4-7 NOV 2012 Cluj-Napoca, Romania

New researcher generation with challenges in civil engineering.

CIVIL ENGINEERING
PhD.INTERNATIONAL CONFERENCE

Proceedings of the First international conference for PhD students in Civil Engineering

Edited by:
Cosmin G. Chiorean

SEnS GROUP

TECHNICAL UNIVERSITY OF CLUJ-NAPOCA

FACULTATEA DE CONSTRUCTII

Proceedings of the First International Conference for PhD Students in Civil Engineering

"*New Researcher Generation with Challenges in Civil Engineering*"

CE-PhD 2012

www.sens-group.ro/CE2012

Cluj-Napoca-ROMANIA

4-7 November 2012

Edited by:

Cosmin G CHIOREAN

Organized by:
Technical University of Cluj-Napoca,
Faculty of Civil Engineering
&
SEnS Group

Published by:
EIKON

Editura Eikon Cluj-Napoca
www.edituraeikon.ro

CIP Description is available at National Library of Romania

ISBN 978-973-757-710-8

Linear Transient Analysis of Laminated Composite Plates using GLPT

Miroslav S. Marjanović*[1], Đorđe M. Vuksanović[2]

[1, 2] University of Belgrade - Faculty of Civil Engineering,
Bulevar kralja Aleksandra 73, 11000 Belgrade, Serbia

Abstract

*The objective of this work is to study the transient response of laminated composite plates under different types of dynamic loading. For this purpose, laminated composite plate is modeled using Reddy's generalized laminated plate theory (**GLPT**). This theory assumes layerwise linear variation of displacements components. Transverse displacement is constant through the thickness of the plate. Linear kinematic relations, as well as Hook's law, are considered. The generalized displacements are expanded in double trigonometric series using Navier-type method. Governing partial differential equations are reduced to a set of ordinary differential equations in time. The equations of motion are solved using constant-average acceleration method. Effect of time step on accuracy of transient response is investigated using a family of simply supported cross-ply (0/90) laminates. Different number of layers, as well as their influence on transient response is considered. Different schemes of transient dynamic loading are investigated. Good agreement is obtained with some results existing from the literature.*

Keywords: laminated composite plate, transient analysis, Navier solution, integration scheme

1. Introduction

Laminar composites play an important role in design and construction of aircrafts, ships, and many other parts in machine industry. They attract great attention in a field of Civil Engineering, too, and their massive use in structural design is expected. Suitability for different design purposes due to their great stiffness to weight ratios is highly valued. Refined mathematical models are of a great importance. ESL theories and layerwise approach arise in the theory of composite laminates [1]. In this work, layerwise approach (GLPT) is applied for calculation of transient response.

Laminated plates (laminates) are composed from arbitrary number of thin orthotropic laminas (plies) of constant thickness. Orthotropic behavior of single ply comes from high-strength fibers, oriented in a certain direction. During exploitation, composite laminates are exposed to different types of static and transient dynamic loading.

Cross-ply rectangular laminated composite plates of different geometries and lamination schemes are investigated. Ply fibers are oriented alternately, with angles of 0^{o} or 90^{o}. Different orientation of plies forms symmetric or anti-symmetric lamination schemes. Linear response is assumed, taking Hook's law into account, as well as linear kinematic relations [1]. Inextensibility of normal (Kirchhoff hypothesis) is imposed. Displacement/stress distributions over thickness coordinate are determined using linear Lagrangian interpolation functions.

2. Layerwise approach

ESL theories play an important role in deriving the global response of the laminated structure, and they are in detail explained in [2]. Single layer models of higher order represent kinematics with improved accuracy [3]. In layerwise approach, it is assumed that C^0-continuity through thickness of the laminate is satisfied. GLPT represents a significant improvement in the analysis of stress/strain of laminar composites [4]. The plate is analyzed as a multilayered in the true sense of word. GLPT allows independent interpolation of in-plane and out-of-plane displacement fields. Piece-wise linear variation of in-plane displacement components, and constant transverse displacement through the thickness are imposed. Cross-sectional warping is taken into account, which

* Corresponding author: Tel./ Fax.: +381 (0)11 3218581
E-mail address: mmarjanovic@grf.rs

is much more kinematically correct representation of displacements [5]. Shear deformation (considerable as a result of plate's anisotropic structure) is included.

2.1 Displacement field

Displacement field (u_1, u_2, u_3) in the point (x, y, z, t) of laminated plate can be written as:

$$u_1(x,y,z,t) = u(x,y,t) + U(x,y,z,t) = u(x,y,t) + \sum_{I=1}^{N} U^I(x,y,t)\Phi^I(z)$$

$$u_2(x,y,z,t) = v(x,y,t) + V(x,y,z,t) = v(x,y,t) + \sum_{I=1}^{N} V^I(x,y,t)\Phi^I(z) \tag{1}$$

$$u_3(x,y,z,t) = w(x,y,t)$$

u, v, w are displacement components in three orthogonal directions in the middle plane of the plate, U^I and V^I are coefficients which will be calculated later, $\Phi^I(z)$ are layerwise continuous functions of the thickness coordinate (linear, quadratic or cubic Lagrangian interpolations), while t denotes arbitrary time point. Linear interpolation is chosen through the thickness coordinate. Lagrangian interpolations $\Phi^I(z)$ are described in detail in [2, 5, 6]. If we assume that all layers in the laminate are of the same thickness h_i (as it is often in practice), linear Lagrangian interpolation functions are:

$$\Phi^1(z) = 1 - \frac{z}{h_i} \qquad \Phi^I(z) = \begin{cases} \frac{z}{h_i} + 2 - I \\ I - \frac{z}{h_i} \end{cases} \quad I = 2,3,\dots,(N-1) \qquad \Phi^N(z) = \frac{z}{h_i} + 2 - N \tag{2}$$

2.2 Single ply kinematics

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \sum_{I=1}^{N} \frac{\partial U^I}{\partial x}\Phi^I \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} + \sum_{I=1}^{N} \frac{\partial V^I}{\partial y}\Phi^I \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \sum_{I=1}^{N}\left(\frac{\partial U^I}{\partial y} + \frac{\partial V^I}{\partial x}\right)\Phi^I$$

$$\gamma_{xz} = \sum_{I=1}^{N} U^I \frac{d\Phi^I}{dz} \qquad \gamma_{yz} = \sum_{I=1}^{N} V^I \frac{d\Phi^I}{dz} \tag{3}$$

Linear strain – displacement relations (3) are assumed as follows: in-plane deformation components are continuous through the plate thickness, while the transverse strains need not to be. Constitutive equations of single ply are used in deriving constitutive equations of laminate, as given in [1].

2.3 Equations of motion, stress resultants and equilibrium equations

When deriving the dynamic equilibrium of virtual strain energy, virtual work of external forces and virtual kinetic energy, as given in [2], it is assumed that loading is acting in the middle plane of the plate. Homogenuous boundary conditions on the surface are imposed. Dynamic version of virtual work principle is given in [6]. Stress resultants are incorporated in (4):

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} dz \qquad \begin{Bmatrix} N^I{}_{xx} \\ N^I{}_{yy} \\ N^I{}_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} \Phi^I dz$$

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} dz \qquad \begin{Bmatrix} Q^I{}_x \\ Q^I{}_y \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} \frac{d\Phi^I}{dz} dz \tag{4}$$

After incorporation of (4), constitutive equations of laminate are derived as given in [1]:

$$\{N^0\} = [A]\{\varepsilon^0\} + \sum_{I=1}^{N}[B^I]\{\varepsilon^I\} \qquad \{N^I\} = [B^I]\{\varepsilon^0\} + \sum_{J=1}^{N}[D^{JI}]\{\varepsilon^I\} \tag{5}$$

$$[A]=\sum_{k=1}^{n}\int_{z_k}^{z_{k+1}}\left[Q_{pq}^{(k)}\right]dz \qquad \left[B^I\right]=\sum_{k=1}^{n}\int_{z_k}^{z_{k+1}}\left[Q_{pq}^{(k)}\right]\Phi^I dz \qquad \left[\overline{B}^I\right]=\sum_{k=1}^{n}\int_{z_k}^{z_{k+1}}\left[Q_{pq}^{(k)}\right]\frac{d\Phi^I}{dz}dz \tag{6}$$

$$\left[D^{JI}\right]=\sum_{k=1}^{n}\int_{z_k}^{z_{k+1}}\left[Q_{pq}^{(k)}\right]\Phi^I\Phi^J dz \qquad \left[\overline{D}^{JI}\right]=\sum_{k=1}^{n}\int_{z_k}^{z_{k+1}}\left[Q_{pq}^{(k)}\right]\frac{d\Phi^I}{dz}\frac{d\Phi^J}{dz}dz$$

Introducing (4, 5, 6) in virtual work principle, we derive Euler-Lagrange equations of equilibrium:

$$N_{xx,x}+N_{xy,y}+\sum_{I=1}^{N}I^I\ddot{U}^I=-I_0\ddot{u} \qquad N_{xy,x}+N_{yy,y}+\sum_{I=1}^{N}I^I\ddot{V}^I=-I_0\ddot{v}$$

$$N_{xx,x}^I+N_{xy,y}^I+Q_x^I+\sum_{J=1}^{N}I^{JI}\ddot{U}^J=-I^I\ddot{u} \qquad N_{xy,x}^I+N_{yy,y}^I+Q_y^I+\sum_{J=1}^{N}I^{JI}\ddot{V}^J=-I^I\ddot{v} \tag{6}$$

$$Q_{x,x}+Q_{y,y}+q=-I_0\ddot{w}$$

3. Navier solution

We have obtained system of 2N+3 partial differential equations, with 2N+3 unknown generalized displacements u, v, w, U^I and V^I. Next, we have to choose appropriate displacement field to satisfy boundary conditions on the edges of the simply supported laminated composite plate. Loading should be expanded in double trigonometric series in a same manner. Assumed displacement field is given in (7). Loading expansion is given in (8). m and n denote number of members in Fourier series. X_{mn}, Y_{mn}, W_{mn}, R^I_{mn}, S^I_{mn} are Fourier coefficients – time functions.

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}X_{mn}(t)\cdot\cos\alpha x\cdot\sin\beta y$$

$$v(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}Y_{mn}(t)\cdot\sin\alpha x\cdot\cos\beta y$$

$$w(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{mn}(t)\cdot\sin\alpha x\cdot\sin\beta y \tag{7}$$

$$U^I(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}R_{mn}^I(t)\cdot\cos\alpha x\cdot\sin\beta y$$

$$V^I(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}S_{mn}^I(t)\cdot\sin\alpha x\cdot\cos\beta y$$

$$q(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q_{mn}(t)\cdot\sin\alpha x\cdot\sin\beta y \tag{8}$$

Note that $\alpha=\frac{m\pi}{a}$ and $\beta=\frac{n\pi}{b}$. If cross-ply laminates are analyzed, some elements in matrix of elastic coefficients are identically zero [1], so equations (6) are compacted to matrix form:

$$\begin{bmatrix} k & k^I \\ k^I & k^{JI} \end{bmatrix}\begin{Bmatrix} X_{mn} \\ Y_{mn} \\ W_{mn} \\ R_{mn}^I \\ S_{mn}^I \end{Bmatrix}+\begin{bmatrix} m & m^I \\ m^I & m^{JI} \end{bmatrix}\begin{Bmatrix} \ddot{X}_{mn} \\ \ddot{Y}_{mn} \\ \ddot{W}_{mn} \\ \ddot{R}_{mn}^I \\ \ddot{S}_{mn}^I \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ -q_{mn} \\ 0 \\ 0 \end{Bmatrix}, \qquad [K]\{\Delta\}+[M]\{\ddot{\Delta}\}=\{F\} \tag{9}$$

Matrices k, k^I and k^{JI} are in detail explained in [1]. Sub matrices of consistent mass matrix [M] are:

$$m=\begin{bmatrix} I_0 & 0 & 0 \\ 0 & I_0 & 0 \\ 0 & 0 & I_0 \end{bmatrix} \qquad m^I=\begin{bmatrix} I^I & 0 \\ 0 & I^I \\ 0 & 0 \end{bmatrix} \qquad m^{JI}=\begin{bmatrix} I^{JI} & 0 \\ 0 & I^{JI} \end{bmatrix}$$

$[K]$ is global stiffness matrix, $\{\Delta\}$ is Fourier coefficients vector and $\{\ddot{\Delta}\}$ is a vector of second derivations of Fourier coefficients. If we observe discrete time point t_n, following matrix equation satisfies equilibrium conditions given above:

$$[K]\{\Delta\}_n + [M]\{\ddot{\Delta}\}_n = \{F\}_n \tag{10}$$

In (10), subscripted n denotes appropriate value in time point t_n. Global stiffness matrix, as well as consistent mass matrix, remains constant in all time points.

4. Transient analysis

Step Pulse $F(t) = F_0$

Sine Pulse $F(t) = F_0 \sin\left(\frac{\pi \cdot t}{T}\right)$

Triangular Pulse $F(t) = F_0\left(1 - \frac{t}{T}\right)$

Blast Pulse $F(t) = F_0 \cdot e^{-\alpha t}$

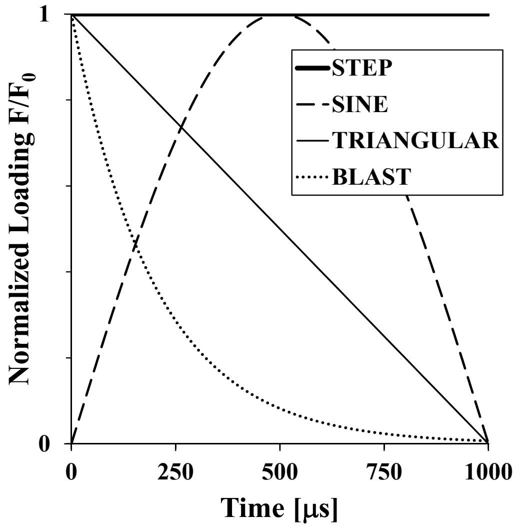

Figure 1 – Different forcing functions

Using this method, different types of dynamic loading can be incorporated easily. Different forcing functions which act on laminated composite plate are analyzed. Required system of differential equations in time is obtained. Only unknowns in this system are previously assumed Fourier coefficients.

System of differential equations in time, that has to be solved in all time points t_n, is given in (10). Superposed dots denote differentiation with respect to time. Due to assumed homogenuous initial conditions (displacements and velocities are zero), X_{mn}, Y_{mn}, W_{mn}, R^I_{mn} and S^I_{mn} and their first derivatives in time are zero. Distributed loading F_0 acts perpendicular to the mid-plane of plate. Forcing functions in following describe the load change through time:

4.1 Temporal discretization

In the Newmark method, accelerations and velocities are approximated using Taylor's series and only terms up to the second derivative are included [2]. Among several well-known Newmark integration schemes, constant-average acceleration method is chosen for this purpose [2]. This provided that introduced approximation error does not grow. Detail revision of Newmark integration schemes is given in [2]. Algorithm of solution process is explained in [8].

Initial values of $\{\Delta\}_0$, $\{\dot{\Delta}\}_0$ and $\{\ddot{\Delta}\}_0$ are needed for obtaining transient response of the structure. First two are known from initial conditions. However, acceleration vector $\{\ddot{\Delta}\}_0$ should be calculated from following expression:

$$\{\ddot{\Delta}\}_0 = [M]^{-1}(\{F\} - [K]\{\Delta\}_0) \tag{12}$$

5. Numerical examples

Preliminary calculations showed that number of members in double trigonometric series does not affect the results severely. According to this, in all following calculations, only first member in double trigonometric series is used. Proposed methodology of obtaining the transient response was investigated on several numerical examples presented in this chapter. Homogeneous initial conditions (zero displacements and velocities) were assumed in all examples.

In all calculations it was assumed that laminated structure is composed from arbitrary number of layers, which have the same mechanical properties, as follows:

$\rho = 8 \times 10^{-6}$ Ns^2/cm^4 $\nu_{12} = 0.20$
$E_1 = 50 \times 10^3$ kN/cm^2 $E_2 = 2 \times 10^3$ kN/cm^2 $G_{12} = G_{13} = 10^3$ kN/cm^2

These properties of single lamina served as the basis for calculating of mechanical properties of whole laminated plate.

Nondimenzionalized center transverse deflection is presented in all examples: $\overline{w} = w \cdot \dfrac{100 E_2 h^3}{q a^4}$

5.1 Influence of time increment

Influence of time increment was investigated with 2 plates with mentioned characteristics: 4-layer composite plate $(0/90)_2$ and 12-layer composite plate $(0/90)_6$.

Different time steps were used: Δt = 50, 100, 125 and 150 μs. Increase of time increment reduced the amplitude of oscillations, but increased the period of oscillations, as showed on Figures 2 and 3. Plate was exposed to equally distributed step loading. Maximum transient center transverse deflections in both cases are 2 times that of the static center transverse deflection. Both plates are squared, with a = b = 25 cm. Overall plate height is h = 1 cm (a/h = 25).

4-layer plate: $\overline{w}_{\max,dynamic} = 1.7943$ $\overline{w}_{\max,static} = 0.8971$

12-layer plate: $\overline{w}_{\max,dynamic} = 1.5544$ $\overline{w}_{\max,static} = 0.7772$

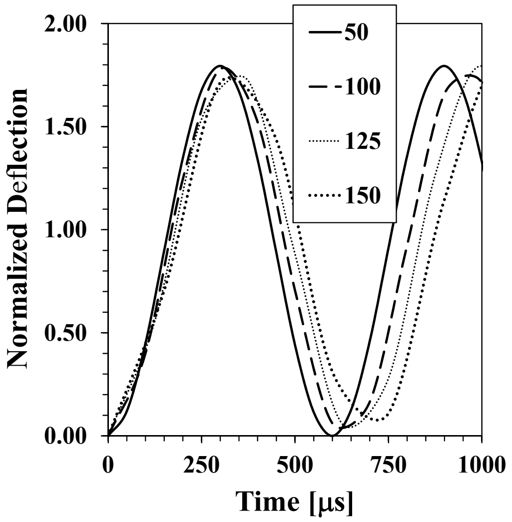

Figure 2 - 4-layer plate, step loading

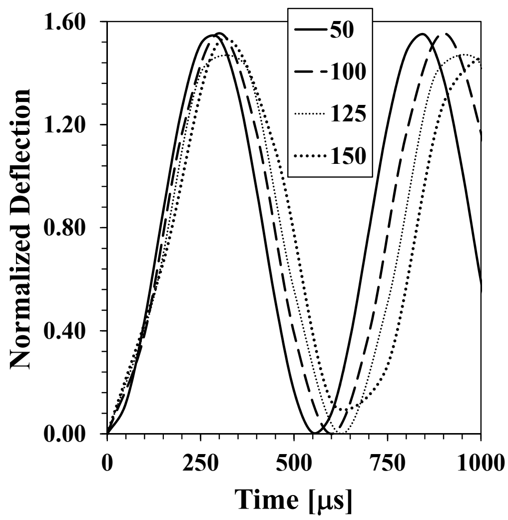

Figure 3 - 12-layer plate, step loading

5.2 Influence of lamination scheme

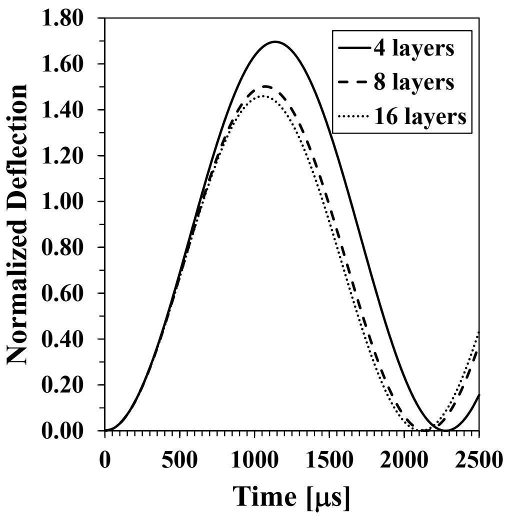

Figure 4 - Simply supported (0/90) laminates, step loading (Δt = 50 μs)

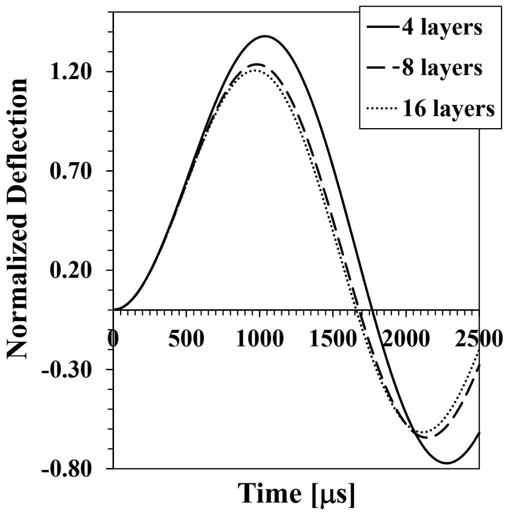

Figure 5 - Simply supported (0/90) laminates, triangular loading (Δt = 50μs)

The effect of the lamination scheme on the transient response of laminated structure is investigated using a simply supported cross ply (0/90) laminates with different numbers of layers. All plates were exposed to uniformly distributed step loading (case 1), and uniformly distributed triangular loading (case 2). Reduction in a number of layers leaded to a more flexible response of plate – it is increasing the amplitude as well as the period, as shown on Figures 4 and 5. Using more cross-ply layers in a same plate thickness, we obtain much stiffer response. Plate is squared, where a = b = 50cm, and overall plate height is h = 1 cm (a/h = 50).

5.3 Influence of different forcing functions

The effect of the applied forcing function on the response of laminated composite plate is investigated using a simply supported 4-layer cross ply $(0/90)_2$ laminate under uniformly distributed loading (see Figure 6). For this purpose, exponential blast loading is chosen as: $F(t) = F_0 \cdot e^{-0.005t}$. Plate is squared, a = b = 25 cm. h = 1 cm (a/h = 25).

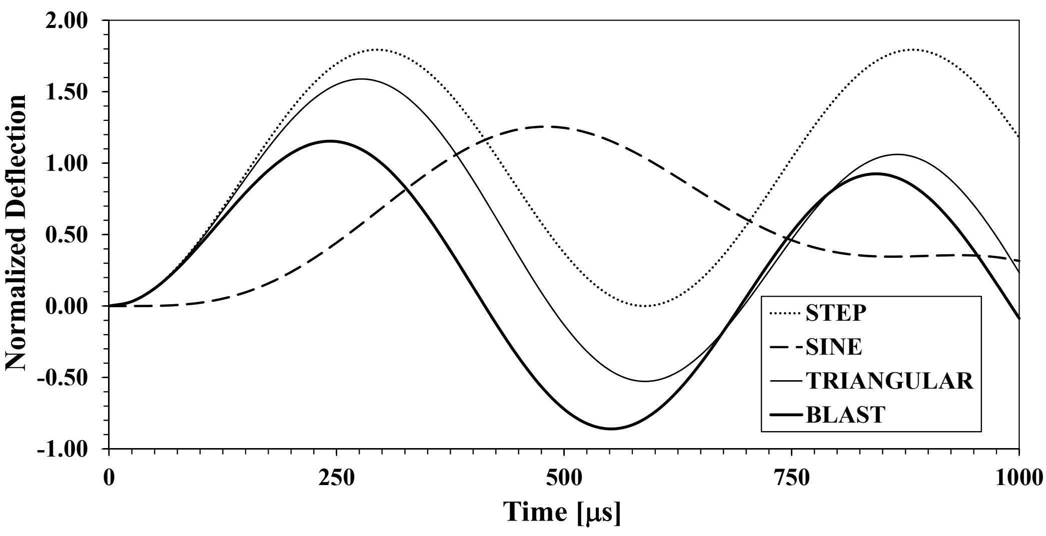

Figure 6 - Simply supported 4-layer cross-ply $(0/90)_2$ laminated plate, subjected to different types of uniformly distributed transient loading (Δt = 25μs, T = 1000μs)

5.4 Influence of plate geometry

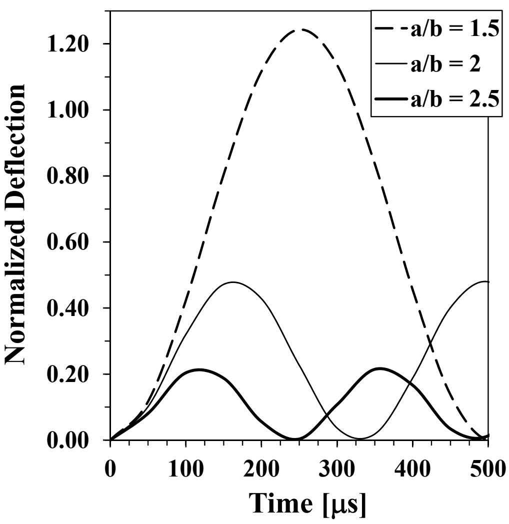

Figure 7 - 2-layer plate, step loading (Δt = 50 μs)

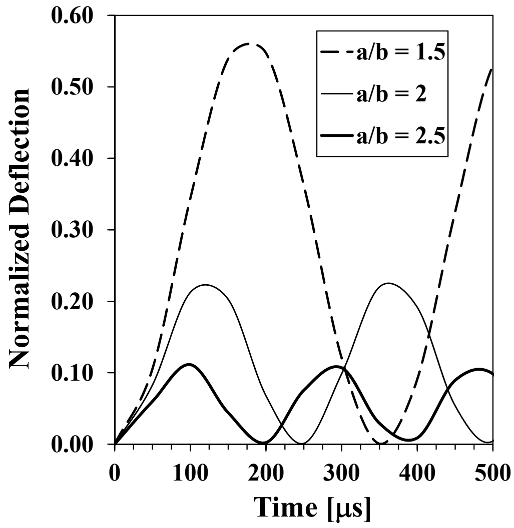

Figure 8 - 8-layer plate, step loading (Δt = 50 μs)

Influence of plate geometry was investigated with: 2-layer (0/90) composite plate and 8-layer $(0/90)_4$ composite plates. Time step of Δt = 50 μs was used. Plates were exposed to equally distributed step loading. Overall plate height is h = 1 cm, and ratio a/h=25. Different ratios a/b, which define the geometry, were used. Results are shown on Figures 7 and 8. In both examples it is obvious that increasing of a/b ratio lead to a stiffer response of plate – it is reducing the amplitude as well as the period of oscillations (see Figures 7 and 8).

6. Conslusions

Generalized Laminated Plate Theory is introduced using dynamic version of virtual work principle. Original MATLAB code is applied for obtaining the transient response. Navier solution, as well as Newmark integration scheme, were used. Calculations showed that the number of members in Fourier series does not affect severely the results of calculation. Influence of time step lenght on the solution accuracy was investigated. It is obvious that larger time step increases the period of oscillation, and reduces the amplitude, as shown in given examples. Stacking sequence affects the response in a following manner - reduction in a number of laminas leads to a more flexible plate behavior (increasing the amplitude as well as the period). Different forcing functions were applied, and plate response due-to several different transient loading types is presented. Also, calculation showed that increasing of a/b plate geometry ratio leads to much stiffer plate response (reducing the amplitude and the period).

Further research will be aimed at obtaining the transient response of laminated plates with different boundary conditions, using FEM, and comparison with presented results will be done. Authors have already derived some results of linear analysis using laminated finite elements, showed in [4].

Acknowledgements

The financial support of the Government of the Republic of Serbia - Ministry of Education and Science, under the Project TP-36048, is gratefully acknowledged.

7. References

[1] Ćetković M. Analytical solution for multilayer plates using general layerwise plate theory. *Facta Universitatis,* **Vol. 3**, pp. 121-36, 2005.

[2] Reddy JN. *Mechanics of Laminated Composite Plates: Theory and Analysis*. CRC Press, 1996.

[3] Vuksanović Đ. Linear analysis of laminated composite plates using single layer higher-order discrete models. *Composite Structures*, **Vol. 48**, pp. 205–11, 2000.

[4] Reddy JN, Barbero EJ, Teply JL: A plate bending element based on a generalized laminate plate theory, International Journal for Numerical Methods in Engineering, **Vol. 28**, pp. 2275-92, 1989.

[5] Ćetković M. *Nonlinear behavior of laminated composite plates*. PhD Thesis. University of Belgrade – Faculty of Civil Engineering, 2010.

[6] Ćetković M, Vuksanović Đ. Bending, free vibrations and buckling of laminated composite and sandwich plates using a layerwise displacement model. *Composite Structures*, **Vol. 88**, pp. 219–27, 2009.

[7] Hinton E, Vuksanović Đ. *Closed form solutions for dynamic analysis of simply supported Mindlin plates*. In: Hinton E, editor: Numerical Methods and Software for Dynamic Analysis of Plates and Shells. Swansea, UK: Pineridge Press, 1988, pp. 1-47.

[8] Hinton E, Vuksanović Đ. *Explicit transient dynamic finite element analysis of initially stressed Mindlin plates*. In: Hinton E, editor: Numerical Methods and Software for Dynamic Analysis of Plates and Shells. Swansea, UK: Pineridge Press, 1988, pp. 205-59.