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Error estimation for the contact length formulas in grinding

Estimación del error en fórmulas que calculan la longitud de contacto en el rectificado

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ABSTRACT

A study of the relative error of the geometric and kinematic contact length formulas found in the literature, both in surface and cylindrical grinding, is performed. For this purpose, the exact formulas have been derived. Under usual grinding conditions, the relative error remains small, < 5%, except for the case of external cylindrical grinding, in which we can find a 2% relative error. It is advisable then to use the exact formulas in order to prevent any kind of error.

KEYWORDS: *contact length in grinding, error estimation formulas.*

RESUMEN

En este trabajo se estudia el error relativo de las fórmulas geométricas y cinemáticas que aparecen en la literatura y que calculan la longitud de contacto en el rectificado cilíndrico. Para este propósito, se obtienen las fórmulas exactas. En condiciones habituales, el error relativo del rectificado es pequeño, <5%, excepto en el caso de rectificado cilíndrico externo, en el que podemos encontrar un error relativo del 2%. Es entonces recomendable hacer uso de las fórmulas exactas a fin de evitar cualquier tipo de error.

PALABRAS CLAVE: *longitud de contacto en el rectificado, fórmulas para la estimación del error.*

INTRODUCTION

Grinding is a machining process in which workpiece material is removed from its surface. For this purpose, hard grits are bonded to the grinding wheel, so that when it rotates at high speed over the workpiece, they produce a high abrasion within the contact area between wheel and workpiece. The length of the contact area is called contact length, while its width is given by the wheel width. The contact length has got a very significant influence in almost all physical phenomena occurring in the grinding process [1], such as surface temperature, grinding wheel wear, residual stresses generation and damping of high frequency vibrations. Therefore, accurate formulas to calculate the contact length are highly desirable. However, the most common formulas for the calculation of the contact length (geometric and the kinematic contact length formulas in surface and cylindrical grinding) are in fact approximated formulas. The aim of this paper is to evaluate the relative error of these formulas for usual grinding conditions. For this purpose, if ℓ_{approx} and ℓ_{exact} are the approximate and exact expressions for the contact lengths respectively, the relative error is defined as.

$$\frac{\Delta\ell}{\ell} = 1 - \frac{\ell_{\text{exact}}}{\ell_{\text{approx}}}, \tag{1}$$

so that a positive relative error means that the approximation overestimates the exact value and a negative one underestimates it. Note that defining the dimensionless contact length as

$$\lambda = \frac{\ell}{R}, \tag{2}$$

where R is the wheel radius, then

$$\frac{\Delta\ell}{R} = \frac{\Delta\lambda}{\lambda} \tag{3}$$

This paper is organized as follows. Sections 2 and 3 performs the derivation of the exact formulas and the relative error formulas for the geometric and kinematic contact length, both in surface and cylindrical grinding. Section 4 shows a set of plots in order to evaluate the relative error of the formulas considered in the previous Sections. The conclusions are summarized in Section 5.

GEOMETRIC CONTACT LENGTH

Surface grinding

According to Figure 1, the geometrical contact length ℓ_g in surface grinding is given by

$$\ell_g = R\theta, \tag{4}$$

where the depth of cut a satisfies

$$a = R(1 - \cos\theta), \tag{5}$$

thus

$$\theta = \cos^{-1}\left(1 - \frac{a}{R}\right). \tag{6}$$

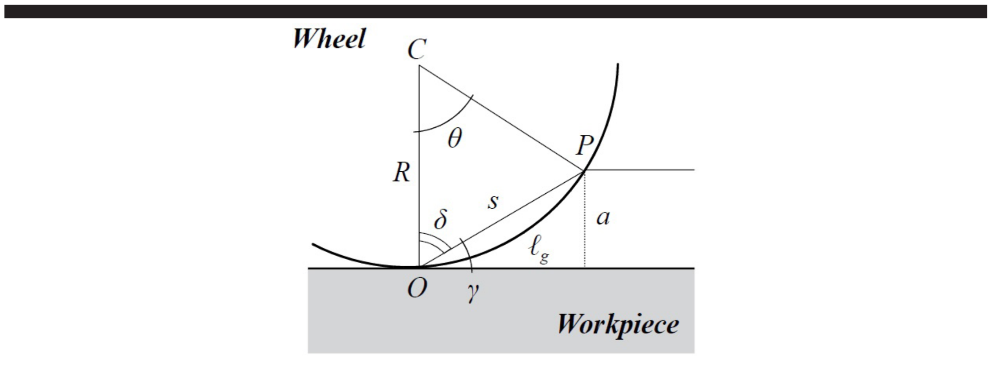

Figure 1. Scheme for geometrical contact length in surface grinding.

Substituting back (6) in (4), we obtain

$$\ell_g = R\cos^{-1}\left(1-\frac{a}{R}\right). \tag{7}$$

Defining the following dimensionless variable

$$\xi = \frac{a}{R}, \tag{8}$$

and taking into account (2), we may rewrite (7) in dimensionless form as

$$\lambda_g = \cos^{-1}\left(1-\xi\right). \tag{9}$$

According to the Maclaurin expansion of the function $\cos^{-1}\left(1-x^2\right)$, we have the following series

$$\cos^{-1}\left(1-x\right) = \sqrt{2}\,x^{1/2} + \frac{x^{3/2}}{6\sqrt{2}} + \frac{3x^{5/2}}{80\sqrt{2}} + O\left(x^{7/2}\right), \tag{10}$$

thus we may approximate the contact length (9) when $\xi<<1$ truncating (10) in the first term, so

$$\lambda_g \approx \sqrt{2\xi}, \tag{11}$$

or in dimensional form

$$\ell_g \approx \sqrt{2Ra}, \tag{12}$$

which is the usual expression given in the literature [2]. Geometrically speaking, this is equivalent to estimate the arc ℓ_g by the chord s when $a =$ R, (that is to say, when $\theta \approx 0$). Indeed, note on the one hand that

$$\sin\gamma = \frac{a}{s}, \tag{13}$$

and on the other hand, since the triangle *COP* in Figure 1 is isosceles

$$\theta + 2\delta = \pi. \tag{14}$$

Also, at vertex *O* we have a right angle, so

$$\gamma + \delta = \frac{\pi}{2}, \tag{15}$$

thus from (14) and (15) we have

$$\gamma = \frac{\theta}{2}. \tag{16}$$

Therefore, from(13) and (16), and applying the sine half angle formula, we have

$$s = \frac{a}{\sin\frac{\theta}{2}} = \frac{a}{\sqrt{\frac{1-\cos\theta}{2}}}. \tag{17}$$

Finally, substituting (6) in (17), we arrive at

$$s = \sqrt{2Ra},$$

as aforementioned. However, sometimes in the literature, (12) seems to be an exact formula [3, 4], probably because the relative error of (12) is quite small for usual grinding conditions, as we will see later on. In order to evaluate the relative error of (12) with respect to the exact formula (7), from (9) and (3) we have

$$\frac{\Delta\ell_g}{\ell_g} = 1 - \frac{\cos^{-1}(1-\xi)}{\sqrt{2\xi}}. \tag{18}$$

Cylindrical grinding

Figure 2 shows the scheme of the contact length $\bar{\ell}_g$ in external cylindrical grinding, and similarly to surface grinding we have

$$\bar{\ell}_g = R\bar{\theta}, \tag{19}$$

where now $\bar{\theta}$ can be calculated applying the Law of Cosines to the triangle *CMP*

$$(r+a)^2 = (R+r)^2 + R^2 - 2R(R+r)\cos\bar{\theta}. \tag{20}$$

Figure 3 shows the geometrical contact length in internal cylindrical grinding. Notice that taking the triangle *CMP* in Figure 3 and applying again the Law of Cosines, we have

$$(r-a)^2 = (r-R)^2 + R^2 - 2R(r-R)\cos(\pi-\bar{\theta}). \tag{21}$$

which is equivalent to (20) making the sustitutions $R \to -R$ and $a \to -a$. Therefore, from now on, we will consider $\pm R$ and $\pm a$ to take into account both cases.

Now, solving for $\bar{\theta}$, we obtain

$$\bar{\theta} = \cos^{-1}\left(1 - \frac{a\left(2r \pm a\right)}{2R\left(r \pm R\right)}\right). \tag{22}$$

Therefore, substituting (22) in (19), we can express the contact length as follows

$$\bar{\ell}_g = R\cos^{-1}\left(1 - \frac{a}{2R}\frac{\frac{2r}{R} \pm \frac{a}{R}}{\frac{r}{R} \pm 1}\right). \tag{23}$$

Notice that from (23) we can recover the contact lenght for surface grinding (7) performing the limit $r \to \infty$ [5]

$$\lim_{r\to\infty} \bar{\ell}_g = \ell_g.$$

When $a << R$ and $a << 2R$, we may approximate (23) truncating the series (10) in the first term, obtaining

$$\bar{\ell}_g \approx \sqrt{aR\frac{2r \pm a}{r \pm R}} \approx \sqrt{ar_e}, \tag{24}$$

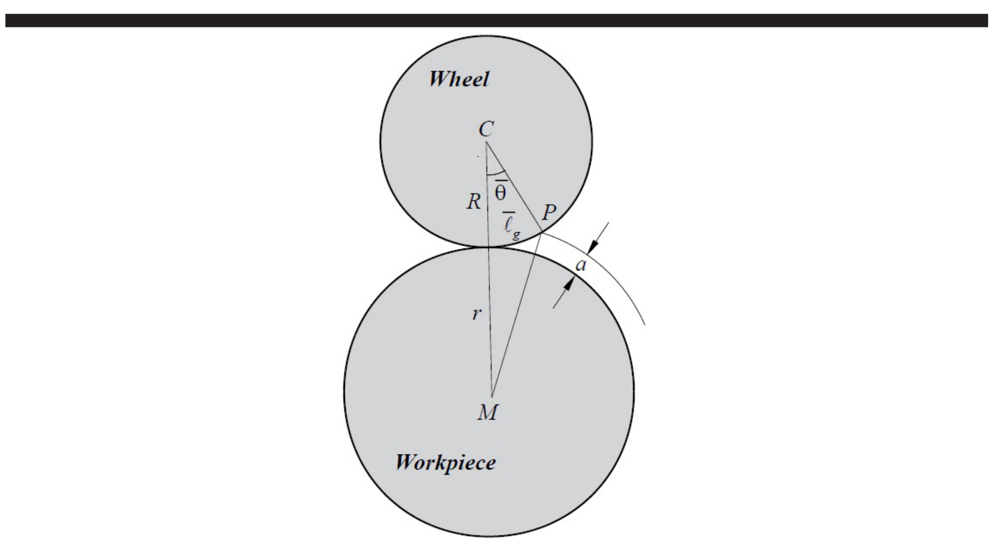

Figure 2. Scheme for the geometrical contact length in external cylindrical grinding.

which is the expression found in the literature [6, 7], being r_e the equivalent radius

$$r_e = \frac{2Rr}{r \pm R}.$$

In order to evaluate the relative error of the approximate expression (24), let us define the dimensionless parameter

$$\rho = \frac{r}{R}, \tag{25}$$

so that, taking into account also (8) we can rewrite in dimensionless form (23) as

$$\bar{\theta} = \bar{\lambda}_g = \cos^{-1}\left(1 - \frac{\xi(2\rho \pm \xi)}{2(\rho \pm 1)}\right), \tag{26}$$

and also (24) as

$$\bar{\lambda}_g \approx \sqrt{\frac{2\xi\rho}{\rho \pm 1}}. \tag{27}$$

Therefore, the relative error is given by

$$\frac{\Delta\bar{\ell}_g}{\ell_g} = 1 - \cos^{-1}\left(1 - \frac{\xi(2\rho \pm \xi)}{2(\rho \pm 1)}\right)\sqrt{\frac{\rho \pm 1}{2\xi\rho}}. \tag{28}$$

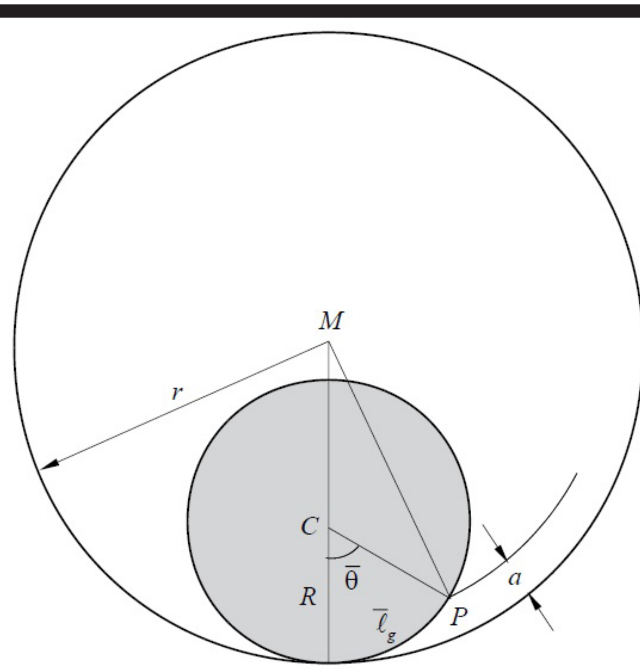

Figure 3. Scheme for the geometrical contact length in internal cylindrical grinding.

KIMEMATIC CONTACT LENGTH

Both in surface and cylindrical grinding, the workpiece is moving with respect to the wheel, thus the path traveled by a grain on the wheel surface in contact with the workpiece is different from the geometrical contact length discussed above. Therefore, there is a kinematic correction to the static geometrical contact length.

Surface grinding

In Figure 4, the Cartesian coordinate system *OXY* is fixed to the workpiece. Initially (at $t = 0$) the center of the wheel is at *C* and in time *t* a grain travels from *O* to *P*. The trajectory described by this grain is the composition of two movements: the shift of the center of the wheel from *C* to *C '* and the rotation of the wheel around its axis an angle θ.

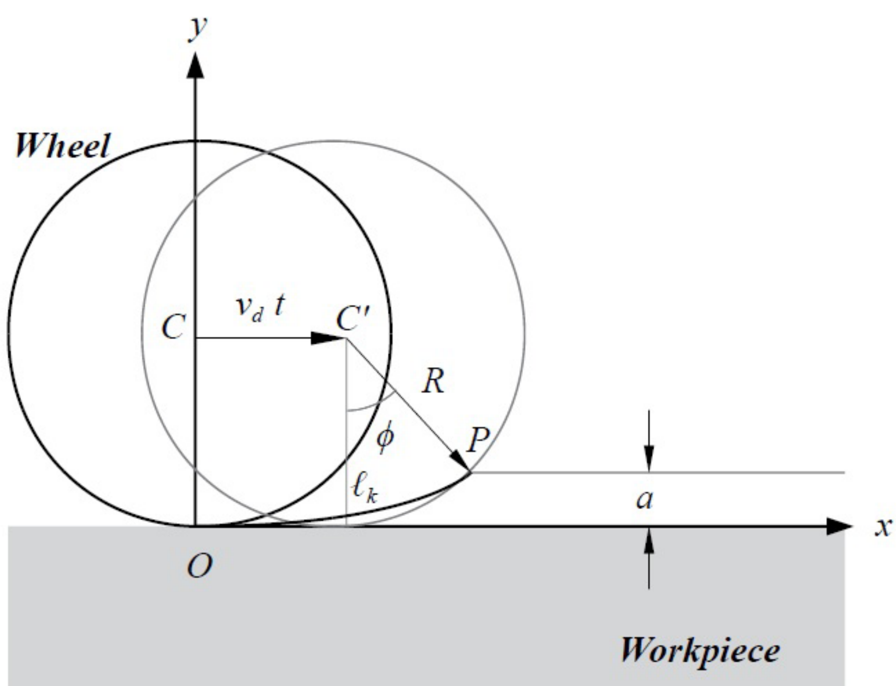

Figure 4. Scheme for the kinematic contact length in surface grinding.

If the wheel rotates at a constant angular velocity ω_θ, we have

$$\theta = \omega_\theta t, \tag{29}$$

The Cartesian coordinates of point *P* are given by the following parametric equations:

$$\begin{aligned} x(t) &= R\sin\omega_\theta t + v_d t, \\ y(t) &= R\left(1-\cos\omega_\theta t\right), \end{aligned} \tag{30}$$

thus

$$x'(t) = v_\theta \cos\omega_\theta t + v_d,$$
$$y'(t) = v_\theta \sin\omega_\theta t, \tag{31}$$

where v_θ is the tangential velocity of the wheel

$$v_\theta = R\omega_\theta. \tag{32}$$

If at time t the point P is located at a distance apart to the abscissa equal to the cutting depth a, then

$$a = R\left(1-\cos\omega_\theta t\right), \tag{33}$$

so

$$t = \frac{1}{\omega_\theta}\cos^{-1}\left(1-\frac{a}{R}\right). \tag{34}$$

The kinematic contact length ℓ_k is then the arc length of the grain trajectory from O to P, thus from (31), and performing the change of variables $u = \omega_\theta t$ we have

$$\ell_k = \int_0^t \sqrt{\left[x'(\tau)\right]^2 + \left[y'(\tau)\right]^2}\,d\tau = \int_0^t \sqrt{v_\theta^2 + v_d^2 + 2v_\theta v_d \cos\omega_\theta\tau}\;d\tau = \frac{1}{\omega_\theta}\int_0^{\omega_\theta t} \sqrt{v_\theta^2 + v_d^2 + 2\cos u}\;du. \tag{35}$$

Taking into account (32) and defining the dimensionless parameter (speed ratio)

$$\chi = \frac{v_d}{v_\theta}, \tag{36}$$

we can rewrite (35) in terms of an *elliptic integral of the second kind* (see the Appendix for details)

$$\ell_k = R\int_0^{\omega_\theta t} \sqrt{1+\chi^2+2\chi\cos u}\;du = 2R\left(1+\chi\right)\mathrm{E}\left(\frac{4\chi}{\left(1+\chi\right)^2}, \frac{\omega_\theta t}{2}\right). \tag{37}$$

Note that if the rotation of the wheelis ω_θ high and the depth of cut a is small, according to (34), we have $t \approx 0$, so we can apply the approximation given in (64) to (37), obtaining

$$\ell_k \approx \omega_\theta t\left(1+\chi\right). \tag{38}$$

Taking into account (32), (34) and (7) in (38), we may approximate the kinematic contact leght ℓ_k in terms of the geometrical contact length ℓ_g as [2]

$$\ell_k \approx \ell_g(1+\chi). \tag{39}$$

Using the approximation given in (12) for ℓ_g we finally get the approximation given in [8]

$$\ell_k \approx \sqrt{2Ra}(1+\chi). \tag{40}$$

In order to give an expression of the relative error of the approximation (40), let us rewrite (37) and (40) in dimensionless form by using the parameter ξ defined in (8)

$$\lambda_k = 2(1+\chi)\,\mathrm{E}\left(\frac{4\chi}{(1+\chi)^2}, \frac{\cos^{-1}(1-\xi)}{2}\right),$$
$$\lambda_k \approx \sqrt{2\xi}\,(1+\chi), \tag{41}$$

so that

$$\frac{\Delta\ell_k}{\ell_k} = 1 - \sqrt{\frac{2}{\xi}}\,\mathrm{E}\left(\frac{4\chi}{(1+\chi)^2}, \frac{\cos^{-1}(1-\xi)}{2}\right). \tag{42}$$

CYLINDRICAL GRINDING

According to Figure 5, in time t the wheel rotates at a constant angular velocity ω_ϕ around the workpiece an angle ϕ, and in the same time, the wheel rotates at a constant angular velocity $\omega_{\bar\theta}$ around its own axis an angle $\bar\theta$, so we have

$$\phi = \omega_\phi t, \tag{43}$$

$$\bar\theta = \omega_{\bar\theta} t. \tag{44}$$

The origin O of the Cartesian axes given in Figure 5 is set in the contact point between wheel and workpiece at $t = 0$. At time t, a grain which is initially at O is displaced to point P because of the movement of the center of the wheel from C to C' and also due to the rotation of the wheel around its own axis. Therefore, the Cartesian coordinates of point P are given by the following equations:

$$x = (R+r)\sin\phi + R\sin\bar\theta,$$
$$y = R(1-\cos\bar\theta) - (R+r)(1-\cos\phi). \tag{45}$$

Substituting (43) and (44) in (45) and simplifying, we have the parametric curve of a point on the wheel surface with respect to the workpiece (initially at O)

$$x(t) = (R+r)\sin\omega_\phi t + R\sin\omega_{\bar\theta} t,$$
$$y(t) = (R+r)\cos\omega_\phi t - r - R\cos\omega_{\bar\theta} t,$$

thus

$$x'(t) = v_\phi \cos\omega_\phi t + v_{\bar\theta}\cos\omega_{\bar\theta} t,$$
$$y'(t) = -v_\phi \sin\omega_\phi t + v_{\bar\theta}\sin\omega_{\bar\theta} t,$$

where we have considered the tangential velocities both of the wheel $v_{\bar\theta}$ and of the center of the wheel v_ϕ

$$v_{\bar\theta} = R\omega_\theta, \tag{46}$$

$$v_\phi = (R+r)\omega_\phi. \tag{47}$$

Consider now that t is the time that a grain is in contact with the workpiece. The arc length that a grain travels during time t is then

$$\bar{\ell}_k = \int_0^t \sqrt{\left[x'(\tau)\right]^2 + \left[y'(\tau)\right]^2}\, d\tau = \int_0^t \sqrt{v_\phi^2 + v_{\bar{\theta}}^2 + 2 v_\theta v_d \cos(\omega_{\bar{\theta}} - \omega_\phi)\tau}\; d\tau.$$

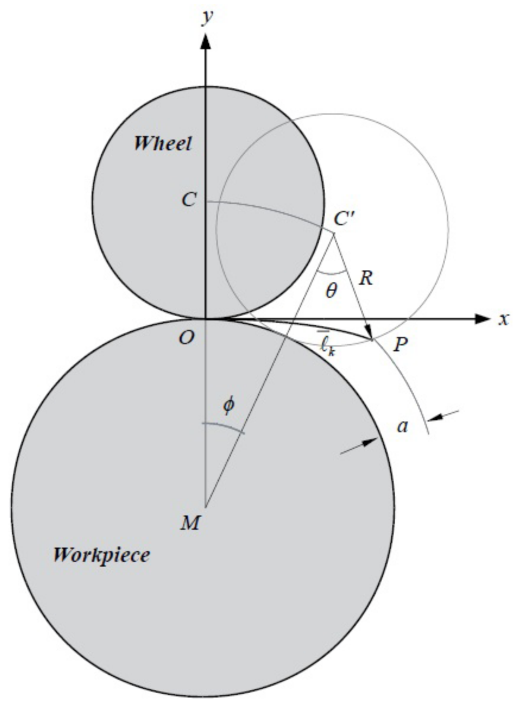

Figure 5. Scheme for the kinematic length in cylindrical grinding.

Defining now the following dimensionless parameter (speed ratio)

$$\bar{\chi} = \frac{v_\phi}{v_{\bar{\theta}}}, \tag{48}$$

and performing the change of variables $u = \left(w_{\bar{\theta}} - w_\phi\right)\tau$, we obtain

$$\bar{\ell}_k = \frac{v_\theta}{\omega_{\bar{\theta}} - \omega_\phi} \int_0^{(\omega_{\bar{\theta}} - \omega_\phi)t} \sqrt{\bar{\chi}^2 + 1 + 2\bar{\chi}\cos u}\; du. \tag{49}$$

According to the result given in the Appendix (63), the integral given in (49) can be expressed in terms of the elliptic integral of the second kind as follows

$$\bar{\ell}_k = \frac{2 v_{\bar{\theta}} \left(1 + \bar{\chi}\right)}{\omega_{\bar{\theta}} - \omega_\phi} \mathrm{E}\left(\frac{4\bar{\chi}}{\left(1 + \bar{\chi}\right)^2}, \frac{(\omega_{\bar{\theta}} - \omega_\phi)t}{2} \right). \tag{50}$$

Applying now the approximation given in (64), we have

$$\bar{\ell}_k \approx v_{\bar{\theta}} t \left(1+\bar{\chi}\right). \tag{51}$$

Taking into account (46), (44) and (19) in (51), and applying the approximation given in (24), we arrive at

$$\bar{\ell}_k = \bar{\ell}_g \left(1+\bar{\chi}\right) \approx \sqrt{a r_e} \left(1+\bar{\chi}\right). \tag{52}$$

which is the expression found in the literature [6, 2]. According to (27), we can rewrite (52) in dimensionless form as

$$\bar{\lambda}_k = \sqrt{\frac{2\xi\rho}{\rho \pm 1}} \left(1+\bar{\chi}\right). \tag{53}$$

Notice also that, considering on the one hand (43), (44) and on the other hand (46), (47), (25) and (48) we have

$$\frac{\omega_{\bar{\theta}} - \omega_{\phi}}{\omega_{\bar{\theta}}} = 1 - \frac{\phi}{\bar{\theta}} \tag{54}$$

$$= 1 - \frac{\bar{\chi}}{\rho \pm 1}. \tag{55}$$

Therefore, from (46) and (55), we can write

$$\frac{\omega_{\bar{\theta}} - \omega_{\phi}}{\omega_{\bar{\theta}}} = \frac{R\left(\rho \pm 1\right)}{\rho \pm 1 - \bar{\chi}}, \tag{56}$$

and taking into account (26) and that (54) is equal to (55), we obtain

$$\frac{\left(\omega_{\bar{\theta}} - \omega_{\phi}\right)t}{2} = \left(1 - \frac{\phi}{\bar{\theta}}\right)\frac{\bar{\theta}}{2} = \frac{1}{2}\left(1 - \frac{\bar{\chi}}{\rho \pm 1}\right) \cos^{-1}\left(1 - \frac{\xi\left(2\rho \pm \xi\right)}{2\left(\rho \pm 1\right)}\right). \tag{57}$$

Therefore, substituting (56) and (57) in (50), we have

$$\bar{\lambda}_k = \frac{2\left(\rho \pm 1\right)}{\rho \pm 1 - \bar{\chi}}\left(1+\bar{\chi}\right) \mathrm{E}\left(\frac{4\bar{\chi}}{\left(1+\bar{\chi}\right)^2}, \frac{\rho \pm 1 - \bar{\chi}}{2\left(\rho \pm 1\right)} \cos^{-1}\left(1 - \frac{\xi\left(2\rho \pm \xi\right)}{2\left(\rho \pm 1\right)}\right)\right). \tag{58}$$

Notice that performing in (58) the limit $\rho \to \infty$, we recover the expression given for surface grinding (41)

$$\lim_{\rho \to \infty} \bar{\lambda}_k = \lambda_k.$$

Finally, from (53) and (58), we arrive at

$$\frac{\Delta \bar{\ell}_k}{\bar{\ell}_k} = 1 - \frac{\sqrt{2}\left(\rho \pm 1\right)^{3/2}}{\left(\rho \pm 1 - \bar{\chi}\right)\sqrt{\xi\rho}} \mathrm{E}\left(\frac{4\bar{\chi}}{\left(1+\bar{\chi}\right)^2}, \frac{\rho \pm 1 - \bar{\chi}}{2\left(\rho \pm 1\right)} \cos^{-1}\left(1 - \frac{\xi\left(2\rho \pm \xi\right)}{2\left(\rho \pm 1\right)}\right)\right). \tag{59}$$

RESULTS

In this Section we present the different graphs for the relative error of the contact length in the cases studied above. In Table 1 the range values for the dimensionless parameters are presented, covering the usual regimes in the industry: shallow cut grinding, creep-feed grinding and high efficient deep grinding. The minimum value of ρ depends on whether we are considering the external (ρ = 0.5) or internal case (ρ = 1.5) in cylindrical grinding.

Figure 6 shows the relative error of the geometric contact length in surface grinding according to equation (18). It is apparent that the relative error grows quiet linearly and the approximation underestimates the exact value (which agrees with the geometrical interpretation given in Section Surface grinding). However, as it is expected, the error is small (< 0.5%).

Table 1. Range values for the dimensionless parameters.

	Min	Max
ξ	0	0.05
χ	0	0.05
ρ	0.5/1.5	5

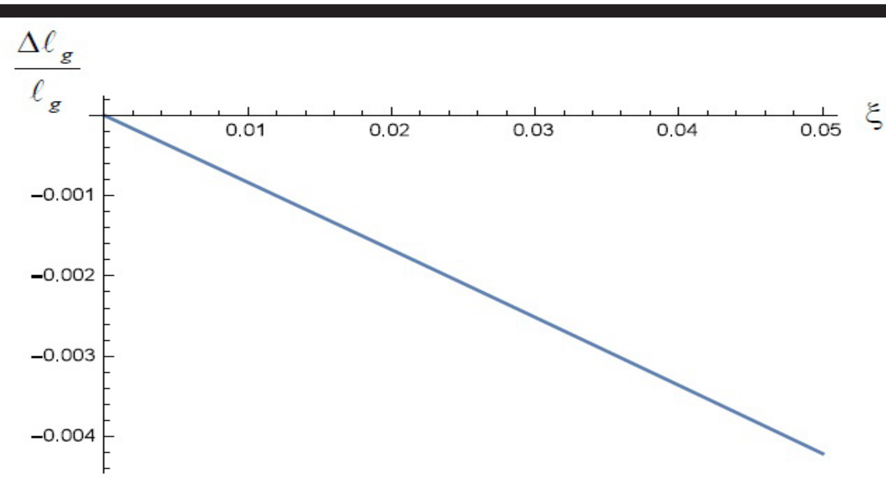

Figure 6. Relative error of the geometric contact length in surface grinding.

Figure 7 and Figure 8 present the relative error of the geometric contact length in external and internal cylindrical grinding respectively, according to equation (28). Since the latter formula depends on both χ and ρ, the plots presented are three-dimensional surfaces. Notice that in the external case, when the radius of the wheel is much bigger than the radius of the workpiece $\rho = 1$, and the depth of cut is large $\xi \approx 0.05$, the relative error is not readily negligible, $\Delta\ell_g/\ell_g \approx 2.5\%$.

Therefore, this is something to be taken into account when the approximated formula (24) is used. On the contrary, the relative error in the internal case remains low, < 0.5%. In both cases, the approximation underestimates the exact value.

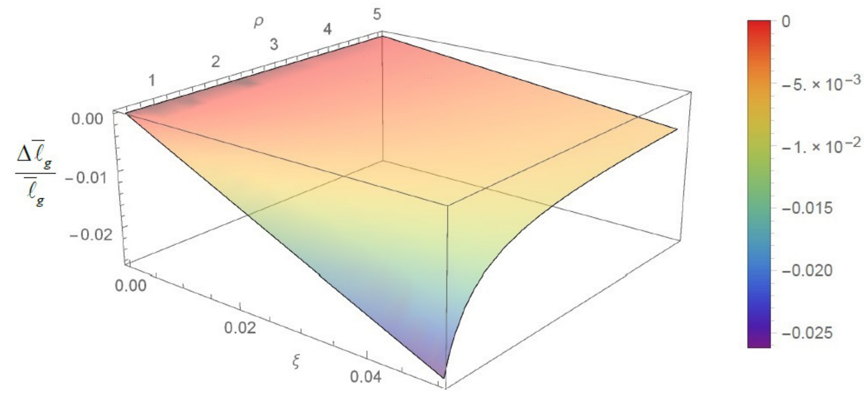

Figure 7. Relative error for the geometric contact length in external cylindrical grinding.

In Figure 9 is plotted the relative error of kinematic contact length in surface grinding by using equation (42). Note that for the value range considered, both for the dimensionless parameters ξ and χ, the error remains low, < 0.5%.

Again, the approximation underestimates the exact value. Figure 10 and Figure 11 show the relative error of kinematic contact length in external and internal cylindrical grinding respectively, by using equation (59). Since the latter formula depends on three parameters (ξ, χ and ρ), we have opted to graph the surfaces for the extreme values of ρ, taken as variables ξ and ρ. The surfaces for the intermediate values of ρ are within the surfaces plotted. It is worth noting that in the external case, when the radius of the wheel is bigger than the radius of the workpiece, $\rho < 1$, and the depth of cut is high, $\xi \geq 0.04$, then the error is not very small, $\Delta\bar{\ell}_g / \bar{\ell}_g \geq 2\%$. On the contrary, the relative error is small in the internal case for the value range considered of the parameters. Once again, the approximation underestimates the exact value.

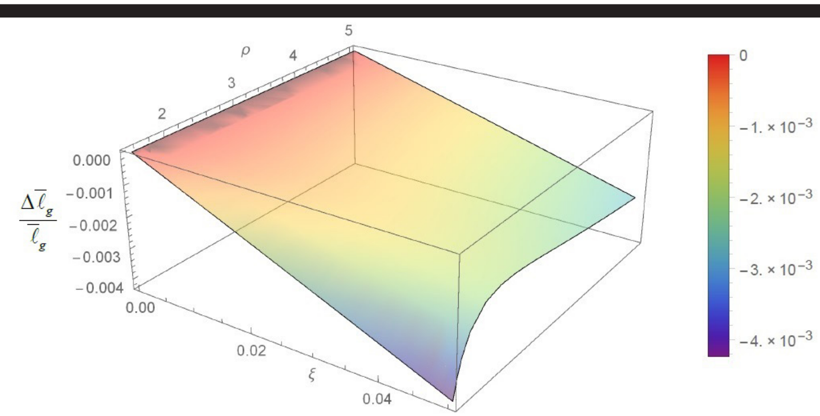

Figure 8. Relative error for the geometric contact length in internal cylindrical grinding.

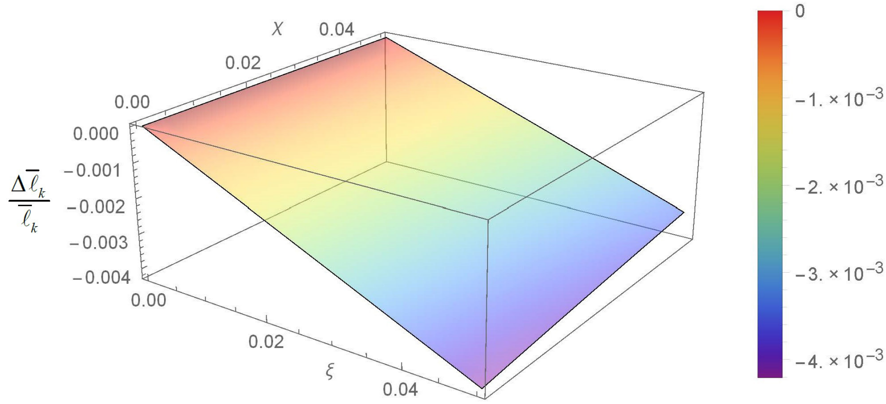

Figure 9. Relative error of kinematic contact length in surface grinding.

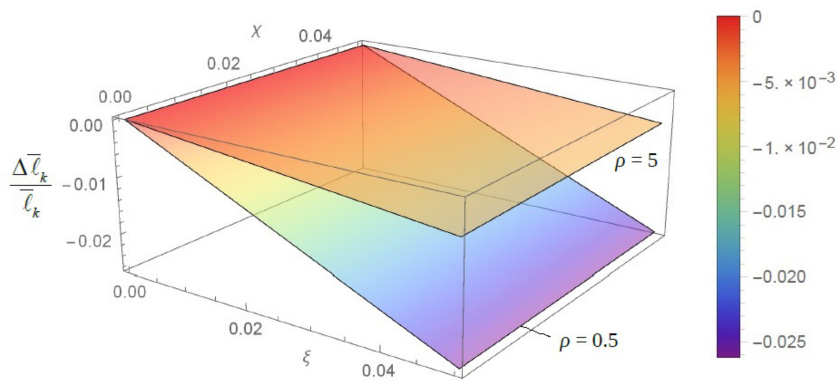

Figure 10. Relative error of kinematic contact length in external cylindrical grinding.

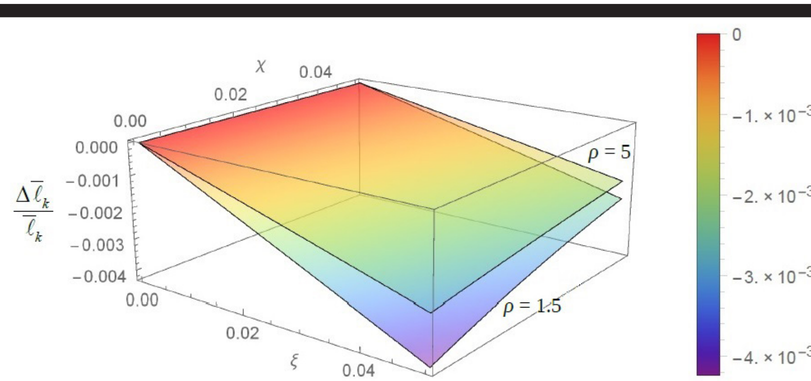

Figure 11. Relative error of kinematic contact length in internal cylindrical grinding.

CONCLUSIONS

We performed a study of the relative error of the geometric and kinematic contact length formulas found in the literature, both in surface and cylindrical grinding. For this purpose, the exact formulas have been derived. It has turned out that for most of the cases the error is very small < 0.5%. However, for external cylindrical grinding, when the radius of the wheel is bigger than the radius of the workpiece, the error is not very small, $\Delta\bar{\ell}_g / \bar{\ell}_g \geq 2\%$, for large depths of cut, $\xi \geq 0.04$.

Despite the fact that the interpretation of the approximated formulas found in the literature is quite readily, for the sake of accuracy, it would be advisable to use the exact formulas, since nowadays the numerical evaluation of elementary and special functions (such as $\cos^{-1}(x)$ and $\mathrm{E}(k,\phi)$) is quite easy and fast by using computer algebra.

APPENDIX: CALCULATION OF THE INTEGRAL

We want to calculate the following integral

$$I(x,\beta) = \int_0^\beta \sqrt{1+x^2+2x\cos\theta}\, d\theta, \quad x>0. \tag{60}$$

We could calculate (60) by using the followin formula (see [10, Eqn. 1.5.20(1)],[11, Eqn. 2.576.1],[12, Eqn. 289.01])

$$\int_0^\beta \sqrt{a+b\cos t}\, dt \overset{?}{=} 2\sqrt{a+b}\ \mathrm{E}\left(\sqrt{\frac{2b}{a+b}}, \frac{\beta}{2}\right), \tag{61}$$

where the *elliptic integral of the second kind* is defined as [9, Eqn. 62:3:2]

$$\mathrm{E}(k,\theta) = \int_0^\phi \sqrt{1-k\sin^2\theta}\, d\theta. \tag{62}$$

However, performing the derivative on the right hand side of (61) we do not get the integrand of (61). Therefore, the scope of this Appendix is just to provide a simple derivation of the proper formula. In order to reduce $I(x,\beta)$ in terms of an elliptic function, let us rewrite (60) as

$$I(x,\beta) = \int_0^\beta \sqrt{(1+x)^2 - 2x(1-\cos\theta)}\, d\theta,$$

so applying the half angle formula for the sine function $2\sin^2(\theta/2) = 1-\cos\theta$ and performing the change of variables $\alpha = \theta/2$, we have

$$I(x,\beta) = \int_0^\beta \sqrt{(1+x)^2 - 4x\sin^2(\theta/2)}\, d\theta = (1+x)\int_0^{\beta/2} \sqrt{1-\frac{4x\sin^2\alpha}{(1+x)^2}}\, d\alpha.$$

Taking into account (62), we finally get

$$I(x,\beta) = \int_0^\beta \sqrt{1+x^2+2x\cos\theta}\, d\theta = 2(1+x)\ \mathrm{E}\left(\frac{4x}{(1+x)^2}, \frac{\beta}{2}\right). \tag{63}$$

Notice that for positive x, the modulus of the elliptic integral in (63) is bounded as follws

$$0 \leq \frac{4x}{(1+x)^2} \leq 1, \quad \forall x \geq 0,$$

as in the geometric interpretation of the elliptic integral [9, Eqn. 62:3:2].

When $\phi \approx 0$, we have the following limiting approximation [9, Eqn. 62:9:2]

$$\mathrm{E}(k,\theta) \approx \phi - \frac{k}{6}\phi^3 + \ldots,$$

thus, for β small

$$I(x,\beta) = \int_0^\beta \sqrt{1+x^2+2x\cos\theta}\, d\theta \approx (1+x)\beta, \quad \beta \to 0. \tag{64}$$

Finally, notice that the proper formula in (61) is just

$$\int_0^\beta \sqrt{a+b\cos t}\, dt = 2\sqrt{a+b}\ \mathrm{E}\left(\frac{2b}{a+b}, \frac{\beta}{2}\right), \quad a > b > 0, 0 \leq \beta \leq \pi, \tag{65}$$

since the following inequality is satisfied

$$1 + x^2 > 2x > 0, \ \forall x > 0.$$

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests regarding the publication of this paper.

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