Self-affinities of Folds and Incomplete Similarity

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***Abstract*—A method to analyze self-affinities is introduced, and applied to the large scale fold geometries of the Quaternary and Tertiary in the inner belt of the Northeast Honshu Arc. Based on this analysis, their geometries are found to be self-affine and can be differently scaled in different directions. We recognize the self-affinities for the amplitude and the wavelength of folds, and discover a crossover from local to global altitude (vertical) variation of the geometries of folds in the Northeast Honshu Arc. Buckingham's Pi-theorem has been applied to similar systems of inhomogeneous viscous Newtonian fluid under similar boundary condition. However, Buckingham's Pi-theorem cannot give us the self-affinities of folds. A general renormalization-group argument is proposed to the applicability of the similarity theory. By this argument, we derive the self-affinities for the amplitude and the wavelength of folds as a parameter for the anisotropic stress field.**

Keywords; Self-affinities, Folds, Incomplete similarity

I. INTRODUCTION

By the analysis of the large scale fold geometries of Quaternary and Tertiary in the inner belt of the northeast Honshu Arc. Kikuchi et al. [1] showed that self-affine in folds is given actually by

$$Y \propto X^H , \quad H \equiv \frac{\nu_y}{\nu_x}, \tag{1}$$

where X and Y are x- and y-variances (horizontal and vertical direction), H is Hurst exponent, ν_x and ν_y are X and Y slopes, respectively (Fig. 1). In a particular case ($\nu_x = \nu_y$), the Hurst exponent H is equal to 1. This case indicates self-similarity for the analyzed fold curve. In a case ($\nu_x \neq \nu_y$), the Hurst exponent H is not equal to 1, so this case indicates self-affinities for the given fold curve. Moreover, Kikuchi et al. [1] pointed out that self-affinity for the crustal deformation is related to the b-value in the Gutenberg-Richter's law as the fractal dimension or the uniformity of the crustal fragmentation. Softening behavior of crusts can lead to localization of fold packets in layered materials and a progression to chaos with fractal geometries [2]. Why do fractal geometries exist, and what controls fractal dimension of fold [3]?

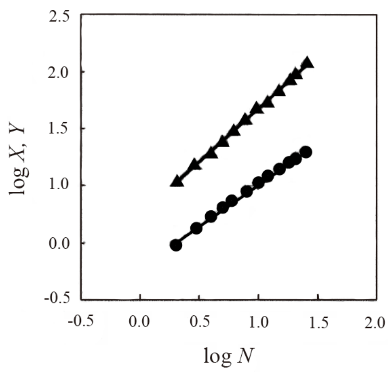

Figure 1: Log-log plots of horizontal and vertical standard deviations (X and Y) and curve length N. Pairs (log X, log N) and (log Y, log N) are linearly approximated by the method of least squares.

Shimamoto [4] examined the conditions of similarity for geometrically similar systems of inhomogeneous viscous Newtonian fluids under similar boundary conditions using the method of dimensional analysis based on Buckingham's Pi-theorem [5]. Then, based on the complete similarity, he clearly derived a relationship between the wavelength of fold and initial thickness of folded layer. There, as Shimamoto [4] did not analyze the amplitude of a fold, his analyses could not derive the self-affinities between amplitude and length for folds. So the background of self-affinities for fold has been unknown from the view point of the dimension analysis. Dimensional analysis postulates physical quantities that tend to infinity or zero. However, often physical quantities are not infinity or zero [6]. Barenblatt [6] gave finite or not zero case of dimensional analysis method. By an application of the general renormalization-group argument based on incomplete similarity theory (IS theory) to a physical system, so for the buckling folds, based on this Barenblatt's method, we can derive the self-affinities of folds expressed by the scaling between the normalized wave length and the length of deformable portion. This paper is an extended paper of the preliminary paper published in the conference proceedings of the 5th Annual International Conference on Geological & Earth Sciences (GEOS 2016) [7], with some modifications.

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II. Pi-theorem (Dimensional analysis) And Application to folding

The Pi theorem stated by Buckingham [5] can be expressed by a complite equation

$$a_0 = f(a_1, a_2, \cdots, a_k, a_{k+1}, \cdots, a_n), \tag{2}$$

whear a_0 is the dependent variable, $a_1, a_2, \cdots, a_n$ are variables and n is total number of variables. In Eq. (2), control variables are $a_1, a_2, \cdots, a_k$. Futuremore, $a_{k+1}, \cdots, a_n$ is represented by control variables

$$\begin{cases} [a_{k+1}] = [a_1]^{p_{k+1}}[a_2]^{q_{k+1}}, \cdots, [a_k]^{r_{k+1}}, \\ \vdots \\ [a_n] = [a_1]^{p_n}[a_2]^{q_n}, \cdots, [a_k]^{r_n}. \end{cases} \tag{3}$$

On the other hands, the dependent variable is represented by control variables

$$[a_0] = [a_1]^p[a_2]^q, \cdots, [a_k]^r. \tag{4}$$

Moreover, we define $(n-k+1)$ dimensionless products form using Eqs. (3) and (4) as

$$\begin{cases} \Pi_1 = \dfrac{a_{k+1}}{a_1^{p_{k+1}} a_2^{q_{k+1}}, \cdots, a_k^{r_{k+1}}}, \\ \vdots \\ \Pi_{n-k} = \dfrac{a_n}{a_1^{p_n} a_2^{q_n}, \cdots, a_k^{r_n}}, \\ \Pi = \dfrac{a_0}{a_1^p a_2^q, \cdots, a_k^r}. \end{cases} \tag{5}$$

Then, we can redefine Eq. (2) by Eq. (5) as follows:

$$\Pi = F(a_1, a_2, \cdots, a_k, \Pi_1, \Pi_2, \cdots, \Pi_{n-k}). \tag{6}$$

In this case, control variable a_1 can be cahnged, $a_2, \cdots, a_k$ and $\Pi = F$ are constant. Besides this mathematical form must be written by

$$\frac{\partial F}{\partial a_1} = \frac{\partial F}{\partial a_2} = \cdots = \frac{\partial F}{\partial a_k} = 0. \tag{7}$$

So we can get a relationship between Π and $(n-k)$ dimensionless products

$$\Pi = \phi(\Pi_1, \Pi_2, \cdots, \Pi_{n-k}), \tag{8}$$

where the Π's are independent power products of the a_i's which are dimensionless in the fundamental units ($i = 1 \cdots k$), and k is the rank of the dimensional matrix of the a_i's.

Shimamoto showed similarity rule for a physical system under the boundary [4], and applied the Pi-theorem to the similarity criteria of slow deformation of inhomogeneous viscous fluid (Fig. 2).

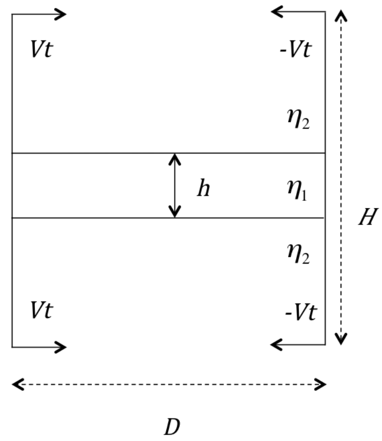

Figure 2: Initial configuration of a single layer and boundary medium (from Kikuchi and Nagahama [7]). The considered variables are initial height of the system ($\bar{H}$), initial width of the system (D), viscosity of the layer (η_1), viscosity of the medium (η_2), boundary velocity (V) and initial thickness of layer (h).

The various quantities related to this problem and their dimensions can be listed as follows:

TABLE I. Notation 1

Name of quantity	Symbol	Dimension
Initial width of system	$\bar{H}$	L
Initial height of system	D	L
Wavelength of fold	l	L
Viscosity of layer	η_1	$ML^{-1}T^{-1}$
Viscosity of medium	η_2	$ML^{-1}T^{-1}$
Boundary velocity	V	$L^{-1}T^{-1}$
Time	t	T
Initial thickness of layer	h	L

Based on [4], the relation among these quantities can be written by

$$f(\bar{H}, h, l, \eta_1, \eta_2, V, t, D) = 0. \tag{9}$$

Since the number of total variables n in Eq. (9) is 8, and dimensional matrix k is 3 (see TABLE Ⅱ).

TABLE II. Dimensional matrix for complete similarity theory

	$\bar{H}$	D	l	η_1	η_2	V	t	h
M	0	0	0	1	1	0	0	0
L	1	1	1	-1	-1	1	0	1
T	0	0	0	-1	-1	-1	1	0

In this case, so $n-k$ is calculated by

$$n-k=5. \tag{10}$$

Therefore, Pi-theorem requires 5 dimensionless products. So, Eq. (9) can be reducible to the form:

$$\Phi_1\left(\frac{h}{\overline{H}},\frac{h}{D},\frac{l}{h},\frac{\eta_1}{\eta_2},\frac{Vt}{D}\right)=0. \tag{11}$$

Moreover, width *D* of the system becomes $(D-2Vt)$ after time *t*. Employing the measure of natural or logarithmic strain quantities can be written by

$$\varepsilon=-\log_e\left(\frac{D-2Vt}{D}\right). \tag{12}$$

Assuming $\overline{H}\gg h$ and $D\gg h$, then $h/\overline{H}=0$, and $h/D=0$. So Eq. (11) can be reduced into

$$\Phi_2\left(\frac{l}{h},\frac{\eta_1}{\eta_2},\varepsilon\right)=0. \tag{13}$$

Solving Eq. (13) for *l*/*h*, Shimamoto [4] obtained

$$\frac{l}{h}=\Phi_3\left(\frac{\eta_1}{\eta_2},\varepsilon\right). \tag{14}$$

Actually, Biot [8] shown that *l*/*h* is given by

$$\frac{l}{h}=2\pi\sqrt[3]{\frac{\eta_1}{6\eta_2}}. \tag{15}$$

Therefore, Eq. (14) is equivalent with Eq. (15). As mentioned above, Shimamoto [4] could elegantly derive the similarity (scaling) for the buckling folding by dimension analysis. However, this solution is related to self-similarity, but not self-affinity.

III. Method of Renormalization Group: Derivation of Self-affinite for Folds

We think limit equal to zero or infinite in Eq. (8). If we can use exponent α in Eq. (8), when $\Pi_{n-k}\to 0$,

$$\Pi=\lim_{\Pi_{n-k}\to 0}\phi(\Pi_1,\Pi_2,\cdots,\Pi_{n-k-1},\Pi_{n-k})$$

$$=\Pi_{n-k}^{\alpha}\phi(\Pi_1,\Pi_2,\cdots,\Pi_{n-k-1}). \tag{16}$$

Here, we difine a new dimensionless parameter by

$$\Pi_*=\frac{\Pi}{\Pi_{n-k}^{\alpha}}. \tag{17}$$

Then, we can get a new equation

$$\Pi_*=\phi(\Pi_1,\Pi_2,\cdots,\Pi_{n-k-1}). \tag{18}$$

Barenblatt [6] classified dimensional analysis: (i) not infinity or not zero case is complete similarity, and (ii) zero or infinite case is incomplete similarity. A phenomenon is defined as similarity in a given dimensionless group.

By this method, now let us consider the folding of a single viscous layer embedded in a thick incompetent viscous medium (Fig. 3).

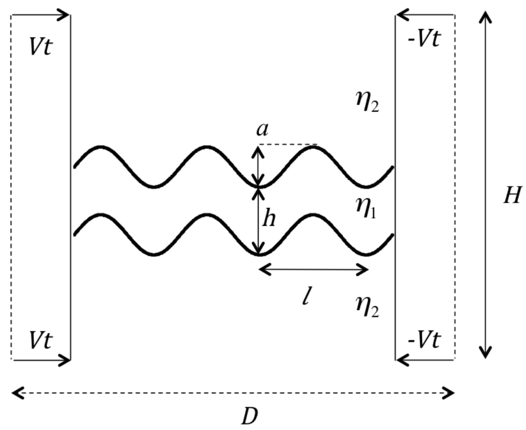

Figure 3: Initial configuration of a single layer and boundary medium by incomplete similarity theory (from Kikuchi and Nagahama [7]). The considered variables are initial height of system ($\overline{H}$), initial width of the system (*D*), viscosity of layer (η_1), viscosity of medium (η_2), boundary velocity (*V*), time (*t*), wave length (*l*), initial thickness of layer (*h*) and amplitude of fold (*a*).

The various quantities related to this problem and their dimensions can be listed as follows:

TABLE III. NOTATION 2

Name of quantity	Symbol	Dimension
Initial width of system	$\overline{H}$	L
Initial height of system	D	L
Wavelength of fold	l	L
Viscosity of layer	η_1	$ML^{-1}T^{-1}$
Viscosity of medium	η_2	$ML^{-1}T^{-1}$
Boundary velocity	V	$L^{-1}T^{-1}$
Time	t	T
Initial thickness of layer	h	L
Amplitude of fold	a	L

The relation among these quantities can be written by

$$f(\overline{H}, D, l, \eta_1, \eta_2, V, t, h, a) = 0. \quad (19)$$

Since $n = 9$, and $k = 3$ in this case (see TABLE Ⅳ).

TABLE IV. Dimensional matrix for IS theory

	$\overline{H}$	D	l	η_1	η_2	V	t	h	a
M	0	0	0	1	1	0	0	0	0
L	1	1	1	-1	-1	1	0	1	1
T	0	0	0	-1	-1	-1	1	0	0

Pi-theorem requires $n - k = 6$ dimensionless products. Hence, Eq. (19) can be reducible to the form:

$$\Phi_4\left(\frac{l}{h}, \frac{h}{D}, \frac{h}{\overline{H}}, \frac{\eta_1}{\eta_2}, \frac{Vt}{D}, \frac{a}{l}\right) = 0. \quad (20)$$

Here, we apply Barenblatt's incomplete similarity theory to Eq. (20), and get

$$\left(\frac{l}{h}\right)^{-\delta} \Phi_5\left(\frac{\eta_1}{\eta_2}, \frac{Vt}{D}, \frac{a}{l}\right) = 0, \quad (21)$$

where δ is constant. Then, using Eq.(12) and solving for a/l,

$$\frac{a}{l} = \left(\frac{l}{h}\right)^{-\delta} \Phi_6\left(\frac{\eta_1}{\eta_2}, \frac{Vt}{l}\right) = \left(\frac{l}{h}\right)^{-\delta} \Phi_7\left(\frac{\eta_1}{\eta_2}, \varepsilon\right)$$
$$= \left(\frac{l}{h}\right)^{-\delta_1} \varepsilon^{-\delta_2} \Phi_8\left(\frac{\eta_1}{\eta_2}\right). \quad (22)$$

Under Barenblatt's theory [6] and Eq. (20), we can derive the following relation between the amplitude and the wavelength of folds as

$$l^{1-\delta_1} \propto a\,. \quad (23)$$

Eq. (23) shows self-affinity for the amplitude and length of fold. Hence, based on the general renormalization-group argument, we can derive the self-affinities for folds.

IV. Discussion and Conclusions

For the self-affinities for the amplitude and the wavelength of folds pointed out by [1], in this paper we introduced incomplete similarity to fold systems, and Eq. (23) can be derived. So the index δ_1 is equivalent to Hurst exponent H by

$$1 - \delta_1 = H. \quad (24)$$

In a particular case ($\delta_1 = 0$), the Hurst exponent H is equal to 1. This case indicates self-similarity for the given fold curve, and a scale invariance of the fold might not be affected by a variety of tectonic processes under the anisotropic stress field. In a case ($\delta_1 \neq 0$), the Hurst exponent H is not equal to 1, so this case indicates self-affinities for the given fold curve, and a scale invariance of the fold might be affected by a variety of tectonic processes under the anisotropic stress field. For example, fault-related folding is caused by compressional asymmetric force and results in shortening [9]. These results imply that anisotropic stress fields by gravitation and tectonic stresses might cause self-affinities of folds. Self-similarity or self-affinities of the fold is affected by a variety of tectonic processes under the isotropy or anisotropic stress field. By an application of the general renormalization-group argument based on incomplete similarity theory for the system of folding, the scaling between the normalized wavelength and the length of deformable portion can be derived. Therefore, this index δ is an important parameter for the anisotropic stress field.

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