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Solving an Open Sensor Exposure Problem using Variational Calculus

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Abstract

Sensor network presents us many new challenging practical and theoretical problems. This paper is concerned with minimal exposure problem in sensor networks. Exposure, proposed by Megerian and others [3] as a new useful metric to describe the sensor coverage of a path in a sensor field, exhibits interesting properties and induces related open problems. In this paper, we present a solution to an open one-sensor exposure problem [2, 3] using variational calculus as our first step in further understanding of the exposure problem in multiple sensor scenarios.

1. Introduction

Sensor network is drawing more and more attention in recent years as a powerful distributed mechanism for observing activities in a large environment. Its application areas include environmental monitoring(e.g., vehicle traffic, area security, forest fire, earth and planetary science), industrial and agricultural sensing and diagnostics (e.g., factory condition, grain field health), battlefield awareness (e.g., enemy motion detection, multi-target tracking) etc.

When a sensor network is deployed as a environment monitoring tool, a key question is how well a sensor network deployment observes its intended environment? In other words, how well a sensor network "covers" its environment? Answers to this question is not only important for deployment optimization purpose, but also important for adversaries of the monitoring task. A sensor network designer would choose to deploy a limited number of sensors in a way to cover its intended environment as better as possible, while an adversary moving in a sensor field would prefer to find and use a less covered path.

In order to answer the coverage question from the theoretical perspective, one first needs to have some model of observation capability of one sensor and collections of sensors. In general, the observation capability of a remote sensor can be described by a sensitivity function $f(s, p)$, where s is its location and p is an arbitrary point in the space. The actual form of this function depends on the physical characteristics of the sensor. For instance, while the accuracy of a remote sensor usually fall as the distance to its target increases, its exact behavior depending on the sensing techniques used. A laser based distance measure technique usually has lower accuracy changing rate than a sound based technique.

Given sensor field composed by a set of n sensors S distributed in space at $s_1, s_2, ..., s_n$, their collective observation power with respect to a point p is in general a function F of the individual sensing capabilities of all sensors.

$$F(S, p) = F(f_1(s_1, p), f_2(s_2, p), ..., f_n(s_n, p))$$

F is called sensor field "intensity" in [3]. The actual form of the sensor field intensity depends on the sensing strategy deployed in the sensor network. Meguerdichian and others proposed two types of sensor field intensities [3]: *all*

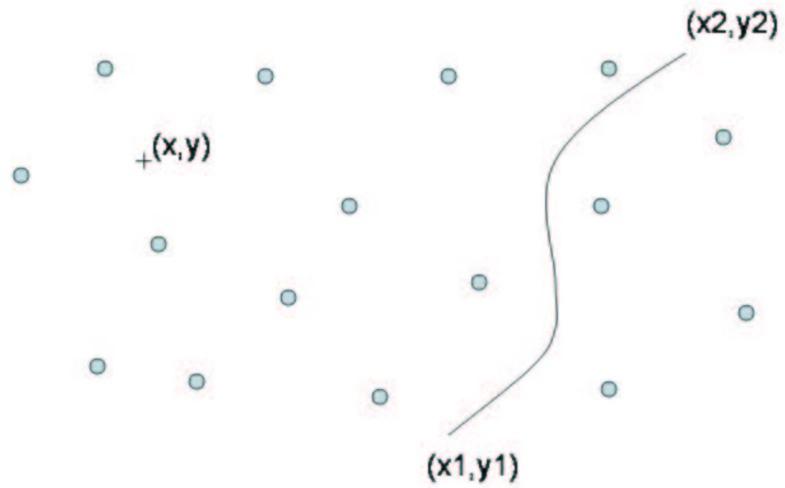

Figure 1: Path exposure in sensor network

sensor field intensity $I_A(S, p)$

$$I_A(S,p) = \sum_{i=1}^{n} f_i(s_i, p) \tag{1}$$

and *closest-sensor field intensity* $I_C(S, p)$[1]

$$I_C(S,p) = \max\{f_i(s_i, p)|i = 1, ..., n\} \tag{2}$$

The em exposure for an object moving in the sensor field during a interval $[t_1, t_2]$ along a path $p(t)$ is defined as [3]:

$$E(p(t), t_1, t_2) = \int_{t_1}^{t_2} I(S, p(t))|\frac{dp(t)}{dt}|dt \tag{3}$$

Finding the minimal exposure path between two arbitrary points in the sensor field is one of fundamental geometric problems for the applications of sensor networks. Before one can fully answer the minimal exposure problem for sensor field containing thousands of sensor, one has to solve the case for one sensor field first. In [3], the authors presented an answer to a special case of the minimal exposure problem for one sensor case: when the sensitivity function is of form $1/distance(s, p)$, and the starting point s and the ending point e has equal distance to the sensor. The minimum exposure path in this case is the shorter arc (between s and e) of the circle centered at s and passing through s and e.

In this paper, we first solve the following open minimal exposure problem [2, 3] via variational calculus:

Given sensor sensitivity function $f(s, p) = 1/d(s, p)$, where $d(s, p)$ is the distance between the sensor location s and an arbitrary location p, what is the minimal exposure path from an arbitrary point A to another arbitrary point B?

Using our theorem, we correct an error in deriving corollary 5 in [3].

Furthermore, we provide a solution to the following more general case:

[1]Note that the form we present here is general than the one presented in [3]

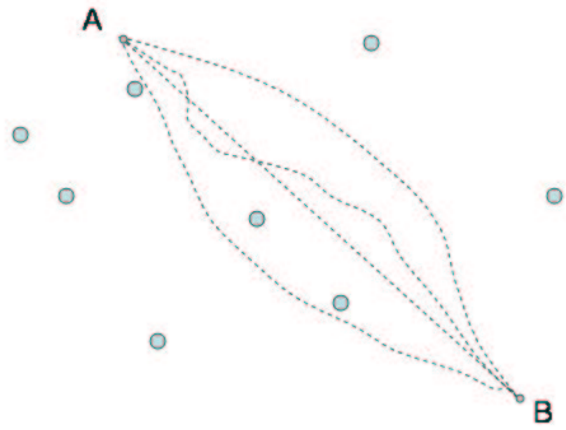

Figure 2: Exposure values are likely path dependent

Given sensor sensitivity function $f(s,p) = 1/(d(s,p))^k$, where $k \geq 0$, and $d(s,p)$ is the distance between the sensor location s and an arbitrary location p, what is the minimal exposure path from an arbitrary point A to another arbitrary point B?

The paper is organized as follows. We first present a short yet hopefully sufficient (for most expected readers) variational calculus preliminaries in section 2. Then in section 3 we present our solution to the minimum exposure problem for the $1/(d(s,p))$ sensor field intensity case and discussion its implication to some claims in [3, 2]. After that we present a general solution to the $1/(d(s,p))^k$ case and discuss its implications to the solution to a more complicated multi-sensor exposure problem in section 4. Then we conclude in section 5.

2. Variational Calculus Preliminary

The definition of exposure by Eq.(3) makes it obviously a path-dependent value. Given two end-points in the field, different paths between them, as shown in Fig.2, are likely to have different exposure values. Finding a minimal or minimum exposure path is one key problem of exposure study.

In order to find a minimum/minimal exposure path, apparently one has to compare the exposure values of all related paths. The difficulty is, the possible paths between two points spans an infinite-dimensional space. In 18th century, Euler (1707-1783) and Lagrange (1736-1813) were concerned with similar problems and developed variational calculus to help solving them. The following is a key theorem [1] we will use in this paper:

THEOREM 2.1. *Let $J[y]$ be a function(al) of the form*

$$J[y] = \int_a^b F(x, y, y')dx$$

defined on the set of functions $y(x)$ which have continuous first derivatives in $[a,b]$ and satisfy the boundary condition $y(a) = A$, $y(b) = B$. Then a necessary condition for $J[y]$ to have an extremum for a given function $y(x)$

is that $y(x)$ satisfy the Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \tag{4}$$

Reader may find a proof of this theorem in most variational calculus or functional analysis book. Note that Eq.(4) is also often called Euler-Lagrange equation. For convenience, we sketch a simple proof of the above theorem here.

A function $y(x)$ makes $J[y]$ an extremum should have the following property: any admissible variations of $y(x)$ makes J to have a higher value. Suppose we give $y(x)$ an increment $\epsilon h(x)$. In order for the function $y(x)+\epsilon h(x)$ to continue to satisfy the boundary condition, i.e, has two end points at A and B respectively, we force $h(a) = h(b) = 0$. Then we have

$$\begin{aligned} \Delta J &= J[y+\epsilon h] - J[y] && (5) \\ &= \int_a^b F(x, y+\epsilon h, y'+\epsilon h')dx - \int_a^b F(x,y,y')dx && (6) \\ &= \epsilon \int_a^b [F_y(x,y,y')h + F_{y'}(x,y,y')h']dx + o(\epsilon^2) && (7) \end{aligned}$$

The last step is achieved by Taylor expansion. The subscripts denote the partial derivatives with respect to the corresponding arguments. A necessary condition for $J[y]$ to have an extremum at $y(x)$ is the principle linear of ΔJ to be zero. That is

$$\delta J = \int_a^b [F_y h + F_{y'} h']dx = 0 \tag{8}$$

Integrating by parts on the second term, we have

$$\delta J = \int_a^b [F_y h - \frac{dF_{y'}}{dx} h]dx + F_{y'} h(x)|_a^b \tag{9}$$

using boundary condition $h(a) = h(b) = 0$, we have

$$\int_a^b [F_y - \frac{dF_{y'}}{dx}]h dx = 0 \tag{10}$$

Then use the "fundamental lemma of calculus of variation" which states that, if

$$\int_a^b M(x)h(x)dx = 0$$

for all h(x) with continuous partial derivatives, then $M(x) = 0$ on (a,b). We have the Euler's equation.

2.1. A simple example

A classical application example of calculus of variations is to find the shortest path joining two points. While Archimedes proved long ago that the shortest path joining two points in a plane is a straight line, one still can gain a glimpse of the power of the variation method. So we present its solution here to better prepare reader for the next few sections.

Let the two points under consideration be $A = (a, c)$ and $B = (b, d)$. Let $y(x)$ be an curve connecting A and B. Then the length of the curve $y(x)$ is

$$\int_a^b \sqrt{(dx)^2 + (dy)^2} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \tag{11}$$

So we have

$$F = \sqrt{1 + (y')^2}$$

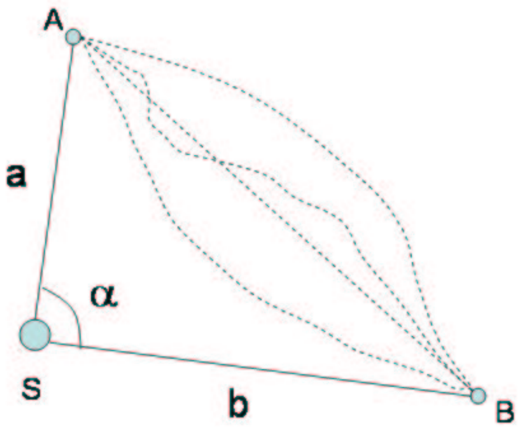

Figure 3: Exposure in one sensor scenario

Applying Euler's equation (4), we get

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = 0 \tag{12}$$

This implies

$$\frac{y'}{\sqrt{1+(y')^2}} = C(constant) \tag{13}$$

This possible only if y' is a constant:

$$y' = \alpha$$

So we can conclude $y(x)$ is a straight line

$$y = \alpha x + \beta$$

where constants α and β can be determined by boundary conditions $y(a) = c$ and $y(b) = d$. EOF

3. Minimal exposure for one sensor field

In this section, we first use the variation method to derive the following theorem:

THEOREM 3.1. *Given sensor sensitivity function $f(s,p) = 1/d(s,p)$, where $d(s,p)$ is the distance between the sensor location s and an arbitrary location p, the minimal exposure path from an arbitrary point A to another arbitrary point B is of the following form in polar coordinates:*

$$\rho(\theta) = ae^{\frac{\ln(b/a)}{\alpha}\theta} \tag{14}$$

where the constant a is the distance from the sensor to A, b is the distance from the sensor to B, and α is the angle formed by (A-sensor-B), as in Fig.(3).

Proof: Note that given two points $A = (x_a, y_a)$ and $B = (x_b, y_b)$, a path $P(A, B)$ between them can be identified by a function pair $(x(t), y(t))$ with boundary conditions $x(t_1) = x_a$, $x(t_2) = x_b$,$y(t_1) = y_a$, $y(t_2) = y_b$. Using this notation, we rewrite Eq. (3) (definition of exposure) in the following equivalent form:

$$E(x(t), y(t), t_1, t_2) = \int_{t_1}^{t_2} I(x(t), y(t))\sqrt{\left(\frac{dx(t)}{dt}\right)^2 + \left(\frac{dy(t)}{dt}\right)^2}dt \tag{15}$$

For convenience, we transform Eq.(15) into polar coordinates (ρ, θ):

$$x(t) = \rho(t)\cos\theta(t) \tag{16}$$

$$y(t) = \rho(t)\sin\theta(t) \tag{17}$$

$$\tag{18}$$

$$E(\rho(t), \theta(t), t_1, t_2) = \int_{t_1}^{t_2} I(\rho(t), \theta(t))\sqrt{\left(\rho\frac{d\theta(t)}{dt}\right)^2 + \left(\frac{d\rho(t)}{dt}\right)^2}dt \tag{19}$$

The right-hand side of Eq.(19) is also equivalent to

$$\int_{\theta(t_1)}^{\theta(t_2)} I(\rho(t), \theta(t))\sqrt{\rho(t)^2 + \left(\frac{d\rho(t)}{d\theta(t)}\right)^2}d\theta(t) \tag{20}$$

This means we can drop the variable t entirely and have the following equivalent exposure expression:

$$E(\rho(\theta), 0, \alpha) = \int_0^\alpha I(\rho, \theta)\sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}d\theta \tag{21}$$

where α is the constant angle formed by A-sensor-B, as shown in Fig.3. The boundary condition in this new coordinates is

$$\rho(0) = a \tag{22}$$

$$\rho(\alpha) = b \tag{23}$$

As the sensitivity function is $f(s, p) = 1/d(s, p) = 1/\rho$, and here we are only concerned with one sensor, we have

$$I(\rho, \theta) = \frac{1}{\rho}$$

Thus, our goal is to find the path function $\rho(\theta)$ that minimizes

$$E(\rho(\theta), 0, \alpha) = \int_0^\alpha \frac{\sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}}{\rho}d\theta \tag{24}$$

Let

$$F = \frac{\sqrt{\rho^2 + (\frac{d\rho}{d\theta})^2}}{\rho} \tag{25}$$

and apply the Euler-Lagrange equation (4), we get

$$\frac{d}{d\theta}\left(\frac{1}{\sqrt{1 + \frac{1}{\rho^2}\left(\frac{d\rho}{d\theta}\right)^2}}\right) = 0 \tag{26}$$

this implies

$$\frac{1}{\sqrt{1 + \frac{1}{\rho^2}\left(\frac{d\rho}{d\theta}\right)^2}} = c(constant) \tag{27}$$

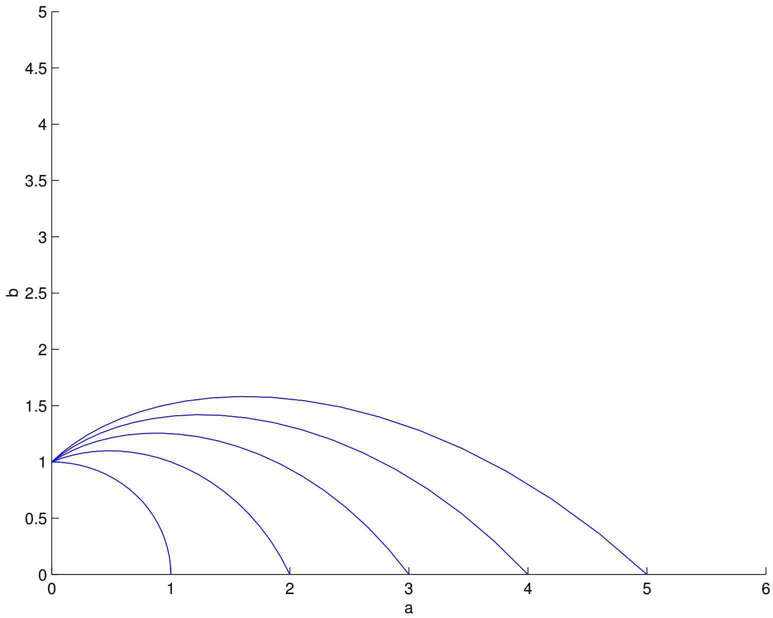

Figure 4: Examples of minimal exposure path

after some manipulation of terms, we can obtain:

$$\frac{d\rho}{d\theta} = \pm c_1 \rho \tag{28}$$

where c_1 is a also constant. Solve this equation we get

$$\rho = c_2 \exp^{\pm c_1 \theta} \tag{29}$$

then use the boundary conditions (22) and (23), we have

$$\rho(\theta) = a e^{\frac{\ln(b/a)}{\alpha}\theta} \tag{30}$$

EOP

A spacial case for theorem (3.1) is by letting $a = b$, in which we get $\rho(\theta) = a$. So we immediate have the following corollary by letting $a = b$:

Corollary 3.1. *Given sensor sensitivity function $f(s, p) = 1/d(s, p)$, where $d(s, p)$ is the distance between the sensor location s and an arbitrary location p. Let A and B be two points having equal distance to the sensor origin. The minimal exposure path from A to B is the arc of the circle centered at s and passing through A and B.*

This corollary is exactly the Lemma 2 derived in [3].

When the sensor is located at the origin $(0, 0)$ with sensitivity $1/d(s, p)$, Fig.(4) shows a few examples of minimal exposure paths between point $(0, 1)$ and $(1, 0)$, $(2, 0)$,$(3, 0)$,$(4, 0)$,$(5, 0)$.

Note that theorem 3.1 also implies some arguments and method in [3, 2] in deriving corollary 5 in those papers are wrong.

4. A more general case

For a more general case, assume the sensor sensitivity (single sensor field intensity) can be approximated by

$$S(s,p) = \frac{1}{d^k(s,p)}, \qquad k \geq 0 \tag{31}$$

Clearly, the higher the k, the faster its sensitivity attenuates with the increase of distance. Different remote sensor sensitivity characteristics may be approximated in the first order by choosing appropriate k.

THEOREM 4.1. *Let sensor sensor field intensity be*

$$S(s,p) = \frac{1}{d^k(s,p)} \qquad k \geq 0, k \neq 1$$

where $d(s,p)$ is the distance between the sensor location s and an arbitrary location p. Given two points A and B with $d(s,A) = a$ and $d(s,B) = b$. Let α be the angle formed by (A-sensor-B). When

$$\alpha \leq \frac{\pi}{k-1} \tag{32}$$

the minimal exposure path between them is of the following form in polar coordinates:

$$\rho(\theta) = a\left(\cos((k-1)\theta) + \sin((k-1)\theta)\frac{\frac{b^{k-1}}{a^{k-1}} - \cos((k-1)\alpha)}{\sin((k-1)\alpha)}\right)^{\frac{1}{k-1}} \tag{33}$$

Proof: Similar to the proof of theorem (3.1), we get

$$F = \frac{\sqrt{\rho^2 + (\frac{d\rho}{d\theta})^2}}{\rho^k} \tag{34}$$

Applying the Euler-Lagrange equation, we get

$$\frac{d}{d\theta}\left(\frac{\rho^2}{\rho^k\sqrt{\rho^2 + (\frac{d\rho}{d\theta})^2}}\right) = 0 \tag{35}$$

this implies

$$\frac{\rho^2}{\rho^k\sqrt{\rho^2 + (\frac{d\rho}{d\theta})^2}} = c(constant) \tag{36}$$

after some manipulation of terms, we can obtain:

$$\frac{d\rho}{d\theta} = \pm\sqrt{c_1\rho^{4-2k} - \rho^2} \tag{37}$$

note that

$$\int \frac{1}{\sqrt{c_1\rho^{4-2k} - \rho^2}} d\rho = \int \frac{1}{\rho^{2-k}\sqrt{c_1 - \rho^{2k-2}}} d\rho = \int \frac{\rho^k - 2}{\sqrt{c_1 - \rho^{2k-2}}} d\rho \tag{38}$$

where c_1 is also a constant. Now do a variable substitution to simplify the expression. Let

$$q = \rho^{k-1} \tag{39}$$

For $k \neq 1$, we have

$$\int \frac{\rho^k - 2}{\sqrt{c_1 - \rho^{2k-2}}} d\rho = \int \frac{1}{(k-1)\sqrt{c_1 - q^2}} dq = -\frac{1}{k-1}\arcsin\left(\frac{q}{\sqrt{c_1}}\right) \tag{40}$$

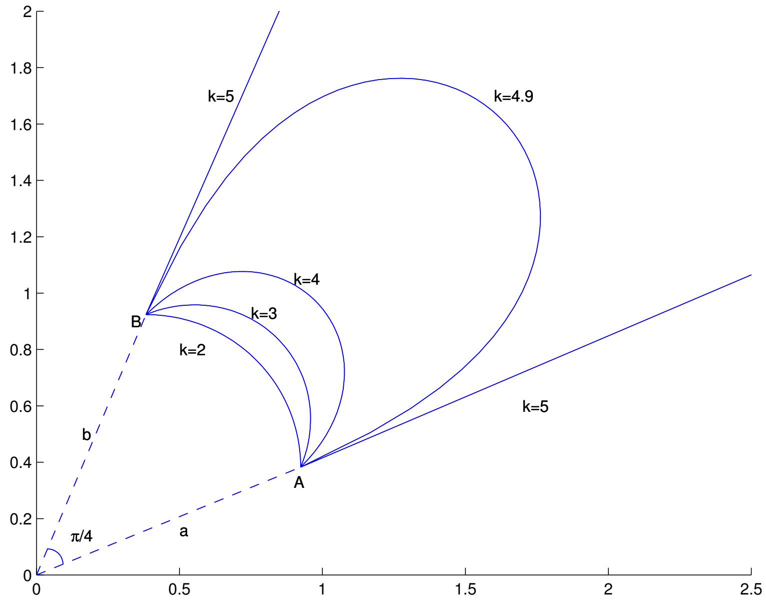

Figure 5: Relation between minimal exposure path and sensitivity attenuation bahavior(k)

Now integrate Eq.(37) with the help of Eq.(40), we get

$$q(\theta) = c_3 \sin\left((k-1)(\theta + c_2)\right) \tag{41}$$

where c_2 is an integral constant. The two constants c_3 and c_2 can be determined by two boundary conditions

$$q(0) = a^{k-1} \tag{42}$$

$$q(\alpha) = b^{k-1} \tag{43}$$

and we get

$$c_2 = \frac{1}{k-1}\arctan\left(\frac{\sin((k-1)\alpha)}{\frac{b^{k-1}}{a^{k-1}} - \cos((k-1)\alpha)}\right) \tag{44}$$

$$c_3 = \frac{a^{k-1}}{\sin((k-1)c_2)} \tag{45}$$

so we have

$$\rho(\theta) = a\left(\frac{\sin((k-1)(\theta+c_2))}{\sin((k-1)c_2)}\right)^{\frac{1}{k-1}} \tag{46}$$

by expanding the numerator and substitute in c_2, Eq.(46) can be rewritten as

$$\rho(\theta) = a\left(\cos((k-1)\theta) + \sin((k-1)\theta)\frac{\frac{b^{k-1}}{a^{k-1}} - \cos((k-1)\alpha)}{\sin((k-1)\alpha)}\right)^{\frac{1}{k-1}} \tag{47}$$

EOP

One can easily check the original boundary conditions $\rho(0) = a$ and $\rho(\alpha) = b$ are satisfied in Eq.(47).

Fig.(5) shows how the minimal exposure paths between points A and B change with the sensor sensitivity attenuation exponent k. Note that in this case ($\alpha = \pi/4$), when k approaches to 5, the length of the minimal exposure path approaches infinity. One can easily see

$$\alpha_c(k) = \frac{\pi}{k-1} \tag{48}$$

is the *critical angle* for the $\frac{1}{r^k}$ sensitivity. It is critical in the sense that once the angle α exceeds α_c, the minimal exposure path always extends infinity.

5. Conclusion

Sensor network presents us many new challenging practical and theoretical problems. In this paper, we solved an open one-sensor exposure problem for sensor networks using variational methods, and pointed out some errors in the literature. We believe our results will facilitate further understanding of the exposure problem in more general, multiple sensor scenarios.

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