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Eigenvectors of Interpoint Distance Matrices

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EIGENVECTORS OF INTERPOINT DISTANCE MATRICES

An honors paper submitted to the Department of Mathematics
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Michelle F Craft
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Michelle Craft
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05/01/15

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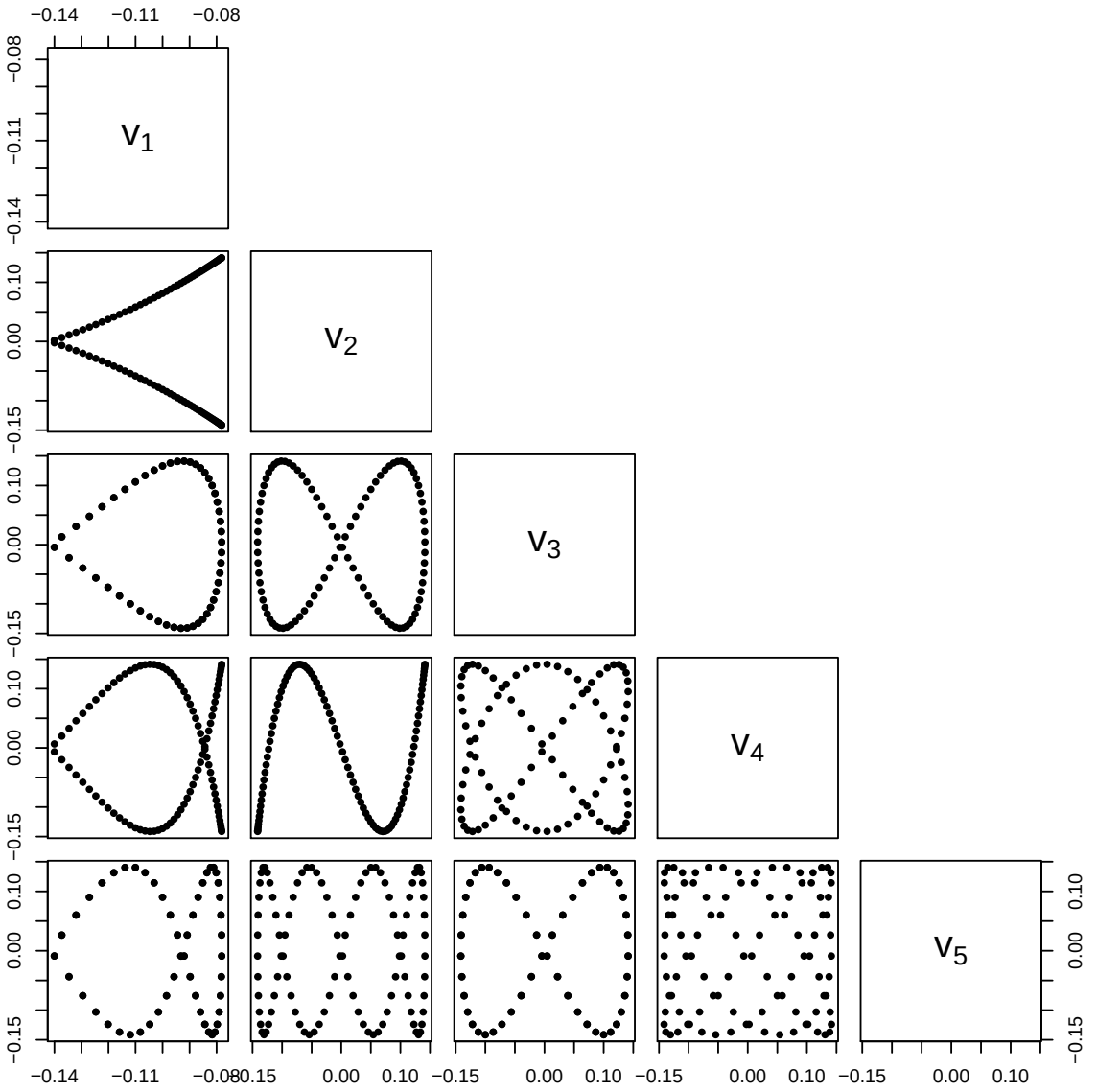
Abstract

In this paper, the eigenvectors of interpoint distance matrices will be discussed. When plotted against each other, the eigenvectors of the distance matrix of evenly spaced points in one dimension produce some interesting patterns. An explanation and description of the patterns will be discussed. After examining many aspects of the general Euclidean interpoint distance matrix of order N, D_N, as well as characteristics of the eigenvectors themselves, some conclusions can be made. Furthermore, research revealed a similarity between our matrices, D_N, and the Discrete Cosine Transform Matrix, DCT-2. This research led to additional conclusion about our matrices D_N and allowed for a classification of the patterns within the graphs of the eigenvectors.

Introduction

This project was proposed by researchers Dr. Elizabeth Hohman and Dr. David Marchette at the Naval Surface Warfare Center at Dahlgren. The project statement was:

Consider the following: given (ordered) one dimensional data $X = x_1, ..., x_N$, compute the interpoint distance matrix $D = (d(x_i, x_j))$, and compute the eigenvectors of D. The eigenvectors can be plotted against each other (see the figure below).



When the points are equally spaced, some interesting patterns emerge in the graphs of pairs of eigenvectors. Explain the pattern. Does it depend on the distance measure used? Is there a corresponding pattern for one dimensional data embedded in a higher dimensional space?

In our study of patterns, we assume that the points are equally spaced. In this case, the eigenvectors of the inter-point distance matrix do not depend on the distance between consecutive points. Therefore, for the remaining of the thesis, for each positive integer N, we fix the points $x_1 = 0, x_2 = 1, ..., x_N = N - 1$, and denote by D_N the matrix that represents the distances between these points.

The figure above shows the first five eigenvectors of the interpoint distance matrix D_N for $N = 100$, graphed against one another. In other words, the plot in row 2 column 1 of the above figure takes $v_1 = (a_1, a_2, ..., a_{100})$ and $v_2 = (b_1, b_2, ..., b_{100})$ of D_{100} against one another by plotting the points $(a_1, b_1), (a_2, b_2), ..., (a_{100}, b_{100})$. Then, the plot in row 3 column 1 would take v_1 against v_3 by plotting the points $(a_1, c_1), (a_2, c_2), ..., (a_{100}, c_{100})$ where $v_3 = (c_1, c_2, ..., c_{100})$.

An interpoint distance matrix is the matrix representation of distances between each pair of observations on a line. So the first line of the matrix will represent the distance from the first observation to itself, then the first observation to the second, and so on. Then, the next line represents the distance from the second observation to the first, then from the second to itself, and so on. This continues for as many observations, N, that the matrix will hold. Throughout the research, we refered to these matrices as our matrices, D_N. For example, for six evenly spaced points in one dimension the interpoint distance matrix D_6 would be:

$$D = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 1 & 2 & 3 & 4 \\ 2 & 1 & 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 & 1 & 2 \\ 4 & 3 & 2 & 1 & 0 & 1 \\ 5 & 4 & 3 & 2 & 1 & 0 \end{bmatrix} \tag{1}$$

These matrices D_N are symmetric, and it is well known that a symmetric $n \times n$ matrix has the real eigenvalues and n orthonormal eigenvectors (see Section 7.3 in Larson (2013)). These n orthonormal eigenvectors are the eigenvectors whose pairs are plotted above.

Background Information

In order to fully understand our matrices D_N there are certain classification definitions that need to be discussed. First, we will define a symmetric matrix, A, to be a matrix that is symmetric across its main diagonal.

Definition. *A matrix A is defined to be symmetric if $A = A^T$.*

Then we define a centrosymmetric matrix to be a matrix, A, that is symmetric about both its main and counterdiagonal.

Definition. *A matrix A is defined to be centrosymmetric if $JAJ = A$ where J is the counteridentity matrix. The counteridentity matrix is defined as the square matrix whose elements are all equal to zero except those on the counterdiagonal, which are all equal to 1.*

A general 3×3 centrosymmetric matrix would have the form:

$$\begin{bmatrix} a & b & c \\ d & e & d \\ c & b & a \end{bmatrix} \tag{2}$$

Both the above definition and the definition below were given by Abu-Jeib (2002, p.430). Below is the definition of a persymmetric matrix which is symmetric about the counterdiagonal.

Definition. *A matrix A is persymmetric if $JAJ = A^T$.*

Finally, we will define Toeplitz matrices, a special case of persymmetric matrices. Let $t_{-(n-1)}, ..., t_{-1}, t_0, t_1, ..., t_{n-1}$ be a sequence of real numbers. Then, a Toeplitz matrix is defined as an $n \times n$ matrix $T_n = [t_{k,j}; k, j = 0, 1, \cdots, n-1]$ where $t_{k,j} = t_{k-j}$ i.e. a matrix of the form:

$$T_n = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{-(n-1)} \\ t_1 & t_0 & t_{-1} & & \\ t_2 & t_1 & t_0 & & \vdots \\ \vdots & & & \ddots & \\ t_{n-1} & & & \cdots & t_0 \end{bmatrix} \tag{3}$$

This definition was given by Robert M. Gray of Stanford University.

Our matrices D_N were classified to be symmetric since they were symmetric about the main diagonal. Then, the matrices were also classified as centrosymmetric since they were symmetric about the main and counterdiagonal, but more specifically $JD_NJ = D_N$. Since D_N is symmetric, we know $D_N = D_N^T$, so $JD_NJ = D_N^T$, and our matrices are also persymmetric. Lastly, our matrices were also classified as Toeplitz since they were the special case of persymmetric where each diagonal surrounding the main diagonal is made up of the same value.

According to Cantoni and Butler (1976), "It is proved that the eigenvectors of a symmetric centrosymmetric matrix of order N are either symmetric or skew symmetric, and that there are $[\frac{N}{2}]$ symmetric and $[\frac{N}{2}]$ skew symmetric eigenvectors" (p.275). Abu-Jeib (2002) defines "A vector x is called symmetric if $Jx = x$ and skew-symmetric if $Jx = -x$." An example of a 3×1 and a 4×1 symmetric vector, respectively, would be:

$$\begin{pmatrix} a_1 \\ a_0 \\ a_1 \end{pmatrix} \qquad \begin{pmatrix} a_1 \\ a_0 \\ a_0 \\ a_1 \end{pmatrix} \tag{4}$$

Then, an example of a 3×1 and a 4×1 skew-symmetric vector, respectively, would be:

$$\begin{pmatrix} a_1 \\ a_0 \\ -a_1 \end{pmatrix} \qquad \begin{pmatrix} a_1 \\ a_0 \\ -a_0 \\ -a_1 \end{pmatrix} \tag{5}$$

The concept that our matrices D_N had symmetric and skew-symmetric eigenvectors was further examined during this research. A presentation by Patricia H. Carter (2010) that was found during the research process also indicated something about the symmetric and skew-symmetric nature of the interpoint distance matrices' eigenvectors. The way the research classifies which eigenvectors are symmertric or skew-symmetric is based on the way the eigenvectors are ordered. Since we used the program R, it automatically orders eigenvalues from largest to smallest, and then the corresponding eigenvector. So the largest eigenvalue of a matrix would be λ_1, and its corresponding eigenvector would be v_1. Then, the second largest eigenvalue would be λ_2, and its corresponding eigenvector would be v_2. This ordering continues for as many λ_N and corresponding v_N that the matrix has.

The presentation by Carter also indicated similarities between our matrices D_N and the Discrete Cosine Transform Matrix (DCT-2). This matrix, DCT-2, has the following form when $N = 6$:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \tag{6}$$

There are four discrete cosine transforms defined as follows (Rao & Yip, 1990, p.11):

(1) DCT-I:

$$[C^I_{N+1}]_{mn} = \left(\frac{2}{N}\right)^{\frac{1}{2}} \left[k_m k_n \cos\left(\frac{mn\pi}{N}\right)\right] \qquad m, n = 0, 1, ..., N; \tag{7}$$

(2) DCT-II:

$$[C^{II}_N]_{mn} = \left(\frac{2}{N}\right)^{\frac{1}{2}} \left[k_m \cos\left(\frac{m(n+\frac{1}{2})\pi}{N}\right)\right] \qquad m, n = 0, 1, ..., N-1; \tag{8}$$

(3) DCT-III:

$$[C^{III}_N]_{mn} = \left(\frac{2}{N}\right)^{\frac{1}{2}} \left[k_n \cos\left(\frac{(m+\frac{1}{2})n\pi}{N}\right)\right] \qquad m, n = 0, 1, ..., N-1; \tag{9}$$

(4) DCT-IV:

$$[C_N^{IV}]_{mn} = \left(\frac{2}{N}\right)^{\frac{1}{2}} \left[\cos\left(\frac{(m+\frac{1}{2})(n+\frac{1}{2})\pi}{N}\right)\right] \qquad m,n = 0,1,...,N-1; \tag{10}$$

All of these transforms are related to one another, but the DCT-II or DCT-2 matrix was the form that seemed to have similarities to the inverse of our matrices D_N. This DCT-2 matrix has known formulas for its eigenvalues and corresponding eigenvector. The m^{th} eigenvalue is given by the formula:

$$\lambda_m = 2 - 2\cos\left(\frac{m\pi}{N}\right) \tag{11}$$

Then, the j^{th} component of the corresponding m^{th} eigenvector is given by:

$$e_{j,m} = \cos\left(\frac{j+\frac{1}{2}}{N} m\pi\right) \tag{12}$$

All this information and classification of our matrices D_N helped in directing the research on these eigenvectors.

Methods

In order to explore why these patterns come about, we looked at different aspects of the matrices and their eigenvectors. Using the R program, we examined different components and patterns of the eigenvectors that could cause these graphs. We also came across the Carter presentation that had information on these matrices as well as their inverses. This information led us to look at the inverse of our matrices D_N and how it compares to the DCT-2 Matrix. Also proposed in the presentation was the fact that some eigenvectors of the matrices D_N were symmetric while others were skew-symmetric (Carter, 2010). Taking this idea, we looked at if or when the eigenvectors were symmetric or skew-symmetric.

Continuing with the research, we looked further into classifying our matrices D_N as well as trying to classify different patterns that appeared in the graphs of the eigenvectors. By classifying further our matrices D_N we were able to research more about characteristics that it or its eigenvectors may have. Then, knowing that whether the eigenvectors were odd or even affected whether they were symmetric or skew-symmetric, this would mean that varrying order would affect the number of symmetric and skew-symmetric vectors which may also affect the patterns. The focus of the first part of this project was to determine how the patterns varied along with order. We also examined interpoint distance matrices when they were normally or uniformly distributed instead of being evenly spaced.

Results

We began this research by looking at the closed form solutions for the eigenvalues and eigenvectors of the matrix D_N for N=2, 3, and 4 calculated by hand which are as follows:

When N=2, $\lambda = 1, -1$ and the corresponding eigenvector to $\lambda = 1$ had the form $v = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ with the corresponding eigenvector to $\lambda = -1$ having the form $v = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

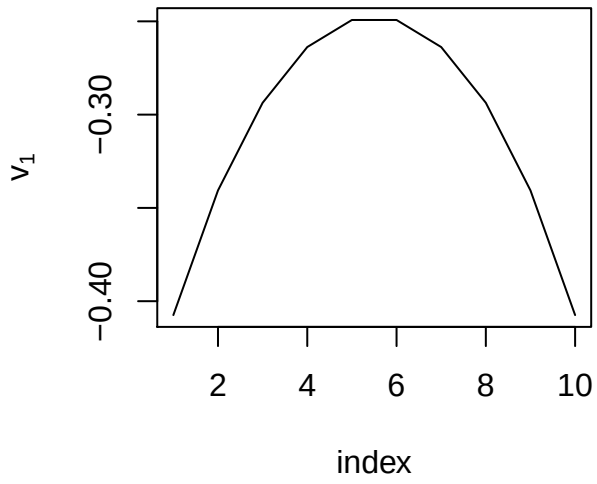
When N=3, $\lambda = 1 \pm \sqrt{3}, -2$ and the corresponding eigenvector to $\lambda = -2$ was $v = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

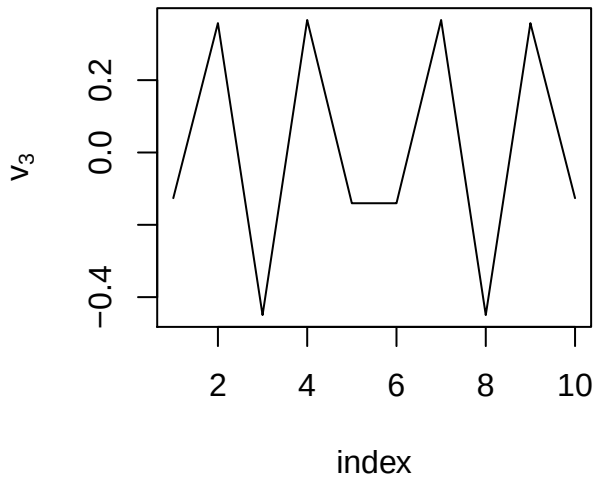
When N=4, we arrived at $\lambda = 2 \pm \sqrt{10}, -2 \pm \sqrt{2}$ and although solutions for the eigenvectors exist, we did not calculate those solutions by hand.

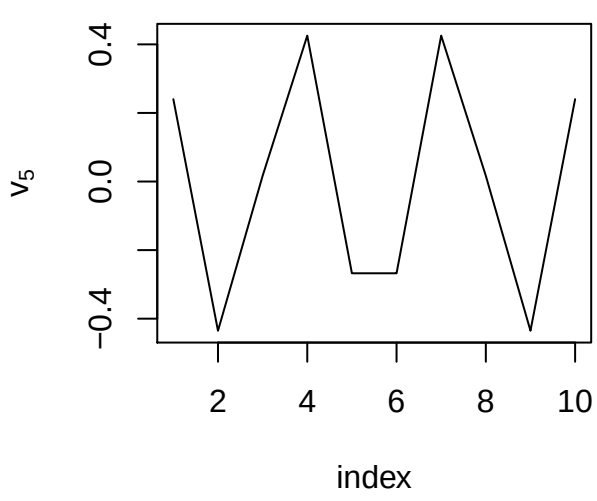
After looking at these, we used R to calculate v_1 and v_2 and see patterns as N increased. In particular, we took N=10 and N=1000 and looked at their v_1 and v_2. It was shown that the coordinates of v_1 for N=10 appear as coordinates 48, 148, 248, 348, 445, 556, 653, 753, 853, and 953 of v_1 for N=1000, respectively. Then, the closest values of coordinates of v_2 for N=10 appear as coordinates 51, 150, 251, 350, 451, 550, 651, 750, 851, 950 of v_2 for N=1000, respectively.

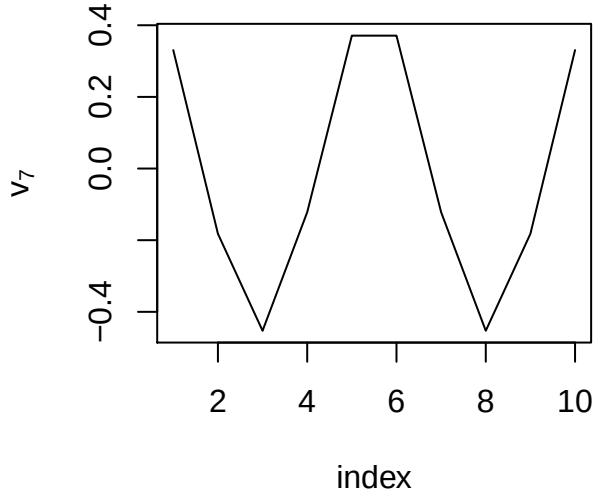
Then, after coming across the Carter powerpoint on interpoint distance matrix eigenvectors, we decided to look at the conjecture that the eigenvectors are either symmetric or skew symmetric. This was calculated using R as listed in Appendix A. This showed v_1 to have symmetry and v_2 to have skew-symmetry. Later on in the research, we discovered the theorem by Cantoni and Butler (1976) that indicated our matrix would have $\frac{N}{2}$ symmetric and $\frac{N}{2}$ skew-symmetric eigenvectors. However, the theorem did not state which eigenvectors were symmetric or skew-symmetric or whether they appeared in a pattern. So, we went on to look at when the eigenvectors of our matrices D_N were either symmetric or skew-symmetric. A way we found to show symmetry versus skew-symmetry more clearly was by the following graphs:

Odd Eigenvectors N=10

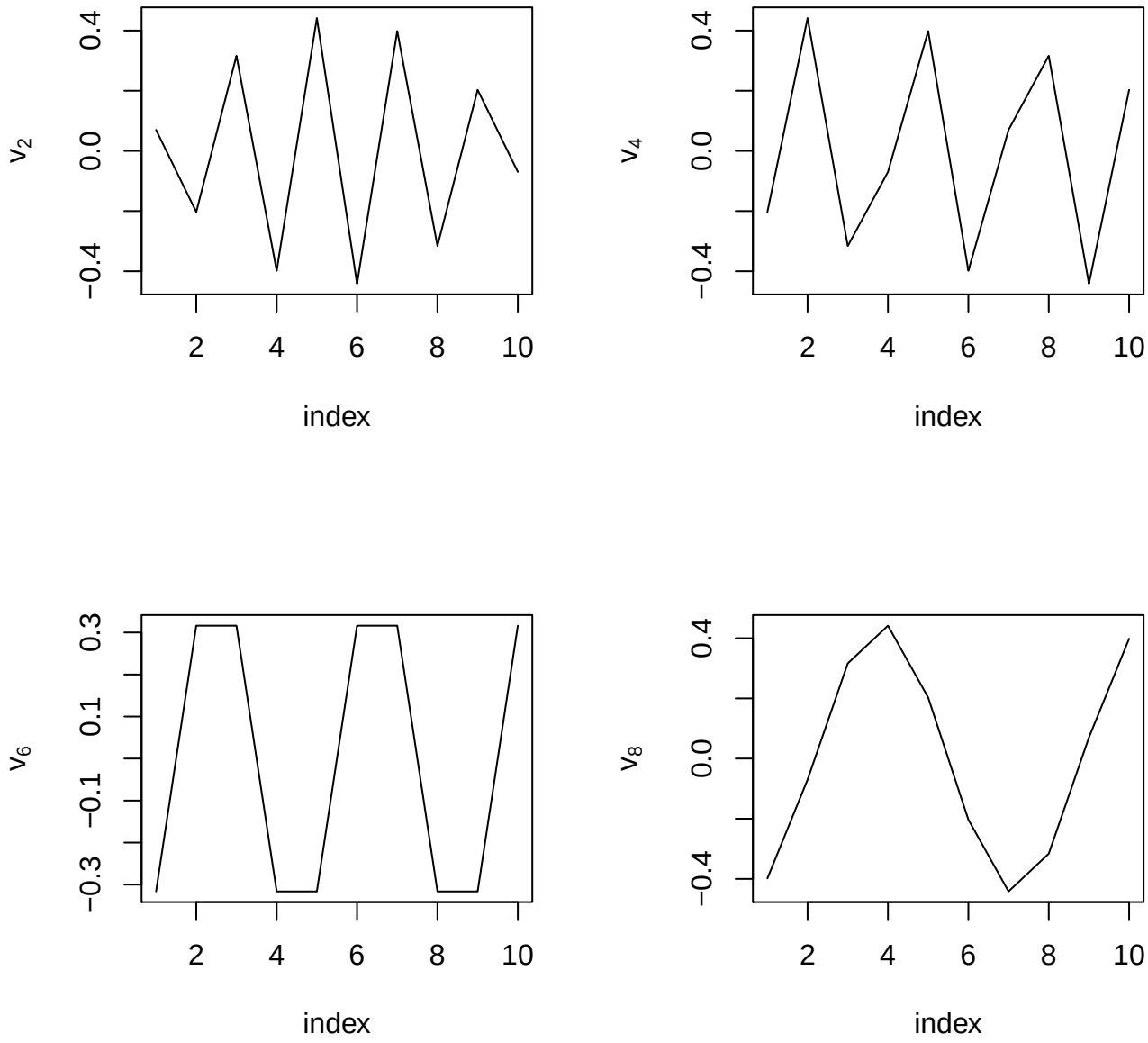






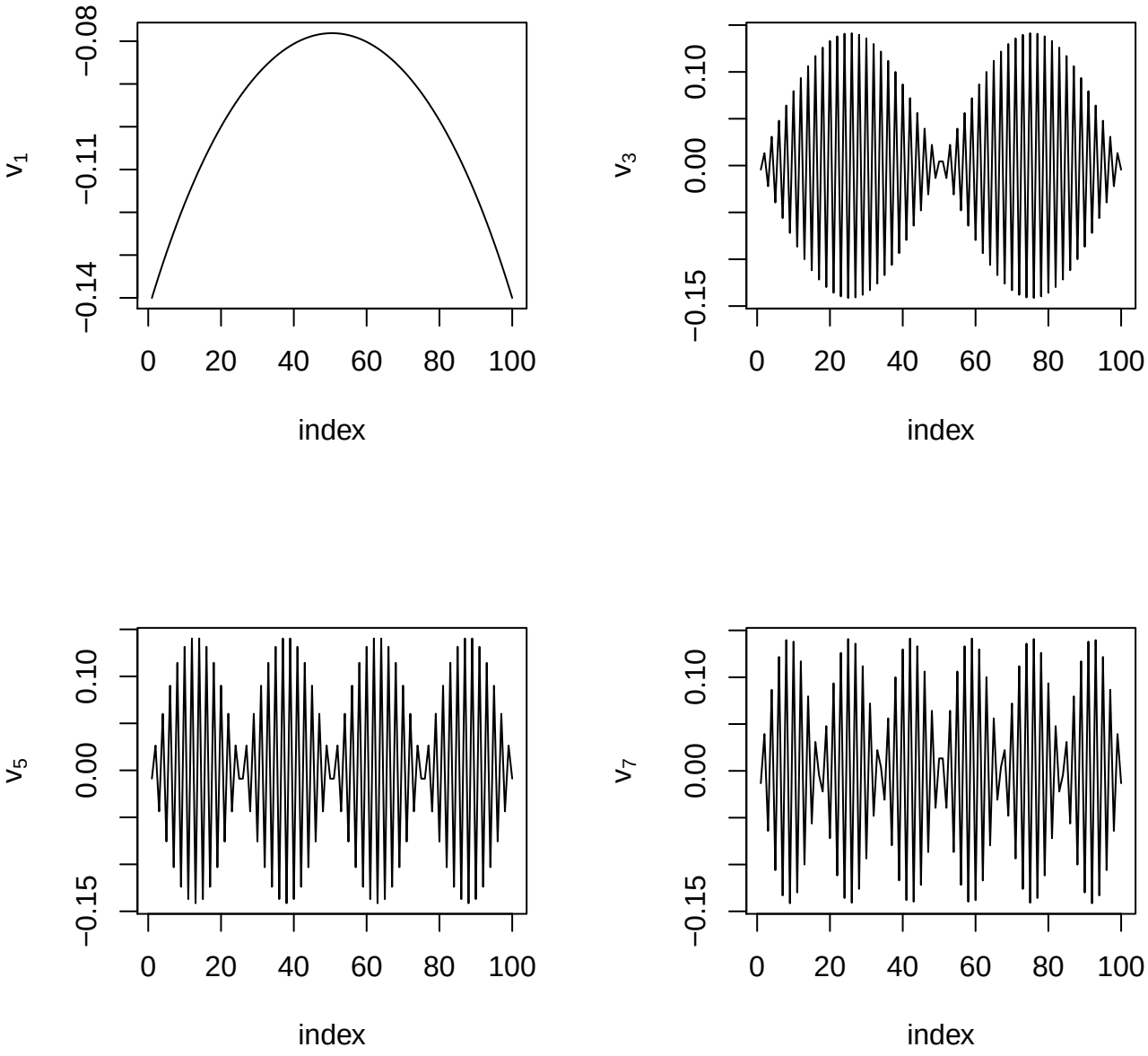


Even Eigenvectors N=10

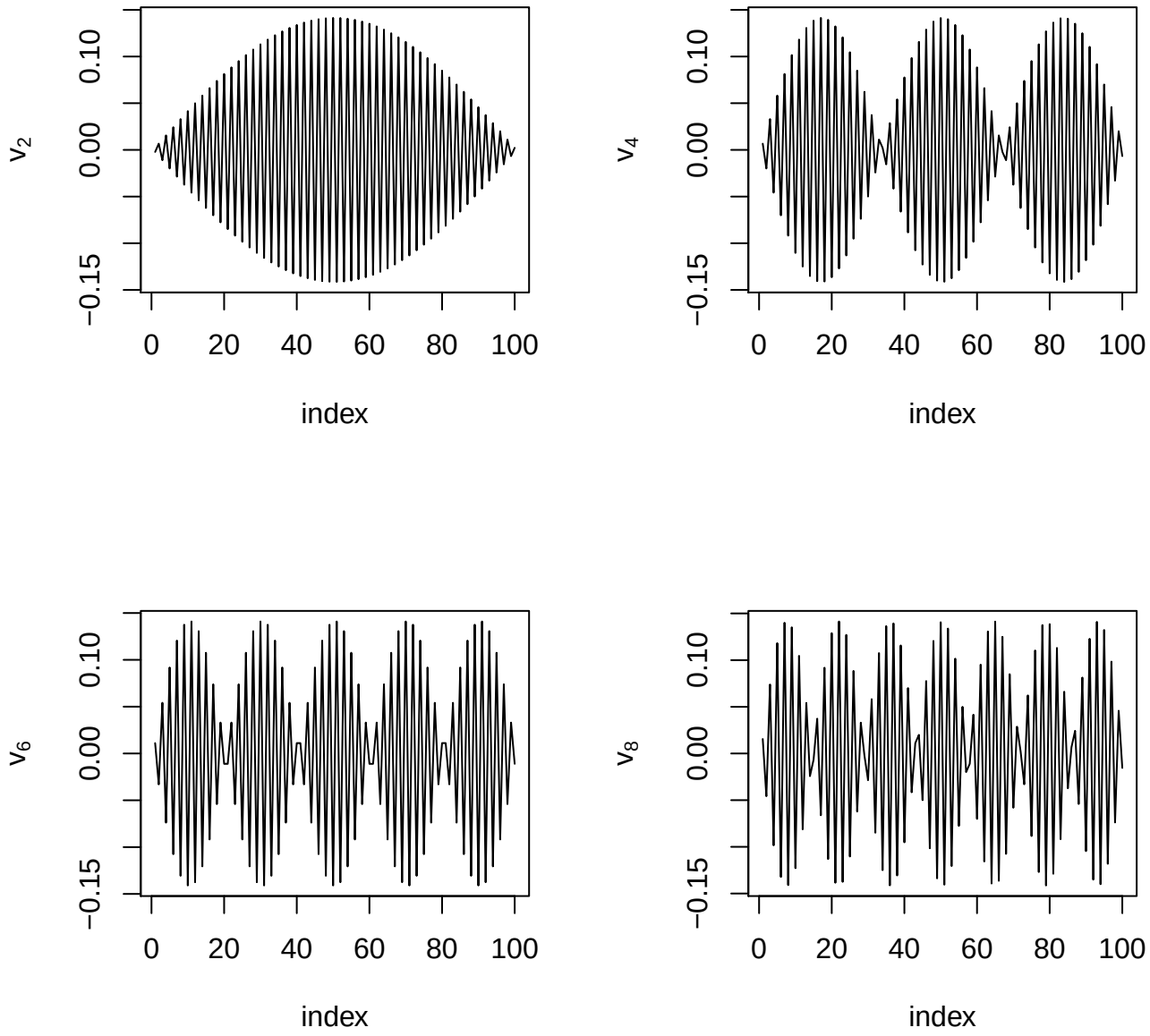


With these graphs it is clear to see that odd eigenvectors create symmetrical graphs while even eigenvectors create skew-symmectrical graphs. Then, when testing this with N=100 the following graphs appear:

Odd Eigenvectors N=100



Even Eigenvectors N=100

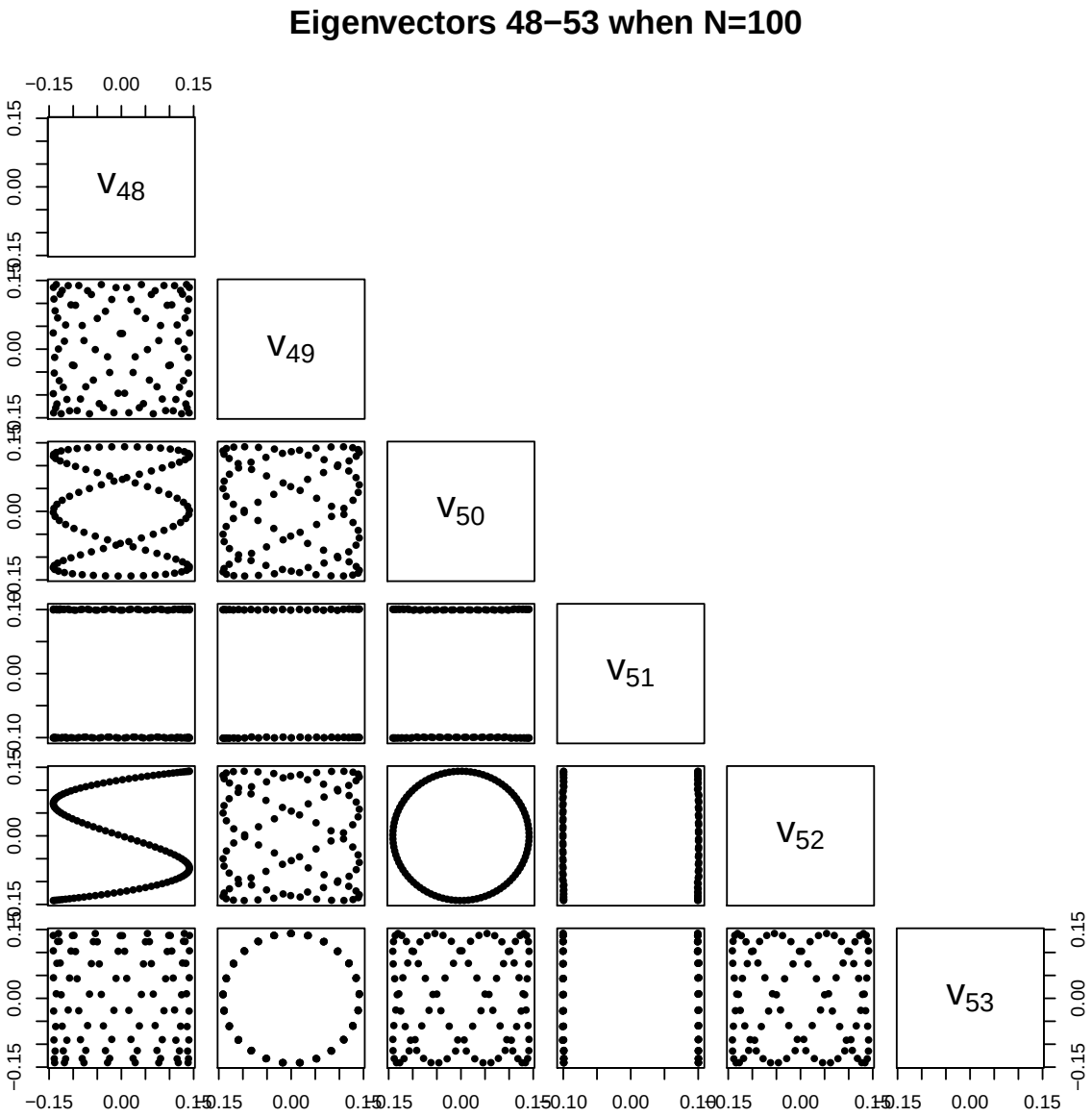


These graphs also show the symmetry with odd eigenvectors and skew-symmetry with even eigenvectors, but furthermore show a repeating pattern of $i-1$ "clusters" of points in the i^{th} eigenvector.

Since it became clear that odd and even eigenvectors were symmetric and skew-symmetric, respectively, we began to consider that perhaps this characteristic may play a part in the graph patterns shown in the problem statement. In order to look further into this, we

began examining matrices of varying orders since they would have varying amounts of skew-symmetric versus symmetric eigenvectors.

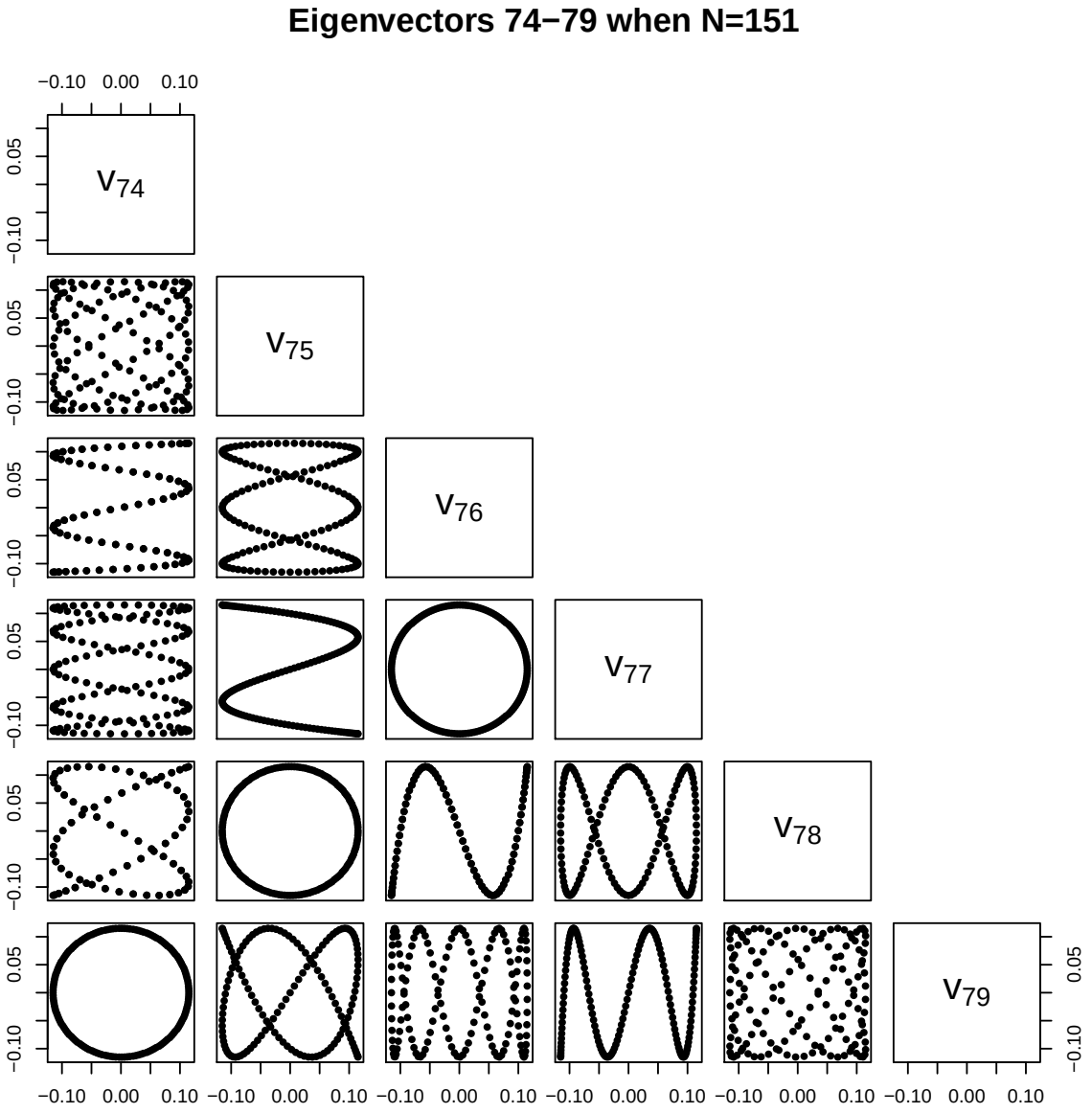
When exploring the eigenvectors of matrix D that had even order, we found specific reoccuring patterns. For example when N=100, the following graphs appear from vectors 48 through 53:



Here, you can see that anything graphed against vector 51 (the "central" vector) are just line graphs. Also interesting is the circle graph that appears for vector 50 versus vector 52 and for vector 49 versus vector 53. These patterns appeared at the central vectors for any matrix of even order. Since matrices of even order would not have an exact center vector, the lines seem to appear around the vector $\frac{N}{2}+1$ which is referred to as the central vector. When

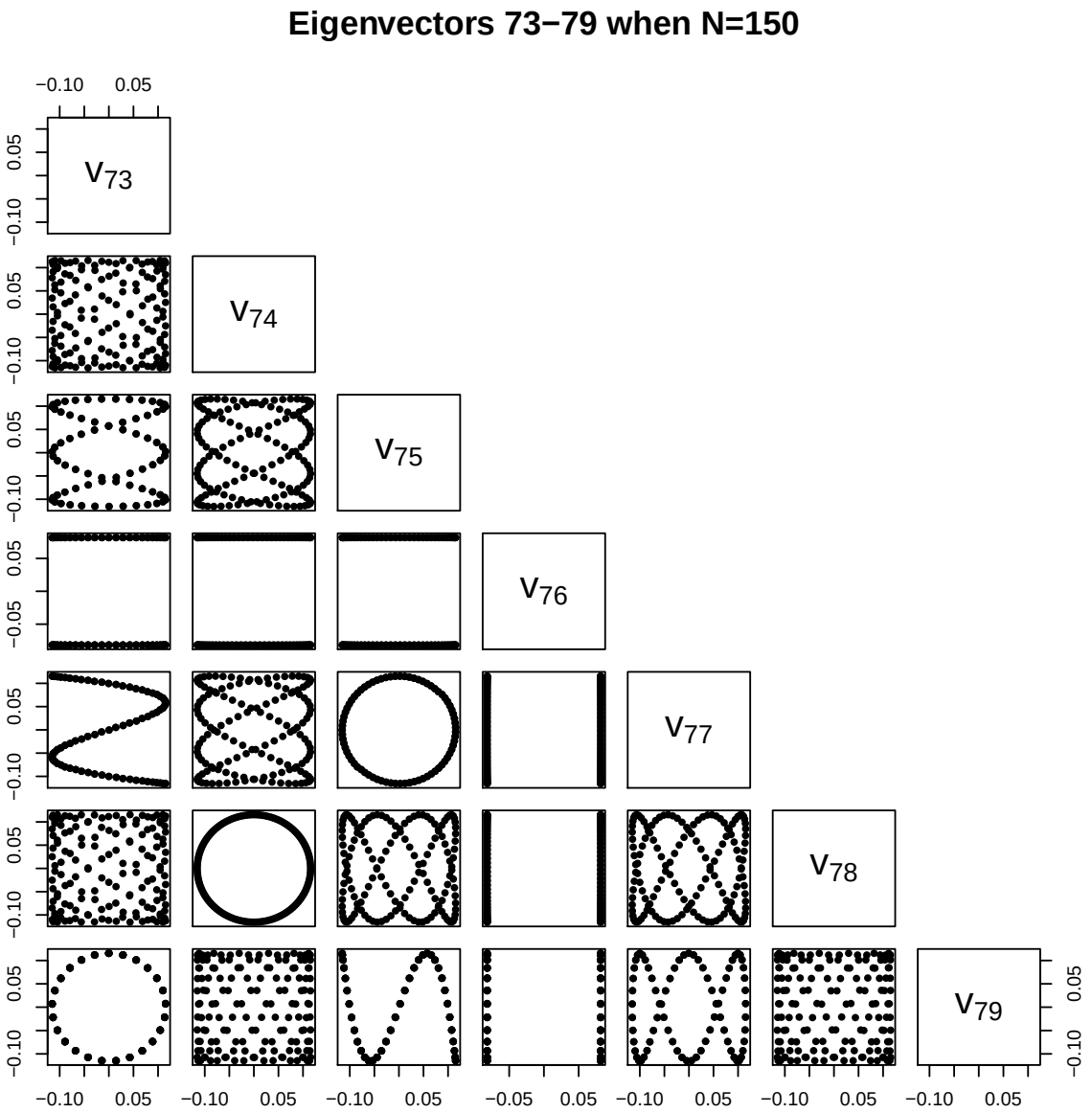
examining the central vector of even-order matrices it was found that if the vector were even, then it would be skew-symmetric and its components would all have the same value. If the vector were odd, then it would be symmetric and its components would be slightly varying values. For example, when N=10 and the central vector is 6 and when N=12 and the central vector is 7. The components of these eigenvectors are listed in Appendix B.

Then, when considering the vectors around the center of odd-order matrices, there were no line graphs that appeared since there is an exact center vector. However, the circle graphs still appear around the vector $\frac{N}{2}+1$ and follow a similar diagonal path. For example, when N=151 the center vectors appear as follows:

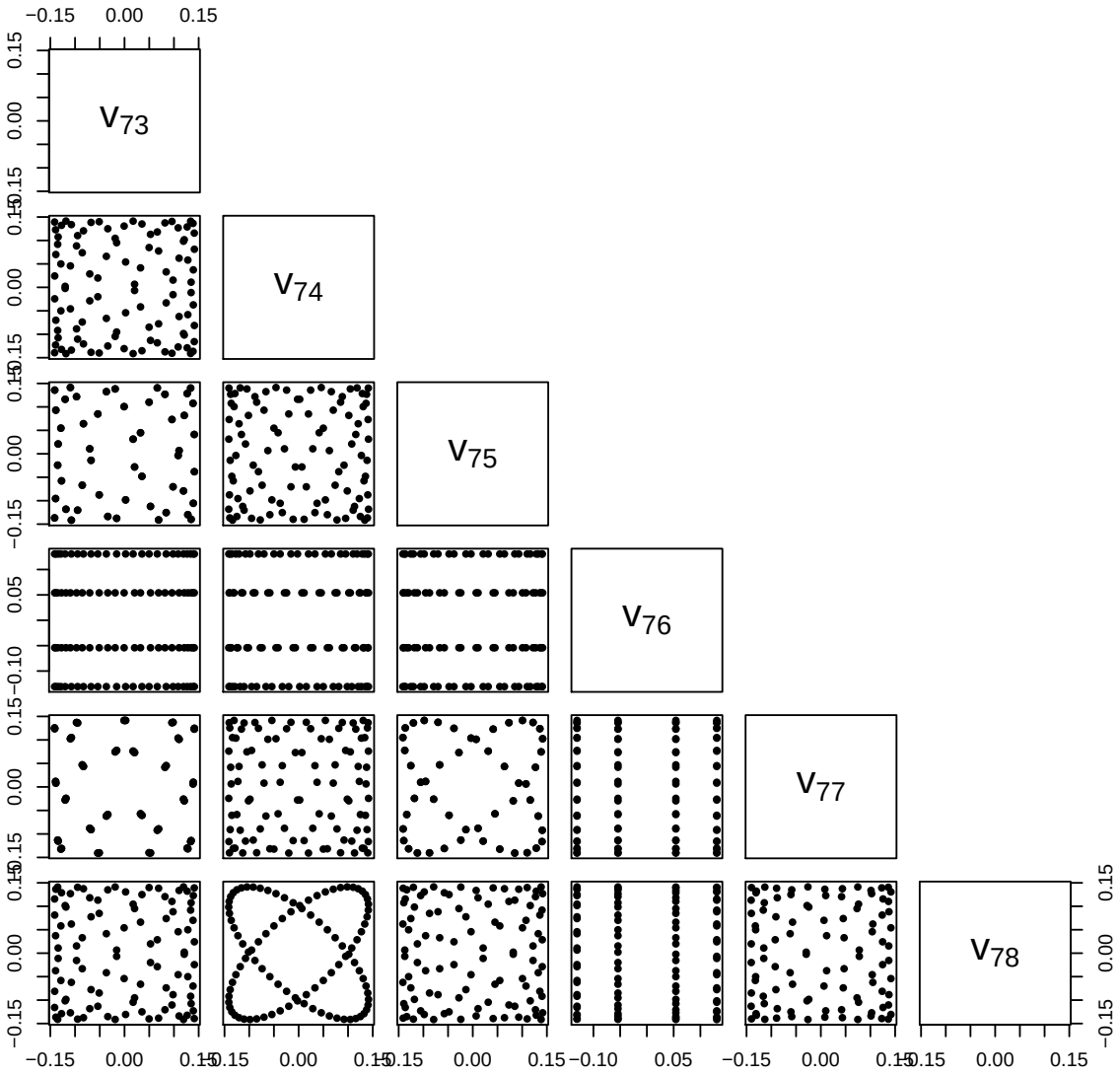


Not only do the circles appear in a similar fashion, but also the graphs around them

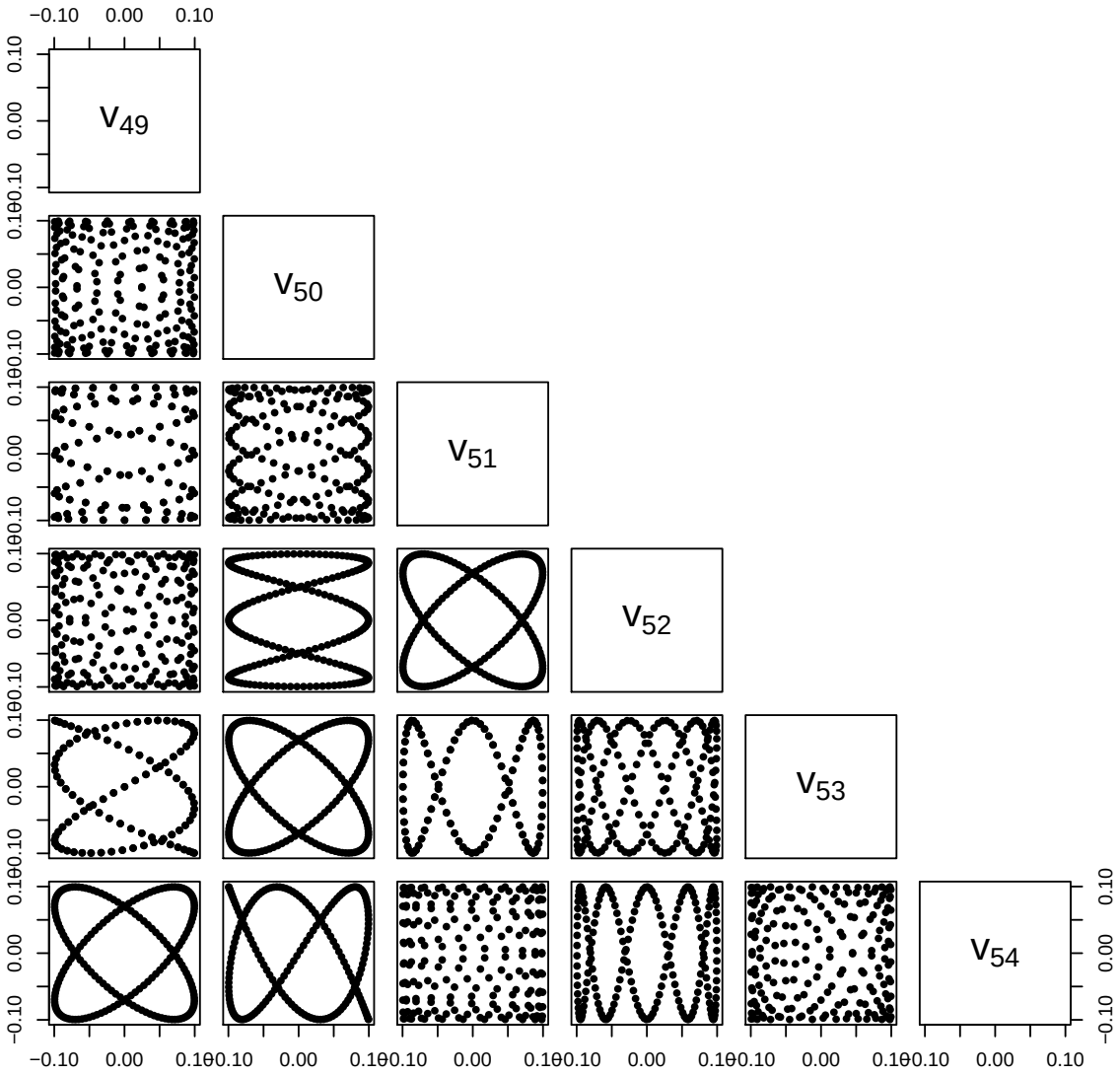
have some similar traits as well. The graphs around the circle graphs seem to reflect and/or rotate when reflected over the circle graphs. This pattern also occurs when looking at the eigenvectors of even-order matrices as shown below when N=150:



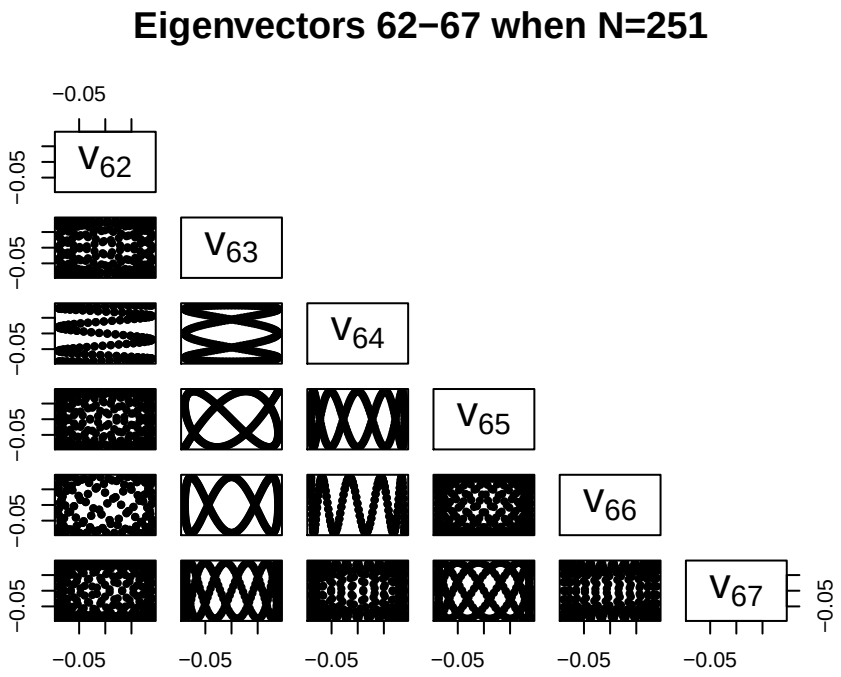
Futhermore, vectors of even-order matrices whose orders are evenly divisible by four also showed lines at what could be thought of as quarter vectors, vector $\frac{N}{4}+1$ and vector $\frac{3N}{4}+1$. Also, an interesting pattern appeared in the place where the circle graphs appear around the central vector:

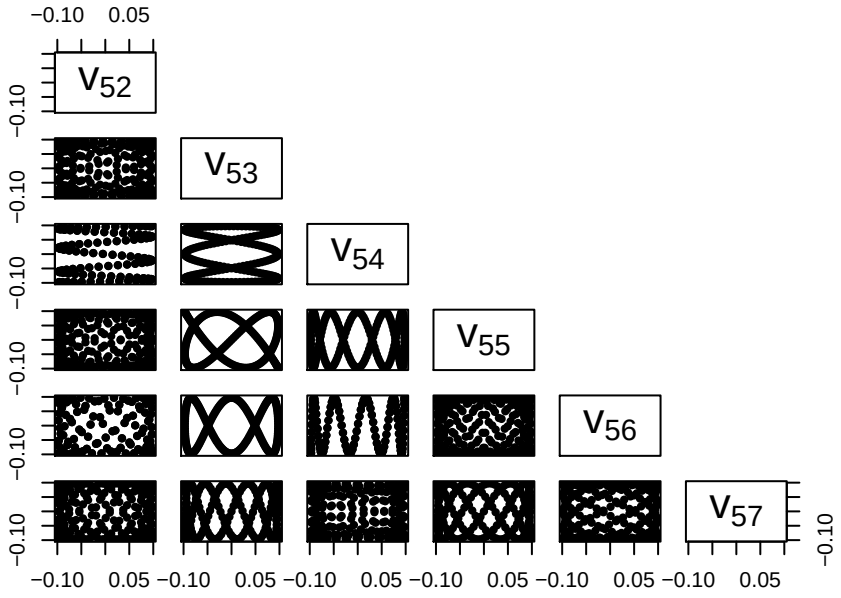


This pattern, which seemed to look like two intersecting ovals, also appeared in the same fashion around what would be the quarter vectors for matrices of even order that are not evenly divisible by four, as shown below:



In regards to the vectors around the quarter sections of matrices with odd order, there are no line graphs and the two instersecting ovals graph does not appear either. Although, there seemed to be certain reoccuring patterns that sometimes were seen as reflections or rotations. For example, when looking at N=251 and N=211, the same graph patterns exist at eigenvector 62 versus 64, eigenvector 63 versus 64, eigenvector 63 versus 65, eigenvector 63 versus 66, eigenvector 63 versus 67, eigenvector 64 versus 65, coordinate 64 versus 66, and coordinate 65 versus 67 when N=251 as coordinate coordinate 52 versus 54, eigenvector 53 versus 54, eigenvector 53 versus 55, eigenvector 53 versus 56, eigenvector 53 versus 57, eigenvector 54 versus 55, eigenvector 54 versus 56, and eigenvector 55 versus 57 when N=211 as shown:

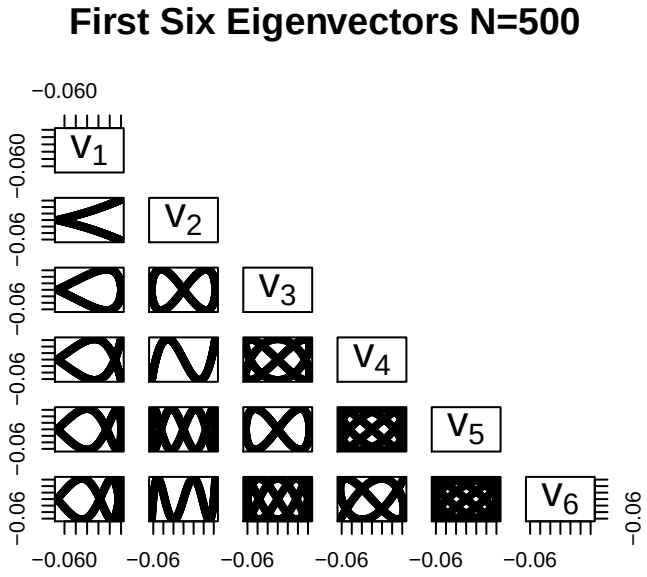


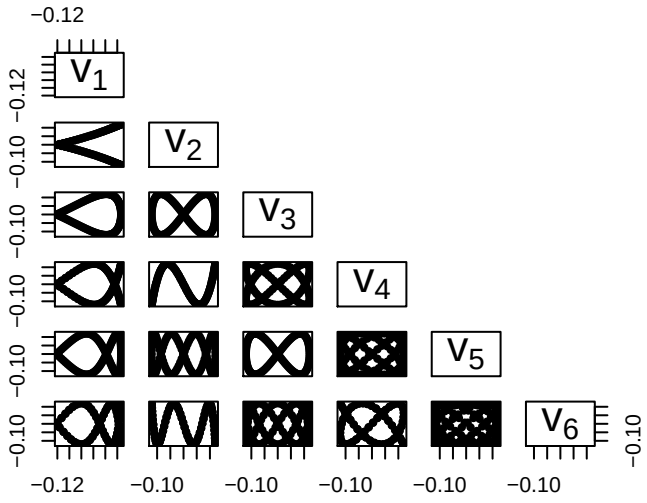


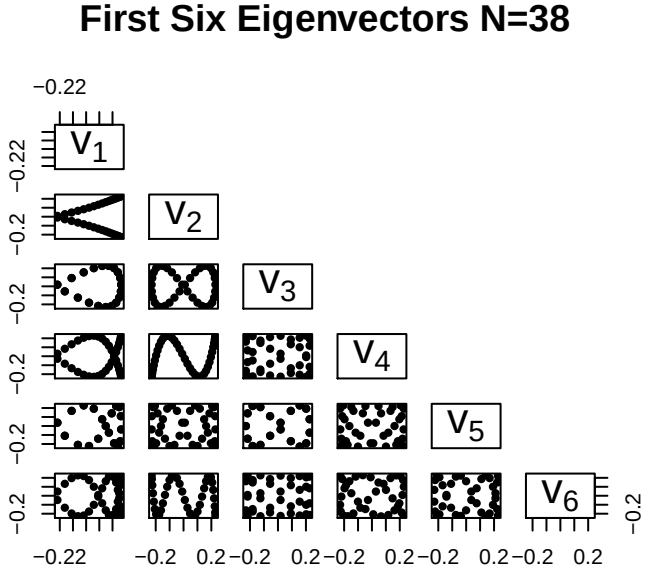
However, with varying order of matrices, not all of these patterns reoccur for every odd ordered matrix. The only consistent pattern that always remains in the same graph location

for every odd order matrix is the pattern shown at coordinate 63 versus 65 when N=251 and at coordinate 53 versus 55 when N=211. This pattern occurs at coordinate $\frac{N+1}{4}$ versus $\frac{N+1}{4}+2$ when $N \equiv 3 \pmod 4$ and at coordinate $\frac{N+3}{4}-1$ versus $\frac{N+3}{4}+2$ when $N \equiv 1 \pmod 4$.

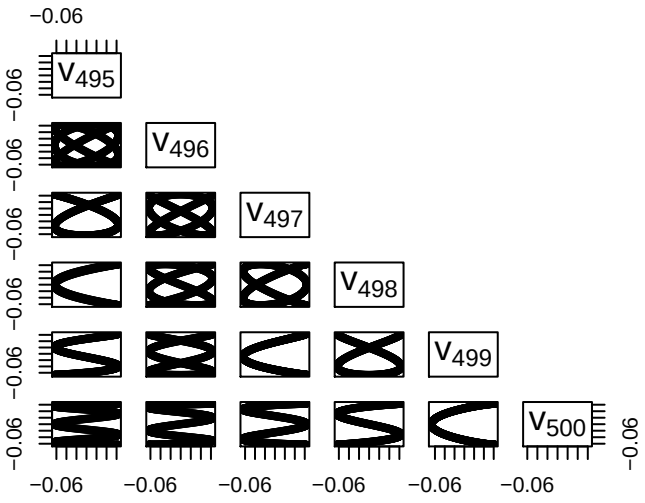
When looking at graphs of the first six eigenvectors of these matrices, the following patterns appeared:

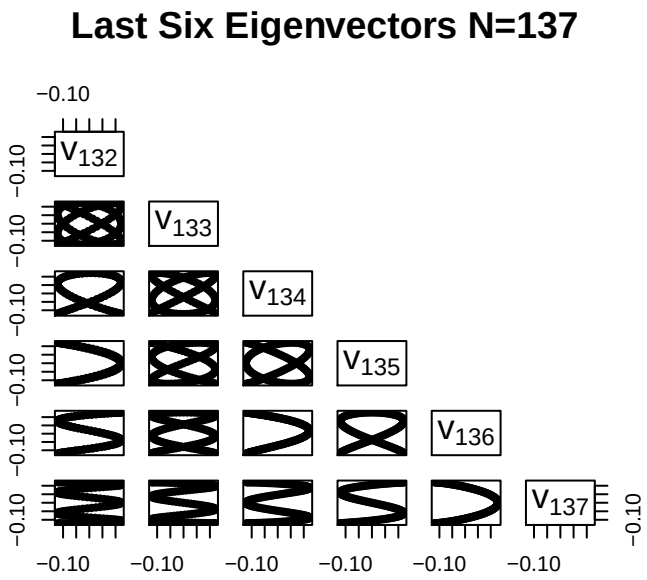


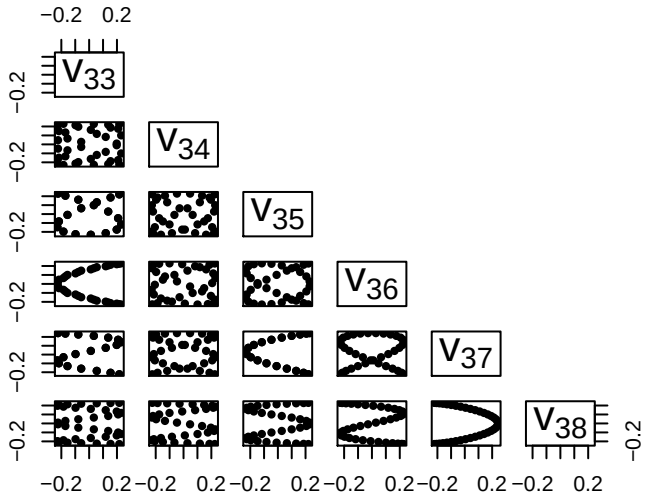




Despite having different orders, all patterns up to v_5 versus v_6 that appear are the same. Although, some patterns do appear as reflections in certain graphs. Similarly, when looking at the last six eigenvectors of these matrices, the following patterns appeared:







Again, it can be seen that the patterns are identical, but sometimes reflected.

These orders were specifically chosen to be examined because they vary in size difference and the specific numbers are very different as well. With N=500, it is an even number whose central eigenvector would be at 251 which is an odd number. With N=137, it is an odd number that has an exact center. Then, with N=38, it is an even number whose central eigenvector would be at 20 which is an even number.

The Carter presentation also gave the idea to look at the inverses of the matrices to

examine the eigenvectors of interpoint distance matrices since the eigenvectors are the same. This, and the fact that the inverse of matrix A has the eigenvalues $\frac{1}{\lambda}$, where λ ranges over the eigenvalue of A, are proved by the following:

Let A be an $n \times n$ matrix with eigenvalues λ and eigenvector v. Then,

$$Av = \lambda v \Rightarrow A^{-1}Av = \lambda A^{-1}v \Rightarrow v = \lambda A^{-1}v \Rightarrow A^{-1}v = \tfrac{1}{\lambda}v$$

However, since we have that the eigenvalue of the inverse matrix will be $\frac{1}{\lambda}$, if the matrix has an eigenvalue of 0, the corresponding inverse matrix's eigenvalue would be undefined. It is known that 0 cannot be an eigenvalue of a nonsingular matrix by the following theorem:

Theorem. *Let A be a nonsingular $n \times n$ matrix. Suppose that λ is an eigenvalue of A. Then there exists a nonzero vector v in $\mathbb{R}^n$ such that $Av = \lambda v$. Thus $A^{-1}Av = A^{-1}(\lambda v)$, and so $v = \lambda(A^{-1}v)$. Hence λ cannot be 0 since v is a nonzero vector.*

Thus, 0 can never be an eigenvalue for our matrices D_N and the idea of looking at its inverse can be pursued.

In the Carter presentation previously mentioned, it was proposed that the formula for inverse matrix for D when N=6 is as follows:

Let $q = \frac{1}{N-1}$

Then, the matrix D has inverse $\frac{-1}{2}M$ where

$$M = \begin{bmatrix} 1-q & -1 & 0 & 0 & 0 & -q \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ -q & 0 & 0 & 0 & -1 & 1-q \end{bmatrix} \tag{13}$$

This was verified using R when N=6 and N=10. The code and output of this can be seen in Appendix C. Although R showed values that were sometimes close to, but not exactly the same as the identity, when checked by hand the inverse by $\frac{-1}{2}M$ does turn out to be correct.

Another aspect of testing whether this was a true inverse or not was going off of the idea that the matrix D is also classified as a Toeplitz matrix. We found an alternative inverse formula for D using the following lemma and theorem:

Lemma. *Let $T = (a_{p-q})_{p,q=1}^{n}$ be an $n \times n$ Toeplitz matrix; then it satisfies the formula*

$$KT - TK = fe_n^T - e_1 f^T J \tag{14}$$

where

$$K = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 & 1 \\ 1 & \ddots & \ddots & \ddots & \vdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & 0 \end{bmatrix}, J = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & .^{\cdot} & .^{\cdot} & 0 \\ 0 & .^{\cdot} & .^{\cdot} & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} \tag{15}$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, f = \begin{pmatrix} 0 \\ a_{n-1} - a_{-1} \\ \vdots \\ a_2 - a_{-n+2} \\ a_1 - a_{-n+1} \end{pmatrix} \tag{16}$$

Theorem. *Let $T = (a_{p-q})_{p,q=1}^{n}$ be a Toeplitz matrix. If each of the systems of equations $Tx = f$, $Ty = e_1$ is consistent with the solutions $x = (x_1, x_2, ..., x_n)^T$, $y = (y_1, y_2, ..., y_n)^T$, repectively, then*

(a) T is invertible;

(b)$T^{-1} = T_1U_1 + T_2U_2$, where

$$T_1 = \begin{bmatrix} y_1 & y_n & \cdots & y_2 \\ y_2 & y_1 & \ddots & \\ \vdots & \ddots & \ddots & y_n \\ y_n & \cdots & y_2 & y_1 \end{bmatrix}, U_1 = \begin{bmatrix} 1 & -x_n & \cdots & -x2 \\ & 1 & \ddots & \vdots \\ & & \ddots & -x_n \\ & & & 1 \end{bmatrix} \tag{17}$$

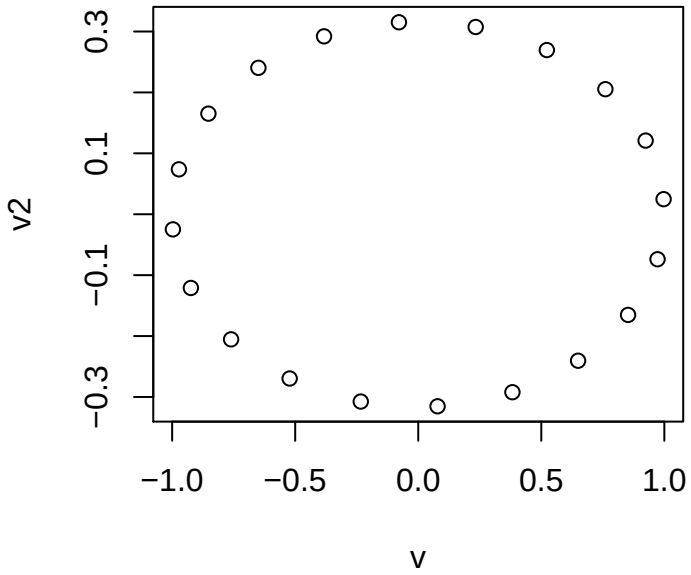
$$T_2 = \begin{bmatrix} x_1 & x_n & \cdots & x_2 \\ x_2 & x_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & x_n \\ x_n & \cdots & x_2 & x_1 \end{bmatrix} U_2 = \begin{bmatrix} 0 & y_n & \cdots & y_2 \\ & 0 & \ddots & \vdots \\ & & \ddots & y_n \\ & & & 0 \end{bmatrix} \tag{18}$$

We then tested using R if this inverse would equal our $\frac{-1}{2}M$ when N=2 (Lv & Huang, 2007). The code that verified this can be found in Appendix D. This showed that this formula gave back the correct inverse as well as that T^{-1} is the same as our inverse matrix $\frac{-1}{2}M$ when N=2. We then also came up with a fairly general way of showing that $\frac{-1}{2}M$ works for all values N in one direction.

$$\begin{bmatrix} \frac{2-N}{2N-2} & \frac{1}{2} & \cdots & \frac{1}{2N-2} \\ \frac{1}{2} & -1 & \frac{1}{2} & \vdots \\ \vdots & \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2N-2} & \cdots & \frac{1}{2} & \frac{2-N}{2N-2} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & \cdots & N-2 & N-1 \\ 1 & 0 & \cdots & N-3 & N-2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ N-2 & N-3 & \cdots & 0 & 1 \\ N-1 & N-2 & \cdots & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \tag{19}$$

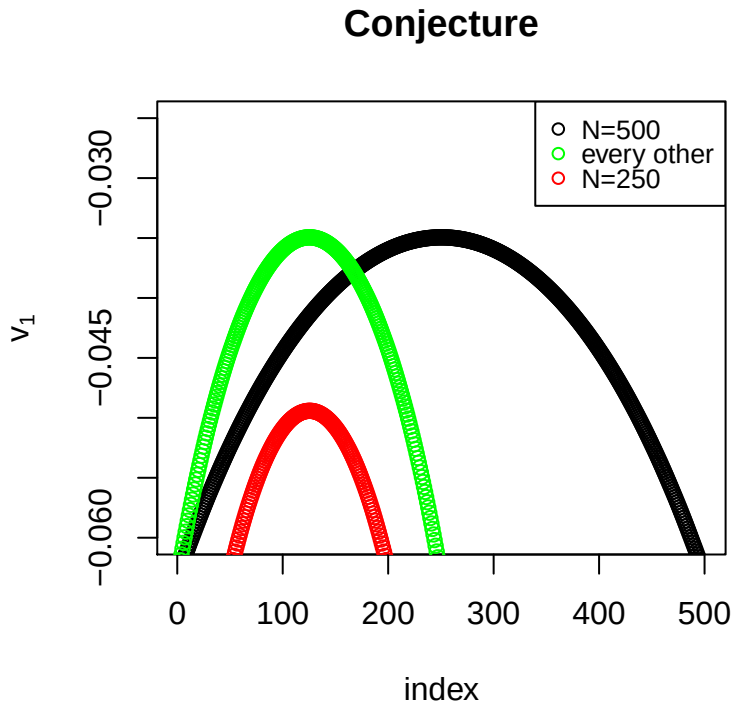
The Carter presentation also discussed this inverse matrix by mentioning that D (or its inverse) is very similar to the discrete cosine transform (DCT-2) matrix. This was important to note since the DCT-2 matrix has known formulas for its eigenvalues and corresponding eigenvectors. These formulas can be seen in the Background Information section as formula 11 and formula 12, respectively.

In order to test this, we first looked at whether the odd eigenvectors of the DCT-2 matrix were also eigenvectors of our matrix D in the case of N=6. The code used in R to test this can be found in Appendix E. This showed that the even vectors of D were the same as the odd vectors of DCT-2. When taking the plot of the first eigenvector of the DCT-2 matrix by the second eigenvector of our matrix, D, when N=20, it gave a circle as follows:

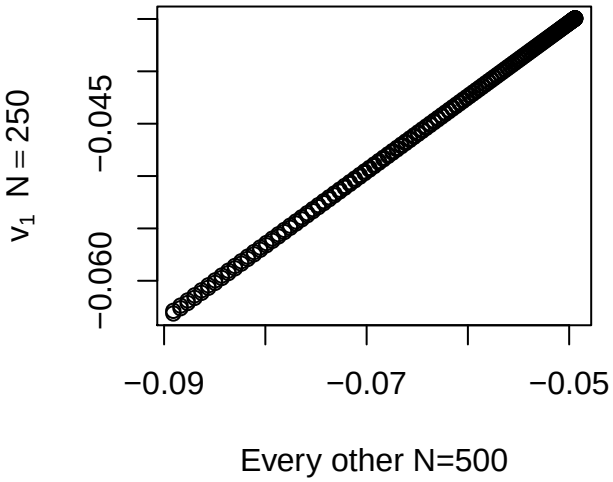


As of now, we are not sure why this graph appears to be circular, but the DCT-2 matrix eigenvector formula proved to be true in giving even eigenvectors for our matrices D_N.

We also looked at the conjecture that when the sequence from 0 to 1 has order N, the i^{th} element in v_1 is also the $2i^{th}$ element in v_1 of the sequence from 0 to 1 with order $2N$. This pattern was examined graphically when N=500 versus when N=250 versus with every other point from the N=500 vector.

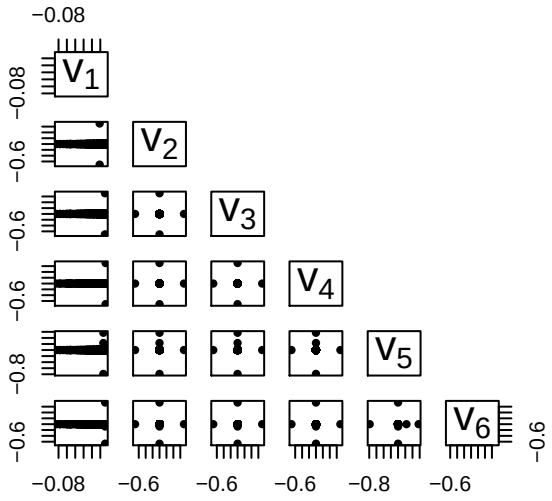


This graph disproved the conjecture, but indicated a proportional relationship between the i^{th} element of v_1 when N=250 and the $2i^{th}$ element of v_1 when N=500. This proportional relationship can be seen when mapping v_1 when N=250 against every other element for v_1 when N=500:

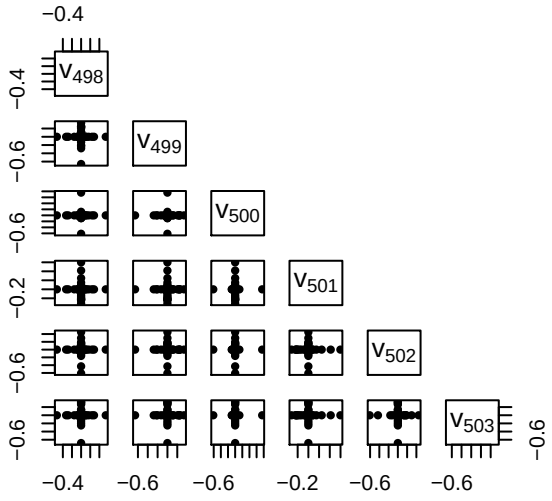


Interestingly enough, the coefficients of a line fit to this graph, seen in Appendix F, show the slope of the line to be $\sqrt{2}$. However, when testing other lengths, the relationship did not hold to always be $\sqrt{2}$.

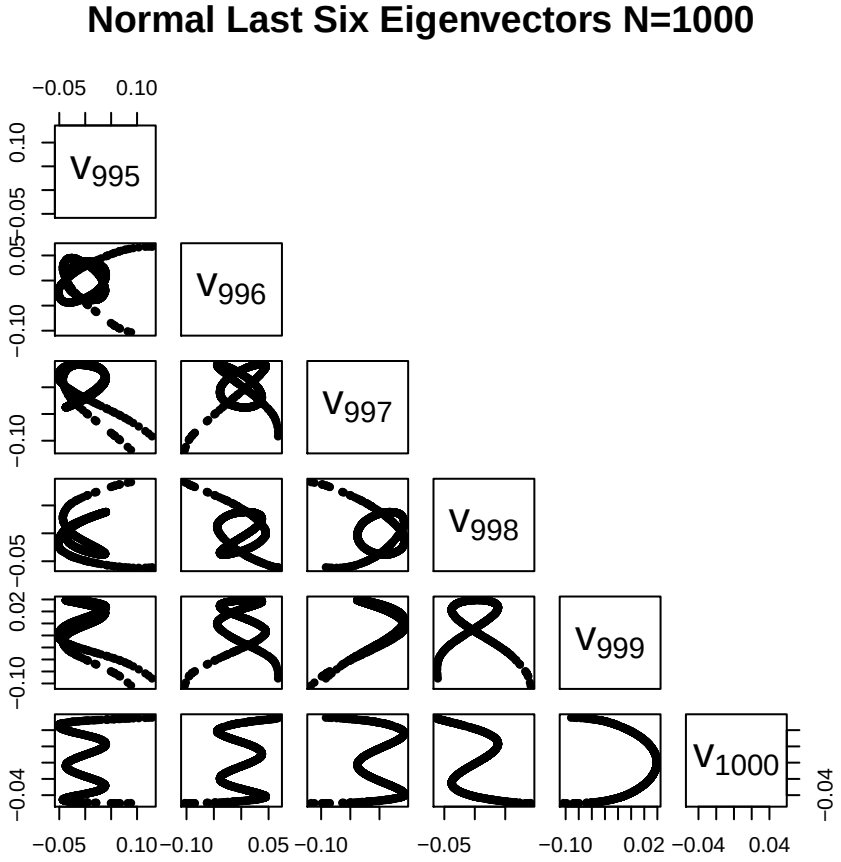
Finally, we looked at interpoint distance matrices when the points were normally or uniformly distributed rather than being evenly spaced. When normally distributed, the first six eigenvector graphs appeared as follows:



Then, around the center vectors showed the following graphs:



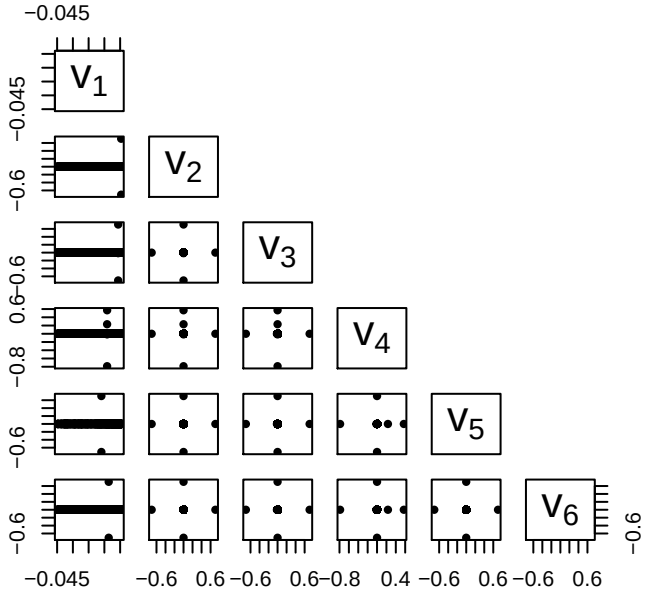
Finally, the last six eigenvectors gave the following graphs:



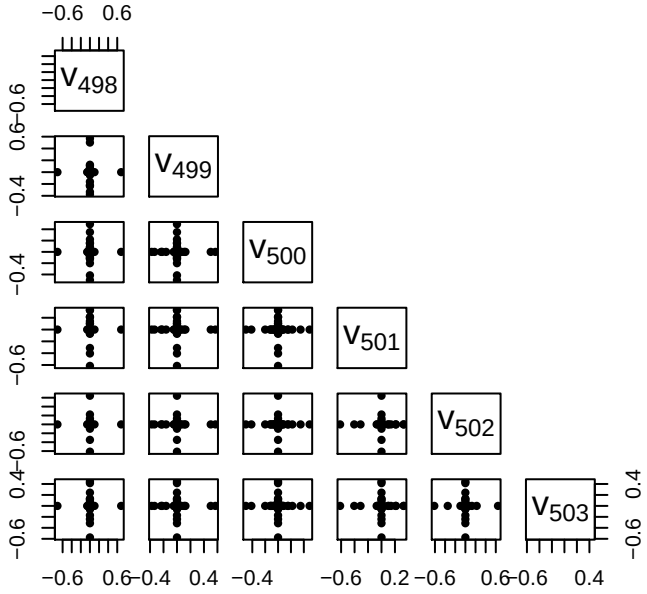
Mostly, the normally distributed points produced eigenvectors that had graphs similar to the ones seen around the first six eigenvectors and those around the center eigenvectors. The somewhat patterns that appear with the last six eigenvectors slightly show resemblance to some graphs that D_N produced. However, these patterns do not appear until the very last eigenvectors and may have some resemblance, but not much, to certain graphs that D_N produced.

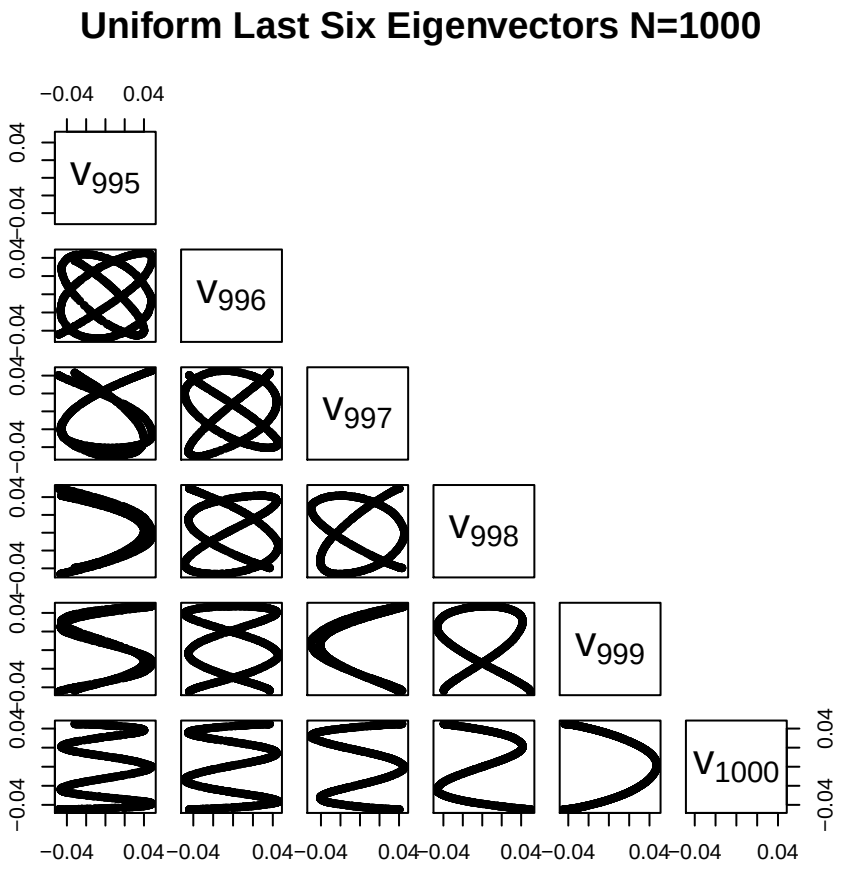
Then, with uniformly distributed points, the following patterns appear around the first six eigenvectors, center eigenvectors, and last six eigenvectors, respectively:

Uniform First Six Eigenvectors N=1000



Uniform Center Eigenvectors N=1000





These graphs appeared to have more definite patterns than when the points were normally distributed. Also, the graphs appear to be more similar in pattern to those of D_N towards the end. However, it is also seen that these similar patterns do not appear until the very last eigenvectors. This is most likely due to the fact that altering the interpoint distance matrix to have uniformly or normally distributed points greatly changes it from our matrix D_N.

Conclusion

When altering the interpoint distance matrix to be uniformly and normally distributed, the patterns that appear with our matrix D_N do not appear. Towards the last eigenvectors of both distributions, some patterns seem to have slight resemblance to those of D_N, this is more evident with uniform distribution rather than normal. Most likely, the majority of these eigenvectors do not show the same patterns as those of D_N because changing the point distribution changes the matrix we are using. When points are uniformly or normally distributed, the interpoint distance matrix is no longer persymmetric, centrosymmetric, or Toeplitz. No longer having these classifications may be the reason the patterns almost do not appear with uniformly and normally distributed interpoint distance matrices.

We also came to the conjecture, when the matrix D has order 2N, the $2i^{th}$ element in its

v_1 is proportional to the i^{th} element of v_1 when D has order N. We looked at two graphical representations of this conjecture which held it to be true.

We also found that the inverse of our matrices D_N can be written as $\frac{-1}{2}M$ where M is the matrix (in N=6 form here):

$$M = \begin{bmatrix} 1-q & -1 & 0 & 0 & 0 & -q \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ -q & 0 & 0 & 0 & -1 & 1-q \end{bmatrix} \tag{20}$$

This led to looking at the DCT-2 matrix and the formula for its eigenvectors. Since the DCT-2 matrix was so similar to the $\frac{-1}{2}M$ inverse matrix, it was possible for eigenvectors to be the same for the inverse matrix. This would mean they would also be eigenvectors for our matrices D_N since we know eigenvectors are the same for matrices and their inverses. Ultimately, we found that every odd eigenvector of the DCT-2 matrix was a corresponding even eigenvector for our matrices D_N.

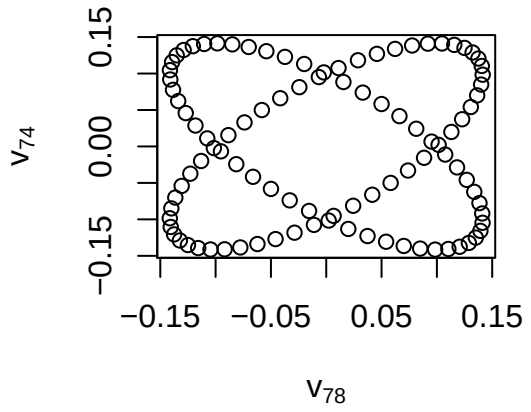
The Carter presentation suggested the eigenvectors would be symmetric when they were even and skew-symmetric when they were odd. Although we only tested v1 and v2, we found with these that the eigenvectors were symmetric for odd eigenvectors and skew-symmetric for even eigenvectors. We further examined this topic to find that graphing the eigenvectors using lines showed symmetry versus skew-symmetry. This also showed odd eigenvectors to be symmetric and even eigenvectors to be skew-symmetric. Then, determining that our matrices D_N were centrosymmetric Toeplitz matrices led to the fact that a proof does exist to show eigenvectors are either symmetric or skew-symmetric and there are number $\frac{N}{2}$ of each.

Considering the order of the matrix would affect the number of symmetric and skew-symmetric eigenvectors, different patterns appeared when testing certain orders. Ultimately, it was found that matrices of even order have lines at vector $\frac{N}{2}+1$ with circle graphs at vectors $\{\frac{N}{2}-n\}_{n=0}^{\frac{N}{2}-1}$ versus vectors $\{\frac{N}{2}+n\}_{n=2}^{\frac{N}{2}}$ where N is the matrix order. Then, looking specifically at values of the vector $\frac{N}{2}+1$, it was shown that if the vector were odd it was symmetric and made up of slightly differing values. If the vector were even, it was skew-symmetric and made up of one identical value. Since the values were only slightly different when vector $\frac{N}{2}+1$ was odd, this probably is due to slight-error and may mean that it would normally be made up of one identical value.

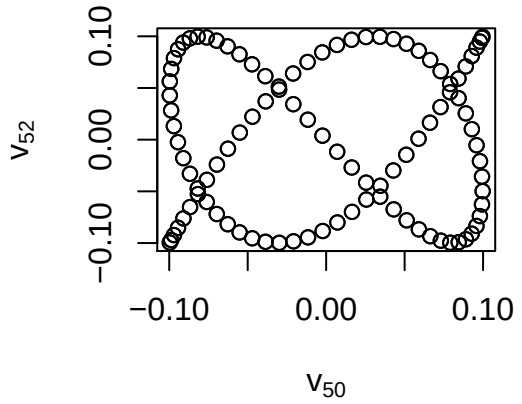
Regarding the eigenvectors of odd order matrices, no line graphs appeared. However, the circle graphs did appear at vectors $\{\frac{N+1}{2}-n\}_{n=0}^{\frac{N+1}{2}-1}$ versus vectors $\{\frac{N+1}{2}+n\}_{n=1}^{\frac{N+1}{2}}$ where N is the matrix order. Another interesting pattern found around the central vectors for both odd and even order matrices was that the graphs around the circle graphs seem to reflect and/or rotate when reflected across the circle graphs.

Looking at even ordered matrices, orders that were evenly divisible by four also showed line graphs at vector $\frac{N}{4}+1$ and vector $\frac{3N}{4}+1$. Also, similar to the place of the circle graphs

mentioned before, the following pattern appeared around the vectors $\{\frac{N+2}{4}-n\}_{n=0}^{\frac{N+2}{4}-1}$ versus vectors $\{\frac{N+2}{4}+n\}_{n=1}^{\frac{N+2}{4}}$:



Finally, when examining odd ordered matrices the following pattern appeared at coordinate $\frac{N+1}{4}$ versus $\frac{N+1}{4}+2$ when $N \equiv 3 \pmod 4$ and at coordinate $\frac{N+3}{4}-1$ versus $\frac{N+3}{4}+2$ when $N \equiv 1 \pmod 4$ with every odd ordered matrix:



Overall, more was found about classifying our matrices D_N and more reoccuring patterns were identified over the course of this research. However, more in depth analysis of these eigenvectors and their graphs beyond my own capabilities may be necessary to find an explicit concluding statement on what causes the patterns in these graphs.

Appendices

Appendix A

The following R code displays v_1 and v_2, respectively of our matrix D when N=20:

```
##  [1] -0.3013348 -0.2739647 -0.2505489 -0.2307495 -0.2142806 -0.2009046
##  [7] -0.1904283 -0.1827006 -0.1776100 -0.1750829 -0.1750829 -0.1776100
## [13] -0.1827006 -0.1904283 -0.2009046 -0.2142806 -0.2307495 -0.2505489
## [19] -0.2739647 -0.3013348
```

```
##  [1]  0.02481094 -0.07382191  0.12101513 -0.16522855  0.20537351
##  [6] -0.24046148  0.26962849 -0.29215636  0.30749037 -0.31525294
## [11]  0.31525294 -0.30749037  0.29215636 -0.26962849  0.24046148
## [16] -0.20537351  0.16522855 -0.12101513  0.07382191 -0.02481094
```

Appendix B

The following R code displays the "central" vectors, v_6 for N=10 and v_7 for N=12, respectively:

```
ev$vector[,6]

##  [1] -0.3162278  0.3162278  0.3162278 -0.3162278 -0.3162278  0.3162278
##  [7]  0.3162278 -0.3162278 -0.3162278  0.3162278
```

```
ev$vector[,7]

##  [1]  0.2674016 -0.3085053 -0.2760716  0.3007468  0.2845236 -0.2927508
##  [7] -0.2927508  0.2845236  0.3007468 -0.2760716 -0.3085053  0.2674016
```

Appendix C

The following R code tested the inverse from the Carter Presentation when N=6 and N=10, respectively:

```
mat1

##      [,1] [,2] [,3] [,4] [,5] [,6]
## [1,]    0    1    2    3    4    5
## [2,]    1    0    1    2    3    4
## [3,]    2    1    0    1    2    3
```

```
## [4,]    3    2    1    0    1    2
## [5,]    4    3    2    1    0    1
## [6,]    5    4    3    2    1    0

invmat1=matrix(c(-0.4,0.5,0,0,0,0.1,0.5,-1,0.5,0,0,0,0,0.5,-1,0.5,0,0,0,0,0.5,-1,0.5,0,0
invmat1

##      [,1] [,2] [,3] [,4] [,5] [,6]
## [1,] -0.4  0.5  0.0  0.0  0.0  0.1
## [2,]  0.5 -1.0  0.5  0.0  0.0  0.0
## [3,]  0.0  0.5 -1.0  0.5  0.0  0.0
## [4,]  0.0  0.0  0.5 -1.0  0.5  0.0
## [5,]  0.0  0.0  0.0  0.5 -1.0  0.5
## [6,]  0.1  0.0  0.0  0.0  0.5 -0.4

invmat1%*%mat1

##               [,1] [,2]          [,3]          [,4]          [,5] [,6]
## [1,]  1.000000e+00    0 -2.775558e-17 -1.665335e-16 -8.326673e-17    0
## [2,]  0.000000e+00    1  0.000000e+00  0.000000e+00  0.000000e+00    0
## [3,]  0.000000e+00    0  1.000000e+00  0.000000e+00  0.000000e+00    0
## [4,]  0.000000e+00    0  0.000000e+00  1.000000e+00  0.000000e+00    0
## [5,]  0.000000e+00    0  0.000000e+00  0.000000e+00  1.000000e+00    0
## [6,] -1.110223e-16    0 -1.110223e-16  0.000000e+00  0.000000e+00    1

mat1%*%invmat1

##               [,1] [,2] [,3] [,4] [,5]          [,6]
## [1,]  1.000000e+00    0    0    0    0 -1.110223e-16
## [2,]  0.000000e+00    1    0    0    0  0.000000e+00
## [3,] -2.775558e-17    0    1    0    0 -1.110223e-16
## [4,] -1.665335e-16    0    0    1    0  0.000000e+00
## [5,] -8.326673e-17    0    0    0    1  0.000000e+00
## [6,]  0.000000e+00    0    0    0    0  1.000000e+00
```

```
## Warning in matrix(c(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 1, 0, 1, 2, 3, 4, 5, 6, :
data length [99] is not a sub-multiple or multiple of the number of rows [10]

mat2

##       [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10]
##  [1,]    0    1    2    3    4    5    6    7    8     9
```

```
##  [2,]    1    0    1    2    3    4    5    6    7    8
##  [3,]    2    1    0    1    2    3    4    5    6    7
##  [4,]    3    2    1    0    1    2    3    4    5    6
##  [5,]    4    3    2    1    0    1    2    3    4    5
##  [6,]    5    4    3    2    1    0    1    2    3    4
##  [7,]    6    5    4    3    2    1    0    1    2    3
##  [8,]    7    6    5    4    3    2    1    0    1    2
##  [9,]    8    7    6    5    4    3    2    1    0    1
## [10,]    9    8    7    6    5    4    3    2    1    0
```

```
invmat2=matrix(c(-0.4444444,0.5,0,0,0,0,0,0,0,0.55555556,0.5,-1,0.5,0,0,0,0,0,0,0,0,0.5,
invmat2
```

```
##             [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9]      [,10]
##  [1,] -0.4444444  0.5  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.5555556
##  [2,]  0.5000000 -1.0  0.5  0.0  0.0  0.0  0.0  0.0  0.0  0.0000000
##  [3,]  0.0000000  0.5 -1.0  0.5  0.0  0.0  0.0  0.0  0.0  0.0000000
##  [4,]  0.0000000  0.0  0.5 -1.0  0.5  0.0  0.0  0.0  0.0  0.0000000
##  [5,]  0.0000000  0.0  0.0  0.5 -1.0  0.5  0.0  0.0  0.0  0.0000000
##  [6,]  0.0000000  0.0  0.0  0.0  0.5 -1.0  0.5  0.0  0.0  0.0000000
##  [7,]  0.0000000  0.0  0.0  0.0  0.0  0.5 -1.0  0.5  0.0  0.0000000
##  [8,]  0.0000000  0.0  0.0  0.0  0.0  0.0  0.5 -1.0  0.5  0.0000000
##  [9,]  0.0000000  0.0  0.0  0.0  0.0  0.0  0.0  0.5 -1.0  0.5000000
## [10,]  0.5555556  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.5 -0.4444444
```

```
invmat2%*%mat2
```

```
##          [,1]      [,2] [,3] [,4] [,5] [,6] [,7] [,8]      [,9]   [,10]
##  [1,] 5.5e+00 4.0000001  3.5  3.0  2.5  2.0  1.5  1.0 0.5000004 4.0e-07
##  [2,] 0.0e+00 1.0000000  0.0  0.0  0.0  0.0  0.0  0.0 0.0000000 0.0e+00
##  [3,] 0.0e+00 0.0000000  1.0  0.0  0.0  0.0  0.0  0.0 0.0000000 0.0e+00
##  [4,] 0.0e+00 0.0000000  0.0  1.0  0.0  0.0  0.0  0.0 0.0000000 0.0e+00
##  [5,] 0.0e+00 0.0000000  0.0  0.0  1.0  0.0  0.0  0.0 0.0000000 0.0e+00
##  [6,] 0.0e+00 0.0000000  0.0  0.0  0.0  1.0  0.0  0.0 0.0000000 0.0e+00
##  [7,] 0.0e+00 0.0000000  0.0  0.0  0.0  0.0  1.0  0.0 0.0000000 0.0e+00
##  [8,] 0.0e+00 0.0000000  0.0  0.0  0.0  0.0  0.0  1.0 0.0000000 0.0e+00
##  [9,] 0.0e+00 0.0000000  0.0  0.0  0.0  0.0  0.0  0.0 1.0000000 0.0e+00
## [10,] 4.0e-07 0.5000004  1.0  1.5  2.0  2.5  3.0  3.5 4.0000001 5.5e+00
```

```
mat2%*%invmat2
```

```
##            [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9]     [,10]
##  [1,] 5.5000000    0    0    0    0    0    0    0    0 0.0000004
```

```
##  [2,] 4.0000001    1    0    0    0    0    0    0    0 0.5000004
##  [3,] 3.5000001    0    1    0    0    0    0    0    0 1.0000003
##  [4,] 3.0000002    0    0    1    0    0    0    0    0 1.5000003
##  [5,] 2.5000002    0    0    0    1    0    0    0    0 2.0000002
##  [6,] 2.0000002    0    0    0    0    1    0    0    0 2.5000002
##  [7,] 1.5000003    0    0    0    0    0    1    0    0 3.0000002
##  [8,] 1.0000003    0    0    0    0    0    0    1    0 3.5000001
##  [9,] 0.5000004    0    0    0    0    0    0    0    1 4.0000001
## [10,] 0.0000004    0    0    0    0    0    0    0    0 5.5000000
```

Appendix D

The following R code compares the theorem of Lv & Huang (2007) to the inverse defined in the Carter Presentation:

```
K=matrix(c(0,1,0,0,0,1,1,0,0),nrow=3)
T=matrix(c(0,1,2,1,0,1,2,1,0),nrow=3)
K%*%T-T%*%K

##      [,1] [,2] [,3]
## [1,]    1   -1    0
## [2,]    0    0    1
## [3,]    0    0   -1

J=matrix(c(0,0,1,0,1,0,1,0,0),nrow=3)
f=c(0,1,-1)
e3=c(0,0,1)
e1=c(1,0,0)
f%*%t(e3)-e1%*%t(f)%*%J

##      [,1] [,2] [,3]
## [1,]    1   -1    0
## [2,]    0    0    1
## [3,]    0    0   -1

b=c(0,1,-1)
solve(T,b)

## [1]  0.25 -1.50  0.75

e1=c(1,0,0)
solve(T,e1)

## [1] -0.25  0.50  0.25
```

```
T1=matrix(c(-.25,.5,.25,.25,-.25,.5,.5,.25,-.25),nrow=3)
U1=matrix(c(1,0,0,-.75,1,0,1.5,-.75,1),nrow=3)
T2=matrix(c(.25,-1.5,.75,.75,.25,-1.5,-1.5,.75,.25),nrow=3)
U2=matrix(c(0,0,0,.25,0,0,.5,.25,0),nrow=3)
Tinv=T1%*%U1+T2%*%U2
Tinv

##       [,1] [,2]  [,3]
## [1,] -0.25  0.5  0.25
## [2,]  0.50 -1.0  0.50
## [3,]  0.25  0.5 -0.25

Tinv%*%T

##      [,1] [,2] [,3]
## [1,]    1    0    0
## [2,]    0    1    0
## [3,]    0    0    1
```

Appendix E

The following R code compares the eigenvalues and eigenvectors of the DCT-2 matrix and our matrix D respectively when N=6:

```
eigen(DCT2)

## $values
## [1] 3.732051e+00 3.000000e+00 2.000000e+00 1.000000e+00 2.679492e-01
## [6] 1.332268e-15
##
## $vectors
##            [,1]       [,2]       [,3]          [,4]       [,5]      [,6]
## [1,] -0.1494292 -0.2886751  0.4082483 -5.000000e-01  0.5576775 0.4082483
## [2,]  0.4082483  0.5773503 -0.4082483 -2.220446e-16  0.4082483 0.4082483
## [3,] -0.5576775 -0.2886751 -0.4082483  5.000000e-01  0.1494292 0.4082483
## [4,]  0.5576775 -0.2886751  0.4082483  5.000000e-01 -0.1494292 0.4082483
## [5,] -0.4082483  0.5773503  0.4082483  3.552714e-15 -0.4082483 0.4082483
## [6,]  0.1494292 -0.2886751 -0.4082483 -5.000000e-01 -0.5576775 0.4082483

mat1=matrix(c(0,1,2,3,4,5,1,0,1,2,3,4,2,1,0,1,2,3,3,2,1,0,1,2,4,3,2,1,0,1,5,4,3,2,1,0),n
eigen(mat1)

## $values
```

```
## [1] 12.1093115 -0.5358984 -0.6798227 -1.0000000 -2.4294888 -7.4641016
##
## $vectors
##           [,1]       [,2]       [,3]       [,4]        [,5]       [,6]
## [1,] 0.4979220 -0.1494292  0.2509572  0.4082483  0.43484956  0.5576775
## [2,] 0.3809911  0.4082483 -0.5877278 -0.4082483 -0.09706644  0.4082483
## [3,] 0.3269854 -0.5576775  0.3026492 -0.4082483 -0.54907556  0.1494292
## [4,] 0.3269854  0.5576775  0.3026492  0.4082483 -0.54907556 -0.1494292
## [5,] 0.3809911 -0.4082483 -0.5877278  0.4082483 -0.09706644 -0.4082483
## [6,] 0.4979220  0.1494292  0.2509572 -0.4082483  0.43484956 -0.5576775
```

Appendix F

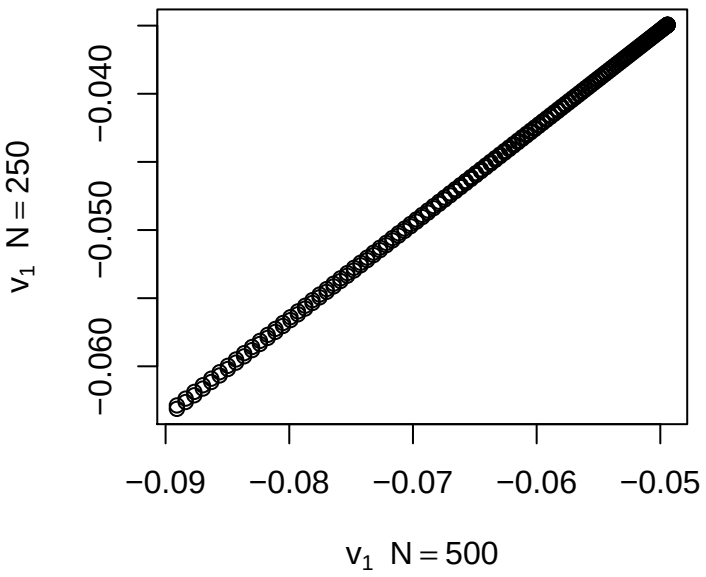
The following R code output shows the correlation and relationship of the following graph:

```
 plot(evy$vector[,1],evnew,xlab=expression(v[1]~~N==500),ylab=expression(v[1]~~N==250))
cor(evy$vector[,1],evnew)

## [1] 0.9999675

 reg=lm(evnew~evy$vector[,1])
 reg

##
## Call:
## lm(formula = evnew ~ evy$vector[, 1])
##
## Coefficients:
##     (Intercept)  evy$vector[, 1]
##      -3.100e-07        7.071e-01
```



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