BOUNDS FOR THE SOLUTION OF ALGEBRAIC MATRIX EQUATIONS ARISING IN MATHEMATICAL CONTROL THEORY

BY RICHARD DAVIES

BSc (2004)
UNIVERSITY OF GLAMORGAN, PONTYPRIDD, WALES

A thesis submitted in partial fulfilment of the requirements for the degree of Doctor of Philosophy

Department of Computing and Mathematical Sciences
Faculty of Advanced Technology
University of Glamorgan
Pontypridd
Wales

November 2007

Acknowledgements

First, I would like to pay many thanks and gratitude to my director of studies, Professor Peng Shi, for the support, guidance, encouragement, advice and immense patience he has shown me during the past three years of my PhD studies. Were it not for his supervision, a project in this field may not have been possible.

I would like, also, to express my thanks to Professor Ron Wiltshire, my second supervisor, for his support and advice, particularly in aiding my career development for the future. Many thanks.

My thanks also go to fellow Department students, both undergraduate and research, for their enthusiasm, drive and inspiration.

I am also grateful for the financial support that I received from the Department over the past three years, which has also helped me greatly in my studies.

Finally, I also wish to pay many thanks and gratitude to my family and friends outside the university for their support and encouragement throughout my time at the University of Glamorgan.

Abstract

The study of solution bounds of algebraic Lyapunov and Riccati equations are highly important in control problems, and have been an attractive research topic over the past three decades. The solution bounds give solution estimates, and can also be applied to solve such problems involving these equations, hence a motivation for the research attraction. Besides, in control applications involving them, the exact solutions are often not required, but rather bounds of their solution, particularly when solving the equation is difficult.

Therefore, many papers have proposed solution bounds for these equations, mainly for a deterministic nominal system, when the exact values of the coefficient matrices of the equations are available. Additionally, some works have focused on solution bounds of these equations for perturbed systems, when only approximate values of the coefficient matrices are available, so they available are perturbed versions of their actual values; as a consequence of these perturbed coefficient matrices, the solution matrix also becomes perturbed, so it becomes of interest to estimate the disturbance range for the solution. Furthermore, fewer works have focused on solution bounds of coupled algebraic Lyapunov and Riccati equations arising from stochastic systems, for both nominal and perturbed cases. In fact, it appears that there is no paper in the literature that studies solution bounds of perturbed coupled algebraic Riccati equations.

Finally, many existing bounds only exist under assumptions which are not always valid, many of which are not realistic in control problems involving each equation. Furthermore, some bounds do not appear to be as tight as others, some bounds require heavy and complicated calculations to determine, and some are not very concise. Therefore, this work seeks to obtain solution bounds for Lyapunov and Riccati equations, which are tighter, less restrictive, possibly simpler in calculation, and more concise than existing results. When possible, all derived results shall be compared with existing results to verify the advantage(s) of the new results.

Contents

1 Introduction and Background Information 1
1.1 An Overview of Control Theory and Dynamical Systems . . . 1
1.2 Linear Control Systems 2
1.2.1 Controllability 3
1.2.2 Observability 3
1.2.3 Stability for Deterministic Systems 4
1.2.4 Stability for Stochastic Systems 5
1.2.5 Optimal Control Design 5
1.3 The Continuous Algebraic Lyapunov Equation 6
1.4 The Continuous Algebraic Riccati Equation 7
1.5 The Discrete Algebraic Lyapunov Equation 7
1.6 The Discrete Algebraic Riccati Equation 8
1.7 The Continuous Coupled Algebraic Lyapunov Equation 9
1.8 The Continuous Coupled Algebraic Riccati Equation 9
1.9 The Discrete Coupled Algebraic Lyapunov Equation 10
1.10 The Discrete Coupled Algebraic Riccati Equation 10
1.11 Solution Bounds of Algebraic Matrix Equations 11
1.12 Outline of the Thesis 14

2 Lemmas, Literature Review and Methodology 16
2.1 Some Useful Lemmas 16
2.2 Discussion of the Solution Bounds Derived so far for a Deterministic Nominal System 18
2.3 Discussion of the Solution Bounds Derived so far for Deterministic Perturbed Systems 20
2.4 Discussion of the Solution Bounds Derived so Far for Stochastic Nominal Systems 21

2.5 Discussion of the Solution Bounds Derived so far for Stochastic Perturbed Systems 23
2.6 Methodology 23
2.7 Solution Bounds by means of the Unified Approach 23

3 Solution Bounds for Perturbed Continuous and Discrete Algebraic Riccati Equations 25
3.1 Matrix Bounds for the Perturbed Continuous Algebraic Riccati Equation 26
3.1.1 Numerical Example 36
3.2 Matrix Bounds for the Perturbed Discrete Algebraic Riccati Equation 37
3.2.1 Numerical Example for the Perturbed DARE 40
3.3 Summary 41

4 Matrix Bounds for the Continuous Algebraic Lyapunov Equation 42
4.1 Lower Matrix Bounds for the CALE 42
4.2 Upper Matrix Bounds for the CALE 46
4.3 Numerical Examples for the CALE 53
4.3.1 Example 1: Q is positive semidefinite and $A + A^T$ is not negative definite 53
4.3.2 Example 2: Q is positive definite and $A+A^T$ is negative definite 55
4.4 Summary 58

5 Matrix Bounds for the Continuous and Discrete Algebraic Riccati Equations 59
5.1 Lower Matrix Bounds for the Solution of the Continuous Algebraic Riccati Equation 59
5.1.1 Numerical Examples 69
5.2 New Upper Matrix Bounds for the Solution of the Discrete Algebraic Riccati Equation 74
5.2.1 Numerical Examples 82
5.3 Summary 86

6 Matrix Bounds of the Continuous Coupled Algebraic Lyapunov and Riccati Equations 87

6.1 Lower Matrix Bounds for the Continuous Coupled Algebraic Lyapunov Equation 87
6.1.1 Numerical Example for the CCALE 92
6.2 Lower Matrix Bound for the Continuous Coupled Algebraic Riccati Equation 93
6.2.1 Numerical Example for the CCARE 97
6.3 Upper Matrix Bounds for the Continuous Coupled Algebraic Riccati Equation 98
6.3.1 Numerical Example for the CCARE 106
6.4 Summary 107

7 Matrix Bounds for the Discrete Coupled Algebraic Riccati Equation 109
7.1 Upper Matrix Bounds for the DCARE 109
7.1.1 Numerical Example for the DCARE 118
7.2 Summary 119

8 Solution Bounds of Coupled Lyapunov and Riccati Equations under Perturbations in the Coefficients 120
8.1 Lower Matrix Bounds for the Continuous Coupled Lyapunov Equation Under Perturbations in the Coefficients 121
8.1.1 Numerical Example 127
8.2 Lower Matrix Bound for the Discrete Coupled Lyapunov Equation Under Perturbations in the Coefficients 128
8.3 Solution Bounds of the Continuous Coupled Algebraic Riccati Equation Under Perturbations in the Coefficients 130
8.4 Solution Bounds of the Discrete Coupled Algebraic Riccati Equation Under Perturbations in the Coefficients 131
8.5 Summary 133

9 Conclusions and Future Work 134
9.1 Conclusions 134
9.2 Future Work 136

A Presentations and Publications 139

Notation

For convenience, the following notations shall be used throughout this thesis.

$\Re$	The real number field
$\Re^{m\times n}$	The set of real $m \times n$ matrices
I	The identity matrix
X, Y, Z	Matrices
$X > (\geq) Y$	Matrix $X - Y$ is positive (semi-)definite
$\lambda_i(X)$	The ith largest eigenvalue of matrix $X \in \Re^{n\times n}$ for $i = 1, 2, \ldots, n$
$\sigma_i(X)$	The ith largest singular value of matrix $X \in \Re^{n\times n}$ for $i = 1, 2, \ldots, n$
$\mathrm{tr}(X)$	The trace of matrix $X \in \Re^{n\times n}$
$\det(X)$	The determinant of matrix $X \in \Re^{n\times n}$
X^T	The transpose of matrix $X \in \Re^{n\times n}$
X^{-1}	The inverse of the nonsingular matrix $X \in \Re^{n\times n}$
$X^{1/2}$	The unique positive semi-definite matrix square root of the non-negative definite matrix $X \in \Re^{n\times n}$ such that $X^{1/2}X^{1/2} = X$
$\Re(\lambda(X))$	Real part of an eigenvalue of matrix $X \in \Re^{n\times n}$

Chapter 1

Introduction and Background Information

1.1 An Overview of Control Theory and Dynamical Systems

Control theory is an area of applied mathematics which is concerned with the analysis and design of controlling devices for dynamical systems [3,69,73,74], so as to influence the behaviour of the system and achieve the desired outcome. Such controlling devices are known as *controllers*.
There are many types of control systems, including:

(1) Linear and non-linear systems: A linear system is one in which the behaviour of the system follows a linear rule. Graphically, this would be a straight line. Similarly, a non-linear system is one that follows a non-linear rule. Graphically, this would be a curve.

(2) Continuous-time and discrete-time systems: A continuous-time system is one that is defined over all time. A discrete-time system is one that is defined at particular instances of time.

(3) Time-invariant and time-variant systems: A time-invariant system is one in which the system parameters do not alter over time. A time-variant system is one in which the system parameters can change over time.

(4) Deterministic and stochastic systems: A deterministic system is a certain system, i.e., the state of the system and other information about the system can be determined definitely. A stochastic system is an uncertain system, i.e., the system state and other information about the system cannot be determined definitely, because the system has an element of uncertainty.

(5) Nominal and perturbed systems: A nominal system is one in which the exact values of the system matrices and other information can be obtained exactly. Often in practice, only approximate values of the system matrices are available (possibly due to cost, inaccessibility, or external disturbances), so they are perturbed versions of the actual ones. Such a system is a perturbed system.

In the field of mathematical control and systems theory, there are a number of control problems whose solution amounts to solving an algebraic matrix equation [1,2,5,6,7,8,21,22,23,55,65,66,69,73,74,76,87,88,90]. In particular, there are 8 algebraic matrix equations that are of concern to this research, which will later be considered in more detail. Before doing so, however, some important concepts arising in this field will briefly be reviewed in the following sections.

1.2 Linear Control Systems

The matrix equations in this project are related to linear time-invariant (LTI) control systems [3,73,74]. The general state-space representation of a continuous LTI control system is

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u} \\ \mathbf{y} &= C\mathbf{x} + D\mathbf{u} \end{aligned} \tag{1.1}$$

Also, the general state-space representation of a discrete LTI control system is

$$\begin{aligned} \mathbf{x}(k+1) &= A\mathbf{x}(k) + B\mathbf{u}(k) \\ \mathbf{y}(k) &= C\mathbf{x}(k) + D\mathbf{u}(k) \end{aligned} \tag{1.2}$$

For each of the systems (1.1) and (1.2), $\mathbf{x}$ is the $n \times 1$ state vector, $\mathbf{u}$ is the $m \times 1$ input vector, $\mathbf{y}$ is the $p \times 1$ output vector, A is the $n \times n$ state matrix,

B is the $n \times m$ input matrix, C is the $p \times n$ output matrix, and D is the $p \times m$ direct transmission matrix. Two important concepts related to these systems that shall be later called upon are controllability and observability.

1.2.1 Controllability

As mentioned at the end of the previous section, one particularly important concept related to control system design and analysis is controllability [3,4,73,74]. A control system is said to be controllable if it is possible to change any initial state of the system to another state within a finite time length for continuous-time systems (or a finite number of 'instances' for discrete-time systems). If controllability can be achieved for all such states, then the system is said to be completely controllable. If controllability cannot be achieved for any system state, then the system is said to be uncontrollable. If complete controllability cannot be achieved then the system is said to be not completely controllable. Controllability is an assumption required to guarantee the existence of a non-negative definite stabilizing solution of the Riccati equations. For both the continuous-time and discrete-time LTI systems given by (1.1) and (1.2) respectively, the controllability matrix, which we denote by C_M, is defined [73,74] by:

$$C_M = [B \; AB \; A^2B \; \dots \; A^{n-1}B]$$

where n is the number of states of the system. A necessary and sufficient condition [73,74] for the system (1.1) to be completely state controllable is that the controllability matrix C_M is of rank n. It is also known (see for example [34]) that controllability implies stabilizability, but the converse is not true in general. A pair (A, B) is stabilizable if there exists a matrix K (with suitable dimensions) such that $A + BK$ is stable.

1.2.2 Observability

Another important concept which arises in control system design and analysis is observability [3,4,73,74]. A control system is said to be observable if it is possible to determine the state of the system directly from the output of the system. If observability can be achieved for all such states, then the system is said to be completely observable. If observability cannot be achieved for any state then the system is said to be unobservable. If complete observability cannot be achieved then the system is said to be not completely observable. In addition to the assumption of controllability being required for the existence of non-negative definite stabilizing solutions of the algebraic Riccati equations, the assumption of observability is also required to guarantee the uniqueness of such solutions, the equations of which will be discussed in more detail later in this chapter. For both the continuous-time and discrete-time LTI systems given by (1.1) and (1.2) respectively, the controllability matrix, which is denoted by O_M, is defined [73,74] by:

$$O_M = [C^T \; A^T C^T \; (A^T)^2 C^T \; \ldots \; (A^T)^{n-1} C^T]$$

where n is the number of states of the system. A necessary and sufficient condition [73,74] for the system (1.1) to be completely observable is that the observability matrix O_M is of rank n. It is known (e.g. from [26,34]) that observability implies detectability, but that the converse is not true in general. A pair (A, C) is observable if (A^T, C^T) is stabilizable.

1.2.3 Stability for Deterministic Systems

An important characteristic of control system design and analysis is the stability of the system. When designing a controller for a system, it is desired that the controller be reliable, so that the system remains under control. It should be noted that controllability of a control system implies that the system is stabilizable, i.e., the system can be made stable. There are three types of stability [74] that will be mentioned:

- Stability: If the state of a system along a trajectory begins at an origin and remains within a region over an indefinite time length, then the system is said to be stable.
- Asymptotic stability: If the state of a system along a trajectory begins at an origin, remains within a region for some time and then decays to the origin over an indefinite time length, then the system is said to be asymptotically stable, the asymptote of stability being the origin.
- Instability: If the state of a system along a trajectory begins at an origin and leaves the region over an indefinite time length, then the system is unstable.

There are other definitions of stability for deterministic systems that can be found in the literature, which need not be discussed here (The interested reader may refer to the references given, as well as other references therein). Also, there are many tests for stability in the literature [73,74] (and references therein) that also need not be elaborated on, but rather be left to the interested reader to follow up. The most general stability analysis method for control systems is the Lyapunov method. Lyapunov's first method involves analysing the stability of the system by first solving the system differential equations to obtain the system state, and then analysing the stability of the system based on the state. In contrast to the first method, the second method of Lyapunov instead determines the stability of the system directly from the system differential equations without solving them; this involves the construction and use of a Lyapunov candidate function, which can be difficult because construction of such a function is not straightforward. For LTI systems, stability analysis involves the solution of the continuous and discrete algebraic Lyapunov matrix equations.

1.2.4 Stability for Stochastic Systems

Unlike deterministic systems, stability is defined in many different ways for stochastic systems. Because the systems are stochastic, one cannot be certain of the system states, or any other information about the system (such as inputs, outputs, etc.). As such, we the stability of the system can only be assessed to a degree of certainty (such as almost-sure stability). There are several types of stability for stochastic systems that are discussed in the literature, for example as in [1,2,8,22,23,29,69,76,87,88], such as mean-square stability, stochastic stability, almost-sure stability, and so on. When analysing the stability of jump linear systems with Markovian parameters, a system of linear matrix equations arise, known as the coupled algebraic Lyapunov equations. Like the Lyapunov matrix equations for a deterministic system, the solution of these equations also involve solving a system of linear equations, although the linear equation systems resulting from the coupled Lyapunov equations are considerably larger due to the involvement of the coupling term.

1.2.5 Optimal Control Design

Another factor in control system analysis and design is how to design and analyze a control system so that it behaves in some optimal way. In particular, the algebraic Riccati equations arise from the well-known linear quadratic regulator (LQR) problem for linear control systems [74] in which the objective is to find the optimal control feedback gains required to 'compel' the system to behave in some optimal way whilst still maintaining system stability. As a real-life example, consider a car travelling on a straight line through a hilly road [93]: The question is, how should the driver press the accelerator pedal in order to minimize the total traveling time? Clearly in this example, the term control law refers specifically to the way in which the driver presses the accelerator and shifts the gears. The "system" consists of both the car and the road, and the optimality criterion is the minimization of the total traveling time. Control problems usually include ancillary constraints. For example the amount of available fuel might be limited, the accelerator pedal cannot be pushed through the floor of the car, speed limits, etc. For deterministic systems, when one is concerned with the optimal design of a controller for a linear system with a quadratic performance index, we have the well-known LQR problem; in the case of stochastic systems, this corresponds to the linear quadratic Gaussian (LQG) problem. Also, for the case of stochastic systems, stability analysis and optimal control design for forced systems with an input matrix give rise to a system of non-linear matrix equations known as the coupled algebraic Riccati equations. Like the algebraic Riccati equations for the deterministic counterpart, a system of non-linear algebraic equations need to be solved to determine the solution matrices, but the number of equations to be solved is greater because of the coupling term in the equations. Another name for such a problem is the linear quadratic optimization problem (see for example [21]).

1.3 The Continuous Algebraic Lyapunov Equation

The continuous algebraic Lyapunov equation (CALE) [74] is

$$A^T P + PA = -Q \tag{1.3}$$

and is related to the n-dimensional continuous-time linear system

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t), \quad \mathbf{x}(0) = x_0$$

where x_0 is the initial state. A is an $n \times n$ stable matrix in which the real parts of the eigenvalues of A are negative, Q is a given $n \times n$ symmetric, positive (semi)definite matrix, and P is the unique $n \times n$ symmetric solution matrix of the CALE (1.3). If Q is positive definite, then P is positive definite. If Q is positive semi-definite, then P is positive semi-definite. If Q is positive semi-definite then P is positive definite if and only if the pair $(A, Q^{1/2})$ is observable, where $Q^{1/2}$ means the nonnegative square root of the matrix Q. It is known that the above system is asymptotically stable if and only if for each positive (semi)definite matrix Q there exists a positive (semi)definite solution P to (1.3).

1.4 The Continuous Algebraic Riccati Equation

The continuous algebraic Riccati equation (CARE) [74] is

$$A^T P + PA - PBB^T P = -Q \tag{1.4}$$

and is related to the n-dimensional continuous-time linear system

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \quad \mathbf{x}(0) = x_0$$

where x_0 is the initial state. A is a constant $n \times n$ matrix, B is an $n \times m$ matrix, Q is a given $n \times n$ symmetric positive (semi)definite matrix, and P is the unique positive (semi)definite symmetric solution matrix of (1.3). It is well-known in the literature (e.g., [34]) that the CARE has a unique symmetric positive (semi)definite stabilizing solution under the assumption that (A, B) is a stabilizable pair and (A, C) is a detectable pair, where $C \in \Re^{p \times n}$ such that $Q = C^T C$.

1.5 The Discrete Algebraic Lyapunov Equation

The discrete algebraic Lyapunov equation (DALE) [73] is

$$P = A^T PA + Q \tag{1.5}$$

and is related to the n-dimensional discrete-time linear system

$$\mathbf{x}(k+1) = A\mathbf{x}(k), \quad \mathbf{x}(0) = x_0$$

where x_0 is the initial state. A is a $n \times n$ stable matrix in which the eigenvalues of A lie within the closed unit circle, Q is a given symmetric positive (semi)definite matrix, and P is the unique symmetric positive (semi)definite solution matrix of the DALE. This work will not be directly concerned with deriving solution bounds of the DALE for a nominal case. However, bounds for the DALE (both nominal and perturbed) will be obtained as special cases of the discrete algebraic Riccati equation (DARE).

1.6 The Discrete Algebraic Riccati Equation

The discrete algebraic Riccati equation (DARE) [73] is

$$P = A^T PA - A^T PB(I + B^T PB)^{-1} B^T PA + Q \tag{1.6}$$

and is related to the n-dimensional discrete-time linear system

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k), \quad \mathbf{x}(0) = x_0$$

where x_0 is the initial state. A is an $n \times n$ constant matrix, B is an $n \times m$ matrix, Q is a given symmetric positive (semi)definite matrix, I_m is the identity matrix of order m, and P is the unique positive (semi) definite solution matrix of the DARE. It is well-known in the literature that their exists a unique positive (semi)definite symmetric stabilizing solution of the DARE under the assumption that (A, B) is a stabilizable pair and that (A, C) is a detectable pair, where $C \in \Re^{p \times n}$ such that $Q = C^T C$. By use of matrix inversion formulae [4,25], the DARE (1.6) may be rewritten in the following forms:

(i) If the solution is positive definite, the DARE may be rewritten as:

$$P = A^T (P^{-1} + BB^T)^{-1} A + Q \tag{1.7}$$

(ii) If the solution is positive semi-definite, the DARE may be rewritten as:

$$P = A^T (I + PBB^T)^{-1} PA + Q \tag{1.8}$$

It is noted that (1.7) can only be employed in the case when the DARE has a positive definite solution, but not a positive (semi)definite solution. In addition to the existence conditions, the DARE has a positive definite solution if at least one of the matrices A or Q is nonsingular, or, at the very least, if the pair (A, C) is observable.

1.7 The Continuous Coupled Algebraic Lyapunov Equation

The continuous coupled algebraic Lyapunov equation (CCALE) [5,69] is

$$A_i^T P_i + P_i A_i + \sum_{j \neq i} d_{ij} P_j = -Q_i \tag{1.9}$$

and is related to the jump linear system

$$\dot{\mathbf{x}} = \tilde{A}(r(t))\mathbf{x}$$

where $\tilde{A}(r(t)) = A_i$ when $r(t) = i$, $A_i = \tilde{A}_i + \frac{1}{2}d_{ii}I \in \Re^{n\times n}$, $Q_i \in \Re^{n\times n}$ is a given symmetric positive semidefinite matrix, and P_i are the unique positive semidefinite solution matrices of the CCALE (1.9). Here, d_{ij} are real constants such that $d_{ii} < 0$, $d_{ij} \geq 0$ for $i \neq j$ and $\sum_{j\in S} d_{ij} = 0$, where $i \in S$, and $S = \{1, 2, \ldots, n\}$ is a finite set. Different conditions for the existence of the CCALE solution can be found from [5,69].

1.8 The Continuous Coupled Algebraic Riccati Equation

The continuous coupled algebraic Riccati equation (CCARE) [1,6,13,14,25,76,88] is

$$A_i^T P_i + P_i A_i - P_i B_i B_i^T P_i + \sum_{j \neq i} d_{ij} P_j = -Q_i \tag{1.10}$$

and is related to the jump linear system

$$\dot{\mathbf{x}} = \tilde{A}(r(t))\mathbf{x} + B(r(t))\mathbf{u}$$

where $\tilde{A}(r(t)) = A_i$ and $B(r(t)) = B_i$ when $r(t) = i$, $A_i = \tilde{A}_i + \frac{1}{2}d_{ii}I \in \Re^{n\times n}$, $B_i \in \Re^{n\times m}$, $Q_i \in \Re^{n\times n}$ is a given symmetric positive semi-definite matrix, and P_i are the unique positive semi-definite solution matrices of the CCARE (1.10). Here, d_{ij} are real constants such that $d_{ii} < 0$, $d_{ij} \geq 0$ for $i \neq j$and $\sum_{j\in S} d_{ij} = 0$, where $i \in S$, and $S = \{1, 2, \ldots, n\}$ is a finite set. In the literature, there are a number of different conditions for the existence of the solutions of the CCARE (1.10). Examples of such conditions can be consulted from [1,6,76,88].

1.9 The Discrete Coupled Algebraic Lyapunov Equation

The discrete coupled algebraic Lyapunov equation (DCALE) [22] is

$$P_i = A_i^T F_i A_i + Q_i \tag{1.11}$$

and is related to the dynamical system

$$\mathbf{x}(k+1) = \tilde{A}(r(k))\mathbf{x}(k)$$

where $\tilde{A}(r(k)) = \tilde{A}_i$ when $r(k) = i$, $F_i = P_i + \sum_{j \neq i} e_{ij} P_j$, $A_i = \sqrt{e_{ii}} \tilde{A}_i \in \Re^{n \times n}$, $Q_i \in \Re^{n \times n}$ is a given symmetric positive semi-definite matrix, and P_i are the unique positive semi-definite solution matrices of the DCALE (1.11). Here, e_{ij} are non-negative constants such that $e_{ij} = (\tilde{e}_{ij}/\tilde{e}_{ii})$ with $\tilde{e}_{ij} \in [0,1]$, $\tilde{e}_{ii} > 0$, and $\sum_{j \in S} \tilde{e}_{ij} = 1$, where $i \in S$, and $S = \{1, 2, \ldots, n\}$ is a finite set. Like the DALE, this thesis will not be concerned with solution bounds of the DCALE for the nominal case; instead, bounds for the DCALE will be obtained as special cases of the bounds for the discrete coupled algebraic Riccati equation (DCARE). However, a solution bound for the DCALE with perturbations in the coefficients will be derived separately. As before, existence conditions for the DCALE solution can be found in references such as [22].

1.10 The Discrete Coupled Algebraic Riccati Equation

The discrete coupled algebraic Riccati equation (DCARE) [2,8,13,14,25,87,88] is

$$P_i = A_i^T F_i A_i - A_i^T F_i B_i (I_m + B_i^T F_i B_i)^{-1} B_i^T F_i A_i + Q_i \tag{1.12}$$

and is related to the jump linear system

$$\mathbf{x}(k+1) = \tilde{A}(r(k))\mathbf{x}(k) + B(r(k))\mathbf{u}(k)$$

where $\tilde{A}(r(k)) = A_i$ and $B(r(k)) = B_i$ when $r(t) = i$, $F_i = P_i + \sum_{j \neq i} e_{ij} P_j$, $A_i \in \Re^{n \times n}$, $B_i \in \Re^{n \times m}$, $Q_i \in \Re^{n \times n}$ is a given symmetric positive (semi)definite matrix, and P_i are the unique positive (semi)definite solution matrices of the DCARE (1.12). Here, e_{ij} are non-negative constants such that $e_{ij} = (\tilde{e}_{ij}/\tilde{e}_{ii})$

with $\tilde{e}_{ij} \in [0,1]$, $\tilde{e}_{ii} > 0$, and $\sum_{j\in S} \tilde{e}_{ij} = 1$, where $i \in S$, and $S = \{1, 2, \ldots, n\}$ is a finite set. Like the DARE (1.6), the DCARE can also be rewritten according to the matrix inversion Lemma (as seen in [22,58]). There are a number of different conditions for the existence of the solutions of the DCARE which can be found in the literature. Examples of such conditions can be found in [2,8,87,88].

This work will not be concerned with deriving solution bounds of the DALE and DCALE for a nominal case, although results for the DALE and DCALE will follow as special cases of the DARE and DCARE respectively, for both the nominal and perturbed cases. As such, this thesis will not review in detail existing works on the discrete Lyapunov equations.

1.11 Solution Bounds of Algebraic Matrix Equations

As has been discussed in both the introduction and abstract, it is either difficult or even impossible to solve the matrix equations (particularly the Riccati equations) when the dimensions of the system matrices are high or become higher. As such, many works have been presented over the past three decades for deriving lower and upper solution bounds of these equations, many of which are summarised in [47,71]. Types of bounds include

(1) Eigenvalue bounds including:
(1.1) extremal eigenvalue bounds (see [11,12,24,27,31,32,45,48,54,59,86,91,92]),
(1.2) eigenvalue summation bounds including the trace (see [13,14,24,27,30,32, 35,39,40,41,42,46,70,77,89]),
(1.3) eigenvalue product bounds including the determinant (see [27,37,38,39, 41,70,85]),
(1.4) norm bounds (see [30,75]).

(2) Matrix bounds (see [9,10,13,14-21,25,26,28,33,34,43,47,49-53,56-58,60-64, 67,72,78-80,86]).

Some of the above references are only examples of such bounds; many of the proposed bounds for these equations are summarised in [44,71]. Of all the types of bounds, the matrix bounds are the most general, since they can offer all other types of bound mentioned above. Furthermore, the derived bounds can be applied to deal with control problems involving the solutions of these equations, such as control problems discussed in [1-3,6,7,19,20,26,67,73,74,76]. Besides, the full, or rather the exact solution is not always required; a reasonably tight bound will suffice, particularly when solving the equation may be difficult. Looking at the literature, it appears that most of the existing works have presented bounds which are only valid under some rather restrictive assumptions, which are not common or realistic in control problems from which these equations arise. In addition, it appears that some existing bounds are not as tight as others in some cases, and the calculation of some bounds are also complex. As such, it is the aim in this work to seek the existence of solution bounds which can overcome these drawbacks. Attention will be paid nearly almost on matrix solution bounds, since they are the most general and can immediately offer all types of eigenvalue bounds mentioned above. The tighter the matrix bound is, the tighter the corresponding eigenvalue bounds are. It appears that only a minority of papers in the literature have proposed matrix bounds. Of course, there are a number of numerical methods in the literature that one may employ to obtain the exact solution of each equation, as mentioned in [13] and presented, for example, in [1,2,5,6,36,76,87,88], although reference [13] mentions that they are usually heavy in terms of their computation, and their efficiency depends on how close the starting matrix is to the actual solution. Furthermore, as mentioned in [13], the numerical solution of coupled Lyapunov and Riccati equations has not been studied as fully in the literature as it has been for the standard Lyapunov and Riccati equations. In light of the numerical algorithms, it is seen that one application of the solution bounds is that they might be used as starting values for these algorithms.

To summarise, there are the following motivations for this work:
(1) solution bounds of the equations can be applied to a number of control applications involving them.
(2) the exact solutions of the equations are sometimes not necessary, but instead a solution bound will suffice, particularly, when solving the equation is difficult. Solving the equation may be difficult because of the dimensions of the system matrices.
(3) the efficiency of numerical solution algorithms depend on how close the starting value is to the exact solution. The bounds may be used as the start-

ing value for such algorithms, and could also result in improved efficiency.
(4) many existing solution bounds in the literature have at least one of the following drawbacks:
(i) they may not be very tight or do not appear to be very tight,
(ii) they have been developed under at least one of the following types of assumptions: 1) they require restrictions on the coefficient matrices, 2) they require restrictions on some free variable or matrix involved, and may effect the tightness of the bound,
(iii) their calculation may be complicated and/or they involve some heavy computational burden(s).
(5) In addition, it is also mentioned in [58,62] that solution bounds of these equations can give rough estimates before actually solving them and can provide a check of whether the solution techniques for them actually results in valid solutions.

It is the aim that this work will determine new solution bounds that overcome at least one of the drawbacks of point 4 above. Throughout, the bounds for the equations are derived on the basis that the equations have a solution. For some of these equations, existence conditions are discussed, whilst some conditions for some equations can be consulted from the relevant literature. Also, throughout this thesis, the trivial lower bound is $P \geq 0$, and the trivial upper bound is $P \leq \infty I$.

Another important factor in this work is comparison of the tightness between parallel bounds of the same measure. For most of the time, a comparison of the tightness between parallel solution bounds of the same measure has not been possible by any mathematical method. When comparing the tightness between such bounds, the following issues need to be taken into account:
(i) The mathematical form of the bound, such as whether it involves matrix inverse, matrix eigenvalues, matrix square roots, and so on,
(ii) the involvement of any free parameters and/or matrices,
(iii) the restriction for validity of the bounds; if one bound does not exist for a case but another does, then there is no point to compare the tightness between them.

Despite the above, arguments, it is always possible, when the bounds exist, to make a comparison by numerical examples.

One example of a real-life application of these solution bounds is in the effect of the advertisement on the sales in the marketing process and the relationship between inventory and production in the production process, which is discussed in [63].

1.12 Outline of the Thesis

The remainder of the thesis has the following structure:

Chapter 2 recalls some useful lemmas that are used in the derivation of the main results. It also reviews existing works on solution bounds for algebraic Lyapunov and Riccati equations arising from deterministic and stochastic systems, for both nominal and perturbed cases, and the methodology in deriving the main results is also discussed; in particular, the unified approach, which has been employed in numerous papers in the literature, is also discussed.

Chapter 3 presents matrix bounds for the continuous and discrete algebraic Riccati equations when their coefficient matrices are subject to small perturbations. It seems that this is the first time that matrix bounds have been proposed for such equations.

Chapter 4 proposes new lower and upper matrix bounds for the continuous algebraic Lyapunov equation, which always work when the CALE solution exists. In particular, the lower matrix bounds are more concise than many existing lower matrix bounds, and are also more efficient in their calculation.

Chapter 5 discusses new lower matrix bounds for the continuous algebraic Riccati equation and new upper matrix bounds for the discrete algebraic Riccati equation. The new lower matrix bounds for the CARE always work when its solution exists, and always provide nontrivial lower matrix bounds for its solution, even when matrix Q is positive semidefinite. The new upper matrix bounds for the DARE are always calculable when its solution exists, whilst all existing upper matrix bounds are only valid under conditions additional to the usual existence conditions for the DARE solution.

Chapter 6 reports nontrivial lower matrix bounds for the continuous coupled algebraic Lyapunov equation which are always valid when its solution exists, a less conservative lower matrix bound for the continuous coupled algebraic Riccati equation, and nontrivial upper matrix bounds for the CCARE which seem to be the first nontrivial upper matrix bounds to exist for such an equation.

Chapter 7 derives two upper matrix bounds for the discrete coupled algebraic Riccati equation, which provide a supplement to what appears to be the only existing nontrivial upper matrix bound for the DCARE in the literature; these bounds also require different validity conditions to the existing upper matrix bound.

Chapter 8 presents some solution bounds for the continuous and discrete coupled algebraic Lyapunov and Riccati equations when all their coefficient matrices are subject to small perturbations. All of these results appear to be the first results to exist for such equations in this area of research.

Chapter 9 gives some concluding remarks regarding the work of this thesis, and also outlines some possible future work as a result of the work that has been undertaken in this thesis.

Following chapter 9 are an appendix and a list of references. Within the appendix are a list of presentations delivered and publications made by the author during the course of this work.

Chapter 2

Lemmas, Literature Review and Methodology

In this chapter, some lemmas that will be used in deriving the main results are recalled. A literature review of existing results in the field is also given, showing what is lacking with the present body of knowledge, as well as a discussion of the methodology used in deriving the main results.

2.1 Some Useful Lemmas

In this section, some useful Lemmas shall be recalled that will be used later in the derivation of the solution bounds for the algebraic Lyapunov and Riccati equations.

Lemma 2.1[4]: For any symmetric matrices X and Y and $1 \leq i,j \leq n$, the following inequalities hold:

$$\lambda_{i+j-n}(X+Y) \geq \lambda_j(X) + \lambda_i(Y) \qquad i+j \geq n+1, \tag{2.1}$$
$$\lambda_{i+j-1}(X+Y) \leq \lambda_j(X) + \lambda_i(Y) \qquad i+j \leq n+1. \tag{2.2}$$

Lemma 2.2[4]: For any symmetric matrix X, the following inequality holds:

$$\lambda_n(X)I \leq X \leq \lambda_1(X)I. \tag{2.3}$$

Lemma 2.3[4]: For any positive (semi-) definite $n \times n$ matrices X and Y such that $X \geq Y > (\geq)0$ and any matrix $A \in \Re^{n\times m}$, the following inequality holds:

$$A^T X A \geq A^T Y A \tag{2.4}$$

with strict inequality if X and Y are positive definite and A is of full rank.

Lemma 2.4[33]: For matrices $A, X, R, Y \in \Re^{n \times n}$ with $R \geq 0$ and $X \geq Y \geq 0$, the following inequality holds:

$$A^T(I + XR)^{-1} X A \geq A^T(I + YR)^{-1} Y A \tag{2.5}$$

with strict inequality if A is nonsingular and $X > Y$.

Lemma 2.5: If a_i and b_i are real non-negative constants, where $i \in S$ and S is a finite set, then

$$\sum_{i \in S} a_i b_i \leq \left(\sum_{i \in S} a_i\right)\left(\sum_{i \in S} b_i\right). \tag{2.6}$$

Proof: As in reference [20], the proof is quite trivial, so it is omitted.

Lemma 2.6: For the CCARE (1.10), we have the following result:

$$\sum_{i \in S} \sum_{j \neq i} d_{ij} P_j \leq (s-1) \max_{i,j \in S, j \neq i} \{d_{ij}\} \sum_{i \in S} P_i. \tag{2.7}$$

Proof: The proof is rather easy, so it is omitted.

Lemma 2.7: For the DCARE (1.12), we have the following result:

$$\sum_{i \in S} \sum_{j \neq i} e_{ij} P_j \leq (s-1) \max_{i,j \in S, j \neq i} \{e_{ij}\} \sum_{i \in S} P_i. \tag{2.8}$$

Proof: Like the proof of Lemma 2.6, the proof of this lemma is also quite easy, and is hence omitted.

Lemma 2.8[68]: For any $X, Y \in \Re^{n \times n}$, one has:

$$\lambda_i(XY) = \lambda_i(YX) \tag{2.9}$$

for $i = 1, 2, \ldots, n$.

Lemma 2.9[68]: For any $X, Y \in \Re^{n\times n}$, one has:

$$\mathrm{tr}(X+Y) = \mathrm{tr}(X) + \mathrm{tr}(Y), \tag{2.10}$$
$$\mathrm{tr}(XY) = \mathrm{tr}(YX), \tag{2.11}$$
$$\mathrm{tr}(X^T) = \mathrm{tr}(X). \tag{2.12}$$

Lemma 2.10[4]: Let $X, Y \in \Re^{n\times n}$ with Y a symmetric positive semidefinite matrix. Then

$$\frac{1}{2}\lambda_n(X+X^T)\mathrm{tr}(Y) \leq \mathrm{tr}(XY) \leq \frac{1}{2}\lambda_1(X+X^T)\mathrm{tr}(Y) \tag{2.13}$$

From (2.3) and (2.11), one can also deduce that for any $X = X^T \geq 0 \in \Re^{n\times n}$

$$\mathrm{tr}(X^2) \leq [\mathrm{tr}(X)]^2. \tag{2.14}$$

Lemma 2.11[26]: For matrices $A, X, R \in \Re^{n\times n}$ with $R > 0$ and $X \geq 0$, one has that

$$A^T(I+XR)^{-1}XA \leq A^TR^{-1}A \tag{2.15}$$

with strict inequality if A is nonsingular.

2.2 Discussion of the Solution Bounds Derived so far for a Deterministic Nominal System

Viewing the existing results in the literature, it appears that most of the proposed lower bounds for the CARE have to assume that Q is nonsingular for them to be able to work [10,45,50,51,57,58], otherwise the lower bounds are not meaningful and/or the lower bounds yield the trivial bound of 0.

For the CALE, many of the existing solution bounds have been developed under the assumptions that $Q > 0$, $A + A^T < 0$ and $AQ + QA^T < 0$, namely [10,28,39,48,50,51,56,60,78]. Reference [64] developed a lower and an upper matrix bound for the CALE which are valid under the assumption that A is a diagonalizable matrix. References [72,86] also developed concise lower and upper matrix bounds for the solution of the CALE. These bounds are always calculated if the solution of the CALE exists and they do not involve any tuning parameter or matrix, although computation of these bounds seem very heavy and complicated. Furthermore, references [77,79,80] also presented solution bounds for the CALE, some of which are also always calculable if the CALE solution exists. In particular, the upper bounds proposed in [80] do not involve any tuning parameter or matrix.

For the DARE, many of the existing solution bounds have been developed under the assumptions that Q and BB^T are nonsingular [43,47,53,57]. Some bounds have been developed under assumptions such as A is nonsingular [9], A is a d-stable matrix [33], or some inequality involving the eigenvalues and/or singular values of the systems matrices must be satisfied, such as in [26,52,63,67]. However, a lower matrix bound was developed for the DARE in [33] which does not need any condition for satisfaction, except for the usual existence conditions for the DARE solution.

In particular, the assumptions $Q > 0$ and $BB^T > 0$ are not common in many control problems because the number of state variables is greater than the number of inputs, and Q is generally a positive semi-definite matrix, rather than a positive definite matrix.

Following the above analysis of the literature, the following conclusions can be drawn for the algebraic Lyapunov and Riccati matrix equations arising from deterministic nominal systems:
(i) No lower and upper matrix bounds for the CALE exist that are always calculated when its solution exists and which are also efficient in their calculations.
(ii) It appears that all existing lower matrix bounds derived for the CARE in the literature have to assume that $Q > 0$ for them to work. This is restrictive, since Q is generally a positive semidefinite matrix. As such, the lower matrix bounds reported in [10,45,50,51,57,58] cannot work for this case.
(iii) All upper matrix bounds for the DARE have been developed under assumptions additional to the usual existence conditions for its solution.

Following this survey of the literature, the thesis addresses:
(i) The derivation of the lower and upper matrix bounds for the CALE that not only always work when its solution exists, but are also more concise and

less computationally complex to calculate,
(ii) The derivation of lower matrix bounds for the CARE that can be applied to the case when Q is positive semidefinite, as well as when Q is positive definite,
(iii) The derivation of an upper matrix bound(s) for the DARE that always works if its solution exists. This bound(s) may be tightened successively by using the DARE (after applying the matrix inversion Lemma (see for example [4,26])).

2.3 Discussion of the Solution Bounds Derived so far for Deterministic Perturbed Systems

Fewer works have focused on the estimation of solution bounds for the matrix equations with perturbations in the coefficients than that for nominal systems. There are a number of papers in the literature (such as [87] and the references therein) that study the sensitivity of the solutions of Lyapunov and Riccati equations when their coefficients undergo small perturbations; however, these works merely assess how much the perturbations effect the solution, rather than what the perturbation of the solution is. In this work, we are interested in seeking bounds on the perturbation of the solution of the equation, like that which has been done in [81-84]. Often in practical situations involving these equations, only rough values of the coefficient matrices are available, so they are perturbed versions of their actual values. Because of this, the solution matrix is also perturbed. References [81-84] derived some solution bounds of the Riccati equations when their coefficients undergo small perturbations, but have to assume that the perturbation ΔP is symmetric non-negative definite, which is restrictive, because although P and $P + \Delta P$ are symmetric non-negative definite, it does not imply that ΔP is symmetric non-negative definite, only that it is symmetric. Furthermore, it seems that nearly all of the proposed bounds are norm bounds, whilst [82,83] present some trace bounds.

Therefore, this thesis derives new solution bounds for the matrix equations under perturbations in the coefficients without the need for the assumption that ΔP is non-negative definite, only that it is symmetric.

2.4 Discussion of the Solution Bounds Derived so Far for Stochastic Nominal Systems

Surveying the literature, there appear to be very few works that focus on the estimation problem for Lyapunov-type and Riccati-type matrix equations arising from stochastic systems. In fact, it appears that only references [13,14,25,61] have focused on this problem. Unlike the Lyapunov and Riccati matrix equations for a deterministic system, these equations are a system of matrix equations. In [13], lower bounds for the eigenvalues and a lower bound for the solution matrix of the unified coupled Riccati equation are presented. In the limiting cases, bounds for the CCARE and DCARE are then obtained. The trivial bound $P_i \geq 0$ is obtained for the CCARE, whilst a nontrivial lower matrix bound is obtained for the DCARE, which has to assume that $1 - e_d > 0$. In [25], a lower and an upper matrix bound are obtained for the UCARE. Using these bounds for the UCARE, bounds for the CCARE and DCARE are obtained as limiting cases of the UCARE. The trivial bounds $P_i \geq 0$ and $P_i \leq \infty I$ are reported for the CCARE. For the DCARE, a nontrivial lower matrix bound is developed, and an upper matrix bound for the DCARE is developed, but has to assume that matrix $B_i B_i^T$ is nonsingular for all $i \in S$. Recently, an improved lower matrix bound was proposed in [61]. For the DCARE, a lower matrix bound is obtained which improves the assumption of [13], but it seems that the bound only works if Q_i is positive definite or if $\sigma_n^2(A_i) > 1$, which contradicts the authors' claim that the bound is improved, since a less restrictive bound for the DCARE already exists in [25], which does not need any condition for satisfaction and is also more straightforward in its calculation. However, one advantage of this work is that a lower matrix bound for the CCARE is developed which is nontrivial when $Q_i > 0$, but not nontrivial when $Q \geq 0$. In [14], upper bounds for the maximal eigenvalue and eigenvalue summations are derived for the UCARE, which are then used to infer upper bounds for the CCARE and DCARE. The drawback of these bounds is that it has to be assumed that $B_i B_i^T$ is nonsingular. Also, the calculation of the bounds derived in [13,14] seem somewhat complicated.

For the CCALE and DCALE, it appears that the only existing solution bounds for their solution are those bounds which already exist for the CCARE and DCALE, since the Lyapunov equations are special cases of the Riccati equations with $B_i = 0$. The lower matrix bounds proposed in [13] and [14] give the trivial lower bound $P_i \geq 0$ for the CCALE. The lower matrix bound for the CCALE derivable from the CCARE in [61] is meaningless, as is the lower bound for the DCALE. Furthermore, the upper bounds for the CCALE and DCALE obtainable from the upper bounds for the CCARE and DCARE in [14] become meaningless, since we then have $B_i = 0$.

Following the above examination of existing results, the following conclusions can be drawn for the coupled Lyapunov and Riccati equations:
(i) only one nontrivial lower bound exists for the CCARE, and is only nontrivial when $Q_i > 0$,
(ii) no nontrivial upper matrix bound exists for the CCARE,
(iii) only one nontrivial upper matrix bound exists for the DCARE, which has to assume that $B_i B_i^T$ is nonsingular,
(iv) only two upper eigenvalue bounds exist for the CCARE and DCARE, both of which have to assume that $B_i B_i^T$ is nonsingular,
(v) there does not exist any nontrivial lower matrix bound for the CCALE,
(vi) there do not exist any upper solution bounds for the CCALE and DCALE.

Therefore, the thesis presents a:
(i) derivation of a nontrivial lower matrix bounds for the CCARE when $Q > 0$, and when $Q \geq 0$,
(ii) derivation of nontrivial upper matrix bounds for the CCARE,
(iii) derivation of less restrictive upper solution bounds for the DCARE,
(iv) derivation of further upper eigenvalue bounds for the CCARE and DCARE, which are possibly tighter, possibly less restrictive, and more concise (and possibly easier to calculate),
(v) derivation of nontrivial lower matrix bounds for the CCALE,
(vi) derivation of upper solution bounds for the CCARE and DCARE.

2.5 Discussion of the Solution Bounds Derived so far for Stochastic Perturbed Systems

Like the deterministic counterpart, often only estimates for the values of the system matrices are available, so the system matrices we obtain are perturbed versions of the actual ones. As such, the coefficient matrices of the coupled Lyapunov and Riccati equations also become perturbed, and as a result their solution matrices also become perturbed. Viewing the literature, it seems that no works have discussed bounds for the solutions of coupled algebraic Lyapunov and Riccati equations when their coefficients are subject to small perturbations.

Therefore, the thesis derives some simple solution bounds for the coupled algebraic Lyapunov and Riccati equations when their coefficients undergo small perturbations.

2.6 Methodology

Solution bounds for the matrix equations take the form of matrix and eigenvalues inequalities. As such, matrix theory and inequalities regarding symmetric and non-negative definite matrices will be used to aid in deriving the solution bounds proposed in this thesis. In some case, some additional scalar inequalities and facts will also be used. These have been discussed in the first section of this chapter. Furthermore, nearly all proposed bounds in this thesis will be matrix bounds, since they are the most general type of bound and can immediately imply all types of eigenvalue bounds.

2.7 Solution Bounds by means of the Unified Approach

In the literature, there are a number of works that have derived solution bounds of Lyapunov and Riccati matrix equations for deterministic and stochastic systems by means of a unified approach. Examples of such works can be found in [11-14,25,44,57,59-61]. Basically, the corresponding equations (e.g., the continuous and discrete Lyapunov equations for a deterministic system) are unified by a single equation. For the Lyapunov equations (1.3) and (1.5), the following unified algebraic Lyapunov equation (UALE) has been utilized [60]:

$$A^T P + PA + \Delta A^T PA + Q = 0 \tag{2.16}$$

where Δ is a constant. When $\Delta = 0$, the UALE (2.16) becomes the CALE (1.3). When $\Delta = 1$ and A is replaced by $A - I$, the UALE (2.9) becomes the DALE (1.5). In addition, solution bounds for the CALE and DALE have also been obtained as limiting cases of the so-called 'generalized Lyapunov equations' in [56,66].

Similarly, the following unified algebraic Riccati equation (UARE) has been employed [57,59] in the past:

$$A^T P + PA + \Delta A^T PA - (\Delta A + I)^T PB(I + \Delta B^T PB)^{-1} B^T P(\Delta A + I) + Q = 0 \tag{2.17}$$

The CARE and DARE are unified by the UARE (2.17) in the same way that the CALE and DALE are unified by the UALE (2.17) respectively. In the literature [57,59,60], it has been seen that the UALE and UARE have provided solution bounds for both the continuous and discrete Lyapunov and Riccati equations that already existed in the literature, as well as being able to provide some new results in some cases [57,60]. However, the use of this approach has resulted in some somewhat conservative results for the CARE (and for the CCARE in the stochastic case), as can be seen in [13,25,57,61]. As such, this approach will not be used in the derivation of the main, contributable results; instead the continuous and discrete equations will be dealt with separately.

Chapter 3

Solution Bounds for Perturbed Continuous and Discrete Algebraic Riccati Equations

In this chapter, solution bounds for the continuous and discrete Riccati equations will be derived when their coefficient matrices undergo small perturbations, with the perturbation in Q being symmetric. This problem is of particular importance, since often in control problems involving the solution of the Riccati equations, only approximate values of the coefficient matrices are available, so they are perturbed versions of the actual ones. As a consequence of these perturbations in the coefficient matrices, the solution of the equation also becomes perturbed, so it becomes of interest to estimate the disturbance range for the solution of the equation. Viewing the literature, it appears that few works have been presented for deriving solution bounds of the Riccati equations when their coefficient matrices undergo perturbations [81-84]. Furthermore, it seems all of the works in this field have been concerned only with bounds for the norm of the perturbation in the solution when the coefficient matrices are subject to small perturbations subject to small perturbations, rather than bounds on the perturbations; see for example [84] and the references therein. In this chapter, bounds for the perturbation of the solution of the continuous and discrete Riccati equations will be derived when their coefficient matrices undergo perturbations. The obtained bounds will use the same ideas that have been used for a nominal system by researchers in the past, i.e., what researchers did to get bounds for nominal systems will be done to get bounds for perturbed systems. The continuous and discrete Lyapunov equations will not be dealt with separately, since they are special cases of the respective Riccati equations when $B = 0$ and A is stable. Finally, it is noted that the results obtained by other researchers for Lyapunov and Riccati matrix equations for nominal systems are not directly applicable to the case of perturbed systems.

3.1 Matrix Bounds for the Perturbed Continuous Algebraic Riccati Equation

In this section, upper matrix bounds for the perturbation of the solution of the continuous algebraic Riccati equation are derived when one, or all of its coefficient matrices undergo small perturbations. The bounds derived will use the same ideas and approaches that other researchers used to get results for the Lyapunov and Riccati equations for a nominal system. The results that follow in this section can also be found in [15].

Consider the perturbed CARE:

$$(A+\Delta A)^T(P+\Delta P)+(P+\Delta P)(A+\Delta A)-(P+\Delta P)(R+\Delta R)(P+\Delta P) = -(Q+\Delta Q) \tag{3.1}$$

where $R = BB^T$ and $\Delta R = B(\Delta B)^T + (\Delta B)B^T + (\Delta B)(\Delta B)^T$. Matrices A, Q, R and P have the same meaning as the CARE for a nominal system. Here ΔA is an $n \times n$ matrix which is the perturbation in ΔA, ΔB is $n \times m$ matrix which is the perturbation in B, ΔQ is a $n \times n$ symmetric matrix which is the perturbation in Q, and ΔP is an $n \times n$ symmetric matrix which is the perturbation in the solution P. Since ΔQ is a small perturbation, it will be assumed, without loss of generality, that $Q + \Delta Q \geq 0$.

Expanding out (3.1) and using the CARE (1.4) gives

$$L^T\Delta P + \Delta PL - \Delta P(R + \Delta R)\Delta P = -M \tag{3.2}$$

where

$$L \equiv A + \Delta A - (R + \Delta R)P \tag{3.3}$$

and

$$M \equiv (\Delta A)^T P + P(\Delta A) - P(\Delta R)P + \Delta Q. \tag{3.4}$$

It is noted that $Q + \Delta Q > (\geq)0$ implies $P + \Delta P > (\geq)0$, but not necessarily that $\Delta P > (\geq)0$. However, ΔP is a symmetric matrix. Since the perturbations are small, it is also assumed, without loss of generality, that $(A+\Delta A, B+\Delta B)$ is a stabilizable pair, $(A+\Delta A, (C+\Delta C)^{1/2})$ is a detectable pair, and $A + \Delta A - (R + \Delta R)(P + \Delta P)$ is an asymptotically stable matrix, so the solution $P + \Delta P$ of (3.1) is a unique, non-negative definite, symmetric stabilizing solution. As such, the solution ΔP of (3.2) will also be unique, as $\Delta P = (P + \Delta P) - P$. If, in addition, M is non-negative definite then ΔP is also non-negative definite, where M is defined by (3.4).

In the following theorem, derive an upper matrix bound for the perturbation ΔP in the solution of the perturbed CARE (3.2) will be derived.

Theorem 3.1: Define

$$W_1 \equiv L - \alpha(R + \Delta R) - I \tag{3.5}$$

where α is a positive constant and L is defined by (3.3). Let ΔP be the symmetric solution of the perturbed CARE (3.2). If there exists a scalar α such that

$$L + L^T < 2\alpha(R + \Delta R) \tag{3.6}$$

then ΔP has the upper bound

$$\Delta P \leq W_1^{-T} \left(\omega[(W_1 + I)^T(W_1 + I) + I] + M + \alpha^2(R + \Delta R)\right) W_1^{-1} \equiv \Delta P_{crul} \tag{3.7}$$

where the constant ψ is defined by

$$\omega \equiv \frac{\lambda_1\{W_1^{-T}[M + \alpha^2(R + \Delta R)]W_1^{-1}\}}{1 - \lambda_1\{W_1^{-T}[(W_1 + I)^T(W_1 + I) + I]W_1^{-1}\}}. \tag{3.8}$$

Proof: Define a positive semi-definite matrix ϕ_1 as:

$$\phi_1 \equiv (\Delta P - \alpha I)(R + \Delta R)(\Delta P - \alpha I)$$
$$\equiv \Delta P(R + \Delta R)\Delta P - \alpha\Delta P(R + \Delta R) - (R + \Delta R)\Delta P + \alpha^2(R + \Delta R) \geq 0. \tag{3.9}$$

Substituting the perturbed CARE (3.2) into (3.9) leads to:

$$[L - \alpha(R + \Delta R)]\Delta P + \Delta P[L - \alpha(R + \Delta R)] + M + \alpha^2(R + \Delta R) \geq 0. \tag{3.10}$$

Via the matrix identity

$$W_1^T \Delta P W_1 = (W_1+I)^T \Delta P (W_1+I) - [L-\alpha(R+\Delta R)]\Delta P - \Delta P[L-\alpha(R+\Delta R)] + \Delta)$$

where W_1 is defined by (3.5), (3.10) can be rewritten as

$$W_1^T \Delta P W_1 \leq (W_1+I)^T \Delta P (W_1+I) + \Delta P + M + \alpha^2(R+\Delta R). \quad (3.11)$$

Along the lines of the proof of Theorem 1 of [59], it is seen that if the condition (3.6) is met, then V is nonsingular, and we then have from (3.11) that

$$\Delta P \leq W_1^{-T}[(W_1+I)^T \Delta P (W_1+I) + \Delta P + M + \alpha^2(R+\Delta R)]W_1^{-1}. \quad (3.12)$$

Applying (2.3) to (3.12) gives

$$\Delta P \leq W_1^{-T}\left[(W_1+I)^T(W_1+I)+I\right]W_1^{-1}\lambda_1(\Delta P) + W_1^{-T}\left[M+\alpha^2(R+\Delta R)\right]W_1^{-1}. \quad (3.13)$$

Introducing (2.2) to (3.13) gives

$$\begin{aligned}\lambda_1(\Delta P) &\leq \lambda_1\left\{W_1^{-T}\left[(W_1+I)^T(W_1+I)+I\right]W_1^{-1}\lambda_1(\Delta P)\right. \\ &\qquad \left. + W_1^{-T}\left[M+\alpha^2(R+\Delta R)\right]W_1^{-1}\right\} \\ &\leq \lambda_1\left\{W_1^{-T}\left[(W_1+I)^T(W_1+I)+I\right]W_1^{-1}\right\}\lambda_1(\Delta P) + \\ &\qquad \lambda_1\left\{W_1^{-T}\left[M+\alpha^2(R+\Delta R)\right]W_1^{-1}\right\}. \end{aligned} \quad (3.14)$$

From (3.14), one has

$$\begin{aligned}\lambda_1(\Delta P)\left[1-\lambda_1\left\{W_1^{-T}\left[(W_1+I)^T(W_1+I)+I\right]W_1^{-1}\right\}\right] &\leq \\ \lambda_1\left\{W_1^{-T}\left[M+\alpha^2(R+\Delta R)\right]W_1^{-1}\right\}. \end{aligned} \quad (3.15)$$

To ensure that we obtain a valid upper bound, we require

$$\begin{aligned} &\lambda_1\left\{W_1^{-T}\left[(W_1+I)^T(W_1+I)+I\right]W_1^{-1}\right\} < 1 \\ \Rightarrow\ & W_1^{-T}\left[(W_1+I)^T(W_1+I)+I\right]W_1^{-1} < I \\ \Rightarrow\ & (W_1+I)^T(W_1+I)+I < W_1^T W_1 \end{aligned}$$

$$\Rightarrow \quad W_1^T W_1 + W_1 + W_1^T + I + I < W_1^T W_1$$
$$L - \alpha(R+\Delta R) - I + L^T - \alpha(R+\Delta R) - I + 2I < 0$$
$$L + L^T < 2\alpha(R+\Delta R).$$

Therefore, it can be seen that if the condition (3.6) is met, then

$$\lambda_1 \left\{ W_1^{-T} \left[(W_1+I)^T (W_1+I) + I \right] W_1^{-1} \right\} < 1.$$

As such, it is found from (3.15) that

$$\lambda_1(\Delta P) \leq \frac{\lambda_1 \left\{ W_1^{-T} \left[M + \alpha^2 (R+\Delta R) \right] W_1^{-1} \right\}}{1 - \lambda_1 \left\{ W_1^{-T} \left[(W_1+I)^T (W_1+I) + I \right] W_1^{-1} \right\}} \equiv \omega_1. \tag{3.16}$$

Substituting (3.16) into (3.13) results in the upper bound (3.7). This completes the proof of the theorem.

Following Theorem 3.1, an iterative algorithm can be proposed for obtaining tighter upper matrix bounds.

Corollary 3.1: The following algorithm can obtain tighter upper matrix bounds for the perturbation in the solution of the perturbed CARE (3.2).

Step 1: Set $X_0 \equiv \Delta P_{cu1}$, where ΔP_{cu1} is defined by.
Step 2: Compute

$$X_{k+1} = W_1^{-T} \left((W_1+I)^T X_k (W_1+I) + X_k + M + \alpha^2 (R+\Delta R) \right) W_1^{-1} \quad k = 0, 1, . \tag{3.17}$$

Then X_k are also upper bounds for the solution of the perturbed CARE (3.2).

Proof: Setting $k = 0$ in (3.17) gives

$$X_1 = W_1^{-T} \left((W_1+I)^T X_0 (W_1+I) + X_0 + M + \alpha^2 (R+\Delta R) \right) W_1^{-1}. \tag{3.18}$$

Applying (2.3) to (3.18) gives

$$X_1 \leq W_1^{-T} \left([(W_1+I)^T (W_1+I) + I] \lambda_1(X_0) + M + \alpha^2 (R+\Delta R) \right) W_1^{-1}. \tag{3.19}$$

By employing the definition $X_0 = \Delta P_{u1}$, the expression for ω_1 in Theorem 3.1 and (2.2), it is found from (3.19) that

$$\lambda_1(X_0) \le \lambda_1 \left\{W_1^{-T}\left(\omega_1[(W_1+I)^T(W_1+I)+I] + \alpha^2(R+\Delta R) + M\right)W_1^{-1}\right\}$$
$$\le \lambda_1 \left\{W_1^{-T}\left([(W_1+I)^T(W_1+I)+I]\right)W_1^{-1}\right\}\omega_1 + \lambda_1\left\{W_1^{-T}\left[\alpha^2(R+\Delta R)+M\right]W_1^{-1}\right\}$$

$$= \left\{1 - \frac{\lambda_1\{W_1^{-T}[M+\alpha^2(R+\Delta R)]W_1^{-1}\}}{\omega_1}\right\}\omega_1 + \lambda_1\{W_1^{-T}[M+\alpha^2(R+\Delta R)]W_1^{-1}\} = \omega_1. \tag{3.20}$$

Using (3.20), (3.19) becomes

$$X_1 \le W_1^{-T}\left([(W_1+I)^T(W_1+I)+I]\omega_1 + M + \alpha^2(R+\Delta R)\right)W_1^{-1} = X_0.$$

Now, assume that $X_k \le X_{k-1}$. Then

$$X_{k+1} = W_1^{-T}\left([(W_1+I)^T X_k (W_1+I) + X_k] + M + \alpha^2(R+\Delta R)\right)W_1^{-1}$$
$$\le W_1^{-T}\left([(W_1+I)^T X_{k-1}(W_1+I) + X_{k-1}] + M + \alpha^2(R+\Delta R)\right)W_1^{-1} = X_k.$$

By mathematical induction, it can be concluded that $X_{k+1} \le X_k \le \ldots \le X_1 \le X_0$. This completes the proof.

Remark 3.1: Theorem 3.1 gives an upper matrix bound for the perturbation of the solution of the CARE when all coefficient matrices undergo perturbations. For the cases when only one of the coefficient matrices undergo a perturbation, the corresponding perturbed CARE is obtained from (3.2) by setting the perturbations in the other coefficient matrices equal to zero. Then, the upper matrix bound is obtained in the same way. For the case when only the matrix A has a perturbation, the perturbed CARE is

$$N_1^T \Delta P + \Delta P N_1^T - \Delta P R \Delta P + S_1 = 0 \tag{3.21}$$

where $N_1 \equiv A + \Delta A - RP$ and $S_1 \equiv (\Delta A)^T P + P(\Delta A)$. Define $W_2 \equiv N_1 - \alpha R - I$, where α is a positive constant. If $S_1 \ge 0$ and there exists some α such that

$$N_1 + N_1^T < 2\alpha R$$

then the solution ΔP of (3.21) has the upper bound

$$\Delta P \le W_2^{-T}\left(\omega_2[(W_2+I)^T(W_2+I)+I] + \alpha^2 R + S_1\right)W_2^{-1} \equiv \Delta P_{u2}$$

where the positive constant ω_2 is defined by:

$$\omega_2 \equiv \frac{\lambda_1\left\{W_2^{-T}\left[S_1+\alpha^2(R+\Delta R)\right]W_2^{-1}\right\}}{1-\lambda_1\left\{W_2^{-T}\left[(W_2+I)^T(W_2+I)+I\right]W_2^{-1}\right\}}.$$

Also, Corollary 3.1 gives an iterative algorithm which can obtain tighter upper matrix bounds for the perturbation in the solution of the CARE when all coefficient matrices are subject to small perturbations. For the cases when only one of the coefficient matrices undergo a perturbation, the iterative algorithm for the case is obtained by setting the perturbations in the other coefficient matrices equal to zero. For the case when only the matrix A has a perturbation, the iterative algorithm for obtaining more precise estimates is as follows:

Step 1: Set $X_0 \equiv P_{u1}$.
Step 2: Compute

$$X_{k+1} = W_2^{-T}\left((W_2+I)^T X_k(W_2+I) + X_k + \alpha^2 R + S_1\right)W_2^{-1} \quad k=0,1,\ldots$$

Then X_k are also upper bounds for the solution of (3.21).

For the individual cases of R or Q only having perturbations, the corresponding matrix bounds and algorithms are obtained from the general case for all perturbations in the same way that the matrix bound and algorithm are obtained for the case when only A has a perturbation.

Corollary 3.2: When $R=0$ and A is a stable matrix, the CARE (1.4) becomes the CALE (1.3). By setting $R=0$ and $\Delta R=0$ in (3.2), the perturbed CALE is:

$$(A+\Delta A)^T\Delta P + \Delta P(A+\Delta A) + (\Delta A)^T P + P(\Delta A) + \Delta Q = 0. \qquad (3.22)$$

Upon setting $R=0$ and $\Delta R=0$ in (3.7), we have the following upper matrix bound for the solution of the perturbed CALE (3.21) when both coefficient matrices A and Q are subject to small perturbations:

$$\Delta P \le (A+\Delta A-I)^{-T}\left(\omega_3[(A+\Delta A)^T(A+\Delta A)+I] + (\Delta A)^T P + P(\Delta A) + \Delta Q\right)(A+\Delta A-I)^{-1}$$

where the constant ω_3 is defined by

$$\omega_3 = \frac{\lambda_1\left\{(A+\Delta A-I)^{-T}\left[(\Delta A)^T P + P(\Delta A) + \Delta Q\right](A+\Delta A-I)^{-1}\right\}}{1-\lambda_1\left\{(A+\Delta A-I)^{-T}\left[(A+\Delta A)^T(A+\Delta A)+I\right](A+\Delta A-I)^{-1}\right\}}.$$

This bound exists if $A + A^T + \Delta A + (\Delta A)^T < 0$. Since ΔA is a small perturbation in A, it may be assumed, without loss of any generality, that $A + \Delta A$ is also stable.

Similarly, setting $R = 0$ and $\Delta R = 0$ in the iterative algorithm of Corollary 3.1 gives an iterative algorithm that can be used to obtain tighter upper matrix bounds for the perturbed CALE.

Following the above results, a different upper matrix bound for ΔP is obtained as follows.

Theorem 3.2: If there exists a positive constant α such that condition (3.6) is met, then the solution ΔP of the perturbed CARE (3.2) satisfies the inequality

$$\Delta P \le W_1^{-T}\left(\omega_1(W_1+2I)^T(W_1+2I) + 2[M+\alpha^2(R+\Delta R)]\right)W_1^{-1} \equiv \Delta P_{cru2} \tag{3.23}$$

where the constant ω_1 is defined by (3.8).

Proof: Using the definition of W_1, (3.10) can be rewritten as

$$(W_1+I)^T\Delta P + \Delta P(W_1+I) + M + \alpha^2(R+\Delta R) \ge 0$$
$$\Rightarrow \quad W_1^T\Delta P + \Delta P W_1 + 2\Delta P + M + \alpha^2(R+\Delta R) \ge 0$$
$$\Rightarrow \quad 2W_1^T\Delta P + 2\Delta P W_1 + 4\Delta P + 2[M+\alpha^2(R+\Delta R)] \ge 0 \tag{3.24}$$

Adding $W_1^T P W_1$ to both sides of (3.24) gives

$$W_1^T P W_1 \le W_1^T P W_1 + 2W_1^T\Delta P + 2\Delta P W_1 + 4\Delta P + 2[M+\alpha^2(R+\Delta R)] \tag{3.25}$$

Using the matrix identity

$$(W_1+2I)^T\Delta P(W_1+2I) = W_1^T P W_1 + 2W_1^T\Delta P + 2\Delta P W_1 + 4\Delta P$$

(3.25) becomes

$$W_1^T \Delta P W_1 \leq (W_1 + 2I)^T \Delta P (W_1 + 2I) + 2[M + \alpha^2(R + \Delta R)] \tag{3.26}$$

With the satisfaction of (3.6), W_1 is nonsingular, and we have from (3.26) that

$$\Delta P \leq W_1^{-T} \left((W_1 + 2I)^T \Delta P (W_1 + 2I) + 2[M + \alpha^2(R + \Delta R)] \right) W_1^{-1} \tag{3.27}$$

Introduction of (2.3) to (3.27) gives:

$$\Delta P \leq W_1^{-T} \left(\lambda_1(\Delta P)(W_1 + 2I)^T (W_1 + 2I) + 2[M + \alpha^2(R + \Delta R)] \right) W_1^{-1} \tag{3.28}$$

Application of (2.2) to (3.28) gives:

$$\begin{aligned} &\lambda_1\{\Delta P\} \leq \lambda_1 \left\{ W_1^{-T} \left(\lambda_1(\Delta P)(W_1 + 2I)^T (W_1 + 2I) + 2[M + \alpha^2(R + \Delta R)] \right) W_1^{-1} \right\} \\ &\leq \lambda_1(\Delta P) \lambda_1 \{ W_1^{-T} (W_1 + 2I)^T (W_1 + 2I) W_1^{-1} \} + 2\lambda_1 \{ W_1^{-T} [M + \alpha^2(R + \Delta R)] W_1^{-1} \} \end{aligned} \tag{3.29}$$

From (3.29), it is found that

$$\lambda_1(\Delta P)[1 - \lambda_1\{W_1^{-T}(W_1 + 2I)^T(W_1 + 2I)W_1^{-1}\}] \leq 2\lambda_1\{W_1^{-T}[M + \alpha^2(R + \Delta R)]W_1^{-1}\} \tag{3.30}$$

To ensure that valid upper bound is obtained, it is required that

$$\begin{gathered} \lambda_1\{W_1^{-T}(W_1 + 2I)^T(W_1 + 2I)W_1^{-1}\} < 1 \\ \Rightarrow \quad W_1^{-T}(W_1 + 2I)^T(W_1 + 2I)W_1^{-T} < I \\ W_1^T W_1 + 2W_1 + 2W_1^T + 4I < W_1^T W_1 \\ L + L^T < 2\alpha(R + \Delta R) \end{gathered}$$

Therefore, it can be seen that if condition (3.6) is satisfied, then $\lambda_1\{W_1^{-T}(W_1 + 2I)^T(W_1 + 2I)W_1^{-1}\} < 1$.
Therefore, (3.30) implies that

$$\begin{aligned} \lambda_1(\Delta P) &\leq \frac{2\lambda_1\{W_1^{-T}[M + \alpha^2(R + \Delta R)]W_1^{-1}\}}{1 - \lambda_1\{W_1^{-T}(W_1 + 2I)^T(W_1 + 2I)W_1^{-1}\}} \\ &= \frac{\lambda_1 \left\{ W_1^{-T} [M + \alpha^2(R + \Delta R)] W_1^{-1} \right\}}{1 - \lambda_1 \left\{ W_1^{-T} [(W_1 + I)^T (W_1 + I) + I] W_1^{-1} \right\}} = \omega_1 \end{aligned} \tag{3.31}$$

Substituting (3.31) into (3.28) leads to the upper bound (3.23). This completes the proof of the theorem.

Following Theorem 3.2, the following iterative algorithm can be proposed for obtaining tighter upper matrix bounds for the solution of the perturbed CARE (3.2).

Corollary 3.3: The following algorithm can obtain tighter upper matrix bounds for the solution of the perturbed CARE (3.2).

Step 1: Set $\overline{X_0} \equiv \Delta P_{cru2}$, where ΔP_{cru2} is defined by (3.23).
Step 2: Calculate

$$\overline{X}_{k+1} = W_1^{-T}[(W_1+2I)^T\overline{X}_k(W_1+2I)+2(M+\alpha^2(R+\Delta R))]W_1^{-1} \quad k = 0, 1, 2, \ldots$$

Then $\overline{X}_k$ are also upper bounds for the solution of the perturbed CARE (3.2).
The proof of this algorithm is similar to that of the first algorithm, and is therefore omitted.

Remark 3.2: Theorem 3.2 and Corollary 3.2 give respectively an upper matrix bound and an iterative algorithm for the perturbation of the solution of the CARE (3.2) when all its coefficient matrices undergo small perturbations. For the case when only one coefficient matrix undergoes a perturbation, the corresponding perturbed CARE, upper matrix bound, and iterative algorithm for obtaining better bounds, is obtained in the same way as in Corollary 3.1.

Remark 3.3: Theorem 3.2 and Corollary 3.2 give respectively an upper matrix bound and an iterative algorithm for the solution of the perturbed CARE (3.2) when all its coefficient matrices undergo small perturbations. When $R = 0$, $\Delta R = 0$, and A is a stable matrix, an upper matrix bound and an iterative algorithm for the solution of the perturbed CALE (3.21) when both of its coefficient matrices undergo small perturbations are obtained from Theorem 3.2 and Corollary 3.2 respectively.

Remark 3.4: It is seen from (3.7) and (3.23) that

$$\begin{aligned}\Delta P_{cru2} &= W_1^{-T}[\omega_1(W_1+I)^T(W_1+I)^T + M + \alpha^2(R+\Delta R) + \omega_1(W_1^T + W_1 + 2I) \\ &\quad + M + \alpha^2(R+\Delta R)]W_1^{-1} \\ &= \Delta P_{cru1} + W_1^{-T}[\omega_1(W_1^T + W_1 + 2I) + \alpha^2(R+\Delta R) + M]W_1^{-1}\end{aligned}$$

As such, if $[\omega_1(W_1^T+W_1+2I)+\alpha^2(R+\Delta R)+M] \geq 0$ then the bound (3.7) is tighter than the bound (3.23), whereas if $[\omega_1(W_1^T + W_1 + 2I) + \alpha^2(R + \Delta R) + M] \leq 0$, then the bound (3.23) is more precise than the bound (3.7).

Remark 3.5: If $R > 0$ and ΔR is a small perturbation, without loss of any generality, we still assume that $R+\Delta R > 0$. If $R+\Delta R > 0$ then there must always exist some positive value of α such that condition (3.6) holds, and so the bounds will always exist when $R + \Delta R > 0$. The bounds will also work for the case when $R + \Delta R \geq 0$, if the condition (3.6) is fulfilled. Viewing the literature, there appear to be no available matrix bounds for the solution of the CARE when its coefficient matrices undergo perturbations. It should also be noted, for the case of perturbation bounds, that matrix bounds are the most general type of solution bound, since they can offer all other types of solution bounds. Therefore, this work is an improvement over existing works on the topic of solution bounds for the perturbed CARE. These bounds can also provide a supplement to existing works.

Remark 3.6: The following procedure can be used to test the satisfaction of the condition (3.6).
Step 1: Select α to be a sufficiently small positive constant and β to be a suitable positive constant.
Step 2: Compute $\lambda_i(2\alpha(R+\Delta R) - L - L^T)$ for $i = 1, 2, \ldots, n$.
Step 3: If $\lambda_i(2\alpha(R+\Delta R) - L - L^T) > 0$ for all i, then the condition (3.6) is met and this procedure can then be stopped; otherwise, set $\alpha = \alpha + \beta$ and go to Step 4.
Step 4: If α is sufficiently large, then stop and give up this procedure, else go to Step 2.
An alternative way of testing the positive definiteness of the matrix $2\alpha(R+\Delta R) - L - L^T$ to the above procedure is to use the determinant criterion [68] for a positive (semi)definite matrix. Furthermore, to reduce computational efforts, one may choose $\alpha = 1$ for checking the condition (3.6) and computing the bounds (3.7) and (3.23).

3.1.1 Numerical Example

In this section, a numerical example will be given to show the effectiveness of the obtained bounds and algorithms for the perturbation of the solution of the perturbed CARE (3.2).

Consider the CARE (1.4) and perturbed CARE (3.2) with (adapted from [54, Example 1]):

$$A = \begin{bmatrix} 4 & 0 \\ 8 & -20 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 16 & 0 \\ 0 & 48 \end{bmatrix}$$

$$\Delta A = \begin{bmatrix} 0.0008 & 0 \\ -0.0012 & 0.0015 \end{bmatrix}, \quad \Delta B = \begin{bmatrix} 0.0023 \\ 0 \end{bmatrix}, \quad \Delta Q = \begin{bmatrix} -0.0024 & 0 \\ 0 & -0.0032 \end{bmatrix}$$

Then the positive definite solution of the CARE (1.4) is:

$$P_{\text{exact}} = \begin{bmatrix} 1.3938 & 0.2456 \\ 0.2456 & 1.1758 \end{bmatrix}$$

With $\alpha = 1$, the condition (3.6) is satisfied, and the bounds ΔP_{cu1} and ΔP_{cu2} provide the following upper matrix bounds for the solution ΔP of (3.2):

$$\Delta P_{cru1} = \begin{bmatrix} 0.2192 & -0.0034 \\ -0.0034 & 0.2011 \end{bmatrix}, \quad \Delta P_{cru2} = \begin{bmatrix} 0.2914 & -0.0066 \\ -0.0066 & 0.1824 \end{bmatrix}.$$

For this case, it is seen that ΔP_{cu2} is tighter than ΔP_{cu1}. Using three iterations of the algorithm of Corollary 3.3 leads to the following tighter upper matrix bounds for the solution ΔP:

$$\overline{X}_1 = \begin{bmatrix} 0.2836 & -0.0109 \\ -0.0109 & 0.1515 \end{bmatrix},$$

$$\overline{X}_2 = \begin{bmatrix} 0.2768 & -0.0141 \\ -0.0141 & 0.1259 \end{bmatrix},$$

$$\overline{X}_3 = \begin{bmatrix} 0.2708 & -0.0164 \\ -0.0164 & 0.1046 \end{bmatrix}.$$

Clearly, as more iterations are carried out, the bounds become tighter.

3.2 Matrix Bounds for the Perturbed Discrete Algebraic Riccati Equation

In this section, derive matrix bounds for the perturbation of the solution of the discrete algebraic Riccati equation are derived when one, or all of its coefficient matrices undergo small perturbations. The bounds derived will use the same ideas and approaches that other researchers used to get results for the Lyapunov and Riccati equations for a nominal system.

Consider the perturbed DARE:

$$P + \Delta P = (A + \Delta A)^T[I + (P + \Delta P)(B + \Delta B)(B + \Delta B)^T]^{-1}(P + \Delta P) \times (A+\Delta A)+(Q+\Delta Q) \quad (3.32)$$

where each coefficient matrix involved has the same meaning as for the perturbed CARE in section 3.1. Here, this equation cannot be as simply expanded as in the perturbed CARE, because the equation is more complicated. Instead, the perturbed DARE (3.32) will be dealt with in its present form. Since ΔQ is a small perturbation, it will be assumed, without loss of generality, that $Q + \Delta Q > 0$. Also, it is noted that $Q + \Delta Q \geq 0$ implies that $P + \Delta P \geq 0$. By extending the method of [67], an upper matrix bound for the perturbation ΔP in (3.32) will be derived as follows. Following this, a lower matrix bound for the perturbation of the solution in the DARE is presented by following the approach of [33].

For brevity throughout, we shall denote $A_1 \equiv A + \Delta A$, $D_1 \equiv (B + \Delta B)(B + \Delta B)^T$ and $Q_1 \equiv Q + \Delta Q$.

Theorem 3.3: Let $P + \Delta P$ be the positive semidefinite solution of the perturbed DARE (3.32). If $b_1 > 0$, then ΔP has the upper bound

$$P + \Delta P = A_1^T[I + (P + \Delta P)_1 D_1]^{-1}(P + \Delta P)_1 A_1 + Q_1 \quad (3.33)$$

where the positive semidefinite matrix ΔP_1 is defined by

$$(P + \Delta P)_1 \equiv A_1^T[\alpha_1^{-1} I + D_1]^{-1} A_1 + Q_1 \quad (3.34)$$

and the positive constant α_1 is defined by

$$\alpha_1 = f(a_1, b_1, c_1)$$

with

$$a_1 = 1 - \lambda_1[A_1^T A_1 + \lambda_1(D_1)Q_1],$$
$$b_1 = 2\lambda_1[A_1^T(B+\Delta B)(I+\lambda_n(Q_1)D_1)^{-1}(B+\Delta B)A_1]$$
$$c_1 = 2\lambda_1(Q_1)$$

Proof: Applying (2.3) and (2.5) to (3.32) gives

$$P + \Delta P \leq \lambda_1(P+\Delta P)A_1^T[I + \lambda_1(P+\Delta P)D_1]^{-1}A_1 + Q_1 \qquad (3.35)$$

Application of (2.3) and (2.5) to (3.35) leads to

$$\begin{aligned} P + \Delta P &\leq \lambda_1(P+\Delta P)A_1^T\{I - \lambda_1(P+\Delta P)(B+\Delta B)[I + \lambda_1(P)(B+\Delta B)^T \\ &\qquad (B+\Delta B)]^{-1}(B+\Delta B)^T\}A_1 + Q_1 \\ &= \lambda_1(P+\Delta P)A_1^T A_1 - \lambda_1^2(P+\Delta P)A_1^T(B+\Delta B)[I + \lambda_1(P+\Delta P)(B+\Delta B)^T \\ &\qquad (B+\Delta B)]^{-1}(B+\Delta B)^T A_1 + Q_1 \end{aligned} \qquad (3.36)$$

Using the fact that $(B+\Delta B)^T(B+\Delta B) \leq \sigma_1^2(B+\Delta B)I$ from (2.3), it is found from (3.36) that

$$\begin{aligned} P + \Delta P \leq \lambda_1(P+\Delta P)A_1^T A_1 - \lambda_1^2(P+\Delta P)A^T B[1 + \lambda_1(P+\Delta P)\sigma_1^2(B+\Delta B)]^{-1} \\ (B+\Delta B)^T A_1 + Q_1 \end{aligned} \qquad (3.37)$$

Multiplying both sides of (3.37) by $[1 + \lambda_1(P)\sigma_1^2(B)]$ results in

$$\begin{aligned} (P+\Delta P)[1 + \lambda_1(P+\Delta P)\sigma_1^2(B+\Delta B)] \leq \lambda_1(P+\Delta P)[1 + \lambda_1(P+\Delta P) \\ \sigma_1^2(B+\Delta B)]A_1^T A_1 - \lambda_1^2(P+\Delta P)A_1^T D_1 A_1 + [1 + \lambda_1(P+\Delta P)\sigma_1^2(B+\Delta B)]Q_1 \end{aligned} \quad (3.38)$$

Introducing (2.1) to (3.38) leads to

$$\begin{aligned} \left(\sigma_1^2(B+\Delta B) - \lambda_1[\sigma_1^2(B+\Delta B)A_1^T A_1 - A_1^T D_1 A_1]\right)\lambda_1^2(P+\Delta P) + [1 - \lambda_1[A_1^T A_1 + \\ \sigma_1^2(B+\Delta B)Q_1]\lambda_1(P+\Delta P) - \lambda_1(Q_1) \leq 0 \end{aligned} \qquad (3.39)$$

Solving the inequality (3.39) leads to

$$\lambda_1(P+\Delta P) \leq f(a_1, b_1, c_1) \quad \text{if} \quad b_1 > 0 \qquad (3.40)$$

Substituting (3.40) into (3.35) and subtracting P leads to (3.34). Substituting (3.34) into (3.32) and subtracting P from both sides results in the bound (3.33). This completes the derivation of the bound, and hence the proof of the theorem.

Reference [33] proposed a lower matrix bound for the solution of the DARE (1.6) which is always computable if its solution exists. Here, the work of [33] will be extended to derive a similar lower matrix bound when the coefficient matrices of the DARE undergoes small perturbations. Since the matrix $(A+\Delta A)^T[I+(P+\Delta P)(B+\Delta B)(B+\Delta B)^T]^{-1}(P+\Delta P)(A+\Delta A)$ is positive semi-definite, one has from (3.32) that $P+\Delta P \geq Q+\Delta Q$. Combining this fact with (2.5) leads to the following lower matrix bound for the solution $P+\Delta P$ of the perturbed DARE (3.32):

$$P + \Delta P \geq (A + \Delta A)^T[I + (Q + \Delta Q)(B + \Delta B)(B + \Delta B)^T]^{-1}(Q + \Delta Q)(A + \Delta A) \\ +(Q + \Delta Q) \equiv (P + \Delta P)_{drL1} \qquad (3.41)$$

The following algorithm can obtain tighter lower matrix bounds for the solution of the perturbed DARE:

Algorithm 3.3:
Step 1: Set $X_0 \equiv (P + \Delta P)_{L1}$.
Step 2: Calculate

$$X_{k+1} = (A+\Delta A)^T[I+X_k(B+\Delta B)(B+\Delta B)^T]^{-1}X_k(A+\Delta A)+(Q+\Delta Q), \quad k = 0, 1, \ldots$$

Then X_k are also lower bounds for the solution of the perturbed DARE (3.32). At each iteration, $\Delta P \geq X_k - P$.

Remark 3.3: When $B = 0$ and A is a stable matrix, the DARE (1.6) becomes the DALE (1.5). As such, the bounds (3.33) and (3.41) becomes the following solution bounds for the DALE when its coefficient matrices are subject to small perturbations:

$$\Delta P \leq \frac{\lambda_1(Q + \Delta Q)}{1 - \sigma_1^2(A + \Delta A)}(A + \Delta A)^T(A + \Delta A) + Q + \Delta Q - P,$$

$$\Delta P \geq (A + \Delta A)^T(Q + \Delta Q)(A + \Delta A) + (Q + \Delta Q) - P.$$

3.2.1 Numerical Example for the Perturbed DARE

In this subsection, a numerical example is considered to show the effectiveness of the obtained lower bound when the coefficient matrices of the DARE undergo small perturbations.

Consider the perturbed DARE (3.32) with (Example 1 from [67]):

$$A = \begin{bmatrix} 1 & 0.1 & 0.2 \\ 0.3 & 0 & -0.1 \\ 0 & 0.4 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 1 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 5 & 0 & 1 \\ 0 & 4 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

$$\Delta A = \begin{bmatrix} -0.002 & 0.0002 & 0.0004 \\ -0.0004 & 0 & -0.0003 \\ 0 & -0.0005 & 0.0005 \end{bmatrix}, \quad \Delta B = \begin{bmatrix} 0.0007 \\ -0.005 \\ -0.004 \end{bmatrix},$$

$$\Delta Q = \begin{bmatrix} -0.006 & 0 & 0.0004 \\ 0 & -0.005 & 0.0003 \\ 0.0004 & 0.0003 & -0.002 \end{bmatrix}$$

The positive definite solution P of the DARE with the above nominal coefficient matrices is

$$P = \begin{bmatrix} 10.2483 & 0.3209 & 2.3770 \\ 0.3209 & 4.2557 & 1.1042 \\ 2.3770 & 1.1042 & 3.3883 \end{bmatrix}$$

The bound (3.41) gives:

$$\Delta P \geq \begin{bmatrix} -1.958472 & -0.180177 & -0.634634 \\ -0.180177 & -0.029057 & 0.051716 \\ -0.634634 & 0.051716 & -0.134289 \end{bmatrix}$$

Using 2 iterations of Algorithm 3.3 gives the following tighter lower matrix bounds for ΔP respectively:

$$\Delta P \geq \begin{bmatrix} -0.782429 & -0.077029 & -0.322591 \\ -0.077029 & -0.018274 & 0.080765 \\ -0.322591 & 0.080765 & -0.050848 \end{bmatrix}$$

$$\Delta P \geq \begin{bmatrix} -0.140878 & -0.033672 & -0.467599 \\ -0.033672 & -0.017999 & 0.062609 \\ -0.467599 & 0.062609 & -0.074143 \end{bmatrix}$$

Clearly, as more iterations are carried out, the bounds become tighter.

3.3 Summary

In this chapter, matrix bounds have been successfully derived for the perturbations in the solutions of the CARE and DARE when their coefficient matrices were subject to small perturbations. These results extend the works of others to the case of small perturbations. Following each bound derivation, iterative methods were proposed for finding more precise estimates. Finally, numerical examples were given to show the effectiveness of the results obtained in this chapter.

Chapter 4

Matrix Bounds for the Continuous Algebraic Lyapunov Equation

In this chapter, new solution bounds for the CALE which are less restrictive than existing results will be derived. The new matrix bounds always exist if the CALE solution exists. The lower matrix bounds developed in this chapter can also be found in reference [18].

Throughout this chapter, consider the CALE (1.3)

$$A^T P + PA = -Q$$

with $Q \geq 0$ and $P \geq 0$.

4.1 Lower Matrix Bounds for the CALE

In this section, two new lower matrix bounds for the CALE will be derived, which are less restrictive than many existing results, as well as being computationally more efficient. Firstly, a lower matrix bound for the CALE is derived as follows.

Theorem 4.1: Define

$$U \equiv A - \alpha I \tag{4.1}$$

where α is a positive constant. Let P be the positive semidefinite solution of the CALE (1.3). Then P has the lower bound

$$P \geq U^{-T}[A^T P_0 A + \alpha^2 P_0 + \alpha Q]U^{-1} \equiv P_{CLL1} \tag{4.2}$$

where the positive semidefinite matrix P_0 is defined by

$$P_0 \equiv \alpha U^{-T} Q U^{-1}. \tag{4.3}$$

Proof: Using the definition of U from (4.1), the following matrix identity can be defined

$$U^T P U = A^T P A - \alpha(A^T P + PA) + \alpha^2 P. \tag{4.4}$$

Substituting the CALE (1.3) into (4.4) gives

$$U^T P U = A^T P A + \alpha^2 P + \alpha Q. \tag{4.5}$$

One has $\Re(\lambda(A - \alpha I)) = \Re(\lambda(A)) - \alpha I$, where the fact $\lambda(X + cI) = \lambda(X) + c$ has been used (see for example [60]). The stability of matrix A means that $\Re(\lambda(A)) < 0$, which implies that $\Re(\lambda(U)) < 0$, so U is also a stable matrix, and hence nonsingular for any value of the positive constant α. Then, pre- and post-multiplying (4.5) by U^{-T} and U^{-1} respectively yields

$$P = U^{-T}[A^T P A + \alpha^2 P + \alpha Q]U^{-1}. \tag{4.6}$$

From (4.6), it is found that $P \geq P_0$, where P_0 is defined by (4.3). Substituting (4.3) into (4.6) leads to the lower bound (4.2). This completes the proof of the theorem.

Following Theorem 4.1, the following iterative algorithm can be proposed to obtain tighter solution estimates for the CALE.

Algorithm 4.1:
Step 1: Set $M_0 \equiv P_0$, where P_0 is defined by (4.3).
Step 2: Calculate

$$M_k = U^{-T}[A^T M_{k-1} A + \alpha^2 M_{k-1} + \alpha Q]U^{-1} \quad k = 1, 2, \ldots \tag{4.7}$$

Then M_k are also lower bounds of the solution of the CALE (1.3). In fact, as $k \to \infty$, $M_k \to P$, where P is the positive semidefinite solution of the CALE.

Proof: Firstly, $P \geq \alpha U^{-T} Q U^{-1} = M_0$. Then, using (4.6) gives

$$\begin{aligned} P &= U^{-T}\left(A^T P A + \alpha^2 P + \alpha Q\right) U^{-1} \\ &\geq U^{-T}\left(A^T M_0 A + \alpha^2 M_0 + \alpha Q\right) U^{-1} \\ &= M_1 \geq \alpha U^{-T} Q U^{-1} = M_0. \end{aligned} \tag{4.8}$$

Now assume $P \geq M_{k-1} \geq M_{k-2}$. Then, by following the routine of (4.8) and remembering (4.7), one finds that

$$\begin{aligned} P &= U^{-T}\left(A^T P A + \alpha^2 P + \alpha Q\right) U^{-1} \\ &\geq U^{-T}\left(A^T M_{k-1} A + \alpha^2 M_{k-1} + \alpha Q\right) U^{-1} = M_k \\ &\geq U^{-T}\left(A^T M_{k-2} A + \alpha^2 M_{k-2} + \alpha Q\right) U^{-1} = M_{k-1}. \end{aligned}$$

By means of mathematical induction, it can be concluded that $0 \leq M_0 \leq M_1 \leq \ldots \leq M_{k-1} \leq M_k \leq P$ for $k = 1, 2, \ldots$. Since $\{M_k\}$ is monotone increasing and bounded (see for example [26]), there exists a matrix $M_\infty \geq 0$ with $M_\infty = \lim_{k\to\infty} M_k$, such that

$$M_\infty = U^{-T}[A^T M_\infty A + \alpha^2 M_\infty + \alpha Q]U^{-1}. \tag{4.9}$$

Here, (4.9) is equivalent to (4.6) with $M_\infty = P$. As such, it can be concluded that $P = \lim_{k\to\infty} M_k$. This concludes the proof.

A different lower bound is derived as follows.

Theorem 4.2: The solution P of the CALE (1.3) satisfies

$$P \geq U^{-T}[2(A + \alpha I)^T P_0 (A + \alpha I) + 2\alpha Q]U^{-1} \equiv P_{CLL2} \tag{4.10}$$

where the positive semidefinite matrix P_0 is defined by (4.3).

Proof: Using the definition of U from (4.1), the CALE (1.3) can be rewritten as

$$\begin{aligned} &(U + \alpha I)^T P + P(U + \alpha I) + Q = 0 \\ \Rightarrow\ &2U^T P + 2PU + 4\alpha P + 2Q = 0 \\ \Rightarrow\ &2\alpha U^T P + 2\alpha PU + 4\alpha^2 P + 2\alpha Q = 0. \end{aligned}$$

$$\Rightarrow \quad U^TPU + 2\alpha U^TP + 2\alpha PU + 4\alpha^2 P + 2\alpha Q = U^TPU \tag{4.11}$$

Using the matrix identity

$$(U + 2\alpha I)^T P(U + 2\alpha I) = U^TPU + 2\alpha U^TP + 2\alpha PU + 4\alpha^2 P$$

(4.11) can be rewritten as

$$U^TPU = (U + 2\alpha I)^T P(U + 2\alpha I) + 2\alpha Q. \tag{4.12}$$

Since U is nonsingular for any $\alpha > 0$, pre- and post-multiplying (4.12) by U^{-T} and U^{-1}, respectively, leads to

$$P = U^{-T}[(A + \alpha I)^T P(A + \alpha I) + 2\alpha Q]U^{-1}. \tag{4.13}$$

From (4.13), it is seen that $P \geq 2P_0$, where P_0 is defined by (4.3). Substituting this bound into (4.13) results in the bound (4.10). This finishes the proof of the theorem.

Having completed the proof of Theorem 4.2, the following iterative algorithm can be proposed to obtain tighter lower matrix bounds for the solution of the CALE (1.3).

Algorithm 4.2:
Step 1: Set $\overline{M}_0 \equiv 2P_0$, where P_0 is defined by (4.3).
Step 2: Calculate

$$\overline{M}_k = U^{-T}[(A + \alpha I)^T \overline{M}_{k-1}(A + \alpha I) + 2\alpha Q]U^{-1} \qquad k = 1, 2, \ldots.$$

Then $\overline{M}_k$ are also lower solution bounds of the CALE (1.3). In fact, as $k \to \infty$, $\overline{M}_k \to P$, where P is the positive semidefinite solution of the CALE.

Proof: The proof of the correctness of this algorithm is similar to that of Algorithm 4.1, and is therefore omitted.

Remark 4.1: It is obvious that the bounds (4.2) and (4.10) always exist if the positive semidefinite solution of the CALE exists. Furthermore, these bounds are also more concise than many of the existing lower matrix bounds for the CALE that have been reported in the literature, and require no condition for satisfaction of the coefficient matrices of the CALE (1.3).

Remark 4.2: By using (4.3), it can be seen from (4.2) and (4.10) that

$$\begin{aligned}
P_{CCLL2} &\equiv U^{-T}[2(A+\alpha I)^T P_0 (A+\alpha I) + 2\alpha Q]U^{-1} \\
&= U^{-T}[2A^T P_0 A + 2\alpha(A^T P_0 + P_0 A) + 2\alpha^2 P_0 + 2\alpha Q]U^{-1} \\
&= U^{-T}[A^T P_0 A + \alpha^2 P_0 + \alpha Q + A^T P_0 A + \alpha^2 P_0 + \alpha Q + 2\alpha(A^T P_0 + P_0 A)]U^{-1} \\
&= P_{CCLL1} + U^{-T}[(A+\alpha I)^T P_0 (A+\alpha I) + \alpha(A^T P_0 + P_0 A) + \alpha Q]U^{-1}.
\end{aligned}$$

Therefore, if $(A+\alpha I)^T P_0 (A+\alpha I) + \alpha(A^T P_0 + P_0 A) + \alpha Q \geq 0$, then P_{CCLL2} is tighter than P_{CCLL1}, whereas if $(A+\alpha I)^T P_0 (A+\alpha I) + \alpha(A^T P_0 + P_0 A) + \alpha Q \leq 0$, then P_{CCLL1} is sharper than P_{CCLL2}.

4.2 Upper Matrix Bounds for the CALE

In this section, two upper matrix bounds for the solution of the CALE will be derived, each followed by an iterative algorithm that can obtain tighter upper matrix bounds. The derivation of these results make use of the method employed in [26]. After the bound developments, it will then be explained why the proposed upper matrix bounds are always calculable if the CALE solution exists.

Theorem 4.3: The solution P of the CALE (1.3) has the upper bound

$$P \leq U^{-T}[A^T R A + \alpha^2 R + \alpha Q]U^{-1} \leq R \equiv P_{CLU1} \tag{4.14}$$

where the positive semidefinite matrix R is selected such that

$$A^T R + RA \leq -Q. \tag{4.15}$$

Proof: From (4.6), suppose that R is an upper bound for P such that

$$P \leq U^{-T}[A^T R A + \alpha^2 R + \alpha Q]U^{-1} \leq R. \tag{4.16}$$

Using (2.4), (4.16) implies

$$A^T R A + \alpha^2 R + \alpha Q \leq U^T R U. \tag{4.17}$$

After some manipulations, (4.17) is equivalent to the condition (4.15). As such, it is seen that satisfaction of the condition (4.15) ensures the validity of the upper bound (4.14). This ends the proof of the theorem.

Now that we have developed the upper matrix bound of Theorem 4.3, the following iterative algorithm will be proposed, which can derive more precise upper matrix bounds for the solution of the CALE (1.3).

Algorithm 4.3:
Step 1: Set $N_0 \equiv R$, where R is a positive semidefinite matrix satisfying (4.15).
Step 2: Calculate

$$N_k = U^{-T}[A^T N_{k-1} A + \alpha^2 N_{k-1} + \alpha Q]U^{-1}, \quad k = 0, 1, \ldots \tag{4.18}$$

Then N_k are also upper bounds for the solution of the CALE (1.3).

Proof: From (4.16), it is obvious that $P \leq N_1 \leq N_0$. It is then found from (4.18) that

$$P \leq U^{-T}[A^T N_1 A + \alpha^2 N_1 + \alpha Q]U^{-1} = N_2.$$

So far, $P \leq N_2 \leq N_1 \leq N_0$. Now assume that $P \leq N_{k-1} \leq N_{k-2}$. Then, another application of (4.18) gives

$$\begin{aligned} P &\leq U^{-T}[A^T N_{k-1} A + \alpha^2 N_{k-1} + \alpha Q]U^{-1} = N_k \\ &\leq U^{-T}[A^T N_{k-2} A + \alpha^2 N_{k-2} + \alpha Q]U^{-1} = N_{k-1}. \end{aligned}$$

By means of mathematical induction, it can be concluded that $P \leq N_k \leq N_{k-1} \leq \ldots \leq N_1 \leq N_0$. Since $N_k \geq 0$, it is obvious that N_k is monotone decreasing and bounded, so there exists $N_\infty = \lim_{k\to\infty} N_k$ such that

$$N_\infty = U^{-T}[A^T N_\infty A + \alpha^2 N_\infty + \alpha Q]U^{-1}. \tag{4.19}$$

(4.19) is equivalent to (4.6) with $N_\infty = P$, where P is the solution of the CALE. This completes the proof of the correctness of this algorithm.

Another upper matrix bound is obtained as follows.

Theorem 4.4: The solution P of the CALE (1.3) satisfies

$$P \leq U^{-T}[(A + \alpha I)^T R(A + \alpha I) + 2\alpha Q]U^{-1} \leq R \equiv P_{CLU2} \tag{4.20}$$

where the positive semidefinite matrix R is chosen to satisfy the condition (4.15).

Proof: From (4.12), suppose that $P \leq R$ such that

$$P \leq U^{-T}[(A+\alpha I)^T R(A+\alpha I)+2\alpha Q]U^{-1}. \tag{4.21}$$

By proceeding along the same lines as in the proof of Theorem 4.3, it is seen that satisfaction of the condition (4.15) validates the existence upper matrix bound (4.20). This completes the proof of the theorem.

Having finished the proof of Theorem 4.4, the following iterative algorithm can be proposed to obtain sharper upper matrix bounds for the solution of the CALE (1.3).

Algorithm 4.4:
Step 1: Set $\overline{N}_0 \equiv R$, where R is a positive semidefinite matrix satisfying (4.15).
Step 2: Calculate

$$\overline{N}_k = U^{-T}[(A+\alpha I)^T \overline{N}_{k-1}(A+\alpha I)+2\alpha Q]U^{-1}, \quad k=0,1,\ldots$$

Then $\overline{N}_k$ are also upper bounds for the solution of the CALE (1.3).

Proof: The proof of the correctness of this algorithm follows along the same lines as that of Algorithm 4.3. Therefore, the proof is left out.

Remark 4.3: According to Lemma 2.1 of [34], any positive semidefinite matrix R satisfying the condition (4.15) is an upper bound for the positive semidefinite solution of the CALE. As such, one can always find such a matrix R, so the upper bounds (4.14) and (4.20) are always computable if the CALE has a symmetric positive semidefinite solution.

Remark 4.4: The following table summarizes lower and upper matrix solution bounds for the CALE that have been proposed in the respective literature:

Bound	Reference
$P \geq S^{-1}[S(Q-M)S]^{1/2}S^{-1} \equiv P_{CLL3}$	[60]
$P \leq \frac{\lambda_1(Q)}{-\lambda_1(A+A^T)} I \equiv P_{CLU3}$	[60]
$P \leq [Q+(A+I)^T P_{L3}(A+I) - A^T P_{L3} A] \equiv P_{CLU4}$	[60]
$P \geq \frac{1}{2}E^{-1}(EQE)^{1/2}E^{-1} \equiv P_{CLL4}$	[10,60]
$P \geq \frac{1}{2}\lambda_n^{1/2}(Q)(A^{-1}QA^{-T})^{1/2} \equiv P_{CLL5}$	[10,60]
$P \geq R_1^{1/2}[R_1^{-1/2}(Q-A^T R_1 A)R_1^{-1/2}]^{1/2}R_1^{1/2} \equiv P_{CLL6}$	[58]
$P_{CLL7} \equiv \frac{Q}{\mu} \leq P \leq \frac{Q}{\sigma} \equiv P_{CLU5}$	[28]
$P_{CLL8} \equiv \lambda_n(N^TQN)N^{-T}WN^{-1} \leq P \leq \lambda_1(N^TQN)N^{-T}WN^{-1} \equiv P_{CLU6}$	[64]
$P_{CLL9} \equiv \lambda_n(G_{cn})M_nM_n^T \leq P \leq \lambda_1(G_{cn})M_nM_n^T \equiv P_{CLU7}$	[72]
$P_{CLL10} \equiv \lambda_n(G_{cm})M_nM_n^T \leq P \leq \lambda_1(G_{cm})M_nM_n^T \equiv P_{CLU8}$	[86]
$P \geq \frac{\lambda_n(Q)}{2\sigma_1(A)} I \equiv P_{CLL11}$	[44]
$P \geq \gamma[Q-\gamma^2 A^T A]^{1/2} \equiv P_{CLL12}$	[66]
$P \geq \frac{1}{\kappa}F^{-1}\left[F(\kappa Q - Q^{-1})F\right]^{1/2}F^{-1} \equiv P_{CLL13}$	[56]
$P \geq \frac{1}{\kappa}\left[\frac{\kappa Q - Q^{-1}}{\lambda_1(AQA^T)}\right] \equiv P_{CLL14}$	[56]
$P_{CLL15} \equiv \lambda_n(X)Y_0 \leq P \leq \lambda_1(X)Y_0 \equiv P_{CLU9}$	[78]

Table 4.1: Existing Matrix Bounds of the CALE

The matrix bounds summarised in Table 4.1 have the following notations:

$$S \equiv (AM^{-1}A^T)^{1/2}, \quad E = (AQ^{-1}A^T)^{1/2}, \quad F = (AQA^T)^{1/2},$$

$$\mu = \lambda_1\{-(A^TQ+QA)Q^{-1}\},$$

$$\sigma = \lambda_n\{-(A^TQ+QA)Q^{-1}\},$$

$$W \equiv \mathrm{diag}\{1/[-2\Re(\lambda_i(A))]\}, \quad A = N\Lambda N, \quad \Lambda \equiv \mathrm{diag}\{\lambda_i(A)\}$$

$$X = -(A^TY_0+Y_0A)^{-1}Q, \quad Y_0 = TR^{-1}T^{-1}, \quad \tilde{A} = TAT^{-1}, \quad R = (\tilde{A}^T\tilde{A})^{-1/2},$$
with $T = I$ or $T = Q^{1/2}$,

$$G_{cn} \equiv \{g_{ij}\} \in \Re^{n\times n},$$

with

$$g_{ij} \equiv \int_0^\infty a_i(t)a_j(t)\,dt,$$

$$e^{A^Tt} = a_1(t)I + a_2(t)A^T + \ldots + a_n(t)(A^T)^{n-1},$$

$$M_n \equiv [D, A^T D, (A^T)^2 D, \ldots, (A^T)^{n-1} D],$$

where $Q = DD^T$,

$$G_{cm} \equiv \{g_{ij}\} \in \Re^{n \times n},$$

with

$$g_{ij} \equiv \int_0^\infty a_i(t) a_j(t)\, dt,$$

$$e^{A^T t} = a_1(t) I + a_2(t) A^T + \ldots + a_n(t)(A^T)^{m-1},$$

$$M_n \equiv [D, A^T D, (A^T)^2 D, \ldots, (A^T)^{m-1} D],$$

where $Q = DD^T$, and m is the degree of the minimal polynomial of A.

Remark 4.5: The bounds P_{CLL3}, P_{CLL4}, P_{CLL5}, P_{CLL6} and P_{CLL7} have to assume that $Q > 0$ for them to work. The bound P_{CLU3} and P_{CLU4} have to assume that $Q > 0$ and $A + A^T < 0$ for them to work. The bound P_{CLU5} has to assume that $Q > 0$ and $\sigma > 0$ for it to be calculated. The bounds P_{CLL8} and P_{CLU6} have to assume that A is diagonalizable for them to be evaluated. The bounds P_{CLL9}, P_{CLL10}, P_{CLL11}, P_{CLU7} and P_{CLU8} need no condition for satisfaction, and always work if the solution of the CALE exists.

For the bound P_{CLL3}, the positive definite matrix M is chosen such that $Q > M$. For the bound P_{CLL6}, the positive definite matrix R_1 is chosen such that $Q > A^T R_1 A$.

In [60], it was shown that bounds P_{CLL4} and P_{CLL5} are merely special cases of bound P_{CLL3}. As such, only bounds P_{CLL5} and P_{CLL6} will be used, when possible, as choices for the bound P_{CLL4} when the numerical examples are performed later in this chapter. The bound P_{CLL6} is derivable for the CALE from [58]. In fact, by setting $M = A^T R_1 A$ in bound P_{CLL3}, the bound P_{CLL6} is obtained. Similarly, setting $R_1 = A^{-T} M A^{-1}$ in bound P_{CLL6} results in the bound P_{CLL3}. It was also shown in [58] that the lower matrix bounds for the CALE which are derivable from [50] and [51], and the lower matrix bound (19) derived in [10], are special cases of the bound P_{CLL6}. To ease the calculation of this bound, some choices for the positive definite matrices R_1 are listed in the table that follows.
The advantage of the presented bounds is that they can always be calculated if the solution of the CALE exists. These bounds can be successively tightened with the aid of the corresponding Algorithms 4.1 to 4.4. Of course, as

R_1	Range of parameter
$\frac{1}{\beta}I$	$\beta > \lambda_1(A^TQ^{-1}A)$
βQ	$0 < \beta < \lambda_1^{-1}(A^TQAQ^{-1})$
βQ^{-1}	$0 < \beta < \lambda_1^{-1}(A^TQ^{-1}AQ^{-1})$

Table 4.2: Simple choices of R_1 and the corresponding range of parameter

mentioned above, the bounds P_{CLL9}, P_{CLL10}, P_{CLL11}, P_{CLU7} and P_{CLU8} need no condition for satisfaction, and always work if the solution of the CALE exists. However, when $Q \geq 0$, the bound P_{CLL11} yields the trivial bound $P \geq 0$. The calculation of the bounds P_{CLL9}, P_{CLL10}, P_{CLU7} and P_{CLU8} seem very complicated. In [86], the bounds P_{CLL10} and P_{CLU8} were reported to reduce the computational burdens required in computing the bounds P_{CLL9} and P_{CLU7}. However, the calculation of these bounds are still very complex. Furthermore, the bounds may not be very tight, and no iterative procedure exists to help tighten these bounds. The presented bounds can always be calculated if the solution of the CALE exists, and are also more concise.

Reference [78] also developed a lower and an upper matrix bound which assumes that $Q > 0$, or at least that the CALE has a positive definite solution. Additionally, a matrix bound improvement procedure was proposed in [79] from which it is possible to derive tighter lower and upper matrix bounds for P by using the lower and upper bounds P_{CLL15} and P_{CLU9} as initial matrices. However, this procedure is only valid under some mild assumption, whilst the presented algorithms always work. Besides, it seems that the calculation of bounds P_{CLL15} and P_{CLU9} are rather complicated, and are not as concise as the bounds P_{CLL1} and P_{CLL2}. Reference [79] extends the work of [78] by using a singular value decomposition to obtain upper solution bounds for the CALE which extend the set of Hurwitz stable matrices for which such bounds are valid. This bound involves an external Lyapunov matrix (ELM) in which a free variable is involved which is determined by some additional procedure. Then, the works of [79] were extended further in [80], which proposes an always valid upper matrix bound for the CALE. The estimate takes the form of an internal Lyapunov Matrix (ILM) in which the bound is expressed completely in terms of the coefficient matrices of the CALE. Examples of these internal matrix bounds, which will be used later in the comparison examples, are:

$$P \le \lambda_1[-Q(A^T AA)^{-1}]A^T A \equiv P_{CLU10}, \quad A \in H^-,$$

$$P \le \lambda_1[-Q(P_1 A)_s^{-1}]P_1 \equiv P_{CLU11}, \quad P_1^2 = A^T A, \quad A \in \tilde{H},$$

$$P \ge \lambda_1[-Q(P_2^{-1} A)_s^{-1}]P_2^{-1} \equiv P_{CLU12}, \quad P_2^2 = AA^T, \quad A \in \tilde{H}$$

Here, the matrix $X_s = X + X^T$ and the matrix sets H^- and $\tilde{H}$ are defined by $H^- \equiv \{A : S_C(\tilde{A}, I) < 0\}$ and $\tilde{H} \equiv \{A : F \in H\}$. Here, $S_C(\tilde{A}, I) < 0$ means any matrix $\tilde{A}$ such that $\tilde{A} + \tilde{A}^T < 0$, and the set $\tilde{H}$ is the set of matrices with Hurwitz unitary parts. Here, one has $H^- \subseteq \tilde{H}$.

Remark 4.6: The tightness of the lower bounds proposed here depend on the choice of the positive constant α, and the tightness of the upper bounds proposed here depend on both the choice of the positive constant α and the positive definite matrix R. It is difficult to say which choice of α gives the best lower bounds, and which choice of α and R give the best upper bounds. Therefore, this problem remains an open question. Besides, for any chosen value of α, one could easily obtain tighter matrix solution bounds by using Algorithms 4.1 to 4.4. It should be noted that, in the case $A + A^T < 0$, a simple choice of $R = P_{CLU3}$ will suffice in the calculation of the upper matrix bounds (4.14) and (4.20). To see why this is true, consider the following analysis, which makes use of (2.3):

$$\begin{aligned} A^T R + RA + Q &= \frac{\lambda_1(Q)}{-\lambda_1(A + A^T)}(A + A^T) + Q \\ &\le \frac{\lambda_1(Q)}{-\lambda_1(A + A^T)}\lambda_1(A + A^T)I + Q \\ &= -\lambda_1(Q)I + Q \le -\lambda_1(Q)I + \lambda_1(Q)I = 0. \end{aligned}$$

Furthermore, it is found that the tightness between existing solution bounds and the presented bounds is hard to be compared by any mathematical method.

Remark 4.7: The only computational burden that may arise in calculating the bounds P_{CLL1} and P_{CLL2} is the inversion of the matrix U. However, it seems that many existing matrix bounds in the literature are even more computationally expensive than these bounds. In particular, the bounds P_{CLL3}, P_{CLL4}, P_{CLL6}, P_{CLL12}, P_{CLL13}, and P_{CLU9} involve matrix inversion and matrix square roots, the bound P_{CLL8} involves matrix inversion, matrix eigenvalues and a matrix decomposition of matrix A, bounds P_{CLL5} and P_{CLL14} involve matrix inversion, matrix square roots and matrix eigenvalues, and bounds P_{CLL9} and P_{CLL10} involve a matrix decomposition, a matrix exponential, evaluation of integrals which may be heavy, and a matrix eigenvalue. The bound P_{CLL7} involves inversion of Q and a matrix eigenvalue, while bound P_{CLL11} involves a singular value and an eigenvalue. Furthermore, P_{CLL11} seems somewhat conservative. Finally, the bounds P_{CLL15} and P_{CLU9} also require a number of computational strains such as the computational strains of bound P_{CLL14}. Therefore, the present bounds are considered to be the least heavy in terms of computational load. In a similar way, the upper bounds P_{CLU1} and P_{CLU2} are also considered to be the least in terms of computational weight and complexity.

Remark 4.8: In light of Remarks 4.3 and 4.5, the upper bounds P_{CLU11} and P_{CLU12} may be chosen as the matrix R, and any value of the positive constant α will suffice, since N_k and $\overline{N}_k$ both tend to P indefinitely.

4.3 Numerical Examples for the CALE

Two numerical examples will now be considered to show the effectiveness of the derived results. The first example will focus on the case that Q is positive semidefinite, whereas the second example will consider the case Q is positive definite.

4.3.1 Example 1: Q is positive semidefinite and $A + A^T$ is not negative definite

Consider the CALE (1.3) with:

$$A = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then, the unique positive definite solution of (1.3) is:

$$P_{\text{exact}} = \begin{bmatrix} 0.5 & 1 \\ 1 & 2.5 \end{bmatrix}.$$

With $\alpha = 1$ and $R = P_{CLU12}$, where P_{CLU12} is defined in Remark 4.5, the lower and upper matrix bounds P_{CLL1}, P_{CLL2}, N_1 and N_2 for the solution P are found by Theorems 4.1 to 4.4, respectively, to be:

$$P_{CLL1} = \begin{bmatrix} 0.375 & 0.75 \\ 0.75 & 1.625 \end{bmatrix}, \quad P_{CLL2} = \begin{bmatrix} 0.5 & 1 \\ 1 & 2.5 \end{bmatrix},$$

$$P_{CLU1} = \begin{bmatrix} 1 & 1.25 \\ 1.25 & 4 \end{bmatrix}, \quad P_{CLU2} = \begin{bmatrix} 0.5 & 1 \\ 1 & 3.5 \end{bmatrix}.$$

In fact, it can be seen that $P_{CLL2} = P_{exact}$. Since Q is singular, the matrix bounds proposed in [2,5,6,8-10,12,15-17] cannot work here. The lower bound P_{CLL11} gives the trivial bound $P \geq 0$. Since the exact solution is found from P_{CLL2}, Algorithm 4.2 need not be used, since it will only return the exact solution at each iteration.

Using Algorithm 4.4 once, the following tighter upper matrix bound for P is obtained:

$$\overline{N}_2 = \begin{bmatrix} 0.5 & 1 \\ 1 & 2.5 \end{bmatrix}.$$

Here, $\overline{N}_2$ is the same as the exact solution. The matrix A cannot be diagonalised, so the lower and upper matrix bounds P_{CLL8} and P_{CLU6} cannot be applied here. Using the bounds P_{CLL9} and P_{CLU7} provides the following lower and upper matrix bounds:

$$P_{CLL9} = \begin{bmatrix} 0.0858 & 0 \\ 0 & 0.0858 \end{bmatrix} \leq P \leq \begin{bmatrix} 2.9142 & 0 \\ 0 & 2.9142 \end{bmatrix} \equiv P_{CLU7}$$

The matrix bounds P_{CLU11} and P_{CLU12} give the following upper solution estimates:

$$P_{CLU11} = \begin{bmatrix} 5 & -5 \\ -5 & 15 \end{bmatrix},$$

$$P_{CLU11} = \begin{bmatrix} 1.5 & 1.5 \\ 1.5 & 4.5 \end{bmatrix}.$$

The matrix A has no minimal polynomial, so the bounds P_{CLL10} and P_{CLU8} give the same estimate as the bounds P_{CLL9} and P_{CLU7}. As it can be seen from the above numerical experiments, the results proposed in this thesis are advantageous over existing results in that they can always be applied.

4.3.2 Example 2: Q is positive definite and $A + A^T$ is negative definite

Consider the CALE (1.3) with:

$$A = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix}, \quad Q = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, the unique positive definite solution of the CALE (1.3) is:

$$P_{\text{exact}} = \begin{bmatrix} 0.5 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}.$$

With $\alpha = 1$ and $R = P_{CLU3}$, where P_{CLU3} is defined in Table 4.1, the lower and upper matrix bounds P_{CLL1}, P_{CLL2}, N_1 and N_2 for the solution P are found by Theorems 4.1 to 4.4, respectively, to be:

$$P_{CLL1} = \begin{bmatrix} 0.3457 & 0.0818 \\ 0.0818 & 0.1226 \end{bmatrix}, \quad P_{CLL2} = \begin{bmatrix} 0.4983 & 0.1050 \\ 0.1050 & 0.1848 \end{bmatrix},$$

$$P_{CLU1} = \begin{bmatrix} 0.5320 & 0.0400 \\ 0.0400 & 0.4327 \end{bmatrix}, \quad P_{CLU2} = \begin{bmatrix} 0.5063 & 0.0801 \\ 0.0801 & 0.3077 \end{bmatrix}.$$

It is seen that P_{CLL2} is the tighter lower matrix bound and $\overline{N}_1$ is the tighter upper matrix bound. Using 2 iterations of Theorem 4.4 gives the following tighter lower matrix bounds for the solution of the CALE (1.3):

$$\overline{M}_2 = \begin{bmatrix} 0.4993 & 0.1013 \\ 0.1013 & 0.1952 \end{bmatrix},$$

$$\overline{M}_3 = \begin{bmatrix} 0.4999 & 0.1003 \\ 0.1003 & 0.1986 \end{bmatrix}.$$

Using 2 iterations of Theorem 4.4 provides the following tighter upper matrix bounds for the solution of the CALE (1.3):

$$\overline{N}_2 = \begin{bmatrix} 0.5007 & 0.0963 \\ 0.0963 & 0.2304 \end{bmatrix},$$

$$\overline{N}_3 = \begin{bmatrix} 0.5001 & 0.0993 \\ 0.0993 & 0.2082 \end{bmatrix}.$$

The bounds P_{CLL11}, P_{CLL4}, P_{CLL5} and P_{CLL7} yield, respectively:

$$P_{CLL11} = \begin{bmatrix} 0.1535 & 0 \\ 0 & 0.1535 \end{bmatrix}, \quad P_{CLL4} = \begin{bmatrix} 0.4811 & 0.0962 \\ 0.0962 & 0.1925 \end{bmatrix},$$

$$P_{CLL5} = \begin{bmatrix} 0.3623 & 0.0264 \\ 0.0264 & 0.1646 \end{bmatrix}, \quad P_{CLL7} = \begin{bmatrix} 0.2971 & 0 \\ 0 & 0.1485 \end{bmatrix}.$$

The upper bound P_{CLU3} provides:

$$P \leq \begin{bmatrix} 0.5578 & 0 \\ 0 & 0.5578 \end{bmatrix}.$$

Using the bounds P_{CLU3} and P_{CLL4}, the bound P_{CLU4} provides the following upper solution estimate:

$$P_{CLU4} = \begin{bmatrix} 0.6334 & -0.1728 \\ -0.1728 & 2.1526 \end{bmatrix}.$$

With $\gamma = 0.25$, the lower matrix bound P_{CLL12} gives:

$$P_{CLL12} = \begin{bmatrix} 0.3303 & 0.0162 \\ 0.0162 & 0.1522 \end{bmatrix}.$$

With $\kappa = 2$, the lower bounds P_{CLL13} and P_{CLL14} result in the following estimations respectively:

$$P_{CLL13} = \begin{bmatrix} 0.3227 & 0.0389 \\ 0.0389 & 0.1756 \end{bmatrix}, \quad P_{CLL14} = \begin{bmatrix} 0.2700 & 0 \\ 0 & 0.1443 \end{bmatrix}.$$

The matrix bounds P_{CLL8}, P_{CLU6}, P_{CLL9} and P_{CLU7} give the following estimates:

$$P_{CLL8} = \begin{bmatrix} 0.1096 & 0.1096 \\ 0.1096 & 0.1827 \end{bmatrix} \leq P \leq \begin{bmatrix} 1.1404 & 1.1404 \\ 1.1404 & 1.9007 \end{bmatrix} = P_{CLU6}$$

$$P_{CLL9} = \begin{bmatrix} 0.031 & -0.0124 \\ -0.0124 & 0.0372 \end{bmatrix} \leq P \leq \begin{bmatrix} 5.302 & -2.1208 \\ -2.1208 & 6.3624 \end{bmatrix} = P_{CLU7}$$

The matrix A has no minimal polynomial, so bounds P_{CLL10} and P_{CLU8} are the same as bounds P_{CLL9} and P_{CLU7}.

The bounds P_{CLL15} and P_{CLU9} yield

$$P_{CLL15} = \begin{bmatrix} 0.0981 & -0.0196 \\ -0.0196 & 0.1569 \end{bmatrix} \leq P \leq \begin{bmatrix} 0.5686 & -0.1137 \\ -0.1137 & 0.9097 \end{bmatrix} = P_{CLU9}, \quad \text{with } T = I,$$

$$P_{CLL15} = \begin{bmatrix} 0.1667 & -0.0333 \\ -0.0333 & 0.1417 \end{bmatrix} \leq P \leq \begin{bmatrix} 0.6666 & -0.1333 \\ -0.1333 & 0.5667 \end{bmatrix} = P_{CLU9}, \quad \text{with } T = Q^{1/2}.$$

The proposed bounds will now be compared with bound P_{L6} by using choices of the matrix R_1 from Table 4.1. Using these choices of R_1 gives the following lower bounds for the solution of the CALE (1.3):

Choice of R_1	Resulting Bound
$R_1 = \frac{1}{\beta}I$ and $\beta = 20$	$P \geq \begin{bmatrix} 0.2998 & 0.0109 \\ 0.0109 & 0.1577 \end{bmatrix}$
$R_1 = \beta Q$ and $\beta = 0.04$	$P \geq \begin{bmatrix} 0.3655 & 0.0216 \\ 0.0216 & 0.1863 \end{bmatrix}$
$R_1 = \beta Q^{-1}$ and $\beta = 0.09$	$P \geq \begin{bmatrix} 0.2861 & 0.0118 \\ 0.0118 & 0.1130 \end{bmatrix}$

Table 4.3: Lower Matrix Bounds for the CALE using the bound P_{CLL6}

The upper matrix bounds P_{CLU10}, P_{CLU11} and P_{CLU12} provide the following upper solution estimates:

$$P_{CLU10} = \begin{bmatrix} 2.4328 & -1.2164 \\ -1.2164 & 6.082 \end{bmatrix},$$

$$P_{CLU11} = \begin{bmatrix} 2.1837 & -0.4377 \\ -0.4377 & 3.5019 \end{bmatrix},$$

$$P_{CLU12} = \begin{bmatrix} 1.9232 & 0.3848 \\ 0.3848 & 1.4101 \end{bmatrix}.$$

Viewing these comparisons, it is seen that the presented bounds are tighter than the majority of existing matrix bounds for this case. As more iterations of Algorithms 4.2 and 4.4 are performed, the presented bounds become tighter.

4.4 Summary

In this chapter, the derivation of new lower and upper matrix bounds for the CALE solution has been presented. These bounds are always valid if the CALE solution exists, and are more concise than many existing parallel bounds. The numerical examples suggest that the derived bounds may be tighter than existing matrix bounds for the CALE proposed in the literature. In particular, it is also believed that the matrix bounds are the lightest in terms of computation.

Chapter 5

Matrix Bounds for the Continuous and Discrete Algebraic Riccati Equations

In this chapter, the lower matrix bounds for the CARE and upper matrix bounds for the DARE will be considered. These new bounds are always computable if the solutions of the CARE and DARE exist.

5.1 Lower Matrix Bounds for the Solution of the Continuous Algebraic Riccati Equation

Consider the CARE (1.4)

$$A^T P + PA - PBB^T P = -Q$$

with $Q = Q^T \geq 0$ and $P = P^T \geq 0$. Viewing the literature, it appears that nearly all lower matrix bounds for the CARE have to assume that $Q$is non-singular for them to be computable. This is a very restrictive assumption, because such an assumption is not common in control and estimation problems involving the solution of this equation. The only existing lower matrix bound that can deal with this case is that of [44]. However, if Q is singular, then $\lambda_n(Q) = 0$ and the lower matrix bound of [44] gives $P \geq 0$, which is trivial. Therefore, this section develops three lower matrix bounds to improve this drawback, and give nontrivial lower solution estimates for the solution of the CARE when Q is singular. It is not necessary to assume that Q is nonsingular for these results. Following the derivation of each matrix bound, an iterative algorithm is also proposed to obtain sharper solution estimates for the CARE. The results of this section can also be found in reference [19].

Before developing the main results, we shall review the following useful result:

The CARE (1.4) has the following upper bound for the maximal eigenvalue of its solution [34]:

$$\lambda_1(P) \leq \lambda_1(P_K) \equiv \eta \tag{5.1}$$

where P_K satisfies the linear equality

$$(A+BK)^T P_K + P_K(A+BK) + Q + K^T K = 0$$

and the matrix $K \in \Re^{m \times n}$ is chosen to make $A+BK$ a c-stable matrix. This eigenvalue upper bound is always computable if the CARE solution exists.

Theorem 5.1: Define

$$V \equiv A - \alpha I - I \tag{5.2}$$

where α is a positive constant. Let P be the positive semi-definite solution of the CARE (1.4). If

$$A + A^T < \lambda_1(BB^T)\eta I \tag{5.3}$$

where η is defined by (5.1), then P has the lower bound

$$P \geq V^{-T}\left(\varphi_1[(V+I)^T(V+I) + (2\alpha + 1 - \lambda_1(BB^T)\eta)I] + Q\right)V^{-1} \equiv P_{crl1} \tag{5.4}$$

where the positive constant α is chosen so that

$$A + A^T < 2\alpha I \tag{5.5}$$

and

$$2\alpha + 1 \geq \lambda_1(BB^T)\eta \tag{5.6}$$

are satisfied, and the non-negative constant φ_1 is defined by

$$\varphi_1 \equiv \frac{\lambda_n[V^{-T}QV^{-1}]}{1 - \lambda_n\{V^{-T}[(V+I)^T(V+I) + (2\alpha+1-\lambda_1(BB^T)\eta)I]V^{-1}\}}$$

Proof: The CARE (1.4) can be rewritten as:

$$P(A - \alpha I) + (A - \alpha I)^T P + 2\alpha P + Q = PBB^T P \tag{5.7}$$

where α is a positive constant.

Using the definition of V from (5.2), the following matrix identity can be defined

$$V^T PV = (V + I)^T P(V + I) - (A - \alpha I)^T P - P(A - \alpha I) + P \tag{5.8}$$

Using (5.8), (5.7) can be rewritten as

$$(V + I)^T P(V + I) + (2\alpha + 1)P + Q = PBB^T P + V^T PV. \tag{5.9}$$

From (2.1) we have $BB^T \leq \lambda_1(BB^T)I$. Then, applying (2.4) to the term PBB^TP gives $PBB^TP \leq \lambda_1(BB^T)P^2$.

Since $0 \leq P \leq \lambda_1(P)I$ from (2.3), one can also have from (2.3) that

$$P^2 = P^{1/2}PP^{1/2} \leq [P\lambda_1^2(P)IP]^{1/2} = \lambda_1(P)P. \tag{5.10}$$

Combining (5.10) with the previous results for PBB^TP gives

$$PBB^T P \leq \lambda_1(BB^T)\eta P. \tag{5.11}$$

Substituting (5.11) into (5.9) gives

$$(V + I)^T P(V + I) + (2\alpha + 1)P + Q \leq \lambda_1(BB^T)\eta P + V^T PV$$

$$\Rightarrow \quad V^T PV \geq (V + I)^T P(V + I) + (2\alpha + 1 - \lambda_1(BB^T)\eta)P + Q. \tag{5.12}$$

Since $\mathrm{Re}\lambda(A - \alpha I) \leq \mu(A - \alpha I) \equiv \frac{1}{2}\lambda_1(A + A^T - 2\alpha I)$, one can see that choosing α to meet the condition (5.5) ensures the nonsingularity of V. Furthermore, if α is chosen to meet condition (5.6), then the term $(2\alpha + 1 - \lambda_1(BB^T)\eta)P$ is non-negative definite, from which the main result follows. Therefore, (5.12) becomes

$$P \geq V^{-T}[(V + I)^T P(V + I) + (2\alpha + 1 - \lambda_1(BB^T)\eta)P + Q]V^{-1} \tag{5.13}$$

Application of (2.3) to (5.13) gives

$$P \geq V^{-T}\left([(V+I)^T(V+I) + (2\alpha + 1 - \lambda_1(BB^T)\eta)I]\lambda_n(P) + Q\right)V^{-1}. \tag{5.14}$$

Introducing (2.1) to (5.14) gives

$$\begin{aligned} &\lambda_n(P) \geq \lambda_n\{V^{-T}\left([(V+I)^T(V+I) + (2\alpha+1-\lambda_1(BB^T)\eta)I]\lambda_n(P) + Q\right)V^{-1}\} \\ &\geq \lambda_n\{V^{-T}[(V+I)^T(V+I) + (2\alpha+1-\lambda_1(BB^T)\eta)I]V^{-1}\}\lambda_n(P) + \lambda_n[V^{-T}QV^{-1}]. \end{aligned} \tag{5.15}$$

Using (2.9), it is found from (5.15) that

$$\begin{aligned} &\lambda_n\{V^{-T}[(V+I)^T(V+I) + (2\alpha+1-\lambda_1(BB^T)\eta)I]V^{-1}\} \\ &= \lambda_n\{[(V+I)^T(V+I) + (2\alpha+1-\lambda_1(BB^T)\eta)I](V^TV)^{-1}\} \\ &= \lambda_n\{[(A-\alpha I)^T(A-\alpha I) + (2\alpha+1-\lambda_1(BB^T)\eta)I][(A-\alpha I-I)^T(A-\alpha I-I)]^{-1}\}. \end{aligned}$$

Let $M \equiv (A-\alpha I)^T(A-\alpha I)$. Then (5.15) can be rewritten as

$$\begin{aligned} &\lambda_n(P) \geq \lambda_n\{V^{-T}\left([(V+I)^T(V+I) + (2\alpha+1-\lambda_1(BB^T)\eta)I]\lambda_n(P) + Q\right)V^{-1}\} \\ &= \lambda_n\{[M+(2\alpha+1-\lambda_1(BB^T)\eta)I][M-(A-\alpha I)^T-(A-\alpha I)+I]^{-1}\}\lambda_n(P)+\lambda_n[V^{-T}QV^{-1}]. \end{aligned}$$

Since $M \geq 0$, it is seen that if condition (5.3) is met, then

$$\lambda_n\{V^{-T}[(V+I)^T(V+I) + (2\alpha+1-\lambda_1(BB^T)\eta)I]V^{-1}\} < 1$$

It is then found from (5.15) that

$$\lambda_n(P) \geq \frac{\lambda_n[V^{-T}QV^{-1}]}{1-\lambda_n\{V^{-T}[(V+I)^T(V+I) + (2\alpha+1-\lambda_1(BB^T)\eta)I]V^{-1}\}} \equiv \varphi_1. \tag{5.16}$$

Substituting (5.16) into (5.14) leads to the lower bound (5.4). This completes the proof of the theorem.

Remark 5.1: Since I is a positive definite matrix of full rank and α is a positive constant, there will always exist a positive constant α such that the conditions (5.3), (5.5) and (5.6) are met. Hence, the lower bound (5.4) is always computable if the CARE has a non-negative stabilizing solution.

Having developed Theorem 5.1, the following iterative algorithm can be proposed to derive sharper lower matrix bounds for the CARE (1.4).

Algorithm 5.1:
Step 1: Set $M_0 \equiv P_{crl1}$, where P_{crl1} is defined by (5.4).
Step 2: Calculate

$$M_k = V^{-T}[(V+I)^T M_{k-1}(V+I)+(2\alpha+1-\sigma_1^2(B)\eta)M_{k-1}+Q]V^{-1} \quad k=1,2,\ldots. \tag{5.17}$$

Then M_k are also lower bounds for the solution of the CARE (1.4).

Proof of Case 1, $Q>0$: Set $k=1$ in (5.17) to get

$$M_1 = V^{-T}[(V+I)^T M_0(V+I)+(2\alpha+1-\sigma_1^2(B)\eta)M_0+Q]V^{-1}. \tag{5.18}$$

Applying (2.5) to (5.18) gives

$$M_1 \geq V^{-T}\left([(V+I)^T(V+I)+(2\alpha+1-\sigma_1^2(B)\eta)I]\lambda_n(M_0)+Q\right)V^{-1}. \tag{5.19}$$

Since $M_0 = P_{crl1}$, applying (2.1) to (5.4) results in

$$\begin{aligned}
&\lambda_n(M_0) \geq \lambda_n\{V^{-T}\left(\varphi_1[(V+I)^T(V+I)+(2\alpha+1-\sigma_1^2(B)\eta)I]+Q\right)V^{-1}\} \\
&\geq \varphi_1\lambda_n\{V^{-T}[(V+I)^T(V+I)+(2\alpha+1-\sigma_1^2(B)\eta)I]V^{-1}\}+\lambda_n[V^{-T}QV^{-1}] \\
&= \varphi_1\left\{1-\frac{\lambda_n[V^{-T}QV^{-1}]}{\varphi_1}\right\}+\lambda_n[V^{-T}QV^{-1}] = \varphi_1,
\end{aligned} \tag{5.20}$$

where (2.1) and (5.15) have been employed. Substituting (5.20) into (5.19) leads to

$$M_1 \geq V^{-T}\left([(V+I)^T(V+I)+(2\alpha+1-\sigma_1^2(B)\eta)I]\varphi_1+Q\right)V^{-1} = M_0.$$

Now assume $M_{k-1} \geq M_{k-2}$. Then

$$\begin{aligned}
&M_k = V^{-T}[(V+I)^T M_{k-1}(V+I)+(2\alpha+1-\sigma_1^2(B)\eta)M_{k-1}+Q]V^{-1} \\
&\geq V^{-T}[(V+I)^T M_{k-2}(V+I)+(2\alpha+1-\sigma_1^2(B)\eta)M_{k-2}+Q]V^{-1} = M_{k-1}.
\end{aligned}$$

By mathematical induction, it can be concluded that $M_k \geq M_{k-1} \geq \ldots \geq M_1 \geq M_0$. This completes the proof of the algorithm for the case $Q>0$.

Proof of Case 2, $Q \geq 0$: Firstly, note that, for this case, $M_0 = V^{-T}QV^{-1}$. Set $k = 1$ in (5.17) to get (5.18). Applying (2.1) to (5.18) leads to (5.19). Since $\lambda_n(M_0) = 0$ for this case, (5.19) becomes

$$M_1 \geq V^{-T}QV^{-1} = M_0.$$

Now assume $M_{k-1} \geq M_{k-2}$. Then

$$M_k = V^{-T}[(V+I)^T M_{k-1}(V+I) + (2\alpha + 1 - \sigma_1^2(B)\eta)M_{k-1} + Q]V^{-1}$$

$$\geq V^{-T}[(V+I)^T M_{k-2}(V+I) + (2\alpha + 1 - \sigma_1^2(B)\eta)M_{k-2} + Q]V^{-1} = P_{k-1}.$$

By mathematical induction, it can be concluded that $M_k \geq M_{k-1} \geq \ldots \geq M_1 \geq M_0$. This completes the proof of the algorithm for the case $Q \geq 0$.

We now obtain a different lower matrix bound as follows.

Theorem 5.2: If the condition (5.11) is fulfilled, then the solution P of the CARE (1.4) satisfies

$$P \geq V^{-T}\left(\varphi_1[(V+2I)^T(V+2I) + (4\alpha - 2\sigma_1^2(B)\eta)I] + 2Q\right)V^{-1} \equiv P_{crl2} \tag{5.21}$$

where the positive constant α is chosen so as to satisfy the condition (5.5) and the condition

$$2\alpha > \sigma_1^2(B)\eta, \tag{5.22}$$

and where the non-negative constant φ_1 is defined by (5.16).

Proof: Using the definition of V from (5.2), (5.7) can be rewritten as

$$\begin{aligned} &P(V+I) + (V+I)^T P + 2\alpha P + Q = PBB^T P \\ \Rightarrow \quad &PV + V^T P + 2P + 2\alpha P + Q = PBB^T P. \end{aligned} \tag{5.23}$$

Multiplying both sides of (5.23) by 2 and then adding $V^T PV$ to both sides of (5.23) gives

$$V^T PV + 2PV + 2V^T P + 4P + 4\alpha P + 2Q = V^T PV + 2PBB^T P \tag{5.24}$$

Using the matrix identity

$$(V+2I)^T P(V+2I) = V^T PV + 2PV + 2V^T P + 4P$$

(5.24) becomes

$$(V+2I)^T P(V+2I)+4\alpha P+2Q = V^T PV + 2PBB^T P \\ \leq 2\sigma_1^2(B)\eta P + V^T PV. \qquad (5.25)$$

(5.25) becomes

$$V^T PV \geq (V+2I)^T P(V+2I) + (4\alpha - 2\sigma_1^2(B)\eta)P + 2Q. \qquad (5.26)$$

Since V is nonsingular under the satisfaction of condition (5.5), (5.26) gives

$$P \geq V^{-T}[(V+2I)^T P(V+2I) + (4\alpha - 2\sigma_1^2(B)\eta)P + 2Q]V^{-1}. \qquad (5.27)$$

Application of (2.3) to (5.27) gives

$$P \geq V^{-T}\left([(V+2I)^T(V+2I) + (4\alpha - 2\sigma_1^2(B)\eta)]\lambda_n(P) + 2Q\right)V^{-1}. \quad (5.28)$$

Using (2.1), (5.28) becomes

$$\lambda_n(P) \geq \lambda_n\{V^{-T}\left([(V+2I)^T(V+2I) + (4\alpha - 2\sigma_1^2(B)\eta)]\lambda_n(P) + 2Q\right)V^{-1}\} \\ \geq \lambda_n\{V^{-T}[(V+2I)^T(V+2I) + (4\alpha - 2\sigma_1^2(B)\eta)I]V^{-1}\}\lambda_n(P) + 2\lambda_n[V^{-T}QV^{-1}]. \qquad (5.29)$$

Following along the same lines as in Theorem 5.1, it can be seen that if condition (5.2) is met, then

$$\lambda_n\{V^{-T}[(V+2I)^T(V+2I) + (4\alpha - 2\sigma_1^2(B)\eta)I]V^{-1}\} < 1.$$

Then, the following can be obtained from (5.29)

$$\lambda_n(P) \geq \frac{2\lambda_n[V^{-T}QV^{-1}]}{1-\lambda_n\{V^{-T}[(V+2I)^T(V+2I) + (4\alpha - 2\sigma_1^2(B)\eta)I]V^{-1}\}} \\ \equiv \frac{\lambda_n[V^{-T}QV^{-1}]}{1-\lambda_n\{V^{-T}[(V+I)^T(V+I) + (2\alpha + 1 - \lambda_1(BB^T)\eta)I]V^{-1}\}} \equiv \varphi_1. \quad (5.30)$$

Substituting (5.30) into (5.28) results in the lower bound (5.21).

Remark 5.2: In fact, the above bound obtained in Theorem 5.2 has taken into account the case when Q is positive definite. When Q is positive semidefinite, $\lambda_n(Q) = 0$. As such, $V^{-T}QV^{-1}$ is also positive semi-definite, which implies that $\lambda_n(V^{-T}QV^{-1}) = 0$. In this case, $\varphi_1 = 0$ and the lower bound (5.22) becomes $P \geq 2V^{-T}QV^{-1} = 2P_{crl1}$. Therefore, for the case when Q is singular, P_{crl2} is always tighter than P_{crl1}.

Following the derivation of the lower bound of Theorem 5.2, the following iterative algorithm to obtain sharper solution estimates can be proposed.

Algorithm 5.2:
Step 1: Set $\overline{M}_0 \equiv P_{crl2}$, where P_{crl2} is defined by (5.22).
Step 2: Calculate

$$\overline{M}_k = V^{-T}[(V+2I)^T\overline{M}_{k-1}(V+2I)+(4\alpha-2\sigma_1^2(B)\eta)\overline{M}_{k-1}+2Q]V^{-1}, \quad k=1,2,..$$

Then $\overline{M}_k$ are also lower solution bounds of the CARE (1.4).

The proof of this algorithm is similar to that of Algorithm 5.1, and is therefore omitted.

Next, a third lower matrix bound will be derived for the solution of the CARE (1.4).

Theorem 5.3: Define

$$W \equiv A - \beta I \tag{5.31}$$

where β is a positive constant. Let P be the positive semi-definite solution of the CARE (1.4). If the condition (5.2) is fulfilled, then P has the lower bound

$$P \geq W^{-T}\left(\omega_3[A^TA + (\beta^2 - \beta\sigma_1^2(B)\eta)I] + \beta Q\right)W^{-1} \equiv P_{crl3} \tag{5.32}$$

where the constant β is chosen such that

$$A + A^T < 2\beta I, \tag{5.33}$$

$$\beta > \sigma_1^2(B)\eta, \tag{5.34}$$

and the non-negative constant φ_2 is defined by

$$\varphi_2 \equiv \frac{\beta\lambda_n[W^{-T}QW^{-1}]}{1-\lambda_n\{W^{-T}[A^TA + (\beta^2 - \beta\sigma_1^2(B)\eta)I]W^{-1}\}}.$$

Proof: Using the definition of W from (5.31), the following matrix identity can be defined:

$$W^T PW = A^T PA - \beta(A^T P + PA) + \beta^2 P. \tag{5.35}$$

Substituting the CARE (1.4) into (5.35) gives:

$$W^T PW = A^T PA - \beta PBB^T P + \beta Q + \beta^2 P. \tag{5.36}$$

The proofs of Theorems 5.1 and 5.2 have shown that $PBB^T P \leq \sigma_1^2(B)\eta P$, where η is defined by (5.1). With this in mind, (5.36) becomes

$$W^T PW \geq A^T PA + (\beta^2 - \beta\sigma_1^2(B)\eta)P + \beta Q. \tag{5.37}$$

Under the satisfaction of (5.33), W is nonsingular, and (5.37) then gives

$$P \geq W^{-T}[A^T PA + (\beta^2 - \beta\sigma_1^2(B)\eta)P + \beta Q]W^{-1}. \tag{5.38}$$

Application of (2.3) to (5.38) gives

$$P \geq W^{-T}[(A^T A + (\beta^2 - \beta\sigma_1^2(B)\eta)I)\lambda_n(P) + \beta Q]W^{-1}. \tag{5.39}$$

where the condition (5.34) has been taken into account. Applying (2.1) to (5.39) gives

$$\lambda_n(P) \geq \lambda_n\{W^{-T}[(A^T A + (\beta^2 - \beta\sigma_1^2(B)\eta)I)]W^{-1}\}\lambda_n(P) + \beta\lambda_n[W^{-T}QW^{-1}]. \tag{5.40}$$

Along the same lines as in Theorem 5.1, it can be seen that if the condition (5.2) is satisfied, then

$$\lambda_n\{W^{-T}[(A^T A + (\beta^2 - \beta\sigma_1^2(B)\eta)I)]W^{-1}\} < 1.$$

From (5.40), one can then obtain

$$\lambda_n(P) \geq \frac{\beta\lambda_n[W^{-T}QW^{-1}]}{1 - \lambda_n\{W^{-T}[A^T A + (\beta^2 - \beta\sigma_1^2(B)\eta)I]W^{-1}\}} \equiv \varphi_2. \tag{5.41}$$

Substituting (5.41) into (5.39) results in the bound (5.32). This completes the proof of the theorem.

Remark 5.3: In fact, the above bound in Theorem 5.3 has taken into account the case when Q is positive definite. As before, when Q is positive semi-definite, the bound (5.34) becomes $P \geq \beta W^{-T} Q W^{-1}$.

Following the development of Theorem 5.3, the following iterative algorithm can be proposed to obtain more precise lower matrix bounds.

Algorithm 5.3:
Step 1: Set $N_0 \equiv P_{crl3}$, where P_{crl3} is defined by (5.32).
Step 2: Calculate

$$N_{k+1} = W^{-T}[A^T N_k A + (\beta^2 - \beta\sigma_1^2(B)\eta)N_k + \beta Q]W^{-1}, \quad k = 1, 2, \ldots.$$

Then N_k are also lower bounds for the solution of the CARE (1.4).

Proof: The proof of this algorithm parallels that of Algorithms 5.1 and 5.2, and hence omitted.

Remark 5.4: From (5.4) and (5.22), it can be seen that

$$\begin{aligned}
P_{crl2} &= V^{-T}\Big(\varphi_1[(V+I)^T(V+I) + (2\alpha + 1 - \lambda_1(BB^T)\eta)I] + Q + \varphi_1[(V+I) \\
&\qquad + (V+I)^T + (2\alpha - \lambda_1(BB^T)\eta)I] + Q\Big)V^{-1} \\
&= P_{crl1} + V^{-T}\Big(\varphi_1[A + A^T - \lambda_1(BB^T)\eta I] + Q\Big)V^{-1}.
\end{aligned}$$

As such, if $\varphi_1[A + A^T - \lambda_1(BB^T)\eta I] + Q \geq 0$ then P_{crl2} is tighter than P_{crl1}, whereas if $\varphi_1[A + A^T - \lambda_1(BB^T)\eta I] + Q \leq 0$ then P_{crl1} is tighter than P_{crl2}. It is find that the tightness between the bound P_{crl3} and P_{crl1} and P_{crl2} cannot be compared mathematically. It is also easy to see that satisfaction of condition (5.22) immediately implies the satisfaction of condition (5.6), so both of the bounds P_{crl1} and P_{crl2} exist under the satisfaction of condition (5.22).

Remark 5.5: Recently, the following lower matrix bound for the CARE (1.4) has been proposed in [58]:

$$P \geq G^{-1}[G(Q - A^T R_1 A)G]^{1/2}G^{-1} \equiv P_{crl4} \tag{5.42}$$

where the positive definite matrix R_1 is chosen such that $Q > A^T R_1 A$, and the positive definite matrix G is defined by $G = (BB^T + R_1^{-1})^{1/2}$. It was shown in [58] that, with suitable choices of R_1, the lower bound (5.42) is tighter than existing lower matrix bounds proposed in [10,45,50,51,57,58] and the corresponding eigenvalue bounds are also sharper than most previous bounds. In [12], some choices of the matrix R_1 were listed to simplify the calculation of (5.42). Some of these choices are re-listed in the table in the 2nd numerical example. It was earlier noted that nearly all existing lower matrix bounds for the CARE have to assume that Q is nonsingular. This assumption is very conservative. Under the satisfaction of the conditions for the bounds, our bounds can always work for the case of Q being singular and nonsingular. Therefore, this work improves the assumption. Also, it is found that the tightness between existing lower matrix bounds and those presented here cannot be compared by any mathematical method. However, they can supplement each other.

Remark 5.6: An iterative technique for solving the CARE (1.4) was proposed in [36]. this technique will be stated as follows: Choose a positive (semi)definite matrix $\hat{P}_0$ such that $A - BB^T\hat{P}_0$ is a stable matrix. Also, let $\hat{P}_k$ be the solution of the following Lyapunov-type matrix equation:

$$\hat{P}_k(A-BB^T\hat{P}_{k-1})+(A-BB^T\hat{P}_{k-1})^T\hat{P}_k = -(Q+\hat{P}_{k-1}BB^T\hat{P}_{k-1}), \quad k = 1,2,\ldots$$

Then, $\lim_{k\to\infty}\hat{P}_k = P$, where P is the unique positive semi-definite solution of the CARE (1.4). If the matrices $A - BB^TP_{crl1}$, $A - BB^TP_{crl2}$ or $A - BB^TP_{crl3}$ are stable, then the proposed lower bounds P_{crl1}, P_{crl2} or P_{crl3} can be chosen as the initial matrix $\hat{P}_0$ and solve the CARE (1.4) by the above iterative algorithm. This too is an application of the solution bounds of the CARE.

5.1.1 Numerical Examples

In this subsection, two numerical examples are given to demonstrate the effectiveness of the derived bounds. The first example will be for the case when Q is singular. The second example will be for the case when Q is nonsingular. Comparisons will be made with existing results when possible.

Example 1: Q is singular

Consider the CARE (1.4) with:

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

For these system matrices, the unique positive definite solution of the CARE (1.4) is:

$$P_{\text{exact}} = \begin{bmatrix} 0.4142 & 0.4142 \\ 0.4142 & 4.4142 \end{bmatrix}.$$

Since the matrix Q is singular and A is not in the range space of Q, the lower matrix bounds proposed in [10,45,50,51,57,58] cannot work for this case. However, the bounds of Theorems 5.1, 5.2 and 5.3 can work, and give tighter results than [9]. With $\eta = 5.6754$ and $\alpha = 3$, the lower matrix bounds for the solution P of the CARE (1.4) are found by Theorems 5.1 and 5.2, respectively, to be:

$$P_{crl1} = \begin{bmatrix} 0.0400 & 0.1200 \\ 0.1200 & 0.3600 \end{bmatrix} \quad \text{and} \quad P_{crl2} = \begin{bmatrix} 0.0800 & 0.2400 \\ 0.2400 & 0.7200 \end{bmatrix}$$

With $\beta = 6$, the lower matrix bound P_{crl3} is found by Theorem 5.3 to be:

$$P_{crl3} = \begin{bmatrix} 0.1224 & 0.2448 \\ 0.2448 & 0.4898 \end{bmatrix}$$

Using two iterations of Algorithm 5.2, the following tighter lower bounds for the solution of the CARE (1.4) can be obtained:

$$\overline{M}_1 = \begin{bmatrix} 0.1109 & 0.2470 \\ 0.2470 & 0.8562 \end{bmatrix}$$

$$\overline{M}_2 = \begin{bmatrix} 0.1228 & 0.2442 \\ 0.2442 & 0.8801 \end{bmatrix}$$

It can be seen that as more iterations of the algorithm are carried out, the bounds become tighter.

Example 2: Q is nonsingular [54, Example 1]

Consider the CARE (1.4) with:

$$A = \begin{bmatrix} 0.5 & 0 \\ 1 & -2.5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

For these system matrices, the unique positive definite solution of the CARE (1.4) is:

$$P_{\text{exact}} = \begin{bmatrix} 0.6989 & 0.1228 \\ 0.1228 & 0.5879 \end{bmatrix}.$$

The minimal and maximal eigenvalues, trace and determinant of the exact solution of the CARE with the above data are $\lambda_n(P_{exact}) = 0.5081$, $\lambda_n(P_{exact}) = 0.7767$, $\mathrm{tr}(P_{exact}) = 0.5081$, and $\det(P_{exact}) = 0.3946$ respectively. With $\eta = 0.8871$ and $\alpha = 1.5$, the lower matrix bound P_{crl1} for the solution P of the CARE (1.4) is found by Theorem 5.1 to be

$$P_{crl1} = \begin{bmatrix} 0.3390 & 0.0490 \\ 0.0490 & 0.2229 \end{bmatrix}.$$

With $\alpha = 2$, the lower matrix bound P_{crl2} is found by Theorem 5.2 to be

$$P_{crl2} = \begin{bmatrix} 0.4081 & 0.0590 \\ 0.0590 & 0.3165 \end{bmatrix}.$$

With $\beta = 3.5$, the lower matrix bound P_{crl3} is found by Theorem 5.3 to be

$$P_{crl3} = \begin{bmatrix} 0.4733 & 0.0778 \\ 0.0778 & 0.3475 \end{bmatrix}.$$

Using two iterations of Algorithm 5.3, the following tighter lower matrix bounds for the solution of the CARE (1.4) can be obtained:

$$N_1 = \begin{bmatrix} 0.5140 & 0.0719 \\ 0.0719 & 0.3625 \end{bmatrix},$$

$$N_2 = \begin{bmatrix} 0.5201 & 0.0708 \\ 0.0708 & 0.3655 \end{bmatrix}.$$

The lower bound derived in [45] gives:

$$P \geq \begin{bmatrix} 0.1651 & 0 \\ 0 & 0.1651 \end{bmatrix}.$$

The lower bound proposed in [57] gives:

$$P \geq \begin{bmatrix} 0.1710 & 0 \\ 0 & 0.1710 \end{bmatrix}.$$

The bounds will now be compared with the lower bound P_{crl4} proposed in [58]. Some choices of the tuning matrix R_1 were listed in [58] to allow simplified calculation of the bound P_{crl4}. These choices are re-listed in the following table:

R_1	G	Range of parameter ϵ
$\left(\frac{1}{\epsilon}I - BB^T\right)^{-1}$	$\frac{1}{\sqrt{\epsilon}}$	$0 < \epsilon < \lambda_1^{-1}(BB^T + AQ^{-1}A^T)$
$\left(\frac{1}{\epsilon}Q - BB^T\right)^{-1}$	$\frac{1}{\sqrt{\epsilon}}Q$	$0 < \epsilon < \lambda_1^{-1}[(BB^T + AQ^{-1}A^T)Q^{-1}]$
$\left(\frac{1}{\epsilon}AA^T - BB^T\right)^{-1}$	$\frac{1}{\sqrt{\epsilon}}AA^T$	$0 < \epsilon < \lambda_1^{-1}[(BB^T + AQ^{-1}A^T)(AA^T)^{-1}]$
$\frac{1}{\epsilon}I$	$(BB^T + \epsilon I)^{1/2}$	$\epsilon > \lambda_1(A^TQ^{-1}A)$
ϵQ	$\left(BB^T + \frac{1}{\epsilon}Q^{-1}\right)^{1/2}$	$0 < \epsilon < \lambda_1^{-1}(A^TQAQ^{-1})$
ϵQ^{-1}	$\left(BB^T + \frac{1}{\epsilon}Q\right)^{1/2}$	$0 < \epsilon < \lambda_1^{-1}(A^TQ^{-1}AQ^{-1})$
$\epsilon(AA^T)^{-1}$	$\left(BB^T + \frac{1}{\epsilon}AA^T\right)^{1/2}$	$0 < \epsilon < \lambda_n(Q)$
$\epsilon(AQA^T)^{-1}$	$\left(BB^T + \frac{1}{\epsilon}AQA^T\right)^{1/2}$	$0 < \epsilon < \lambda_n^2(Q)$

Table 5.1: Simple choices of R_1 together with the corresponding matrices G of P_{crl4}

With $R_1 = \left(\frac{1}{\epsilon}I - BB^T\right)^{-1}$ and $\epsilon = 0.1$, P has the lower bound P_{crl4} given by:

$$P \geq \begin{bmatrix} 0.2911 & 0.0322 \\ 0.0322 & 0.4863 \end{bmatrix}.$$

With $R_1 = \left(\frac{1}{\epsilon}Q - BB^T\right)^{-1}$ and $\epsilon = 0.1$, P has the lower bound P_{crl4} given by:

$$P \geq \begin{bmatrix} 0.1964 & 0.0100 \\ 0.0100 & 0.1849 \end{bmatrix}.$$

With $R_1 = \left(\frac{1}{\epsilon}AA^T - BB^T\right)^{-1}$ and $\epsilon = 0.02$, P has the lower bound P_{crl4} given by:

$$P \geq \begin{bmatrix} 0.6473 & -0.0454 \\ -0.0454 & 0.0367 \end{bmatrix}.$$

With $R_1 = \frac{1}{\epsilon}I$ and $\epsilon = 4$, P has the lower bound P_{crl4} given by:

$$P \geq \begin{bmatrix} 0.3419 & -0.0255 \\ -0.0255 & 0.5436 \end{bmatrix}.$$

With $R_1 = \epsilon Q$ and $\epsilon = 0.1$, P has the lower bound P_{crl4} given by:

$$P \geq \begin{bmatrix} 0.2031 & 0.1709 \\ 0.1709 & 0.4620 \end{bmatrix}.$$

With $R_1 = \epsilon Q^{-1}$ and $\epsilon = 0.5$, P has the lower bound P_{crl4} given by:

$$P \geq \begin{bmatrix} 0.3348 & 0.0771 \\ 0.0771 & 0.5661 \end{bmatrix}.$$

With $R_1 = \epsilon(AA^T)^{-1}$ and $\epsilon = 0.5$, P has the lower bound P_{crl4} given by:

$$P \geq \begin{bmatrix} 0.3358 & -0.0186 \\ -0.0186 & 0.4164 \end{bmatrix}.$$

With $R_1 = \epsilon(AQA^T)^{-1}$ and $\epsilon = 0.5$, P has the lower bound P_{crl4} given by:

$$P \geq \begin{bmatrix} 0.3342 & -0.0075 \\ -0.0075 & 0.2680 \end{bmatrix}.$$

From the above numerical results, one can see that the presented lower bounds are tighter than many existing results for some cases.

5.2 New Upper Matrix Bounds for the Solution of the Discrete Algebraic Riccati Equation

Consider the DARE (1.8)

$$P = A^T(I + PBB^T)^{-1}PA + Q$$

with Q a given positive semidefinite matrix. Viewing the literature, there appear to be many upper matrix bounds available for the solution of the DARE [9,26,33,43,47,52,53,57,63,67]. However, these upper bounds are only valid under conditions which are more conservative than the fundamental existence conditions for the solution of the DARE. Therefore, this section develops an upper matrix bound for the solution of the DARE which is always calculated if its solution exists. The derivation of these bounds make use of the fact that if (A, B) is a controllable (or stabilizable) pair, then there should always exist a matrix K such that $A + BK$ is a d-stable matrix. This is a well-known fact in control theory, and has been employed in the solution of a number of control problems in the literature; one particular example is in [34], where this idea has been used to derive an upper matrix-type bound for the CARE which always works if its solution exists. The results derived in this section can also be found in [16].

Theorem 5.4: Let P be the positive semi-definite solution of the DARE (1.6). If $\sigma_1^2(A + BK) < 1$ then P has the upper bound

$$P \leq \kappa(A + BK)^T(A + BK) + Q + K^TK \equiv P_{dru1} \tag{5.43}$$

where the positive constant κ is defined by

$$\kappa \equiv \frac{\lambda_1(Q + K^TK)}{1 - \sigma_1^2(A + BK)} \tag{5.44}$$

and the matrix $K \in \Re^{m\times n}$ is chosen to stabilize $A + BK$.

Proof: Define a positive semi-definite matrix Λ as

$$\begin{aligned}\Lambda &\equiv [K + (I + B^TPB)^{-1}B^TPA]^T(I + B^TPB)[K + (I + B^TPB)^{-1}B^TPA] \\ &= K^TK + K^TB^TPBK + A^TPBK + K^TB^TPA + A^TPB(I + B^TPB)^{-1}B^TPA \geq 0\end{aligned} \tag{5.45}$$

where $K \in \Re^{m\times n}$. Using the DARE (1.6), (5.45) becomes

$$P \le A^T PA + K^T K + K^T B^T PBK + A^T PBK + K^T B^T PA + Q \qquad (5.46)$$

By use of the matrix identity

$$(A + BK)^T P(A + BK) = A^T PA + A^T PBK + K^T B^T PA + K^T B^T PBK$$

(5.46) becomes

$$P \le (A + BK)^T P(A + BK) + Q + K^T K \qquad (5.47)$$

By making use of (2.3), (5.47) becomes

$$P \le \lambda_1(P)(A + BK)^T(A + BK) + Q + K^T K \qquad (5.48)$$

Introducing (2.1) to (5.48) gives

$$\begin{aligned}\lambda_1(P) &\le \lambda_1\{\lambda_1(P)(A + BK)^T(A + BK) + Q + K^T K\} \\ &\le \sigma_1^2(A + BK)\lambda_1(P) + \lambda_1(Q + K^T K)\end{aligned} \qquad (5.49)$$

If $\sigma_1^2(A+BK) < 1$ then (5.49) infers $\lambda_1(P) \le \kappa$, where κ is defined by (5.44).

Substituting (5.44) into (5.48) results in the bound (5.43). This completes the proof of the theorem.

Having developed the upper bound P_{dru1} of Theorem 5.1, the following iterative algorithm to derive tighter upper matrix bounds for the solution of the DARE (1.6) is suggested. Before doing so, first consider the modified DARE (1.8). Using this transformed DARE, together with (2.5), provides the following iterative algorithm to obtain sharper upper matrix solution bounds of the DARE (1.6).

Algorithm 5.4:
Step 1: Set $M_0 \equiv P_{dru1}$, where P_{dru1} is defined by (5.43).
Step 2: Calculate

$$M_k = A^T(I + M_{k-1}BB^T)^{-1}M_{k-1}A + Q \qquad k = 1, 2, \ldots \qquad (5.50)$$

Then M_k are upper solution bounds of the DARE (1.6). In fact, as $k \to \infty$, $M_{k+1} = M_k$ and $M_\infty = \lim_{k\to\infty} M_k = P$, where P is the positive semidefinite solution of the DARE (1.6).

Proof: Firstly, it will be shown that $M_1 \leq M_0$. Setting $k = 1$ in (5.50) gives:

$$M_1 = A^T(I + M_0BB^T)^{-1}M_0A + Q \tag{5.51}$$

Applying (2.3) and (2.5) to (5.51) gives

$$M_1 \leq \lambda_1(M_0)A^T[I + \lambda_1(M_0)BB^T]^{-1}A + Q \tag{5.52}$$

Now, let $N \equiv \lambda_1^{1/2}(M_0)I$. By Applying the matrix inversion formula ((2) in [26]) to (5.52), and following along the lines of the proof of Theorem 5.4, itis found that

$$\begin{aligned}
M_1 &\leq A^T(\lambda_1(M_0))A - \lambda_1(M_0)A^TNB[I + B^TN^2B]^{-1}B^TNA + Q \\
&= A^TN^2A - A^TN^2B[I + B^TN^2B]^{-1}B^TN^2A + Q \\
&= A^TN^2A - [K + (I + B^TN^2B)^{-1}B^TN^2A]^T(I + B^TN^2B)[K+ \\
&\quad (I+B^TN^2B)^{-1}B^TN^2A]+K^TK+K^TB^TN^2BK+A^TN^2BK+K^TB^TN^2A \\
&\leq (A + BK)^TN^2(A + BK) + (Q + K^TK) \\
&= \lambda_1(M_0)(A + BK)^T(A + BK) + (Q + K^TK).
\end{aligned} \tag{5.53}$$

Application of (2.1) to (5.53) gives

$$\begin{aligned}
\lambda_1(M_0) &= \lambda_1\{\kappa(A + BK)^T(A + BK) + (Q + K^TK)\} \\
&\leq \kappa\sigma_1^2(A + BK) + \lambda_1(Q + K^TK) \\
&= \kappa\left\{1 - \frac{\lambda_1(Q + K^TK)}{\kappa}\right\} + \lambda_1(Q + K^TK) = \kappa
\end{aligned} \tag{5.54}$$

where the condition $\sigma_1^2(A + BK) < 1$ and (5.44) have been employed. Substituting (5.54) into (5.53) gives

$$M_1 \leq \kappa(A + BK)^T(A + BK) + (Q + K^TK) \equiv M_0$$

Therefore, it has been completely proven that $M_1 \leq M_0$. Assume now that $M_{k-1} \leq M_{k-2}$. By (5.50) and use of (2.5), it is implied that

$$\begin{aligned}
M_k &= A^T(I + M_{k-1}BB^T)^{-1}M_{k-1}A + Q \\
&\leq A^T(I + M_{k-2}BB^T)^{-1}M_{k-2}A + Q = M_{k-1}.
\end{aligned}$$

One can conclude, by means of induction, that $M_k \leq M_{k-1} \leq \ldots \leq M_1 \leq M_0$. Clearly, we have $M_k \geq 0$ for any k. Along the lines of Theorem 1 in [26], it can be seen that M_k is monotone decreasing and bounded, so there exists $M_\infty \geq 0$, with $M_\infty = \lim_{k\to\infty} M_k$, such that

$$M_\infty = A^T(I + M_\infty BB^T)^{-1} M_\infty A + Q.$$

Here, M_∞ is merely the DARE solution with P replaced by M_∞. Hence, it can be concluded that this algorithm can obtain the exact solution of the DARE.

Even though K is chosen to stabilize $A + BK$, it is not always possible to fulfill the condition $\sigma_1^2(A + BK) < 1$. To get around this problem, a free matrix D will be utilized in the following theorem and corollary.

Firstly, the DARE (1.6) is modified, using the similarity transformation, to obtain the following modified DARE:

$$\overline{P} = \overline{A}^T \overline{P}\overline{A} - \overline{A}^T \overline{P}\overline{B}(I + \overline{B}^T \overline{P}\overline{B})^{-1} \overline{B}^T \overline{P}\overline{A} + \overline{Q} \tag{5.55}$$

where $\overline{P} = D^{-T} P D^{-1}$, $\overline{A} = DAD^{-1}$, $\overline{B} = DB$ and $\overline{Q} = D^{-T} Q D^{-1}$, and D is a nonsingular matrix.

Theorem 5.5: The solution P of the DARE (1.6) has the following upper matrix bound on its solution:

$$P \leq \mu(A + BK)^T D^T D(A + BK) + Q + K^T K \equiv P_{dru2} \tag{5.56}$$

where K is chosen to stabilize $A + BK$, the nonsingular matrix D is chosen so that $\sigma_1^2[D(A + BK)D^{-1}] < 1$, and the positive constant μ is defined by

$$\mu \equiv \frac{\lambda_1[D^{-T}(Q + K^T K)D^{-1}]}{1 - \sigma_1^2[D(A + BK)D^{-1}]}. \tag{5.57}$$

Proof: By applying the method of Theorem 5.4 to the modified DARE (5.55), the following upper bound for $\overline{P}$ can be obtained:

$$\overline{P} \leq \mu(\overline{A} + \overline{BK})^T(\overline{A} + \overline{BK}) + (\overline{Q} + \overline{K}^T \overline{K}) \tag{5.58}$$

where μ is defined by (5.57), and $\overline{K} = KD^{-1}$. Reverting to the original matrices in (5.58) and then applying (2.4) leads to the upper bound (5.56).

This finishes the proof of the theorem.

Corollary 5.1: Based on the analysis of Theorem 5.5, the following upper eigenvalue bounds for the solution of the DARE (1.6) are obtained:

$$\lambda_i(P) \le \lambda_i\{\mu(A+BK)^T D^T D(A+BK) + Q + K^T K\}, \quad i = 1, 2, \ldots, n$$

$$tr(P) \le tr\{\mu(A+BK)^T D^T D(A+BK) + Q + K^T K\},$$
$$det(P) \le det\{\mu(A+BK)^T D^T D(A+BK) + Q + K^T K\}.$$

Following the development of Theorem 5.5, the following iterative algorithm is suggested to derive more accurate upper matrix solution bounds of the DARE.

Algorithm 5.5:
Step 1: Set $\overline{M}_0 \equiv P_{dru2}$, where P_{dru2} is defined by (5.56).
Step 2: Calculate

$$\overline{M}_k = A^T(I + \overline{M}_{k-1}BB^T)^{-1}\overline{M}_{k-1}A + Q \qquad k = 1, 2, \ldots$$

Then $\overline{M}_k$ are upper solution bounds of the DARE (1.6). In fact, as $k \to \infty$, $\overline{M}_{k+1} = \overline{M}_k$ and $\overline{M}_\infty = \lim_{k\to\infty}\overline{M}_k = P$, where P is the positive semidefinite solution of the DARE (1.6).
Step 3: If $\overline{M}_{k+1} = \overline{M}_k$, then stop this procedure, and take M_{k+1} as the more precise estimate.

Proof: The proof of this algorithm parallels that of Algorithm 5.4, and is therefore omitted.

Remark 5.7: When A is stable, $K = 0$ and $D = I$, the results obtained in this chapter decompose into the upper matrix bounds for the DARE reported in [33]. Therefore, this work can be considered to be a generalization of the upper matrix bounds presented in [33].

The following upper bound was reported in [53]:

$$P \le \frac{\lambda_1(Q)}{1 + \eta\sigma_n^2(B) - \sigma_1^2(A)} A^T A + Q \tag{5.59}$$

where $\eta \equiv \lambda_1(A^T[Q^{-1} + BB^T]^{-1}A + Q)$. The calculation of bound (5.59) has to assume that Q is nonsingular and (i) BB^T is nonsingular and $1 + \eta\sigma_n^2(B) > \sigma_1^2(A)$ or (ii) BB^T is singular and $\sigma_1^2(A) < 1$. It can be seen that when $K = 0$ and $D = I$, the bound (5.56) is identical to the bound (5.59) when BB^T is singular. In these cases, the resulting bounds only work when $\sigma_1^2(A) < 1$. Furthermore, when $K = 0$ and $D = I$ in (5.56), and when BB^T is nonsingular, the bounds (5.56) and (5.61) become, respectively, the bounds P_{U1} and P_{U2}, where:

$$P_{U1} \equiv \frac{\lambda_1(Q)}{1 - \sigma_1^2(A)} A^T A + Q$$

$$P_{U2} \equiv \frac{\lambda_1(Q)}{1 + \eta\sigma_n^2(B) - \sigma_1^2(A)} A^T A + Q$$

In this case, we have $P_{U2} \leq P_{U1}$, so the bound (5.59) gives the tighter solution estimate than the bound (5.58) for this case. Furthermore, we have, for this case, that the bound P_{U1} only works when $\sigma_1^2(A) < 1$, whilst the bound P_{U2} only works when $1 + \eta\sigma_n^2(B) > \sigma_1^2(A)$ and Q is nonsingular. Of course, non-zero values of the matrices K and D can still be used to deal with the cases that $\sigma_1^2(A) < 1$ and $\sigma_1^2(A + BK) < 1$ are not satisfied.

When BB^T is nonsingular, one can choose $K = -B^T(BB^T)^{-1}A$ and $D = I$ in the bound (5.56), which results in the following upper matrix bound for the DARE:

$$P \leq A^T(BB^T)^{-1}A + Q. \tag{5.60}$$

The bound (5.60) is the same as the upper matrix bound (25) for the DARE proposed in [43]. Hence, the bound (5.62) can be considered to be a special case of the upper matrix bounds (5.45) and (5.58) for these particular choices of K and D. For this particular case it can also be seen that the remaining upper matrix bounds in the literature [9,26,47,52,57,63,67] are tighter than the bounds P_{dru1} and P_{dru2}, provided that the restrictions for validity are fulfilled for these bounds. A general comparison of the bounds P_{dru1} and P_{dru2} with P_{dru3} is not possible by any mathematical method, due to the type of bound, the involvement of the matrices K and D, and the assumptions required to calculate these bounds.

For the remaining upper matrix bounds existing in the literature [9,26,47,52, 57,63,67], one can see that the presented bounds in this chapter also cannot be compared with these existing ones by any mathematical method. However, comparison via a numerical example is always possible. Such comparisons are given, when possible, in the numerical examples that follow later in this section.

Remark 5.8: The condition $\sigma_1^2[D(A+BK)D^{-1}] < 1$ is equivalent to $\lambda_1[D^{-T}(A+BK)^T D^T D(A+BK)D^{-1}] < 1$, which is equivalent to

$$D^{-T}(A+BK)^T D^T D(A+BK)D^{-1} < I. \tag{5.61}$$

Using (2.4), (5.61) is equivalent to the condition

$$(A+BK)^T P_D (A+BK) < P_D \tag{5.62}$$

where $P_D = D^T D$. Since the pair (A, B) is assumed to be controllable, there will always exist a matrix K stabilizing $A+BK$. Then, since $A+BK$ is stable, there will always exist a symmetric matrix P_D yielding (5.62) by the Stein Theorem [68]. Therefore, the upper bounds of Theorem 5.5 and Corollary 5.1 are always calculated if the solution of the DARE exists. In fact, such a free matrix D may be constructed via the following procedure:
Step 1: Choose a matrix K such that $A+BK$ is stable.
Step 2: Select a positive definite matrix P_D satisfying the inequality:

$$(A+BK)^T P_D (A+BK) < P_D. \tag{5.63}$$

In a similar way to [34], one way of choosing such a matrix P_D to yield (5.63) is to use an LMI satisfying

$$(A+BK)^T P_D (A+BK) - P_D < -M$$

where M is a positive definite matrix.
Step 3: Having selected a positive definite matrix P_D which satisfies (5.63), the constant μ and upper bound P_{dru2} defined by (5.57) and (5.56) respectively, are:

$$\begin{aligned}\mu &= \frac{\lambda_1[D^{-T}(Q+K^T K)D^{-1}]}{1-\lambda_1[D^{-T}(A+BK)^T D^T D(A+BK)D^{-1}]} \\ &= \frac{\lambda_1[D^{-1}D^{-T}(Q+K^T K)]}{1-\lambda_1[D^{-1}D^{-T}(A+BK)^T D^T D(A+BK)]}\end{aligned}$$

$$= \frac{\lambda_1[(D^TD)^{-1}(Q+K^TK)]}{1-\lambda_1[(D^TD)^{-1}(A+BK)^TD^TD(A+BK)]}$$

$$= \frac{\lambda_1[P_D^{-1}(Q+K^TK)]}{1-\lambda_1[P_D^{-1}(A+BK)^TP_D(A+BK)]}$$

and

$$P \leq \mu(A+BK)^TP_D(A+BK)+Q+K^TK$$

where (2.9) and the fact [68] that $(XY) = Y^{-1}X^{-1}$ for nonsingular X, Y have been taken into account.

Having chosen a matrix K to stabilize $A+BK$, one, other method for finding such a matrix D is to let D be symmetric, find an expression for $P_D = D^TD$, and then use trial-and-error to see which possibilities of D satisfy the inequality $P_D-(A+BK)^TP_D(A+BK) > 0$. One can use the determinant criterion [68] for a positive definite matrix to aid one in finding such a possibility of D.

Remark 5.9: The tightness of the upper bounds developed here depend on the choice of the matrices K and D. It is hard to say which choice of K and D give the best upper bound for the DARE (1.6). Therefore, the choice of matrices K and D which give the optimal upper bound remains an open question. However, the choice of K and D which give the optimal bounds could be considered as an optimization problem. Besides, for a matrix K chosen to stabilize $A+BK$ and a matrix P_D selected to yield the condition (5.63), one may easily obtain tighter upper matrix bounds by successive use of Algorithms 5.4 or 5.5. It should also be noted that if $\sigma_1^2(A+BK) < 1$ is satisfied for a matrix K chosen to stabilize $A+BK$, then a simple choice of $D = I$ will suffice in the calculation of the bounds, and Theorem 5.4 can be used. Furthermore, when A is a stable matrix and $\sigma_1^2(A) < 1$, the upper matrix bounds reported in [33] may be referred to.

Remark 5.10: In the literature, there exists many methods to construct a matrix K such that the matrix $A+BK$ has arbitrarily assigned eigenvalues. For example, one may use pole placement techniques to stabilize $A+BK$, which are discussed, for example, in [73].

Remark 5.11: When $B = 0$ and A is a d-stable matrix, the DARE becomes the DALE (1.5). Then, with $K = 0$, the bounds (5.43) and (5.56) become the following upper matrix bounds for the DALE, respectively:

$$P \leq \frac{\lambda_1(Q)}{1-\sigma_1^2(A)} A^T A + Q, \tag{5.64}$$

$$P \leq \frac{\lambda_1(D^{-T}QD^{-1})}{1-\sigma_1^2(DAD^{-1})} A^T D^T D A + Q. \tag{5.65}$$

The bound (5.64) is the same as the bound (11) proposed in [47]. The bound (5.65) is a generalisation of the bound (5.64) with the free matrix D involved. The bound (5.64) is valid if $\sigma_1^2(A) < 1$, whilst the bound (5.65) requires the condition $A^T D^T D A < D^T D$ for a nonsingular matrix D, as defined above. This second condition is always satisfied.

Remark 5.12: A possible alternative to computing the bounds (5.43) and (5.56) is to consider the following discrete Lyapunov-type equation:

$$P_K = (A + BK)^T P_K (A + BK) + Q + K^T K. \tag{5.66}$$

Subtracting (5.47) from (5.66) gives

$$P_K - P \geq (A + BK)^T (P_K - P)(A + BK) \tag{5.67}$$

(5.67) implies

$$P_K - P = (A + BK)^T (P_K - P)(A + BK) + M \tag{5.68}$$

where M is a positive semidefinite matrix. Since $A+BK$ is stable and $M \geq 0$, (5.68) has a positive semidefinite solution by the Stein Theorem [68,73]. As such, we have $P_K - P \geq 0$, which implies that $P_K \geq P$, i.e., P_K is an upper matrix bound for the solution of the DARE. However, this approach may require solving high order linear algebraic equations like the DALE, in which case the bounds (5.43) and (5.56) may be preferrable, as in the case of the DALE. Furthermore, P_K may be used as the initial matrix for Algorithm 5.4.

5.2.1 Numerical Examples

In this subsection, numerical examples are given to demonstrate the effectiveness of the upper matrix bounds, and make comparisons, when possible, with existing results.

Example 1 (Example 1, [9])

Consider the DARE (1.6) with:

$$A = \begin{bmatrix} 1.45 & -0.45 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.25 & 0.19 \\ 0.19 & 0.1444 \end{bmatrix}.$$

Then, the unique positive definite solution of the DARE (1.6) is:

$$P_{\text{exact}} = \begin{bmatrix} 1.5989 & -0.1802 \\ -0.1802 & 0.2690 \end{bmatrix}.$$

with $\lambda_n(P_{exact}) = 0.2450$, $\lambda_1(P_{exact}) = 1.6229$, $\text{tr}(P_{exact}) = 1.8679$ and $\det(P_{exact}) = 0.3976$.

When $K = \begin{bmatrix} -1 & 0.4 \end{bmatrix}$, the upper bounds P_K for the solution of the DARE (1.6) is found from (5.66) to be

$$P_K = \begin{bmatrix} 1.6879 & -0.0559 \\ -0.0559 & 0.3086 \end{bmatrix}.$$

Using one iteration of Algorithm 5.4 gives the following tighter upper matrix solution bound for the DARE:

$$M_1 = \begin{bmatrix} 1.6035 & -0.1803 \\ -0.1803 & 0.2716 \end{bmatrix}.$$

For M_1, we have $\lambda_n(M_1) = 0.2476$, $\lambda_1(M_1) = 1.6275$, $\text{tr}(M_1) = 1.8751$ and $\det(M_1) = 0.4030$. The resulting bounds are very close to the real values.

Since the matrix BB^T is singular for this case, the upper matrix bounds of [43,47,57] cannot work for this case. Since $\sigma_n^2(B) = 0$ and $\sigma_1^2(A) > 1$, the upper matrix bound of [53] cannot work either. Since $\lambda_n(Q) = 0$ and $\sigma_n^2(A) < 1$, the upper matrix bound of [52] cannot work here. Because $\eta \equiv \lambda_1\{A^T[I - B(\lambda_1^{-1}(Q)I + B^TB)^{-1}B^T]A\} > 1$, the upper matrix bound of [63] also cannot work for this case. For this example, the matrix A is not stable, so the upper matrix bound of [33] cannot be applied here. Furthermore, $b_1 \equiv 2\sigma_1^2(B) - 2\lambda_1[\sigma_1^2(B)A^TA - A^TBB^TA] = 0$, so the upper matrix bound of [67] also cannot be used. Since the results of [9] are merely a special case of those in [26] with $M = \alpha I$, where α is a positive constant, only consider the results of [26] will be considered.

With $M = \begin{bmatrix} 4 & 0.1 \\ 0.1 & 0.5 \end{bmatrix}$, the upper matrix bound proposed in [26] gives:

$$P \leq \begin{bmatrix} 2.49 & -0.341 \\ -0.341 & 0.3064 \end{bmatrix}$$

if there exists a positive semi-definite matrix M such that $A^T(I+MBB^T)^{-1}MA+Q \leq M$.

Example 2 (Example from [59])

Consider the DARE (1.6) with:

$$A = \begin{bmatrix} 1.1 & 0 \\ 0.1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}.$$

Then, the unique positive definite solution of the DARE (1.6) is:

$$P_{\text{exact}} = \begin{bmatrix} 3.3340 & 1 \\ 1 & 4 \end{bmatrix}.$$

with $\lambda_n(P_{exact}) = 2.6130$, $\lambda_1(P_{exact}) = 4,7210$, $\text{tr}(P_{exact}) = 7.3340$ and $\det(P_{exact}) = 12.336$.

When $K = \begin{bmatrix} -0.5 & 0.1 \end{bmatrix}$, the upper bounds P_K for the solution of the DARE (1.6) is found from (5.66) to be

$$P_K = \begin{bmatrix} 3.3456 & 1.0377 \\ 1.0377 & 4.1438 \end{bmatrix}$$

Using one iteration of Algorithm 5.4 gives the following tighter upper matrix solution bound for the DARE:

$$M_1 = \begin{bmatrix} 3.3358 & 1 \\ 1 & 4 \end{bmatrix}$$

For M_1, we have $\lambda_n(M_1) = 0.2476$, $\lambda_1(M_1) = 1.6275$, $\text{tr}(M_1) = 1.8751$ and $\det(M_1) = 0.4030$. The resulting bounds are very close to the exact values.

By using Algorithm 5.4 again, the following tighter bound for the solution of the DARE is determined:

$$P \leq M_2 = \begin{bmatrix} 3.3340 & 1 \\ 1 & 4 \end{bmatrix}$$

which is the same as the exact solution of the DARE (1.6). Further use of Algorithm 5.4 will only give the exact solution of the DARE, so the iterations can be terminated.

For this example, the matrix BB^T is singular, so the upper matrix bounds of [43,47,57] cannot work. Also, $\sigma_1^2(A) > 1$, so the upper matrix bound of [53] also cannot work. The matrix A is not stable for this example, so the upper matrix bounds of [33] cannot be applied here. Furthermore, since the matrix A is singular, the upper matrix bounds of [9] cannot work either. However, it is found that the upper matrix bounds of [26,52,63,67] can work for this case.

For this case, the upper matrix bound derived in [52] gives:

$$P \leq \begin{bmatrix} 3.3341 & 1 \\ 1 & 4 \end{bmatrix}.$$

The upper matrix bound proposed in [63] gives the estimate:

$$P \leq \begin{bmatrix} 3.3377 & 1 \\ 1 & 4 \end{bmatrix}.$$

For this example, the upper matrix bound presented in [67] gives:

$$P \leq \begin{bmatrix} 3.3349 & 1 \\ 1 & 4 \end{bmatrix}.$$

With $M = \begin{bmatrix} 3.5 & 1 \\ 1 & 4 \end{bmatrix}$, the upper matrix bound proposed in [26] gives:

$$P \leq \begin{bmatrix} 3.4133 & 1 \\ 1 & 4 \end{bmatrix}.$$

if there exists a positive semi-definite matrix M such that $A^T(I+MBB^T)^{-1}MA+Q \leq M$.

5.3 Summary

The successful derivation of new matrix bounds for the continuous and discrete Riccati matrix equations have been addressed in this chapter. More precisely, the following results have been obtained:
(1) New lower matrix bounds for the CARE solution. These bounds always exist if the CARE has a unique non-negative definite stabilizing solution, and overcome the restriction of nearly all existing lower matrix bounds, that is that Q is positive definite. A comparison in the second numerical example suggests that these bounds can be tighter than some lower matrix bounds existing in the literature, although the examples also suggest it may be possible and worthwhile seeking to improve the tightness of such bounds.
(2) New upper matrix bounds for the DARE solution. The upper bounds of Theorem 5.5 and Corollary 5.1 always exist if the DARE has a unique non-negative definite stabilizing solution, whilst all upper matrix bounds for the DARE reported in the literature [9,26,33,43,47,52,53,57,63,67] are only valid under assumptions in addition to the usual existence conditions for the DARE solution.

Chapter 6

Matrix Bounds of the Continuous Coupled Algebraic Lyapunov and Riccati Equations

In this chapter, the estimation problem of solution bounds for coupled Lyapunov and Riccati matrix equations arising from the analysis and design of stochastic control systems will be considered. Many of the techniques employed for Lyapunov and Riccati equations from a deterministic system shall be applied to the stochastic counterpart here.

6.1 Lower Matrix Bounds for the Continuous Coupled Algebraic Lyapunov Equation

Consider the CCALE (1.9)

$$A_i^T P_i + P_i A_i + \sum_{j \neq i} d_{ij} P_j = -Q_i$$

with $Q_i = Q_i^T \geq 0$, $\forall i$, $i \in S$ and $S = \{1, 2, \ldots\}$. Recall that the constants d_{ij} are such that $d_{ii} < 0$, $d_{ij} \geq 0$ for $i \neq j$, and $\sum_{j \in S} d_{ij} = 0$. Viewing the literature, the only available lower matrix bounds for the CCALE are those which are derivable from the lower matrix bounds for the CCARE (1.10) when $B_i = 0$, $\forall i$. The lower matrix bounds proposed in [13] and [25] yield the trivial lower bound $P_i \geq 0$. Recently, an improved lower matrix bound was reported in [61] which yields a nontrivial lower matrix bound for the CCARE when $Q_i > 0$; for the case of the CCALE when $B_i = 0$, this bound becomes meaningless. Therefore, it appears that no nontrivial lower matrix bound exists for the CCALE. By extension of the method of Section 5.1, this section therefore presents two nontrivial lower matrix bounds to improve this drawback and yield nontrivial lower matrix bounds for the solution of the CCALE. The bounds do not require any condition on the coefficient matrices for them to work, other than that the solution of the CCALE exists.

Theorem 6.1: Define matrices V_i by

$$V_i = A_i - \alpha_i I \tag{6.1}$$

where α_i are positive constants. Let P_i be the positive semi-definite solutions of the CCALE (1.9) for $i = 1, 2, \ldots, n$. Then P_i has the lower bound

$$P_i \geq V_i^{-T}\left(A_i^T P_{i0} A_i + \alpha_i \sum_{j\neq i} d_{ij} P_{j0} + \alpha_i^2 P_{i0} + \alpha_i Q_i\right) V_i^{-1} \equiv P_{(i)CCLL1} \tag{6.2}$$

where the positive semi-definite matrix P_{i0} is defined by

$$P_{i0} \equiv \alpha_i V_i^{-T} Q_i V_i^{-1}. \tag{6.3}$$

Proof: Using the definition of the matrix V_i from (6.1), the following matrix identity can be defined

$$V_i^T P_i V_i = A_i^T P_i A_i - \alpha_i (A_i^T P_i + P_i A_i) + \alpha_i^2 P_i \tag{6.4}$$

Using the CCALE (1.9), (6.4) becomes

$$V_i^T P_i V_i = A_i^T P_i A_i + \alpha_i \sum_{j\neq i} d_{ij} P_j + \alpha_i^2 P_i + \alpha_i Q_i \tag{6.5}$$

Pre- and post-multiplying both sides of (6.5) by V_i^{-T} and V_i^{-1} respectively gives

$$P_i = V_i^{-T}\left(A_i^T P_i A_i + \alpha_i \sum_{j\neq i} d_{ij} P_j + \alpha_i^2 P_i + \alpha_i Q_i\right) V_i^{-1}. \tag{6.6}$$

From (6.6), one has $P_i \geq P_{i0}$, where P_{i0} is defined by (6.3). Substituting (6.3) into (6.6) results in the bound (6.2). This completes the proof of the theorem.

Following the development of Theorem 6.1, we can develop the following computational algorithm to derive sharper lower matrix bounds for the solution of the CCALE (1.9).

Algorithm 6.1:
Step 1: Set $S_i^{(0)} \equiv P_{i0}$, where P_{i0} is defined by (6.3).
Step 2: Calculate

$$S_i^{(k)} = V_i^{-T}\left(A_i^T S_i^{(k-1)} A_i + \alpha_i \sum_{j\neq i} d_{ij} S_j^{(k-1)} + \alpha_i^2 S_i^{(k-1)} + \alpha_i Q_i\right) V_i^{-1}, \quad k = 1, 2, \ldots \tag{6.7}$$

Then $S_i^{(k)}$ are also lower bounds for the solution of the CCALE (1.9). In fact, as $k \to \infty$, $S_i^k \to P_i$, where P_i is the positive semidefinite solution of the CCALE.

Proof: Firstly, $P_i \geq \alpha_i V_i^{-T} Q_i V_i^{-1} = S_i^{(0)}$. Then, using (6.7), we have

$$\begin{aligned} P_i &= V_i^{-T}\left(A_i^T P_i A_i + \alpha_i \sum_{j\neq i} d_{ij} P_j + \alpha_i^2 P_i + \alpha_i Q_i\right) V_i^{-1} \\ &\geq V_i^{-T}\left(A_i^T S_i^{(0)} A_i + \alpha_i \sum_{j\neq i} d_{ij} S_j^{(0)} + \alpha_i^2 S_i^{(0)} + \alpha_i Q_i\right) V_i^{-1} = S_i^{(1)} \geq S_i^{(0)} \end{aligned} \tag{6.8}$$

Now assume $P_i \geq S_i^{(k-1)} \geq S_i^{(k-2)}$. Then, by following the routine of (6.8) and remembering (6.7), we get

$$\begin{aligned} P_i &= V_i^{-T}\left(A_i^T P_i A_i + \alpha_i \sum_{j\neq i} d_{ij} P_j + \alpha_i^2 P_i + \alpha_i Q_i\right) V_i^{-1} \\ &\geq V_i^{-T}\left(A_i^T S_i^{(k-1)} A_i + \alpha_i \sum_{j\neq i} d_{ij} S_j^{(k-1)} + \alpha_i^2 S_i^{(k-1)} + \alpha_i Q_i\right) V_i^{-1} = S_i^{(k)} \\ &\geq V_i^{-T}\left(A_i^T S_i^{(k-2)} A_i + \alpha_i \sum_{j\neq i} d_{ij} S_j^{(k-2)} + \alpha_i^2 S_i^{(k-2)} + \alpha_i Q_i\right) V_i^{-1} = S_i^{(k-1)} \end{aligned}$$

By means of mathematical induction, it can be concluded that $0 \leq S_i^{(0)} \leq S_i^{(1)} \leq \ldots \leq S_i^{(k-1)} \leq S_i^{(k)} \leq P_i$ for $i = 1, 2, \ldots, s$ and $k = 1, 2, 3, \ldots$. Since

$S_i^{(k)}$ is monotone increasing and bounded, there exists $S_i^{(\infty)}$ such that

$$S_i^{(\infty)} = V_i^{-T}\left(A_i^T S_i^{(\infty)} A_i + \alpha_i \sum_{j \neq i} d_{ij} S_j^{(\infty)} + \alpha_i^2 S_i^{(\infty)} + \alpha_i Q_i\right) V_i^{-1}. \tag{6.9}$$

(6.9) is equivalent to (6.6) with $S_i^{(\infty)} = P_i$, so it can be concluded that $P_i = \lim_{k\to\infty} S_i^{(k)}$. This finishes the proof of the algorithm.

A different lower bound is obteined as follows.

Theorem 6.2: The positive semi-definite solution P_i of the CCALE (1.9) has the following lower bound

$$P_i \geq V_i^{-T}\left(2(V_i + 2\alpha_i I)^T P_{i0}(V + 2\alpha_i I) + 4\alpha_i \sum_{j \neq i} d_{ij} P_{j0} + 2\alpha_i Q_i\right) V_i^{-1} \equiv P_{(i)CCLL2} \tag{6.10}$$

where the matrix V_i is defined by (6.1) and the positive semi-definite matrix $\overline{P}_{i0}$ is defined by (6.3).

Proof: Using the definition of V_i from (6.1), the CCALE (1.9) can be rewritten as

$$\begin{aligned} (V_i + \alpha_i I)^T P_i + P_i(V_i + \alpha_i I) + \sum_{j \neq i} d_{ij} P_j + Q_i &= 0 \\ V_i^T P_i + P_i V_i + 2\alpha_i P_i + \sum_{j \neq i} d_{ij} P_j + Q_i &= 0 \end{aligned} \tag{6.11}$$

Multiplying both sides of (6.11) by $2\alpha_i$ and adding $V_i^T P_i V_i$ gives

$$V_i^T P_i V_i + 2\alpha_i P_i V_i + 2\alpha_i V_i^T P_i + 4\alpha_i^2 P_i + 2\alpha_i \sum_{j \neq i} d_{ij} P_j + 2\alpha_i Q_i = V_i^T P_i V_i \tag{6.12}$$

By realizing that

$$(V_i + 2\alpha_i I)^T P_i (V_i + 2\alpha_i I) = V_i^T P_i V_i + 2\alpha_i P_i V_i + 2\alpha_i V_i^T P_i + 4\alpha_i^2 P_i$$

(6.12) becomes

$$V_i^T P_i V_i = (V_i + 2\alpha_i I)^T P_i (V_i + 2\alpha_i I) + 2\alpha_i \sum_{j \neq i} d_{ij} P_j + 2\alpha_i Q_i \tag{6.13}$$

Pre- and post-multiplying both sides of (6.13) by V_i^{-T} and V_i^{-1} respectively leads to

$$P_i = V_i^{-T}\left((V_i + 2\alpha_i I)^T P_i (V_i + 2\alpha_i I) + 2\alpha_i \sum_{j\neq i} d_{ij} P_j + 2\alpha_i Q_i\right) V_i^{-1}. \quad (6.14)$$

From (6.14) we obtain $P_i \geq 2\alpha_i V_i^{-T} Q_i V_i^{-1} = 2P_{i0}$, where P_{i0} is defined by (6.3). Substituting this bound into (6.14) leads to the bound (6.10). This finishes the proof of the theorem.

Having completed the proof of Theorem 6.2, the following computational algorithm can be developed for deriving more precise lower solution bounds of the CCALE (1.9).

Algorithm 6.2:
Step 1: Set $\overline{S}_i^{(0)} = 2P_{i0}$, where P_{i0} is defined by (6.3).
Step 2: Calculate

$$\overline{S}_i^{(k)} = V_i^{-T}\left((A_i + \alpha_i I)^T \overline{S}_i^{(k-1)} (A_i + \alpha_i I) + 2\alpha_i \sum_{i\neq j} d_{ij} \overline{S}_j^{(k-1)} + 2\alpha_i Q_i\right) V_i^{-1},$$

$$k = 1, 2, 3, \ldots$$

Then $\overline{S}_i^{(k)}$ are also lower solution bounds for the CCALE (1.9). In fact, as $k \to \infty$, $\overline{S}_i^{(k)} \to P_i$, where P_i is the positive semidefinite solution of the CCALE.

The proof of this algorithm is similar to that of Algorithm 6.1, and is therefore omitted.

Remark 6.1: The only existing meaningful lower matrix bound for the CCALE seems to be $P_i \geq 0$. This is trivial, and the least sharp bound possible. Our bounds for the CCALE are always calculated if the solutions of the CCALE exist, and always yield nontrivial lower matrix bounds for the CCALE, even when $Q_i \geq 0$. Also, these bounds are always tighter than $P_i \geq 0$, and are concise.

Remark 6.2: From (6.2) and (6.10), it can be seen that

$$P_{(i)CCLL2} = V_i^{-T}\left(A_i^T P_{i0} A_i + \alpha_i \sum_{j\neq i} d_{ij} P_{j0} + \alpha_i^2 P_{i0} + \alpha_i Q_i + A_i^T P_{i0} A_i + \alpha_i \sum_{j\neq i} d_{ij} P_{j0} + \right.$$
$$\left. 2\alpha_i (A_i^T P_{i0} + P_{i0} A_i) + \alpha_i^2 P_{i0} + \alpha_i Q_i \right) V_i^{-1}$$
$$= P_{(i)CCLL1} + V_i^{-T}\left((A_i + \alpha_i I)^T P_{i0} (A_i + \alpha_i I) + \alpha_i \left(A_i^T P_{i0} + P_{i0} A_i + Q_i + \sum_{i\neq j} d_{ij} P_{j0}\right)\right) V_i^{-1}$$

Therefore, if $(A_i+\alpha_i I)^T P_{i0}(A_i+\alpha_i I)+\alpha_i \left(A_i^T P_{i0} + P_{i0} A_i + Q_i + \sum_{i\neq j} d_{ij} P_{j0}\right) \geq 0$, then $P_{(i)CCLL2}$ is tighter than $P_{(i)CCLL1}$, whereas if $(A_i + \alpha_i I)^T P_{i0}(A_i + \alpha_i I) + \alpha_i (A_i^T P_{i0} + P_{i0} A_i + Q_i + \sum_{i\neq j} d_{ij} P_{j0}) \leq 0$, then $P_{(i)CCLL1}$ is sharper than $P_{(i)CCLL2}$.

6.1.1 Numerical Example for the CCALE

In this subsection, a numerical example is given to show the effectiveness of the derived lower matrix bounds for the solution of the CCALE, and when possible, give comparisons with existing bounds.

Consider the CCALE (1.9) with:

$$A_1 = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -3 & 0 \\ 1 & -4 \end{bmatrix}, \quad (d_{ij})_{i,j\in S} = \begin{bmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}$$
$$Q_1 = \begin{bmatrix} 6 & 0.5 \\ 0.5 & 12 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 13 & 26.5 \\ 26.5 & 62.5 \end{bmatrix}, \quad S = \{1, 2\}$$

Then the exact solutions P_{1exact} and P_{2exact} of the CCALE with these system matrices are:

$$P_{1exact} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \quad \text{and} \quad P_{2exact} = \begin{bmatrix} 4 & 5 \\ 5 & 8 \end{bmatrix}$$

The only available lower matrix bounds for the solution of the CCALE is, to the best of our knowledge, $P_i \geq 0$. This is a trivial bound. However, the bounds P_{L1} and P_{L2} can be applied to this case to give nontrivial solution estimates. By Theorem 6.1 the bound $P_{(i)CCLL1}$ is found, for P_1 and P_2, to be:

$$P_1 \geq \begin{bmatrix} 1.1277 & 0.4152 \\ 0.4152 & 1.2023 \end{bmatrix} \quad \text{and} \quad P_2 \geq \begin{bmatrix} 2.5898 & 3.1499 \\ 3.1499 & 4.2088 \end{bmatrix}$$

By Theorem 6.2 the bound $P_{(i)CCLL2}$ is found, for P_1 and P_2, to be:

$$P_1 \geq \begin{bmatrix} 1.6627 & 0.6198 \\ 0.6198 & 2.0602 \end{bmatrix} \quad \text{and} \quad P_2 \geq \begin{bmatrix} 3.7861 & 4.7848 \\ 4.7848 & 6.8175 \end{bmatrix}$$

For this case, it is seen that P_{L2} gives the better solution estimate for both P_1 and P_2. Using 2 iterations of Algorithm 6.2, the following tighter lower matrix bounds for the solution matrices P_1 and P_2 are obtained:

$$\overline{S_1}^{(2)} = \begin{bmatrix} 1.9388 & 0.9315 \\ 0.9315 & 2.7347 \end{bmatrix}$$

$$\overline{S_2}^{(2)} = \begin{bmatrix} 3.9233 & 4.9780 \\ 4.9780 & 7.5367 \end{bmatrix}$$

$$\overline{S_1}^{(3)} = \begin{bmatrix} 1.9847 & 0.9880 \\ 0.9880 & 2.9130 \end{bmatrix}$$

$$\overline{S_2}^{(3)} = \begin{bmatrix} 3.9722 & 5.0151 \\ 5.0151 & 7.8226 \end{bmatrix}$$

From the above calculations, it can be seen that as more iterations are performed, the bounds become tighter.

6.2 Lower Matrix Bound for the Continuous Coupled Algebraic Riccati Equation

Consider the CCARE (1.10)

$$A_i^T P_i + P_i A_i - P_i B_i B_i^T P_i + \sum_{j \neq i} d_{ij} P_j = -Q_i$$

with $Q_i = Q_i^T \geq 0$, $\forall i \in S$, and $S = \{1, 2, \ldots\}$. Recall that the constants d_{ij} are such that $d_{ii} < 0$, $d_{ij} \geq 0$ for $i \neq j$, and $\sum_{j \in S} d_{ij} = 0$. In [13] and [25], the lower bound $P_i \geq 0$ was reported for the CCARE (1.10), which is obvious. Recently, an improved lower matrix bound for the CCARE was reported in [61], which provides a nontrivial lower matrix bound for the CCARE (1.10) when $Q_i > 0$. In this section, Lee's method [55] is extended to derive a less conservative, possibly sharper lower matrix bound for the CCARE which is always calculated if $Q_i > 0$. A numerical example is then given to demonstrate the effectiveness of the derived bound, and a comparison is also made with the lower bound derived in [61].

Theorem 6.3: Let P_i be the positive definite solutions of the CCARE (1.10). Then P_i has the lower bound

$$P_i \geq G_i^{-1} \left[G_i \left(\sum_{j \neq i} d_{ij} \overline{P}_{(j)0} + Q_i - A_i^T R_i A_i \right) G_i \right]^{1/2} G_i^{-1} \equiv P_{CCRL1} \quad (6.15)$$

where the symmetric positive definite matrix R_i is chosen such that $Q_i > A_i^T R_i A_i$, and the positive definite matrices $\overline{P}_{i0}$ and G_i are defined by

$$\overline{P}_{(i)0} \equiv G_i^{-1}[G_i(Q_i - A_i^T R_i A_i)G_i]^{1/2} G_i^{-1}, \quad (6.16)$$

$$G_i \equiv (R_i^{-1} + B_i B_i^T)^{1/2}. \quad (6.17)$$

Proof: Define a positive semi-definite matrix Π as:

$$\begin{aligned}\Pi &\equiv \left(R_i^{-1/2} P_i + R_i^{1/2} A_i\right)^T \left(R_i^{-1/2} P_i + R_i^{1/2} A_i\right) \\ &= P_i R_i^{-1} P_i + P_i A_i + A_i^T P_i + A_i^T R_i A_i \geq 0. \end{aligned} \quad (6.18)$$

where R_i is a symmetric positive definite matrix. Substituting the CCARE (1.10) into (6.18) gives:

$$P_i R_i^{-1} P_i + P_i B_i B_i^T P_i \geq \sum_{j \neq i} d_{ij} P_j + Q_i - A_i^T R_i A_i. \quad (6.19)$$

From (6.19), it is obvious that

$$P_i (R_i^{-1} + B_i B_i^T) P_i \geq Q_i - A_i^T R_i A_i. \quad (6.20)$$

Pre- and post-multiplying both sides of (6.20) by G_i, where G_i is defined by (6.17), leads to

$$G_i P_i G_i^2 P_i G_i = (G_i P_i G_i)^2 \geq G_i (Q_i - A_i^T R_i A_i) G_i. \quad (6.21)$$

Solving the inequality (6.21) with respect to P_i leads to the positive definite matrix $\overline{P}_{(i)0}$, which is defined by (6.16). Substituting (6.16) into (6.19) gives:

$$P_i G_i^2 P_i \geq \sum_{j \neq i} d_{ij} \overline{P}_{(j)0} + Q_i - A_i^T R_i A_i. \quad (6.22)$$

Pre- and post-multiplying both sides of (6.22) by G_i^{-1} leads to

$$(G_iP_iG_i)^2 \geq G_i\left(\sum_{j\neq i} d_{ij}\overline{P}_{(j)0}Q_i - A_i^TR_iA_i\right)G_i. \tag{6.23}$$

Solving (6.23) for P_i yields the lower bound (6.15). This ends the proof of the theorem.

Corollary 6.1: Based on the lower matrix bound (6.15), the following eigenvalue lower bounds for the solution of the CCARE (1.10) can be defined:

$$\lambda_i(P_i) \geq \lambda_n\left\{G_i^{-1}\left[G_i\left(\sum_{j\neq i} d_{ij}\overline{P}_{(j)0} + Q_i - A_i^TR_iA_i\right)G_i\right]^{1/2}G_i^{-1}\right\}, \quad i = 1, 2, \ldots, n$$

$$tr(P_i) \geq tr\left\{G_i^{-1}\left[G_i\left(\sum_{j\neq i} d_{ij}\overline{P}_{(j)0} + Q_i - A_i^TR_iA_i\right)G_i\right]^{1/2}G_i^{-1}\right\},$$

$$det(P_i) \geq det\left\{G_i^{-1}\left[G_i\left(\sum_{j\neq i} d_{ij}\overline{P}_{(j)0} + Q_i - A_i^TR_iA_i\right)G_i\right]^{1/2}G_i^{-1}\right\}.$$

Remark 6.3: The lower matrix bound (6.15) gives a nontrivial lower solution estimate for the CCARE which always works if $Q_i > 0$. Although this bound may be somewhat restrictive, it is still tighter than the lower matrix bounds for the CCARE presented in [13] and [25], and may be tighter than the following lower matrix bound for the CCARE, which was reported in [61]:

$$P_i \geq \frac{2\lambda_n(Q_i)}{-\lambda_n(A_i + A_i^T) + [\lambda_n(A_i + A_i^T)^2 + 4\sigma_1^2(B_i)\lambda_n(Q_i)]^{1/2}}I \equiv P_{CCRL2} \tag{6.24}$$

The bound (6.24) only provides a meaning, nontrivial lower solution estimate when $Q_i > 0$. When $Q_i \geq 0$, the bound becomes meaningless. As with [58], a table is given in the numerical example of this section which list choices of R_i to ease the calculation of bound (6.15). It is also found impossible to compare mathematically the tightness between the bounds P_{CCRL1} and P_{CCRL2}.

Remark 6.4: Theorem 6.3 gives a lower matrix bound for the solution of the CCARE (1.10). When $B_i = 0$ and A_i is a stochastically stable matrix, the CCARE (1.10) becomes the CCALE (1.9). As such, setting $B_i = 0$ and A_i to be stochastically stable in the bound (6.15) gives the following lower matrix bound for the CCALE (1.9):

$$P_i \geq R_i^{1/2} \left[R_i^{-1/2} \left(\sum_{j \neq i} d_{ij} \underline{P}_{(j)0} + Q_i - A_i^T R_i A_i \right) R_i^{-1/2} \right] R_i^{1/2} \equiv P_{CCLL3} \tag{6.25}$$

where the symmetric positive definite matrix R_i is chosen such that $Q_i > A_i^T R_i A_i$, and the positive definite matrix $\underline{P}_{(i)0}$ is now defined by

$$\overline{P}_{(i)0} \equiv R_i^{1/2} [R_i^{-1/2} (Q_i - A_i^T R_i A_i) R_i^{-1/2}]^{1/2} R_i^{1/2}$$

The bound (6.25) is always calculated if $Q_i > 0$. Furthermore, it appears that the tightness between the bound P_{CCLL3} and the bounds P_{CCLL1} and P_{CCLL2} cannot be compared mathematically.

Remark 6.5: As in [58], choices of the matrix R_1 can be listed to simplify the calculation of the bound (6.15). These choices, together with the range of the parameter ϵ_i, are listed in the following table:

R_i	Range of parameter ϵ_i
$\left[\frac{1}{\epsilon_i} I - B_i B_i^T\right]^{-1}$	$0 < \epsilon_i < \lambda_1^{-1}(B_i B_i^T + A_i Q_i^{-1} A_i^T)$
$\left[\frac{1}{\epsilon_i} Q_i - B_i B_i^T\right]^{-1}$	$0 < \epsilon_i < \lambda_1^{-1}[(B_i B_i^T + A_i Q_i^{-1} A_i^T) Q_i^{-1}]$
$\frac{1}{\epsilon_i} I$	$\epsilon_i > \lambda_1(A_i^T Q_i^{-1} A_i)$
$\epsilon_i Q_i$	$0 < \epsilon_i \lambda_1^{-1}(A_i^T Q_i A_i Q_i^{-1})$
$\epsilon_i Q_i^{-1}$	$0 < \epsilon_i < \lambda_1^{-1}(A_i^T Q_i^{-1} A_i Q_i^{-1})$
$\epsilon(A_i A_i^T)^{-1}$	$0 < \epsilon_i < \lambda_n(Q_i)$
$\epsilon(A_i Q_i^{-1} A_i^T)^{-1}$	$0 < \epsilon_i < 1$
$\epsilon(A_i Q_i A_i^T)^{-1}$	$0 < \epsilon_i < \lambda_n^2(Q_i)$

Table 6.1: Choices of R_i and the corresponding range of parameter ϵ_i

Next, a numerical example is given to show the effectiveness of the derived bound (6.15).

6.2.1 Numerical Example for the CCARE

Consider the CCARE (1.10) with:

$$A_1 = \begin{bmatrix} -3 & 1 \\ 0 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 4 & 0 \\ 2 & -3 \end{bmatrix}, \quad (d_{ij})_{i,j\in S} = \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 79 & 134 \\ 134 & 268 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 189 & 116 \\ 116 & 87 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad S = \{1, 2\}.$$

Then the exact solutions P_{1exact}, P_{2exact} and P_{3exact} of the CCALE with these system matrices are:

$$P_{1exact} = \begin{bmatrix} 5 & 4 \\ 4 & 9 \end{bmatrix}, \quad P_{2exact} = \begin{bmatrix} 5 & 3 \\ 3 & 4 \end{bmatrix}.$$

Also, $\operatorname{tr}(P_{1exact}) = 14$ and $\operatorname{tr}(P_{2exact}) = 12$.The trivial bound $P_i \geq 0$ for the CCARE was reported in [13] and [25]. The only nontrivial bound available for the CCARE is the bound (6.24), which was proposed in [61]. As such, the bound P_{CCRL1} will be compared only with bound (6.24).

Using some of the listed choices for R_i in Table 6.1, the lower bound P_{CCRL1} for the CCARE (1.10) can now be calculated.

(i) With $R_1 = 0.5(A_1Q_1^{-1}A_1^T)^{-1}$ and $R_2 = 0.5(A_2Q_2^{-1}A_2^T)^{-1}$, one has:

$$P_1 \geq \begin{bmatrix} 3.2114 & 1.9519 \\ 1.9519 & 5.2693 \end{bmatrix}, \quad P_2 \geq \begin{bmatrix} 2.8046 & 1.3280 \\ 1.3280 & 3.2349 \end{bmatrix}.$$

(ii) With $R_1 = R_2 = 0.1I$, the bound P_{CCRL1} yields:

$$P_1 \geq \begin{bmatrix} 2.0519 & 1.7794 \\ 1.7794 & 4.1189 \end{bmatrix}, \quad P_2 \geq \begin{bmatrix} 2.8177 & 1.6067 \\ 1.6067 & 2.1258 \end{bmatrix}.$$

For this example, the lower bound P_{CCRL4} reported in [61] gives:

$$P_1 \geq 0.9524I, \quad P_2 \geq 0.8239I.$$

In view of these numerical experiments, our lower matrix bound is tighter than the lower matrix bound (6.24) reported in [61] for this case, suggesting that the presented bound may be tighter than the lower bound (6.24) reported in [61].

6.3 Upper Matrix Bounds for the Continuous Coupled Algebraic Riccati Equation

In this section, the upper matrix bounds for the CCARE (1.10) with $Q_i = Q_i^T \geq 0$ are derived. Viewing the literature, it appears that only [14] has derived an upper solution bound for the CCARE, which is an upper summation bound for the maximal eigenvalue of the solutions P_i of the CCARE, which has to assume that the matrix $B_iB_i^T$ is nonsingular for it to be valid. Furthermore, based on [13,14,25,61], there does not yet appear to exist a non-trivial upper matrix bound for the CCARE in the literature. Here, it is the aim to derive upper matrix bounds for the CCARE which are non-trivial under the satisfaction of a required condition. Based on these matrix bounds, individual eigenvalue bounds for the solutions P_i can then be defined. The derivation makes particular use of (2.7) and which is a preliminary result concerning the coupling term for the CCARE, as well as some other lemmas. The results of this section can also be found in [20].

Theorem 6.4: Let P_i be the positive semi-definite solution matrices of the CCARE (1.10). If there exist positive constants α_i such that

$$a < 1 \tag{6.26}$$

and

$$A_i + A_i^T < 2\alpha_i B_i B_i^T \tag{6.27}$$

then P_i has the upper bound

$$P_i \leq V_i^{-T}\left(\eta\left[(V_i+I)^T(V_i+I)+I+\left(\sum_{j\neq i} d_{ij}\right)I\right]+Q_i+\alpha_i^2 B_i B_i^T\right)V_i^{-1} \equiv P_{iCCRU1} \tag{6.28}$$

where the matrix V_i and positive constant η are defined, respectively, by:

$$V_i \equiv A_i - \alpha_i B_i B_i^T - I \tag{6.29}$$

$$\eta = \frac{\sum_{i\in S}\lambda_1[V_i^{-T}(Q_i+\alpha_i^2 B_i B_i^T)V_i^{-1}]}{1-a} \tag{6.30}$$

and where the positive constant a is defined by

$$a = \sum_{i\in S}\lambda_1\{V_i^{-T}[(V_i+I)^T(V_i+I)+I]V_i^{-1}\}+\sum_{i\in S}\lambda_1(V_i^{-T}V_i^{-1})(s-1)\max_{i,j\in S, j\neq i}\{d_{ij}\} \tag{6.31}$$

Proof: By use of the identity

$$P_iB_iB_i^TP_i = (P_i - \alpha_i I)B_iB_i^T(P_i - \alpha_i I) + \alpha_i B_iB_i^TP_i + \alpha_i P_iB_iB_i^T - \alpha_i^2 B_iB_i^T$$

the CCARE (1.10) can be rewritten as

$$(A_i-\alpha_iB_iB_i^T)^TP_i+P_i(A_i-\alpha_iB_iB_i^T)+\sum_{j\neq i} d_{ij}P_j+Q_i+\alpha_i^2B_iB_i^T = (P_i-\alpha_iI)B_iB_i^T(P_i-\alpha_iI) \tag{6.32}$$

With the aid of the identity

$$V_i^TP_iV_i = (V_i + I)^TP_i(V_i + I) - (A_i - \alpha_iB_iB_i^T)^TP_i - P_i(A_i - \alpha_iB_iB_i^T) + P_i$$

where V_i is defined by (6.29), (6.32) can be re-written as

$$\begin{aligned}(V_i + I)^TP_i(V_i + I) + P_i + \sum_{j\neq i} d_{ij}P_j + Q_i + \alpha_i^2B_iB_i^T &= V_i^TP_iV_i \\ &+(P_i - \alpha_iI)B_iB_i^T(P_i - \alpha_iI) \geq V_i^TP_iV_i \end{aligned} \tag{6.33}$$

Since $\Re(\lambda(A_i - \alpha_iB_iB_i^T)) \leq \mu(A_i - \alpha_iB_iB_i^T) \equiv \frac{1}{2}\lambda_1(A_i + A_i^T - 2\alpha_iB_iB_i^T)$, it can be seen that if the condition (6.27) is fulfilled, where the positive constant a is defined by (6.31) then V_i is nonsingular and it is implied from (6.33) that

$$P_i \leq V_i^{-T}\left[(V_i + I)^TP_i(V_i + I) + P_i + \sum_{j\neq i} d_{ij}P_j + Q_i + \alpha_i^2B_iB_i^T\right]V_i^{-1} \tag{6.34}$$

Application of (2.3) to (6.34) gives

$$P_i \leq V_i^{-T}\left[[(V_i + I)^T(V_i + I) + I]\lambda_1(P_i) + \sum_{j\neq i} d_{ij}\lambda_1(P_j)I + Q_i + \alpha_i^2B_iB_i^T\right]V_i^{-1} \tag{6.35}$$

Using (2.1), one has from (6.35) that

$$\begin{aligned}\lambda_1(P_i) &\leq \lambda_1\left\{V_i^{-T}\left[[(V_i + I)^T(V_i + I) + I]\lambda_1(P_i) + \sum_{j\neq i} d_{ij}\lambda_1(P_j)I + Q_i + \alpha_i^2B_iB_i^T\right]V_i^{-1}\right\} \\ &\leq \lambda_1\{V_i^{-T}[(V_i + I)^T(V_i + I) + I]V_i^{-1}\}\lambda_1(P_i) + \lambda_1(V_i^{-T}V_i^{-1})\sum_{j\neq i} d_{ij}\lambda_1(P_j) + \\ &\qquad \lambda_1[V_i^{-T}(Q_i + \alpha_i^2B_iB_i^T)V_i^{-1}]\end{aligned}$$

$$(6.36)$$

Summing (6.36) over $i \in S$ and using (2.6) and (2.7) gives

$$\begin{aligned}
\sum_{i\in S}\lambda_1(P_i) &\le \sum_{i\in S}\left(\lambda_1\{V_i^{-T}[(V_i+I)^T(V_i+I)+I]V_i^{-1}\}\lambda_1(P_i)\right)+ \\
&\quad \sum_{i\in S}\left\{\lambda_1(V_i^{-T}V_i^{-1})\sum_{j\neq i}d_{ij}\lambda_1(P_j)\right\}+\sum_{i\in S}\lambda_1[V_i^{-T}(Q_i+\alpha_i^2B_iB_i^T)V_i^{-1}] \\
&\le \sum_{i\in S}\lambda_1\{V_i^{-T}[(V_i+I)^T(V_i+I)+I]V_i^{-1}\}\sum_{i\in S}\lambda_1(P_i)+\sum_{i\in S}\lambda_1(V_i^{-T}V_i^{-T})\sum_{i\in S}\sum_{j\neq i}d_{ij}\lambda_1(P_j) \\
&\qquad +\sum_{i\in S}\lambda_1[V_i^{-T}(Q_i+\alpha_i^2B_iB_i^T)V_i^{-1}] \\
&\le \sum_{i\in S}\lambda_1\{V_i^{-T}[(V_i+I)^T(V_i+I)+I]V_i^{-1}\}\sum_{i\in S}\lambda_1(P_i)+ \\
&\quad \sum_{i\in S}\lambda_1(V_i^{-T}V_i^{-T})(n-1)\max_{i,j\in S,j\neq i}\{d_{ij}\}\sum_{i\in S}\lambda_1(P_i)+\sum_{i\in S}\lambda_1[V_i^{-T}(Q_i+\alpha_i^2B_iB_i^T)V_i^{-1}] \\
&= \left(\sum_{i\in S}\lambda_1\{V_i^{-T}[(V_i+I)^T(V_i+I)+I]V_i^{-1}\}+\sum_{i\in S}\lambda_1(V_i^{-T}V_i^{-T})(n-1)\max_{i,j\in S,j\neq i}\{d_{ij}\}\right)\sum_{i\in S}\lambda_1(P_i) \\
&\qquad +\sum_{i\in S}\lambda_1[V_i^{-T}(Q_i+\alpha_i^2B_iB_i^T)V_i^{-1}] \\
&= a\sum_{i\in S}\lambda_1(P_i)+\sum_{i\in S}\lambda_1[V_i^{-T}(Q_i+\alpha_i^2B_iB_i^T)V_i^{-1}] \qquad (6.37)
\end{aligned}$$

where the positive constant a is defined by (6.31). Under the satisfaction of condition (6.26), (6.37) implies that

$$\sum_{i\in S}\lambda_1(P_i)\le\eta \qquad (6.38)$$

where η is defined by (6.30). From (6.38), it is immediately obvious that $\lambda_1(P_i) \le \eta$, which, on substituting into (6.35), leads to the upper bound (6.28). This finishes the proof of Theorem 6.5.

Having successfully derived the upper bound (6.28) in Theorem 6.5, the following iterative algorithm to obtain tighter upper matrix bounds for the CCARE can be proposed.

Algorithm 6.3:
Step 1: Set $X_i^{(0)} = P_{CCRU1}$, where P_{CCRU1} is defined by (6.28).
Step 2: Compute

$$X_i^{(k+1)} = V_i^{-T}\left[(V_i+I)^T X_i^{(k)}(V_i+I) + X_i^{(k)} + \sum_{j\neq i} d_{ij}X_j^{(k)} + Q_i + \alpha_i^2 B_i B_i^T\right] V_i^{-1} \tag{6.39}$$

$$k = 0, 1, 2, \ldots$$

Then $X_i^{(k)}$ are upper solution bounds of the CCARE (1.10).

Proof: Letting $k = 0$ in (6.39) gives

$$X_i^{(1)} = V_i^{-T}\left[(V_i+I)^T X_i^{(0)}(V_i+I) + X_i^{(0)} + \sum_{j\neq i} d_{ij}X_j^{(0)} + Q_i + \alpha_i^2 B_i B_i^T\right] V_i^{-1} \tag{6.40}$$

Application of (2.3) to (6.40) leads to

$$X_i^{(1)} \leq V_i^{-T}\left[\lambda_1(X_i^{(0)})[(V_i+I)^T(V_i+I)+I] + \left(\sum_{j\neq i} d_{ij}\lambda_1(X_j^{(0)})\right) I + Q_i + \alpha_i^2 B_i B_i^T\right] V_i^{-1} \tag{6.41}$$

Using the definition of $X_i^{(0)} = P_{iu1}$ and the upper bound (6.28), it is found from (6.41) that

$$\begin{aligned}
\lambda_1(X_i^{(0)}) &\leq \lambda_1\left\{V_i^{-T}\left(\eta[(V_i+I)^T(V_i+I)] + \left(\sum_{j\neq i} d_{ij}\right)\eta I + Q_i + \alpha_i^2 B_i B_i^T\right) V_i^{-1}\right\} \\
&\leq \eta\lambda_1\{V_i^{-T}[(V_i+I)^T(V_i+I)+I]V_i^{-1}\} + \lambda_1(V_i^{-T}V_i^{-1})\left(\sum_{j\neq i} d_{ij}\right)\eta \\
&\quad +\lambda_1[V_i^{-T}(Q_i+\alpha_i^2 B_i B_i^T)V_i^{-1}] \\
&\leq \eta\left[\sum_{i\in S}\lambda_1\{V_i^{-T}[(V_i+I)^T(V_i+I)+I]V_i^{-1}\} + \left(\sum_{i\in S}\lambda_1(V_i^{-T}V_i^{-1})\right)(s-1)\max_{i,j\in S, j\neq i}\{d_{ij}\}\right] \\
&\quad +\sum_{i\in S}\lambda_1[V_i^{-T}(Q_i+\alpha_i^2 B_i B_i^T)V_i^{-1}] \\
&= \beta\eta + \sum_{i\in S}\lambda_1[V_i^{-T}(Q_i+\alpha_i^2 B_i B_i^T)V_i^{-1}] + \sum_{i\in S}\lambda_1[V_i^{-T}(Q_i+\alpha_i^2 B_i B_i^T)V_i^{-1}] \\
&= \eta\left(1 - \frac{\sum_{i\in S}\lambda_1[V_i^{-T}(Q_i+\alpha_i^2 B_i B_i^T)V_i^{-1}]}{\eta}\right) + \sum_{i\in S}\lambda_1[V_i^{-T}(Q_i+\alpha_i^2 B_i B_i^T)V_i^{-1}] = \eta
\end{aligned} \tag{6.42}$$

where (2.1) has been employed, and (6.30) has been taken into account. Substituting (6.42) into (6.41) leads to

$$X_i^{(1)} \le V_i^{-T}\left[\eta[(V_i+I)^T(V_i+I)+I]+\left(\sum_{j\neq i} d_{ij}\eta\right)I+Q_i+\alpha_i^2B_iB_i^T\right]V_i^{-1}=X_i^{(0)}$$

Assume now that $X_i^{(k)} \le X_i^{(k-1)}$. Then

$$X_i^{(k+1)}=V_i^{-T}\left[(V_i+I)^TX_i^{(k)}(V_i+I)+X_i^{(k)}+\sum_{j\neq i}d_{ij}X_j^{(k)}+Q_i+\alpha_i^2B_iB_i^T\right]V_i^{-1}$$

$$\le V_i^{-T}\left[(V_i+I)^TX_i^{(k-1)}(V_i+I)+X_i^{(k-1)}+\sum_{j\neq i}d_{ij}X_j^{(k)}+Q_i+\alpha_i^2B_iB_i^T\right]V_i^{-1}=X_i^{(k)}$$

By use of mathematical induction, one can draw the conclusion that $X_i^{(k)} \le X_i^{(k-1)} \le \ldots \le X_i^{(1)} \le X_i^{(0)}$. This brings the proof of the algorithm to an end.

Another upper matrix bound for the CCARE (1.10) can be derived as follows.

Theorem 6.5: If there exist positive constants α_i such that (6.46) holds and such that

$$b<1 \tag{6.43}$$

then P_i has the upper bound

$$P_i \le V_i^{-T}\left[\varphi\left[(V_i+2I)^T(V_i+2I)+\left(2\sum_{j\neq i}d_{ij}\varphi\right)I\right]+2(Q_i+\alpha_i^2B_iB_i^T)\right]V_i^{-1}\equiv P_{iCCRU2} \tag{6.44}$$

where the matrix V_i is defined by (6.29), and the positive constants b and φ are defined, respectively, by

$$b\equiv\sum_{i\in S}\lambda_1[V_i^{-T}(V_i+2I)^T(V_i+2I)V_i^{-1}]+2\sum_{i\in S}\lambda_1(V_i^{-T}V_i^{-1})(s-1)\max_{i,j\in S,j\neq i}\{d_{ij}\} \tag{6.45}$$

$$\varphi\equiv\frac{2\sum_{i\in S}\lambda_1[V_i^{-T}(Q_i+\alpha_i^2B_iB_i^T)V_i^{-1}]}{1-b}. \tag{6.46}$$

Proof: Using the definition of V_i from (6.29), one has from (6.32) that

$$(V_i+I)^TP_i+P_i(V_i+I)+\sum_{j\neq i}d_{ij}P_j+Q_i+\alpha_i^2B_iB_i^T=(P_i-\alpha_iI)B_iB_i^T(P_i-\alpha_iI). \tag{6.47}$$

Multiplying both sides of (6.47) by 2 and adding $V_i^T P_i V_i$ gives

$$V_i^T P_i V_i + 2V_i^T P_i + 2P_i V_i + 4P_i + 2\sum_{j\neq i} d_{ij} P_j + 2(Q_i + \alpha_i^2 B_i B_i^T) =$$
$$V_i^T P_i V_i + 2(P_i - \alpha_i I) B_i B_i^T (P_i - \alpha_i I). \quad (6.48)$$

By realizing that

$$(V_i + 2I)^T P_i (V_i + 2I) = V_i^T P_i V_i + 2V_i^T P_i + 2P_i V_i + 4P_i$$

(6.48) becomes

$$V_i^T P_i V_i \leq (V_i + 2I)^T P_i (V_i + 2I) + 2\sum_{j\neq i} d_{ij} P_j + 2(Q_i + \alpha_i^2 B_i B_i^T). \quad (6.49)$$

Since V_i is nonsingular, pre- and post-multiplying (6.49) by V_i^{-T} and V_i^{-1} respectively results in

$$P_i \leq V_i^{-T} \left[(V_i + 2I)^T P_i (V_i + 2I) + 2\sum_{j\neq i} d_{ij} P_j + 2(Q_i + \alpha_i^2 B_i B_i^T) \right] V_i^{-1}. \quad (6.50)$$

Applying (2.3) to (6.50) leads to

$$P_i \leq V_i^{-T} \left[\lambda_1(P_i)(V_i + 2I)^T (V_i + 2I) + 2\sum_{j\neq i} d_{ij} \lambda_1(P_j) I + 2(Q_i + \alpha_i^2 B_i B_i^T) \right] V_i^{-1}. \quad (6.51)$$

Utilizing (2.1), (6.51) becomes

$$\lambda_1(P_i) \leq \lambda_1 \left\{ V_i^{-T} \left[\lambda_1(P_i)(V_i + 2I)^T (V_i + 2I) + 2\sum_{j\neq i} d_{ij} \lambda_1(P_j) I + 2(Q_i + \alpha_i^2 B_i B_i^T) \right] V_i^{-1} \right\}$$
$$\leq \lambda_1[V_i^{-T}(V_i + 2I)^T (V_i + 2I) V_i^{-1}] \lambda_1(P_i) + 2\lambda_1(V_i^{-T} V_i^{-1}) \sum_{j\neq i} d_{ij} \lambda_1(P_j) +$$
$$2\lambda_1[V_i^{-T}(Q_i + \alpha_i^2 B_i B_i^T) V_i^{-1}]. \quad (6.52)$$

Summing (6.52) over $i \in S$ and using (2.6) and (2.7), it is found that

$$\sum_{i\in S} \lambda_1(P_i) \leq \sum_{i\in S} \left(\lambda_1[V_i^{-T}(V_i + 2I)^T (V_i + 2I) V_i^{-1}] \right) \lambda_1(P_i)$$

$$
\begin{aligned}
&+2\sum_{i\in S}\left\{\lambda_1(V_i^{-T}V_i^{-1})\sum_{j\neq i}d_{ij}\lambda_1(P_j)\right\}+2\sum_{i\in S}\lambda_1[V_i^{-T}(Q_i+\alpha_i^2B_iB_i^T)V_i^{-1}]\\
&\le\sum_{i\in S}\left(\lambda_1[V_i^{-T}(V_i+2I)^T(V_i+2I)V_i^{-1}]\right)\sum_{i\in S}\lambda_1(P_i)+2\sum_{i\in S}\lambda_1(V_i^{-T}V_i^{-1})\sum_{i\in S}\sum_{j\neq i}d_{ij}\lambda_1(P_j)+\\
&\qquad+2\sum_{i\in S}\lambda_1[V_i^{-T}(Q_i+\alpha_i^2B_iB_i^T)V_i^{-1}]\\
&\le\sum_{i\in S}\left(\lambda_1[V_i^{-T}(V_i+2I)^T(V_i+2I)V_i^{-1}]\right)\sum_{i\in S}\lambda_1(P_i)+\\
&2\sum_{i\in S}\lambda_1(V_i^{-T}V_i^{-1})(n-1)\max_{i,j\in S,j\neq i}\{d_{ij}\}\sum_{i\in S}\lambda_1(P_j)+2\sum_{i\in S}\lambda_1[V_i^{-T}(Q_i+\alpha_i^2B_iB_i^T)V_i^{-1}]\\
&=b\sum_{i\in S}\lambda_1(P_i)+2\sum_{i\in S}\lambda_1[V_i^{-T}(Q_i+\alpha_i^2B_iB_i^T)V_i^{-1}]. \qquad (6.53)
\end{aligned}
$$

where b is defined by (6.45). If condition (6.43) is satisfied, then (6.53) leads to

$$\sum_{i\in S}\lambda_1(P_i)\le\varphi \qquad (6.54)$$

where φ is defined by (6.46). From (6.54), it is immediately found that $\lambda_1(P_i)\le\varphi$, which, when substituted into (6.51), results in the upper bound (6.44). This ends the proof of the theorem.

As with Theorem 6.4, the following iterative algorithm can be suggested to derive more precise upper matrix bounds for the solution of the CCARE (1.10), based on the development of Theorem 6.5.

Algorithm 6.4:
Step 1: Set $Y_i^{(0)}=P_{CCRU2}$, where P_{CCRU2} is defined by (6.44).
Step 2: Compute

$$Y_i^{(k+1)}=V_i^{-T}\left[(V_i+2I)^TY_i^{(k)}(V_i+2I)+2\sum_{j\neq i}d_{ij}Y_j^{(k)}+2(Q_i+\alpha_i^2B_iB_i^T)\right]V_i^{-1}$$
$$k=0,1,2,\dots$$

Then $Y_i^{(k)}$ are upper matrix bounds for the solution of the CCARE (1.10).

The proof of this algorithm parallels the proof of Algorithm 6.3, so it is left out.

Remark 6.5: The following procedure may be used to test the satisfaction of the condition (6.26).
Step 1: Set α_i to be sufficiently small positive constants and δ_i to be appropriate positive constants.
Step 2: Calculate a for $i = 1, 2, \ldots, s$, where a is defined by (6.50).
Step 3: If $a < 1$ is met then the condition (6.45) is fulfilled, and this procedure can be stopped; otherwise, set $\alpha_i = \alpha_i + \delta_i$ and go to **Step 4**.
Step 4: If α_i are sufficiently large, then stop and give up this procedure; otherwise, go to **Step 2**.
To simplify the the calculation and save time, one can choose $\alpha_i = 1$ for checking the condition (6.26) and computing the bound (6.28). A similar procedure can be adopted to test the satisfaction of condition (6.43) for computing the bound (6.44), and also for testing the condition (6.27).

Remark 6.6: When $B_i = 0$ and A_i is stochastically stable, the CCARE (1.10) becomes the CCALE (1.9). As such, the bounds (6.28) and (6.44) become the following upper matrix bounds for the solution of the CCALE:

$$P_i \leq (A_i - I)^{-T} \left(\overline{\eta} \left[A_i^T A_i + I + \left(\sum_{j \neq i} d_{ij} \right) I \right] + Q_i \right) (A_i - I)^{-1} \tag{6.55}$$

$$P_i \leq (A_i - I)^{-T} \left(\overline{\varphi} \left[(A_i + I)^T (A_i + I) + 2 \left(\sum_{j \neq i} d_{ij} \right) I \right] + 2Q_i \right) (A_i - I)^{-1} \tag{6.56}$$

where the positive constants $\overline{\eta}$ and $\overline{\varphi}$ are defined, respectively, by:

$$\overline{\eta} = \frac{\sum_{i \in S} \lambda_1 [(A_i - I)^{-T} Q_i (A_i - I)^{-1}]}{1 - \overline{a}}$$

$$\overline{\varphi} = \frac{2 \sum_{i \in S} \lambda_1 [(A_i - I)^{-T} Q_i (A_i - I)^{-1}]}{1 - \overline{b}}$$

with

$$\overline{a} = \sum_{i \in S} \lambda_1 \{ (A_i - I)^{-T} [A_i^T A_i + I] (A_i - I)^{-1} \} + (s - 1) \max_{i,j \in S, j \neq i} \{ d_{ij} \} \times \sum_{i \in S} \lambda_1 [(A_i - i)^{-T} (A_i - I)^{-1}]$$

$$\bar{b} = \sum_{i \in S} \lambda_1 \{(A_i - I)^{-T}(A_i + I)^T (A_i + I)(A_i - I)^{-1}\} + 2(s-1) \max_{i,j \in S, j \neq i} \{d_{ij}\}$$
$$\times \sum_{i \in S} \lambda_1 [(A_i - I)^{-T}(A_i - I)^{-1}]$$

The bounds (6.55) and (6.56) are valid if $\bar{a} < 1$ and $\bar{b} < 1$ respectively.

6.3.1 Numerical Example for the CCARE

Having developed two upper matrix bounds for the solution of the CCARE, a numerical example will now be given to demonstrate the effectiveness of these derived bounds.

Consider the CCARE (1.10) with:

$$A_1 = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -3 & 2 \\ 2 & -4 \end{bmatrix}, \quad S = \{1, 2\}$$

$$B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad (d_{ij})_{i,j \in S} = \begin{bmatrix} -0.1 & 0.1 \\ 0.1 & -0.1 \end{bmatrix}$$

$$Q_1 = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

For these system matrices, the positive definite solution matrices P_1 and P_2 are:

$$P_1 = \begin{bmatrix} 4.4860 & 1.7725 \\ 1.7725 & 1.1080 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.6207 & 0.2185 \\ 0.2185 & 0.2873 \end{bmatrix}$$

In this example, one can see that there do not exist any positive constants α_i such that the condition (6.28) is met, so the bound (7.28) cannot be applied here. However, when $\alpha_1 = 3$ and $\alpha_2 = 0.1$, $\gamma = 0.9920$. As such, the conditions (7.27) and (7.43) are satisfied, so the bound (7.44) can work for this example. With these values of α_i, the bound (7.15) gives the following non-trivial upper matrix bounds for the solutions of P_1 and P_2:

$$P_1 \leq \begin{bmatrix} 262.7582 & -53.5631 \\ -53.5631 & 93.9401 \end{bmatrix}$$

$$P_2 \leq \begin{bmatrix} 185.0088 & -83.8766 \\ -83.8766 & 273.6983 \end{bmatrix}$$

By three applications of Algorithm 6.2, one can obtain the following tighter upper matrix bounds for the solution matrix P_1:

$$Y_1^{(1)} = \begin{bmatrix} 72.9429 & -10.0907 \\ -10.0907 & 31.3256 \end{bmatrix}$$

$$Y_1^{(2)} = \begin{bmatrix} 18.3831 & -4.1882 \\ -4.1882 & 12.2208 \end{bmatrix}$$

$$Y_1^{(3)} = \begin{bmatrix} 8.8792 & -0.2084 \\ -0.2084 & 4.9861 \end{bmatrix}$$

and the following tighter upper matrix bounds for the solution matrix P_2:

$$Y_2^{(1)} = \begin{bmatrix} 64.7340 & -58.3609 \\ -58.3609 & 125.2727 \end{bmatrix}$$

$$Y_2^{(2)} = \begin{bmatrix} 26.6389 & -32.2844 \\ -32.2844 & 60.1278 \end{bmatrix}$$

$$Y_2^{(3)} = \begin{bmatrix} 12.1749 & -16.5995 \\ -16.5995 & 29.4911 \end{bmatrix}$$

Indeed, it can be seen that as more iterations are performed, the bounds become more precise.

6.4 Summary

In this chapter, the focus was placed on the derivation of matrix solution bounds for continuous coupled Lyapunov and Riccati Equations arising in control analysis and design of jump linear systems. The following results have been successfully derived:

(1) New lower matrix bounds for the CCALE, which are always valid if the CCALE has non-negative definite solution matrices. These bounds were derived by extending the method of section 5.1 used to derive lower matrix bounds for the CALE. Likewise, iterative algorithms were also developed to obtain tighter lower matrix bounds. A numerical example was then given to

show the performance of the results.
(2) A new lower matrix bound for the solution of the CCARE by extending the method of [55]. This bound provides a supplement to the lower matrix bound reported in [58], which appears to be the only existing non-trivial lower matrix bound for the CCARE proposed in the literature. Furthermore, our lower matrix bound is also less conservative than the lower matrix bound reported for the CCARE in [61].
(3) Upper matrix bounds for the CCARE. The upper matrix bounds proposed for the CCARE appear to be the first non-trivial upper matrix bounds proposed for the CCARE so far. These bounds were derived by extending the method employed for the CARE in [62]. Finally, it is mentioned that the results presented here are merely basis results to pave the way for further research into this area; the results proposed here may easily be extended and generalised to obtain new, less restrictive and possibly more concise results.

Chapter 7

Matrix Bounds for the Discrete Coupled Algebraic Riccati Equation

In this chapter, matrix bounds for the DCARE are considered. More precisely, some new upper matrix bounds for the DCARE will be derived, which can provide a supplement to what appears to be the only existing upper matrix bound for the DCARE in the literature [25]. The first bound is derived by extending the method employed in section 5.2, and the second bound is derived by extending the method of [53].

Consider the DCARE

$$P_i = A_i^T F_i A_i - A_i^T F_i B_i (I + B_i^T F_i B_i)^{-1} B_i^T F_i A_i + Q_i$$

Throughout, it is assumed that $Q_i \geq 0$. Recall that $F = P_i + \sum_{j \neq i} e_{ij} P_j$, where the non-negative constants e_{ij} are are explained in Theorem 7.1 below.

7.1 Upper Matrix Bounds for the DCARE

The first upper matrix bound is derived as follows.

Theorem 7.1: Let P_i be the positive semi-definite solution matrices of the

DCARE (1.12). If there exists a matrix $K_i \in \Re^{m\times n}$ such that

$$\left(\sum_{i\in S}\sigma_1^2(A_i + B_iK_i)\right)\left((s-1)\max_{i,j\in S,j\neq i}\{e_{ij}\}+1\right) < 1 \tag{7.1}$$

then P_i has the upper bound

$$P_i \leq \tilde{\alpha}\left(1+\sum_{j\neq i}e_{ij}\right)(A_i+B_iK_i)^T(A_i+B_iK_i)+(Q_i+K_i^TK_i) \equiv P_{dcru1} \tag{7.2}$$

where the positive constant $\tilde{\alpha}_i$ is defined by

$$\tilde{\alpha} \equiv \frac{\sum_{i\in S}\lambda_1(Q_i+K_i^TK_i)}{1-\left(\sum_{i\in S}\sigma_1^2(A_i+B_iK_i)\right)\left((s-1)\max_{i,j\in S,j\neq i}\{e_{ij}\}+1\right)}. \tag{7.3}$$

and the constants e_{ij} are such that [61] $e_{ij} = (\tilde{e}_{ij}/\tilde{e}_{ii})$ with $\tilde{e}_{ij} \in [0,1]$, $\tilde{e}_{ii} > 0$ and $\sum_{j\in S}\tilde{e}_{ij} = 1$.

Proof: By use of the matrix identity

$$A_i^TF_iB_i(I+B_i^TF_iB_i)^{-1}B_i^TF_iA_i = \left[K_i+(I+B_i^TF_iB_i)^{-1}B_i^TF_iA_i\right]^T(I+B_i^TF_iB_i)$$
$$\left[K_i+(I+B_i^TF_iB_i)^{-1}B_i^TF_iA_i\right] - K_i^TK_i - K_i^TB_i^TF_iB_iK_i - A_i^TF_iB_iK_i - K_i^TB_i^TF_iA_i$$

the DCARE (1.12) can be rewritten as

$$\begin{aligned}P_i + \left[K_i+(I+B_i^TF_iB_i)^{-1}B_i^TF_iA_i\right]^T(I+B_i^TF_iB_i)\left[K_i+(I+B_i^TF_iB_i)^{-1}B_i^TF_iA_i\right] \\ = A_i^TF_iA_i + K_i^TK_i + K_i^TB_i^TF_iB_iK_i + A_i^TF_iB_iK_i + K_i^TB_i^TF_iA_i + Q_i\end{aligned} \tag{7.4}$$

Upon realization of the matrix identity

$$(A_i+B_iK_i)^TF_i(A_i+B_iK_i) = A_i^TF_iA_i+K_i^TB_i^TF_iB_iK_i+A_i^TF_iB_iK_i+K_i^TB_i^TF_iA_i$$

(7.4) becomes

$$P_i \leq (A_i+B_iK_i)^TF_i(A_i+B_iK_i)+Q_i+K_i^TK_i \tag{7.5}$$

Applying (2.3) to (7.5) gives

$$P_i \leq \lambda_1(F_i)(A_i+B_iK_i)^T(A_i+B_iK_i)+Q_i+K_i^TK_i \tag{7.6}$$

Application of (2.2) to (7.6) results in

$$\begin{aligned}\lambda_i(P_i) &\leq \lambda_1\{\lambda_1(F_i)(A_i + B_iK_i)^T(A_i + B_iK_i) + Q_i + K_i^TK_i\} \\ &\leq \lambda_1(F_i)\sigma_1^2(A_i + B_iK_i) + \lambda_1(Q_i + K_i^TK_i) \qquad (7.7)\end{aligned}$$

Summing (7.7) over $i \in S$ and using (2.1), (2.6) and (2.8) gives

$$\begin{aligned}\sum_{i\in S}\lambda_i(P_i) &\leq \sum_{i\in S}\lambda_1(F_i)\sigma_1^2(A_i + B_iK_i) + \sum_{i\in S}\lambda_1(Q_i + K_i^TK_i) \\ &\leq \sum_{i\in S}\sigma_1^2(A_i + B_iK_i)\sum_{i\in S}\lambda_1(F_i) + \sum_{i\in S}\lambda_1(Q_i + K_i^TK_i) \\ &= \sum_{i\in S}\sigma_1^2(A_i + B_iK_i)\sum_{i\in S}\lambda_1\left(P_i + \sum_{i\neq j}e_{ij}P_j\right) + \sum_{i\in S}\lambda_1(Q_i + K_i^TK_i) \\ &\leq \sum_{i\in S}\sigma_1^2(A_i + B_iK_i)\left[\sum_{i\in S}\lambda_1(P_i) + \sum_{i\in S}\sum_{i\neq j}e_{ij}\lambda_1(P_j)\right] + \sum_{i\in S}\lambda_1(Q_i + K_i^TK_i) \\ &\leq \sum_{i\in S}\sigma_1^2(A_i + B_iK_i)\left[\sum_{i\in S}\lambda_1(P_i) + (s-1)\max_{i,j\in S, j\neq i}\{e_{ij}\}\sum_{i\in S}\lambda_1(P_i)\right] + \sum_{i\in S}\lambda_1(Q_i + K_i^TK_i) \\ &= \sum_{i\in S}\sigma_1^2(A_i + B_iK_i)\left[1 + (s-1)\max_{i,j\in S, j\neq i}\{e_{ij}\}\right]\sum_{i\in S}\lambda_1(P_i) + \sum_{i\in S}\lambda_1(Q_i + K_i^TK_i) \qquad (7.8)\end{aligned}$$

If there exists a matrix K satisfying the condition (7.1), then (7.8) infers $\sum_{i\in S}\lambda_1(P_i) \leq \tilde{\alpha}$, where $\tilde{\alpha}$ is defined by (7.3). From this, it is readily deduced that $\lambda_1(P_i) \leq \tilde{\alpha}$, from which it can be further deduced that

$$\begin{aligned}\lambda_1(F_i) &\leq \lambda_1(P_i) + \sum_{j\neq i}e_{ij}\lambda_1(P_j) \\ &\leq \tilde{\alpha}\left(1 + \sum_{j\neq i}e_{ij}\right) \qquad (7.9)\end{aligned}$$

where $\tilde{\alpha}$ is defined by (7.3). Substituting (7.9) into (7.6) leads to the bound (7.2). This completes the proof of the theorem.

Remark 7.1: The following upper matrix bound for the solution of the DCARE (1.12) was reported in [25]:

$$P_i \leq A_i^T(B_iB_i^T)^{-1}A_i + Q_i \equiv P_{dcru2} \qquad (7.10)$$

This bound is only valid when $B_iB_i^T$ is nonsingular. Furthermore, this bound appears to be the only existing nontrivial upper matrix bound for the DCARE. When $K_i = -B_i^T(B_iB_i^T)^{-1}A_i$, the bound P_{dcru1} becomes the bound P_{dcru2}, which is defined by (7.10). As such, the bound P_{dcru2} could be considered as a special case of the bound P_{dcru1}.

Remark 7.2: By using the matrix inversion formula [e.g.,26], the DCARE can be rewritten as

$$P_i = A_i^T(I + F_iB_iB_i^T)^{-1}F_iA_i + Q_i$$

Then, by using the upper bound (7.2) of Theorem 7.2 as an initial matrix, one can obtain more precise upper solution bounds for the DCARE (1.12) by using the following iterative equations:

$$P_i \leq M_i^{(k)} \quad k = 1, 2, \ldots \tag{7.11}$$

where

$$\begin{aligned} M_i^{(k)} &= A_i^T \left(I + \left(M_i^{(k-1)} + \sum_{j \neq i} e_{ij} M_j^{(k-1)}\right) B_i B_i^T\right)^{-1} \\ &\quad \times \left(M_i^{(k-1)} + \sum_{j \neq i} e_{ij} M_j^{(k-1)}\right) A_i + Q_i \qquad (7.12) \\ M_i^{(0)} &= P_{dcru1} \end{aligned}$$

Moreover, one can use the above iterative equations to obtain the exact solutions P_i to the DCARE (1.12). The proof of the correctness of these iterative equations is parallel to the proof of the correctness of Algorithm 5.4, so it is omitted.

Numerical Example for the DCARE

Having derived the upper matrix bound (7.2) for the solution of the DCARE, the effectiveness of this bound will now be demonstrated via a numerical example.

Consider the DCARE with:

$$A_1 = \begin{bmatrix} 0 & 0.1 \\ -2.5 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0.2 \\ -4 & 4.5 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0.3 \\ 5 & -5 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 3.6 & -3.8 \\ -3.8 & 4.87 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 10 & -3 \\ -3 & 8 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 5 & -4.5 \\ -4.5 & 4.5 \end{bmatrix}$$

$$(\tilde{e}_{ij})_{i,j\in S} = \begin{bmatrix} 0.67 & 0.17 & 0.16 \\ 0.3 & 0.47 & 0.23 \\ 0.26 & 0.1 & 0.64 \end{bmatrix}, \quad S = \{1, 2, 3\}$$

For these matrices, the positive definite solution matrices P_1, P_2 and P_3 are:

$$P_1 = \begin{bmatrix} 9.5188 & -10.7165 \\ -10.7165 & 12.9885 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 13.96 & -7.3172 \\ -7.3172 & 13.1578 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 6.555 & -6.1335 \\ -6.1335 & 6.4682 \end{bmatrix}.$$

For this example, $B_i B_i^T$ is singular for $i = 1, 2, 3$, so the upper bounds reported in [13] and [25] cannot work here. However, the presented bounds may be applied to do so. With $K_1 = [\,2.5 \ \ -3\,]$, $K_2 = [\,2 \ \ -2.25\,]$ and $K_3 = [\,-1.25 \ \ 1.25\,]$, the condition (7.1) is met, and the upper bound of Theorem 2.3 gives the following upper solution estimates for P_1, P_2 and P_3:

$$P_1 \leq \begin{bmatrix} 9.85 & -11.3 \\ -11.3 & 15.1135 \end{bmatrix}$$

$$P_2 \leq \begin{bmatrix} 14 & -7.5 \\ -7.5 & 20.1536 \end{bmatrix}$$

$$P_3 \leq \begin{bmatrix} 6.5625 & -6.0625 \\ -6.0625 & 17.7793 \end{bmatrix}$$

Using the iterative equations (7.11) and (7.12) gives the following tighter upper matrix bounds for the DCARE:

$$M_1^{(1)} = \begin{bmatrix} 9.605 & -10.8623 \\ -10.8623 & 13.2375 \end{bmatrix}$$

$$M_2^{(1)} = \begin{bmatrix} 13.9744 & -7.3796 \\ -7.3796 & 13.4417 \end{bmatrix}$$

$$M_3^{(1)} = \begin{bmatrix} 6.56 & -6.101 \\ -6.101 & 6.8257 \end{bmatrix}$$

Indeed, as more iterations are carried out the bounds become tighter.

In what follows, another upper matrix bound for the solution of the DCARE is derived. The bound is derived by extending the method of [53]. The following result can also be found in [17]. Before developing this result, however, we shall first review the following useful result:

Lemma 7.1 [25]: For the DCARE, the following lower matrix solution bound:

$$P_i \geq A_i^T(I + F_{Q_i}B_iB_i^T)^{-1}F_{Q_i}A_i + Q_i \equiv P_{dcrL1(i)} \tag{7.13}$$

where the positive semidefinite matrix F_{Q_i} is defined by

$$F_{Q_i} = Q_i + \sum_{j \neq i} e_{ij}Q_j$$

It is implied from (7.13) that

$$\lambda_1(F_i) \geq \lambda_1\left(P_{dcrL1(i)} + \sum_{j \neq i} e_{ij}P_{dcrL1(j)}\right) \equiv \kappa_i \tag{7.14}$$

The next result can now be derived, which is a new upper matrix bound for the solution of the DCARE.

Theorem 7.2: Let P_i be the positive semidefinite solution matrix of the DCARE (1.12). If

$$\varphi \equiv \left(\sum_{i \in S}\left[1 + \sigma_n^2(B_i)\kappa_i\right]^{-1}\sigma_1^2(A_i)\right)\left(1 + (s-1)\max_{i,j \in S, j \neq i}\{e_{ij}\}\right) < 1 \tag{7.15}$$

where κ_i is defined by (7.14), then P_i has the upper bound

$$P_i \leq A_i^T\left[\left(\eta + \sum_{j \neq i} e_{ij}\eta\right)^{-1} I + B_iB_i^T\right]^{-1} A_i + Q_i \equiv P_{dcru1} \tag{7.16}$$

where the positive constant η is defined by

$$\eta \equiv (1-\varphi)^{-1}\sum_{i \in S}\lambda_1(Q_i). \tag{7.17}$$

Proof: By application of the matrix identity [e.g.,26]

$$(I+ST)^{-1} = I - S(I+TS)^{-1}T$$

the DCARE (1.12) can be rewritten as

$$P_i = A_i^T(I + F_iB_iB_i^T)^{-1}F_iA_i + Q_i \tag{7.18}$$

Applying (2.3) combined with (2.5) to (7.18) leads to

$$\begin{aligned}
P_i &\le \lambda_1(F_i)A_i^T[I + \lambda_1(F_i)B_iB_i^T]^{-1}A_i + Q_i \\
&\le \lambda_1(F_i)\lambda_1([I + \lambda_1(F_i)B_iB_i^T]^{-1})A_i^TA_i + Q_i \\
&= \frac{\lambda_1(F_i)}{\lambda_n[I + \lambda_1(F_i)B_iB_i^T]}A_i^TA_i + Q_i
\end{aligned} \tag{7.19}$$

where the fact $\lambda_1(X^{-1}) = 1/\lambda_n(X)$ has been used for nonsingular matrix X. Using (2.2) and (7.14), it is found from (7.19) that

$$\begin{aligned}
\lambda_n[I + \lambda_1(F_i)B_iB_i^T] &\ge \lambda_n(I) + \lambda_1(F_i)\sigma_n^2(B_i) \\
&\ge 1 + \sigma_n^2(B_i)\kappa_i
\end{aligned} \tag{7.20}$$

Substituting (7.20) into (7.19) yields

$$P_i \le \frac{\lambda_1(F_i)}{1+\sigma_n^2(B_i)\kappa_i}A_i^TA_i + Q_i \tag{7.21}$$

Applying (2.1) to (7.21) gives

$$\begin{aligned}
\lambda_1(P_i) &\le \lambda_1\left\{\frac{\lambda_1(F_i)}{1+\sigma_n^2(B_i)\kappa_i}A_i^TA_i + Q_i\right\} \\
&\le \frac{\sigma_1^2(A_i)\lambda_1(F_i)}{1+\sigma_n^2(B_i)\kappa_i} + \lambda_1(Q_i) \\
&= \left[1+\sigma_n^2(B_i)\kappa_i\right]^{-1}\sigma_1^2(A_i)\lambda_1\left(P_i + \sum_{j\ne i} e_{ij}P_j\right) + \lambda_1(Q_i) \\
&\le \left[1+\sigma_n^2(B_i)\kappa_i\right]^{-1}\sigma_1^2(A_i)\left[\lambda_1(P_i) + \sum_{j\ne i} e_{ij}\lambda_1(P_j)\right] + \lambda_1(Q_i)
\end{aligned} \tag{7.22}$$

Summing (7.22) over $i \in S$ and using (2.6) and (2.8) gives

$$\begin{aligned}\sum_{i\in S}\lambda_1(P_i) &\le \sum_{i\in S}\left\{\left[1+\sigma_n^2(B_i)\kappa_i\right]^{-1}\sigma_1^2(A_i)\left[\lambda_1(P_i)+\sum_{j\ne i}e_{ij}\lambda_1(P_j)\right]\right\}+\sum_{i\in S}\lambda_1(Q_i)\\ &\le \left(\sum_{i\in S}\left[1+\sigma_n^2(B_i)\kappa_i\right]^{-1}\sigma_1^2(A_i)\right)\left(\sum_{i\in S}\lambda_1(P_i)+\sum_{i\in S}\sum_{j\ne i}e_{ij}\lambda_1(P_j)\right)+\sum_{i\in S}\lambda_1(Q_i)\\ &\le \left(\sum_{i\in S}\left[1+\sigma_n^2(B_i)\kappa_i\right]^{-1}\sigma_1^2(A_i)\right)\left(1+(s-1)\max_{i,j\in S,j\ne i}\{e_{ij}\}\right)\sum_{i\in S}\lambda_1(P_i)+\sum_{i\in S}\lambda_1(Q_i)\end{aligned} \tag{7.23}$$

If the condition (7.15) is fulfilled, then one has from (7.23) that

$$\sum_{i\in S}\lambda_1(P_i)\le\eta \tag{7.24}$$

where η is defined by (7.16). From (7.24), it is immediately obvious that

$$\lambda_1(P_i)\le\eta \tag{7.25}$$

From (7.25), we have that

$$\lambda_1(P_i)\le\lambda_1(P_i)+\sum_{j\ne i}e_{ij}\lambda_1(P_j)\le\left(\eta+\sum_{j\ne i}e_{ij}\eta\right) \tag{7.26}$$

Substituting (7.26) into the first line of (7.19) leads to the upper bound

$$P_i\le\left(\eta+\sum_{j\ne i}e_{ij}\eta\right)A_i^T\left[I+\left(\eta+\sum_{j\ne i}e_{ij}\eta\right)B_iB_i^T\right]^{-1}A_i+Q_i$$

which is equivalent to the bound (7.16). This completes the proof of the theorem.

Having developed Theorem 7.2, we now present an iterative algorithm which can obtain tighter upper matrix solution bounds using the bound of Theorem 7.2 as an initial iteration. The derivation of this algorithm will follow partly along the lines of the derivation of Theorem 4 of [25].

Algorithm 7.1

Step 1: Set $X_i^{(0)} = P_{u1}$, where P_{u1} is defined by (7.16).
Step 2: Calculate

$$X_i^{(k)} = A_i^T(I + F_i^{(k-1)}B_iB_i^T)^{-1}F_i^{(k-1)}A_i + Q_i, \quad k = 1, 2, \dots \tag{7.27}$$
$$F_i^{(k)} = X_i^{(k-1)} + \sum_{j \neq i} e_{ij}X_j^{(k-1)}$$

Then $X_i^{(k)}$ are upper solution bounds for the solution of the DCARE (1.12).

Proof: First, it will be shown that $X_i^{(1)} \leq X_i^{(0)}$. Setting $k = 1$ in (7.27) gives:

$$X_i^{(1)} = A_i^T(I + F_i^{(0)}B_iB_i^T)^{-1}F_i^{(0)}A_i + Q_i. \tag{7.28}$$

Applying (2.3) and (2.5) to (7.28) results in

$$X_i^{(1)} \leq \lambda_1(F_i^{(0)})A_i^T[I + \lambda_1(F_i^{(0)})B_iB_i^T]^{-1}A_i + Q_i. \tag{7.29}$$

From the derivation of Theorem 7.2, it was found that

$$\lambda_1(F_i^{(0)}) \leq \left(\eta + \sum_{j \neq i} e_{ij}\eta\right). \tag{7.30}$$

Substituting (7.30) into (7.29) yields

$$X_i^{(1)} \leq \left(\eta + \sum_{j \neq i} e_{ij}\eta\right) A_i^T \left[I + \left(\eta + \sum_{j \neq i} e_{ij}\eta\right) B_iB_i^T\right]^{-1} A_i + Q_i \equiv X_i^{(0)}.$$

Hence, it has been proved that $X_i^{(1)} \leq X_i^{(0)}$. Assume now that $X_i^{(k-1)} \leq X_i^{(k-2)}$. Using (7.27) gives

$$\begin{aligned} X_i^{(k)} &= A_i^T(I + F_i^{(k-1)}B_iB_i^T)^{-1}F_i^{(k-1)}A_i + Q_i \\ &\leq A_i^T(I + F_i^{(k-2)}B_iB_i^T)^{-1}F_i^{(k-2)}A_i + Q_i = X_i^{(k-1)}. \end{aligned}$$

By induction, one can conclude that $X_i^{(k)} \leq X_i^{(k-1)} \leq \dots \leq X_i^{(1)} \leq X_i^{(0)}$. Clearly, we have $X_i^{(k)} \geq 0$ for any k. Along the lines of Theorem 4 in [25], it can be seen that $X_i^{(k)}$ is monotone decreasing and bounded, so there exists $X_{i\infty} \geq 0$, with $X_{i\infty} = \lim_{k \to \infty} X_i^{(k)}$, such that

$$X_{i\infty} = A_i^T(I + F_{i\infty}B_iB_i^T)^{-1}F_{i\infty}A_i + Q_i.$$

Here, $X_{i\infty}$ is merely the DCARE solution with P_i replaced by $X_{i\infty}$. Hence, it can be concluded that this algorithm can obtain the exact solution of the DCARE.

Remark 7.3: In [25], the upper matrix bound (7.10) for the solution of the DCARE (1.12) was proposed. This bound has to assume that $B_iB_i^T$ is nonsingular for it to be computable and non-trivial. When $B_iB_i^T$ is nonsingular, the bound (7.10) always works, and the bound (7.16) works if the condition (7.15) is met. In this case it is seen that the bound (7.16) is tighter than (7.10) which was proposed in [25]. For the case that $B_iB_i^T$ is singular, our result still works if the condition (7.15) is fulfilled, whilst bound (7.10) proposed in [25] can never work.

From the analysis of Theorem 7.2, the following result is presented.
Corollary 7.1: The following eigenvalue upper bounds exist for the solution of the DCARE (1.12):

$$\lambda_k(P_i) \leq \lambda_k \left\{ A_i^T \left[\left(\eta + \sum_{j \neq i} e_{ij}\eta \right)^{-1} I + B_iB_i^T \right]^{-1} A_i + Q_i \right\}, \quad k = 1, 2, \ldots, n.$$

$$tr(P_i) \leq tr \left\{ A_i^T \left[\left(\eta + \sum_{j \neq i} e_{ij}\eta \right)^{-1} I + B_iB_i^T \right]^{-1} A_i + Q_i \right\},$$

$$det(P_i) \leq det \left\{ A_i^T \left[\left(\eta + \sum_{j \neq i} e_{ij}\eta \right)^{-1} I + B_iB_i^T \right]^{-1} A_i + Q_i \right\}.$$

7.1.1 Numerical Example for the DCARE

In this subsection, a numerical example is given to show the performance of the derived bound (7.16).

Consider the DCARE (1.12) with:

$$A_1 = \begin{bmatrix} 0.2 & 0 \\ 0.3 & 0.4 \end{bmatrix}, \; A_1 = \begin{bmatrix} 0.1 & 0.2 \\ 0 & 0.3 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$Q_1 = Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (\tilde{e}_{ij})_{i,j \in S} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

For this case, $B_i B_i^T$ is singular, so the bound (7.10) cannot work. The bound (7.16) gives the following solution estimates for P_1 and P_2:

$$P_1 \leq \begin{bmatrix} 1.8785 & 0.1142 \\ 0.1142 & 1.1523 \end{bmatrix}$$

$$P_2 \leq \begin{bmatrix} 1.0025 & 0.0049 \\ 0.0049 & 2.7938 \end{bmatrix}$$

By using 1 iteration of Algorithm 7.1, the following tighter upper matrix solution bounds for P_1 and P_2 are obtained:

$$P_1 \leq \begin{bmatrix} 1.1898 & 0.0977 \\ 0.0977 & 1.1276 \end{bmatrix}$$

$$P_2 \leq \begin{bmatrix} 1.0023 & 0.0049 \\ 0.0049 & 1.3651 \end{bmatrix}$$

Indeed, it is seen that as more iterations are performed the bound become tighter.

7.2 Summary

In this chapter, the focus was placed on the derivation of matrix solution bounds for continuous coupled Lyapunov and Riccati Equations arising in control analysis and design of jump linear systems. The following results have been successfully derived:

(1) Upper matrix bounds for the DCARE have been developed by extending the methods of [16] and [53]. These bounds provide a supplement and different validity conditions to what appears to be the only existing nontrivial upper matrix bound for the DCARE, which was reported in [25].

Chapter 8

Solution Bounds of Coupled Lyapunov and Riccati Equations under Perturbations in the Coefficients

This chapter will focus on the problem of deriving solution bounds for coupled Lyapunov and Riccati equations arising from stochastic systems when their coefficient matrices undergo small perturbations.

Often in practice, only approximate values of the coefficient matrices are available and not the exact ones, possibly due to external system disturbances, such as noise, time delay, etc. In this case, the available values of the coefficient matrices are perturbed versions of their actual values, so it becomes of interest to derive lower and upper bounds for the perturbation in the solution of each equation, so as to give an estimate for the disturbance range.

Firstly, some solution bounds for the CCARE will be derived when all its coefficient matrices undergo small perturbations. Based on these results, we shall then deduce solution bounds of this equation for the cases when only one coefficient matrix undergoes a small perturbation. These solution bounds for the perturbed CCARE can also be used to provide solution bounds for the CCALE when its coefficient matrices are perturbed.

Secondly, some solution bounds for the DCARE will be derived when all its coefficient matrices undergo small perturbations. Like with the perturbed CCARE, bounds are then given for the cases of individual perturbations, and also bounds for the DCALE when its coefficient matrices are perturbed.

For each equation, numerical examples will also be given to show the effectiveness of the derived bounds.

8.1 Lower Matrix Bounds for the Continuous Coupled Lyapunov Equation Under Perturbations in the Coefficients

In this section, lower matrix bounds are proposed for the CCALE when its coefficient matrices are subject to small perturbations. Before deriving these bounds, however, some conditions are given under which the perturbations in the solution matrices of the CCALE are non-negative definite.

Consider the CCALE (1.9) with $Q_i \geq 0$. Now, consider the CCALE when its coefficient matrices undergo small perturbations:

$$(P_i+\Delta P_i)(A_i+\Delta A_i)+(A_i+\Delta A_i)^T(P_i+\Delta P_i)+\sum_{j\neq i} d_{ij}(P_j+\Delta P_j) = -(Q_i+\Delta Q_i) \tag{8.1}$$

where $\Delta A_i, \Delta Q_i, \Delta P_i \in \Re^{n\times n}$ are the perturbations in A_i, Q_i and P_i respectively. Here, ΔQ_i is a symmetric matrix, so $P_i + \Delta P_i$ is symmetric, hence ΔP_i is symmetric. Since ΔA_i and ΔQ_i are small perturbations, it can be assumed without loss of generality that $A_i + \Delta A_i$ is stable and $Q_i + \Delta Q_i$ is positive semidefinite. Expanding out (8.1) and using the CCALE (1.9), we have the following perturbed CCALE:

$$\Delta P_i(A_i + \Delta A_i) + (A_i + \Delta A_i)^T\Delta P_i + \sum_{j\neq i} d_{ij}\Delta P_j = -M_i \tag{8.2}$$

where $M_i \equiv \Delta Q_i + P_i\Delta A_i + (\Delta A_i)^T P_i$. For this work, it will also be assumed that $\Delta Q_i + P_i\Delta A_i + (\Delta A_i)^T P_i$ is positive semidefinite, so that the solution ΔP_i of the perturbed CCALE (8.2) is non-negative definite (Lyapunov stability theorem [69]). Then, the main results follow.

In the following theorem, a lower matrix bound for the solution ΔP_i of the perturbed CCALE (8.2) is presented.

Theorem 8.1: Define V_i by

$$V_i \equiv A_i + \Delta A_i - \alpha_i I \tag{8.3}$$

where α_i are positive constants. Let ΔP_i be the positive semidefinite solution matrices of the perturbed CCALE (8.2). Then ΔP_i has the lower bound

$$\Delta P_i \geq V_i^{-T} \left((A_i + \Delta A_i)^T \Delta P_{(i)0} (A_i + \Delta A_i) + \alpha_i \sum_{j \neq i} d_{ij} \Delta P_{(j)0} + \alpha_i^2 \Delta P_{i0} + \alpha_i M_i \right) V_i^{-1}$$

$$\equiv \Delta P_{CCLL1} \tag{8.4}$$

where the positive semidefinite matrix $\Delta P_{(i)0}$ is defined by

$$\Delta P_{(i)0} = \alpha_i V_i^{-T} M_i V_i^{-1}. \tag{8.5}$$

Proof: Using the definition of V_i from (8.3) allows for the following matrix identity:

$$V_i^T \Delta P_i V_i = (A_i + \Delta A_i)^T \Delta P_i (A_i + \Delta A_i) - \alpha_i [(A_i + \Delta A_i)^T \Delta P_i + \Delta P_i (A_i + \Delta A_i)] + \alpha_i^2 P_i \tag{8.6}$$

Using the perturbed CCALE (8.2), (8.6) becomes

$$V_i^T \Delta P_i V_i = (A_i + \Delta A_i)^T \Delta P_i (A_i + \Delta A_i) + \alpha_i \sum_{j \neq i} d_{ij} \Delta P_i + \alpha_i^2 \Delta P_i + \alpha_i M_i \tag{8.7}$$

Now, $\lambda_k(V_i) = \lambda_k(A_i + \Delta A_i - \alpha_i I) = \lambda_k(A_i + \Delta A_i) - \alpha_i$. Since we have assumed the stability of $A_i + \Delta A_i$, we have that $\Re(\lambda_k(V_i)) < 0 \ \forall k = 1, 2, \ldots, n$, $i = 1, 2, \ldots, s$. Therefore, V_i is nonsingular for any value of the positive constant α_i. Then, pre- and post-multiplying both sides of (8.7) by V_i^{-T} and V_i^{-1} respectively gives

$$\Delta P_i = V_i^{-T} \left((A_i + \Delta A_i)^T \Delta P_i (A_i + \Delta A_i) + \alpha_i \sum_{j \neq i} d_{ij} \Delta P_i + \alpha_i^2 \Delta P_i + \alpha_i M_i \right) V_i^{-1} \tag{8.8}$$

From (8.8), it is immediately obvious that $\Delta P_i \geq \Delta P_{(i)0}$, where $\Delta P_{(i)0}$ is defined by (8.5). Substituting (8.5) into (8.8) yields the lower bound (8.4).

This completes the proof.

Having derived the lower bound (8.4) of Theorem 8.1, the following iterative algorithm to obtain sharper lower matrix bounds for the solution of the perturbed CCALE (8.2) is proposed.

Algorithm 8.1:
Step 1: Set $\Delta P_i^{(0)} \equiv \Delta P_{(i)0}$, where $\Delta P_{(i)0}$ is defined by (8.5).
Step 2: Compute

$$\Delta P_i^{(t)} = V_i^{-T}\left((A_i + \Delta A_i)^T \Delta P_i^{(t-1)}(A_i + \Delta A_i) + \alpha_i \sum_{j\neq i} d_{ij}\Delta P_i^{(t-1)} \right.$$
$$\left. + \alpha_i^2 \Delta P_i^{(t-1)} + \alpha_i M_i\right) V_i^{-1} \quad (8.9)$$

where $t = 1, 2, \ldots$. Then $\Delta P_i^{(t)}$ are lower solution bounds for the perturbed CCALE (8.2). As $t \to \infty$, $\Delta P_i^{(t)} \to \Delta P_i$, where ΔP_i is the solution of the perturbed CCALE (8.2).

Proof: Firstly, $\Delta P_i \geq \alpha_i V_i^{-T} M_i V_i^{-1} = \Delta P_i^{(0)}$. Then, using (8.9) gives

$$\Delta P_i = V_i^{-T}\left((A_i + \Delta A_i)^T \Delta P_i (A_i + \Delta A_i) + \alpha_i \sum_{j\neq i} d_{ij}\Delta P_i + \alpha_i^2 \Delta P_i + \alpha_i M_i\right) V_i^{-1}$$
$$\geq V_i^{-T}\left((A_i + \Delta A_i)^T \Delta P_i^{(0)} (A_i + \Delta A_i) + \alpha_i \sum_{j\neq i} d_{ij}\Delta P_i^{(0)} + \alpha_i^2 \Delta P_i^{(0)} + \alpha_i M_i\right) V_i^{-1}$$
$$= \Delta P_i^{(1)} \geq \alpha_i V_i^{-T} M_i V_i^{-1} = \Delta P_i^{(0)} \quad (8.10)$$

Now assume that $\Delta P_i \geq \Delta P_i^{(t-1)} \geq \Delta P_i^{(t-2)}$. Following the method of (8.10) and remembering (8.9) provides

$$\Delta P_i = V_i^{-T}\left((A_i + \Delta A_i)^T \Delta P_i (A_i + \Delta A_i) + \alpha_i \sum_{j\neq i} d_{ij}\Delta P_i + \alpha_i^2 \Delta P_i + \alpha_i M_i\right) V_i^{-1}$$
$$\geq V_i^{-T}\left((A_i + \Delta A_i)^T \Delta P_i^{(t-1)} (A_i + \Delta A_i) + \alpha_i \sum_{j\neq i} d_{ij}\Delta P_i^{(t-1)} + \alpha_i^2 \Delta P_i^{(t-1)} + \alpha_i M_i\right) V_i^{-1}$$
$$= \Delta P_i^{(t)}$$

$$\geq V_i^{-T}\left((A_i+\Delta A_i)^T \Delta P_i^{(t-2)}(A_i+\Delta A_i)+\alpha_i \sum_{j\neq i} d_{ij}\Delta P_i^{(t-2)}+\alpha_i^2 \Delta P_i^{(t-2)}+\alpha_i M_i\right) V_i^{-1}$$
$$= \Delta P_i^{(t-1)}$$

By induction, one can conclude that

$$0 \leq \Delta P_i^{(0)} \leq \Delta P_i^{(1)} \leq \ldots \leq \Delta P_i^{(t-1)} \leq \Delta P_i^{(t)} \leq \Delta P_i.$$

for $i = 1, 2, \ldots, s$ and $t = 1, 2, \ldots$. Since $\Delta P_i^{(t)}$ is monotone increasing and bounded, there exists $\Delta P_i^{\infty} \geq 0$ with $\Delta P_i^{(\infty)} = \lim_{t\to\infty} P_i^t$ such that

$$\Delta P_i^{(\infty)} = V_i^{-T}\left((A_i+\Delta A_i)^T \Delta P_i^{(\infty)}(A_i+\Delta A_i)+\alpha_i \sum_{j\neq i} d_{ij}\Delta P_i^{(\infty)} + \alpha_i^2 \Delta P_i^{(\infty)}+\alpha_i M_i\right) V_i^{-1} \quad (8.11)$$

(8.11) is equivalent to (8.8) with $\Delta P_i = \Delta P_i^{(\infty)}$, which is the solution of the perturbed CCALE (8.2). This ends the proof of the correctness of the algorithm.

A second lower matrix bound is derived as follows.

Theorem 8.2: The positive semidefinite solution matrices ΔP_i of the perturbed CCALE (8.2) are such that

$$\Delta P_i \geq V_i^{-T}\left(2(V_i+2\alpha_i I)^T \Delta P_{(i)0}(V_i+2\alpha_i I)+4\alpha_i \sum_{j\neq i} d_{ij}\Delta P_{(j)0}+2\alpha_i M_i\right) V_i^{-1}$$
$$\equiv \Delta P_{CCLL2} \quad (8.12)$$

where the matrix V_i is defined by (8.3) and the positive semidefinite matrix $P_{(i)0}$ is defined by (8.5).

Proof: By the definition of V_i from (8.3), the perturbed CCALE (8.2) can be rewritten as

$$(V_i+\alpha_i I)^T \Delta P_i + \Delta P_i (V_i+\alpha_i I) + \sum_{j\neq i} d_{ij}\Delta P_j = -M_i$$
$$V_i^T \Delta P_i + \Delta P_i V_i + 2\alpha_i \Delta P_i + \sum_{j\neq i} d_{ij}\Delta P_j + M_i = 0 \quad (8.13)$$

Multiplying both sides of (8.13) by $2\alpha_i$ and adding $V_i^T \Delta P_i V_i$ gives

$$V_i^T \Delta P_i V_i = V_i^T \Delta P_i V_i + 2\alpha_i (V_i^T \Delta P_i + \Delta P_i V_i) + 4\alpha_i^2 \Delta P_i + 2\alpha_i \sum_{j \neq i} d_{ij} \Delta P_j + 2\alpha_i M_i \tag{8.14}$$

By recognizing that

$$(V_i + 2\alpha_i)^T \Delta P_i (V_i + 2\alpha_i) = V_i^T \Delta P_i V_i + 2\alpha_i (V_i^T \Delta P_i + \Delta P_i V_i) + 4\alpha_i^2 \Delta P_i$$

(8.14) becomes

$$V_i^T \Delta P_i V_i = (V_i + 2\alpha_i)^T \Delta P_i (V_i + 2\alpha_i) + 2\alpha_i \sum_{j \neq i} d_{ij} \Delta P_j + 2\alpha_i M_i \tag{8.15}$$

Pre- and post-multiplying both sides of (8.15) by V_i^{-T} and V_i^{-1} respectively leads to

$$\Delta P_i = V_i^{-T} \left((V_i + 2\alpha_i)^T \Delta P_i (V_i + 2\alpha_i) + 2\alpha_i \sum_{j \neq i} d_{ij} \Delta P_j + 2\alpha_i M_i \right) V_i^{-1} \tag{8.16}$$

From (8.16), one has $\Delta P_i \geq 2\Delta P_{(i)0}$, where $\Delta P_{(i)0}$ is defined by (8.5). Upon substituting this lower bound into (8.16), one arrives at the lower bound (8.12). This concludes the proof.

Following the development of Theorem 8.2, the following iterative algorithm to derive more precise lower matrix bounds for the solution ΔP_i of the perturbed CCALE (8.2) is proposed.

Algorithm 8.2:
Step 1: Set $\overline{\Delta P_i}^{(0)} = 2\Delta P_{(i)0}$, where $\Delta P_{(i)0}$ is defined by (8.5).
Step 2: Compute

$$\overline{\Delta P_i}^{(t)} = V_i^{-T} \left((V_i + 2\alpha_i I)^T \overline{\Delta P_i}^{(t-1)} (V_i + 2\alpha_i I) + 2\alpha_i \sum_{j \neq i} d_{ij} \overline{\Delta P_j}^{(t-1)} + 2\alpha_i M_i \right) V_i^{-1}$$

for $t = 1, 2, \ldots$. Then, $\overline{\Delta P_i}^{(t)}$ are also lower matrix solution bounds of the perturbed CCALE (8.2). As $t \to \infty$, where ΔP_i is the solution of the perturbed CCALE (8.2).

The proof of this algorithm parallels that of Algorithm 8.1. Therefore, its proof is omitted.

Remark 8.1: As mentioned above, this appears to be the first attempt to present lower matrix bounds for the CCALE when its coefficient matrices are subject to small perturbations. Using these matrix bounds, one can infer eigenvalue lower bounds for the perturbation in the solution of the CCALE. Furthermore, these bounds are always determined if the perturbation in the solution of the CCALE is non-negative definite. Conditions for the non-negative definiteness of the solution perturbations are as discussed above.

Remark 8.2: A systematic method to determine which choice of α_i yields the best lower matrix bounds for the perturbed CCALE (8.2) is not be found. However, one might consider this as an optimization problem. Besides, one may easily get tighter lower matrix solution estimates with the aid of Algorithms 8.1 and/or 8.2.

Remark 8.3: From (8.4) and (8.12), it is found that

$$\begin{aligned}\Delta P_{CCLL2} &= V_i^{-T}\Big(2(V_i^T\Delta P_{(i)0}V_i + 2\alpha_i(V_i^T\Delta P_{(i)0} + \Delta P_{(i)0}V_i \\ &\quad +4\alpha_i^2\Delta P_{(i)0}) + 4\alpha_i\sum_{j\neq i} d_{ij}\Delta P_{(j)0} + 2\alpha_i M_i\Big)V_i^{-1} \\ &= \Delta P_{CCLL1} + V_i^{-T}\Big(V_i^T\Delta P_{(i)0}V_i + 3\alpha_i(V_i^T\Delta P_{(i)0} + \Delta P_{(i)0}V_i) + 7\alpha_i^2\Delta P_{(i)0} \\ &\quad +3\alpha_i\sum_{j\neq i} d_{ij}\Delta P_{(j)0} + \alpha_i M_i\Big)V_i^{-1}\end{aligned}$$

Therefore, if

$$V_i^T\Delta P_{(i)0}V_i+3\alpha_i(V_i^T\Delta P_{(i)0}+\Delta P_{(i)0}V_i)+7\alpha_i^2\Delta P_{(i)0}+3\alpha_i\sum_{j\neq i} d_{ij}\Delta P_{(j)0}+\alpha_i M_i \geq 0$$

then ΔP_{CCLL2} is tighter than ΔP_{CCLL1}, whereas if

$$V_i^T\Delta P_{(i)0}V_i+3\alpha_i(V_i^T\Delta P_{(i)0}+\Delta P_{(i)0}V_i)+7\alpha_i^2\Delta P_{(i)0}+3\alpha_i\sum_{j\neq i} d_{ij}\Delta P_{(j)0}+\alpha_i M_i \geq 0$$

then ΔP_{CCLL1} is tighter than ΔP_{CCLL2}.

Remark 8.4: The bounds of Theorems 8.1 and 8.2, and Algorithms 8.1 and 8.2 give lower solution bounds for the perturbed CCALE (8.2) when both its coefficient matrices A_i and Q_i undergo small perturbations. For the case when only A_i undergoes a small perturbation, the corresponding perturbed CCALE is obtain from (8.2) by setting the perturbation ΔQ_i to be equal to zero. Then, the corresponding lower matrix bounds and iterative algorithms are obtained in the same way. For the case that only Q_i undergoes a small perturbation, the corresponding results are obtained in the same way as when only A_i undergoes a small perturbation.

Remark 8.5: From Theorems 8.1 and 8.2, one can see that $\Delta P_i^{(1)} = \Delta P_{CCLL1}$ and $\overline{\Delta P_i}^{(1)} = \Delta P_{CCLL2}$.

8.1.1 Numerical Example

In this subsection, a numerical example is provided to illustrate the effectiveness of the derived results.

Consider the CCALE (1.9) and perturbed CCALE (8.2) with:

$$A_1 = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -3 & 0 \\ 1 & -4 \end{bmatrix}, \quad (d_{ij})_{i,j\in S} = \begin{bmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}$$

$$\Delta A_1 = \begin{bmatrix} 0.02 & 0 \\ 0.001 & 0.01 \end{bmatrix}, \quad \Delta A_2 = \begin{bmatrix} 0.02 & 0.002 \\ 0 & 0.03 \end{bmatrix}, \quad S = \{1,2\}$$

$$Q_1 = \begin{bmatrix} 6 & 0.5 \\ 0.5 & 12 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 13 & 26.5 \\ 26.5 & 62.5 \end{bmatrix}$$

$$\Delta Q_1 = \begin{bmatrix} 0.04 & 0.002 \\ 0.002 & 0.009 \end{bmatrix}, \quad \Delta Q_2 = \begin{bmatrix} 0.07 & -0.006 \\ -0.006 & 0.09 \end{bmatrix}$$

Then, the positive definite solutions P_{1exact} and P_{2exact} of the CCALE (1.9) are:

$$P_{1exact} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \quad P_{2exact} = \begin{bmatrix} 4 & 5 \\ 5 & 8 \end{bmatrix},$$

Also, the positive definite solutions ΔP_{1exact} and ΔP_{2exact} of the perturbed CCALE (8.2) are:

$$\Delta P_{1exact} = \begin{bmatrix} 0.0382 & 0.0196 \\ 0.0196 & 0.0244 \end{bmatrix}, \quad \Delta P_{2exact} = \begin{bmatrix} 0.0581 & 0.0486 \\ 0.0486 & 0.0759 \end{bmatrix},$$

Throughout the following, $\alpha_1 = \alpha_2 = 1$ are used. Then, the lower bound ΔP_{CCLL1} of Theorem 8.1 gives the following solution estimates for ΔP_1 and ΔP_2:

$$\Delta P_1 \geq \begin{bmatrix} 0.0225 & 0.0108 \\ 0.0108 & 0.0118 \end{bmatrix}, \quad \text{and} \quad \Delta P_2 \geq \begin{bmatrix} 0.0363 & 0.0305 \\ 0.0305 & 0.0406 \end{bmatrix}$$

Also, the lower bound ΔP_{CCLL2} of Theorem 8.2 gives the following solution estimates for ΔP_1 and ΔP_2:

$$\Delta P_1 \geq \begin{bmatrix} 0.0354 & 0.0178 \\ 0.0178 & 0.0206 \end{bmatrix} \quad \text{and} \quad \Delta P_2 \geq \begin{bmatrix} 0.0549 & 0.0466 \\ 0.0466 & 0.0657 \end{bmatrix}$$

Here, it is seen that ΔP_{CCLL2} gives the tighter solution estimate for both ΔP_1 and ΔP_2. Furthermore, one can obtain tighter lower matrix bounds here by utilizing Algorithm 8.2.

8.2 Lower Matrix Bound for the Discrete Coupled Lyapunov Equation Under Perturbations in the Coefficients

In this section, the focus is placed on the problem of estimating the perturbations in the solution matrices of the DCALE when its coefficient matrices undergo small perturbations. More precisely, a lower matrix bound for the perturbations in the solution matrices of the DCALE when its coefficient matrices are subject to small perturbations is derived.

First, consider the DCALE (1.11) with $Q_i > 0$. Next, consider the DCALE when its coefficient matrices are subject to small perturbations:

$$P_i + \Delta P_i = (A_i + \Delta A_i)^T (F_i + \Delta F_i)(A_i + \Delta A_i) + Q_i + \Delta Q_i \tag{8.17}$$

where

$$\Delta F_i = \Delta P_i + \sum_{j \neq i} e_{ij} \Delta P_j$$

and $\Delta A_i, \Delta Q_i, \Delta P_i \in \Re^{n \times n}$ are the perturbations in A_i, Q_i, P_i respectively. Here, ΔQ_i is a symmetric matrix, so $Q_i + \Delta Q_i$ is symmetric, implying that

$P_i + \Delta P_i$ is symmetric, and ΔP_i is symmetric. Expanding out (8.17) and using the DCALE (1.11) results in the following perturbed DCALE:

$$\Delta P_i = (A_i + \Delta A_i)^T \Delta F_i (A_i + \Delta A_i) + N_i \tag{8.18}$$

where $N_i \equiv \Delta Q_i + (\Delta A_i)^T F_i A_i + A_i^T F_i \Delta A_i + (\Delta A_i)^T F_i (\Delta A_i)$. Since ΔA_i is a small perturbation, it will be assumed, without loss of generality, that $A_i + \Delta A_i$ is asymptotically stable. Also, if N_i is non-negative definite then $\Delta P_i \geq 0$. Then, the lower matrix bound follows.

Theorem 8.3: The non-negative definite solution ΔP_i of the perturbed DCALE (8.1) has the lower matrix bound:

$$\Delta P_i \geq \left(\phi_i + \sum_{j \neq i} e_{ij} \phi_j\right) (A_i + \Delta A_i)^T (A_i + \Delta A_i) + N_i \equiv \Delta P_{dL1} \tag{8.19}$$

where the non-negative constant ϕ_i is defined by

$$\phi_i \equiv \frac{\lambda_n(N_i)}{1 - \sigma_n^2(A_i + \Delta A_i)}. \tag{8.20}$$

Proof: Applying (2.3) to (8.18) leads to

$$\Delta P_i \geq \lambda_n(\Delta F_i)(A_i + \Delta A_i)^T (A_i + \Delta A_i) + N_i \tag{8.21}$$

Now, $\Delta F_i \geq \Delta P_i$, so one has from (8.21) that

$$\Delta P_i \geq \lambda_n(\Delta P_i)(A_i + \Delta A_i)^T (A_i + \Delta A_i) + N_i \tag{8.22}$$

Applying (2.1) to (8.22) results in

$$\begin{aligned} \lambda_1(\Delta P_i) &\geq \lambda_n\{\lambda_i(\Delta P_i)(A_i + \Delta A_i)^T (A_i + \Delta A_i) + N_i\} \\ &\geq \sigma_n^2(A_i + \Delta A_i)\lambda_1(\Delta P_i) + \lambda_n(N_i) \end{aligned} \tag{8.23}$$

From (8.23), one has $\lambda_n(\Delta P_i) \geq \phi_i$, where ϕ_i is defined by (8.20). Hence, it can be found, using (2.1), that

$$\begin{aligned} \lambda_n(\Delta F_i) &= \lambda_n\left\{\Delta P_i + \sum_{j \neq i} e_{ij} \Delta P_j\right\} \\ &\geq \lambda_n(\Delta P_i) + \sum_{j \neq i} e_{ij} \lambda_n(P_j) \\ &\geq \phi_i + \sum_{j \neq i} e_{ij} \phi_j \end{aligned} \tag{8.24}$$

Substituting (8.24) into (8.21) yields the lower bound (8.19). This concludes the proof of the theorem.

8.3 Solution Bounds of the Continuous Coupled Algebraic Riccati Equation Under Perturbations in the Coefficients

This section will concentrate on the problem of estimating the solution of the Continuous Coupled Algebraic Riccati Equation (CCARE) when its coefficient matrices undergo small perturbations; more precisely, a lower trace bound for the perturbation of the solution of the CCARE when its coefficient matrices undergo small perturbations will be developed.

Consider the CCARE (1.10) with $Q_i = C_i^T C_i$ when its coefficient matrices undergo small perturbations:

$$(A_1 + \Delta A_i)^T(P_i + \Delta P_i) + (P_i + \Delta P_i)(A_1 + \Delta A_i) - (P_i + \Delta P_i)(B_i + \Delta B_i)$$
$$(B_i + \Delta B_i)^T(P_i + \Delta P_i) + \sum_{j\neq i} d_{ij}(P_j + \Delta P_j) = -(C_i + \Delta C_i)^T(C_i + \Delta C_i) \tag{8.25}$$

where $\Delta A_i, \Delta P_i \in \Re^{n\times n}$, $\Delta B_i \in \Re^{n\times m}$ and $\Delta C_i \in \Re^{m\times n}$ are the perturbations in A_i, P_i, B_i and C_i respectively. Since ΔA_i, ΔB_i and ΔC_i are small perturbations it can be assumed, without loss of generality, that under the existence conditions for the CCARE (1.10) [1,20,26,76], the perturbed CCARE (8.25) has a unique positive semidefinite stabilizing solution, i.e., $(P_i + \Delta P_i) \geq 0$. Expanding out (8.25) results in the following perturbed CCARE:

$$L_i^T \Delta P_i + \Delta P_i L_i - \Delta P_i(R_i + \Delta R_i)\Delta P_i + \sum_{j\neq i} d_{ij}\Delta P_j = M_i \tag{8.26}$$

where

$$L_i \equiv A_i + \Delta A_i - (R_i + \Delta R_i)P_i$$
$$M_i \equiv \Delta Q_i + P_i\Delta A_i + (\Delta A_i)^T P_i - P_i \Delta R_i P_i$$

Since P_i and $(P_i + \Delta P_i)$ are unique, there also exists a unique ΔP_i, since $(P_i + \Delta P_i) - P_i = \Delta P_i$. In the sequel, a lower trace bound for the solution ΔP_i of (8.26) will be derived. Throughout, it will also be assumed that

$M_i \geq 0$ and L_i is a stochastically stable matrix, which implies that $\Delta P_i \geq 0$.

Theorem 8.4: Let ΔP_i be the solution of the perturbed CCARE (8.25). If $b_i^2 + 4a_ic_i \geq 0$, then ΔP_i has the lower bound

$$\operatorname{tr}(\Delta P_i) \geq \frac{b_i + \sqrt{b_i^2 + 4a_ic_i}}{2a_i} \tag{8.27}$$

where

$$\begin{aligned} a_i &= \lambda_1(R_i + \Delta R_i) \\ b_i &= \lambda_n(L_i + L_i^T) \\ c_i &= \operatorname{tr}(M_i) \end{aligned}$$

Proof: Taking the trace of both sides of (8.26) and using (2.10) gives:

$$\operatorname{tr}(L_i^T \Delta P_i) + \operatorname{tr}(\Delta P_i L_i) - \operatorname{tr}[\Delta P_i(R_i + \Delta R_i)\Delta P_i] + \sum_{j\neq i} d_{ij}\operatorname{tr}(\Delta P_j) = -\operatorname{tr}(M_i) \tag{8.28}$$

By using (2.11) and (2.12), (8.28) is equivalent to

$$\operatorname{tr}[(R_i + \Delta R_i)(\Delta P_i)^2] \geq 2\operatorname{tr}(\Delta P_i L_i) + tr(M_i) \tag{8.29}$$

where $\sum_{j\neq i} d_{ij} tr(\Delta P_j) \geq 0$ has been used. Application of (2.13) and (2.14) to (8.29) yield:

$$\begin{aligned} \lambda_1(R_i + \Delta R_i)[\operatorname{tr}(\Delta P_i)]^2 &\geq 2\lambda_n(\overline{L_i})\operatorname{tr}(\Delta P_i) + \operatorname{tr}(M_i) \\ \lambda_1(R_i + \Delta R_i)[\operatorname{tr}(\Delta P_i)]^2 - \lambda_n(L_i + L_i^T)\operatorname{tr}(\Delta P_i) - \operatorname{tr}(M_i) &\geq 0 \end{aligned} \tag{8.30}$$

where $\overline{L}_i = \frac{1}{2}(L_i + L_i^T)$. Solving (8.30) for $\operatorname{tr}(\Delta P_i)$ leads to the lower bound (8.27). This completes the proof.

8.4 Solution Bounds of the Discrete Coupled Algebraic Riccati Equation Under Perturbations in the Coefficients

Here, bounds on the perturbations of the solution matrices of the DCARE are proposed when its coefficient matrices undergo small perturbations. More precisely, a lower matrix bound for the perturbation of the solution of the DCARE is proposed when its coefficient matrices are subject to small perturbations.

First, Consider the DCARE (1.12) with $Q_i = C_i^T C_i \geq 0$. Now, consider the perturbed DCARE:

$$P_i + \Delta P_i = (A_i + \Delta A_i)^T [I + (F_i + \Delta F_i)(B_i + \Delta B_i)(B_i + \Delta B_i)^T]^{-1} \times \\ (F_i + \Delta F_i)(A_i + \Delta A_i) + Q_i + \Delta Q_i - P_i \quad (8.31)$$

with

$$\Delta F_i = \Delta P_i + \sum_{j \neq i} \Delta P_j$$

where $\Delta A_i, \Delta P_i \in \Re^{n\times n}$, $\Delta B_i \in \Re^{n\times m}$ and $\Delta C_i \in \Re^{m\times n}$ are the perturbations in A_i, P_i, B_i and C_i respectively, and $Q_i + \Delta Q_i = (C_i + \Delta C_i)^T (C_i + \Delta C_i)$. Since ΔA_i, ΔB_i and ΔC_i are small perturbations it can be assumed, without loss of generality, that under the existence conditions for the DCARE (1.12) [2,7,76], the perturbed DCARE (8.31) has a unique positive semidefinite stabilizing solution, i.e., $(P_i + \Delta P_i) \geq 0$.

Theorem 8.5: Let $P_i + \Delta P_i$ be the symmetric non-negative definite solution of the perturbed DCARE (8.31). Then ΔP_i has the lower bound

$$\Delta P_i \geq (A_i + \Delta A_i)^T [I + (F_i + \Delta F_i)_{Q_i} (B_i + \Delta B_i)(B_i + \Delta B_i)^T]^{-1} \times \\ (F_i + \Delta F_i)_{Q_i} (A_i + \Delta A_i) + Q_i + \Delta Q_i - P_i \quad (8.32)$$

where

$$(F_i + \Delta F_i)_{Q_i} = Q_i + \Delta Q_i + \sum_{j \neq i} e_{ij} (Q_j + \Delta Q_j)$$

Proof: From (8.31), it is obvious that

$$P_i + \Delta P_i \geq Q_i + \Delta Q_i \quad (8.33)$$

Using (2.5) with substitution of (8.33) into (8.31) leads to the bound (8.32). This finishes the proof.

8.5 Summary

In this chapter, some results for bounds on the solutions of coupled Lyapunov and Riccati equations when their coefficients are subject to small perturbations have been presented. These appear to be the first results proposed for such equations when their coefficients are subject to small perturbations.

Chapter 9

Conclusions and Future Work

9.1 Conclusions

In this thesis, new solution bounds of the Lyapunov and Riccati matrix equations have been derived for deterministic and stochastic counterparts, including when their coefficient matrices undergo small perturbations. The main achievements are:

- The development of solution bounds for the CARE when its coefficient matrices undergo small perturbations. These bounds extend the results obtained by other researchers for the nominal case. These new bounds are also less restrictive than existing perturbation bounds.

- The successful derivation of lower matrix bounds for the continuous ALE solution. These bounds are always computed if the solution of the CALE exists, and always yield nontrivial lower solution estimates, even when the matrix Q is positive semi-definite. Also, the bounds can be algorithmically improved to give tighter solution estimates.

- The derivation of improved lower matrix bounds for the continuous ARE solution. These bounds are always determined if the stabilizing solution of the CARE exists, and always give nontrivial solution estimates even when Q is positive semi-definite. It should also be noted that another advantage of these lower bounds is that they remove the assumption that Q is positive definite, which is a strong assumption in control and estimation problems involving the solution of this equation. The resulting bounds can be tightened with the aid of computational algorithms.

- Derivation of new, improved upper matrix bounds for the discrete ARE solution. The new matrix bound (5.56) is always calculated if the stabilizing solution of the DARE exists. In comparison to existing upper matrix bounds for the DARE, the new upper bound is also tighter for some cases, and is less complicated in its calculation. Furthermore, it is also seen that the new bound is more general than the upper matrix bound proposed in [33]. As a final note, the new upper bound, as well as existing bounds, can be tightened successively by back-substituting it into the DARE.

- The development of new lower matrix bounds for the CCALE and CCARE. The lower matrix bounds for the CCALE are always determined if the solutions of the CCALE exist, and always yield nontrivial solution estimates for its solutions. The lower matrix bound for the CCARE always works if Q_i is positive definite, and is also the only non-trivial lower bound for the CCARE other than that proposed in [61].

- The successful derivation of new upper solution bounds for the CCARE and DCARE. The new upper bounds for the CCARE do not necessarily have to assume that $B_iB_i^T$ is nonsingular for them to work, and also yield non-trivial solution estimates for the CCARE, whilst the upper bound derived in [25] has to assume $B_iB_i^T$ is nonsingular for it to be non-trivial, and the upper bound resulting from the UCARE (unified coupled algebraic Riccati equation) in [25] yields the trivial bound $P \leq \infty I$ for the solution of the CCARE. Similarly, the new upper matrix bound for the DCARE also does not necessarily have to assume that $B_iB_i^T$ is nonsingular for it to work, whilst the upper bounds proposed in [14] and [25] do have to assume that $B_iB_i^T$ is nonsingular for them to work. The bounds for both equations have been developed by extending methods that have been used to derive solution bounds for the algebraic Riccati equations arising from deterministic systems.

- The presentation of some solution bounds for the coupled Lyapunov and Riccati equations when their coefficients are subject to small perturbations. The bounds for the coupled Riccati equations appear to be the only existing bounds for such Riccati equations when their coefficients are subject to small perturbations.

9.2 Future Work

Although many achievements have been made on solution bounds for the Lyapunov and Riccati matrix equations, there is still room for more work in this field:

- The upper matrix bounds developed for the CALE always work if the solution of the CALE exists, the only existence assumption required being that A is a stable matrix. However, the computation of these bounds still seem somewhat difficult and not as concise as some existing upper matrix bounds, such as those reported in [72,86], which do not need any condition for satisfaction, and which do not involve any free variable of matrix. However, a comparison by numerical examples suggests that our bounds may be tighter. Nonetheless, it seems that more research is required to derive upper matrix bounds which are always computable if the solution of the CALE exists, which are possibly the tightest bounds, and also ones which are easiest in terms of computation and do not involve any free variable or matrix.

- For the lower matrix bounds derived for the CARE, the assumption $Q > 0$ has been relaxed, and replaced with the less restrictive assumption that $Q \geq 0$. These lower matrix bounds are always calculated if the stabilizing solution of the CARE exists, and always yield nontrivial lower solution estimates, even when Q is positive semi-definite. However, viewing the numerical examples, the derived bounds do not appear to be very tight. Therefore, more research is required to derive lower matrix bounds which are tighter, always work if the solution of the CARE exists, and always yield nontrivial solution estimates. Furthermore, it appears that existing lower matrix bounds for the CARE in the literature also do not seem give results as tight as the upper bounds. Hence, further work is required to seek the existence of lower matrix bounds for the CARE which yield sharper results.

- New, improved solution bounds of the Lyapunov and Riccati matrix equations have been derived for the nominal case. The methods employed to get these bounds could be extended for the case when the coefficient matrices of these equations undergo small perturbations.

- The solution bounds derived in this thesis involve some tuning parameter or tuning matrix, or both. Although solution bounds have been successfully derived involving these parameters/matrices, there remains the question as to which parameter/matrix or both yields the optimal solution bound. It is therefore expected that future research will be conducted to determine which parameter/matrix or both gives the optimal solution bound.

- To date, there does not appear to exist, or at least there has not appeared to be, any mathematical method to compare the tightness between parallel solution bounds of any of the matrix equations, both in the case of existing bounds, or in the case of the bounds derived in this thesis with existing bounds in the literature. Therefore, it is hoped that further research will determine a mathematical method(s) to compare the tightness between the bounds of the same measure for the solution of each matrix equation.

- In this work, some new results have been presented for solution bounds of coupled Lyapunov and Riccati equations for a nominal jump linear system, although many of these results are rather conservative and restrictive. Furthermore, the solution bounds and iterative algorithms developed for the coupled Lyapunov and Riccati equations are not explicit, in that the bounds involved in the algorithms require finding bounds for each solution matrix before a bound can be found for another solution matrix. Hence, further work would be required if one wishes to find less conservative and restrictive bounds for these equations, including bounds which may also be tighter and more explicit.

- In this work, only a few, simple results have been reported for solution bounds of coupled Lyapunov and Riccati equations when their coefficient matrices are subject to small perturbations; there is much more that could be done on this topic. Hence, another possibility for future research is to derive further solution bounds for coupled Lyapunov and Riccati equations when their coefficient matrices undergo small perturbations.

- Some of the solution bounds for the perturbed Riccati equations considered in this thesis may be restrictive. In particular, the solution bounds have been developed under the assumption that the perturbation in the solution, ΔP, is non-negative definite, which is not always the case, since $Q + \Delta Q$ may be non-positive definite or indefinite. Another drawback of these bounds is that their computation also involves finding the exact solution P of the equation for the nominal values, without the perturbations present. For the case of the perturbed discrete Riccati equations, we may use a nominal solution bound to find the perturbed solution bound without having to find the exact nominal solution matrix P. Also, these results have, at the very least, been developed under the assumption that $Q + \Delta Q \geq 0$ which, as we have discussed above, is not always the case. Therefore, future work also involves finding less restrictive solution bounds for the perturbed equations, and also with the preferrability that the exact solution may not need to be found.

Appendix A

Presentations and Publications

During the course of my PhD studies, I delivered several presentations and was also able to successfully publish some of my work in refereed journals. The presentations I delivered were:

(1) R. Davies, "Bounds for the solution of algebraic matrix equations arising in mathematical control theory", Fourth Annual Doctoral Seminar, University of Glamorgan, 2005.

(2) R. Davies, "New Upper Solution Bounds of the Discrete Algebraic Riccati Matrix Equation", Fifth Annual Doctoral Seminar, University of Glamorgan, 2006.

My journal publications are detailed below:

(1) R. Davies, P. Shi, and R. Wiltshire, "New upper solution bounds for perturbed continuous algebraic Riccati equations applied to automatic control", Chaos, Solitons and Fractals, vol. 32, pp. 487-495, 2007.

(2) R. Davies, P. Shi, and R. Wiltshire, "New upper solution bounds of the discrete algebraic Riccati matrix equation", J. of Computational and Applied Mathematics, accepted for publication in 2007.

(3) R. Davies, P. Shi, and R. Wiltshire, "New lower solution bounds of the continuous algebraic Riccati matrix equation", Linear Algebra and Its Ap-

plications, accepted for publication in 2007.

(4) R. Davies, P. Shi, and R. Wiltshire, "Upper solution bounds of the continuous and discrete coupled algebraic Riccati equations", Automatica, accepted for publication in 2007.

(5) R. Davies, P. Shi, and R. Wiltshire, "New lower matrix bounds for the solution of continuous algebraic Lyapunov equation", Asian J. of Control, accepted for publication in 2007.

I also have a conference publication detailed below:

(1) R. Davies, P. Shi, and R. Wiltshire, "Upper matrix bound for the solution of the discrete coupled algebraic Riccati equation", Proc. of the 1st Research Student Workshop, University of Glamorgan, pp. 44-52, 2007.

Bibliography

[1] H. Abou-Kandil, G. Freiling, and G. Jank, "Solution and asymptotic behaviour of coupled Riccati equations in jump linear systems", IEEE Trans. Automatic Control, vol. 39, pp. 1631-1636, 1994.

[2] H. Abou-Kandil, G. Freiling, and G. Jank, "On the solution of discrete-time Markovian jump linear quadratic control problems", Automatica, vol. 31, pp. 765-768, 1995.

[3] S. Barnett, "Introduction to Mathematical control theory", Oxford Applied Mathematics & Computing Science Series, Oxford University Press, 1985.

[4] D.S. Bernstein. "Matrix Mathematics: Theory, facts and formulas with application to linear systems theory", Princeton University Press, 2005.

[5] I. Borno, "Parallel computation of the solutions of coupled algebraic Lyapunov equations", Automatica, vol. 31, pp. 1345-1347, 1995.

[6] I. Borno and Z. Gajic, "Parallel algorithms for optimal control of weakly coupled and singularly perturbed jump linear systems", Automatica, vol. 31, pp. 985-988, 1995.

[7] E.K. Boukas, A. Swierniak, K. Simek, and H. Yang, "Robust stabilization and guaranteed cost control of large scale linear systems with jumps", Kybernetika, vol. 33, pp. 121-131, 1997.

[8] H.J. Chizeck, A.S. Willsky, and D. Castano, "Discrete-time

Markovian-jump linear quadratic optimal control", Int. J. of Control, vol. 43, pp. 213-231, 1986.

[9] H.H. Choi, "Upper matrix bounds for the discrete algebraic Riccati matrix equation", IEEE Trans. on Automatic Control, vol. 46, pp. 504-508, 2001.

[10] H.H. Choi and T.Y. Kuc, "Lower matrix bounds for the continuous algebraic Riccati and Lyapunov matrix equations", Automatica, vol. 38, pp. 1147-1152, 2002.

[11] A. Czornik, "On bound for the solution of the unified algebraic Riccati equation", Matematyka Stosowana, vol. 40, pp. 13-19, 1997.

[12] A. Czornik, and A. Nawrat, "On upper for the unified algebraic Riccati equation", Systems and Control Letters, vol. 32, pp. 235-239, 1997.

[13] A. Czornik, and A. Swierniak, "Lower bounds on the solution of coupled algebraic Riccati equation", Automatica, vol. 37, pp. 619-624, 2001.

[14] A. Czornik, and A. Swierniak, "Upper bounds on the solution of coupled algebraic Riccati equation", J. Inequalities & Applications, vol. 6, pp. 373-385, 2001.

[15] R. Davies, P. Shi, and R. Wiltshire, "New upper solution bounds for perturbed continuous algebraic Riccati equations applied to automatic control", Chaos, Solitons and Fractals, vol. 32, pp. 487-495, 2007.

[16] R. Davies, P. Shi, and R. Wiltshire, "New upper solution bounds of the discrete algebraic Riccati matrix equation", J. of Computational and Applied Mathematics, accepted for publication, 2007.

[17] R. Davies, P. Shi, and R. Wiltshire, "Upper matrix bound for the solution of the discrete coupled algebraic Riccati equation", Proc. of the 1st Research Student Workshop, University of Glamorgan, pp. 44-52, 2007.

[18] R. Davies, P. Shi, and R. Wiltshire, "New lower matrix bounds for the solution of continuous algebraic Lyapunov equation", Asian Journal of control, accepted for publication, 2007.

[19] R. Davies, P. Shi, and R. Wiltshire, "New lower solution bounds of the continuous algebraic Riccati matrix equation", Linear Algebra and Its Applications, accepted for publication, 2007.

[20] R. Davies, P. Shi, and R. Wiltshire, "Upper solution bounds of the continuous and discrete coupled algebraic Riccati equations", Automatica, accepted for publication, 2007.

[21] V. Dragan, "The linear quadratic optimization problem for a class of singularly perturbed stochastic systems", Int. J. of Innovative Computing, Information and Control, vol. 1, pp. 53-63, 2005.

[22] Y. Fang, and K.A. Loparo, "Stochastic stability of jump linear systems", IEEE Trans. on Automatic Control, vol. 47, pp. 1204-1208, 2002.

[23] Y. Fang, and K.A. Loparo, "Stabilization of continuous-time jump linear systems", IEEE Trans. on Automatic Control, vol. 47, pp. 1590-1603, 2002.

[24] Y. Fang, K.A. Loparo, and X. Feng, "New estimates for solutions of Lyapunov equations", IEEE Trans. on Automatic Control, vol. 42, pp. 408-411, 1997.

[25] L. Gao, A. Xue and Y. Sun, "Matrix bounds for the coupled algebraic Riccati equation", Proc. of the 4th World Congress on Intelligent Control and Automation, pp. 10-14, 2002.

[26] L. Gao, A. Xue and Y. Sun, "Comments on "Upper matrix bounds for the discrete algebraic Riccati matrix equation"", IEEE Trans. on Automatic Control, vol. 47, pp. 1212-1213, 2002.

[27] J. Garloff, "Bounds for the eigenvalues of the solution of dis-

crete Riccati and Lyapunov equations and the continuous Lyapunov equation", Int. J. of Control, vol. 43, pp. 423-431, 1986.

[28] J.C. Geromel and J. Bernussou, "On bounds of Lyapunov's matrix equation", IEEE Trans. on Automatic Control, vol. 24, pp. 482-483, 1979.

[29] Y. Ji, and H.J. Chizeck, "Controllability, stabilizability and continuous-time Markovian jump linear quadratic control", IEEE Trans. on Automatic Control, vol. 35, pp. 777-788, 1990.

[30] T. Kang, B.S. Kim and J.G. Lee, "Spectral norm and trace bounds of algebraic matrix Riccati equations" IEEE Trans. on Automatic Control, vol. 41, pp. 1828-1830, 1996.

[31] V.R. Karanam, "A note on eigenvalue bounds in algebraic Riccati equation", IEEE Trans. on Automatic Control, vol. 28, pp. 109-111, 1983.

[32] J.H. Kim and Z. Bien, "Some bounds of the solution of the algebraic Riccati equation", IEEE Trans. on Automatic Control, vol. 37, pp. 1209-1210, 1992.

[33] S.W. Kim and P.G. Park, "Matrix bounds of the discrete ARE solution", Systems & Control Letters, vol. 36, pp. 15-20, 1999.

[34] S.W. Kim, and P.G. Park, "Upper bounds of the continuous ARE solution", IEICE Trans. on Fundamentals of Electronics, Communications and Computer Sciences, vol. E83-A, pp. 380-385, 2000.

[35] S.W. Kim, P.G. Park and W.H. Kwon, "Lower bounds for the trace of the solution of the discrete algebraic Riccati equation", IEEE Trans. on Automatic Control, vol. 38, pp. 312-314, 1993.

[36] D.L. Kleinmann, "On an iterative technique for Riccati equation computations", IEEE Trans. on Automatic Control, vol. 13, pp. 114-115, 1968.

[37] N. Komaroff, "Simultaneous eigenvalue lower bounds for the Lyapunov matrix equation", IEEE Trans. on Automatic Control, vol. 33,

pp. 126-128, 1988.

[38] N. Komaroff, "Simultaneous eigenvalue lower bounds for the Riccati matrix equation", IEEE Trans. on Automatic Control, vol. 34, pp. , 1989.

[39] N. Komaroff, "Upper summation and product bounds for solution eigenvalues of the Lyapunov matrix equation", IEEE Trans. on Automatic Control, vol. 37, pp. 1040-1042, 1992.

[40] N. Komaroff and B. Shahian, "Lower summation bounds for the discrete Riccati and Lyapunov equations", IEEE Trans. on Automatic Control, vol. 37, pp. 1078-1080, 1992.

[41] N. Komaroff, "Upper bounds for the solution of the discrete Riccati equation", IEEE Trans. on Automatic Control, vol. 37, pp. 1370-1373, 1992.

[42] N. Komaroff, "Diverse bounds for the eigenvalues of the continuous algebraic Riccati equation", IEEE Trans. on Automatic Control, vol. 39, pp. 532-534, 1994.

[43] N. Komaroff, "Iterative matrix bounds and computational solutions to the discrete algebraic Riccati equation", IEEE Trans. on Automatic Control, vol. 39, pp. 1676-1678, 1994.

[44] W.H. Kwon, Y.S. Moon and S.C. Ahn, "Bounds in algebraic Riccati and Lyapunov equations: a survey and some new results", Int. J. of Control, vol. 64, pp. 377-389, 1996.

[45] W.H. Kwon, and A.E. Pearson, "A note on the algebraic matrix Riccati equation", IEEE Trans. on Automatic Control, vol. 22, pp. 143-144, 1977.

[46] B.H. Kwon, M.J. Youn and Z. Bien, "On bounds of the Riccati and Lyapunov equations", IEEE Trans. on Automatic Control, vol. 30, pp. 1134-1135, 1985.

[47] C.H. Lee, "On the matrix bounds for the solution matrix of the discrete algebraic Riccati equation", IEEE Trans. on Circuits and Systems-Part I, vol. 43, pp. 402-407, 1996.

[48] C.H. Lee, "Eigenvalue upper and lower bounds of the solution for the continuous algebraic matrix Riccati equation", IEEE Trans. on Circuits and Systems-Part I, vol. 43, pp. 683-686, 1996.

[49] C.H. Lee, "Upper and lower matrix bounds of the solution for the discrete Lyapunov equation", IEEE Trans. on Automatic Control, vol. 41, pp. 1338-1341, 1996.

[50] C.H. Lee, "New results for the bounds of the solution for the continuous Riccati and Lyapunov equations", IEEE Trans. on Automatic Control, vol. 42, pp. 118-123, 1997.

[51] C.H. Lee, "On the upper and lower bounds of the solution for the continuous Riccati matrix equation", Int. J. of Control, vol. 66, pp. 105-118, 1997.

[52] C.H. Lee, "Upper and lower bounds of the solutions of the discrete algebraic Riccati and Lyapunov matrix equations", Int. J. of Control, vol. 68, pp. 579-598, 1997.

[53] C.H. Lee, "Upper matrix bound of the solution for the discrete Riccati equation", IEEE Trans. on Automatic Control, vol. 42, pp. 840-842, 1997.

[54] C.H. Lee, "Eigenvalue upper bounds of the solution of the continuous Riccati equation", IEEE Trans. on Automatic Control, vol. 42, pp. 1268-1271, 1997.

[55] C.H. Lee, "Simple stabilizability criteria and memoryless state feedback control design for time-delay systems with time-varying perturbations", IEEE Trans. on Circuits and Systems-Part I, vol. 45, pp. 1211-1215, 1998.

[56] C.H. Lee, "Further results on the measurement of solution bounds of the generalized Lyapunov equations", J. of Franklin Institute,

vol. 340, pp. 461-480, 2003.

[57] C.H. Lee, "Matrix bounds of the solutions of the continuous and discrete Riccati equations-a unified approach", Int. J. of Control, vol. 76, pp. 635-642, 2003.

[58] C.H. Lee, "Solution bounds of the continuous Riccati matrix equation", IEEE Trans. on Automatic Control, vol. 48, pp. 1409-1413, 2003.

[59] C.H. Lee, "A unified appraoch to eigenvalue bounds of the solution to the Riccati equation", J. of Chinese Institute of Engineers, vol. 27, pp. 431-437, 2004.

[60] C.H. Lee, "Solution bounds of the continuous and discrete Lyapunov matrix equations", J. of Optimization Theory and Applications, vol. 120, pp. 559-578, 2004.

[61] C.H. Lee, "An improved lower matrix bound of the solution of the unified coupled Riccati equation", IEEE Trans. on Automatic Control, vol. 50, pp. 1221-1223, 2005.

[62] C.H. Lee, "New upper solution bounds of the continuous algebraic Riccati matrix equation", IEEE Trans. on Automatic Control, vol. 51, pp. 330-334, 2006.

[63] C.H. Lee and Y.C. Chang, "Solution bounds for the discrete Riccati equation and its applications", J. of Optimization Theory and Applications, vol. 99, pp. 443-463, 1998.

[64] C.H. Lee and F.C. Kung, "Upper and lower matrix bounds of the solutions for the continuous and discrete Lyapunov equations", J. of Franklin Institute, vol. 334B, pp. 539-546, 1997.

[65] C.H. Lee, T.H.S. Li and F.C. Kung, "A new approach for the robust stability of perturbed systems with a class of non-commensurate time delays", IEEE Trans. On Circuits and Systems, vol. 40, pp. 605-608, 1993.

[66] C.H. Lee, T.H.S. Li and F.C. Kung, "On the estimation of solution bounds of the generalized Lyapunov equations and the robust root clustering for the linear perturbed systems", Int. J. of Control, vol. , pp. 996-1008, 2001.

[67] C.H. Lee and T.S. Lien, "New solution bounds for the discrete algebraic matrix Riccati equation", Optimal Control Applications & Methods, vol. 20, pp. 113-125, 1999.

[68] H. Lütkepohl, "Handbook of matrices", John Wiley & Sons, 1997.

[69] M. Mariton, "Jump linear systems in automatic control", Marcel Dekker, Inc., New York, 1990.

[70] T. Mori, "On some bounds in the algebraic Riccati equations", IEEE Trans. on Automatic Control, vol. 30, pp. 162-164, 1985.

[71] T. Mori and I.A. Derese, "A brief summary of the bounds on the solution of the algebraic matrix equations in control theory", Int. J. Control, vol. 39, pp. 247-256, 1984.

[72] T. Mori, N. Fukuma and M. Kuwahara, "Explicit solution and eigenvalue bounds in the Lyapunov matrix equation", IEEE Trans. on Automatic Control, vol. 31, pp. 656-658, 1986.

[73] K. Ogata, "Discrete-time control systems", Prentice-Hall International, 2nd Edition, 1995.

[74] K. Ogata, "Modern control engineering", 3rd edition, Prentice Hall, 1997.

[75] R.V. Patel and M. Toda, "On norm bounds for algebraic Riccati and Lyapunov equations", IEEE Trans. onAutomatic Control, vol. 23, pp. 87-88, 1978.

[76] M.A. Rami, and L. El Ghaoui, "LMI optimization for non-

standard Riccati equations arising in stochastic control", IEEE Trans. on Automatic Control, vol. 41, pp. 1666-1671, 1996.

[77] S. Savov, and I. Popchev, "New upper estimates for the solution of the continuous algebraic Lyapunov Equation", IEEE Trans. on Automatic Control, vol. 49, pp. 1841-1842, 2004.

[78] S. Savov, and I. Popchev, "The continuous algebraic Lyapunov's equation. lower and upper matrix bounds", Comptes rendus de l' Académie bulgare des Sciences, tome 58, pp. 1039-1042, 2005.

[79] S. Savov, and I. Popchev, "Solution bounds validity for CALE extension via singular value decomposition approach", Comptes rendus de l' Académie bulgare des Sciences, tome 59, pp. 933-938, 2006.

[80] S. Savov, and I. Popchev, "Valid inner upper matrix bound for the continuous Lyapunov equation", Comptes rendus de l' Académie bulgare des Sciences, tome 59, pp. 1009-1012, 2006.

[81] P. Shi, and C.E. de Souza, "Bounds on the solution of the algebraic Riccati equation under perturbations in the coefficients", Systems & Control Letters, vol. 15, pp. 175-181, 1990.

[82] P. Shi, and C.E. de Souza, "Trace bounds for perturbed algebraic Riccati equation", Tech. Rep., EE9073, Department of Electrical and Computer Engineering, University of Newcastle, NSW 2308, Australia, 1990.

[83] P. Shi, and C.E. de Souza, "Lower bounds of the solution of the discrete algebraic Riccati equation under perturbations in the coefficients", Tech. Rep., EE9056, Department of Electrical and Computer Engineering, University of Newcastle, NSW 2308, Australia, 1990.

[84] J.G. Sun, "Perturbation theory for algebraic Riccati equations", SIAM J. of Matrix Analysis and Applications, vol. 19, pp. 39-65, 1998.

[85] M.T. Tran, and M.E. Sawan, "A note on the discrete Lyapunov and Riccati matrix equations", Int. J. Control, vol. 39, pp. 337-341, 1984.

[86] I. Troch, "Improved bounds for the eigenvalues of solutions of the Lyapunov equation", IEEE Trans. on Automatic Control, vol. 32, pp. 744-747, 1987.

[87] J.B.R. do Val, J.C. Geromel, and O.L.V. Costa, "Uncoupled Riccati iterations for the linear quadratic control problem of discrete-time Markov jump linear systems", IEEE Trans. on Automatic Control, vol. 43, pp. 1727-1733, 1998.

[88] J.B.R. do Val, J.C. Geromel, and O.L.V. Costa, "Solutions for the linear-quadratic control problem of Markov jump linear systems", J. of Optimization Theory and Applications, vol. 103, pp. 283-311, 1999.

[89] S.D. Wang, T.S. Kuo and C.F. Hsu, "Trace bounds on the solution of the algebraic Riccati and Lyapunov equations", IEEE Trans. on Automatic Control, vol. 31, pp. 654-656, 1986.

[90] S.S. Wang, and T.P. Lin, "Robust stability of uncertain time-delay systems", Int. J. of Control, vol. 46, pp. 963-976, 1987.

[91] K. Yasuda and K. Hirai, "Upper and lower bounds on the solution of the algebraic Riccati equation", IEEE Trans. on Automatic Control, vol. 24, pp. 483-487, 1979.

[92] E. Yaz, "Bounds for the eigenvalues of the solution matrix of the algebraic Riccati equation", Int. J. of Systems Science, vol. 16, pp. 815-820, 1985.

[93] http://en.wikipedia.org/wiki/Optimal_Control.