2014

Time Required for a Sphere to Fall Through a Funnel

J. Sridharan

Brett Altschul

S. Crittenden

AIP Advances

Time required for a sphere to fall through a funnel

Cite as: AIP Advances **4**, 127137 (2014); https://doi.org/10.1063/1.4904948
Submitted: 31 July 2014 . Accepted: 08 December 2014 . Published Online: 18 December 2014

J. Sridharan, B. Altschul, and S. Crittenden

AIP ADVANCES **4**, 127137 (2014)

Time required for a sphere to fall through a funnel

J. Sridharan,[1] B. Altschul,[1] and S. Crittenden[1,a]
[1]*Department of Physics and Astronomy, University of South Carolina, Columbia, SC, 29208, USA*

(Received 31 July 2014; accepted 8 December 2014; published online 18 December 2014)

We experimentally test a recently proposed theory of the behavior of a single frictional, inelastic, spherical particle falling under gravity through a symmetric funnel. We find that, while many qualitative results of the theory are supported by the data, the quantitative behavior of a real sphere falling through a real funnel differs from the predictions. The behavior above a 45° funnel angle, the duration, and the dependence of the duration on the initial horizontal position all show significant deviations from the predicted results. In particular, for drop positions near the gap, the duration of the fall is often significantly less than predicted for 50° and 60° funnel angles; and at a 60° funnel angle, where the data best matches the model, the R^2 goodness of fit is only 0.27. The fit can be significantly improved for 60° funnel angle by relaxing the most stringent approximation of the theory, which asserts that the transition from slipping to rolling is governed by a single constant parameter, β, independent of impact speed and angle. We conclude that, although the theory captures most of the key features of the dynamics of a ball falling through a funnel, it does not do so with quantitative accuracy, indicating that for commonly encountered balls and drop heights, a more realistic model of particle collisions is required. [http://dx.doi.org/10.1063/1.4904948]

I. INTRODUCTION

We experimentally test the recently proposed theory by Zhang *et al.*[1] describing the behavior of a single frictional, inelastic, spherical particle falling under gravity through a one-dimensional symmetric funnel. Two recent studies[1,2] have suggested that this simple system exhibits interesting and counterintuitive behavior. They find that, while the interesting effects exist in the ideal frictionless case, they are enhanced when the friction and the rotation of the ball it causes are considered. They find, among other things, that the amount of time the ball spends in the funnel is much larger for funnel angles greater than 45° than for smaller angles. Further, the duration depends sensitively on the initial drop position for these smaller angles, yielding a duration vs. drop position plot essentially indistinguishable from noise; whereas for larger angles, the duration vs. position varies smoothly over large regions.

The study of such a simple system is motivated by the widespread use of funnels in industry, where an improved understanding of the mechanics would allow for such things as the minimization of time spent in the funnel and the reduction of wear on the machine and the dropped objects. In addition, examining the change in behavior as the number of particles is increased to form a dense granular flow may shed light on the many results for that end-case. For instance, dense flow through an hourglass,[3] a closed-loop hourglass,[4,5] a funnel, of a two-dimensional layer of balls,[6] and more[7–16] have been studied by a variety of researchers. However, as pointed out by Zhang, the dilute flow case has received much less attention, even though the study of other systems with single or a few particles has led to a variety of interesting results.[17–26]

Zhang applies a general mathematical model due originally to Walton[23,27,28] to the specific case of a single particle falling through a funnel. They conclude that various interesting effects occur,

[a]Electronic mail: crittenden@physics.sc.edu.

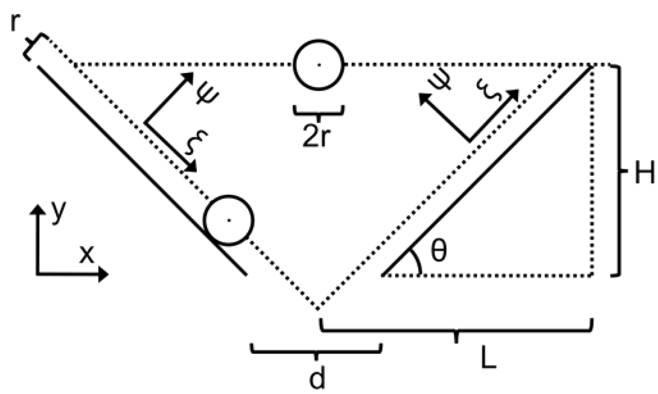

FIG. 1. Schematic of the system. The angle of the funnel walls is measured from the horizontal. Two lines a perpendicular distance r from the walls mark the limits of motion for the center of the ball. The origin of the x-y coordinate system lies at their intersection, which in general is below the level of the gap. Two additional coordinate systems, one for each wall as indicated, are used when considering impacts.

including a qualitative change in the dependence of the duration of the fall with position as the angle of the funnel wall passes through 45°. The effects are most pronounced for balls with coefficients of restitution and dynamic friction close to unity, but they still exist for balls with physically realizable values of the parameters. Aldahri[29] tested an earlier version of the model,[2] which does not include friction, and concluded that it did not reproduce the behavior of a real ball. However, the zero friction case is quite different from the frictional one, which is the point of Zhang's second paper. Here we test the frictional version of the model.

II. MODEL

We provide a concise description of the model here and refer to Zhang[1] for the complete details. We then describe the characteristic behavior it predicts.

We consider a spherical, frictional, inelastic ball of radius r with uniform density, falling under gravity g through a symmetric funnel with walls aligned at an angle θ to the horizontal and a gap of width d at the bottom of the funnel. The particle is released with zero initial velocity and zero initial angular velocity with its center a height H above the bottom of the funnel and a horizontal location x measured from the central axis of the funnel (see Fig. 1).

Following Zhang, we track the center of the ball, so the funnel is defined by two lines parallel to and at a distance r from the actual funnel walls. The origin of the coordinate system is at the intersection of these two lines and lies below the gap. When reporting results, we scale times and horizontal distances by the free-fall time of the ball through the center of the funnel, $\sqrt{2H/g}$, and the horizontal distance from the center to the intersection of the horizontal line at height H and the funnel wall, L. Note that these scalings are slightly different from Zhang's. They scale time by $\sqrt{H/g}$. We use x, y for the laboratory coordinate system and ξ, ψ for the coordinate system defined by the funnel wall, as in Fig. 1.

A quick estimate reveals that the Reynolds number for our case can be ~ 5,000 and so the flow of air around the ball is often turbulent. However, Stokes drag for the ball we used results in a drag force approximately 1,000 times smaller than gravity. We are therefore justified in ignoring air resistance. Consequently, the motion of the ball between impacts is either parabolic or linear rolling along the funnel. The only part of the system that is not immediately clear is the collision model. Continuing to follow Zhang, we represent the inelasticity of the ball with a constant coefficient of perpendicular restitution, e. Thus, the components of momentum perpendicular to the funnel wall after and before impact are related by

$$\psi_f = -e\psi_i, \tag{1}$$

where $\psi_{\{i,f\}}$ are the initial and final, respectively, perpendicular components of velocity in the funnel coordinate system.

The ball is further characterized by its constant coefficient of dynamic friction, μ. Because μ is not zero and the ball is extended, it will emerge from the impact rotating. It will also have a different momentum tangent to the funnel wall than it did before impact. The post-impact values of the tangential and rotational velocities are central to the model's representation of the complicated physics of collisions of real objects. For example, when an object moving at some angle to a surface impacts that surface, it will not immediately bounce off, but will spend some time in contact. While in contact, it may roll with or without slipping, or it may simply slide, and it will certainly compress. The model considers only two cases: either the collision involves slipping, or it does not. The compression of the ball and funnel wall and the amount of time spent in contact, as well as the motion of the ball during the slipping is wrapped up in one parameter β, which affects the final tangential and rotational velocities.

The parameter β is most usefully seen as an effective tangential coefficient of restitution,[28] yielding a model defined by three fixed parameters, the perpendicular and tangential coefficients of restitution, e and β, and the coefficient of dynamic friction, μ.

In the simplest physical case, slipping ceases when the combined rotational and translational velocity of the ball yields a speed of zero at the point of contact with the surface. However, for real, deformable balls, slipping may cease before or after this due to varying velocities between the deformed part of the ball in contact with the surface and the rest of the ball. β condenses all the complicated physics of the actual impact into a single parameter, asserting that slipping will generally occur and will cease when

$$\psi_f + r\omega_f = -\beta(\psi_i + r\omega_i), \tag{2}$$

For a perfectly rigid ball and surface, $\beta = 0$, while for a perfectly sticky ball-surface interaction, $\beta = 1$, i.e., the rotational velocity simply changes sign, a case that can be physically realized with Super Balls on rough surfaces.[27] A perfectly slippery surface corresponds to $\beta = -1$, i.e., the tangential velocity is utterly unaffected by the impact.

Critically, Zhang's model, following both Lun's and Walton's default case, uses a constant β for all impacts.

Impacts are assumed to be instantaneous. Since β determines when rolling will occur and conceptually depends on the frictional properties of the ball-surface interaction, it yields a critical value of an effective, velocity dependent, coefficient of friction, which determines whether the impact will involve rolling,

$$\mu_c = \frac{(1+\beta)|\xi_i + r\omega_i|}{(1+e)(\frac{mr^2}{I}+1)|\psi_i|}, \tag{3}$$

where ω is the angular velocity, m the mass, I the moment of inertia, and ξ_i and ψ_i the position immediately before impact in the funnel coordinate system. When $\mu \geq \mu_c$ slipping does not occur and the tangential and rotational velocities after impact are

$$\xi_f = \xi_i - \mu(1+e)|\psi|\mathrm{sgn}(\xi_i) \tag{4}$$

and

$$\omega_f = \omega_i - \frac{\mu(1+e)|\psi_i|\mathrm{sgn}(\xi_i)mr}{I}. \tag{5}$$

If $\mu < \mu_c$, slipping occurs, so that

$$\xi_f = \frac{mr^2 - I\beta}{I+mr^2}\xi_i - \frac{Ir}{I+mr^2}(1+\beta)\omega_i, \tag{6}$$

and

$$\omega_f = -\frac{mr(1+\beta)}{I+mr^2}\xi_i + \frac{I - mr^2\beta}{I+mr^2}\omega_i. \tag{7}$$

Again, we refer to Zhang and Walton[27] for the detailed derivation of these final results.

As noted by Zhang, a particle evolving according to this model may suffer inelastic collapse, wherein an infinite number of collisions occur in finite time. In simulating the motion, we handle this case by imposing an ad hoc cutoff time, here 1 ms. If the time between collisions is less than that, we assume that the ball will in fact begin rolling along the funnel wall. Finally, the ball may impact the funnel and emerge with insufficient perpendicular speed to lose contact with the funnel again. In this case it will immediately roll, with or without slipping. We assume slipping ceases when the velocity of the surface of the ball matches that of the center of mass.

The problem having been reduced to simple kinematics, it is relatively straightforward to determine the motion of the ball without having to simulate its motion throughout its entire trajectory using regular time steps. Instead, calculations are needed only at the impacts. An analysis of each impact and the resulting velocities allows for direct calculation of the next impact point, be it with the wall or the gap at the bottom of the funnel.

Figure 2 shows the results of the model. We model a 0.5 inch diameter ball, weighing 1.3 g, a 1 inch gap, and an initial drop height 44 cm above the gap. Following Zhang et al., β here is set to 0, $e = 0.99$, and $\mu = 1$ and our results match theirs. As can be seen, the time required from release to exiting the funnel varies with horizontal drop position and with the angle of the funnel walls. The angles chosen represent two qualitatively different regimes of behavior. For all angles above 45°, there exist regions where the duration varies apparently smoothly with position, whereas for all angles less than that, the variation in duration is so sensitive to position that there is no apparent correlation between adjacent positions. Figure 2 is an extreme case. Real balls will have e and μ less than 1, while β, representing as it does every complication not considered, is difficult to estimate.

For our ball and funnel, $e = 0.88$ and $\mu = 0.41$. The results of the model for various β values are shown in Fig. 3. Some differences are immediately obvious. For the extreme case shown in Fig. 2, the durations for high angles are much larger than for low ones, whereas for the more realistic case, the times are comparable. Further, as β grows, the shape of the duration vs. position graph changes drastically. The broad peak in the center shifts outward and shrinks, eventually disappearing. For $-1 < \beta < 0$, allowed by the model, the results are similar; The broad peak emerges from the left and marches across the plot as β moves from -1 to 0. The prediction for 40° is similar to Fig. 2(a) but scaled by a factor of 1/3.

As shown in Fig. 4, the apparent randomness of the predictions for 40° in Fig. 2 is an artifact of insufficient sampling and the complexity of the duration of fall with position. In fact, as is obvious for 60°, the duration for 40° consists of many smoothly varying segments separated by discontinuous changes. The leftmost arc of Fig. 4(d) corresponds, for our experimental parameters, to a width of about 2 μm.

III. EXPERIMENTAL SETUP

Balls of various diameters and materials were tested for their coefficients of restitution. Because it had the highest coefficient of restitution, we selected a Plexiglas ball 0.5 inches in diameter, weighing 1.3 g. Four of these were glued to the bottom of a base, upon which was set a 200 g mass. This was set on a brushed aluminum surface (McMaster-Carr, 6061 multipurpose aluminum, $6'' \times 24'' \times 0.25''$), and a string was attached, which ran over a pulley. Mass was added to the pulley until the block/ball system moved at a constant velocity across the plate, from which μ was determined. The coefficient of restitution was determined from a comparison of consecutive bounce heights for a ball dropped onto the horizontal aluminum surface (e varies like the square root of the ratio of the bounce heights) using a consumer-grade Casio EX-FZ100 high-speed camera.

The funnel was constructed from aluminum plates whose bottom ends were placed into two slots milled into a block of wood set on the lab bench. The plates were held at a fixed angle by another pair of wooden blocks clamped to the bench. The system was carefully leveled so that the ball, when dropped, did not bounce laterally out of the funnel. Often the ball moved less than a couple centimeters toward the edges of the plates when dropped from a height of ~40 cm.

Since this arrangement does not in fact have a gap, the location of the putative gap was marked by two pieces of tape on the front and back edges of the funnel, where the ball would never encounter them, for the 50° and 60° angles and by two pieces of tape on the aluminum surface whose edges

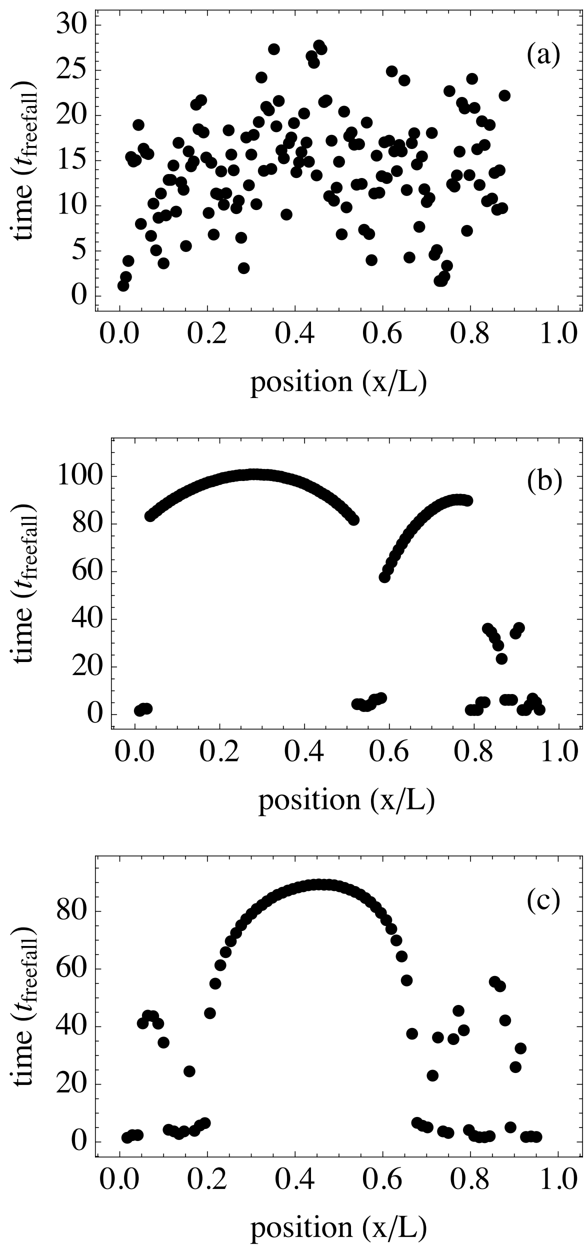

FIG. 2. Duration of fall as a function of initial drop position for $e = 0.99$, $\mu = 1$, and $\beta = 0$. The behavior is symmetric about the origin, so results are shown only for a ball dropped on the right funnel wall. Time is scaled by the time to fall directly through the gap when dropped from $x = 0$, $t_{freefall}$. (a), (b), and (c) are for funnel angles of 40°, 50°, and 60°, respectively. Durations are calculated for values of x/L corresponding to the experimentally acquired data.

defined the gap for the 40° angle. During analysis, the ball was tracked until it passed below these marks and was considered to have fallen through the gap.

The lab bench used has an additional, higher shelf. A rigid, massive block set on this shelf was used as a support for a piece of wood with a small divot on its side to enable reproducible placement of the ball. The wood was taped to the block. The ball was colored black with a marker and held in place with a glass microscope slide. A white background was placed behind the funnel to make tracking the position of the ball easy. The drop distance was set to 44 cm.

The motion of the dropped ball was then captured with the camera recording at 240 frames per second and analyzed in Adobe Premiere Pro CS5.5. Even for the fastest observed motion, the ball

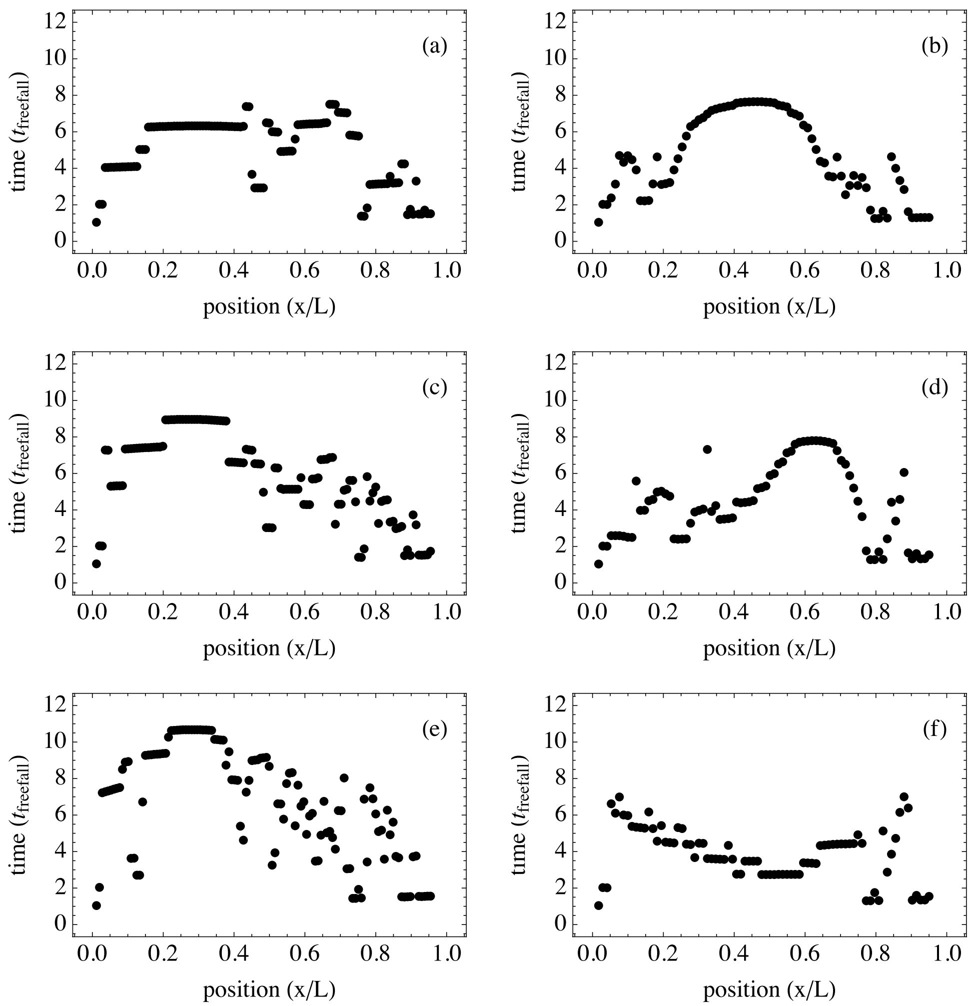

FIG. 3. Model predictions for $\beta \in \{0, 0.3, 0.6\}$. β increases down the page. The left column is for 50° and the right for 60°. For 40°, the changes with β are difficult to distinguish due to the complexity of the duration of fall dependence and so are not shown, but are qualitatively similar to the previous Fig. 3(a), scaled by $1/3$.

never moved more than one diameter per frame, so time and location resolution were high. However, the resolution of the camera is insufficient to observe the initial motion of the ball from rest, so four frames, corresponding to $\frac{4}{240} = 17$ ms were added to all measured fall times, based on our estimate of our ability to determine the initial motion. When adding four frames, the observed free fall times were within a few percent of expectations.

The drop position was set to begin in the center of the funnel, and moved 3 mm after every five ball drops, until it was too close to the funnel wall to be practicable or, in the case of 40° funnel angle, until the end of the 2 ft funnel wall was reached. We tested three angles, 40°, 50°, and 60°, one below and two above the transition angle identified by Zhang.

IV. RESULTS

Fig. 5 shows the results overlaid on the model predictions for our measured values of e and μ and for β set to zero. While there are many qualitative similarities, there are also clear differences.

For 40°, the variation with position is smoother and, on average, the duration of fall is smaller than predicted. Due to the highly variable predictions and our inability to drop the ball with micron-scale lateral precision (see Fig. 4(d)), experimental data here should be equivalent to an average of the

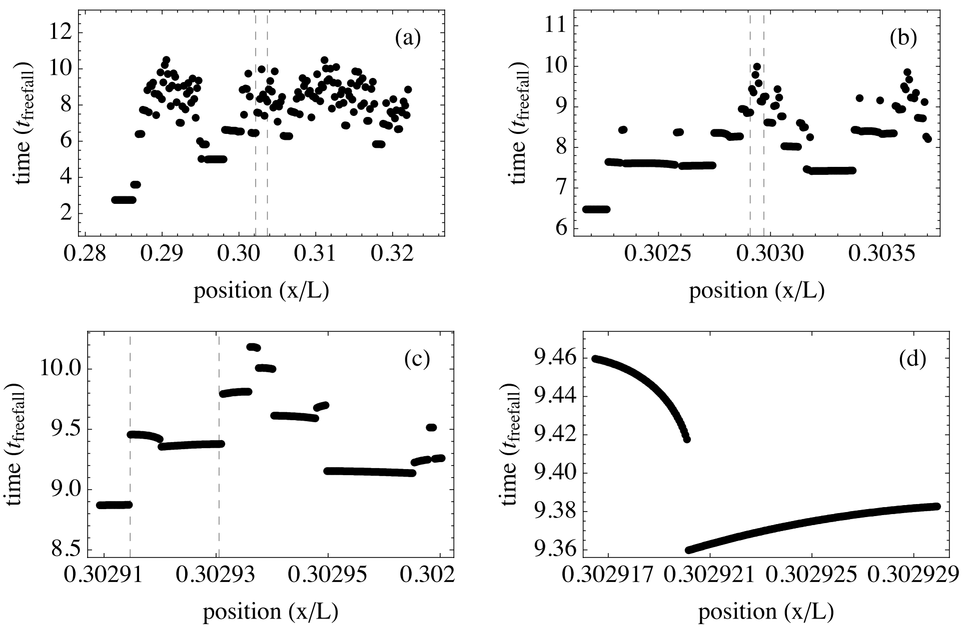

FIG. 4. Successively finer examinations of the duration of fall for 40°. Sufficiently close inspection of any region of x/L shows smoothly varying arcs of duration of fall separated by discontinuous jumps. Dashed lines indicate the region plotted in the succeeding image, i.e. (b) is the central region of (a), etc. The right-most arc of (d) corresponds, for our experiment, to a width of about 2μm.

model predictions. If so, we would then expect measured results intermediate to the predicted ones, which we clearly have obtained. Further, there is a dip in duration at $x/L = 0.7$, which our results reproduce, as well as a flat region at $x/L = 0.8$, which we again experimentally observe.

For 50°, there is good quantitative agreement above $x/L = 0.7$ and for $0.5 < x/L < 0.55$ and moderate agreement between the extremes of those regions. However, the behavior for $x/L \leq 0.45$ deviates significantly from the model prediction, often being 30% too low and dropping to 50% too low at $x/L = 0.45$. Also, the experimental uncertainty drops precipitously on the right half of the plot, indicating that the duration varies much less rapidly with position there. For $0.15 < x/L < 0.45$ the experimental uncertainty is high, although the model suggests that there ought to be essentially no variation in duration at all in this region.

For 60°, there is again good quantitative agreement far from the gap, this time for $x/L > 0.6$, with progressively poorer agreement moving backward from there to $x/L = 0.2$. Nevertheless, the data reproduce many features of the prediction: a peak near $x/L = 0.1$, a second peak near $x/L = 0.5$, along with a third, much thinner one, two flat regions near the outside edge at $x/L \sim 1$, and a fourth very small peak between them.

The small uncertainties in duration of fall for half of the 50° and all of the 60° drop positions imply that the three-dimensional nature of our funnel is not the source of the lack of fit to the two-dimensional funnel of the model. Errors due to motion toward the front and back of the funnel would be expected to be random and independent of drop position, and our uncertainties are much smaller than the differences between the model and data. Similarly, the errors cannot be due to small imperfections in the ball or funnel surface, because these also would appear as large uncertainties is the measured durations of fall.

There is no particular reason to assert a priori what β ought to be for a Plexiglas ball bouncing on an aluminum plate. As Fig. 3 suggests, the central peak moves to the right and becomes less prominent as β increases, eventually disappearing. Running the calculation with β ranging from -1 to 1 shows the central peak appear from the left, move across the plot, and then decay past $\beta = 0$. Clearly, even in just the three 60° examples in Fig. 3, as β varies, the duration vs. position varies continuously but drastically. Having examined the results for β between -1 and 1 in steps of 0.1, none of them appear to be especially better fits than $\beta = 0$. Performing an R^2 goodness of fit measurement for β varying

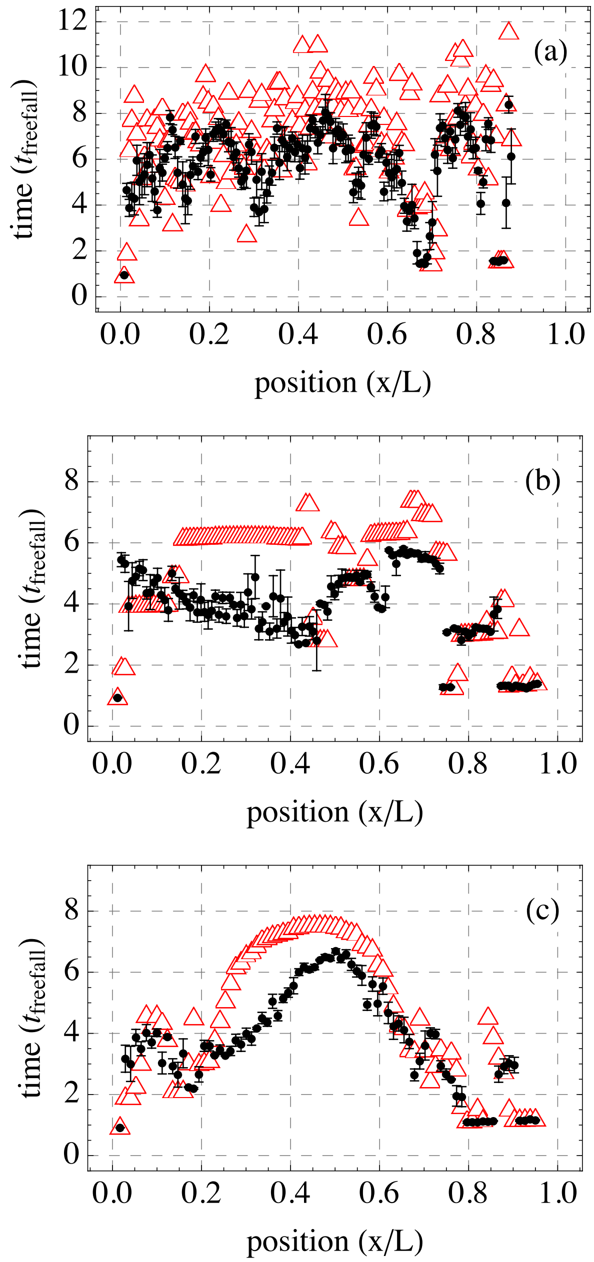

FIG. 5. Experimental results for (a) 40°, (b) 50°, and (c) 60°. Comparing with Fig 3, clear qualitative agreement with the model is evident. Equally clear is the lack of quantitative agreement. The duration of fall at 50° for $0.15 < x/L < 0.45$ is particularly different, being roughly 1/3 smaller than predicted. Agreement is also quantitatively poor for 60° within the same region. For 40°, the values are generally less overall, but the sensitive dependence of duration of fall on position makes interpretation difficult.

from −1 to 1 in steps of 0.02, the best fit is for $\beta = 0.06$ where $R^2 = 0.27$. Thus it is clear that the only available free parameter, β, is not sufficient on its own to define a model that reproduces the data quantitatively.

Because the ball was coated with black ink during drop measurements but not during friction measurements, we also compared the results to model predictions with μ varying between 0 and 1. By the time μ reaches 0.3, the model duration vs. position plot is visually indistinguishable from $\mu = 1$, and for $\mu < 0.3$, the model results move away from the measured ones. Thus, an incorrect μ will not explain the data. The size of the experimental apparatus makes setting the angle with an accuracy

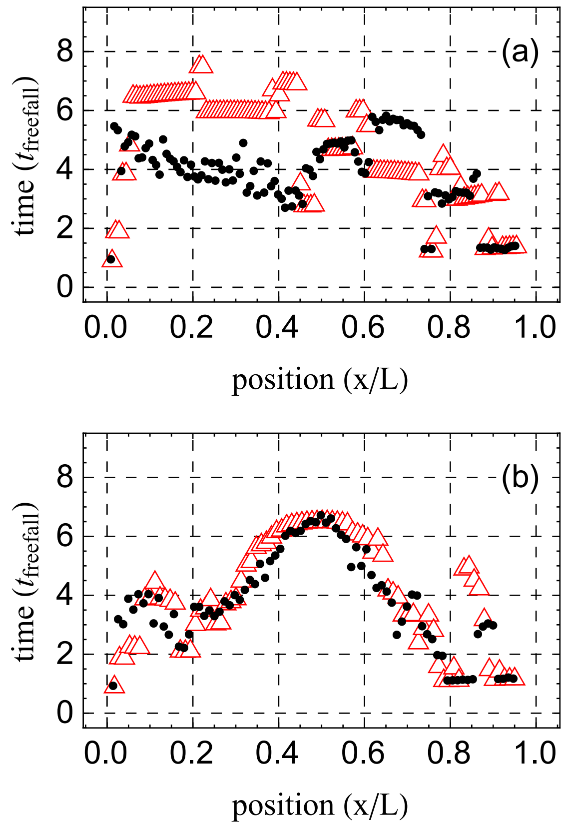

FIG. 6. Effect of varying *beta* for impact speeds greater than 40% of the exit speed when dropped directly through the gap. (a) 50° and (b) 60°. $\beta = 0$ for impact speeds below 40% freefall maximum, $\beta = 0.1$ for those above 80% freefall maximum, and β varies linearly for speeds between these two values. In addition, e is adjusted from 0.88 to 0.86 to visually obtain the best fit for the 60° data. The fit for 60° is clearly improved, although not for 50°, indicating that the dynamics are simply too complicated to be represented by the simple model, even with our modification.

much better than a degree easy. The video rate allows time measurements with 4 ms ($1/75^{th}$ of $t_{freefall}$) accuracy and retiming of a few of the videos yielded a mean difference in measured duration of fall of ~ 1 frame. The deviations from predictions for 50° and $x/L < 0.5$ are approximately 150 frames. The divot used to reliably place the ball often caused some initial rotation, so we examined the model predictions for $\omega_i > 0$. There was no observable large-scale change in the predictions for rotational velocities below ~30 rev/s.

Varying e does cause drastic changes in the predicted results, but, again, none are visually better than those employing our measured value.

We conclude that the cause of the significant deviations between the model and experiment are not due to either experimental inaccuracies or to improper model parameters.

However, as indicated in Fig. 6, we were able to obtain a significantly better fit for the 60° funnel angle by simultaneously adjusting e from 0.88 to 0.86 (a 2% change) and replacing a constant β with one that varied in a simple way with impact velocity. As discussed by Walton[27] and Lun,[28] setting β constant is a severe approximation and a more accurate approach would require it to vary with both impact speed and impact angle.

Determining exactly how it should vary requires detailed finite element analysis of the impact for the particular ball and surface at each translational and rotational speed and impact angle, which defeats the purpose of a simple model. Inspection of Fig. 5 suggests that the fit is poor for more energetic drops ($x/L < 0.5$) but good for less energetic ones. Since the undeformed ball-surface case is being considered here, this deviation at high initial ball energy suggests that at high impact speeds, deformation becomes significant, and β should deviate from zero. As indicated in Fig. 3, increasing

β drives the freefall time down for low x/L values, thus we modified the model to 'turn on' β as the impact velocity rose above 40% of the maximal velocity for the freefall through the gap at $x/L = 0$.

Specifically, if v_{ff} is the speed of the ball as it exits the funnel when dropped directly through the gap and v_i is the speed at impact, then, if $v_i/v_{ff} < 0.4, \beta = 0$. If $v_i/v_{ff} > 0.8, \beta = 0.1$, and β rises linearly from 0 to 0.1 between those two points. Simultaneously adjusting e to 0.86 yields Fig. 6, which clearly results in a better fit for 60°, although the results for 50° are, if anything, worse. The prediction for 40°, due to its chaotic apperance is essentially visually unchanged, although the actual numbers change, and so is not shown. We arrived at 40%, 80%, and $e = 0.86$ by varying each over reasonable ranges and visually comparing with the experimental results. The complicated shape of the predicted duration of fall makes least squares fits too coarse a measure of goodness of fit to give reliable results.

While we could replace β with a yet more complicated function of the speed and impact angle, our only guide would be whatever reproduces the data, since, as recognized by the original developers of the model,[27,28] the correct approach is to employ finite element analysis of every impact and it is not our goal to produce an accurate model, but to determine if a simple model suffices for predicting fall duration. Clearly it doesn't, even when modified from the constant $beta$ version in the next least complex way.

We emphasize, however, that the model does broadly predict the results at 40°, and at 60° and 50° for $x/L < 0.5$. Furthermore, as indicated in Fig. 5, it predicts discontinuous jumps in duration, which are observed in the data, mostly in the predicted locations. When the drop position is moved outward one step, the general pattern of bounces is observed to be similar but slightly shifted relative to the previous drop position. This happens because the first bounce in the new position occurs only slightly higher up the funnel wall. The subsequent bounces are similarly shifted slightly relative to the previous drop. This is true, in particular, for the final bounce into the gap. As the drop position moves, the position of the ball as it enters the gap also slowly shifts, approaching one of the gap edges. Eventually, it reaches the gap edge and just barely hits the funnel wall, whereupon it has at least one more bounce to complete before it can exit, leading to a significant jump in the duration of the fall. The reverse can, of course, also happen. If an impact position previously outside the gap shifts into the gap, the duration can decrease discontinuously. This kind of discontinuous variation in the duration of the fall through the funnel is a potentially important effect in engineering and other applications as an infinitesimal change in drop position can result in a change in duration of fall of 80% (see Fig. 5(c) near $x/L = 0.85$).

V. CONCLUSION

We find that the model proposed by Zhang is broadly accurate in predicting the behavior of real balls falling in real funnels. It captures a good deal of the dependence of the duration on drop position. However, for 50° and 60° it is less successful for positions less than half the maximum ($x/L < 0.5$), missing the magnitude by ~ 50% for 50° and not predicting observed regions of smooth duration variation with drop position.

The fit can be improved by employing a less severe approximation of the tangential coefficient of restitution, β, at least in some cases. However, it seems clear from the experimental data that, for commonly occurring balls and drop heights, the simple model of collision dynamics is insufficiently accurate to obtain quantitative agreement for the duration of fall of a real ball through a funnel. A more comprehensive approach that takes into account the variability of contact time, ball deformation, and other energy loss mechanisms with impact speed and angle will clearly be required.

Nevertheless, although the model's accuracy is mostly qualitative, it does capture many interesting features. Notably, it does predict quantized changes in duration for small changes in drop position, which we observed experimentally.

ACKNOWLEDGMENTS

We thank Art Illingworth and Allen Frye in the college of Arts and Sciences Machine Shop for their, as always, expert emergency machining service.

[1] Q. Zhang, Y. Fang, and J. J. Wylie, Phys. Rev. E **83**, 015303 (2011).
[2] Y. Fang, M. Gao, J. J. Wylie, and Q. Zhang, Phys. Rev. E **77**, 041302 (2008).
[3] K. L. Schick and A. A. Verveen, Nature (London) **251**, 599 (1974).
[4] X.-l. Wu, K. J. Maloy, A. Hansen, M. Ammi, and D. Bideau, Phys. Rev. Lett. **71**, 1363 (1993).
[5] C. T. Veje and P. Dimon, Phys. Rev. E **56**, 4376 (1997).
[6] C. T. Veje and P. Dimon, Phys. Rev. E **54**, 4329 (1996).
[7] G. W. Baxter, R. P. Behringer, T. Fagert, and G. A. Johnson, Phys. Rev. Lett. **62**, 2825 (1989).
[8] O. Moriyama, N. Kuroiwa, M. Matsushita, and H. Hayakawa, Phys. Rev. Lett. **80**, 2833 (1998).
[9] T. Le Pennec, M. Ammi, J. C. Messager, and A. Valance, Eur. Phys. J. B **7**, 657 (1999).
[10] E. Longhi, N. Easwar, and N. Menon, Phys. Rev. Lett. **89**, 045501 (2002).
[11] S. Horluck, M. van Hecke, and P. Dimon, Phys. Rev. E **67**, 021304 (2003).
[12] D. Helbing, A. Johansson, J. Mathiesen, M. H. Jensen, and A. Hansen, Phys. Rev. Lett. **97**, 168001 (2006).
[13] E. Manger, T. Solberg, B. H. Hjertager, and D. Vareide, Int. J. Multiphase Flow **21**, 561 (1995).
[14] B. Perez and I. Sanchez, Mech. Res. Comm. **38**, 244 (2011).
[15] M. A. Aguirre, J. G. Grande, L. A. Pugnaloni, and J.-C. Geminard, Phys. Rev. E **83**, 061305 (2011).
[16] L. Staron, P.-Y. Lagree, and S. Popinet, Eur. Phys. J. E **37**, 5 (2014).
[17] A. Mehta and J. M. Luck, Phys. Rev. Lett. **65**, 393 (1990).
[18] J. M. Luck and A. Mehta, Phys. Rev. E **48**, 3988 (1993).
[19] S. McNamara and W. R. Young, Phys. Fluids A **4**, 496 (1992).
[20] J. J. Wylie and Q. Zhang, Phys. Rev. E **74**, 011305 (2006).
[21] R. Yang and J. J. Wylie, Phys. Rev. E **82**, 011302 (2010).
[22] M. Gao, J. J. Wylie, and Q. Zhang, Commun. Pure Appl. Anal. **8**, 275 (2009).
[23] O. R. Walton, Lawrence Livermore National Laboratory Report No. UCID-20297-88-1 (1988) (unpublished).
[24] S. F. Foerster, M. Y. Louge, H. Chang, and Kh. Allia, Phys. Fluids **6**, 1108 (1994).
[25] S. Luding, Phys. Rev. E **52**, 4442 (1995).
[26] N. V. Brilliantov, F. Spahn, J. M. Hertzsch, and T. Poschel, Phys. Rev. E **53**, 5382 (1996).
[27] O. R. Walton, *Particular Two-Phase Flow* (Butterworth-Heinemann, 1993), Chapter 25.
[28] C. K. K. Lun and S. B. Savage, J. App. Mech. **54**, 47 (1987).
[29] T. H. Aldahri, The Behavior of a Falling Particle in a Funnel, Master's project, University of Western Ontario, Physics Department (2012).