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Dynamic Steiner Tree Problem

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DYNAMIC STEINER TREE PROBLEM

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ABSTRACT

This paper proposes a new problem, which we call the *Dynamic Steiner Tree Problem*. This is related to multipoint connection routing in communications networks, where the set of nodes to be connected changes over time. This problem can be divided into two cases, one in which rearrangment of existing routes is not allowed and a second in which rearrangment is allowed. In the first case, we show that there is no algorithm whose worst error ratio is less than ½ log n where n is the number of nodes to be connected. In the second case, we present an algorithm whose error ratio is bounded by a constant and rearrangment is relatively small.

Makoto Imase is with NTT Software Laboratories. This work described here was performed while on leave at Washington University.

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1. Introduction

With the growth of interest in flexible multipoint communications networks for supporting a wide class of applications, the importance of routing technique for multipoint connections is being emphasized [1,2]. Routing a multipoint connection is typically treated as the problem of finding the shortest subtree of the network containing a set of nodes, called a terminal node set. If the terminal node set is known in advance and does not change, this problem is a classical problem in graph theory, the Steiner Tree Problem (ST), which has been studied extensively [3,4] including the implementation of a distributed algorithm [5].

In order to support some services, for example video broadcasts and multi-person conferences [1], we need facilities for adapting to changes in the terminal node set. There are relatively few studies dealing with this problem [6] in spite of its practical importance.

This problem, which we call the *Dynamic Steiner Tree Problem* (DST), comes in two flavors, one in which rearrangement of existing routes is not allowed and a second in which rearrangement is allowed. In the second case, we consider the number of rearrangements in addition to the cost of a generated tree.

This paper considers algorithms for DST mainly focusing on the worst case error ratio of the generated trees relative to minimum Steiner trees (T_{OPT}). After the formal definitions of these problems in Section 2, Section 3 shows that in the non-rearrangeable case there is no algorithm whose worst error ratio is less than $\frac{1}{2}\log n$, where n is the cardinality of a terminal node set. Section 4 proposes an algorithm for the rearrangeable case whose error ratio is bounded by a constant with relatively little rearrangement.

2. Definition

In DST we are given a graph $G = (V, E)$, a cost function $\text{cost} : E \rightarrow \Re^+$ and a sequence of requests $R = \{r_o, r_1, \ldots, r_k, \ldots, r_K\}$, where each r_k is a pair (v_k, ρ_k),

$v_k \in V$, $\rho_k \in \{add, remove\}$. Each request is to add a node to, or to remove a node from, a connection. We let S_k =

$$\{v_i \mid (v_i, add) = r_i \text{ for some } i\ (0 \le i \le k),\quad (v_i, remove) \ne r_j \text{ for any } j\ (i < j \le k)\},$$

and refer to S_k as *a terminal node set at Step k*.

The object is to find a minimum cost tree connecting terminal nodes in S_k for each k without knowledge of r_j for any $j > k$. This problem can be divided into two cases. In the first case, once a particular set of edges has been used in a route, no rearrangement is allowed as the algorithms proceeds. In the second case, rearrangement is allowed.

Problem 1 (DTS-N) *Given an instance* (G, cost, R), *find a sequence of trees,* $\{T_1, T_2, \ldots, T_K\}$, *satisfying the following conditions and minimize some cost function, for example* $\sum_k \text{cost}(T_k)$:

1. *Each* T_k *spans* S_k.
2. *If* r_k *is an add request,* T_k *includes* T_{k-1} *as a subgraph.*
3. *If* r_k *is a remove request,* T_{k-1} *includes* T_k.

Conditions 2 and 3 imply that edges and nodes are added to a tree only for an add request and removed only for a remove request.

In the rearrangeable case, we also minimize the rearrangements to get T_k from T_{k-1}. In other words, we want to find an algorithm minimizing the cost of generated trees and the modification. Two operations, *a primitive path deletion* and *a primitive path insertion*, which correspond to point-to-point call connection and deletion, are defined to measure the number of rearrangements. In deleting a subgraph p from T, if p is a (simple) path and every intermediate node in it is neither a terminal node nor a Steiner node, the operation is called primitive path deletion. A Steiner node is a node in T having degree larger than 2. In adding a subgraph p of G to T, if p is a path and every intermediate node in it is not contained in T, the operation is called a primitive path insertion. Any modification to T is restricted to primitive path deletions and insertions. Then *the number of rearrangements* α_k at step k is defined as follows:

- If r_k is an add request, α_k is the number of primitive path deletions necessary to get T_k from T_{k-1}.
- If r_k is an remove request, α_k is the number of primitive path insertions necessary to get T_k from T_{k-1}.

Problem 2 (DST-R) *Given an instance* (G, cost, R), *find a sequence of trees,* $\{T_k\}$ $(k = 1, 2, \ldots K)$ *where each* T_k *spans* S_k *and minimize some cost function and some function of the* $\{\alpha_k\}$, *for example* $\sum \alpha_k$.

3. Non-rearrangeable Case

We first consider the worst case performance in terms of the error ratio for DST-N restricting our attention to the case where each request is an operation.

We begin by defining graphs G_κ, $\kappa \in Z_0^+$ and a constant cost function c_κ on the edges of G_k. $G_0 = K_2$, the complete graph with two nodes, where its single edge has a cost of 1. We name the two nodes $v_{0,0}$ and $v_{0,1}$ and refer to them as level 0 nodes. Graph G_κ for $\kappa > 0$ is defined using graph $G_{\kappa-1}$. For each edge (u, v) in $G_{\kappa-1}$ we introduce a pair of nodes α, β and replace (u, v) with two paths (u, α, v) and (u, β, v). We refer to the nodes α and β as level κ sister nodes. Each edge in G_κ is then assigned a cost of $2^{-\kappa}$. Note that $v_{0,0}$ and $v_{0,1}$ will be connected by (simple) paths of cost 1. (See Fig. 1.)

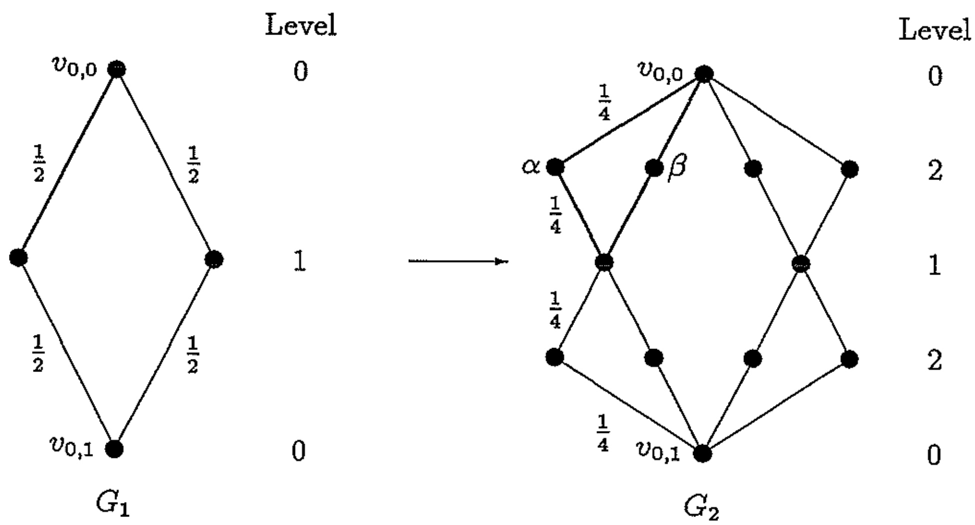


Figure 1: Example of G_k

We say that two nodes $u, v \in V(G_\kappa)$ are i-adjacent, $0 \leq i \leq \kappa$ if the level of both u and v is no more than i and there is a path from u to v which has no intermediate node from level j, $j \leq i$. That is the corresponding nodes in graph G_i would actually be adjacent. Note that exactly one of two i-adjacent nodes must be a level i node and that the distance between i-adjacent nodes is 2^{-i}.

Lemma 3.1 *In graph G_i, $i > 1$,*

i. *each level i node α is adjacent to exactly two nodes, node v at level $i-1$ and node u at level j, $0 \leq j < i-1$. In addition node α has a sister node β at level i which is also adjacent to both u and v.*

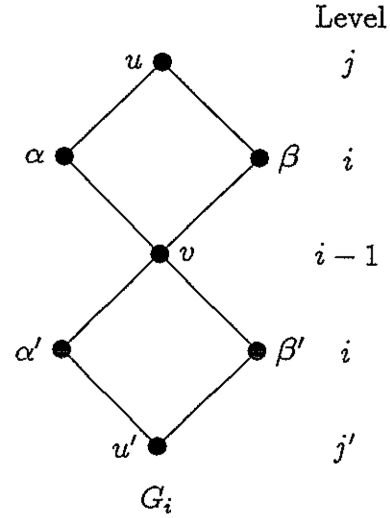


Figure 2: Level i Sister Nodes

ii. *each level $i-1$ node v is adjacent to exactly four level i nodes, consisting of two sister pairs.*

Thus, in graph G_κ, $\kappa \geq i$, (i) and (ii) hold if the term adjacent is replaced by i-adjacent.

Proof This lemma follows by a simple induction on i based on the recursive definition of G_i. Figure 2 illustrates the properties described here for two sister pairs α, β and α', β' in graph G_i. ∎

A sequence of node sets $N = \{N_0, N_1, \ldots, N_\kappa\}$ is called a κ-sequence for graph G_κ if there exists a path p from $v_{0,0}$ to $v_{0,1}$ such that N_i is the set of level i nodes in path p. Note that $N_0 = \{v_{0,0}, v_{0,1}\}$, that $|N_i| = 2^{i-1}$ for $0 < i \leq k$ and that $\cup_{i=0}^{i=\kappa} N_i = V(p)$, the set of nodes in path p. We now define a minimal tree sequence $\hat{T} = \{\hat{T}_0, \hat{T}_1, \ldots, \hat{T}_\kappa\}$ for graph G_κ, with respect to a κ-sequence N. $\hat{T}_0$ is any tree that spans the nodes in N_0 such that no proper subgraph of $\hat{T}_0$ also spans N_0. $\hat{T}_i$, $0 < i \leq \kappa$ must contain tree $\hat{T}_{i-1}$ as a subgraph and all of the nodes in N_i from the κ-sequence N. The requirement that $\hat{T}_i$ is minimal means that no subgraph of $\hat{T}_i$ also satisfies these properties.

In the next lemma we consider a minimal tree sequence $\{\hat{T}_i\}$ constructed by an algorithm, which generates each $\hat{T}_i$ based only on knowledge of $\hat{T}_{i-1}$ and N_i. We choose the terminal nodes for each N_i based on $\hat{T}_{i-1}$, $i > 0$ in order to make the cost of each $\hat{T}_i$ as large as possible. Restricting N to a κ-sequence insures that there exists a minimum Steiner tree for the entire terminal set with cost 1.

Lemma 3.2 *Given a graph G_κ, $\kappa \in Z_0^+$, and any algorithm A which constructs a minimal tree sequence, where each tree $\hat{T}_i$ is generated based only on knowledge of*

$\hat{T}_{i-1}$ *and* N_i*, there exists a* κ*-sequence* N *such that the minimal tree sequence for* N *satisfies the inequality*

$$\text{cost}(\hat{T}_i) \quad \geq \quad 1 + \frac{1}{2}i \tag{1}$$

for all i*,* $0 \leq i \leq \kappa$*.*

Proof: We prove this lemma for each $\hat{T}_i$, $0 \leq i \leq \kappa$ by induction on i. Inequality (1) clearly holds for $i = 0$ since the distance from $v_{0,0}$ to $v_{0,1}$ in G_κ is 1. If we choose N_1 so that it contains the level 1 node not in $\hat{T}_0$ then $\text{cost}(\hat{T}_1) \geq 1.5$. Note that there is a path of cost 1 in G_κ that contains all nodes in N_0 and N_1, so that N_0, N_1 is an initial segment of some κ-sequence.

Let us now assume that (1) holds for every j, $0 \leq j < i$ where $1 < i \leq \kappa$ and that $N_0, N_1, \ldots, N_{i-1}$ is an initial segment for some κ-sequence. Since $\hat{T}_{i-1}$ is a minimal tree each leaf node must be in one of the N_j and there are no cycles in $\hat{T}_{i-1}$. Consider a node v from N_{i-1}. Node v is i-adjacent to exactly four nodes at level i consisting of two sister pairs by Lemma 3.1 (ii). If $\hat{T}_{i-1}$ contains both nodes of a level i sister pair then $\hat{T}_{i-1}$ has a cycle or a leaf node at a level greater than $i-1$. But this contradicts the minimality of $\hat{T}_{i-1}$, hence $\hat{T}_{i-1}$ can contain at most one of the nodes from each sister pair adjacent to v.

For each node $v \in N_{i-1}$ we select one node from each sister pair of v, that is not in $\hat{T}_{i-1}$, to place in N_i. Note that if there is a path of cost 1 which contains all nodes from every N_j, $0 \leq j < i$ then there is a path of cost 1 which also contains the nodes in N_i. The cost of a shortest path from each of the nodes in N_i to a node in T_{i-1} will be 2^{-i} since each is i-adjacent to a node in N_{i-1} and every leaf node of $\hat{T}_{i-1}$ must be at a level no greater than $i-1$. All the nodes we select for N_i are distinct since each is i-adjacent to only one node at level $i-1$ by Lemma 3.1 (i). Since $|N_{i-1}| = 2^{i-2}$ we will have selected 2^{i-1} level i nodes to include in N_i, and N_i will contain all level i nodes along a path from $v_{0,0}$ to $v_{0,1}$. Thus, it follows that $\text{cost}(\hat{T}_i) \geq \text{cost}(\hat{T}_{i-1}) + \frac{1}{2}$ and that $N_0, N_1, \ldots, N_i$ is an initial segment for some κ-sequence. ■

Using Lemma 3.2 we derive a lower bound for the best possible worst case error ratio given any algorithm for DST-N.

Theorem 3.1 *Given any algorithm* A *for* DST-N *there is an instance of the problem (*G*,cost,*R*) such that for all* i*,* $0 < i \leq K$

$$\frac{\text{cost}(T_i)}{\text{cost}(T_{opt})} \;\; \geq \;\; 1 + \frac{1}{2}\lfloor \lg(n_i - 1) \rfloor \tag{2}$$

where $\{T_i, 0 \leq i \leq K\}$ *is the sequence of trees generated by algorithm* A *and* $n_i = |S_i|$*. Here* T_{OPT} *is a minimum Steiner tree for* S_i*. Furthermore this bound holds even if each request is restricted to node addition.*

Proof: We consider an instance of the dynamic Steiner problem (G_κ, c_κ, R) where $R = \{(u_i, add), 0 \leq i \leq 2^\kappa\}$, all u_i are distinct with $u_0 = v_{0,0}$ and for $i > 0$ $u_i \in N_j$, $j = \lceil \lg i \rceil$ for a κ-sequence N. At steps i, $2^j \leq i < \min(2^{j+1}, 2^\kappa + 1)$ for $0 \leq j \leq \kappa$ the solution, generated by any algorithm A, must contain some minimal tree $\hat{T}_j$. Note that a minimum Steiner tree for nodes in the sequence N has cost 1 by the definition of a κ-sequence. By Lemma 3.2 we can construct a κ-sequence so that $\text{cost}(\hat{T}_j) \geq 1 + 1/2j$. This gives us a lower bound on the worst case performance of any algorithm without rearrangement and the theorem follows. ■

We present a simple dynamic greedy algorithm (DGA) for DST-N which has performance within two times the best possible bound in the case where each request is restricted to node addition. For each add request DGA joins the new node by a shortest path to a nearest node already in the connection. In the case of a remove request a terminal node is dropped by simply deleting the portion of the connection which serves only that terminal node. See Fig. 3 for complete details.

```
T_0 := ({v_0}, φ); S_0 := {v_0}; k = 1;
do k ≤ K →
      if r_k is an add request →
            Choose the shortest path p_k from v_k to T_{k-1};
            T_k := T_{k-1} ∪ p_k;
            S_k := S_{k-1} ∪ {v_k} and increment k by one;
      | r_k is a remove request →
            S_k := S_{k-1} − {v_k};
            T_k := T_{k-1};
            do V(T_k) − S_k contains node w with degree 1 →
                  T_k := T_k − w;
            od
      fi
od
```

Figure 3: Dynamic Greedy Algorithm (DGA)

Theorem 3.2 *For any instance (G, cost, R), of* DST-N *let $\{T_k,\ 0 \leq k \leq K\}$ be a sequence of trees generated by* DGA *and let $n_k = |S_k|$. Then if each r_k is an add request*

$$\frac{\text{cost}(T_k)}{\text{cost}(T_{opt})} \leq \lg(n_k) \tag{3}$$

for $0 < k \leq K$.

Proof: Let $v_0, v_{\alpha_1}, \ldots, v_{\alpha_i}, \ldots, v_{\alpha_k}$ be a preodering of the tree T_{opt} for the node set S_k with root v_0, that is, we list a node the first time we visit it during a depth-first traversal of T_{opt} from v_0. First construct a new graph L_k which consists of a path from v_0 to v_0' joining v_{α_i} and $v_{\alpha_{i+1}}$ for $0 \le i < k$ by an edge with cost $\text{dis}(v_{\alpha_i}, v_{\alpha_{i+1}}; T_{opt})$. In the case of $i = k$, join v_{α_i} and v_0' with cost $\text{dis}(v_{\alpha_i}, v_0; T_{opt})$. Then the cost of this path, $cost(L_k)$, is equal to twice of the cost of T_{opt}, because the depth-first traversal visits every edge exactly twice.

In L_k, let l_{v_i} be the distance from v_i to the nearest node whose subscript is less than i, that is,

$$l_{v_i} := \min_{j<i} \text{dis}(v_i, v_j; L_k).$$

Since $\text{dis}(v_i, v_j; G) \le \text{dis}(v_i, v_j; T_{opt}) = \text{dis}(v_i, v_j; L_k)$ and the node v_j is contained in S_{i-1},

$$\text{cost}(p_i) \le l_{v_i} \qquad \text{for } \ 1 \le i \le k$$

where p_i is the path selected by DGA to join v_i to tree T_{i-1}. If we show that

$$\sum_{i=1}^{k} l_{v_i} \le \frac{\lg(k+1)}{2} \text{cost}(L_k) \tag{4}$$

the proof will be completed because $n_i = k + 1$ and

$$\text{cost}(T_k) = \sum_{i=1}^{k} \text{cost}(p_i) \le \sum_{i=1}^{k} l_{v_i} \quad \text{and} \quad \text{cost}(L_k) = 2\text{cost}(T_{opt}).$$

In the remainder of the proof we show that (4) holds for any path L_k.

Given nodes, v_i and v_j, the path between them in L_k is denoted by $L(v_i, v_j)$. Let $\text{sum}(L(v_i, v_j))$ be the summation of l_x where x is an intermediate node in $L(v_i, v_j)$. If the subscript of every intermediate node is larger than i and j, this path is called *regular*. Note that L_k is also denoted by $L(v_0, v_0')$ and is regular. Let v_l be the intermediate node in $L(v_i, v_j)$ with the smallest subscript. If $L(v_i, v_j)$ is regular, $l_{v_l} = \min\{\text{dis}(v_l, v_i), \text{dis}(v_l, v_j)\}$. Furthermore both $L(v_i, v_l)$ and $L(v_l, v_j)$ are regular.

Thus, for a regular path $L(v_i, v_j)$, the summation $\text{sum}(L(v_i, v_j))$ is calculated by a recursive function as follows:

$$\text{sum}(L(v_i, v_j)) = \begin{cases} \min\{\text{dis}(v_l, v_i), \text{dis}(v_l, v_j)\} + \text{sum}(L(v_i, v_l)) + \text{sum}(L(v_l, v_j)) \\ \qquad\qquad\qquad if \ v_l \in L(v_i, v_j) \\ 0 \qquad\qquad\quad\; if \ v_l \notin L(v_i, v_j) \end{cases} \tag{5}$$

where v_l is the intermediate node with the smallest subscript.

Let $L(v_i, v_j)$ be a regular path whose end nodes are v_i and v_j and let m be the number of intermediate nodes. The following inequality will be shown by induction on m,

$$\text{sum}(L(v_i, v_j)) \le \frac{\lg(m+1)}{2} \text{dis}(v_i, v_j) \tag{6}$$

which implies (4).

1. $m = 0$
   From the second case of Eq. 5, it is clear that (6) is valid.

2. $m > 0$
   Assume for any regular path with m' intermediate nodes ($m' < m$), (6) holds. Let v_l be the intermediate node whose subscript is less than that of any other intermediate node and let x, x_1 and x_2 be $\text{dis}(v_i, v_j)$, $\text{dis}(v_i, v_l)$ and $\text{dis}(v_l, v_j)$, respectively, then $x = x_1 + x_2$. From Eq. 5,

$$\text{sum}(L(v_i, v_j)) = \min\{x_1, x_2\} + \text{sum}(L(v_i, v_l)) + \text{sum}(L(v_l, v_j)) \tag{7}$$

   As we can assume $x_1 \leq x_2$ without loss of generality,

$$\text{sum}(L(v_i, v_j)) = x_1 + \text{sum}(L(v_i, v_l)) + \text{sum}(L(v_l, v_j)). \tag{8}$$

   Let m_1 and m_2 be the number of intermediate nodes in $L(v_i, v_l)$ and $L(v_l, v_j)$, respectively, then $m_1 + m_2 = m - 1$, which implies $m_1 < m$ and $m_2 < m$. By the inductive hypothesis and Eq. (8)

$$\begin{aligned} \text{sum}(L(v_i, v_j)) &\leq x_1 + \frac{\lg(m_1+1)}{2} x_1 + \frac{\lg(m_2+1)}{2} x_2 \\ &= x_1 + \frac{\lg(m_1+1)}{2} x_1 + \frac{\lg(m-m_1)}{2}(x - x_1). \end{aligned}$$

   Let $g(m_1, x_1)$ be the right side of this inequality, where $0 \leq m_1 < m$ and $0 \leq x_1 \leq x/2$, then

$$\text{sum}(L(v_i, v_j)) \leq \max\{g(m_1, 0), g(m_1, x/2)\}$$

   because g is a linear function of x_1.

$$\begin{aligned} g(m_1, 0) &= \frac{x}{2}\lg(m-m_1) \leq \frac{x}{2}\lg m < \frac{x}{2}\lg(m+1) && (9) \\ g(m_2, x/2) &= \frac{x}{4}(2 + \lg(m_1+1) + \lg(m-m_1)) && (10) \\ &= \frac{x}{2}\lg 2\sqrt{(m_1+1)(m-m_1)} \leq \frac{x}{2} lg(m+1) && (11) \end{aligned}$$

   Thus, $\text{sum}(L(v_i, v_j))$ is not larger than $x\lg(m+1)/2$. ∎

We now consider the general case of DST-N where we allow both the addition and removal of nodes. In this case the situation is even worse. In the next theorem we show that any algorithm for DST-N has worst case performance that is unbounded as a function of the number of terminal nodes in the solution tree.

Theorem 3.3 *Let A be any algorithm for* DST-N, *and let $\{T_i,\ 0 \leq i \leq K\}$ be a sequence of trees generated by A for an instance (G, cost, R) of* DST-N. *Given any $M, \ell \in Z^+$ there exists an instance (G, cost, R) of* DST-N, *and an integer $j \in Z^+$ such that*

$$\frac{\text{cost}(T_i)}{\text{cost}(T_{opt})} \geq M \tag{12}$$

for $j < i \leq j + \ell$ independent of the number of terminal nodes at step i.

Proof: Let graph G contain cycle C_{M+2} with $M+2$ additional nodes, let each edge in C_{M+2} have cost 1, and let every node in the cycle be connected to a distinct node not in the cycle by an edge of cost ϵ. Let the set R consist of an initial sequence of $M+2$ add requests, one for each node in C_{M+2}. Let the next M steps remove each node of degree 2 in T_{M+1} to create T_j, where $j = 2M+1$. Thus, the cost of T_j is $M+1$ while an optimal solution has a cost of 1. For the remaining steps we alternately add and remove one of the nodes connected to a leaf node of T_j. We assume that the value of ϵ is sufficiently small so that (12) holds. ∎

If we relax our definition of DST-N so that the solution at each step need not be a tree, Theorem 3.3 no longer holds. However, $1 + \frac{1}{2}\lceil \lg(n_i - 1) \rceil$ is still is a lower worst case bound on the error ratio even with this modification.

4. Rearrangeable Case

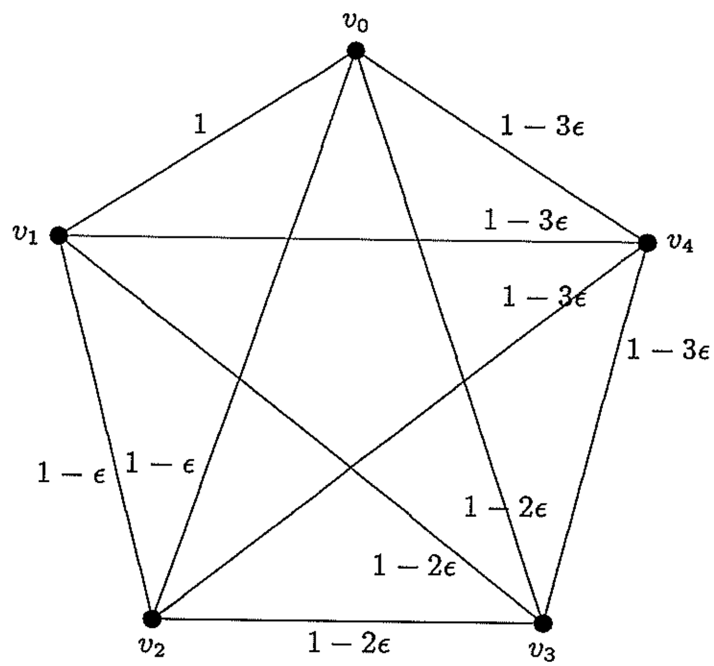
For a given graph $G = (V, E)$ and a cost function cost, we can define a complete graph $G' = (V', E')$ with cost′ where $V = V'$, $E = \{(u,v) | u, v \in V\}$ and $\text{cost}'(u,v) = dis(u,v;G)$. The optimum Steiner tree for G' is the same as that for G, and vice versa. Further cost function *cost′* satisfies the triangle inequality, that is, $\text{cost}'(u,v) \leq \text{cost}'(u,w) + \text{cost}'(w,v)$ for any $u, v, w \in V$. In this section in order to simplify the explanations we assume the input network G and cost is a complete network satisfying the triangle inequality. We call this graph a *distance graph.* The results obtained remain valid even without this assumption.

4.1. Edge-Bounded Tree

If we do not consider the number of rearrangements, problem DST-R is treated as ST for each instance (G, cost, S_k), which is an NP-hard problem. As a starting point, we apply the Minimum Spanning Tree Approximation Algorithm(MSTA) for ST [7]. MSTA is one of the most well known heuristics for the Steiner Tree Problem, because, in spite of its simplicity, it has the best worst case behavior among all the known heuristics; the minimum spanning tree T_{MST} produced by MSTA has a cost that is never more than twice optimal. T_{MST} is the minimum spanning tree for the subgraph induced by S_k.

However if we apply MSTA to DST-R, the number of rearrangements would be very large. For example, in the case of the graph shown in Fig 4, the number of rearrangements for each Step k is k, which implies we may have to change every edge.

While MSTA generates T_{MST} as a solution, the algorithm proposed is based on a δ edge-bounded tree defined below.



$$cost(v_i, v_j) = 1 - (i-1)\epsilon \quad \text{where} \quad i > j.$$
$$R = (v_0, add), (v_1, add), \ldots, (v_K, add),$$

Figure 4: Example for Maximum Number of Rearrangements

Definition 1 *Let u and v be nodes in T. If u and v satisfy the following condition, they are called a δ edge-bounded pair.*

$$\textit{For all} \quad e \in p(u, v; T), \quad \mathrm{cost}(e) \leq \delta \cdot \mathrm{cost}(u, v), \tag{13}$$

where $p(u, v; T)$ is the set of edges on the path between u and v in T.

Further if every pair of nodes in T is δ edge-bounded, T is called a δ edge-bounded tree.

A tree generated by MSTA is a δ edge-bounded tree with $\delta = 1$. Thus, the δ edge-bounded tree can be viewed as a generalization of T_{MST}.

To explain our algorithm, another definition needs to be introduced.

Definition 2 *For a node set V, if tree $T_e = (V_e, E_e)$ satisfies the following conditions, T_e is called "an extension tree of V".*

- $V_e \supseteq V$.
- *For any node v in $V_e - V$, the degree, in T_e, of v is larger than 2.*

Lemma 4.1 *If T=(V,E) is a δ edge-bounded tree then*

$$\mathrm{cost}(T) \leq \delta \cdot \mathrm{cost}(T_{MST}) \leq 2\delta \cdot \mathrm{cost}(T_{OPT}) \tag{14}$$

where T_{MST} is the tree generated by MSTA *and T_{OPT} is the optimal Steiner tree for V.*

Proof: In (14) the right inequality is valid from [7]. Since T and T_{MST} are trees and their node sets are the same, the cardinality of E is equal to the cardinality of the edge set of T_{MST}, which is denoted by E_{MST}. Let us assume that there exists a one to one mapping function f from E_{MST} to E such that if $f(e') = e$ then edge e is contained in $p(u, v; T)$ where u and v are the two endpoints of e' in T_{MST}. Then from the definition of a δ edge-bounded tree, $cost(f(e')) \leq \delta \cdot cost(e')$. Thus,

$$w(T) = \sum_{e \in E} cost(e) = \sum_{e' \in E_{MST}} cost(f(e')) \leq \sum_{e' \in E_{MST}} \delta \cdot cost(e') = \delta \cdot w(T_{MST}) \tag{15}$$

We complete this proof by showing the existence of f.

For an edge $e' = (u, v) \in E_{MST}$, let $\Gamma(e')$ be the set of edges on the path between u and v in T, that is, $p(u, v; T)$. From P.Hall's Theorem [9, p.45 Th.5.1.1], there exists a one-to-one mapping function f if and only if

$$|\Gamma(S)| \geq |S| \qquad \text{for any subset } S \subseteq E_{MST}. \tag{16}$$

where $\Gamma(S) = \cup_{e' \in S} \Gamma(e')$.

Let S be an arbitrary subset of E_{MST}. Consider the graphs $G_0 = (V(S), S)$ and $G_1 = (V(\Gamma(S)), \Gamma(S))$ where $V(S)$ is the set of nodes incident with an edge in S. Let α_0 and α_1 be the number of connected components of G_0 and G_1, respectively. As G_0 and G_1 are forests, the number of edges, $|S|$ and $|\Gamma(S)|$, are related to the number of nodes and the number of connected components as follows:

$$|S| = |V(S)| - \alpha_0 \quad \text{and} \quad |\Gamma(S)| = |V(\Gamma(S))| - \alpha_1.$$

It is clear that $V(S) \subseteq V(\Gamma(S))$ and $\alpha_0 \geq \alpha_1$ from the definitions of G_0, G_1 and $\Gamma(e)$. Thus

$$|S| = |V(S)| - \alpha_0 \leq |V(\Gamma(S))| - \alpha_0 \leq |V(\Gamma(S))| - \alpha_1 = |\Gamma(S)|.$$

Therefore, (16) is valid. ■

Lemma 4.2 *If $T_e = (V_e, E_e)$ is a δ edge-bounded extension tree tree for V,*

$$\mathrm{cost}(T_e) \leq 2\delta \cdot \mathrm{cost}(T_{MST}) \leq 4\delta \cdot \mathrm{cost}(T_{OPT}) \tag{17}$$

where T_{MST} is the tree generated by MSTA *and T_{OPT} is the optimal Steiner tree for V.*

Proof: Let T_{MST} be (V, E_{MST}). Let us assume that there exists a mapping function g from E_{MST} to power set 2^{E_e} such that

Condition 1	If $e_e \in g((u,v))$, e_e is contained in $p(u,v;T_e)$.
Condition 2	For all $e \in E_{MST}$, $|g(e)| \leq 2$.
Condition 3	For all $e_e \in E_e$, there is some $e \in E$ such that $e_e \in g(e)$.

If $e_e \in g(e)$, $cost(e_e) \leq \delta \cdot cost(e)$ from Condition 1 and the definition of a δ edge-bounded tree. Let $cost(g(e))$ be $\sum_{e_e \in g(e)} cost(e_e)$, then from Condition 2

$$cost(g(e)) \leq 2\delta \cdot cost(e).$$

Further $E_e \subseteq \cup_{e \in E_{MST}} g(e)$ from Condition 3. Therefore

$$w(T_e) = \sum_{e_e \in E_e} cost(e_e) \leq \sum_{e \in E_{MST}} cost(g(e)) \leq \sum_{e \in E_{MST}} 2\delta \cdot cost(e) = 2\delta \cdot w(T_{MST})$$

In the remainder of the proof we show the existence of a mapping function g by induction on $|V|$, the number of nodes in T_{MST}.

1. $(|V| = 2)$

T_{MST} has only one edge, which is denoted by e. It is clear that $T_e = T_{MST}$. Let $g(e) = \{e\}$, then it satisfies Condition 1, 2 and 3.

2. $(|V| = n+1)$

By the inductive hypothesis for any tree $T'_{MST} = (V', E'_{MST})$ with n nodes and any extension tree T'_e for V', there exists a mapping function g' satisfying the conditions. Let $T_{MST} = (V, E_{MST})$ be a tree with $n+1$ nodes and $T_e = (V_e, E_e)$ be an extension tree for V.

Since T_{MST} is a tree, there is a node with degree 1, which is denoted by v. Let w be the node adjacent to v. Consider

$$T'_{MST} = (V', E'_{MST}) \quad \text{where} \quad V' = V - \{v\} \quad and \quad E'_{MST} = E - \{(v,w)\}.$$

T'_{MST} has n nodes. We will construct an extension tree for V', $T'_e = (V'_e, E'_e)$, from T_e and define a mapping function g by modifying $g' : E'_{MST} \rightarrow 2^{E'_e}$. There are three cases to consider, depending on the degree of v in T_e

Case 1 ($degree(v;T_e) > 2$)

Every node in $V_e - V$ has degree larger than 2 in T_e and the degree of v is also larger than 2. Thus, every node in $V_e - V'$ has degree larger than 2, which implies T_e is an extension graph of T'_{MST}. From the inductive hypothesis, there exists a mapping $g' : E'_{MST} \rightarrow 2^{E_e}$. Now define the function $g : E_{MST} \rightarrow 2^{E_e}$ as follows:

$$g(e) = \begin{cases} g'(e) & \text{if } e \neq (v,w) \\ \phi & \text{if } e = (v,w) \end{cases}$$

This function satisfies Condition 1, 2 and 3.

Case 2 ($degree(v;T_e) = 2$)

Let x and y be the nodes adjacent to v in T_e and it can be assumed that the path from v to w goes through x. (It is possible that $x = w$.) Let

$$T_e' = (V_e', E_e') \quad \text{where} \quad V_e' = V_e - \{v\} \quad \textit{and} \quad E_e' = E_e - \{(x,v),(v,y)\} + \{(x,y)\}.$$

Note that this modification does not change the degree of any node in $V_e - \{v\}$. The degree of every node in $V_e' - V'$ is not less than 3 because $V_e' - V' \subset V_e - V$. Thus, T_e' is an extension graph of T_{MST}', and there exists a function $g' : E_{MST}' \to 2^{E_e'}$. Now define the function g as follows:

$$g(e) = \begin{cases} g'(e) & \text{if } e \neq (v,w) \text{ and } (x,y) \notin g'(e) \\ \{(v,x)\} & \text{if } e = (v,w) \\ g'(e) - \{(x,y)\} + \{(v,y)\} & \text{if } (x,y) \in g'(e) \end{cases}$$

If $e = (v,w)$, Condition 1 is satisfied, since the path between v and w goes through x. If path $p(u,v;T_e')$ contains edge (x,y), $p(u,v;T_e)$ contains (x,v) and (v,y). Thus, if $(x,y) \in g'(e)$ Condition 1 holds for $g(e)$. Clearly Condition 1 holds when $g(e) = g'(e)$. It is also clear that g satisfies Conditions 2 and 3.

Case 3 (*degree*$(v;T_e) = 1$)

Let x be the nodes adjacent to v in T_e. We divide this case into the following two subcases:

Case 3.1 ($x \in V'$ or *degree*$(x;T_e) \geq 4$ if $x \notin V'$). We define T_e' and g' as follows:

$$T_e' = (V_e', E_e') \quad \text{where} \quad V_e' = V_e - \{v\} \quad \textit{and} \quad E_e' = E_e - \{(v,x)\}.$$

$$g(e) = \begin{cases} g'(e) & \text{if } e \neq (v,w) \\ \{(v,x)\} & \text{if } e = (v,w) \end{cases}$$

Case 3.2 ($x \notin V'$ and *degree*$(x;T_e) = 3$) Let y, z be the two adjacent nodes to x except v and assume the path between v and w goes through y. We define T_e' and g' as follows:

$$T_e' = (V_e', E_e') \quad \text{where} \quad V_e' = V_e - \{v,x\} \quad \textit{and} \quad E_e' = E_e - \{(v,x),(x,y),(x,z)\} + \{(y,z)\}.$$

$$g(e) = \begin{cases} g'(e) & \text{if } e \neq (v,w) \text{ or } (y,z) \notin g'(e) \\ \{(v,x),(x,y)\} & \text{if } e = (v,w) \\ g'(e) - \{(y,z)\} + \{(x,z)\} & \text{if } (x,y) \in g'(e) \end{cases}$$

We can verify that g satisfies Condition 1, 2 and 3 in a manner similar to that used for Case 2. ■

4.2. Algorithm

This section proposes an algorithm (EBA(δ)) generating a sequence $\{T_k\}$ such that T_k is a δ edge-bounded extension tree tree for S_k.

Figure 5 presents the details of EBA(δ). For an add request (v_k, add), EBA(δ) joins v_k to T_{k-1} by the shortest edge and investigates whether every pair v_k and w in T_{k-1} is δ edge-bounded. If not, replace the maximum cost edge in $p(v_k, w; T_{k-1})$ by edge (v_k, w). For a remove request $(v_k, remove)$, if the degree of v_k is larger than 2, T_k is the same as T_{k-1}. Otherwise v_k is deleted from T_k and if $T_k - v_k$ is disconnected then the two components are joined by an edge (u, v) such that the cost of the maximum edge in $p(x_0, x_1; T_{k-1} - v_k + (u, v))$ is minimized, where x_0 and x_1 are the nodes adjacent to v_k in T_{k-1}. Further if the resulting graph has a non-terminal node with degree 2 or 1, repeat this step.

It is clear that the generated trees are extension trees for S_k. We begin with the following property to show they are δ edge-bounded trees.

Lemma 4.3 *For any α (≥ 1) and δ (≥ 1), if a pair of nodes u_0 and u_1 is α edge-bounded in some intermediate tree T generated by* EBA(δ) *then it is also α edge-bounded in any intermediate tree generated after T.*

Proof: Let T_a be an intermediate tree. The elementary modifications to T_a in algorithm EBA(δ) are divided into the following four cases:

1. add a new node v and join v to T_a by a minimum cost edge e_a between them.
2. remove the maximum cost edge $e_d = (x_0, x_1)$ in $p(v_0, v_1; T_a)$ and join the two components by edge $e_a = (v_0, v_1)$, where $\text{cost}(e_d) > \delta\text{cost}(e_a)$.
3. remove node w with degree 1 and the edge incident with w.
4. remove node w with degree 2 and the two incident edges, (x_0, w) and (x_1, w), and join the two components by the edge $e_a = (v_0, v_1)$ that minimizes $g(v_0, v_1)$, where function $g(v_0, v_1)$ is the cost of the maximum edge in $p(x_0, x_1; T - w + (v_0, v_1))$.

Let T_b be the tree resulting from the application of one of the above operations to T_a. In order to prove the lemma, we show the following property holds for all four cases.

Property If u_0 and u_1 are a α edge-bounded pair in T_a then they are a α edge-bounded pair in T_b.

Case 1: The path between u_0 and u_1 in T_b is the same as the path in T_a because there is no deletion for edges. Thus u_0 and u_1 are a α edge-bounded pair in T_b by the assumption of the property.

Case 2: If $p(u_0, u_1; T_a)$ does not contain the deleted edge e_d,
$p(u_0, u_1; T_a) = p(u_0, u_1; T_b)$ and the property is valid.

```
EBA(δ)
{    T_0 := ({v_0}, φ), S_0 := {v_0} and k = 1;
     do k ≤ K →
          if r_k is an add request →
               T_k := add(v_k, T_{k-1}) and S_k := S_{k-1} ∪ {v_k};
          | r_k is a remove request →
               S_k := S_{k-1} − {v_k} and T_k := remove(T_{k-1}, S_k);
          fi
          k:=k+1;
     od}

add(v, T)
{    Let W be a set of edges between v and T.
     Select the minimum cost edge (v, w_1) from W;
     T := T + (v, w_1) and W := W − (v, w_1);
     do W ≠ φ →
          Select the minimum cost edge (v, w_i) from W and W := W − (v, w_i)
          Find a maximum cost edge, e, in p(v, w_i; T);
          if cost(e) > δ · cost(v, w_i) → T := T − e + (v, w_i); fi
     od  return(T)}

remove(T, S)
{    W := V(T) − S where V(T) is the node set of T;
     do W contains a node of degree 2 or 1 →
          Let w ∈ W be a node of degree 2 or 1;
          if degree(w) = 1 → T := T − w and W := W − {w};
          | degree(w) = 2 →
               Let x_0 and x_1 be the nodes adjacent to w;
               Let C_0 and C_1 be the connected components of T − w;
               Select two nodes v_a ∈ C_0 and v_b ∈ C_1 which minimizes g(v_a, v_b),
               where g(u, v) = max{cost(e) | e ∈ p(x_0, x_1; T − w + (u, v))};
               T := T − w + (v_a, v_b) and W := W − {w};
          fi
     od  return(T)}
```

Figure 5: Edge Bounded Algorithm (EBA(δ))

If $p(u_0, u_1; T_a)$ contains e_d, $p(u_0, u_1; T_b)$ is a subset of $p(u_0, u_1; T_a) \cup p(v_0, v_1; T_a) \cup \{e_a\}$ (see Fig 6).

Suffices to show that for every edge e in $p(u_0, u_1; T_a) \cup p(v_0, v_1; T_a) \cup \{e_a\}$,

$$\text{cost}(e) \leq \alpha \cdot \text{cost}(u_0, u_1). \tag{18}$$

If $e \in p(u_0, u_1; T_a)$, (18) holds since the pair u_0 and u_1 is α edge-bounded in T_a. Note that this implies $\text{cost}(e_d) \leq \alpha \cdot \text{cost}(u_0, u_1)$. If $e \in p(v_0, v_1; T_a)$, (18) holds since e_d is the maximum cost edge in $p(v_0, v_1; T_a)$. Finally, (18) holds for edge e_a since $\delta \cdot \text{cost}(e_a) < \text{cost}(e_d)$.

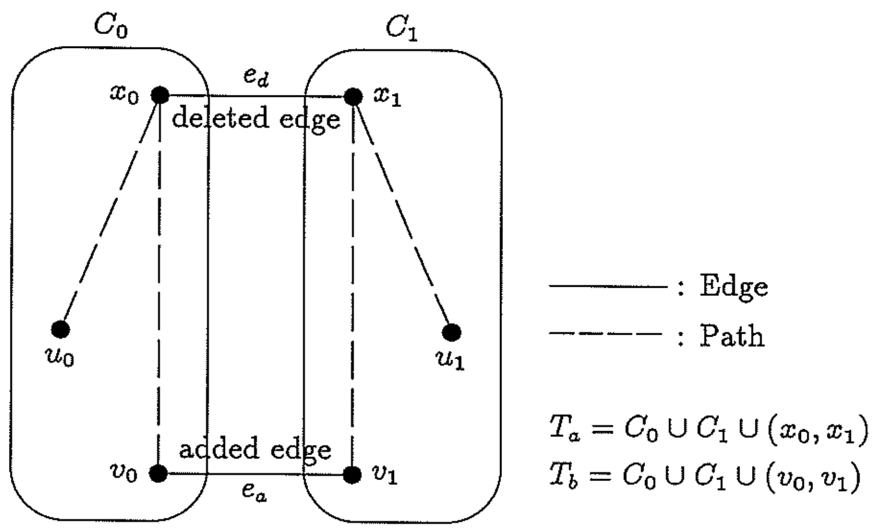


Figure 6: Elementary Modification 2

Case 3: The property is valid because $p(u_0, u_1; T_b) = p(u_0, u_1; T_a)$.

Case 4: Let the deleted two edges be (x_0, w) and (w, x_1). The two components of $T - w$ are denoted by C_0 and C_1. If nodes u_0 and u_1 are contained in the same component, the property is valid because $p(u_0, u_1; T_b) = p(u_0, u_1; T_a)$.

When u_0 and u_1 are contained in the different components, we can assume u_0, x_0 and v_0 are contained in C_0 and u_1, x_1 and v_1 are contained in C_1 without loss of generality(see Fig 7).

From the choice of (v_0, v_1), the following inequality holds:
$max\{\text{cost}(e) | e \in p(x_0, x_1; T_b)\} \leq$

$$max\{\text{cost}(e) | e \in p(x_0, u_0; C_0) \cup \{(u_0, u_1)\} \cup p(u_1, x_1; C_1)\} \tag{19}$$

As u_0 and u_1 are a α edge-bounded pair in T_a, the right side of Inequality 19 is not larger than $\alpha \cdot \text{cost}(u_0, u_1)$. Thus

$$\text{cost}(e) \leq \alpha \cdot \text{cost}(u_0, u_1) \quad \text{for any } e \in p(x_0, x_1; T_b) \cup p(u_0, x_0; C_0) \cup p(x_1, u_1; C_1).$$

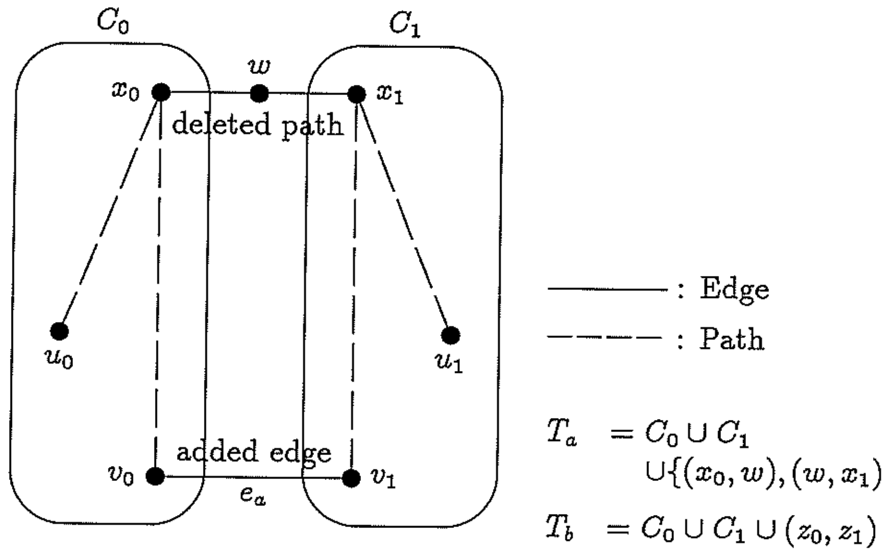


Figure 7: Elementary Modification 3

On the other hand,

$$p(u_0, u_1; T_b) \subseteq p(x_0, x_1; T_b) \cup p(u_0, x_0; C_0) \cup p(x_1, u_1; C_1).$$

Consequently, any edge in $p(u_0, u_1; T_b)$ is not larger than $\alpha \cdot \text{cost}(u_0, u_1)$. ■

This lemma is useful for estimating the number of rearrangements in addition to showing the following theorems. In the following theorems and lemmas, where not stated explicity, we assume that an instance of DST-R is given with $R = \{r_0, r_1, \ldots, r_K\}$.

Theorem 4.1 *Any tree T_k, generated by* EBA(δ)*, is a δ edge-bounded extention tree for S_k. This implies the following inequalities:*

$$\text{cost}(T_k) \leq 2\delta \cdot \text{cost}(T_{MST}) \leq 4\delta \cdot \text{cost}(T_{OPT}), \tag{20}$$

for $0 \leq k \leq K$.

Proof: It is clear that T_k is an extension tree for S_k. It will be shown that T_k is δ edge-bounded by induction on k. T_1 is clearly a δ edge-bounded tree. We assume T_{k-1} is δ edge-bounded. From Lemma 4.3, every pair of nodes in T_{k-1} is δ edge-bounded in T_k. The remaining pair to be considered is v_k and $w \in T_{k-1}$ when $r_k = (v_k, add)$. EBA(δ) examines each pair and if it not δ edge-bounded then modifies the tree so that it becomes 1 edge-bounded. This modification maintains the δ edge-bounded property for other pairs as Lemma 4.3 shows. Thus T_k is a δ edge-bounded tree and the theorem follows from Lemma 4.2 ■

Theorem 4.2 *If every request r_k is an add operation, every generated tree T_k satisfies*

$$\text{cost}(T_k) \leq \delta \cdot \text{cost}(T_{MST}) \leq 2\delta \cdot \text{cost}(T_{OPT}). \tag{21}$$

Proof: If every request is an add request, the set of nodes of T_k is the same as S_k. T_k is δ edge-bounded. Thus the theorem is valid from Lemma 4.1. ■

4.3. Total Number of Changes

In this section we estimate the total number of rearrangements for EBA(δ) in case of $\delta \geq 2$. Note that if an edge (u, v) is contained in T_i, then the pair u and v is a 1 edge-bounded pair in any T_j $(j \geq i)$ from Lemma 4.3. We begin by finding other 1 edge-bounded pairs.

If r_i is an add request, let L_i be the set of endpoints for the edges added at Step i. The cardinality of L_i is one more than the number of edges added since v_i is one of the endpoints for each edge. Therefore the number of rearrangements at Step i, α_i, is $|L_i| - 2$ because the number of edges deleted is one less than the number of the edges added. If r_i is a remove request, let $L_i = \phi$.

Lemma 4.4 *Every pair of nodes in L_i is 1 edge-bounded in T_j $(i \leq j \leq K)$.*

Proof: Let $L_k = \{v_k, w_1, w_2, \ldots, w_\alpha\}$. Since T_k contains edge (v_k, w_j) for any $w_j \in L_k$, a pair of v_k and w_j is 1 edge-bounded which implies it is 1 edge-bounded in any $T_j (j \geq k)$ from Lemma 4.3. We consider a pair of w_i and w_j. Without loss of generality we can assume that

$$cost(w_i, v_k) \leq cost(w_j, v_k). \tag{22}$$

Consider the substep at which (w_j, v_k) is added to the tree and let T_a and T_b be the intermediate trees just before and after this substep. Thus

$$T_b = T_a - (x_0, x_1) + (w_j, v_k)$$

where (x_0, x_1) is the edge deleted at this substep. Note that T_a and T_b contain an edge (w_i, v_k).

Since a pair of w_i and w_j is δ edge-bounded in T_a and (x_0, x_1) is an edge in $p(w_i, w_j; T_a)$, $cost(x_0, x_1) \leq \delta cost(w_i, w_j)$. Since (x_0, x_1) is replaced with (w_j, v_k), $cost(x_0, x_1) > \delta cost(w_j, v_k)$. Thus $cost(w_j, v_k) < cost(w_i, w_j)$.
From Inequality 22, $cost(w_i, v_k) \leq cost(w_j, v_k) < cost(w_i, w_j)$. Therefore, the pair of nodes w_i and w_j is 1 edge-bounded in T_a and in any T_j $(j \geq k)$. ■

Lemma 4.5 *If $\delta \geq 2$,*

$$|L_i \cap L_j| \leq 1 \quad \text{for all } i \text{ and } j,\ 0 < i < j \leq K\ . \tag{23}$$

Proof: We assume $|L_i \cap L_j| \geq 2$ and derive a contradiction. Let w_1 and w_2 be nodes in $L_i \cap L_j$ and assume $j > i$. Note that edges (v_j, w_1) and (v_j, w_2) are added at Step j, where $r_j = (v_j, add)$. Without loss of generality we can assume

$$\text{cost}(v_j, w_1) \leq \text{cost}(v_j, w_2) \tag{24}$$

From the triangle inequality,

$$\text{cost}(w_1, w_2) \leq \text{cost}(w_1, v_j) + \text{cost}(v_j, w_2) \leq 2\text{cost}(v_j, w_2).$$

Consider the substep at which (v_j, w_2) is added to an intermediate tree and let T_a be the intermediate tree just before this substep. It is clear that

$$p(v_j, w_2; T_a) = \{(v_j, w_1)\} \cup p(w_1, w_2; T_a) \tag{25}$$

Since $w_1, w_2 \in L_i$, a pair w_1 and w_2 is 1 edge-bounded by Lemma 4.4. Thus

$$cost(e) \leq \text{cost}(w_1, w_2) \leq 2\text{cost}(v_j, w_2) \qquad \text{for any } e \in p(w_1, w_2; T_a) \tag{26}$$

From (24), (25) and (26), pair v_j and w_2 is 2 edge-bounded, which contradicts that (v_j, w_2) is added at this substep. ■

Let K_a be the number of add requests in R, and K_r be the number of remove requests. We can derive the following theorem by using Lemma 4.4.

Theorem 4.3 *For any instance (G, cost, R), if $\delta = 2$ then the total number of rearrangements in* EBA*(δ) is bounded by the following inequality:*

$$\sum_{i=0}^{K} \alpha_i \leq \frac{1}{2} K_a(\sqrt{4K_a - 3} - 1) + K_r. \tag{27}$$

Proof: Let R_a and R_r be the sets of add and remove requests in R, respectively. Consider Step i for which the request, r_i, is a remove request. The number of rearrangements α_i is not larger than the number of deleted nodes at this step. A deleted node is not contained in S_k, which implies there is a request to remove it from a connection. Thus the total number of rearrangements for R_r, $\sum_{r_i \in R_r} \alpha_i$, is not larger than the number of remove requests, $K_r (= |R_r|)$.

Consider the bipartite graph $G_b = (V_1, V_2; E)$ where $V_1 = R_a$, $V_2 = \cup_{r_i \in R_a} L_i$ and $E = \{(r_i, v_2) | v_2 \in L_i, v_2 \in V_2, r_i \in V_1\}$. If there are multiple requests to add the same node, we considers them as different nodes. It is clear that $|V_1| = |V_2| = K_a$. From Lemma 4.5, this graph does not contain a complete bipartite graph $K_{2,2}$ (that is, a cycle with 4 edges) as a subgraph. The following inequality is shown in [8, p.74 Th.10]:

$$z(n, n, 2, 2) \leq \frac{1}{2} n(1 + \sqrt{4n - 3})$$

where $z(n,n,2,2)$ is the maximum number of edges in a bipartite graph whose two node sets both have n nodes and that does not include $K_{2,2}$. Since the number of edges in G_b is equal to $\sum L_i$ and α_i is equal to $|L_i| - 2$,

$$\sum_{r_i \in R_a} \alpha_i \leq \frac{1}{2} K_a(\sqrt{4K_a - 3} - 1). \tag{28}$$

■

We do not know if the $K\sqrt{K}$ growth permitted by Theorem 4.3 can actually be achieved. We conjecture that the total number of rearrangements is not larger than $K(= K_a + K_b)$.

One of the interesting questions which still remains regarding DST is whether there exist algorithms for which both the worst case error ratio and the maximum number of rearrangements at each step are bounded by a pair of constants.

Other areas of interest include issues such as average case performance of algorithms for DST, distributed implementation of algorithms for DST, and the application of these algorithms to multipoint communication networks.

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