a widely-recognized difficulty in this measurement: multiple choice testing omits most of the interesting data. To remedy this, the task-based interview is increasingly employed as a research tool. The investigator /teacher talks with the learner while the learner solves the problem. The sessions are often audio or video taped. One 20-minute session may take 50 person-hours of analysis. The investigator/teacher must be trained to give no clues yet to build rapport with the student.

This was the method employed by one of the most significant long-term studies of how children learn mathematics, the Long Term Study (U. Illinois) headed by Davis. The study had four goals and a philosophy as follows:

1. How do students think about mathematical problems?
2. Where, and why, do they seem to encounter special difficulties?
3. What seems to help?

4. What are the natural parameters for learning mathematics — for example, how old should a student be before undertaking the study of calculus?

But perhaps, even more than our questions, it has been our philosophy that has determined the methods of this study. We do not see "mathematics" as a collection of algorithms, to be memorized by rote and practiced. Nor do we see mathematics as something to be "taught" to students, with control in the hands of the teacher (which is not to deny the importance of the teacher, but merely to redefine the teacher's proper role). Instead, we see "mathematics" as a collection of ideas and methods which a student builds up in his or her own head. In recent years this has come to be called the "constructivist" approach to the study of the learning of mathematics, an approach that has developed through work of Piaget, Papert, Lawler (1986), Easley, Ginsburg, Thompson, Steffe, Duckworth, Erlwanger, and others. (Davis 1984, p. 92)

In the Kumon classroom, the small amount of "teaching" that occurs is one-on-one, giving rise over the years to considerable knowledge about how children learn. Some of this knowledge is shared within the program by seminars, internal publications, videotapes, etc., but virtually none finds its way into the professional (English) literature.

"Rote" versus "meaningful" learning

If "higher-order" thinking is to be for everyone, how shall it be taught? Educators from John Dewey (Dewey 1938, p. 27) to the NCTM have decried the unthinking drill remote from anything useful or interesting in the life of the learner. Is Kumon a mere throwback to such methods? Is its popularity based on what Papert calls "local improvement" thereby winning the endorsement of the community, but failing to prepare the student for a lifetime of real mathematical thinking and learning? (1980, p. 140) The demand for "meaningful" learning instead of "rote memorization," as the choice is often portrayed, stems from the premises (adapted from Davis 1984, cf. Resnick and Klopfer 1989) that:

Math is too complex to be taught by rote. No system of rote tricks or formulae can encompass the range of possible problems. Math in the real world — engineering, finance, analysis, research, and so forth requires creative pattern-seeking and imaginative thinking.

Students constantly "invent" new algorithms anyway. Their discoveries race ahead of the standard algorithms of adults. In fact, it is very difficult to accurately assess their actual algorithms.

We need to recognize, encourage, and support originality and creativity. Sometimes originality is what *most* needs to be learned — even more than the mathematics itself. For example, original and insightful proofs are central to high school geometry.

Rote learning makes classes dull. It bores teacher and student, until opportunities for real learning sink in the slough of tedium.

The much-heralded "new math" endeavored to lift math learning from the doldrums of rote to the peaks of meaning from 1956 to 1970. (Contrary to folk history, its roots *do* predate Sputnik!) On the face of it, the layering of set theory and abstract algebra did little to make math meaningful to children, and deeply frustrated parents and teachers who found their own grasp of math precariously obsolete (cf. Bruner 1966, Papert 1980, pp. 140ff.). Furthermore, its recommendations were regrettably insensitive to the daily realities of typical teaching environments. Much to their credit, the *Standards* more strongly reflect and accommodate these environments.

Behind its face though, "new math" was really dozens of programs over a wide diversity of models, most of which were generally ignored by American school systems: "In those few schools where they were extensively and carefully implemented, the best of the new curricula produced very pronounced gains in student performance" (Davis 1984, pp. 312f.). Its worthiest new element is the paradigm teaching strategy. Paradigms are visual metaphors for abstract concepts (cf. "Art, Math & Other Things" in Holt 1983). For example, a meter-tall set of balance scales can be used to "guess" the number of marbles in sealed envelopes. An equation is portrayed as "two envelopes plus six loose marbles balance with twenty loose marbles." The paradigm of maintaining balance leads the students to understand the field properties that maintain the balance of equations.

The condemnation of "rote" without "meaning" does not usurp the natural need for repetition and practice to gain the skills of mathematics. Davis and McKnight (1980), Davis et al. (1990), Hollingsworth (1991) and others indicate clearly that semantic, conceptual content of mathematics is not of itself sufficient to master the requisite algorithms. In Davis and McKnight's (1980) study of third and fourth graders, they discovered that a child could demonstrate conceptual knowledge with MAB blocks (blocks portraying 1's, 10's, 100's, and 1000's) and making change while continuing to fail to apply these concepts to the problem of subtracting 25 from 7002. The former should cause the learner to recognize that the answer is about 7,000, but a faulty "borrowing" algorithm led to an answer of about 5,000. Both syntactic facility with algorithms *and* semantic grasp of principles are needed. Adults have linkages between these domains that render such contradictory behavior unacceptable, but fourth graders haven't formed these self-correcting linkages yet. (This is further discussed in connection with Piaget.)

Dr. S. K. Lo holds a doctorate in physics and an M.B.A. She moved to the US from Hong Kong and later became a Kumon instructor in an oft-repeated pattern. She heard about Kumon from a friend, enrolled her own children, and got increasingly interested in sharing it with other children. She comments on the importance of training minds as we train athletes: "When we do a lot of foul shots [in basketball practice], we say that is what you have to do, but when we do math, we say it's boring. How come there is such a difference in mental attitude? To me it is a mental exercise." (Hamer 1991). Hollingsworth echos this sentiment in discussing the use of Kumon at the University of Tulsa School for Gifted Children:

The student may repeat some prior worksheets to gain speed and accuracy, much as an athlete might repeat certain exercises in training. The comparison to

physical training may take some attitude adjustment. We understand the value of repetition in sports and music but we have failed to see how similar the situation is for math. At some juncture in a child's education we, as parents and teachers, should be saying, "Yes, math is hard but you can do it with practice. Math is like learning to play the piano or getting better at basketball. It takes practice, and that means doing some things over and over again until numbers become a part of you." Instead we say, "You understand the concepts and can do the problems; I know you are bored by having to repeat work." We thus unwittingly provide the perfect excuse for future failure.

Of course, every discipline has its drills and practices. The success of such practice depends upon the degree to which the learner devotes alert attention to it, integrates it with growing knowledge structures, and continues in the learning. Davis notes (1984) that most children are capable of learning at a rate much faster than that of their classrooms. This gap between potential speed and actual speed accounts for much of the dullness of classroom learning, and is a significant strength of individualized activities like Kumon. The learner who is being challenged right at her performance ability is much less likely to get bored.

Bloom and the taxonomy of educational objectives

In the era of "scientific management" following World War II, designers of educational tests attempted to better qualify their objectives. Benjamin Bloom led a committee of the American Psychological Association in identifying the *hierarchy* of objectives in three distinct *domains* of education and testing: cognitive, affective, and psychomotor. As a testament to the committee's success, test designers and educators still rely on this hierarchy 40 years later although extensions are occasionally proposed (Chancellor 1991).

The cognitive domain (Bloom et al. 1956) is ordered by (1) knowledge: specifics, ways of dealing with specifics, and universals (abstractions), and intellectual abilities and skills: (2) comprehension, (3) application, (4) analysis, (5) synthesis, and (6) evaluation. The categories are hierarchical in nature. Higher-order thinking is associated with the latter intellectual abilities. Consider an original geometric proof as an example of *synthesis*. The student must *synthesize* statements, supporting theorems, and definitions to assemble the proof. This in turn requires an *analysis* of the problem, perhaps searching for "subproofs" that lead to the solution. It certainly involves *applying* the relevant theorems and constructions. To successfully apply theorems, the student must *comprehend* them, which of course implies *knowledge* of them. Thus synthesis naturally depends upon the lower intellectual skills.

Test designers and educators recognize that the lower-level skills are easier to teach and test, and have consequently received more than their share of attention over the years. The plea of the NCTM Standards, the NAEP, President Bush, and the 50 Governors is in effect a demand to honestly cultivate more analytical, synthetical, and evaluative skills in our students. Facility-oriented programs like Kumon are suspected of limiting development to just knowledge and comprehension, thus displacing the acquisition of the higher skills that open doors to advanced study and adult success. Davis and McKnight's work indicates that a mix of facility-learning (with honest mastery) *and* conceptual development is better than either alone in leading the

student to higher-order thinking. Advocates of the Kumon philosophy further claim that concepts should be positioned well behind the mastery of facilities to not undermine the successful acquisition of the former.

The affective domain (Krathwohl et al. 1964) relates to feelings and values in the learner. It is ordered by increasing internalization of the values being presented to the learner: receiving (attending), responding, valuing, organizing (conceptualizing), and characterization (incorporation into patterns of living). For learning to have longitudinal, transformative power in the learner, it must be accompanied by changes in the affective domain as well as the cognitive. Although less attention is given to this domain (partly due to the difficulty of measurement), its role is widely recognized. The success of methodologies like Kumon's (and Escalante's: Mathews 1988, Escalante 1990) depend deeply on the affective dimensions of the teaching and learning — on the ongoing enthusiasm of both child and parent.

The psychomotor domain embraces muscular coordination. It is evident in activities such as handwriting, dance, and athletics. It may be categorized by gross and fine performance, and by continuous or discrete tasks (Gagne 1984, Chapter 10). Although this may bear more of a relation to learning in the other domains than is generally recognized, it was the last domain to be developed, and has received the least attention from educators. Borrowing again from the musical analogy, the psychomotor elements of learning are difficult to separate from the affective or the cognitive. The touch of the finger to the key or string is mediated by elements of all three domains.

Learning theories and mind

Before diving into specific learning theories, let me note that one cannot explore theories of education very far without bumping into theories of mind. This is even more true of cognitive science, discussed in the next section. Even an introduction to mind is beyond the scope of this paper. Nonetheless, a cautionary *caveat* is unavoidable. Educational theories necessarily build upon widely varying assumptions about the nature of mind: its genesis, growth, processes, limits, and content (cf. the "strange loop" of Hofstadter 1970, the self of Turkle 1984, and Bruner 1983 and 1986). Alas, even the *locus* of mind or consciousness remains deeply disputed (Maxfield in Smith 1990, Phillips and Soltis 1985). Broadly-held theories consider consciousness an epiphenomenon (a "side effect") of the electrical and chemical processes of the brain. Johnson-Laird likens it to the operating system of all conscious and unconscious processes (1988). A strong contingent of dissenters (for instance, Penfield 1950, Holt 1983, Gilder 1989, Diestfrey 1991) differentiate *mind* (that which thinks, recognizes, and learns) from brain/nervous system. Bechtel and Ecanow (1982) warn that computer models are *useful* but not *complete* for understanding human mental activity. They prefer the phrase *mind-brain*.

Between cognitive research and actual cognition yaws a deceptive chasm whose floor is littered with the bones of researchers who thought the leap from theory to thought narrower than it proved to be. This paper doesn't add to this debate (which I expect to last for centuries), but simply employs *mind* to name that which emerges through all the internal and external processes of individual human development. The educational theories briefly considered below illumine the development of the contents and processes of mind in this sense. For our present purposes,

even the most compelling cognitive models are better understood as insightful analogies, rather than ontological necessities.

"Classical" educational theories and programmed instruction

Pavlov and von Bechterev launched half a century of research on *conditioning*. From the famous dogs to myriads of rats and pigeons, learning was related to responses conditioned by associations (classical), and by patterns of reward and punishment (operant). Thorndike and Skinner extended this behavioral modelling to more sophisticated connections between the conditioning and outcome (Gagne 1984, Wilson et al. 1969, Chapter 4, Phillips and Soltis 1985, Chapter 2). This view of learning continued in some degree in the programmed instruction texts that blossomed in the 1960's, reemerged as programmed learning on centralized mainframe computers in the 1970's, and flowered in the 1980's on microcomputers (e.g. Kearsley 1987). The Math Skills Ladder developed by SRA (Science Research Associates, educational subsidiary of IBM) is representative of this genre. Assigned to advanced students for enrichment, and to less advanced students for remediation, it emphasized individualized learning and the fine-grained stepwise decomposition of subject matter.

The most widely-used of the centrally computerized programmed instruction packages was developed by Patrick Suppes of Computer Curriculum Corporation (CCC). Every teacher has probably dreamed of *fully* debugging a course, a text, or a set of materials. I know that I did. ("If I teach whole number factoring properly now, it will better lead to simplifying algebraic radicands later.") Suppes' team developed 41 blocks of thoroughly coordinated, tested, and debugged materials. A grade of 80% let the learner advance a block; 60%, back a block. Standardized test scores of economically or educationally disadvantaged students improved significantly (Copeland 1984, p. 23). Although average response time was thought important at first, later versions de-emphasized it. Also, they discovered that the most effective learning interval was ten minutes per day.

The Math Skills Ladder, the CCC program, and the myriads of similar products are not unlike Kumon in objective and device. They aim at a modest portion of the mathematics curriculum, and at a largely self-guided assimilation of the content. One very significant difference in Kumon is the depth of the debugging process. Twenty thousand Kumon instructors (currently) make recommendations to fine-tune the materials, and offer hints on how to diagnose problem areas and remediate them (i.e. help the learner correct herself). The process of fine-tuning has spanned three and a half decades. To put that in perspective, the Kumon materials predate Sputnik, and have witnessed the parade of "old basics," "new math," "back-to-basics," and the Educational Goal for 2000. Second, Kumon is administered by instructors (*cum* coach), and by parents (*cum* coaches). The work never proceeds far without the watchful eye of an interested human. Third, considerable emphasis *is* placed on completion times and 100% (not 80%) is needed to advance to the next level.

These latter two practices hint at the fundamental difference: Kumon methods are intended to assure *mastery* (instead of *competence*) before tackling more challenging concepts. This is math "known in the bones" as our grandparents put it, in "muscle memory" as coaches and (sometimes) music teachers might say it, and "syntactically rather than semantically processed"

as the computer scientist defines it. Also, the time constraints help students learn to stay on task, to concentrate, and to focus. Further research is needed to assess the significance of these systemic differences in the longitudinal (mathematical and non-mathematical) successes of Kumon-taught students.

Gagne (Gagne and Fleishman 1959) added to the traditional behavioral model significant refinements such as "differential sensory detection....retained, or stored by means of a neural mechanism called memory," — anticipating cognitive science in one aspect. He recognized that the learner builds internal models of external events and meanings, recorded in five categories of learning — intellectual skills, cognitive strategies, verbal information, motor skills, and attitudes. This record is made more readily available by what he called *overlearning* — practice extending well beyond competence. He wrote, "...such abilities as numerical ability, manual dexterity, and reasoning ability are fairly stable attributes over lengthy periods of time. This probably results from the fact that they have been learned, relearned, and practiced so many times during the individual's lifetime."

Gagne also describes the "progressive part" method of learning. Learning may be considered progressively longer chains of association. The learner is most successful who exercises the links already known in connection with a "progressive part" that is yet to be learned. For example, an actor memorizing a play rehearses the lines already learned while trying to add a few with each session. This not only increases the retention of the former, but builds a chain of association from the former to the latter, increasing the likelihood of associating the two when needed. Each new volume of Kumon materials begins with review of earlier concepts. This supports overlearning, but also leads to chains of association from old ideas to the "progressive part" — the new ideas to be gained.

Gagne and others lamented in the 1960's that so little attention is given by educational theorists to the wide variations in the speed and methods of learning from student to student. He wrote in 1967, "At the present time it seems fair to say that we know considerably more about learning, its varieties and conditions, than we did ten years ago. But we do not know much more about individual differences in learning than we did thirty years ago." In 1991, these differences are the subject of significant research (especially via what Gagne called the "information processing" model of learning, or cognitive science in general), but it still isn't clear that schools are generally better able to respond to such individual differences.

Gagne suggested (as had Dewey before him) the importance of concrete experiences with weights, blocks, and familiar manipulable objects to assist the learning of mathematical concepts. In later works (Gagne 1984), this is generalized to "discovery learning" — sudden insights or reorganizations of the student's field of experience. He further defined a hierarchy of learning that leads to such "discovery" namely: signal processing (Pavlov), stimulus-response (Skinner's discriminated operant), chains of *several* stimulus-response pairs, verbal associations, discriminations among similar stimuli, concept-level response to widely varying classes of stimuli, rules (chains of two or more concepts — if A then B), problem-solving (combining two or more rules). The latter is what he called *thinking*. "Each of the eight varieties of learning conditions,... establishes a different kind of capability in the learner. This capability may be narrowly specific, as in responding to a signal [*syntactically*, I would suggest], or it may range in

generalizability to the kind of competence attained in rule learning and problem solving." (Gagne 1970, p. 237)

He aligned with the rebellion mid-century led by Thorndike against rote learning and unthinking memorization of formulae. Math was to be made "relevant" to the activities of the child, such as shopping, making change, etc. This rebellion folded into diverse experiments of "new math" discussed earlier.

Piaget: genetic epistemology

Piaget is popularly known for his four meticulously defined stages of cognitive development, but his major explorations relate more to the *genesis* of knowledge. He preferred to be known, not as a psychologist or philosopher, but as a genetic epistemologist. He put on record the processes by which structures of knowing are built, and demonstrated these to be systematically traceable and understandable across many cultures and languages.

His earliest work was in the biological sciences, and he drew upon biological models for inspiration throughout his long and fruitful career (40+ books, continuing into his 80's) (Boden 1979). Any biological entity constantly *assimilates* (absorbs) some of its environment and *accommodates* (adapts to) the rest of it. Piaget visualized the development of mind as the outcome of a ceaseless "equilibration" between these two methods of relating to the mind's environment. The mind assimilates some of its environment and accommodates (through internal reformation) other elements of it (Slavin 1991, Chapter 2).

The earliest growth of higher animals follows a well-defined pattern: the fertilized egg develops differentiated and specialized cells, then organs, then systems for digestion, vision, energy transfer, and structural support. Piaget described the analogous growth of cognitive "organs" called *schemata*. The *coordination* of these schemata create thought systems (*operations*) capable of comprehending the world in its glorious depth and complexity.

His methods mirrored those of the biologists of his early training. His theories grew from exhaustive observations of a few subjects rather than statistical data on thousands. (Most of his neonatal observations were based on his own three children.) He sought to separate the universal from the incidental by combining this careful observation with the systematic formulation of broad, unifying theories and the constant testing of the empirical against the theoretical. His methods, institutionalized in the Center of Genetic Epistemology in Geneva, might at first seem naive compared to the statistical methods of our advanced era. However, American psychologist Jerome Bruner, lamenting the after-the-game scorekeeping that he feels characterizes too much of statistically-oriented psychological research, acknowledges researchers' abiding debt to Piaget and his powerful insights.

Piaget categorized the development of cognition into four stages (Piaget 1976, Inhelder and Piaget 1976, Wilson et al. 1969, Wadsworth 1979, Copeland 1984). The period from birth to two years is the *sensorimotor* stage (Stage 0). The newborn already has structures that differentiate, for example, things to suck which provide milk, and things to suck which don't. By 8 months, cognitive structures have grown to the point that the child can distinguish herself

from her environment. Clear evidence of intention appears by the end of the first year. Progressively, the child adopts better, and more subtle, thought mechanisms for achieving goals. Increasingly sophisticated coordinations of these primitive schemata prepare the child for internal representation of surroundings in the next stage.

Stage I, the *preoperational* stage, extends from 2 to 7 years. The predominant event is the increasing use of symbols (words and others) to stand for things. Intuitive notions of the world are developed such as cause and effect, conservation of size and so forth. These grow into articulated intuitions from 5 1/2 years to 7, and then interactions with symbols (alphabetic and mathematical). This is the stage where so much of the learning is syntactic (not Piaget's word, of course) in character.

From 7 to 11 years, in Stage II: *concrete operational*, the child conceives the world via concrete operations. These are thought-frames which coordinate complex schemata, but which remain anchored to concrete, tangible experience. As Slavin notes, it is no coincidence that *worldwide*, these are the years of elementary school. The Stage I child used intuitive *regulations* to succeed in tasks like slinging, building, and catapulting rather than *operations* employed by Stage II children (the latter distinguished by being reversible, conscious, discussable). The concrete operations of Stage II include class inclusion, serial ordering, and conservation of volume. The child can categorize objects by size or color. She understands that the volume of a tall flask of liquid is not reduced by pouring it into a shallow pan. The rich tapestry of such *concrete* operations sets the mental stage for the *formal* understanding to follow.

Stage II transformations are more or less serial in nature, one following the other. The degree of formal logic is limited to *relations* between propositions ("size" *relates* to "weight"). By contrast, the logic of Stage III children exhibits all the breadth of what the logician denotes by propositional calculus (A implies B). The children grasp implications integritously, isolate variables, and construct experiments to accept or reject hypotheses. They naturally test implications and their contrapositives before accepting the truth of an explanation. If presented with the assertion that metal things sink, they naturally cast about for a counter-example (like an aluminum boat).

Fully mature, adult modes of thought are denoted as Stage III, *formal operational* (although perhaps half of all adults may not reach fully formal operations). The Stage III individual is capable of reasoning from premises to conclusion and back again to test the consistency of the implication. Contradictions of logic that didn't trouble the Stage II child now demand resolution. The abstract and the possible are more prominent than the tangible and present. The tangibility of the *possible* is at the heart of imaginative problem-solving — in mathematics or any realm.

Even the six-month old baby discovers *cause*, but *represents* little of this knowledge internally; it just isn't a tangible *concept* in the baby. Piaget was tracing the *concepts* of cause, force, effort, ego. He wrote (1976, p. vii) "...although psychologists have tried primarily to determine in which situation a child is cognizant, they have too often neglected the other complementary question of how this happens. From the epistemological point of view, on the other hand, the interiorization of the actions is at the source of both logical-mathematical and causal operatory structures and thus necessitates careful examination." Each individual constructs her own

universe of cognition, her own cognitive structure, that is both the instrument and the outcome of intellectual growth. A powerful influence in this recursive construction is *reflection*. Children do not simply retain exact images of all that they see. Their perceptions are powerfully mediated by the range of transformations available to them at each given stage, and thus may differ substantially from the original events. "Children retain in their memory only what they have understood." (ibid., p. 205) Reflection is assimilating and accommodating what is remembered.

A typical Piagetian experiment illustrates this. Six-year olds quickly learn how to sling a wooden ball at a target, but believe that they are releasing the ball when it is directly between them and the target (which for reference, is called the *12 o'clock* position of the circle made by the sling). In fact, of course, the ball is released either at 9 o'clock or 3 o'clock depending on the direction of rotation. Piaget notes that success in hitting the target is completely separate from the recognition of *how* it is done. It is as though the child is blind to the actual process until the necessary cognitive structures have independently matured. Only children who have reached Stage III (about 11 years) are able to explain (or even *see*) the trajectory of the ball correctly. The conscious awareness of the act is called *cognizance* (*conscience* in the original French). It dawns only through what Piaget calls reflexive abstraction. Reflection is here used in two complementary meanings: (1) reflecting/projecting physical knowledge back to conceptual representations, and (2) reflecting upon and reorganizing the internal representation to form an improved one.

Piaget's theory of cognitive growth has widely been interpreted to mean that formal or abstract subjects cannot be taught until the child is of "appropriate" chronological age. The Kumon practice of letting a child advance through the materials as quickly as she can runs afoul of this popular interpretation. Developmental research subsequent to Piaget suggests that the *sequence* of learning may correspond to his model, but the *chronological* boundaries of readiness are rife with exceptions.

The Houston Kumon office brought an eleven-year old Texas farmer's son who mastered algebra through linear equations, and a ten-year old Japanese girl who has completed the entire series of worksheets (through college mathematics and physics) to the NCTM conference in New Orleans. Hundreds of teachers gathered to observe and question them. Armed only with pencil and paper, Keith was able to keep up with (and sometimes beat) the teachers armed with calculators. Shoko was given a stream of pre-calculus and analysis problems to solve. She worked accurately and quickly, consistently producing correct answers with all appropriate intermediate steps.

The teachers naturally (per Piaget) asked, "Sure, but do they *understand* it?" It is a thoroughly valid question. Even as experienced math teachers, do *we* understand analysis? It's not likely that these children (chronologically positioned at Stage IIB) entertained the *same internal representations* of algebra and calculus as did the NCTM conferees. The internal schemata of Keith and Shoko were likely more symbolic and relational (more *syntactic*) than those of their questioners. However, this doesn't minimize the value of such a rich repertoire of *reliable* syntactic representations, nor the eventual contribution these constructs will make to the demands of high-level problem-solving in later years. Rather, it may raise significant questions as to whether later problem-solving skills might be rooted in just such syntactic facility.

Late in his career, Piaget (1976) reflected deeply upon the contribution of early unconscious or intuitive knowing to later formal operations.

The practical success of the stage I children does not suddenly occur; it results from their varied attempts to complete the tasks....*often without realizing it* [my italics]. In such a situation it is obvious that each of the successive actions is itself partly conscious....However, what is missing in this partial cognizance of each specific action is, in general, the influence that the earlier actions can exert on the the subsequent ones,...The only solution surely is to distinguish... between cognizance of specific actions...and the coordination of the actions into an intelligible whole, from which through reflexive abstraction is born a conceptualization that is attributable to the objects and that constitutes the source of causal coordinations. (ibid., p. 162-4)

Further addressing the evasive boundary of cognizance he writes:

For example, what precedes conception ("subception") and is defined as an "unconscious perception" could well be accompanied by a certain consciousness at the moment it actually occurs, but this remains temporary in the sense that it does not seem to be integrated in the subsequent states ...it is more a question of the degree of integration than one of the sudden passage from the unconscious to the conscious.... In addition to enabling us to analyze how a child gains cognizance as such, this research has shown us that *action in itself constitutes autonomous and already powerful knowledge* [my italics]. Even if this knowledge (just knowing how to do something) is not conscious in the sense of a conceptualized understanding, it nevertheless *constitutes the latter's source* [again, my italics], since on almost every point the cognizance lags, and often markedly so, behind this initial knowledge, which is thus of remarkable efficacy despite the lack of understanding. (ibid., p. 341-7)

When Keith rapidly processes systems of equations to produce the right answers, is this an action embodying "autonomous and already powerful knowledge?" Piaget at least leaves open the door to the possibility that such early syntactic facilities might be contributors to — or the very source of — later higher cognizant processes. Dreyfus (1986) identifies a degree of expert learning when "knowhow" is *embodied* in the learner. Does this delay or interfere with the acquisition of "knowwhy?" Piaget's experiments indicate the usefulness of successful experiences with concrete manipulatives (pre-formal regulations) before higher-order coordinations can be made. It is intriguing (though unproven) to extend this to infer that successes in the syntactic domain can lay the groundwork for coordinations that open up the semantic domain.

One Kumon teacher half-joked about calculus-competent Kumon students like Shoko: even if they don't understand all the concepts of calculus, it's much easier to teach the relevant concepts when they already know how to get the answers! In the absence of the pressure to get answers, ideas and experiments can flow freely in the regular school classroom. This, however, *does* put considerable pressure on the classroom teacher to take skillful advantage of such knowledge in

her charges. This is all the more difficult if only a portion of the students in the classroom are involved in Kumon, and of course even those who are, may be at different levels in Kumon.

As mentioned above, Piaget's pioneering researches have been subsequently questioned in various particulars (particularly the inability of learners to comprehend concepts until certain chronological ages), but remain a powerful watershed of learning theory. More research is needed to understand the growth from the "pre-cognizant" regulations of pre-formal children to the conceptual understanding of adults. The successes of Kumon in particular, and the Japanese, Chinese, German, and other children in general (Stigler 1990, Husen 1967) *hint* at least at a possible (but unproven) linkage between early syntactic facilities and later problem-solving skills.

Bruner: enactment, iconic, and symbolic learning

Across the Atlantic from Geneva (and across Harvard Yard from B. F. Skinner), Jerome Bruner, with George Miller, led the rebellion against behaviorism in founding the the Center for Cognitive Studies in 1960 (Bruner in Hirst 1988). Blind for his first two years, Bruner stands as one of America's most insightful theorists. Eclectic in theory and practice, and acknowledging the debt that all learning theorists owe to Piaget, he stressed problem solving, intuition, and learning through discovery and inquiry. He defined education as *assisting* growth and development by many diverse means. "There are so many aspects of growth that any theory can find something that it can explain well." He therefore filtered theories of cognitive growth by the practical test of increasing independence of *response* from immediate *stimulus*. (A more mature response is one mediated by more knowledge.) Such growth depends upon internalizing events into a consultable world model (like Gagne's), and is identified by the enlarged capacity to use words or symbols to say to oneself or to others what one has done or will do (Bruner 1966).

He identified (1966) three mechanisms of learning which appear in the following order in the life of the child, but persist through adulthood:

(1) enacting: This is knowledge encoded in doing. To reproduce such learning, the learner must "act it out" as in music and sports. This is obviously related to Piaget's sensorimotor learning. Young children can crawl on "all fours" with great skill, but cannot correctly identify the sequence in which they move their arms and legs (Piaget 1976, Chapter 1). Gradually such learning may be replaced by summary images (*icons*), and later by representation in words and other *symbols*, leading to the next two categories.

(2) iconic: This is knowledge which is encoded in diagrammatic representations, and decoded and employed by the application of "syntactic transformations" (Bruner 1966, p. 11).

(3) symbolic: Eventually information is encoded as symbols such as words, numbers, and conceptual maps. Such symbols are powerfully evocative and give rise to the conjectural realms of possibilities. Such realms are where problems are solved. "...the heart of the educational process consists of providing aids and

dialogues for translating experience into more powerful systems of notation and ordering." (ibid., p. 21)

Bruner's colleague at Harvard, George Miller, proposed "seven plus or minus two" as the range of human attention or immediate memory (Miller 1956). Bruner adds, "[C]ompacting or condensing [through icons and, later, symbols] is the means by which we fill our seven slots with gold rather than dross." (Bruner 1966, p. 12) Such learning corresponds closely to the gestalt theory. Gestalt is a German word without a simple English equivalent, but it has the sense of form/structure/relation — kindred to my use of *syntax* and Bruner's use of *icon*.

A key feature of the growth from enacting to icon to symbol is what Bruner calls the *generative* power of the last — its ability, derived from the economy of representation, to evoke metaphors in the solution space, the imaginative realm of potential answers. This is best illustrated by an example which Bruner borrowed from Dr. J. Richard Hayes (Bruner 1966, p. 46-8):

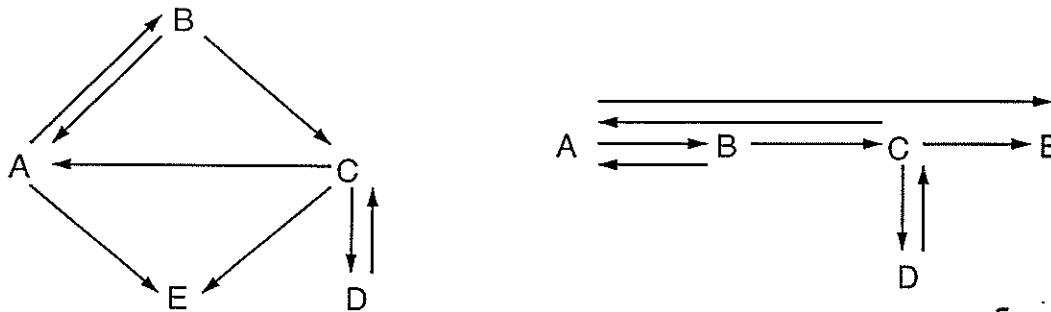
Suppose the domain of knowledge consists of available plane service within a twelve-hour period between five cities in the Northeast — Concord, New Hampshire, Albany, New York, Danbury, Connecticut, Elmira, New York, and Boston, Massachusetts. One of the ways in which the knowledge can be imparted is by asking the student to memorize the following list of connections:

Boston to Concord
Danbury to Concord
Albany to Boston
Concord to Elmira
Albany to Elmira
Concord to Danbury
Boston to Albany
Concord to Albany

Now we ask, "What is the shortest way to make a round trip from Albany to Danbury?" The amount of information processing required to answer this question under such conditions is considerable. We increase economy by "simplifying terms" in certain characteristic ways. One is to introduce an arbitrary but learned order — in this case, an alphabetical one. We rewrite the list:

Albany to Boston
Albany to Elmira
Boston to Albany
Boston to Concord
Concord to Albany
Concord to Danbury
Concord to Elmira
Danbury to Concord

Search then becomes easier, but there is still a somewhat trying sequential property to the task. Economy is further increased by using a diagrammatic notation, and again there are varying degrees of economy in such recourse to the iconic mode. Compare the diagram on the left and the one on the right.



The latter contains at a glance the information that there is only one way from Albany to Danbury and return, that Elmira is a "trap," and so on....The effective power of any particular way of structuring a domain of knowledge for a particular learner refers to the generative value of *his* set of learned propositions....[R]ote learning of a set of connections between cities resulted in a rather inert structure from which it was difficult to generate pathways through the set of cities....The power of a representation can also be described as its capacity, in the hands of a learner, to connect matters that, on the surface, seem quite separate. This is especially crucial in mathematics.

Cultivating in the learner the generative power derived from appropriately economical representation applies to all education. Skill in the rapid manipulation of symbols according to syntactic rules *can* lead to facility in more sophisticated use of symbols in solving non-syntactic problems. Thus symbol manipulation is not the *end* of learning to manipulate symbols, any more than letter recognition by the early reader is the end of learning to recognize letters. The *lack* of the manipulative facility however might constitute just as formidable a barrier to higher mathematics as to reading readiness.

Bruner decries those who invoke Piaget's discoveries to deny a child exposure to formal subjects (like geometric proof) until a certain age. He feels that children can learn any subject in a manner appropriate to their current cognitive development (1986).

The "curriculum revolution" has made it plain even after only a decade that the idea of "readiness" is a mischievous half-truth. It is a half-truth because it turns out that one *teaches* readiness or provides opportunities for its nurture, one does not simply wait for it. Readiness, in these terms, consists of mastery of those simpler skills that permit one to reach higher skills. Readiness for Euclidean geometry can be gained by teaching intuitive geometry or by giving children an

opportunity to build increasingly elaborate constructions with polygons. Or, to take the aim of the new, "second-generation" mathematics project [See the report of the Cambridge Conference of School Mathematics, *Goals for School Mathematics* (Boston: Houghton Mifflin, 1963), one example of the so-called "new math"], if you wish to teach the calculus in the eighth grade, then begin it in the first grade by teaching the kinds and skills necessary for its mastery. (1966, p. 29)

If our assessment of this goal, namely calculus by the eighth grade, is limited by the *implementation* of "new math" in this country, most observers would dismiss it out of hand. As discussed earlier, "new math" may have done little for the *mastery* of the prerequisites for calculus, and (in too many instances) loaded the curriculum with theoretical and conceptual content without sufficient intellectual motivation. Even where adopted, it often failed to significantly change what teachers daily did in the classroom. But to conclude that the *goal* was unworkable misses the point. In fact, many Kumon children *do* learn calculus by the eighth grade. And many European and Asian (virtually all Japanese) children learn calculus before leaving high school. "Since most subjects can be translated into forms that place emphasis upon *doing*, [my emphasis] or upon the development of appropriate imagery, or upon symbolic-verbal encoding [syntactic learning, in our phrase], it is often possible to render the end result to be achieved in a simpler, more manageable form so that the child can move more easily and deeply to full mastery." (Bruner 1966, p. 30)

For example, Papert (1980, p. 185) notes the degree to which this was accomplished in skiing. For years, learners on the snowy slopes labored through an elaborate ritual of pole planting, edge setting, unweighting etc. and few achieved crisp parallel turns without protracted effort. Today the "graduated length method" (GLM) lets learners acquire a comfortable degree of skiing skill in a fraction of the time. Papert describes it not so much as a revolution in skiing as a revision of *vision*. As seen from the earlier instructional examples for addition, Kumon is attempting to do the same for the learning of mathematics. Its method emphasizes the cultivation, by many small steps, of what Bruner calls "appropriate imagery, or ... symbolic-verbal encoding ... so that the child can move more easily and deeply to full mastery."

The degree to which Kumon achieves a "ramp" of gentle enough slope that every learner can scale the heights of mathematics remains an open question. It has certainly worked for many students, and sometimes it has succeeded where other methods had not. But learning remains far too diverse to conclude that this ramp is the only or universally best road. Bruner's observations open the possibility of improved paths to learning, but don't suggest specific mechanisms by which such learning is achieved and measured. His later pioneering work in cognitive science did suggest such mechanisms of learning, and these will be considered in the third section. First though, we must view two more aspects of this many-faceted cathedral of education theory.

Socioconstructivism

For almost all students, learning is *not* a silent march along a lonely trail — no matter how well-prepared the path. Sooner or later, one stumbles on a question (or an attitude!) that requires outside help. Then the role of parent, friend, coach, or instructor is indispensable. The sociocon-

structivist enlarges this episodic support into a community that continually supports and stimulates active learning.

Socioconstructivism is the learning theory most directly associated with the vision of the NCTM Standards (Davis et al. 1990, Cobb et al. 1991). In supporting this vision, Davis observes that modern science, technology, manufacturing — even farming — depend increasingly on mathematics. Our society is not only more mathematical, but more interconnected, than ever. It is not enough just to know mathematics; today one must know how to communicate mathematically with others. One must also learn how to learn math, to think through the analysis of unanticipated mathematical problems. He includes in this category all problems that relate to the patterns that humans find or impose on their environments.

To support a *communicable* grasp of math, socioconstructivists see the *mathematical community* as the ideal learning environment. A narrow view of Piaget might suggest that understanding is inherently individual, and that social actions serve as no more than catalyst to the internal processes of assimilation and accommodation. The socioconstructivist rejects this narrowness, recognizing that every individual does indeed construct her own cognitive structures, her own schemata, but that her community significantly influences this internal construction. After all, it is precisely the *environment* of the learner which is assimilated and accommodated in Piaget's model of equilibration. The environment that nurtures, encourages, values, validates, and stirs the learner can powerfully affect the learning. Davis, Maher and Noddings in introducing *Constructivist Views on the Teaching and Learning of Mathematics* state:

Many people other than mathematicians engage in mathematical activity, and their interests properly vary over a wide range. While engaged in mathematical activity, all of them have to hypothesize, try things out, execute mathematical procedures, communicate and defend results, and reflect on the methods selected and results generated. From a constructivist perspective these activities are all part of what it means to engage in mathematics. Learning mathematics requires construction, not passive reception, and to know mathematics requires constructive work with mathematical objects in a mathematical community.... [L]earners have to construct their own knowledge — individually and collectively. Each learner has a tool kit of conceptions and skills with which he or she must construct knowledge to solve problems presented by the environment. The role of the community — other learners and teacher — is to provide the setting, pose the challenges, and offer the support that will encourage mathematical construction.

In their phrase, the completion of worksheets might be considered relatively weaker "acts of construction." On the other hand, if a small group discovers the Pythagorean Theorem by the consideration of area relationships (perhaps through tessellating tiles), this is a stronger "act of construction." Both types of experience — structured practice and open-ended discovery — are necessary for the construction of powerful mathematical thinking.

Active involvement by students with the learning process has another significant benefit. It adjusts the balance of power in the classroom away from the teacher as the repository of right answers, and toward mathematical thinking itself as the route to veracity. It combines socializa-

tion with ratiocination in a manner reminiscent of Friere (1968). Most work is done in small groups, in pairs, or occasionally as a class. Much less emphasis is placed on individual work.

Students of varying ability are mixed (heterogeneous classes); stronger students are expected to strengthen their knowledge by communicating it with others. This also serves to empower students more equally than systems that visibly place some learners above others (Cobb et al. 1991). The *Standards* include many examples of classroom scenarios illustrating the teaching of mathematics by socioconstructivist methods.

This is an aspect of learning that Kumon simply doesn't address. As discussed earlier, the "Kumon community" does provide supportive elements, but the learning is fundamentally individual. In geometric phrase, the learning community is almost perfectly the complement of the Kumon angle. It would add to the Kumon experience an opportunity to articulate and stretch one's understanding, and to gain the values and empowerment that this affords. Similarly, Kumon could provide opportunities for the individual practice and individual responsibility necessary to solidify the skills and knowledge gained from the classroom experience.

Creativity

In virtually all teaching, individual or social, mathematics or dance, creativity is arguably a legitimate goal. It plays a vital role in later mathematical subjects. The sparks of creativity are needed to build original proofs in geometry or analysis and in nearly any kind of problem solving that goes beyond the application of formulae. The socioconstructivist intends that creative problem solving be the touchstone of the mathematical community. Does the Kumon style of learning tend to stifle the growth of such creativity? Or does it cultivate it through the growth of confidence and self-reliance? Is it orthogonal to creativity? Again, here is an excellent area for scholarly research, especially via longitudinal study of children who have supplemented their school curriculum with many years of Kumon.

A cautionary hint is offered by Teresa Amabile (1989), a chemist who returned to Stanford for her doctorate in psychology, and later taught at Brandeis. After 15 years of research with hundreds of children, she concludes that the blend of vision, values, and passion associated with "creativity" derives from three things: "(1) skill in the domain, (2) creative working and thinking skills, and (3) intrinsic motivation." (ibid. p. 35). Of these the latter is the most important.

She observes that during the 1960's, many argued that rote learning destroyed creativity, as if too much knowledge stifled it. In the 1980's, the "back to basics" advocates argued that command of a specified body of knowledge is a necessary preliminary to creativity. "From the evidence we have about creativity, it appears that both viewpoints have merit. It is not possible to have too much knowledge to be creative. An increase of domain skills (which include knowledge) can only increase the chances of coming up with some new combination of ideas. It is possible for creativity to be stifled by knowledge that is stored in the wrong way." (ibid., p. 87) She offers as an example of the latter, the memorization of multiplication tables *without* any sense of multiplication as a "shortcut for addition." Without the link-building recognition of the relationship between multiplication and addition, the knowledge is not accessible for creative application later. Finally, she reinforces the central role of parents in cultivating creativity by encouraging

and admiring children's work, by discouraging inter-personal competition, by arranging creative events (like costume parties), by not over-structuring children's time, and by allowing free expression of emotions.

In another dimension, the findings of Stigler et al. (1990) and Geary, Fan, and Bow-Thomas (1991) also suggest that the strong development of facility is not incompatible with complex problem-solving, nor with such imaginative endeavors as teaching Martians how to subtract! These observations both encourage and warn us. The structured practice of Kumon is not likely an obstacle to the blossoming of creativity, any more than is daily practice on the piano. However, syntactic learning is not the end of math or music. Activities and learning experiences outside of the worksheets are clearly needed to nurture creativity.

3 Kumon and Cognitive Science

Cognitive science's contributions to education theory

Cognitive science sprouted at the intersection of psychology and computer science. It seeks useful models of human cognition among models of computation. The theory is largely the creation of Piaget, Minsky, Papert, Herbert Simon, and Driiaan de Groot, with contributions by Kurt Lewin, George Miller, and especially in the area of mathematics learning, Davis. The latter writes about the comparison of human and electronic processing:

We certainly agree that many aspects of information processing in computers have a "mechanical" nature that seems quite different from human thought. Yet considering human thought and computer information processing side by side can be extremely valuable, if only because the computer operates in a highly explicit way that forces us into greater clarity in analyzing information processing. When you are programming a computer you are teaching a very stupid student. Effective pedagogy becomes a necessity.

Davis has been actively committed to the earlier-mentioned Madison Project since 1957. This project, associated with University High School (University of Illinois, Champaign-Urbana), explored long range (at least 5 years) mathematics learning, and the specific subject of mathematics education. Mathematics education as a distinct discipline dates from the work of David Smith and J. W. A. Young (University of Chicago and Teachers College at Columbia). It challenges the historical assumption that some people "do" math and some don't. It asks exactly what obstacles impede a person's progress in mathematics. What errors are being made, and why are they made? Computer-suggested models help to provide answers for the cognitive scientist in mathematics education.

The model of human "processing" includes a central processing area with Miller's seven or so slots as working space (the "workbench"). Human memory is vast, but only the items on the workbench can be processed, considered, acted upon, transformed. A secondary area with somewhat more storage stands just outside immediate attention (what Davis calls the "on deck circle"). Access to an item outside these two areas requires a pointer — a label or symbol that permits the whole item to be retrieved. The phrase "first man on the moon" is a pointer to a collection of memories like "Neil Armstrong," "Buzz Aldrin," "the Eagle has landed," "one small step for a man; one giant leap for mankind" and a picture of an astronaut saluting a flag against the black vacuum of space. "Going to the Tiger's game" calls up memories of an aged stadium, two teams, nine field positions against the greensward, etc. Such a collection of associated memories, relations, events, and expectations is called a *frame* (from Minsky). Schank's scripts are similar in concept.

The act of walking into a restaurant serves as a pointer to summon up the *restaurant frame* (which includes waiters, waitresses, Maitre D', menus, food, tip, payment, and so on). This

frame has embedded within it pointers to other frames. For example, a menu is a small frame in its own right. We expect descriptions of food on the left, prices on the right, grouping by position within the meal (appetizer, entre, dessert). We fill in the general menu frame with the particulars of the specific *instance* of menu in our hand: beefburgers are \$3.75; lobster is \$11.75, etc. If we are currently haunted by the *budget frame*, the prices may quickly call up the frame for decimal addition!

This introduces another element of the frame concept. That decimal addition will require more than just the frame into which we fill the particular addends; it also requires the *procedure* of addition. This procedure may call other procedures (such as carrying or regrouping) and so it goes. As humans mature, the frames and the procedures associated with them become more elaborate individually and much more numerous. How do we deal with the ensuing complexity?

Consider the computer programmer charged with writing a word processor — a large and complicated program. She might begin by writing a single procedure to retrieve a word. Such a procedure may be called several times with a bit of additional control to retrieve a whole sentence. Another procedure might build on these to retrieve a paragraph. Other structures may be created to indent the beginning of a paragraph, to add tabs, and so forth. Out of such humble beginnings, elegant word processors are formed. In later stages of development, the program architect is using the structures "MarkSentence" and "FormatParagraph" as encapsulated thought units or *abstract data types* (cf. Wirth 1984). These data models include useful *storage* structures for letters, words, and sentences as well as the *procedures* to perform routine services on them (like reformatting a paragraph to fit neatly on the page).

In short, the software engineer manages complexity by *layering* it. Any block may be elaborated to lower levels, or considered as a unit in the construction of more complex structures. In building the higher-level constructs, she no longer considers the internal structure of the building blocks, just as a skyscraper architect doesn't consider the internal amenities of elevators. Analogously, the beginning piano student visualizes a chord sequentially, note by note. Later it is seen as a single symbol evoking a single response from the hands. It has become an encapsulated module, an abstract data type, in the building of a musical composition — an elaborate structure whose complexity would be unmanageable if every note required separate consideration.

Visually-moderated sequences and integrated sequences

Davis calls the beginner's step by step process a *visually-moderated sequence* (VMS). He likens it to directions to a new house: Drive until you see the windmill; follow the stream to the bridge; then turn left over the bridge until you see the blue house; turn into the driveway. One image or symbol leads to the next. One frame points to the next. One procedure calls the next. When the drive to the new home has been performed enough times, it becomes automatic. We no longer depend on the visual cues (the windmill and bridge) although they are still present. The drive has become an *integrated sequence*. The "verbs" of looking, finding, turning, crossing have melded into a single "noun" — the drive home. Once, an integrated sequence has been formed, it can then serve as a unit for building a more complex visually-moderated sequence which in turn may become another integrated sequence, and so on. The thought units of high schoolers (like solving a linear equation) are unmanageably complex sequences for first graders. Long division

is a visually-moderated sequence for nearly all of us! "The process by which the bits and pieces of a VMS sequence are united into a single entity — an integrated sequence — must be one of the fundamental processes in learning mathematics." (Davis 1984, p. 37)

Davis compares this process of constant integration to sports learning. Coaches analyze batting with a detail that is incomprehensible to the layman — stances, wrists, pointing of toes and so forth. Yet batters have to apply this learning in less than half a second. This calls for the systematic building of integrated sequences that may be called upon at short notice. The analogy may be extended further. When the batter is in a slump, the coach is called upon to debug the procedures by observing the processes, and making suggestions. The coach can't bat for the batter, and can't really *teach* the batter how to bat — the integrated sequence is each batter's own — but the coach *can* note what has changed, what might be out of line with earlier successes, what problems are offered by particular pitchers, and what methods have worked for other hitters.

The process of knowing mathematics is much more than simply knowing how to perform mathematical operations. It extends to the ability to detect and represent mathematics in daily activities. The adaptability and elegance with which we do this depends in some measure on the internalized knowledge representation structures (KRS's) we build to represent the mathematics that we claim to know. (Later, this paper suggests that this ability may be the very *medium* of successful mathematical thinking.) The iconic air travel example from Bruner illustrates the power of appropriate representation.

Debugging

Student errors are not so random as once imagined. On the contrary, student errors turn out to be quite regular and systematic, and it is often possible to predict exactly which wrong answers are most likely for a particular student (Davis 1984, p. 43). These may be caused by buggy procedures or frames, or by summoning up the wrong one for a particular problem. For example, after years of addition problems, children tend to ignore the operation sign. Early efforts in multiplication (e.g. $2 * 4$) often result in a call to the addition frame because the sign has not been important before. The answer is given as 6. If the teacher asks specifically what is $2 \text{ plus } 4$, the answer often suddenly changes to 8 (that is, ignoring the second question) as the earlier error is seen, and the correct frame is retrieved.

The flow of incoming symbols is constantly guiding the retrieval of potential frames for handling them. Relating this process to Piaget's model, Davis writes, "When the judgment is made that the instantiated frame is an acceptable match to the input data, we can say that 'assimilation' occurs. If the judgment is that the match is unacceptable, we have a precondition for 'accommodation' to take place, although more steps are needed before accommodation can be considered complete. (Maybe all that is needed is the retrieval of a new candidate frame)." The frame retrieved, new or old, then becomes the dominant reference point — the original data is thereafter ignored.

Experienced Kumon instructors, like many elementary teachers, are well-versed in such potential roadblocks to the learner. For example, the multiplication fact of $8 * 7$ often causes

more trouble than $8 * 9$. To some degree the sequence of worksheets anticipates and provides for these. Some learners have specific numbers that cause recurring difficulty. The Kumon instructor may recommend specific worksheets to remove such obstacles.

The stored episode (in the above example, $2 * 4$, and the importance of the operation sign) is made available for later reflection, and possibly consideration from outside the episode itself by creating new *meta-language*. To a child who has never paid attention to the sign connecting the two numbers, this is a radical new view of mathematics. New meta-language is needed to think about the combination of numbers *and* operators — perhaps terms to refer to operators, addends, multipliers, etc. Certainly the child doesn't need words like *multiplicand!* "First guy," "second guy," and "guy who tells what to do" will work fine as meta-language for binary operations. Such reflection leads to meta-cognition — an element of Gagne's cognitive strategies which he calls *executive control* — cognizance of the process from outside of the process. Comparable thought assistants are elsewhere called self-critics and deemons. They aid the learner in self-debugging (Davis 1984, Turkle 1984). They are reminders peering over the mental shoulder muttering, "Hmm, that doesn't look right."

There is a significant gap between algorithmic processing and the recognition of meaning; students who merely memorize definitions tend to do very poorly. This is especially true in the development of this meta-cognition in problem-solving. "What is required in the learning of mathematics is not the verbatim repeating of verbal statements, but the synthesis of appropriate mental frames to represent the concepts and procedures of mathematics" (Davis 1984, pp. 200ff.).

Why do some errors persist when the student and teacher both work hard to correct them? Perhaps, the vastness of memory and relative smallness of the active workbench lead to awkward partitioning, faulty retrieval, and flawed control mechanisms. In such a case, Davis notes how paper can become an extension of memory; "especially if the written notations are arranged shrewdly on the paper, this can effectively enlarge workbench memory in a significant way, as nearly everyone knows who has ever done much mathematics."

Overloading the workbench

Skemp, Bruner, and Davis all warn, in different ways, of overloading the workbench. On an overloaded workbench, things fall off into the "cobweb zone" and are lost. Bruner said that symbolic compaction lets us fill the *available slots* with "gold rather than dross." This compaction is analogous to the layering of structures by the software engineer. Underlying structures may, and *must*, be treated as opaque units.

Consider a simple algebra problem:

$$\text{Solve } 8x^2 + 10x - 75 = 0$$

The process necessary to solve this generates "calls" to several procedures: (1) factoring the whole numbers 8 and 75, (2) multiplying binomials $(2x - 5)$ and $(4x + 15)$, (3) solving the two related linear equations, (4) dividing whole numbers, (5) multiplying positive and (6) negative mixed numbers (to check the answers $2 \frac{1}{2}$ and $-3 \frac{3}{4}$), and possibly other procedures as well.

Obviously a failure of any of these procedures dooms the solution, but a subtler point is intended in this example. What happens if several of these processes are grasped in themselves, but only through subordinate procedure calls. Interim factoring results might be put on hold (on paper normally), while the binomial multiply procedure is called to verify trial factors. As the available slots in the short-term workspace are overrun, a critical loss of focus may occur. Rather abruptly, problems of this type enter a domain where they are no longer graspable in their entirety. If subordinate procedures are retrievable, but performed without full mastery (as is far too often the case), the difficulty compounds, and the likelihood of a successful solution declines.

Contrast this with the case where the various "procedures" necessary to the solution have been genuinely *mastered*. Each may be done *directly* without conscious elaboration to further procedure calls. Factoring, fraction operations, and binomial multiplication are done quickly with confidence in the accuracy of the result. Focal attention is not derailed onto rusty sidetracks.

But the most important point remains. I never cease to be surprised at the amount of ambiguity, uncertainty, and flawed logic that students can tolerate and still arrive at correct answers. When I have tutored students one-on-one, and (too rarely, I confess) probed unrelentingly for the details of the process, bugs of years' standing have surfaced. A superficial example is the algebra student who can factor polynomials with aplomb, but who cannot convert .15 to a ratio. Another example is a gifted student (one of the best math students in the state), who could not add negative fractions correctly. Sooner or later these bugs take a *disheartening* toll on even well-motivated students. Why do so few students "last" through four years of high school math? As problems grow more complex, and nested procedures more numerous, the emotional cost to access, manage, and debug subordinate processes grows exponentially (literally and figuratively!). Perhaps, we should be more impressed by, and grateful for, those who do "last!"

In Davis' phrasing, one student factors 75 through a visually-moderated sequence, while another, the mastery student, treats it as an integrated sequence. The former calls the relevant procedure and then works through each of its subprocedures. The latter calls the procedure and retrieves the answer in one step. (Note, too, that a calculator is more of a distraction than a help at this point. Punching in trial factors just adds another procedure layer.)

This analogy extends to tasks outside of mathematics. Envision, for example, the student writing a history paper. As more books are required, the library carrel fills to its limit. Suppose the next sentence to be written refers to the Treaty of Ghent. With an assist from memory, the sentence can be written in less than a minute. Otherwise, the student may need to open a dictionary to verify the spelling of "Ghent" and then a history text or two to recall that it was signed in 1814. The train of thought must now be "re-railed" to complete the sentence, if indeed the main point has not already wandered off.

Admittedly, these examples are simple, but they suggest the cluttering of concepts that can crowd the next new concept off the learner's workbench. At some point, the student lacking mastery becomes frustrated and discouraged, perhaps convinced of her inability to "do" higher-level math. The *new* concepts required for any single advance (for example, trigonometric ratios) may be well within the ability of such a student, but the processing overhead of despised fractions, buggy multiplication, and distracting factoring have trammled progress to a standstill.

This is not to minimize the significant concepts required to handle trigonometry and calculus. But it does ask whether the *additional* concepts constitute as significant a "barrier to entry" as the frustrating tedium of chronically-clogged operations. It also suggests a possible explanation for the success of Kumon students in higher-level courses. More of the mental desktop is clear for the focused attention necessary to acquire the incremental concept (like trigonometric ratios or infinite series). Parents and teachers regularly note that Kumon students also gain a faculty for extended concentration. Such self-discipline may not *enlarge* the workbench, but it certainly helps *manage* it.

Syntax and semantics revisited

As explained before, syntax defines "lawful" or "correct" combinations within a set of symbols. Semantics defines the meaning of such combinations to the knowledgeable recipient. Among the subtler imponderables of computer science is the need for so many programming languages — each with its own rules of syntax — to talk to computers. These syntactically different statements (drawn from five different programming languages)

```
LET Total = 3 + 7.  
Total = 3 + 7;  
Total := 3 + 7;  
(setf Total (+ 3 7))  
LDC 3; ADC 7; STA Total;
```

all have the same semantic meaning, namely: "Perform the operation of addition on the two constants; then change the space in memory labelled *Total* to the internal representation of the result, the integer 10." The semantics are identical; only the syntax is different.

The child's earliest forays into verbal symbolism are often amusing precisely because the syntax is correct, but the semantics fail. "I want wallets for breakfast!" is syntactically lawful. The semantic content is unappetizing at best (but we made her waffles anyway). Similarly the program statement

```
a := a;
```

is syntactically valid, but has no useful semantic content; the variable *a* is unchanged. However, it is precisely this constant frolic in syntax that lets the child *test* the language and discover intuitively, largely unconsciously, the *forms* of language that are admissible and those that are not. In the time the adult requires to formulate a self-conscious question in a second language, the child can make eight mistakes, but find a new truth or two in that language! Modern language teachers differentiate the *learning* of language that labors on in the classroom from the *acquiring* of language that goes on naturally outside the classroom. (This is observable, for example, in African children who routinely learn a tribal, national, and international tongue.) Perhaps, it is more accurate to say that Kumon is a method for *acquiring* mathematics, rather than for *learning* mathematics. This analogy seems especially *apropos* to younger Kumon learners (like the 38-month old girl who can quickly solve a page of linear algebra). (For a

comparison of syntax and semantics in language, see Bruner et al. 1966, Chapter 2. Also see "Talk" in Holt, 1983.)

The line that separates acquisition of syntax from comprehension of semantics is not as sharp as the definitions suggest. Computer scientists have formulated sophisticated *syntax* rules (axiomatic semantics) that completely express the *semantic* content of a well-defined symbol set. In other words, they have completely abstracted semantics into syntax. The informational and operational content of the strings of symbols are fully encapsulated by purely syntactical combinations of "meta-symbols." (Pagan 1981, Chapter 4) This is comparable to Kumon's introduction (shown before) of multiplication. Multiplication merely represents strings of addition. Familiar symbols are used to build bridges to new symbols *and* new meanings.

As a model of learning, syntactic processing is closely related to Bruner's iconic learning, and is very significant even in adulthood. Indeed, the line between symbols and icons is not sharply demarcated. Consider our reactions when closely following another car. Do we process the sudden brake light flash as an icon with a direct (undecoded) link to braking (a conditioned, but unmediated stimulus-response), or do we substitute for the brake-light-symbol its understood meaning, and act upon that meaning?

Another example: reduce the fraction $\frac{3}{6}$

When doing so, do we act upon it as the symbol of a relation between 3 and 6 which are in turn symbols of infinite equivalence classes, or do we make a purely *syntactic* transformation from $\frac{3}{6}$ to $\frac{1}{2}$ without any consciousness of the *semantics* of $\frac{3}{6}$? For most adults, this operation is a procedure call that no longer needs conscious elaboration. Even more to the point, when we *first* learned to reduce fractions, what was the balance between syntactic action and semantic awareness?

For all that we teachers teach of the concepts of mathematics and logic, don't we often suspect that the vast majority of mathematical operations are both initially learned and subsequently performed syntactically? As much as we might wish that children see mathematics in the same way that we do as mature mathematicians, it seems that they generally don't *learn* it that way. Piaget's experiments clearly indicate that the cognizance of concepts plods well behind operational skills. Even the most rarefied conceptual content may stand to benefit from an insightful balance between syntactic and semantic modes of learning.

Syntactic learning, neural nets, and "muscle memory"

A *neural net* is a computer program that learns by being fed a stream of patterns which are described as positive and negative exemplars. It may form links between input variables and output conclusions that surprise us. It may even uncover unstated assumptions. One neural net employed as an "expert system" for loan applications quickly found links between the applicant's race and eventual loan approval that embarrassed, and implicated, its trainers.

Children too learn in as unexpected ways, and, like neural nets, they demand a rich stream of examples and feedback. With little more than this, they learn to ride bicycles, hit baseballs, and

speaking their mother tongue. Fortunately, the learning that is most demanding, complex, intricate, and sensitive to minute error is achieved long before we professional educators have a chance to intervene. Perhaps no processing of the human mind is more demanding than perceiving the three-dimensional world with a pair of two-dimensional, narrow field receptors pouring out a billion bits per second of signal data (Duff 1986). Certainly, nothing learned in a calculus class approaches the number and speed of calculations necessary to grab a firefly on wing or swat a softball. Learning how to judge distance by the angular differential between the eyes is far more complex than multiplying fractions, yet nearly all children learn it (as do owls and cats!) (Spelke in Chapter 4 of Osherson, Kosslyn, and Hollerbach 1990). Do we *teach* the child how to visualize depth? No. Do we worry about her *conceptual representation* of depth? No. We test it, satisfy ourselves that the facility is reliable, and let her have a driver's license! Are we asking the right questions of children's learning in other areas?

Do you remember the flash of insight when you abruptly understood *why* fractions could be divided by inverting and multiplying? For me, it was on the threshold of algebra in junior high school — more than three years after learning the algorithm. My fourth grade teacher was an exceptionally capable teacher — the strongest of my first dozen years. (I am doubtless prejudiced by the fact that she was also the first to recognize and cultivate my mathematical interests; still, her exceptional teaching abilities have been widely recognized in our city). She *must* have explained to us the logic behind inverting and multiplying. It seems that the *receptacle* to receive such a high and wonderful mystery was simply not in place at that time. Consequently I suppose, I can *recall* no such explanation in the elementary years.

I've used the summation of digits (we called it "casting out 9's") to check addition, subtraction, multiplication, and division from elementary years. I first understood why it works during my work in algebraic number theory in college. I must have been capable of understanding it well before then, but I didn't! Also in elementary school, I learned how to extract square roots with the traditional "double-shift-add" algorithm (which the Kumon materials still teach). I did not figure out why this algorithm works until a student asked me about it during my second year of teaching; I figured it out during lunch period that day.

This also illustrates a frequent point in Kumon circles that concepts are easy to grasp if you wait long enough after facilities are acquired! One Kumon instructor observes that his students who have advanced beyond the four basic operations with fractions never have trouble with the concept of fraction addition. Is this because Piagetian structures have grown up during the interim? Or has the early facility nurtured the seeds of later understanding? Or do the children simply gain the understanding elsewhere? Perhaps all three intertwine to produce the result.

The musical analogy, *reprise*

The learning that inheres in our fingers is deeply "overlearned" in Gagne's phrase. Hence its permanence and stability. That's why we can swim, ride bikes, or even play guitars comfortably after years without practice once we have overlearned the skill. In a humbling aside, I confess that my best guitar student *never* really learned to read music, much less the structures and progressions of chords, but through many hours with instrument in hand, he far surpassed his teacher in the *music itself*, eventually appearing on an album.

Tap Masters and Pitch Masters are supplemental computer-assisted instruction (CAI) programs designed to teach rhythm and pitch to elementary school children. They demand progressively higher concentration to succeed in the exercises (imitating and reading pitch and rhythm from standard musical notation). Obviously, correct speed is demanded by this even more than by Kumon. In a study of four schools from third grade to high school, Thomas (1985) reported significant gains in musical ability and in *reading comprehension* (Tests: Metropolitan, McGraw Hill Comprehensive Tests of Basic Skills, Iowa, one school, $p < .005$, the other three, $p < .0005$) after six to twelve weeks of the *musical* training. These gains are in spite of the fact that the experimental groups lost some reading instruction time (compared to the control group) while spending ten minutes per day with the CAI. These results also challenge widely-held assumptions about the speed at which children are capable of learning, about the universality of the ability to learn music, and about the carryover in strengthened abilities "to grasp, remember, and expand reasoning processes."

The study involving the two high schools (in Bellevue, Washington) was reported by Dale in more detail. The Pitch Master CAI was administered for seven weeks to randomly-selected students. During this time students spent from 20 minutes to 8 hours and 35 minutes with the CAI. Both reading comprehension and mathematical skills were pre- and post-tested. Significant (analysis of variance) interaction was discovered between improvements in CTBS Math Application scores and the musical training. Dale cautions that all four of these studies consider short term results. It is not clear to what degree we might expect transferable general improvement from better concentration, reduced anxiety, increased confidence, and, conjecturally, better exercise of the frame retrieval "muscles."

Kinesthetic analogies

Batters talk of curve balls (curving from top to bottom) as "falling off the table," though of course even forkballs obey the laws of projectile and fluid dynamic motion. But these major league pros know how to hit the ball! Analogously, the belief that the words and pictures of children accurately convey their understanding or lack of understanding is oft misleading. There is truth in this assumption, but under a narrow definition of understanding — namely conscious, effable understanding. To make this assumption more concrete: would you want to be flown by a pilot who had only *conceptual* (or *verbalizable*) understanding, or only *facility* understanding of controlling a plane in a thunderstorm? Put another way, would you rather fly with a pilot whose learning consisted of *talking about flying* with other pilots, or one who logged a thousand hours in the air? Yet again, would you rather fly in a jet built by an engineer who knew formulas by rote, or one who interacts broadly with other engineers? The point is labored but hopefully clear: *many* aspects of learning are helpful, and indispensable.

Consider kids' quickness when playing video games. Their conceptualizations of the game's goals and means may err, but they seem much more successful than their concepts deserve. As Piaget said (1976), "...[S]uccess in action precedes understanding of it and cognizance of it... [I]n cases of lasting conflict between observable features of the object and certain preconceived ideas, the former are distorted, and these distortions affect the subject's cognizance." This continues until Piaget's Level IIB where the learner manages to make more accurate observations, although often still at a loss to explain her successes because of yet undislodged *preconceptions*.

The invisible agents of intuition

Every mathematician or computer scientist encounters the ineffable power of intuition. A long-considered problem may suddenly emerge with just the idea needed to solve it. Papert postulates the activity of unconscious agents processing, testing, and recombining potential solutions until a compellingly elegant one emerges from the fray, and is reintroduced into consciousness. A cultivated and informed intuition is a powerful ally to mathematical thinking (cf. Maxfield in Smith 1990, Tell 1980).

Virtually all educators recognize that mere computation of correct answers is very limited without the support of some conceptual model. Beware! Extending this to mean that right answers without complete conceptualization are valueless would entail abandoning much of quantum theory! Notwithstanding the elegance of its modern elaboration, how many physicists have a contradiction-free *concept* of the electron? For that matter, Bohr's *intuitive* grasp of the structure of the atom was leagues ahead of his flawed mathematics. In 1949, Einstein was still marvelling at Bohr's ability to *sense* the truth of the model (Gribbin 1984). One can imagine bemused extraterrestrials overhearing a debate about the Copenhagen interpretation of quantum theory: "(Chuckle), those earthlings get the right answers, but look how tangled their concepts are!" Bohr himself said, "Anyone who is not shocked by quantum theory has not understood it."

Heisenberg's own encounter with the inconceivable is described in a moving passage from his autobiographical *Physics and Beyond*. As he narrates: [May of 1925 on the rocky island of Heliogoland, one morning at three o'clock] "I could no longer doubt the mathematical consistency and coherence of the kind of quantum mechanics to which my calculations pointed. At first, I was deeply alarmed. I had the feeling that, through the surface of atomic phenomena, I was looking at a strangely beautiful interior, and felt almost giddy at the thought that I now had to probe this wealth of mathematical structures nature had so generously spread out before me." (Gribbin 1984, p. 103)

Einstein commented frequently on the significant role of intuition in his discoveries, especially general relativity (Clark 1971, pp. 53, 118, 222, 588-590). Of the latter, he said:

Once the validity of this mode of thought [i.e. general relativity] has been recognized, the final results appear almost simple; any intelligent undergraduate can understand them without much trouble. But the years of searching in the dark for *a truth that one feels*, but cannot express; the intense desire and the alternations of confidence and misgiving, until one breaks through to clarity and understanding, are only known to him who has himself experienced them [my italics].

This not only illustrates the inevitably *individual* effects of intuition, but reminds us of the rough and ill-mapped path to understanding that every learner travels.

Pattern processing

The human mind can perform feats of pattern processing that boggle the imagination. *Pattern* in this context includes the visual but extends to other senses as well. Manipulatives work to give concrete sensory experience, but are less effective in sharpening or refining the syntactic facility, which is nearer kin to pattern processing. Adult life does *not* generally consist of handling manipulatives (with some arguable exceptions like the planner's spreadsheets and the engineer's computer-aided design station). But much of adult life *does* involve syntactic processing. We must feed the visual just as much as we feed the manual. Patterns are processed on more levels than we consciously cognize. The subliminal detection and recall of patterns are well established, and consistently exploited by visual media. Count the number of scene changes during the next TV advertisement. Some change more than once per second. The eye dulls quickly when incoming data dawdles. Even modern movies pump images out every few seconds.

My daughter enjoys a computer game called "Tetris." It is based on the rapid recognition, rotation, and placement of different shapes. Her speed is stunning. Recognition, decision, and execution take place in a few tenths of a second. This is not at all exceptional, but "child's play" for computer games (cf. Loftus and Loftus 1983, Greenfield 1984). In games like "Pac Man," the objects change behavior and strategies at higher levels, requiring the "rote" memorization of a mountain of complex data to continue advancing.

Obviously children are very good at whatever type of learning such skill involves — rapid manipulation and memorization of complex gestalt-like symbol patterns — indeed, better than their parents! I can't match my daughter's speed. In fact, even at a time when I had played more computer games than she had, she was able to progress quickly past me in significantly fewer hours than I had invested. Later in this paper, I consider the significance of the tight loop from eye to finger tips (Penfield and Rasmussen, qtd. in Kimura 1990). This stimulus-response loop must not only be tight and flexible, but highly trainable in childhood.

A rather bucolic example of pattern processing is the determination of the gender of new chicks. John Lunn wrote ("Chick Sexing," *American Scientist*, 36 (1948), pp. 280f. qtd. in Dreyfus and Dreyfus 1986):

With the depression at its lowest point in the early 1930's...one of America's big industries, that of producing eggs and raising chickens, was faced with an important question [determining pullets from cockerels]... Some of the producers had heard of the practice of chick sexing, developed in Japan. This new technique of accurately determining the sex of the day-old chick sounded like one of the mysteries of the Orient. Nevertheless, the hatcheries felt that, perhaps, here was the answer that would save their industry. Five young Japanese experts were sent for.... They gave an amazing demonstration; the American investigators were astounded at the accuracy of the Japanese sexors. One expert, Hikosoboro Yogo, during the demonstration, reached a speed of 1,400 chicks an hour with an accuracy of 98 percent.

Dreyfus notes that even professional poultry men cannot tell the sex of a day-old chick. Fortunately the industry was desperate enough to dispense with formality (or conceptual rationale), and start tossing chicks down the "his" and "hers" troughs. A number of Americans were "trained" (working beside the Japanese for three months), and soon began showing comparable results. "The secret was not in Japanese fingers but in a culture that trusts intuition." (Dreyfus and Dreyfus 1986, p. 197) Continuing in Lunn's words:

[American] Ben Salewski, who learned to sex chickens in 1936 in Washington, claimed that he could sense the sex of the chick by touch...and no doubt there are other experts who use similar intuition in sexing. He has developed an accuracy of 99.5 percent and cruises along at the rate of 900 to 1000 chicks per hour.

How universal are such remarkable capabilities? Does every learner possess them in some measure? Are we really connecting with such processes in our present teaching? Do Kumon's methods of syntactical learning cultivate such abilities?

One popular game at Kumon centers involves arranging magnetic disks with numbers from 1 to 100 on a metal board. The children consistently beat their parents, and are delighted in so doing! Similarly, children regularly outperform their parents in both speed and accuracy on worksheets. This too is highly motivating for the students. This is not just the (potentially patronizing) empowerment suggested by "letting" a class discover how to divide fractions. It is the genuine empowerment that comes from measurable achievement at a level beyond that of the teacher. This does not suggest that the children have passed the teacher's understanding of the salient mathematics, but that the students have been allowed to learn in a manner at which they excel. The confidence inspired by this empowerment *lasts* precisely because it is not given by the adult, and therefore cannot be taken away by an adult. Each student has earned it purely as a result of her own persistent efforts.

4 Issues Deserving Further Exploration

As explained in the introduction, this paper raises many questions. This section explores several worthy candidates for further study by researchers in education, cognitive science, and computer science. The descriptions are general and suggestive; specific details and experiments for future investigation are left to the reader.

Learning concepts

Various studies indicate that we depend too much class time *talking* to our students (e.g. NRC 1991). Writing of both colleges and high schools (but equally or more applicable to elementary schools), they reported:

To believe that one can teach mathematics successfully by lectures, one must believe what most mathematicians know to be untrue — that mathematics can be learned by watching someone else do it correctly. Research shows clearly that this method of teaching does little to help beginning students learn mathematics....It is widely recognized that lectures place students in a passive role, failing to engage them in their own learning....Yet despite its recognized ineffectiveness for most mathematics students, lecturing continues as the dominant form of instruction in mathematics classrooms because it is inexpensive...

We move far too slowly for most of the class, causing boredom which blocks conceptual learning (Davis 1984, cf. Slavin 1991, Chapter 11). The concepts are best formed, and actually only formed, *by the learner*. We can lead the horse to water, and we can talk to him, but we can't make him learn. No more can we *make* a child conceptualize what is in front of her. A learner's internal concepts that differ from our own and from those of her peers are tolerable as long as they do not block further steps or limit her eventual achievement. How do we know they don't? This is assured only by *measuring* (often one-on-one) what can be measured: actual performance in a variety of situations. Certainly we can't rely on mere observation — the most important part is invisible. Asking the learner helps sometimes, but a class size of even twenty-five precludes more than a slender sample of real thinking. Also, neither children or adults are very good at verbalizing their *actual* algorithms. (Cf. Wood 1989, Chapter 2) Sometimes they are afraid that they will sound inappropriate or wrong. If what Johnny *really* thinks is, "Let 1's stab the 0's and leave 9's as corpses," he knows better than to tell us about it! We are conditioned as educators to say something like, "Oh no — you must visualize the 0 as a place holder for groups of 10's." *We* don't visualize it like that! Even if we did, *why should Johnny?* (Cf. Dewey 1938, Chapter 1)

Recognizing the need to teach the concepts while we teach the method, we confront the question of which concepts are we to teach? As shown before, the Kumon approach to the earliest work with addition views numbers as ordinal. That is, $3 + 1$ is understood as the successor to 3. Later $3 + 2$ can be mastered as the successor to the successor to 3, and so on. There is suggestive

evidence that children perform early addition in this manner (as successor operations) even when taught other algorithms (Suppes and Groen 1967).

Approached from this standpoint, the commutativity of addition is a non-trivial discovery — a curious result worthy of amazement and appreciation. If numbers are regarded primarily as cardinal, commutativity is obvious and expected. A different concept has taken the lead. Clearly, successful adult adders mix the concepts naturally, but which is the one to teach earliest? And how can we substantiate or measure the success of such a choice? Again we run aground on the difference between final adult knowledge and the untidy process that creates it.

What if we focus on right and wrong exemplars and on syntactic processes and let the internal linkages and structures remain hidden? In fact, even when we teach the the linkages, the real structures are often hidden anyway and stubbornly resist our best efforts to detect and correct them (Suppes and Groen 1967, Papert 1980, Davis 1984). If the examples, practice, checking, and corrective feedback are finely grained, we should detect potentially damaging flaws in the internal structure or representation in time to adjust them and assure the practical soundness of the structures that remain. The linkages and concepts must be learned, but perhaps we overestimate our ability to teach them verbally.

Moreover, concepts are only half the picture and arguably the easier half. Very few children get through a dozen years of school without at least the primitive concepts of mathematical operations. What is regrettably absent in many case is the grander connections that string the pearls of isolated concepts into a necklace of insight. Does facility learning accomplish this? Not by itself. Does it clear the decks for this type of learning? That depends on the teaching that accompanies and follows the facility learning. As said before, facility learning is knowing that is "in the bones." How often we hear from the golfing pro that it's all in the wrists. Maybe we've worked too hard at convincing ourselves that it's all in the head. Perhaps it's both places.

Recall the illustrative examples (in the second section) of middle school (H) and tenth grade (L) problems routinely and accurately solved by Kumon students. The reader is asked to consider to what degree these could be successfully handled by her own students, and the degree to which such problems are handled through "rote memorization of formulae." In fairness though, the absence of longitudinal studies of Kumon students give no basis for identifying the locus of such learning (regular school, parents, worksheets, etc.) nor for correlating the contribution of Kumon-based learning to the eventual understanding evident in such work.

Most of us educators are occasionally pricked by the successes of methods that apparently sneak around our cherished theories. James' timeless challenge still haunts us: "Show me thy faith without thy works, and I will show thee my faith *by my works*." More than 500 preschool Kumon students have literally "shown their works" up to linear algebra. Perhaps, it's time to more deeply investigate the extent and power of syntactic learning styles and their relationship to later sophisticated problem-solving.

What about calculators?

The NCTM *Curriculum and Evaluation Standards* state:

Because technology is changing mathematics and its uses, we believe that appropriate calculators should be available to *all students at all times*.... [T]he availability of calculators does not eliminate the need for students to learn algorithms. Some proficiency with paper-and-pencil computational algorithms is important, but such knowledge should grow out of the problem situations that have given rise to the need for such algorithms....There is no evidence to suggest that the availability of calculators makes students dependent on them for simple calculations. [my italics]

In the most recent National Assessment of Educational Progress (NAEP), forty-eight percent of the public school eighth graders report that they "almost always" use calculators to work problems in mathematics class. Is this improving their grasp of mathematical concepts? Again, opinions are rife, and the research is ambiguous (Fielker 1990, Kenelly 1990, Pagni 1991, Mullis et al. 1991). In the following tables (14.8 and 14.10 in Mullis et al. 1991), the regular use of calculators is negatively related to "Average Proficiency" as measured by NAEP.

Average scores for Grade 8 Public Schools compared to students' report on the frequency with which they use calculators to work problems in mathematics class:

Almost always: 254	Sometimes: 267	Never: 272
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Average scores for Grade 8 Public Schools compared to students' report on the frequency with which they use calculators to work problems on mathematics tests:

Almost always: 253	Sometimes: 258	Never: 274
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The standard errors are in the range 1.3 to 2.6; these differences are strongly significant, but significant of what? It could be argued that NAEP is unintentionally biased in favor of students who do not use calculators, but such an argument seems hard to support. NAEP's five content areas are numbers and operations; estimation and measurement; geometry; data analysis, statistics, and probability; and algebra and functions. Its three ability levels are conceptual understanding, procedural knowledge, and problem solving. Furthermore, calculators *were* provided during the test for some questions.

The medium of mathematics

"It was Dante, I believe who commented that the poor workman hated his tools. It is more than a little troubling to me that so many of our students dislike two of the major tools of thought — mathematics and the conscious deployment of their native language in its written form, both of them devices for ordering thought about things and thoughts about thoughts. I should hope that in the new era that

lies ahead, we will give a proper consideration to making these tools more lovable. Perhaps the best way to make them so is to make them more powerful in the hands of their users." (Bruner 1966, p. 112)

We must teach students the *medium* of mathematics as well as its message. An artist is not made by her knowledge of brush and oils, but knowing the medium — a knowing born of love *and* of hours of "rote drill" — is an indispensable prerequisite to painterly expression. Mastery precedes masterpiece.

What then is the medium of mathematics? I believe (with Kimura) that it is *abstraction through symbolism* — a medium close in character to the computer scientist's *abstract data type*. It is the clever combination of representation and procedure that precisely encapsulates the insight of centuries, or the algebra lesson of this morning. For example, the concept of addition can be adequately represented by stones, cattle, fingers, or feet, but consider what power for communication and extension inheres in the compact notation of $2 + 3 = 5$. The matrix (with its operations) is likewise a symbolic abstraction of vast evocative extent in the realm of transformations in n -space. So is the Cartesian coordinate system in analytical geometry.

Command of the medium will never replace the powerful unifying metaphors that organize and unite the representations and procedures. This is just as true of mathematics as it is of science, art, music, or history. The methods of Kumon may clear the deck of the mental workspace by efficient organization of the requisite processes and representations. Its trustworthy processes are akin to the trusty fingers of the young artist who has sketched her chestnut mare a hundred times. On that day when vision finally takes wing and the mare is suddenly haloed by the flame of setting sun, the artist is ready to take wing with it on sketchpad. Did the hundred preliminary sketches teach her this vision? No. But, could she have captured, or even been ready to receive, the vision without them. Again, no.

In his autobiographical *In Search of Mind*, Bruner wrote after long consideration of Bach's *Art of the Fugue*

I decided that the exercises that constituted Bach's *Art of the Fugue* bore the same kind of relationship to his *B-minor Mass* that, say, spherical geometry bore to Copernicus's theory of the movement of heavenly bodies, or that combinatorial mathematics bears to gene coding, and so on ad infinitum. He was solving the formal little puzzles of the fugue form, mastering what he would later use to such overwhelming effect in the Mass. *Practice in form* [my emphasis] must in itself then have the same utility in the arts that "playing with mathematics" has for the sciences. And if for music,... why not for the novel, for painting, for poetry, for any art form. (p. 207)

If the medium of mathematics is elegant abstraction, then one substrate of this medium is pencil and paper. What if this substrate could be transformed into a silicon sketchpad?

The Kumon Machine — an instructional computer built from "silicon paper"

In many different incarnations, the computer continues to reappear as an educational panacea. The six national educational goals mentioned earlier are accompanied by a national agenda including more computers in the classroom and a network of special schools. Former Secretary of Education Terrel H. Bell suggests that the federal government provide a computer to every three students in the nation (Bencivenga 1991). Counting hardware and software, this works out to \$900 per pupil. Given the 43 million students, this amounts to about \$40 billion to be spread over five years. The NCTM *Standards* recommend more work with computers in the classroom. But is jostling with keyboard and mouse (and a plethora of odd little commands like Ctrl-Alt-Del) a catalyst or impediment to mathematical exploration?

Papert hoped for the day when the computer would be viewed as pencil — as natural as doodling and as versatile at the artist's sketchpad (1980). In the mid 1970's, Alan Kay at Xerox Corporation's Palo Alto Research Center proposed a radical divergence from the computer paradigms of his day in calling for the *Dynabook* — a notebook-sized supercomputer with the knowledge base of a book (or lots of books) and enough intelligence to help the user find whatever is needed (Kay 1977 and 1984, Markoff 1991). At \$50,000 a copy, the manufacture of the device looked unprofitable if not impossible. But the vision fueled a generation of Silicon Valley dreams.

A decade and a half later, we may be on the threshold of its realization. At Washington University in St. Louis, a project is under way to translate the Kumon method into silicon. Still in the exploratory stages, the team hopes to realize Kay's vision of a genuinely easy to use instructional computer. In the words of Principal Investigator, Dan Kimura (1991(b)):

We are developing a pen-based multimedia computer system for school children. Its goal is to evaluate the technological feasibility of a computerized math curriculum for K-12 based on the Kumon Math Method. At the center of the project is a design of a visual programming language, called Hyperflow, which integrates pen operations, particularly gesturing, into a dataflow-based, traditional visual programming language. Hyperflow is intended to be used not only as an end user language, but also as a system implementation language.

Penfield and Rasmussen noted the disproportionate share of the human motor cortex given to the control of the fingers and eyes (Kimura 1990). The eye-fingertip loop enables remarkable dexterousness. Artists and musicians exploit this to wonderful effect. The focal point of the pianist, violinist, guitarist, cellist, and many others is precisely at the fingertip. Computers which demand input from mouse and keyboard depend upon the arms and shoulders more than the fingertips. The Kumon Machine attempts to realize in silicon the paper and pencil paradigm that most tightly tracks the thinking of the learner.

The machine can present examples, worksheets, and tests using graphics, text, and sound. It receives both commands and data from the stylus ("pen"). A sophisticated telescription and voice compaction system will enable a Kumon instructor to talk with a student while simultaneously observing the pencil work being done on a remote Kumon Machine. Detailed records of the progress and problems of the learner are constantly maintained to help the resident expert

system select activities and remediation. Secondly, such records (rendered anonymous) should provide a wealth of data about the detailed cognitive processes of learning mathematics — not from the keyboard but from the natural medium of pencil and (silicon) paper. This could prove a fertile field for further research.

Larry Cuban's thoughtful analysis (1989) of the encounter between new technology and educators spans the introduction of film, radio, and instructional television which all promised to transform the conduct of education. He writes, "Each of the machine technologies went through a cycle of expectations, rhetoric, policies, and limited use." Each new technology was ushered in with extravagant claims of the transformation of learning, and was scanned warily as a threat to the profession of education. Shortly thereafter, academic studies established the effectiveness of the new technologies in producing genuine learning. Later surveys revealed limited use of the technologies, and eventually administrators and teachers were blamed for the lack of use. Cuban predicted the same cycle for computers in 1985, and seems at least partly vindicated so far (cf. Bass 1987 and O'Neil 1990). He wrote in 1989 that the failures to institutionalize pedagogical revolutions stem not from funding or demonstrations of effectiveness.

It was the high personal costs that teachers and administrators had to pay when they tried to implement different ways of teaching and learning within organizational structures and belief systems that prevented anything more than a passing embrace of neoprogressive notions.... This is how I can begin to explain the remarkable durability of teacher-centered instruction and the graded school, and , of course, the limited penetration of microcomputers into the classroom.

The last three words point to both problem and solution. The teacher is necessarily the final gatekeeper for the conduct of learning in the classroom (cf. Mann 1990). The Kumon machine may find its way into *some* regular schools (as has the Kumon method itself for about 30,000 children). The great majority of users will more likely be supplementing their regular schooling (as do the million other Kumon students), bypassing public gatekeepers altogether.

Models of learning

The expert system is being designed with the support of professional educators and the accumulated knowledge of many Kumon instructors. The system depends upon individual profiles of the strengths and weaknesses of each learner. One of the stumbling blocks of artificially intelligent tutoring systems is the accuracy and usefulness of the learning model (Kearsley 1987 and Clancey 1987). Representations of current knowledge and skills, desired knowledge and skills, and the process by which one moves from the former to the latter are part of such a model. The expert system must make constant assumptions about such learning. A profile of potential "bugs" is maintained, both general, and specific to the learner's history.

We have considered the collection of affective or attitudinal data as well. Hints about the learner's feelings may appear in character size (typically getting larger and messier when frustration sets in). Also changes in the learner's average solution time (offset by problem difficulty), doodling, and general lapses in input, may signal a loss of interest or distraction, or stumbling block. Any increase in the rate of mistakes may signal this. Again, such data may be

made available to the expert system and, in recorded form, to the instructor or parent. An amusing possibility is such program-supplied propositions as: "If you finish this worksheet in the Standard Completion Time, I won't tell anyone about your dropping me during the long division worksheet!" This was the spirit of the best dun letter seen by the author. It had the billing computer say to the delinquent customer: "Right now, only you and I know about this unpaid invoice. If you pay it by X, all is well. If not, I have to tell the humans about it!"

However whimsical are such matters of the human-machine interface, they point to the very significant issue of the human mediation of computer-*assisted* learning. Kumon instructors (or possibly parents) may modify various elements of the system's "expertise" to meet the special needs of the learner. The Kumon Machine does *not* eliminate the specific correction, guidance, and coaching of the instructor. Likewise, the support and guidance of the parent remain indispensable, although the need for correcting worksheets is eliminated. (Even now, such correction is often done by students.)

The syntactic curriculum of the Kumon method affords an unprecedented opportunity to investigate a type of axiomatic semantics for learning (that is, a syntactic definition of learning mathematical processes). It draws upon the educational experience of the unusually well-debugged Kumon materials. Also the breadth of Kumon's popularity suggests a wide cultural and national test bed for this methodology.

Neural Nets

The Kumon Machine may make use of neural nets to recognize handwritten characters and symbols, and possibly to perform other tasks. The current interest in neural nets represents a renaissance of the work in connectionism of thirty years ago. The introduction of back propagation learning, and the dramatic improvement in single chip processors have put the millions of repetitions necessary for this approach within reach of the "palmtop" computer. The basic scheme is elegantly simple — grossly suggested by the neural connections of the brain (Duff 1986, Barlow 1990, Montgomery 1989). A network is some fixed number of inputs is connected to some fixed number of outputs. These two layers may be connected through many intermediate layers. The connections are weighted; not all "votes" are counted equally. As learning takes place, the weights are adjusted to reflect the desired outcomes.

Luger and Stubblefield (1989) report a neural net used to synthesize speech that started with indistinguishable grunts and gurgles, then gained a kind of childish speech, and eventually progressed to intelligibility only by the use of positive and negative exemplars. As the desired results increasingly follow from some combination of weights, the weights are strengthened and stabilized. The fortifying of such well-travelled paths helps the "learner network" to resist occasional "noise" inputs such as faulty exemplars. Etymologically, the word *learn* derives from the Old English word for footprint, which derives from the Latin *lira* or furrow, a well-worn path.

What if the network is constantly perturbed by instability in a lower level? Then the higher-level weights may not stabilize or may require more intermediate layers to stabilize. Either case imposes a penalty on the stability of the higher layers. If the lower layer is sound, we can begin

building higher layers. Imagine the student learning two-digit addition from two teachers employing different methods. Uncertainty about the basic processes will interfere with the assimilation of later operations. It remains to test to what degree the neural network emulates human learning in this respect.

Many problems remain open for further research at the intersection of cognitive science and learning theory. The Kumon Machine may provide a unique educational laboratory for the exploration of these new vistas.

5 Summary and conclusions

The invention of the hammer didn't make carpenters obsolete — the eye that guides it is far more important than the tool. The educator that guides the learner through computer-*assisted* instruction is no more expendable than the student. Amid the enthusiasm for the promise of intelligent CAI, it is well to remember that the hardware is the *easy* part. The change of national culture and support — the overcoming of what Papert called our national "mathophobia" — is the *hard* part.

In even a broader context, the poverty of the educational environment (decaying facilities, unsupportive parents, absent role models, exhausted or uncaring teachers) remains the most intractable dimension of the learning challenge (see Mullis et al. 1991, Kozol 1991). Strongly supportive and caring teaching with weak materials can "win out" over stunning curricular materials and *care-less* teaching; nonetheless, curricula and materials *are* important, and their constant testing and refinement is important (Bruner 1966, p. 42, Mathews 1988, Escalante and Dirmann 1990, Mullis et al. 1991). What can Kumon bring to this quest for improvement, and, in turn, how can its methods be improved by other methodologies?

What challenges does Kumon pose to our educational practices?

In the eyes of the professional educator, the commercial success of Kumon is both "good news" and "bad news." The parents of a million students pay (\$65 in the US) every month. No contract binds them beyond the month for which they have already paid. Re-enrollment only costs \$30 (again, US). This *laissez-faire* obligation notwithstanding, the Kumon Institute of Education has revenues of \$300 million per year, and this does not count the earnings of the 20,000 Kumon instructors (who receive part of the monthly fee). If the program were not producing some degree of at least perceived success, it would vanish, as have several of its imitators. However, its very success as a commercial venture casts a shadow of opprobrium on its pedagogical methods. Its origin in Japan wins a mix of respect and resentment. Its reliance on drill and worksheets (even if computerized), automatically alienates it from many.

For all the talk about individual learning and the clear indications that each learner must build her own cognitive structures, we still rely far more on lecturing and board work (Davis 1984, Mullis et al. 1991, Parker and Kurtz 1990). As important as are the skills of communicating mathematics, they cannot overcome individual accountability for learning some math to communicate. The anecdotal successes of Kumon students are many, but perhaps dismissible given the self-selection of the population by parents interested enough (and rich enough) to pay the monthly fee. The successes in public and private regular schools, even though sparingly documented, are harder to dismiss. Likewise, of course, the comparisons with our international competitors should help rouse a generation lulled by US-based norms.

Kumon's successes challenge us to integrate conceptual presentation of subjects with the syntactic types of learning at which children so excel. They challenge us to provide more

effectively for deeply individual learning. We need to teach children the concentration, responsibility, and self-esteem that pave the way for a lifetime of (still, often individual) learning. We need to help teachers and parents be better "coaches" of mathematical learning — including the skills of perceptive diagnosis. We need to adopt longer-term perspectives on teaching and measuring. When we teach the order of operations (perhaps with fractions), we must be conscious of the implications of today's pedagogy for tomorrow's algebra, trigonometry, and calculus. These are areas where Kumon's model offers useful lessons. None of this is new or unknown. More research into Kumon is certainly needed, but this should not delay the application of what we already recognize as useful.

What challenges to Kumon are posed by other educational systems?

To put the shoe on the other foot, the challenges to Kumon may be divided into curricular, pedagogical, and professional. The curricular are addressed first.

The Kumon curriculum is aimed straight toward calculus. Along this straight and narrow way, it omits significant areas of the NCTM *Standards* — geometry, proportion, measurement, estimating (except indirectly in long division — which is generally less than 25 hours of Kumon's curriculum), and logic. As an aside, the Japanese Ministry of Education, Science, and Culture increased the emphasis on estimating two years ago (Morrow 1991). Perhaps Kumon will now add more on estimation. As a *supplement* to regular schools, these omissions are less significant. They certainly suggest that the Kumon curriculum would not be acceptable as a *substitute* for the curriculum required by nearly all communities, and especially not for the widely-endorsed NCTM *Standards*.

However, the curricular gap is only part of the issue. The character of mathematics *pedagogy* is perhaps the principal thrust of the *Standards*. It is summarized by the following excerpts. First is the vision of future job opportunities and lifelong learning:

Traditional notions of basic mathematical competence have been outstripped by ever-higher expectations of the skill and knowledge....Henry Pollack [notes from a talk given at the Mathematical Sciences Education Board, Frameworks Conference, May, 1987, at Minneapolis], a noted industrial mathematician, recently summarized the mathematical expectations for new employees in industry:

The ability to set up problems with the appropriate operations

Knowledge of a variety of techniques to approach and work on problems

Understanding of the underlying mathematical features of a problem

The ability to work with others on problems

The ability to see the applicability of mathematical ideas to common and complex problems

Preparation for open problem situations, since most real problems are not well formulated

Belief in the utility and value of mathematics

Notice the difference between the skills and training inherent in these expectations and those acquired by students working independently to solve explicit sets of drill and practice exercises.... Problem-solving — which includes the ways in which problems are represented, the meanings of the language of mathematics, and the ways in which one conjectures and reasons — must be central to schooling so that students can explore, create, accommodate to changed conditions, and actively create new knowledge over the course of their lives.

Few disagree with the sentiments of these goals. The disagreement lies in the pedagogy that instills these outcomes. No one style of teaching, certainly not collective learning nor individual drill, *fully* satisfies all these demands. Even the lecture method is occasionally harnessed in the plowing of fruitful fields. As a pedagogy, Kumon methods capitalize on *some* facets of learning such as concentration, self-reliance, and the remarkable ability of children to process syntactic information, while utterly omitting others.

Japan's standardized curriculum and the traditionally slim books (a hundred pages per grade) that "teach" it, are more or less ignored by Kumon. Indeed the single-minded drive for calculus tends to bypass all semantic content prior to algebra. The successes of Kumon suggest that a surprising amount of mathematics is learnable through syntax. As valuable as this learning is, and as useful as are the self-control, individual responsibility, and self-confidence so nurtured, syntax is not the ultimate of mathematics any more that it is of language. The semantic demand may be delayed, but not bypassed. Hinting perhaps at a national shift, the newest middle and high school Japanese textbooks include significantly more of the conceptual motivation, semantic development, and elaboration of topics.

Unsurprisingly, Kumon's strongest success is in the elementary grades (about 75% of its enrollees); preschool, and middle school account for about 10% each. These are, in Piagetian paraphrase, the *pre-formal* years. It is thus expected that the syntactic (or facility-based) methods are successful during these years. As students mature to Piaget's Level III, the *formal* years, the demands on pedagogy mature proportionately. The semantic content of mathematics and the abstraction that it permits are increasingly necessary to the success of students. This is compellingly true for geometry — a subject largely omitted by the Kumon materials. Although Kumon has always positioned itself as only "home study," that is, supplemental to regular school, it might be more successful in these advanced topics if it more deeply addressed their semantic content.

Semantic knowledge has utility beyond its intrinsic value to the student. It can help children learn more *efficiently* as soon as they are prepared to *honestly* assimilate it. In the past Kumon has "piggybacked" on the central curriculum of the public and private programs it supplements. As Kumon is accepted as a central part of the curriculum, the need to address the broader types of learning identified by NCTM and others increases commensurately. This now falls to each

implementer, but much benefit could accrue from coordinating this effort more widely. The Kumon Institute of Education, or perhaps national Kumon headquarters in countries with widespread adoption, could play a supportive role in this coordination.

Compared to the very fine-grained decomposition of learning in the worksheets, the instructional interface is rather coarse (twice per week). The Kumon Machine may represent a partial answer (for those able to afford it), since it will provide a fine-grained response of one sort (even stroke by stroke if it helped!). For the majority of Kumon students though, this may not be available. More must be done to empower parents (answers, hints, guidance, coaching on coaching), and perhaps even more to empower students in this area. (It must be acknowledged, though, that being responsible for one's own answers and corrections — as being able to beat parents on timed activities — can be unbelievably motivating!) This feedback gap is one problem that will not be exacerbated, but mitigated, by incorporation of Kumon into regular schools, since teachers can theoretically respond to students daily. This is complicated of course by the fact that individual students within one classroom may span many levels of Kumon materials.

All educators from Dewey and Thorndike to the present recognize the value of building bridges between skills of problem-solving and the "real world." In light of this, the pervasiveness of sports in NCTM materials (e.g. batting averages, NBA player statistics, team standings) is almost amusing. Kumon materials show an equally disproportionate use of business problems (e.g. sales, production, transportation) *Whose* "real world" do we mean – NBA stars or profitable production schedules? All would agree that many traditional algebra problems (clock hands, reversed digits, age multiples in a hypothetical family, etc.) are *no one's* real world. Kumon needs to filter out these types of problems. We could use more problems from the "real world" of business. Then perhaps employers like the electronics plant mentioned earlier wouldn't have to search so far for statistical quality control workers.

Commercial successes and corporate practices tend to interpose a barrier to the research and interchange that would likely benefit both Kumon and the educational *profession*. Anything *that* successful with parents and children is naturally suspect. However, on the other hand, many outcomes of what I called "the world's largest mathematics learning laboratory" are frustratingly concealed behind the corporate wall. Information is unavailable about criteria essential to the serious researcher: longitudinal studies of individual performance in higher-level course work, standardized test results, correlation of Kumon study with international tests such as those by Husen et al. and Stigler et al. Although voluminous anecdotal evidence suggests the rapid advancement of students beyond their grade level after two years of Kumon, the supporting statistics are not available. Given the various efforts to mimic the materials and methods of Kumon, some degree of corporate caution is natural, but an opening of doors in both directions is most desirable.

The cultural milieu

Kumon grew up in Japan, where the educational curriculum is a national treasure (the National Course of Study, cf. Becker *et al.* 1990), where a government-appointed committee to encourage more relaxation ended up working nights and weekends to complete its report, and where many mothers remain at home to attend to the education of their children. American children average

180 days per year of school; Japanese children average 243. Many children in Japan are expected to attend the *juku* after regular school to improve their chances for admission to the more prestigious *high schools*. Occupations in the scientific fields are highly esteemed. To what degree is its success transferable to different cultures, especially to the United States?

The education of their children is valued by Japanese parents to a depth not comprehended by most Americans (White 1987). The quality of local schools is a primary criterion in locating the home. Parents expect to make many sacrifices for their children's learning, and consequently expect much from it. Family travel plans will be changed, or even cancelled, to accommodate homework demands. US Kumon parents are sometimes nonplussed by requests that the children continue to spend the 10 to 30 minutes per day even during family travels (and *of course* during summer vacation).

Also, the "hidden workforce" of well-educated mothers significantly contributes to the education of Japanese children. Between two-career and single-parent households, this workforce is rarely available in the US, rendering more difficult the steady support of the learning effort. (Ironically, the growing number of Japanese two-career households is slowly fomenting a similar shift there.)

Some resistance to Kumon stems from its Japanese origin — for reasons both economic and ethnic (Lewis and Guidry 1990). Some parents are irritated by the many stray Japanese characters in margins, on answer sheets, and on certificates. Although not readily substantiated, this prejudice seems to have been less of a barrier to American minorities. Such resistance may diminish as more American Kumon instructors open centers, and as Kumon becomes a part of more local school districts.

Finally, there is the issue of graphic art. Although the earliest worksheets (8A through 4A) are somewhat lively and colorful, the rest are black and white. American children and teachers have come to expect richer, and sometimes better, use of color, pictures, and layout.

Where from here?

Even in elementary schools, where Kumon is most popular, its program is regarded as perhaps 35 to 40 percent of the child's mathematics curriculum. In fact, the worksheets 5A through I can be completed by the majority of students in about 750 hours (averaging three repetitions per worksheet — see Appendix B). This is the curriculum from drawing two lines per page to quadratic equations. It could be acquired by the average student in the time spent in seven average months of televiewing (assuming the US average of 28 hours of TV per week).

One Kumon instructor suggests that we use 750 hours of the math classes of kindergarten through fourth grade in just this way — in effect, syntactic learning through algebra. Then during fifth through eighth grades, we could cultivate a rich repertoire of *applications*: physics, chemistry, measurement, estimation, geometric relationships, and so forth. The formulae for volume, area, forces, momentum and so forth would be second nature. This opens a pathway to powerful concepts of advanced chemistry and physics in subsequent years. It also builds a stable platform of mathematics *facility* that frees the learner to cultivate heuristic problem solving and the sharing of mathematical thinking.

This suggestion could lead to geometry and trigonometry in middle school and calculus by ninth or tenth grade. If the compaction suggested by Kumon methods did prove true, it could make room in the curriculum for various innovations, such as computers in the classroom to open new realms of simulation and graphics. Perhaps, even the very Kumon Machines that taught line-drawing, adding, and quadratic equations could be recruited for these explorations.

The "bottom line:" mathematics is still hard work

As Escalante says, mathematics *is* fun, but it is not won without hard work — by learner and teacher. It involves detecting, challenging, and correcting deeply-seated misconceptions — again in learner and teacher. As he, Kozol, and many others observe, children perform to our level of expectation. The plowing, planting, nurturing, stirring and overturning that constitute education require the persistent long-term attention and care of both teacher and learner. Teaching or testing mere mechanical operations amounts to scattering a few seeds on the hard dry surface of possibilities. Our tightening competitive world demands, and our children deserve, that we plant deeply and cultivate persistently the mathematical potential of every student.

The human mind is capable of remarkable feats of recognition, association, adaptation, discernment, flexibility, retention, and insight — far more than we've yet tapped. The learning theories of Piaget, Bruner, and many others have provided indispensable insights into the process of learning. They have unquestionably advanced the descriptive task of mapping the regions of thought, but no one theory can fairly plant the flag of conquest over the kingdom of human cognition. Too many variables yet elude our grasp. On the most cautionary side of this contest, Escalante warns (p. 422):

It is not at all clear to me that the fields of psychology and education mix well. Teachers *teach*. They provide students with the understanding and the ability to *do* something, to achieve a goal. And this — teaching — is something most people can learn to do well. In my opinion, psychology as applied in education is still in the experimental stage. We have mystified the learning process (for teachers as well as students) by accepting psychological theories into the field of education....Teaching is a noble profession and it has been for centuries. It can be improved; it should not be alloyed.

From a very different perspective, Davis, Maher, and Noddings, editors of a book published by NCTM in support of the constructivist approach describe in the Introduction the "major disagreement on how to proceed in order to make things better," that is, US students' performance in mathematics. They state: (1990, p. 2)

If we regard doing mathematics as the following of explicit rules, then a certain kind of instruction should be employed, and a certain kind of research activity seems to be called for. If we regard the doing of mathematics as involving complex processes that call for the use of heuristics and analyses, then another kind of learning activity becomes appropriate, and another mode of inquiry is

needed.... The approaches are incompatible and differ substantially in their emphasis. We can expect a lively debate in the next few years over these issues.

Humbly analyzing past proclamations of any one methodology "supported by current research," we find little correlation between confidence of proclamation and ensuing achievement by students. At best, the diverse models of human cognition and the diverse methods of learning are valuable precisely in their *pluralism*; none has won the exclusive franchise on descriptive (much less predictive) power. This is as it should be. No single food completely nourishes a growing body; even less could a single instructional method properly nurture *one* developing mind, and much less again, all of them.

We *do* "know from the research" that many approaches to teaching mathematics can be successful if honestly supported by parents, teachers, and communities. We know too that students exhibit many different learning styles; the tack that succeeds for one fails miserably for another. It would be hard to find a successful adult user of mathematics who has not experienced "drill and practice" and "heuristic problem-solving" and many other methods of learning — of *constructing* mathematical meaning.

The several examples given from other countries suggest that operational speed and extensive drill are not incompatible with excellence in higher-level problem-solving. Every successful pianist has practiced scales long and hard, and every one has doubtless been inspired to sudden insight by virtuoso (dare we say heuristic) performances of others. We know too, as Escalante and many others confirm, that confidence, concentration, and *self-discipline* are powerful predictors of success in mathematics or any endeavor.

Perhaps it's time to stop using the phrase "we know from the research" to close debate, and allow research to open the conversation. We know the power of evocative metaphors to further the deep learning of concepts — metaphors such as Papert's gears and turtles, vectors in physics, matrices in linear algebra, and balance scales in algebra. Could the stubborn retreat to "we know from the research" betray a dearth of evocative metaphors for mathematics *educators*? The learning community is one such powerful metaphor. So is the artist at easel, the pianist at the keyboard, and the software architect elaborating a vision in clever and extendable *types*. So too is Hirst's mind at play in its gymnasium — trained facility and spontaneous delight seamlessly blended as by a graceful athlete. As we deeply and honestly reflect on these visions, we uncover our *own* new organizing metaphors, our own accommodations to the endlessly new process of teaching.

This paper, like Kumon, necessarily raises more questions than it answers — questions worthy of the best thinking of educational professionals and academic researchers. The issues of social, economic, and political support tower over the environment where children "receive" education. Obviously, we must *care* enough as parents and educators to provide the best learning possible in the present environment, and to improve that environment as we are honestly able. Also we must continue to learn from children and grow more sensitive to how they learn. Then we can more confidently adopt the mix of instructional methodologies that best responds to their diverse needs and that best cultivates their surprising strengths.

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7 Appendices

Appendix A

Biographical sketch

Thomas H. Fuller, Jr. received a B.A. (Math) from Amherst College and an M.S. (Math Education) from Old Dominion University. He is working toward his Ph.D. in Computer Science at Washington University in St. Louis. He taught four years in public schools (inner city and suburban) from remedial eighth grade math to physics, three years in industrial education, and is in his third year at Principia College in the Department of Mathematics and Computer Science. In between and during these years, he taught adults in GED and similar programs. He has worked as an individual tutor off and on for twenty-five years. (Systematic tutoring of a single individual over months or years often affords rare glimpses into the nature of educational success and failure.) He was nominated by his high school Principal for the JayCee "Young Educator of the Year" award.

Fuller held several marketing, instructional, and training development positions at Burroughs Corporation (now Unisys). This included the development and publishing of courses in many media (classroom, print, audio, video, computer-based training, and interactive videodisk). He was "lent" by several of his employers to help develop certification testing in manufacturing, administered by the Educational Testing Service in twenty-seven countries.

For three years, he managed a division of The Christian Science Publishing Society in Boston. He has published articles in *Production and Inventory Management Journal* and two booklets in the *Auerbach Series on Manufacturing*. He served as volume editor, and contributor to *APICS Material Requirements Planning Reprints*. In 1987, Dow Jones-Irwin published his book, *Microcomputers in Production and Inventory Management*.

Fuller is a Fellow of the American Production and Inventory Control Society. He is certified in data processing by the Institute for Certification of Computer Professionals. He served on the course development committee for APICS' Applied Manufacturing Education Series (AMES), and co-taught the first tutorial on AMES. He has served on the MRP and JIT committees, on the Curricula and Certification Council, and was chairman of the newest committee — Systems and Technology. He addresses meetings, seminars, and conferences, including the First World Congress of Production and Inventory Control in Vienna. He has taught manufacturing concepts and methods in the United States and in Europe. In 1986, he led the APICS Manufacturing Study Mission in Japan.

Appendix B

The Kumon mathematics curriculum by worksheets and hours

		Level	Content	Work sheets	Time per sheet			Time per unit	
					Min (minutes)	Max (minutes)	Avg reps	Min (hours)	Max (hours)
Primary	5A	3 yr old	Line drawing	200	1	2	3	10	20
	4A	4 yr old	Numerals and sets	200	1	2	3	10	20
	3A	5 yr old	Introduction to addition	200	1	2	3	10	20
	2A	K	Mental addition	200	1	2	3	10	20
	A	1st	Add,subtract I	200	1	2	3	10	20
	B	2nd	Add,subtract II	200	3	5	3	30	50
	C	3rd	Multiply,divide I	200	4	6	3	40	60
	D	4th	Multiply, divide II	200	4	7	3	40	70
	E	5th	Fractions I	200	4	7	3	40	70
	F	6th	Fractions II	200	6	9	3	60	90
Total hours through F								260	440
Middle	G	7th	Pre-algebra,algebra	200	7	10	2	47	67
	H	8th	Linear equations	200	10	15	2	67	100
	I	9th	Quadratic equations	200	15	20	2	100	133
Total hours through I								473	740
High	J	10th	Higher degree equations	200	10	20	2	67	133
	K	10th	Log,trig functions	200	15	30	2	100	200
	L	10th	Plane geometry/trig	200	20	40	2	133	267
	M	11th	Series,limits,derivatives	200	20	40	2	133	267
	N	11th	Linear algebra,matrices	200	15	30	2	100	200
	O	12th	Differential calculus	200	30	60	2	200	400
	P	12th	Diff,integral calculus	140	30	60	2	140	280
Total hours through P								1347	2487
College	Q		Probability,statistics	110	30	60	2	110	220
	R		Differential equations	200	30	60	2	200	400
Total hours through R								1657	3107