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GRAPH HOMOMORPHISM RECONFIGURATION AND FROZEN H-COLOURINGS

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ABSTRACT. For a fixed graph H, the *reconfiguration problem* for H-colourings (i.e. homomorphisms to H) asks: given a graph G and two H-colourings φ and ψ of G, does there exist a sequence $f_0, \ldots, f_m$ of H-colourings such that $f_0 = \varphi$, $f_m = \psi$ and $f_i(u)f_{i+1}(v) \in E(H)$ for every $0 \leq i < m$ and $uv \in E(G)$? If the graph G is loop-free, then this is the equivalent to asking whether it possible to transform φ into ψ by changing the colour of one vertex at a time such that all intermediate mappings are H-colourings. In the affirmative, we say that φ *reconfigures* to ψ. Currently, the complexity of deciding whether an H-colouring φ reconfigures to an H-colouring ψ is only known when H is a clique, a circular clique, a C_4-free graph, or in a few other cases which are easily derived from these. We show that this problem is PSPACE-complete when H is an odd wheel.

An important notion in the study of reconfiguration problems for H-colourings is that of a *frozen H-colouring*; i.e. an H-colouring φ such that φ does not reconfigure to any H-colouring ψ such that $\psi \neq \varphi$. We obtain an explicit dichotomy theorem for the problem of deciding whether a given graph G admits a frozen H-colouring. The hardness proof involves a reduction from a CSP problem which is shown to be NP-complete by establishing the non-existence of a certain type of polymorphism.

1. INTRODUCTION

All graphs in this paper are finite and undirected (although, some of the problems that we consider are also interesting for directed graphs; see [8]). We allow loops but no multiple edges; a vertex is said to be *reflexive* if it has a loop. A *homomorphism* from a graph G to a graph H, sometimes called an *H-colouring* of G, is a mapping $\varphi : V(G) \to V(H)$ such that $\varphi(u)\varphi(v) \in E(H)$ whenever $uv \in E(G)$. We write $G \to H$ to indicate that G admits a homomorphism to H and $\varphi : G \to H$ to refer to a particular homomorphism φ. The vertices of H are often referred to as *colours* and, for $v \in V(G)$, the image $f(v)$ of v is often referred to as the *colour* of v. Given graphs G and H, let $\mathrm{Hom}(G, H)$ denote the set of all homomorphisms from G to H.

The most fundamental decision problem for graph homomorphisms is the H-COLOURING problem, which asks whether a given graph G admits a homomorphism to H. This problem is easily solvable in polynomial time if H is bipartite or contains a loop. Hell and Nešetřil [19]

famously proved that, in all other cases, the problem is NP-complete. More recently, new proofs of this result using techniques from universal algebra [12] and Fourier analysis [21] have been discovered, as well as a shorter purely combinatorial proof [25].

Our focus in this paper is on two decision problems for graph homomorphisms arising from an area of research known as "combinatorial reconfiguration." A standard reconfiguration problem asks, given two solutions to a fixed combinatorial problem, whether it is possible to transform one of the solutions into the other by applying a sequence of allowed modifications. Results in combinatorial reconfiguration can provide interesting insights into the structure of the solution space of a combinatorial problem and, sometimes, ideas from combinatorial reconfiguration can even be used to establish the existence of a solution with special properties. For example, Wrochna [31] used ideas from his own paper on combinatorial reconfiguration [30] to prove that every graph H without cycles of length 4 is "multiplicative" in the sense that $G \times F \to H$ implies that $G \to H$ or $F \to H$ (see Definition 3.18 for a definition of the $\times$ product for graphs). For more background on combinatorial reconfiguration, one should consult the broad surveys of van den Heuvel [28] and Nishimura [22].

Given $\varphi, \psi \in \mathrm{Hom}(G, H)$, a *reconfiguration sequence* taking φ to ψ is a sequence $f_0, \ldots, f_m \in \mathrm{Hom}(G, H)$ such that $f_0 = \varphi$, $f_m = \psi$ and, for $0 \le i < m$, we have $f_i(u)f_{i+1}(v) \in E(H)$ for all $uv \in E(G)$. If there exists a reconfiguration sequence taking φ to ψ, then we say that φ *reconfigures* to ψ. Given a fixed graph H, the following decision problem is known as H-RECOLOURING:

> **Instance:** A graph G and $\varphi, \psi \in \mathrm{Hom}(G, H)$.
> **Question:** Does φ reconfigure to ψ?

One may observe that, if φ reconfigures to ψ, then there exists a reconfiguration sequence taking φ to ψ in which any two consecutive elements of the sequence differ in only one vertex. From this, it is not hard to see that, if G is loop-free and $\varphi, \psi \in \mathrm{Hom}(G, H)$, then φ reconfigures to ψ if and only if φ can be transformed into ψ by changing the colour of one vertex at a time while maintaining that the mapping is an H-colouring. For general G, one requires the extra condition that, if $u \in V(G)$ is reflexive, then the colour of u must always be changed to a neighbour of its current colour.

Problems in combinatorial reconfiguration can usually be viewed as questions about the structure of a so called "reconfiguration graph": the vertices of this graph are solutions and the edges correspond to a single application of a reconfiguration step. In our case, the reconfiguration graph, denoted $\mathbf{Hom}(G, H)$, has $\mathrm{Hom}(G, H)$ as its vertex set where two homomorphisms f and g are adjacent whenever $f(u)g(v) \in E(H)$ for all $uv \in E(G)$. Thus H-RECOLOURING asks whether $\varphi, \psi \in \mathrm{Hom}(G, H)$ are in the same component of $\mathbf{Hom}(G, H)$ [10, 13].

It is trivial that H-RECOLOURING is solvable in polynomial time if $H = K_1$ or $H = K_2$, and it is also not hard to see that H-RECOLOURING is contained in PSPACE for general fixed H. Cereceda, van den Heuvel and Johnson [13] proved that, surprisingly, K_3-RECOLOURING is solvable in polynomial time despite the fact that the K_3-COLOURING problem is NP-complete (i.e. it is NP-complete to decide if a graph admits a proper 3-colouring). On the other hand, Bonsma and Cereceda [4] proved that K_k-RECOLOURING is PSPACE-complete for all $k \ge 4$. They achieved this by showing that the decision problem SLIDING TOKENS, which was shown to be PSPACE-complete in [18], reduces to K_k-RECOLOURING. Thus, the reconfiguration problem for homomorphims to cliques admits the following dichotomy theorem.

Theorem 1.1 (Cereceda, van den Heuvel and Johnson [13]; Bonsma and Cereceda [4])**.** *The* K_k*-*RECOLOURING *problem is solvable in polynomial time if* $k \leq 3$ *and is PSPACE-complete if* $k \geq 4$.

There has been some recent work on extending Theorem 1.1 to larger classes of graphs. Wrochna [30] extended the polynomial side of Theorem 1.1 to the following remarkably general result: *If H does not contain a 4-cycle, then H-*RECOLOURING *is solvable in polynomial time.* Brewster, McGuinness, Moore and Noel [9] generalised both the polynomial and PSPACE-complete sides of Theorem 1.1 to circular (p, q)-cliques (see [9] for a definition): the problem is polynomial if $p/q < 4$ and PSPACE-complete otherwise. Intriguingly, the class of graphs for which H-RECOLOURING is known to be solvable in polynomial time coincides exactly with the class of graphs which are known to be multiplicative. The fact that circular (p, q)-cliques with $p/q < 4$ are multiplicative was first proved by Tardif [27]; a new proof, using ideas from reconfiguration, was recently given by Wrochna [31]. More recently, Brewster, Lee and Siggers [8] have investigated the complexity of H-RECOLOURING for digraphs in which every vertex is reflexive (where digraph homomorphisms are the same as graph homomorphisms, except that they are required to preserve the directions of the arcs).

Our first theorem extends the list of "hardness" results for H-RECOLOURING to a new family of graphs. The *wheel* of length m, denoted W_m, is the graph obtained from a cycle of length m by adding a vertex adjacent to all vertices of the cycle. We prove the following.

Theorem 1.2. *For* $k \geq 1$, W_{2k+1}*-*RECOLOURING *is PSPACE-complete.*

An important first step in each of the polynomial-time algorithms for H-RECOLOURING in [9, 13, 30] is to determine the set of vertices v of G such that $\varphi'(v) = \varphi(v)$ for every homomorphism φ' which reconfigures to φ; i.e. to determine the set of vertices which cannot change their colour under any reconfiguration sequence starting with φ. We say that such a vertex is *frozen* by φ and that the homomorphism φ itself is *frozen* if every vertex of G is frozen by φ. Clearly, a frozen homomorphism corresponds to an isolated vertex in **Hom**(G, H). We consider the following decision problem, which we call FROZEN H-COLOURING:

> **Instance:** A graph G.
> **Question:** Does there exist a frozen H-colouring of G?

Clearly, FROZEN H-COLOURING is in NP. As it turns out, there are many graphs H for which there does not even exist a graph G admitting a frozen H-colouring; as a simple example, consider a complete multipartite graph in which every part has size at least two. We say that a connected graph H is *thermal* if there does not exist a graph G such that $V(G) \neq \emptyset$ and G has a frozen H-colouring. If H is thermal, then FROZEN H-COLOURING is trivially solvable in polynomial time (as the answer is always "no"). We prove a dichotomy theorem for FROZEN H-COLOURING which is stated in full generality in Section 3. Here, we state the theorem only for connected graphs.

Theorem 1.3. *Let H be a connected graph. If H has at most two vertices or H is thermal, then* FROZEN H-COLOURING *is solvable in polynomial time. Otherwise,* FROZEN H-COLOURING *is NP-complete.*

In the next section, we prove Theorem 1.2 by showing that, for $k \geq 2$, the K_{2k+1}-RECOLOURING problem can be reduced to the W_{2k+1}-RECOLOURING problem, and therefore the latter is PSPACE-complete by Theorem 1.1. This proof involves an intermediate reduction to and from a reconfiguration problem for so called "edge-coloured homomorphisms." In

Section 3, we prove a dichotomy theorem for FROZEN H-COLOURING. The hardness proof involves showing that a certain constraint satisfaction problem, which we show is NP-complete using algebraic techniques, reduces to the FROZEN H-COLOURING problem. We conclude the paper in Section 4 with some remarks and open problems.

2. ODD WHEEL RECOLOURING IS HARD

2.1. **Preliminaries.** Let us first take the time to fix some notation and terminology and make some preliminary observations. Note that W_3 is simply K_4 and so W_3-RECOLOURING is PSPACE-complete by Theorem 1.1. Thus, it suffices to prove Theorem 1.2 for $k \geq 2$. The following two definitions are standard.

Definition 2.1. Let C_m denote the cycle with vertex set $\{0, \ldots, m-1\}$ where ij is an edge if $i \equiv j \pm 1 \mod m$.

Definition 2.2. Let P_m be the path on m vertices obtained from C_m by deleting the edge from 0 to $m - 1$.

We write the vertex set of W_{2k+1} as $\{0, \ldots, 2k\} \cup \{\alpha\}$ where the vertices of $\{0, \ldots, 2k\}$ form a cycle on $2k + 1$ vertices with adjacencies are as in C_{2k+1} and α is adjacent to every vertex of $\{0, \ldots, 2k\}$. We always view the colours in $\{0, \ldots, 2k\}$ modulo $2k + 1$.
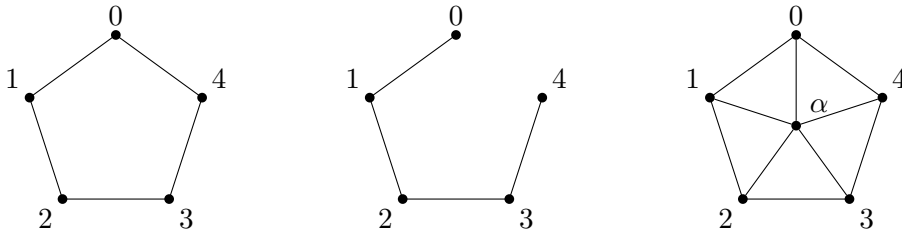

FIGURE 1. The graphs C_5, P_5 and W_5.

In the proof of Theorem 1.2, it will be convenient to reduce K_{2k+1}-RECOLOURING to an intermediate reconfiguration problem for so called "edge-coloured homomorphisms," defined below, and then to reduce this problem to W_{2k+1}-RECOLOURING.

Definition 2.3. An *edge-coloured graph* is a tuple $G = (G_1, \ldots, G_k)$ for some $k \geq 1$, where $G_1, \ldots, G_k$ are graphs with the same vertex set. The *vertex set* of G is taken to be $V(G) := V(G_1)$.

Definition 2.4. Given edge-coloured graphs $G = (G_1, \ldots, G_k)$ and $H = (H_1, \ldots, H_k)$, a *homomorphism* from G to H, or an *H-colouring* of G, is a function $f : V(G) \to V(H)$ such that f is a homomorphism from G_i to H_i for all $1 \leq i \leq k$.

One should think of an edge-coloured graph $G = (G_1, \ldots, G_k)$ as a multigraph obtained from superimposing the graphs $G_1, \ldots, G_k$ on top of each other (keeping any multiple edges that arise) and colouring the edges of G_i with colour i for $1 \leq i \leq k$. In this sense, an edge-coloured homomorphism is a mapping which preserves "coloured adjacencies." Problems regarding edge-coloured homomorphisms are well studied; see, e.g., [1, 5–7].

Given an edge-coloured graph $H = (H_1, H_2, \ldots, H_k)$, the H-RECOLOURING problem extends the definition for graphs in the natural way: the input is an edge-coloured graph G and a

pair of H-colourings φ and ψ of G and the goal is to decide whether there exists H-colourings $f_0, \ldots, f_m$ such that $f_0 = \varphi$, $f_m = \psi$ and $f_i(u)f_{i+1}(v) \in E(H_j)$ for every $0 \le i < m$, $1 \le j \le k$ and $uv \in E(G_j)$. The proof of Theorem 1.2 is divided into the following two lemmas; first, we give a definition.

Definition 2.5. For $m \ge 1$, let Z_m be the graph obtained from a clique on vertex set $\{0, \ldots, m\} \cup \{\alpha\}$ by adding a loop at every vertex except for α.
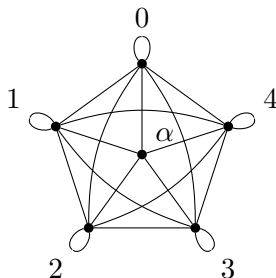

FIGURE 2. The graph Z_5.

Lemma 2.6. (W_{2k+1}, Z_{2k+1})-RECOLOURING *reduces to* W_{2k+1}-RECOLOURING.

Lemma 2.7. K_{2k+1}-RECOLOURING *reduces to* (W_{2k+1}, Z_{2k+1})-RECOLOURING.

We prove Lemma 2.6 in the next subsection and then we prove Lemma 2.7 in the subsection which follows it.

For readers familiar with the "indicator construction" in Hell and Nešetřil's [19] proof of the H-COLOURING dichotomy, parts of our reductions will look familiar. We mimic the classical reduction of K_{2k+1}-COLOURING to C_{2k+1}-COLOURING. Let G be a graph and replace each edge $uv \in E(G)$ with a copy of P_{2k}, identifying the ends of the path with u and v, to create a graph G'. The key to the reduction is that for any distinct pair of vertices in C_{2k+1} there is a homomorphism from P_{2k} to C_{2k+1} with the ends of the path mapping to the two prescribed vertices, but there is no homomorphism mapping the ends of the path to the same vertex. Thus, $G' \to C_{2k+1}$ if and only if $G \to K_{2k+1}$.

However, this reduction does not work to reduce K_{2k+1}-COLOURING to C_{2k+1}-COLOURING as the homomorphisms from P_{2k} to C_{2k+1} do not admit the necessary reconfiguration sequences between them.[1] To add more flexibility, we may view the C_{2k+1}-colourings of P_{2k+1} as W_{2k+1}-colourings which allows us to find the reconfiguration sequences that we require. Unfortunately, this now becomes too flexible in the sense that we can reconfigure to W_{2k+1}-colourings which map the endpoints of P_{2k} to the same vertex (which corresponds to allowing non-proper K_{2k+1}-colouring of G). We eliminate unwanted homomorphisms from P_{2k} to W_{2k+1} from reconfiguration sequences using a second indicator $F_k(x, y)$. In what we believe is a novel modification of the indicator construction for reconfiguration, we exclude the unwanted homomorphisms by ensuring they are frozen (isolated vertices in $\mathbf{Hom}(F_k(x, y), W_{2k+1})$) and thus cannot appear in a reconfiguration sequence, or they are pendant vertices in $\mathbf{Hom}(G', W_{2k+1})$ and can be removed from any reconfiguration sequence.

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[1]The fact that this reduction does not work should come as no surprise since C_{2k+1}-RECOLOURING is solvable in polynomial time while K_{2k+1}-RECOLOURING is PSPACE-complete.

2.2. **Freezing Gadgets and Proof of Lemma 2.6.** The following gadget will be used to keep control over the set of vertices which are mapped to α. See Figure 3 for an illustration of the "main parts" of this gadget.

Definition 2.8 (Freezing Gadget)**.** Let x and y be vertices. Define $F_k(x, y)$ to be the graph with vertex set

$$\{x, y\} \cup \left\{z_0^x, \dots, z_{4k-3}^x\right\} \cup \left\{z_0^y, \dots, z_{4k-3}^y\right\} \cup \{b^x, b^y\} \cup \{w_0, \dots, w_{2k}\} \cup \{\alpha'\}$$

where

- $z_0^x \cdots z_{4k-3}^x$ and $z_0^y \cdots z_{4k-3}^y$ are cycles,
- z_1^x and z_{2k}^x are adjacent to x and z_1^y and z_{2k}^y are adjacent to y,
- z_0^x and z_{2k-1}^x are adjacent to b^x and z_0^y and z_{2k-1}^y are adjacent to b^y,
- y is adjacent to b^x and x is adjacent to b^y,
- $w_0 \cdots w_{2k}$ is a cycle, and
- α' is joined to every vertex of $F_k(x, y)$, except for itself, x, y and their neighbours.
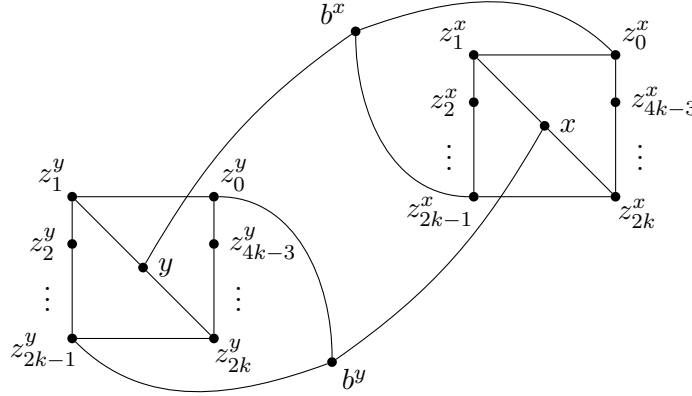

FIGURE 3. The graph $F_k(x, y) \setminus \{w_0, \dots, w_{2k}, \alpha'\}$.

As $F_k(x, y)$ contains a copy of W_{2k+1}, the homomorphisms of $W_{2k+1} \to W_{2k+1}$ play a key role in our analysis. Observe that under any homomorphism $\varphi : W_{2k+1} \to W_{2k+1}$, for $k \geq 2$, the outer cycle $W_{2k+1} \setminus \{\alpha\}$ must map to an odd cycle every vertex of which is adjacent to $\varphi(\alpha)$. If follows that $\varphi(\alpha) = \alpha$ and the outer cycle maps to the outer cycle, so φ is frozen. (For readers familiar with the concept of a *core*, these observations are immediate since W_{2k+1} is a core, any $\varphi : W_{2k+1} \to W_{2k+1}$ must be an automorphism.) Applying these arguments to the copy of W_{2k+1} on the vertex set $\{w_0, \dots, w_{2k}, \alpha'\}$ we get the following.

Observation 2.9. The only vertices that can map to α under a homomorphism $F_k(x, y) \to W_{2k+1}$ are α' (which must), x, y, and the neighbours of x and y.

A second useful observation is the following.

Observation 2.10. For every $c_1, c_2 \in V(W_{2k+1})$ there exists a W_{2k+1}-colouring f of $F_k(x, y)$ such that $f(x) = c_1$ and $f(y) = c_2$.

The next two lemmas illustrate the key properties of the freezing gadget which we will apply; first, a definition and a remark.

Definition 2.11. Given an H-colouring f of G, a vertex $v \in V(G)$ is said to be *fixed* by f if $f(v) = g(v)$ for every neighbour g of f in $\mathbf{Hom}(G, H)$.

Remark 2.12. Fixed vertices and frozen vertices are similar, but not quite the same. A vertex $v \in V(G)$ is frozen by f if $g(v) = f(v)$ for every g in the same component of $\mathbf{Hom}(G, H)$ as f. Thus, if a vertex $v \in V(G)$ is frozen by f, then it is also fixed by f, but the converse does not hold. However, f itself is frozen if and only if every vertex of G is fixed by f.

Lemma 2.13. *If f is a W_{2k+1}-colouring of $F_k(x, y)$ with $f(x) = f(y) = \alpha$, then f is frozen.*

Proof. Let f be such a homomorphism. Our goal is to show that every vertex of $F_k(x, y)$ is fixed by f. Since $\{w_0, \ldots, w_{2k}, \alpha'\}$ induces a copy of W_{2k+1} in $F_k(x, y)$ and every homomorphism from W_{2k+1} to itself is frozen, we have that every vertex of $\{w_0, \ldots, w_{2k}, \alpha'\}$ is frozen, and thus fixed. Moreover, every homomorphism from W_{2k+1} to itself maps α to itself, and therefore $f(\alpha') = \alpha$. Thus, since we are assuming that $f(x) = f(y) = \alpha$, every vertex of $F_k(x, y)$ apart from x, y and α' has a neighbour of colour α, and is therefore mapped into $\{0, \ldots, 2k\}$.

Consider the cycle $z_0^y z_1^y \cdots z_{2k-1}^y b^y$ of length $2k + 1$. The cycle of length $2k + 1$ has only one homomorphism to itself, up to composing with an automorphism. So, without loss of generality, we may assume that

$$f(z_i^y) = i \quad \text{for } 0 \le i \le 2k - 1,$$

$$f(b^y) = 2k.$$

Now, consider the cycle $z_{2k-1}^y z_{2k}^y \cdots z_{4k-3}^y z_0^y b^y$, which is also of length $2k + 1$. We already know that $f(z_{2k-1}^y) = 2k - 1$, that $f(b^y) = 2k$ and that $f(z_0^y) = 0$. Applying uniqueness of the homomorphism from C_{2k+1} to itself once again, we see that this implies

$$f(z_{2k+i}^y) = 2k - i - 2 \quad \text{for } 0 \le i \le 2k - 3.$$

Thus, every vertex $u \in \{z_0^y, \ldots, z_{4k-3}^y, b^y\}$ has neighbours of colours α, $f(u) - 1$ and $f(u) + 1$. Therefore, every such vertex is fixed by f. Also, y is joined to z_1^y and z_{2k}^y which are mapped by f to colours 1 and $2k - 2$, respectively. The only vertex of W_{2k+1} joined to these two colours is α, and so y is fixed by f. By symmetry, the vertices $z_0^x, \ldots, z_{4k-3}^x, b^x$ and x are also fixed and so f is indeed frozen. $\square$

Lemma 2.16 below shows the $F_k(x, y)$ gadget admits reconfiguration sequences which are required in later proofs. In order to prove the lemma, we use the concept of mixing. A graph G is H-*mixing* if for any two H-colourings f and g of G, there is a reconfiguration sequence taking f to g; that is, $\mathbf{Hom}(G, H)$ is connected. The following two propositions are used in the proof; see [10, Propositions 3.12 and 7.12] for proofs.

Proposition 2.14. *Every tree T is C_{2k+1}-mixing.*

Proposition 2.15. *For $r \ge 3$, if $r \le 2k$, then C_{2r} is C_{2k+1}-mixing.*

Lemma 2.16. *Let $c_1, c_1', c_2 \in V(W_{2k+1})$ such that $(c_1, c_2) \ne (\alpha, \alpha)$ and $(c_1', c_2) \ne (\alpha, \alpha)$. If f and g are W_{2k+1}-colourings of $F_k(x, y)$ with $f(u) = g(u)$ for all $u \in \{w_0, \ldots, w_{2k}, \alpha'\}$ such that $(f(x), f(y)) = (c_1, c_2)$ and $(g(x), g(y)) = (c_1', c_2)$, then there exists a reconfiguration sequence $f_0 f_1 \cdots f_m$ and an index $0 \le j \le m - 1$ such that*

- *$f_0 = f$ and $f_m = g$,*
- *$(f_i(x), f_i(y)) = (c_1, c_2)$ for $0 \le i \le j$, and*

- $(f_i(x), f_i(y)) = (c_1', c_2)$ *for* $j + 1 \le i \le m$.

Proof. First, suppose that $c_1 = c_1'$. In this case, we need to show that we can reconfigure f to g without changing the colour of x or the colour of y. As a first step, if $c_1 \ne \alpha$, then we change the colours of the neighbours of x to α one by one (note that by Observation 2.9, if $c_1 \ne \alpha$, then no such vertex has a neighbour of colour α); otherwise, we leave the colours of the neighbours of x unchanged. Similarly, if $c_2 \ne \alpha$, then we change the colours of the neighbours of y to α one by one. Let h be the resulting colouring.

Now, let F' be the subgraph of $F_k(x, y)$ induced by the set of vertices $u \in V(F_k(x, y)) \setminus \{x, y, w_0, \ldots, w_{2k}\}$ such that $h(u) \ne \alpha$. By definition of h and the fact h and g agree on x and y, we see that each vertex of F' is also mapped into $\{0, \ldots, 2k\}$ by g. By assumption, at least one of c_1 or c_2 must be different from α. If exactly one of c_1 or c_2 is equal to α, then F' is the disjoint union of a cycle of length $4k - 2$ and a path of length $4k - 3$ and if, neither c_1 nor c_2 is equal to α, then F' is the disjoint union of four paths of length $2k - 2$. In either case, by Propositions 2.14 and 2.15, every component of F' is C_{2k+1}-mixing. Thus, there is a reconfiguration sequence taking h to a colouring h' such that the colour of each vertex outside of $V(F')$ remains constant throughout this sequence and h' and g agree on the vertices of F'. As a final step, we change the colour of each vertex which is a neighbour of x or y, one by one, to match its colour under g. This is possible since h' and g agree on the neighbourhood of every such vertex. This completes the proof in the case that $c_1 = c_1'$.

Now, suppose that $c_1 \ne c_1'$. In this case, all that we need to do is to show that f can be reconfigured to a W_{2k+1}-colouring f' such that $f'(x) = c_1'$ and $f'(y) = c_2$ without changing the colour of y during the sequence. If this is true, then we can simply apply the result of the previous case to reconfigure f' to g and we will be done.

Consider first the case that $c_1, c_1' \ne \alpha$. In this case, no neighbour of x is adjacent to a vertex which is mapped to α by f. Thus, we can change the colours of the neighbours of x to α, one by one, and then change the colour of x from c_1 to c_1'. Letting f' be the resulting colouring, we see that we are done in this case.

Next, suppose that $c_1 = \alpha$ and $c_1' \ne \alpha$. By hypothesis, this implies that $c_2 \ne \alpha$. We begin by changing the colours of the neighbours of y to α one by one and letting h be the resulting W_{2k+1}-colouring. Let F' be the subgraph of $F_k(x, y)$ induced by the set of vertices $u \in V(F_k(x, y)) \setminus \{y, w_0, \ldots, w_{2k}\}$ such that $h(u) \ne \alpha$. Then the components of F' are the cycle $z_0^x \cdots z_{4k-3}^x$ and the path $z_{2k+1}^y \cdots z_{4k-3}^y z_0^y b^y z_{2k-1}^y \cdots z_2^y$. By Proposition 2.15, by only changing colours of vertices on F', we can reconfigure h to a W_{2k+1}-colouring h' such that

$$h'(z_0^x) = c_1' + 2,$$

$$h'(z_i^x) = c_1' + i \quad \text{for } 1 \le i \le 2k,$$

$$h'(z_{2k+i}^x) = c_1' + 2k - i \quad \text{for } 1 \le i \le 2k - 3,$$

$$h'(b^y) = c_1' + 1.$$

Now, we are simply done by changing the colour of x from $c_1 = \alpha$ to c_1'.

Finally, suppose that $c_1 \ne \alpha$ and $c_1' = \alpha$. By reversing the roles of f and g, and applying the previous case, we are done. (Equivalently, $\mathbf{Hom}(F_k(x, y), W_{2k+1})$ is an undirected graph and we can follow the path from the previous case in the opposite direction.)   □

We are now in position to prove Lemma 2.6.

Proof of Lemma 2.6. Let $G = (G_1, G_2)$ be an edge-coloured graph and let f and g be two (W_{2k+1}, Z_{2k+1})-colourings of G. Let G' be the graph obtained from G_1 by adding the gadget $F_k(x, y)$ for every $xy \in E(G_2)$ disjointly from G_1 except at x and y. Using Observation 2.10, we can let f' and g' be any W_{2k+1}-colourings of G' obtained by extending f and g to the vertices of $V(G') \setminus V(G)$, respectively. Clearly, this construction can be completed in polynomial time and $V(G') = O(|V(G)| + |E(G_2)|)$. We claim that f reconfigures to g as (W_{2k+1}, Z_{2k+1})-colourings of G if and only if f' reconfigures to g' as W_{2k+1}-colourings of G'.

First, suppose that f' reconfigures to g' and let $f'_0, \ldots, f'_m$ be any reconfiguration sequence taking f' to g'. For $0 \leq i \leq m$, let f_i be the restriction of f'_i to $V(G)$. Clearly $f_0 = f$ and $f_m = g$ and so we are done if f_i is a (W_{2k+1}, Z_{2k+1})-colouring of G for $1 \leq i \leq m$. Since G_1 is a subgraph of G', we have that f_i is a W_{2k+1}-colouring of G_1 and so the only way in which it could fail to be a (W_{2k+1}, Z_{2k+1})-colouring is if there exists $xy \in E(G_2)$ such that $f_i(x) = f_i(y) = \alpha$. For any such pair, the gadget $F_k(x, y)$ is present in G'. Thus, by Lemma 2.13, the vertices x and y are frozen by f'_i. In particular, this implies that $f(x) = f'_0(x) = \alpha$ and $f(y) = f'_0(y) = \alpha$, which contradicts the assumption that f is a (W_{2k+1}, Z_{2k+1})-colouring of G. This completes the proof of this direction.

Now, we suppose that f reconfigures to g and show that f' reconfigures to g'. We may assume that f and g differ on only one vertex, say $x \in V(G)$, from which the general case will follow by induction on the length of the reconfiguration sequence taking f to g.

As a first step, let us show that f' reconfigures to a W_{2k+1}-colouring h' of G' such that $h'(u) = g(u)$ for every $u \in V(G)$ (including x). Given a vertex y such that $xy \in E(G_2)$, let $c_1 := f(x)$, $c'_1 := g(x)$ and $c_2 := f(y)$ (note that $g(y) = c_2$ since we are assuming that f and g differ only on x). Since f and g are (W_{2k+1}, Z_{2k+1})-colourings of G, we have that $(c_1, c_2), (c'_1, c_2) \neq (\alpha, \alpha)$. Thus, by Lemma 2.16, there is a reconfiguration sequence of (W_{2k+1}, Z_{2k+1})-colourings of $F_k(x, y)$ taking the restriction of f' to $F_k(x, y)$ to a colouring which maps x to c'_1 and y to c_2 such that the colour of y does not change during this sequence and the colour of x changes exactly once. For every y such that $xy \in E(G_2)$, one at a time, we perform the "first half" of this reconfiguration sequence, stopping before the colour of x changes. After this, we can safely change the colour of x from $f(x)$ to $g(x)$, thereby obtaining the desired W_{2k+1}-colouring h' of G'.

Now, we have that h' and g' agree on all vertices of G, but that they may differ on some of the the freezing gadgets. However, since g' is a (W_{2k+1}, Z_{2k+1})-colouring (i.e. it does not map both sides of any edge of G_2 to α), we can apply Lemma 2.16 (in the case $c_1 = c'_1$) to each freezing gadget one by one to make the colourings of these gadgets match their colourings under g' without changing the colour of any vertex of $V(G)$. Thus, f' reconfigures to h' which reconfigures to g' and the proof is complete. $\square$

2.3. Proof of Lemma 2.7.
Finally, we prove Lemma 2.7, which completes the proof of Theorem 1.2.

Proof of Lemma 2.7. Let G be a graph and let φ and ψ be homomorphisms from G to K_{2k+1}. We construct an instance of (W_{2k+1}, Z_{2k+1})-RECOLOURING in five steps, as follows:

Step 1: Let G^* be the graph obtained from G by subdividing each edge of G exactly $2k - 2$ times. Vertices of $V(G)$ are called *original* vertices and vertices of $V(G^*) \setminus V(G)$ are called *subdivision* vertices. (That is, replace each edge with a path of length $2k - 1$.)

Step 2: For each original vertex v, add $2k$ new vertices, say $\ell^v_1, \ldots, \ell^v_{2k}$ such that $v\ell^v_1 \cdots \ell^v_{2k}$ forms a cycle. These new vertices are called *locking* vertices. Let $\tilde{G}$ be the resulting graph.

Step 3: Add a set W of $2k + 2$ vertices disjoint from $V(G^*)$ such that W induces a copy of W_{2k+1} and every vertex which is not adjacent to an original vertex in $\tilde{G}$ is joined to the vertex of W corresponding to vertex α. Let G'_1 be the resulting graph.

Step 4: Extend φ and ψ, respectively, to the subdivision vertices, the locking vertices, and W to obtain φ' and ψ' mapping G'_1 to W_{2k+1}. We may assume φ' and ψ' agree on W and do not map any vertex of $V(G'_1) \setminus W$ to α. The existence of these extensions follows from the fact that, for any two distinct vertices of C_{2k+1}, there is a homomorphism from P_{2k} to C_{2k+1} mapping the endpoints to these vertices.

Step 5: Let G'_2 be the graph with vertex set $V(G'_1)$ where xy is an edge of G'_2 if there exists distinct original vertices u, v such that $xu \in E(G^*)$ and $y \in \{\ell^v_1, \ell^v_{2k}\}$.

It is clear that these steps can be performed in polynomial time and that the resulting edge-coloured graph $G' = (G'_1, G'_2)$ satisfies $|V(G')| = O(|V(G)| + |E(G)|)$. Also, since φ' and ψ' do not use colour α on $V(G') \setminus W$, it is clear that they are (W_{2k+1}, Z_{2k+1})-colourings of G'. Our goal is to show that φ reconfigures to ψ (as K_{2k+1}-colourings) if and only if φ' reconfigures to ψ' (as (W_{2k+1}, Z_{2k+1})-colourings).

First suppose that φ' reconfigures to ψ' and let $f'_0, \ldots, f'_m$ be any reconfiguration sequence taking φ' to ψ'. For $0 \le i \le m$, let f_i be the restriction of f'_i to $V(G)$. Clearly, we have $f_0 = \varphi$ and $f_m = \psi$ and that f_i differs from f_{i+1} on at most one vertex for $0 \le i \le m - 1$. Since every original vertex is joined to a vertex of W of colour α, we have that $f_i(v) \in \{0, \ldots, 2k\}$ for all $v \in V(G)$ and $0 \le i \le m$. Of course, if each of $f_0, \ldots, f_m$ is a K_{2k+1}-colouring, then we have found a reconfiguration sequence taking φ to ψ and we are done. The following claim allows us to handle the possibility that some of the f_i map adjacent vertices of G to the same colour.

Claim 2.17. *For $1 \le i \le m - 1$, if f_i is not a K_{2k+1}-colouring, then there exists a vertex $v \in V(G)$ such that f_{i-1} and f_{i+1} agree on $V(G) \setminus \{v\}$.*

Proof. Suppose that f_i is not a K_{2k+1}-colouring and let $uv \in E(G)$ such that $f_i(u) = f_i(v)$. (By Lemma 2.13 $f_i(v) \ne \alpha$.) Observe that there is no homomorphism from P_{2k} to C_{2k+1} mapping the endpoints to the same vertex. Therefore, at least one vertex, say x, on the copy of P_{2k} in G'_1 obtained from subdividing the edge uv must be mapped to colour α by f_i. By Step 3, we have that x must be either a neighbour of u or a neighbour of v. Without loss of generality, x is a neighbour of v.

Now recall that, for every original vertex $w \ne v$ and $y \in \{\ell^w_1, \ell^w_{2k}\}$, we have that $xy \in E(G'_2)$. Since x maps to α and f'_i is a (W_{2k+1}, Z_{2k+1})-colouring of G', we must have that $f'_i(y) \in \{0, \ldots, 2k\}$. Since all of the vertices $\ell^w_2, \ldots, \ell^w_{2k-1}$ are adjacent to a vertex of colour α, this implies that every vertex of the cycle $w\ell^w_1 \cdots \ell^w_{2k}$ is mapped into $\{0, \ldots, 2k\}$. Since there is a unique homomorphism from C_{2k+1} to itself, up to automorphism, this implies that every $w \in V(G) \setminus \{v\}$ is adjacent to a vertex of colour α, a vertex of colour $f_i(w) + 1$ and a vertex of colour $f_i(w) - 1$. Thus, every such vertex is fixed by f'_i, which proves the claim. $\qquad\square$

Thus by the claim, $f_0, \ldots, f_{i-1}, f_{i+1}, \ldots, f_m$ is a reconfiguration sequence without the improper colouring f_i. Let $g_0, \ldots, g_t$ be the subsequence of $f_0, \ldots, f_m$ consisting only of the mappings f_j which are K_{2k+1}-colourings. By Claim 2.17, we have that g_i and g_{i+1} differ on at most one vertex for $0 \le i \le t - 1$, and so this sequence certifies that φ reconfigures to ψ. This completes one direction of the proof.

Suppose now that there is a reconfiguration sequence $f_0, \ldots, f_m$ taking φ to ψ. Our goal is to show that φ' reconfigures to ψ'. We may assume that $m = 1$ since the general case will follow easily by induction on m. So, we assume that there exists a unique vertex $v \in V(G)$ such that $\varphi(v) \neq \psi(v)$.

By construction, no vertex of $G' \backslash W$ is mapped to α by φ'. So, as a first step, we may change the colour of each neighbour of v to α, one at a time. Note that this preserves the property of being a (W_{2k+1}, Z_{2k+1})-colouring. At this point, we can safely change the colour of v from $\varphi(v)$ to $\psi(v)$. Thus, we have reached a (W_{2k+1}, Z_{2k+1})-colouring, say γ', of G' which maps every vertex of $V(G)$ to its colour under ψ.

The final step is to show that γ' reconfigures to ψ'. By construction, γ' maps every neighbour of v, including ℓ_1^v and ℓ_{2k}^v, to α. So, we can apply Proposition 2.14 to reconfigure the colouring of the path $\ell_2^v \cdots \ell_{2k-1}^v$ to match its colouring under ψ'. After this, we can change the colours of ℓ_1^v and ℓ_{2k}^v to match their colours under ψ'. At this point, none of the vertices of the form ℓ_1^u or ℓ_{2k}^u for $u \in V(G)$ are mapped to α. Thus, by definition of G_2', we can now change the colour of every subdivision vertex x which is in the neighbourhood of some original vertex to α, one by one. The set of subdivision vertices which are not currently mapped to α induces a disjoint union of paths and so we can apply Proposition 2.14 to change the colouring of these vertices to match its colouring under φ'. We can now change the colour of every subdivision vertex x which is the neighbour of an original vertex from α to $\psi'(x)$. Now, we have reached a (W_{2k+1}, Z_{2k+1})-colouring which only possibly differs from ψ' on vertices of the form ℓ_i^u for $u \in V(G)$. We complete the reconfiguration sequence by going through the vertices $u \in V(G)$, one by one, changing the colours of ℓ_1^u and ℓ_{2k}^u to α, applying Proposition 2.14 to make the colouring of the path $\ell_2^u \cdots \ell_{2k-1}^u$ match its colouring under ψ', and then changing the colours of ℓ_1^u and ℓ_{2k}^u to match ψ' as well. Thus, φ' reconfigures to ψ' and we are done. $\qquad \square$

3. Frozen Homomorphisms and Constraint Satisfaction

3.1. **The Structure of Thermal Graphs and Frozen Homomorphisms.** Our goal in this section is to prove a dichotomy theorem for FROZEN H-COLOURING. We begin by building up a sequence of basic observations regarding the structure of thermal graphs and frozen homomorphisms. We will mostly focus on frozen homomorphisms to a connected graph F (which we think of as being a component of H). Given a graph F and $v \in V(F)$, let $N_F(v)$ be the set of neighbours of v (in particular, if v has a loop, then $v \in N_F(v)$).

Definition 3.1. Given a connected graph F and a set $S \subseteq V(F)$, we say that a vertex $\alpha \in S$ is *redundant* for S if there exists a vertex $\beta \in V(F) \setminus \{\alpha\}$ such that

$$N_F(\alpha) \cap S \subseteq N_F(\beta) \cap S.$$

Definition 3.2. Given a connected graph F, let S_F be a (possibly empty) subset of $V(F)$ such that no vertex in S_F is redundant for S_F and, subject to this, S_F is maximal under subset inclusion.

The following lemma implies that S_F is, in fact, unique.

Lemma 3.3. *If F is a connected graph and $T \subseteq V(F)$ is a set such that no element of T is redundant for T, then $T \subseteq S_F$.*

Proof. Suppose not and let $S := S_F \cup T$. By maximality of S_F, there must exist $\alpha \in S$ and $\beta \in V(F) \setminus \{\alpha\}$ such that

$$N_F(\alpha) \cap S \subseteq N_F(\beta) \cap S.$$

In particular, this implies that both

$$N_F(\alpha) \cap S_F \subseteq N_F(\beta) \cap S_F$$

and

$$N_F(\alpha) \cap T \subseteq N_F(\beta) \cap T.$$

This either contradicts the choice of S_F or the choice of T, depending on whether $\alpha \in S_F$ or $\alpha \in T$. Thus, the lemma is proved. $\qquad \square$

As a corollary of Lemma 3.3, we get that any frozen homomorphism from a graph G to a connected graph F must map into the set S_F.

Corollary 3.4. *Let F be a connected graph. Then for every frozen homomorphism f from a graph G to F, we have $f(V(G)) \subseteq S_F$.*

Proof. By Lemma 3.3, if $f(V(G)) \nsubseteq S_F$, then there exists a vertex $\alpha \in f(V(G))$ which is redundant for $f(V(G))$. Let u be a vertex with $f(u) = \alpha$ and let $\beta \in V(F) \setminus \{\alpha\}$ such that $N_F(\alpha) \cap f(V(G)) \subseteq N_F(\beta) \cap f(V(G))$. Then the function $g : V(G) \to V(F)$ defined by

$$g(v) := \begin{cases} \beta & \text{if } v = u, \\ f(v) & \text{otherwise.} \end{cases}$$

is a homomorphism from G to F which differs from f only on u, contradicting the assumption that f is frozen. $\qquad \square$

Next, we obtain a characterisation of thermal graphs.

Corollary 3.5. *A connected graph F is thermal if and only if $S_F = \emptyset$.*

Proof. If F is not thermal, then there exists a graph G with $V(G) \neq \emptyset$ such that there is a frozen homomorphism f from G to F. By Corollary 3.4, we have $f(V(G)) \subseteq S_F$, which implies that $S_F \neq \emptyset$.

Now, suppose that $S_F \neq \emptyset$ and let F' be the subgraph of F induced by S_F. Let $f : V(F') \to V(F)$ be defined by $f(\alpha) = \alpha$ for all $\alpha \in V(F')$. If f is not frozen, then there exists a vertex $\alpha \in S_F$ which can change its colour from α to some $\beta \neq \alpha$. However, in order for this to be possible, we need $N_F(\alpha) \cap S_F \subseteq N_F(\beta) \cap S_F$, which implies that α is redundant for S_F. This contradiction completes the proof. $\qquad \square$

In other words, Corollary 3.5 says that a connected graph F is thermal if and only if, for every subset S of $V(F)$, there exists a vertex $\alpha \in S$ and a vertex $\beta \in V(F) \setminus \{\alpha\}$ such that $N_F(\alpha) \cap S \subseteq N_F(\beta) \cap S$. This is reminiscent of (but different from) the notion of "dismantlability" which appears in the study of pursuit games on graphs (see, e.g., Nowakowski and Winkler [23]). A connected graph F is said to be *dismantlable* if, for every subset S of $V(F)$, there exists distinct $\alpha, \beta \in S$ such that $\alpha\beta \in E(F)$ and $N_F(\alpha) \cap S \subseteq N_F(\beta) \cap S$.

We remark that, given Lemma 3.3 and Corollary 3.5, it is easy to decide whether a connected graph F is thermal in polynomial time. Simply begin by initialising $S \leftarrow V(F)$ and, while there exists a vertex $\alpha \in S$ which is redundant for S, set $S \leftarrow S \setminus \{\alpha\}$. The graph F is thermal if and only if the set S eventually becomes empty. The running time of this algorithm is clearly polynomial in $|V(F)|$.

Definition 3.6. Given a connected graph F and a vertex $\alpha \in V(F)$, we say that a set $D \subseteq S_F$ is *distinguishing* for α if α is the unique vertex of $V(F)$ such that $D \subseteq N_F(\alpha)$.

Building on the Corollary 3.4, we prove the following.

Lemma 3.7. *Let F be a connected graph. Then a homomorphism from a graph G to F is frozen if and only if for all $v \in V(G)$ the set $f(N_G(v))$ is distinguishing for $f(v)$.*

Proof. Suppose first that f is frozen. By Corollary 3.4, we know that $f(N_G(v)) \subseteq S_F$ for all $v \in V(G)$. Clearly, $f(N_G(v)) \subseteq N_F(f(v))$ since f is a homomorphism. If there exists some $\beta \neq f(v)$ such that $f(N_G(v)) \subseteq N_F(\beta)$, then we can change the colour of v from $f(v)$ to β, contradicting the fact that f is frozen. So, $f(N_G(v))$ is distinguishing for $f(v)$.

Now, suppose that f is a homomorphism such that $f(N_G(v))$ is distinguishing for $f(v)$ for every $v \in V(G)$. If f were not frozen, then we could change the colour of some $v \in V(G)$ from $f(v)$ to some $\beta \neq f(v)$. However, both $f(v)$ and β would have to be adjacent to every colour in $f(N_G(v))$, contradicting the fact that this set is distinguishing for $f(v)$. The result follows. $\square$

As the next lemma demonstrates, one trivial example of a distinguishing set is a set of the form $N_F(\alpha) \cap S_F$ where $\alpha \in S_F$.

Lemma 3.8. *Let F be a connected graph and let $\alpha \in S_F$. Then the set $N_F(\alpha) \cap S_F$ is distinguishing for α.*

Proof. If not, then there must be a vertex $\beta \in V(F)$ such that $N_F(\alpha) \cap S_F \subseteq N_F(\beta)$ which implies that $N_F(\alpha) \cap S_F \subseteq N_F(\beta) \cap S_F$ and so α is redundant for S_F. This is a contradiction. $\square$

The following lemma provides an important distinction between K_1 and K_2 and other non-thermal graphs.

Lemma 3.9. *If F is a connected non-thermal graph with at least three vertices and $D \subseteq S_F$ is distinguishing for some $\alpha \in V(F)$, then $|D| \geq 2$.*

Proof. Clearly, since F has more than one vertex, the empty set is not distinguishing for any $\alpha \in V(F)$. So, suppose that there is a set of cardinality one, say $\{\delta\}$ where $\delta \in S_F$, which is distinguishing for some $\alpha \in V(F)$. This means that α is the unique neighbour of δ.

Since F is connected and has at least three vertices, we see that α must have a neighbour $\beta \notin \{\delta, \alpha\}$. We have $N_F(\delta) \cap S_F$ is either equal to $\emptyset$ or $\{\alpha\}$ depending on whether or not $\alpha \in S_F$. In either case, $N_F(\delta) \cap S_F$ is contained in $N_F(\beta) \cap S_F$. Thus, δ is redundant for S_F which is a contradiction. $\square$

3.2. General Dichotomy Theorem for Frozen $\boldsymbol{H}$-Colouring.
Using the terminology of the previous subsection, we can now state the general dichotomy theorem for FROZEN H-COLOURING. First, a definition.

Definition 3.10. Given a graph H, the *freezer* of H is the subgraph of H induced by the union of the sets S_F over all components F of H.

To be clear, when we say that a graph is bipartite, we mean that it is loop-free and contains no odd cycle.

Theorem 3.11. FROZEN H-COLOURING *is solvable in polynomial time if either*

- *every component of H is either thermal or contains at most two vertices,*
- *H contains a component isomorphic to K_2 and the freezer of H is bipartite, or*
- *H contains a component consisting of a single reflexive vertex.*

Otherwise, FROZEN H-COLOURING *is NP-complete.*

Clearly, Theorem 1.3 is equivalent to Theorem 3.11 in the case that H is connected. The polynomial side of Theorem 3.11 is easy, as we show now.

Proof of the polynomial side of Theorem 3.11. Let G be any graph such that $V(G) \neq \emptyset$. Clearly, no frozen H-colouring of G can map any vertex of G into a thermal component of H. Therefore, G admits a frozen H-colouring if and only if G admits a frozen H'-colouring where H' is obtained from H by deleting all thermal components. So, from here forward, we can assume that H has no thermal components.

Suppose that H contains an isolated vertex. If this isolated vertex is the only vertex of H, then G admits a frozen H-colouring if and only if G has no edges, and so the FROZEN H-COLOURING problem is clearly solvable in polynomial time. If H contains an isolated vertex and has more than one vertex, then G admits a frozen H-colouring if and only if G admits a frozen H'-colouring where H' is obtained from H by deleting all isolated vertices. Also, if H has a component which is just a single reflexive vertex x, then G admits a frozen H-colouring if and only if either x is the only vertex of H or G has no isolated vertices. So, from here forward, we can assume that H has no components F with $|V(F)| = 1$.

Thus, we can assume that every component of H has at least two vertices and that H has no thermal components. By hypothesis, this implies that H contains a component isomorphic to K_2 and that the freezer of H is bipartite. We claim that G admits a frozen H-colouring if and only if G is bipartite and contains no isolated vertices (which are all conditions which can clearly be checked in polynomial time). First, if G is bipartite and contains no isolated vertices, then any homomorphism from G to K_2 is frozen. On the other hand, if G admits a frozen H-colouring, then G cannot have an isolated vertex (as H has at least two vertices). Also, by applying Corollary 3.4 to every component of H, we see that the image of any frozen homomorphism from G to H must be contained in the vertex set of the freezer of H. Therefore, since the freezer of H is bipartite, G must be bipartite too. This completes the proof.   □

3.3. **Relational Structures and Polymorphisms.** Before moving on to the proof of Theorem 3.11, we pause to introduce a tool from the theory of constraint satisfaction problems which is useful in showing that problems are NP-complete.

Definition 3.12. For an integer $k \geq 1$, a *k-relation* $\mathcal{H}$ consists of a finite set $V = V(\mathcal{H})$ and a *k-ary relation* $R(\mathcal{H}) \subseteq V^k$.

A k-relation is an example of a relational structure. Most of the following definitions are simplified versions of more general definitions that are well known for relational structures.

Definition 3.13. Given two k-relations $\mathcal{G}$ and $\mathcal{H}$, a *homomorphism* from $\mathcal{G}$ to $\mathcal{H}$ is a function $f : V(\mathcal{G}) \to V(\mathcal{H})$ such that

$$(r_1, \ldots, r_k) \in R(\mathcal{G}) \Rightarrow (f(r_1), \ldots, f(r_k)) \in R(\mathcal{H}).$$

Given a k-relation $\mathcal{H}$, the following decision problem is known as the *constraint satisfaction problem* for $\mathcal{H}$ and is denoted $\text{CSP}(\mathcal{H})$:

      Instance: A k-relation $\mathcal{G}$.

Question: Does there exist a homomorphism from $\mathcal{G}$ to $\mathcal{H}$?

The famous dichotomy conjecture for constraint satisfaction problems of Feder and Vardi [15] says that, for every relational structure $\mathcal{H}$ (a well known generalisation of a k-relation), the problem $\mathrm{CSP}(\mathcal{H})$ is either solvable in polynomial time or is NP-complete. This problem has been open for nearly 20 years and motivated the discovery of numerous important connections between theoretical computer science, combinatorics and universal algebra. Very recently, several groups have announced proofs of the dichotomy conjecture for CSPs [11, 14, 24, 32] (one of which has now been withdrawn; see [29]). One of the key discoveries in the study of the complexity of constraint satisfaction problems is that the complexity of $\mathrm{CSP}(\mathcal{H})$ is intimately tied to the types of polymorphisms (which we define next) that $\mathcal{H}$ possesses.

Definition 3.14. Let $\mathcal{H}$ be a k-relation. An *m-ary polymorphism* of $\mathcal{H}$ is a function $\varphi : V(\mathcal{H})^m \to V(\mathcal{H})$ which preserves $R(\mathcal{H})$; that is, such that for any set of m k-tuples

$$(r_{1,1}, \ldots, r_{1,k}), (r_{2,1}, \ldots, r_{2,k}) \ldots, (r_{m,1}, \ldots, r_{m,k}) \in R$$

we have

$$(\varphi(r_{1,1}, \ldots, r_{m,1}), \varphi(r_{1,2}, \ldots, r_{m,2}), \ldots, \varphi(r_{1,k}, \ldots, r_{m,k})) \in R$$

In the proof of Theorem 3.11 in the next subsection, we will apply a consequence of the following theorem. It has been observed be several people, (see Barto and Stanovský [3]) that this theorem, or actually a more general version stated for relational structures, follows from Barto, Kosik, and Niven [2] by the proof of a similar statement by Siggers in [26].

Theorem 3.15 ([3, 26]). *If $\mathcal{H}$ is a k-relation which does not admit a 4-ary polymorphism φ such that*

$$\varphi(a, r, e, a) = \varphi(r, a, r, e)$$

for all $a, r, e \in V(\mathcal{H})$, then $\mathrm{CSP}(\mathcal{H})$ is NP-complete.

Say that a k-relation $\mathcal{H}$ is *totally symmetric* if whenever $(r_1, \ldots, r_k) \in R(\mathcal{H})$ and $(r'_1, \ldots, r'_k)$ is a k-tuple such that $\{r'_1, \ldots, r'_k\} = \{r_1, \ldots, r_k\}$ we have $(r'_1, \ldots, r'_k) \in R(\mathcal{H})$. A *constant tuple* is an element of V^k for some k of the form $(v, \ldots, v)$. We use Theorem 3.15 to prove the following lemma which will be applied in the proof of Theorem 3.11. This lemma is of independent interest; indeed, we recently found out that it has been proved independently, with quite a different proof, by Ham and Jackson [17].

Lemma 3.16. *If $\mathcal{H}$ is a non-empty totally symmetric k-relation, for $k \geq 3$, containing no constant tuple, then $\mathrm{CSP}(\mathcal{H})$ is NP-complete.*

Proof. Let m be the minimum of $|\{r_1, \ldots, r_k\}|$ over all $(r_1, \ldots, r_k) \in R(\mathcal{H})$; i.e. m is the smallest number of distinct symbols which appear in an element of the relation $R(\mathcal{H})$. Up to relabelling the elements of $V(\mathcal{H})$, we can assume that there is $(r_1, \ldots, r_k) \in R(\mathcal{H})$ such that $\{r_1, \ldots, r_k\} = \{1, \ldots, m\}$. Since $R(\mathcal{H})$ contains no constant tuple, we know that $m \geq 2$.

Now, suppose to the contrary that $\mathcal{H}$ is not NP-complete. Then, by Theorem 3.15, there exists a 4-ary polymorphism on $\mathcal{H}$ such that

(3.17) $$\varphi(a, r, e, a) = \varphi(r, a, r, e)$$

for all $a, r, e \in V(\mathcal{H})$. If $k > m$, then, since $R(\mathcal{H})$ is totally symmetric, we have that $R(\mathcal{H})$ contains the following four k-tuples (written as column vectors for convenience when applying

φ):

$$\begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \\ m \\ \vdots \\ m \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \\ 4 \\ \vdots \\ m \\ \vdots \\ m \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \\ m \\ \vdots \\ m \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \\ 4 \\ \vdots \\ m \\ \vdots \\ m \end{bmatrix}.$$

Applying (3.17) with $a = 2$, $r = e = 1$ we see that the image of the sequence of entries in the first row under φ is the same as the image of the sequence of entries in the second row under φ, i.e., $\varphi(2, 1, 1, 2) = \varphi(1, 2, 1, 1)$. Also, letting $a = e = 1$ and $r = 2$ shows that $\varphi(1, 2, 1, 1) = \varphi(2, 1, 2, 1)$. Since φ is a polymorphism, we have that the k-tuple

$$(\varphi(2, 1, 1, 2), \varphi(1, 2, 1, 1), \varphi(2, 1, 2, 1), \varphi(3, 3, 3, 3), \dots, \varphi(m, m, m, m), \dots, \varphi(m, m, m, m))$$

is in $R(\mathcal{H})$. However, as we saw above, the first three entries of this tuple are the same and, clearly, so are the last $k - m$ entries. Therefore, this is a k-tuple in $R(\mathcal{H})$ with at most $m - 1$ distinct entries, contradicting our choice of m.

The argument in the case $m = k$ is similar except that, this time, we consider the following four k-tuples:

$$\begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \\ 5 \\ \vdots \\ m \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \\ 5 \\ \vdots \\ m \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 4 \\ 5 \\ \vdots \\ m \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \vdots \\ m \end{bmatrix}.$$

Note that all four of these tuples are in $R(\mathcal{H})$ because of total symmetry and the fact that $k \geq 3$. Applying φ to the sequence of entries in the first row yields the same result as applying φ to the sequence of entries in the second row by (3.17). Thus, since φ is a polymorphism, we again get that $R(\mathcal{H})$ contains an element with at most $m - 1$ distinct entries, contradicting our choice of m. This completes the proof. □

We observe now that Lemma 3.16 implies that H-colouring is NP-complete for any non-bipartite graph H. As this is the hard part of the the H-colouring dichotomy of [19], this further attests to the possible independent interest of the lemma. (We must point out, however, that the proof of Lemma 3.16 relies on results that are stronger than the H-colouring dichotomy.)

Indeed, given a non-bipartite graph H with odd-girth $g \geq 3$, let d be smallest odd integer with $g \leq 3d$. One can show that where C_{3d} is the cycle of girth $3d$ with vertices x, y and z mutually distance d apart, the set

$$\mathcal{H} = \{(\phi(x), \phi(y), \phi(z)) \mid \phi : C_{3d} \to H\}$$

is a non-empty totally symmetric 3-graph on $V(H)$, containing no constant. So by Lemma 3.16 CSP($\mathcal{H}$) is NP-complete. For an instance $\mathcal{G}$ of CSP($\mathcal{H}$) one can construct an instance G of

H-colouring by replacing any tuple $e = (x_e, y_e, z_e)$ with a copy of C_{3d} where all vertices are new except for the copies of x, y and z, which we take to be x_e, y_e and z_e respectively. It is not hard to see that

$$\mathcal{G} \to \mathcal{H} \iff G \to H,$$

and so that H-colouring is NP-complete.

3.4. **The Reduction.** We turn our attention now to the proof of Theorem 3.11. We remark that the main construction used in the reduction is partly inspired by a construction used by Fiala and Paulusma [16] to prove a dichotomy for the problem of deciding whether a graph G admits an H-colouring f such that $f(N_G(v)) = N_H(f(v))$ for all $v \in V(G)$. However, rather than using a reduction from the 2-colouring problem for hypergraphs as was done in [16], we will use a reduction from $\mathrm{CSP}(\mathcal{H})$ for some relational structure $\mathcal{H}$. The following definitions will be useful in the proof.

Definition 3.18. Given graphs $J_1, \ldots, J_t$, let $J_1 \times \cdots \times J_t$ be the graph with vertex set $V(J_1) \times \cdots \times V(J_t)$ where $(u_1, \ldots, u_t)$ is adjacent to $(v_1, \ldots, v_t)$ if and only if u_i is adjacent to v_i in J_i for $1 \le i \le t$. We call $\times$ the *categorical product*.

Definition 3.19. Given a graph J and an integer t, we let J^t denote the t-fold categorical product of J with itself; i.e. $J^t := \underbrace{J \times \cdots \times J}_{t}$.

Definition 3.20. Given a graph G and an integer $t \ge 1$ and $1 \le i \le t$, the function $\varphi_i : V(J^t) \to V(J)$ which maps each vertex to its ith coordinate is called the *projection map* of J^t onto the ith coordinate.

In proving Theorem 3.11, we will make use of three simple facts about the categorical product and its projection maps given by the following lemma.

Lemma 3.21. *Given a graph J and an integer $t \ge 1$, the following statements are true:*
(a) For $1 \le i \le t$, the projection map φ_i is a homomorphism from J^t to J.
(b) If J contains no isolated vertices, then for every vertex $u \in J^t$ and every neighbour γ of $\varphi_i(u)$ in J there exists a neighbour v of u in J^t with $\varphi_i(v) = \gamma$.
(c) If every component of J is non-bipartite, then every component of J^t is non-bipartite.

Proof. If $uv \in E(J^t)$, then, in particular, the ith coordinate of u is adjacent to the ith coordinate of v in J and so $\varphi_i(u)\varphi_i(v) \in E(J)$. Thus, φ_i is a homomorphism and (a) is proved.

Next, we prove (b). Let $u = (\alpha_1, \ldots, \alpha_t)$ be a vertex of $V(J^t)$. Since none of $\alpha_1, \ldots, \alpha_t$ are isolated in J, we get that u is adjacent to v where $v = (\gamma_1, \ldots, \gamma_t)$ and γ_j is a neighbour of α_j for $1 \le j \le t$. Since γ_i can be chosen to be an arbitrary neighbour of α_i, we see that (b) is proved.

The contrapositive of (c) is now immediate by (a). Indeed, if J^t contains an odd cycle, then it must map to an odd cycle in J via any of the projections.

$\square$

Finally, we present the proof of Theorem 3.11.

Proof of Theorem 3.11. The polynomial side was already proved earlier in this section. So, we prove only the NP-complete side. Let H be a graph containing no component consisting of a single reflexive vertex such that either

- H has a component isomorphic to K_2 and the freezer of H is non-bipartite, or
- H does not have a component isomorphic to K_2 and H has at least one non-thermal component with at least three vertices.

Our goal is to show that FROZEN H-COLOURING is NP-complete. Let H' be a subgraph of H such that

- If H has a component isomorphic to K_2, then H' is obtained from H by deleting all thermal components and all components F such that $F[S_F]$ is bipartite.
- If H does not have a component isomorphic to K_2, then H' is obtained from H by deleting all isolated vertices and all thermal components.

Note that every component of H' has at least three vertices. Let $\mathcal{D}$ be the collection of all $D \subseteq V(H')$ such that D is distinguishing for some vertex α in some component F of H'. We construct a k-relation $\mathcal{H}$ with vertex set $V(\mathcal{H}) = V(H')$ as follows:

- Define $k := \max\left(\{3\} \cup \{|D| : D \in \mathcal{D}\}\right)$.
- Let $R(\mathcal{H})$ consist of all of the k-tuples $(r_1, \ldots, r_k)$ such that $\{r_1, \ldots, r_k\} \in \mathcal{D}$.

Note that $k \geq 3$ and $\mathcal{H}$ is totally symmetric by construction; also $\mathcal{H}$ contains no constant tuple by Lemma 3.9 and the fact that every component of H' has at least three vertices. Therefore, by Lemma 3.16, we have that $\mathrm{CSP}(\mathcal{H})$ is NP-complete. Our goal is to reduce $\mathrm{CSP}(\mathcal{H})$ to FROZEN H-COLOURING.

To this end, let $\mathcal{G}$ be any instance of $\mathrm{CSP}(\mathcal{H})$. For each $r = (r_1, \ldots, r_k) \in R(\mathcal{G})$, we create a vertex w_r corresponding to r and let $W := \{w_r : r \in R(\mathcal{G})\}$. We first construct the incidence graph of $\mathcal{G}$: a bipartite graph B with bipartition $(V(\mathcal{G}), W)$ where a vertex $v \in V(\mathcal{G})$ is adjacent to w_r with $r = (r_1, \ldots, r_k)$ if $v \in \{r_1, \ldots, r_k\}$.

Let J be the freezer of H', let $\tilde{J} := J^{|V(J)|}$ and let $\tilde{z}$ be a vertex of $\tilde{J}$ with all of its coordinates distinct. We construct a graph G from B by adding a copy $\tilde{J}_v$ of $\tilde{J}$ for each $v \in V(\mathcal{G})$ and identifying v with the copy $\tilde{z}_v$ of $\tilde{z}$ in $\tilde{J}_v$. Note that $|V(G)| = O\left(|V(\mathcal{G})| + |R(\mathcal{G})|\right)$ and that G can be constructed from $\mathcal{G}$ in polynomial time.

We claim that $\mathcal{G}$ admits a homomorphism to $\mathcal{H}$ if and only if G has a frozen H-colouring. First, we suppose that $\mathcal{G}$ admits a homomorphism f to $\mathcal{H}$ and construct a frozen H-colouring g of G. First, let $g(v) = f(v)$ for all $v \in V(\mathcal{G})$. Next, for every $r = (r_1, \ldots, r_k) \in R(\mathcal{G})$, we have that $\{f(r_1), \ldots, f(r_k)\} \in \mathcal{D}$. So, we can let α_r be the unique vertex of H such that $\{f(r_1), \ldots, f(r_k)\} \subseteq N_H(\alpha_r)$ and let $g(w_r) = \alpha_r$. Now, for each $v \in V(\mathcal{G})$, we colour the vertices of $\tilde{J}_v$ with the projection map onto the coordinate of $\tilde{z}_v$ corresponding to the colour $g(v)$ (note that this is clearly consistent with the colour already chosen for v). We note here that by Lemma 3.21(b) the projection map is locally surjective. Using Lemmas 3.7 and 3.8, together with the fact that J is the freezer of H', shows this projection is frozen.

We claim that the neighbourhood of every vertex of G is mapped to a set in $\mathcal{D}$ which will imply that g is frozen by Lemma 3.7. As we have already seen, the neighbourhood of each $w_r \in W$ maps (by f and so by g) onto a set of $\mathcal{D}$. Given $u \in V(\tilde{J}_v)$ for some $v \in V(\mathcal{G})$ (including the case $u = v$), let F be the component of H' containing $g(u)$. By Corollary 3.4, we have that $g(u) \in S_F$. By definition, the graph J contains the graph $F[S_F]$ as a component. So, applying Lemma 3.21 (b) and the fact that J has no isolated vertices (by definition of H'), we see that, for every $\beta \in N_F(g(u)) \cap S_F$, there is a neighbour of u of colour β. Thus, $g(N_G(u)) \in \mathcal{D}$ by Lemma 3.8. Therefore, g is frozen.

Now, for the other direction, suppose that there exists a frozen H-colouring g of G. We claim that the restriction of g to $V(\mathcal{G})$ is a homomorphism from $\mathcal{G}$ to $\mathcal{H}$. Let $r = (r_1, \ldots, r_k) \in$

$R(\mathcal{G})$ be arbitrary; our goal is to show that $g(\{r_1, \ldots, r_k\}) \in \mathcal{D}$. Let F be the component of H containing $g(w_r)$. Then F must be non-thermal and, by Lemma 3.7, we must have that $g(\{r_1, \ldots, r_k\})$ is distinguishing for $g(w_r)$. If F is a component of H', then we have $g(\{r_1, \ldots, r_k\}) \in \mathcal{D}$ and we are done. So, we assume that F is a non-thermal component of H which is not a component of H'. Clearly F cannot have only one vertex as H does not contain a component consisting of a single reflexive vertex and w_r is not an isolated vertex of G. Therefore, the only possibility is that H contains a component isomorphic to K_2 and $F[S_F]$ is bipartite. For an arbitrary neighbour v of w_r, we have that the component of $\tilde{J}_v$ containing $\tilde{z}_v$ is mapped by g into S_F by Corollary 3.4. However this is impossible since $F[S_F]$ is bipartite and no component of $\tilde{J}$ is bipartite. Indeed, by construction of H' (recall that we are in the case that H contains a component isomorphic to K_2), we have that every component of J is non-bipartite and so every component of $\tilde{J}$ is non-bipartite by Lemma 3.21 (c). This completes the proof.   □

4. Concluding Remarks

The complexity of the H-Recolouring problems are still wide open for general H. For H-Recolouring, it is very tempting to conjecture that the problem is always either PSPACE-complete or solvable in polynomial time (indeed, there is no known example which would refute such a conjecture).

Related to the H-Recolouring problem of finding paths between homomorphisms in $\mathbf{Hom}(G, H)$, is the problem of determining the diameter of components of $\mathbf{Hom}(G, H)$. Given a fixed graph H if, for every graph G, every component of $\mathbf{Hom}(G, H)$ has diameter bounded by a polynomial in $|V(G)|$ (where the polynomial depends only on H), then the H-Recolouring problem is in NP as every "yes" instance has a polynomial sized certificate. This raises the following natural question.

Question 4.1. *For which graphs H does there exist an integer $k = k(H)$ such that, for every graph G, every component of $\mathbf{Hom}(G, H)$ has diameter at most $|V(G)|^k$?*

It seems plausible that, if H-Recolouring is in NP (in particular, if it is in P), then the components of $\mathbf{Hom}(G, H)$ "should" have polynomial diameter, since a reconfiguration sequence is the most natural certificate. Therefore, answering Question 4.1 could be an important step towards clarifying the complexity of the H-Recolouring problem. We should mention that Bonsma and Cereceda [4] proved that, for $k \geq 4$, there exists graphs G such that $\mathbf{Hom}(G, K_k)$ contains a component of superpolynomial diameter. We should also mention that the complexity of the problem of finding the shortest path between two vertices of $\mathbf{Hom}(G, H)$ has been considered (in the case that H is a clique) by Johnson, Kratsch, Kratsch, Patel and Paulusma [20].

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