On some fractional nonlocal integrated semi groups

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Abstract

Some classes of fractional abstract differential equations with α-integrated semi groups are studied in Banach space. The existence of a unique solution of the nonlocal Cauchy problem is studied. Some properties are given.

Key wards: α-Integrated semi groups-Nonlinear fractional abstract differential equations- Nonlocal Cauchy problem.

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1 Introduction

Consider the following abstract fractional differential equation:

$$\frac{d^\beta u(t)}{dt^\beta} = Au(t) + f(t, B(t)u(t)) + s(t)\sum_{i=1}^{k} c_i u(t_i), \tag{1.1}$$

with the initial condition

$$u(0) = u_0, \tag{1.2}$$

where $0 \leq t_1 < \cdots < t_k \leq T$, $c_1, ..., c_k$ are real numbers, A is a linear closed operator defined on a dense set S in a Banach Space E,

$$B(t)u = (B_1(t)u, ..., B_r(t)u), B_i(t), i = 1, ..., r$$

is a family of linear closed operators defined on dense sets $S_1, ..., S_r \supset S$, respectively in E to E , f is a given abstract function defined on JXE^r to E, $0 < \beta \leq 1$,u_0 is a given element in S and s is a real function , which has continuous derivative

$$r(t) = \frac{d^{\alpha\beta}s(t)}{dt^{\alpha\beta}}, \quad on \quad J = [0, T], \quad s(0) = 0.$$

It is assumed that A generates α- times integrated semi groups Q(t), $t \geq 0$ with the following Properties:
$C_1 : Q(t) : t \geq 0$ is family of strongly continuous operator.
C_2: There exist positive constants M and c such that $\|\|Q(t)\|\| \leq Me^{ct}$, where $\| \cdot \|$ is the norm in E.
C_3 : The interval (c, ∞) is contained in the resolvent set $\rho(A)$ of A and,
$C_4 : (I\lambda - A)^{-1} = \lambda^\alpha \int_0^\infty e^{-\lambda t}Q(t)dt$, for all $\lambda > c$,
(I is the identity operator), $0 < \alpha \leq 1$,([1-9]).
Let $C_S(J)$ be the set of all continuous functions u on J with values in S. By a strong Solution of the Cauchy problem (1.1), (1.2), we mean a function u such that:

$$u \in C_S(J),$$

$$\frac{d^\beta u(t)}{dt^\beta} \in C_E(J),$$

u satisfies the following equation :

$$u(t) = u_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t - \theta)^{\beta-1}[Au(\theta) + f(\theta, B(\theta)u(\theta)]d\theta$$

$$+\frac{1}{\Gamma(\beta)} \int_0^t (t - \theta)^{\beta-1}s(\theta) \sum_{i=1}^k c_i u(t_i)d\theta, \tag{1.3}$$

Where $\Gamma(.)$ is the gamma function. In section 2, we shall consider the linear case. In other words when f depends only on t. In this case the solution can be obtained in a closed form. Also the stability of solutions can be established. In section 3, we shall solve equation (1.3) under suitable conditions on f and the operators $B_1, \cdots, B_r$.

It is assumed that:
$C_5 : \|B_i(t_2)Q(t_1)h\| \leq \frac{K}{t_1^c}\|h\|$, for all $t_1 > 0, h \in E$,
$C_6 : B_1(t)h, \cdots, B_r(t)h$ are uniformly Holder in $t \in J$ for all $h \in \bigcap_i S_i$.
It is assumed also that there exists a function g such that:

$$C_7 : f(t, B(t)u(t)) = \frac{1}{\Gamma(\alpha\beta)} \int_0^t (t - \theta)^{\alpha\beta-1}g(\theta, B(\theta)u(\theta))d\theta$$

(This means that f(0,w)=0 , $\frac{d^{\alpha\beta} f}{dt^{\alpha\beta}} = g$ exists) , where g is continuous on JXE, with the following properties:
C_8 : g satisfies a uniform Holder condition in $t \in J$ and a Lipschitz condition with respect to $B_1(t)u, \cdots, B_r(t)u$. There are many important applications of the theory of integrated semi groups and the nonlocal Cauchy problem for fractional differential equation. The applications can be found in the theory of quantum mechanics and the theory of elasticity. [1-8]."