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Roman dominating function (RDF) if every vertex u with $f(u) = 0$ is adjacent to at least one vertex v with $f(v) = 2$. Henning et al. [4] introduced the concept of perfect Roman domination. A perfect Roman dominating function (PRDF) on a graph G is a function $f : V \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to exactly one vertex v for which $f(v) = 2$. The weight of a PRDF is the sum $f(V) = \sum_{v \in V} f(v)$, and the minimum weight of a PRDF on G is called the perfect Roman domination number of G, denoted by $\gamma_R^p(G)$.

In 2003, Cockayne et al. [5] introduced the concept of weak Roman domination. A function $f : V \longrightarrow \{0, 1, 2\}$ is a weak Roman dominating function (WRDF) if every vertex u with $f(u) = 0$ has a neighbor v with $f(v) > 0$, such that the function $f' : V \longrightarrow \{0, 1, 2\}$ defined by $f'(u) = 1$, $f'(v) = f(v) - 1$ and $f'(w) = f(w)$ if $w \in V \setminus \{u, v\}$, has no undefended vertex. The weight of a WRDF is the value $f(V) = \sum_{v \in V} f(v)$. The weak Roman domination number, denoted by $\gamma_r(G)$, is the minimum weight of a WRDF in G.

Let $f : V \longrightarrow \{0, 1, 2\}$ be a function and let u be a vertex with $f(u) = 0$. We say that u is undefended with respect to f if it is not adjacent to a vertex with positive weight. Furthermore, f is called a weak Roman dominating function (WRDF) if every vertex u for which $f(u) = 0$ is adjacent to a vertex v for which $f(v) > 0$, such that the function $f' : V \longrightarrow \{0, 1, 2\}$, defined by $f'(u) = 1$, $f'(v) = f(v) - 1$ and $f'(w) = f(w)$ if $w \in V \setminus \{u, v\}$, has no undefended vertex.

The concept of strong equality of domination parameters was introduced by Haynes and Slater in [6]. Suppose $\mu(G)$ and $\psi(G)$ are two graph parameters and that every ψ-set (set that achieves the value of $\psi(G)$) is also a μ-set. If $\mu(G) \leq \psi(G)$ for every graph G, then we say that μ strongly equals ψ, written $\mu \equiv \psi$, if every ψ-set is also a μ-set for every graph G.

In this paper, we study the strong equality of perfect Roman and weak Roman domination numbers, and provide a constructive characterization of trees T with $\gamma_R^p(T) \equiv \gamma_r(T)$.

2. Strong Equality of $\gamma_R^p(T)$ and $\gamma_r(T)$ in Trees

In this section, we provide a constructive characterization of trees T with $\gamma_R^p(T) \equiv \gamma_r(T)$. For this purpose, we introduce a family of labeled trees, which are closed under four operations $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ and $\mathcal{O}_4$, defined below.

We will use the concept of the status of a vertex, which is used to assign labels to the vertices. During the construction, the label or status of a vertex v, denoted $sta(v)$, is an element of the set $\{A, B, C\}$. We denote by $(T, sta(T))$ a tree with the status function sta. Let $A(T) = \{v \in V(T) : sta(v) = A\}$. The sets $B(T)$ and $C(T)$ are defined analogously.

Now we are ready to define the family of labeled trees. Let r be a vertex labeled A. Let $\mathcal{T}$ be the family of labeled trees such that:

(i) contains (K_1, A), and

(ii) is closed under the four operations $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ and $\mathcal{O}_4$, which extend the tree T' by attaching a tree to the vertex $v \in V(T')$.

Recall that a labeling of a tree is called a status. To achieve our aim, we need some more notation.

one-to-one correspondence. For this reason, it is handy to set $f = (V_0, V_1, V_2)$. If every vertex u with $f(u) = 0$ has at least one neighbor v with $f(v) = 2$, then the function $f : V(G) \to \{0, 1, 2\}$ is referred to as a *Roman dominating function* (RDF). The weight of an RDF f is $w(f) = f(V(G)) = \sum_{u \in V(G)} f(u)$. We denote by $\gamma_R(G)$ the minimum weight of an RDF of G. It is called the the *Roman domination number* of G. For some further results on Roman dominating function in graphs, we recommend the reader to consult the papers [4–7]. RDFs are useful in the study of generalized k-core percolation [8], where $V_2 = \alpha$-removable, $V_0 = \beta$-removable, and $V_1 = $ non-removable vertices.

Henning, Klostermeyer and MacGillivray [9] introduced the concept of perfect Roman domination in graphs. An RDF $f = (V_0, V_1, V_2)$ is referred to as a *perfect Roman dominating function* (PRDF) when each vertex u with $f(u) = 0$ has only one neighbor v such that $f(v) = 2$ [9]. The minimum weight of an RDF is represented by $\gamma_R^P(G)$ and is called the *perfect Roman domination number*. In other words,

$$\gamma_R^P(G) = \min\{w(f) \mid f \text{ is a PRDF in } G\}.$$

Note that for any graph G, $\gamma_R^P(G)$ is valid. This is because $(\varnothing, V(G), \varnothing)$ is a PRDF for G. Note also that every graph G of order n satisfies $\gamma_R^P(G) \leq n$, as one can define a perfect Roman dominating function f by letting $f(u) = 1$ for any vertex u of G. A PRDF of G with minimum weight is also called a $\gamma_R^P(G)$-function (see Figure 1). For some recent results on perfect Roman domination number of graphs we refer the readers to the papers [10,11].
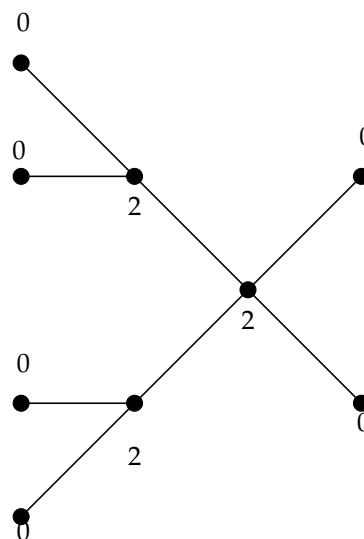

Figure 1. $\gamma_R^P(T)$-function of a tree T.

For graph $G = (V, E)$ and function $f : V \longrightarrow \{0, 1, 2\}$, a vertex u with $f(u) = 0$ is undefended with respect to f if all its neighbors have only zero weight. Henning and Hedetinemi [12] defined the weak Roman dominating function as follows. The function f is a *weak Roman dominating function* (WRDF) if each vertex u with $f(u) = 0$ is adjacent to a vertex v with $f(v) > 0$ such that the function $f' : V \longrightarrow \{0, 1, 2\}$ defined by $f'(u) = 1$, $f'(v) = f(v) - 1$ and $f'(w) = f(w)$ if $w \in V \setminus \{u, v\}$, has no undefended vertex. Let the weight of f be $w(f) = \sum_{v \in V} f(v)$. The *weak Roman domination number*, denoted by $\gamma_r(G)$, is the minimum weight of a WRDF in G, that is:

$$\gamma_r(G) = \min\{w(f) \mid f \text{ is a WRDF in } G\}.$$

It was shown by Henning and Hedetinemi [12] that the problem of computing $\gamma_r(G)$ is NP-hard. This is true even when the graph in consideration is bipartite or chordal. Hence, finding the weak Roman domination number for some special classes of graphs as well as finding some good bounds for

this invariant is of great importance. For some recent results on the weak Roman domination number of graphs we refer the interested reader to the papers [4,13].

Note that $\gamma_r(G) \leq \gamma_R^P(G)$ for every graph G. Clearly, if G is a graph with $\gamma_r(G) = \gamma_R^P(G)$, then every $\gamma_R^P(G)$-function is a $\gamma_r(G)$-function. However, not every $\gamma_r(G)$-function is an $\gamma_R^P(G)$-function (see Figure 2), even when $\gamma_r(G) = \gamma_R^P(G)$. For example, consider the path P_3. We say that $\gamma_R^P(G)$ and $\gamma_r(G)$ are strongly equal, denoted by $\gamma_r(G) \equiv \gamma_R^P(G)$, if every $\gamma_r(G)$-function is an $\gamma_R^P(G)$-function. In this paper, we provide a constructive characterization of trees T such that $\gamma_r(T) \equiv \gamma_R^P(T)$. Other interesting examples of characterizations for strong equalities have been discussed in [4,6,7,14,15].
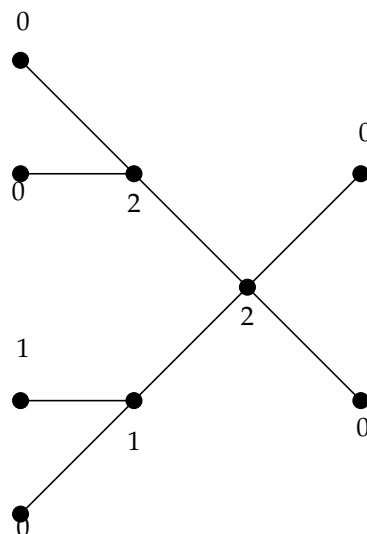

Figure 2. $\gamma_r(T)$-function over a tree T, which is not a perfect Roman dominating function (PRDF).

2. Constructive Characterization of Strong Equality

We start with the following essential lemmas.

Lemma 1. *Let T be a tree with $\gamma_r(T) \equiv \gamma_R^P(T)$. Then T has no weak support vertex with degree two.*

Proof. Let T be a tree with $\gamma_r(T) \equiv \gamma_R^P(T)$ and u be a weak support vertex of it with leaf neighbor v. Let f be a $\gamma_r(T)$-function and $N(u) - \{v\} = \{w\}$. If $f(w) = 2$, then, clearly $f(u) + f(v) = 1$ and so the function g with $g(u) = 1$, $g(v) = 0$ and for $w \in V \setminus \{u, v\}$, $g(w) = f(w)$ is a $\gamma_r(T)$-function, which is not a PRDF. This leads to a contradiction. Thus, we may assume that $f(w) \neq 2$. We first assume that $f(w) = 1$. Then, it is true that $f(u) = 1$, $f(v) = 0$ or $f(v) = 1$, $f(u) = 0$. Hence, f is not a PRDF, which leads to a contradiction. Now, assume that $f(w) = 0$. Then, $1 \leq f(u) + f(v) \leq 2$. If $f(u) + f(v) = 1$, then $f(u) = 1$, $f(v) = 0$ or $f(v) = 1$, $f(u) = 0$. Accordingly, f is not a PRDF, which is a contradiction. Thus, $f(u) + f(v) = 2$. Therefore, function f' by $f'(w) = f'(v) = 1$ and $f'(u) = 0$ and for $x \notin \{w, v, u\}$, $f'(x) = f(x)$ is a $\gamma_r(T)$-function, which is not a PRDF. This leads to a contradiction. Hence, T has no weak support vertex with degree two. $\square$

Lemma 2. *Fix a tree T. Assume that $\gamma_r(T) \equiv \gamma_R^P(T)$ and u is a strong support vertex with $\ell_u = 2$. If f is a $\gamma_r(T)$-function, then $f(u) = 2$. Also, if $\deg(u) = 3$ then any non-leaf neighbor of it has weight 0.*

Proof. Let $L(u) = \{x, y\}$. We first assume that $f(u) = 1$. Since f is a PRDF, we deduce that $f(x) = f(y) = 1$. Then by re-assigning to the vertex u the value 2 and to the vertices x, y the value 0, we get a new WRDF with weight less than $\gamma_r(T)$, which is a contradiction. Next, assume that $f(u) = 0$, then $f(x) = f(y) = 1$. If for any vertex $v \in N(u)$ we have $f(v) \neq 0$, then by re-assigning to the vertex u the value 1 and to the vertex x the value 0, we get a new $\gamma_r(T)$-function that is not a PRDF. This is a contradiction. Hence, we can assume that $w \in N(u)$ and $f(w) = 0$. Since f is a $\gamma_R^P(T)$-function, there exists a vertex $z \neq u$ such that $z \in N(w)$ and $f(z) = 2$. Thus, by re-assigning to the vertex u the value 2 and to the vertices x, y the value 0, we obtain a new $\gamma_r(T)$-function that is not a PRDF. This is a contradiction. Hence $f(u) = 2$. Assume that $\deg(u) = 3$ and $N(u) = \{x, y, w\}$. If $f(w) \neq 0$, then by re-assigning to the vertices u and x the value 1, the $\gamma_r(T)$-function is not a PRDF. This is a contradiction. Therefore, $f(w) = 0$. □

In the case of $\gamma_R^P(T) \equiv \gamma_r(T)$, a family of trees can be defined as follows:
Let F be the tree depicted in Figure 3 and $\mathcal{T}$ be the collection of trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k = T (k \geq 1)$ of trees, where T_1 is a star $K_{1,t}$ with $t \geq 3$, and T_{i+1} can be obtained recursively from $T' = T_i$ by one of the following six operations $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_6$ defined below and illustrated in Figure 4.

- **Operation $\mathcal{O}_1$** : Add a star $K_{1,t}$, $(t \geq 3)$ and link its central vertex v to a vertex u of T' such that $\gamma_r(T' - u) > \gamma_r(T')$.
- **Operation $\mathcal{O}_2$** : Add a new vertex v to T' and link it to a strong support vertex u of T' with at least three leaf neighbors.
- **Operation $\mathcal{O}_3$** : Add a new vertex v to T' and link it to a strong support vertex u of T' with $\deg(u) = 3$.
- **Operation $\mathcal{O}_4$** : Add a star $K_{1,3}$, and link a leaf v to a leaf u of T' that is adjacent to a strong support vertex of T'.
- **Operation $\mathcal{O}_5$** : Add a star $K_{1,3}$ and join a leaf to vertex u of T' such that every $\gamma_r(T')$-function of T' assigns to the vertex u the value 0.
- **Operation $\mathcal{O}_6$** : Add a tree F (as depicted in Figure 3) and join its support vertex v with degree two to a vertex u of T' such that $\deg(u) \geq 2$ and every $\gamma_r(T')$-function of T' assigns to the vertex u the value 2.
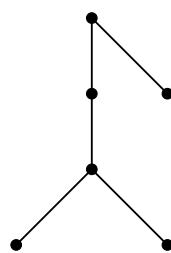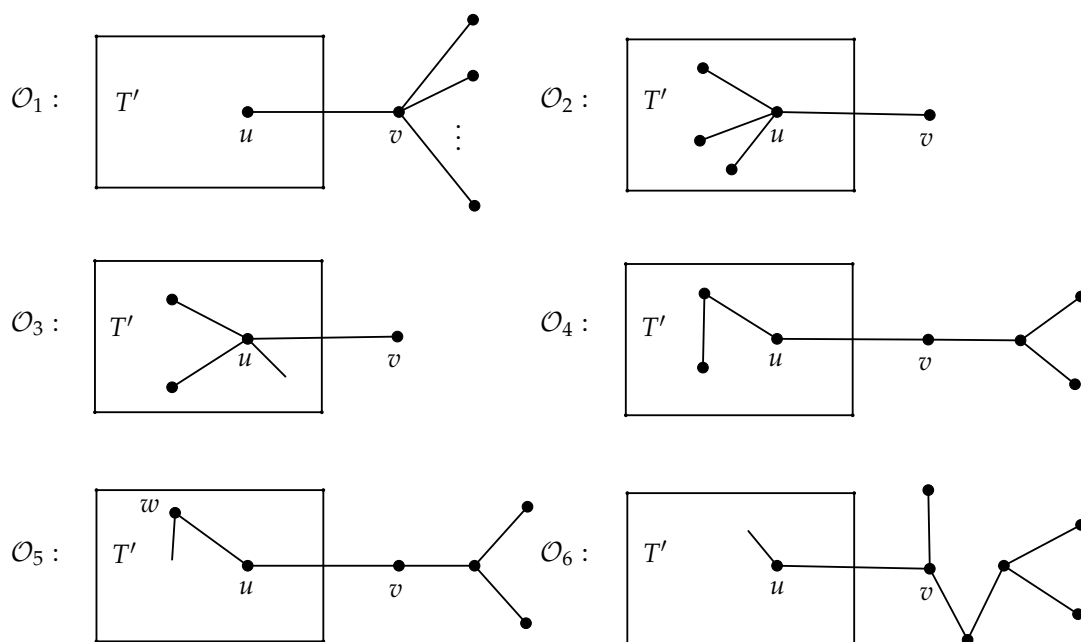

Figure 3. The tree F.

Figure 4. The operations $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_6$.

Next, we will show that for every tree in the family $\mathcal{T}$ (see Figure 5 for an example of a tree in this family), we have $\gamma_R^p(T) \equiv \gamma_r(T)$.
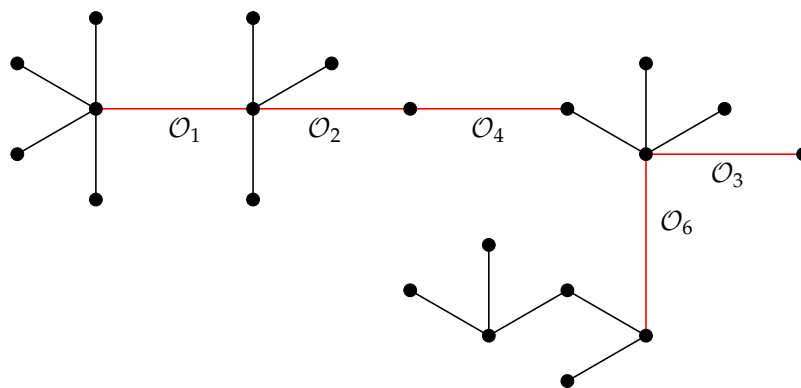

Figure 5. An example of a tree T in the family $\mathcal{T}$, obtained from tree $K_{1,4}$ by Operations $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_4, \mathcal{O}_3$ and $\mathcal{O}_6$.

Lemma 3. *Suppose that T is a tree in the family $\mathcal{T}$, then $\gamma_R^p(T) \equiv \gamma_r(T)$.*

Proof. We proceed by induction on the order $n \geq 4$ of a tree $T \in \mathcal{T}$. If $n = 4$, then $T = K_{1,3}$ and clearly $\gamma_R^p(T) \equiv \gamma_r(T)$. Suppose that $n \geq 5$ and that for every tree in $\mathcal{T}$ of order n', where $5 \leq n' \leq n$, we have $\gamma_R^p(T) \equiv \gamma_r(T)$. Assume that $T \in \mathcal{T}$ has order n. Thus, T can be obtained from a sequence of trees $T_1, \dots, T_k$, where $k \geq 1$ and where $T_1 = K_{1,3}$ and $T = T_k$. For each $i \leq k - 1$, the tree T_i can be constructed by T_{i-1} via one of the operations $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_6$.

Set $T' = T_{k-1}$. We know that $T' \in \mathcal{T}$ has less than n vertices. Applying the inductive hypothesis to T', we know that $\gamma_R^p(T') \equiv \gamma_r(T')$. Suppose that f is a γ_r-function of T, where the sum of the values assigned to all leaves under f is the minimum. We restrict the function f to the tree T' and denote it by

f'. For any vertex $w \in V(T')$, the equality $f'(w) = f(w)$ holds. Six different situations are considered in the sequel.

Case 1. T is derived by T' through Operation $\mathcal{O}_1$.

Assume T is obtained from T' by adding a star $K_{1,3}$, with central vertex v and leaves x, y, z, and adding the edge uv such that $\gamma_r(T' - u) > \gamma_r(T')$. Then u can not get the value 0 under any $\gamma_r(T')$-function. Hence every PRDF on T' can be extended to a PRDF on T by assigning the weight 2 to v and the weight 0 to the three neighbors of v. Hence, by the statement above, and inductive hypothesis, we obtain

$$\gamma_r(T) \leq \gamma_R^P(T) \leq \gamma_R^P(T') + 2 = \gamma_r(T') + 2. \tag{1}$$

Conversely, the vertex v is a strong support vertex of T and so we can see that $f(v) = 2$ and $f(x) = f(y) = f(z) = 0$. If $f(u) = 0$, then $f|_{T'-u}$ is a WRDF on $T' - u$ and so

$$\gamma_r(T') < \gamma_r(T' - u) \leq w(f|_{T'-u}) = w(f) - 2 = \gamma_r(T) - 2. \tag{2}$$

Now assume that $f(u) \neq 0$, then $f|_{T'}$ is a WRDF on tree T' and so

$$\gamma_r(T') \leq w(f|_{T'}) = w(f) - 2 = \gamma_r(T) - 2. \tag{3}$$

Therefore, we must have equalities throughout the inequality chain (1). In particular, $\gamma_r(T) = \gamma_R^P(T)$.

Now, assume that $\gamma_r(T) \not\equiv \gamma_R^P(T)$. Then there is a $\gamma_r(T)$-function g such that g is not a PRDF. Since v is a strong support vertex, hence $g(v) = 2$. If $g(u) \neq 0$, then the function $g|_{T'}$ is a $\gamma_r(T')$-function that is not a PRDF. This is a contradiction. Hence, $g(u) = 0$. Thus, the function $g|_{T'-u}$ is a WRDF on tree $T' - u$ and so $\gamma_r(T') < \gamma_r(T' - u) = w(g|_{T'-u}) = w(g) - 2 = \gamma_r(T) - 2 = \gamma_r(T')$. This is a contradiction. Therefore, $\gamma_r(T) \equiv \gamma_R^P(T)$.

Case 2. T is derived by T' through Operation $\mathcal{O}_2$.

Assume that f' is a $\gamma_R^P(T')$-function. We have $f'(u) = 2$ and therefore f' is extended to a PRDF on T by giving the weight 0 to v. Hence, by inductive hypothesis, we obtain

$$\gamma_r(T) \leq \gamma_R^P(T) \leq \gamma_R^P(T') = \gamma_r(T'). \tag{4}$$

Conversely, the vertex u is a strong support vertex of T and so we can see that $f(u) = 2$ and $f(v) = 0$. Hence, f' is a WRDF on T' and so $\gamma_r(T') \leq w(f') = w(f) = \gamma_r(T)$. Therefore, we must have equalities throughout the inequality chain (4). In particular, $\gamma_r(T) = \gamma_R^P(T)$. Assume that $\gamma_r(T) \not\equiv \gamma_R^P(T)$. Thus, there is a $\gamma_r(T)$-function g such that g is not a PRDF. Since u is a strong support vertex, then $g(u) = 2$ and $g(v) = 0$. Hence, the function $g|_{T'}$ is a $\gamma_r(T')$-function that is not a PRDF. This is a contradiction. Therefore, $\gamma_r(T) \equiv \gamma_R^P(T)$.

Case 3. T is derived by T' through Operation $\mathcal{O}_3$.

Assume that f' is a $\gamma_R^P(T')$-function and $L(u) = \{x, y\}$. Hence, we assume that $f'(u) = 2$ and $f(x) = f(y) = 0$. Then f' can be extended to a PRDF on T by assigning the weight 0 to v. Hence, by the inductive hypothesis, we obtain

$$\gamma_r(T) \leq \gamma_R^P(T) \leq \gamma_R^P(T') = \gamma_r(T'). \tag{5}$$

Conversely, the vertex u is a strong support vertex of T and so we obtain that $f(u) = 2$ and $f(v) = 0$. Hence, f' is a WRDF on T' and so $\gamma_r(T') \leq w(f') = w(f) = \gamma_r(T)$. Therefore, we must have equalities throughout the inequality chain (5). In particular, $\gamma_r(T) = \gamma_R^P(T)$. Assume that $\gamma_r(T) \not\equiv \gamma_R^P(T)$. Thus, there is a $\gamma_r(T)$-function g such that g is not a PRDF. Since u is a strong support vertex, then $g(u) = 2$ and $g(v) = 0$. Hence, the function $g|_{T'}$ is a $\gamma_r(T')$-function that is not a PRDF. This is a contradiction. Therefore, $\gamma_r(T) \equiv \gamma_R^P(T)$.

Case 4. T is derived by T' through Operation $\mathcal{O}_4$.

Assume T is obtained from T' by adding a star $K_{1,3}$, with central vertex x and leaves v, y, z, and adding the edge uv where u is a leaf that is adjacent to a strong support vertex w of T'. Assume that f' is a $\gamma_r(T')$-function. Note that w is a strong support vertex, we assume that $f'(w) = 2$ and $f'(u) = 0$. By the inductive hypothesis, f' is a $\gamma_R^P(T')$-function. Then, we extend f' to a PRDF on T by giving the weight 2 to x and the weight 0 to the vertices v, y and z. Hence, by the statement above, and the inductive hypothesis, we obtain

$$\gamma_r(T) \leq \gamma_R^P(T) \leq \gamma_R^P(T') + 2 = \gamma_r(T') + 2. \tag{6}$$

Conversely, the vertex x is a strong support vertex of T. Also, $f(x) = 2$ and $f(v) = f(y) = f(z) = 0$ hold. Hence, f' is a WRDF on T' and therefore

$$\gamma_r(T') \leq w(f') = w(f) - 2 = \gamma_r(T) - 2. \tag{7}$$

Therefore, we must have equalities throughout the inequality chain (6). In particular, $\gamma_r(T) = \gamma_R^P(T)$.

Assume that $\gamma_r(T) \not\equiv \gamma_R^P(T)$. Then there is a $\gamma_r(T)$-function g such that g is not a PRDF. Since x is a strong support vertex, we may assume that $g(x) = 2$ and $g(v) = 0$. Then the function $g|_{T'}$ is a $\gamma_r(T')$-function on tree T' that is not a PRDF. This is a contradiction. Therefore, $\gamma_r(T) \equiv \gamma_R^P(T)$.

Case 5. T is derived by T' through Operation $\mathcal{O}_5$.

If T is constructed from T' via adding a star $K_{1,3}$ with central vertex x and leaves v, y, z, and joining leaf v of the star to vertex u of T' such that every $\gamma_r(T')$-function of T' gives to the vertex u the value 0. Assume that g' is a $\gamma_r(T')$-function. By the hypothesis, $g'(u) = 0$. On the other hand, g' is a $\gamma_R^P(T')$-function since $\gamma_r(T') \equiv \gamma_R^P(T')$. Thus, g' can be extended to a PRDF on T by assigning the weight 2 to x and the weight 0 to v, y and z. Hence, by the statement above, and inductive hypothesis, we obtain

$$\gamma_r(T) \leq \gamma_R^P(T) \leq \gamma_R^P(T') + 2 = \gamma_r(T') + 2. \tag{8}$$

Conversely, the vertex x is a strong support vertex of T and so we can see that $f(x) = 2$ and $f(v) = f(y) = f(z) = 0$. Hence, f' is a WRDF on T' and therefore

$$\gamma_r(T') \leq w(f') = w(f) - 2 = \gamma_r(T) - 2. \tag{9}$$

Accordingly, we know these equalities across the inequality chain (8). Accordingly, $\gamma_r(T) = \gamma_R^P(T)$.

Assume that $\gamma_r(T) \not\equiv \gamma_R^P(T)$. Then there is a $\gamma_r(T)$-function g such that g is not a PRDF. Since x is a strong support vertex, we may assume that $g(x) = 2$ and $g(v) = 0$. Then, the function $g|_{T'}$ is a $\gamma_r(T')$-function on tree T' and so $g(u) = 0$. Since g is not a PRDF, we deduce that $g|_{T'}$ is not a PRDF, which leads to a contradiction. Therefore, $\gamma_r(T) \equiv \gamma_R^P(T)$.

Case 6. T is derived by T' through Operation $\mathcal{O}_6$.

Suppose that T is derived from T' via adding a tree F and joining its support vertex v with degree two to a vertex u of T' such that $\deg(u) \geq 2$ and every $\gamma_r(T')$-function of T' assigns to the vertex u the value 2. Let $N(v) = \{u, x, w\}$, $L(v) = \{w\}$, $N(x) = \{v, r\}$ and $L(r) = \{y, z\}$.

Suppose that g' is a $\gamma_R^P(T')$-function. Then $g'(u) = 2$. Hence, we extend g' to a PRDF on T by assigning the weight 0 to v, x, y, z, the weight 1 to w and the weight 2 to r. Hence, by the statement above, and the inductive hypothesis, we obtain

$$\gamma_r(T) \leq \gamma_R^P(T) \leq \gamma_R^P(T') + 3 = \gamma_r(T') + 3. \tag{10}$$

Conversely, the vertex r is a strong support vertex of T and so we can assume that $f(r) = 2$ and $f(x) = f(y) = f(z) = 0$. Without loss of generality, we may assume that $f(v) = 0$ or $f(v) = 2$. If $f(v) = 0$, then $f(w) \geq 1$ and also f' is a WRDF on T' and therefore

$$\gamma_r(T') \leq w(f') \leq w(f) - 3 = \gamma_r(T) - 3.$$

Next, suppose that $f(v) = 2$, then $f(w) = 0$ and we may assume that $f(u) \neq 2$. Then the function g' such that $g'(u) = \max\{f(u), 1\}$ and for $w \neq u$, we have $g'(w) = f(w)$, is a WRDF on T' and so

$$\gamma_r(T') \leq w(g') \leq w(f) - 3 = \gamma_r(T) - 3.$$

Accordingly, we must have equalities throughout the inequality chain (10). In particular, $\gamma_r(T) = \gamma_R^P(T)$.

Now, suppose that $\gamma_r(T) \not\equiv \gamma_R^P(T)$. Then there is a $\gamma_r(T)$-function g such that g is not a PRDF. Since r is a strong support vertex, we may assume that $g(r) = 2$ and $g(x) = g(y) = g(z) = 0$. If $g(v) = 0$, then we can assume that $g(w) = 1$. Then $g|_{T'}$ is $\gamma_r(T')$-function that is not a PRDF, which leads to a contradiction. Hence, we assume that $g(v) \neq 0$. We may assume that $g(v) = 2$ and $g(w) = 0$. If $g(u) \neq 0$, then $g|_{T'}$ is $\gamma_r(T')$-function with weight $\gamma_r(T) - 4 = \gamma_r(T') - 1$. This is a contradiction. Hence, we assume that $g(u) = 0$. Thus, function g' by $g'(u) = 1$ and $g'(a) = g(a)$ for $a \neq u$ is a $\gamma_r(T')$-function such that $g'(u) \neq 2$, which is a contradiction. Therefore, $\gamma_r(T) \equiv \gamma_R^P(T)$. This completes the proof. □

Next we have our main results as follows.

Theorem 1. *Let T be a tree. We have $\gamma_r(T) \equiv \gamma_R^P(T)$ if and only if T is K_1 or $T \in \mathcal{T}$.*

Proof. Suppose T is a tree of order n. Moreover, we have $\gamma_r(T) \equiv \gamma_R^P(T)$. When $n = 1$, $T = K_1$ holds. Without loss of generality, we suppose T is a tree or order $n \geq 2$. The induction method will be used in terms of n in the following. We can safely assume that $n \geq 3$ because a path P_2 has a $\gamma_r(P_2)$-function which is not a PRDF. In the case of $n = 3$, $T = P_3$. Furthermore, $\gamma_r(T) \not\equiv \gamma_R^P(T)$. What remains to be shown is the case of $n \geq 4$. Assume that every tree T' of order $2 \leq n' < n$ having $\gamma_r(T) \equiv \gamma_R^P(T)$ lies in $\mathcal{F}$. Suppose that T has n vertices and $\gamma_r(T) \equiv \gamma_R^P(T)$. Let f be a $\gamma_r(T)$-function. It is also safe to view T as a tree with a diameter of at least three because any star having at least four vertices is a member of $\mathcal{F}$. In the case of $diam(T) = 3$, T becomes a double star $S_{a,b}$. We can safely suppose that $a \leq b$. If $b = 2$, then $T = P_4$ and clearly $\gamma_r(T) \not\equiv \gamma_R^P(T)$, which is a contradiction. Hence, we assume that $b \geq 3$. If $a = 2$, then T has a $\gamma_r(P_2)$-function that is not a PRDF, which is a contradiction. Hence, we assume that $a \geq 3$. Then $T \in \mathcal{F}$ because it is obtained from a star $K_{1,b}$ by using Operation $\mathcal{O}_1$. Therefore, we assume that $diam(T) \geq 4$ and T has a strong support vertex u with $\ell_u \geq 4$.

Let $v \in L(u)$, $T' = T - v$ and f be a $\gamma_R^P(T)$-function. Then $f(u) = 2$ and $f|_{T'}$ is a PRDF for tree T' and so $\gamma_r(T') \leq \gamma_R^P(T') \leq \gamma_R^P(T) = \gamma_r(T)$. Now, assume that g' is a $\gamma_r(T')$-function, then g' can be extended to a WRDF on T by assigning the weight 0 to the vertex v, and then $\gamma_r(T) \leq \gamma_r(T')$. Accordingly, $\gamma_r(T) = \gamma_r(T')$. On the other hand, $\gamma_R^P(T) = \gamma_r(T) = \gamma_r(T') \leq \gamma_R^P(T')$ and so $\gamma_R^P(T) = \gamma_R^P(T')$. Therefore, $\gamma_r(T') = \gamma_R^P(T')$. If $\gamma_r(T')$ and $\gamma_R^P(T')$ are not strongly equal, every $\gamma_r(T')$-function that is not a PRDF can be extended to a $\gamma_r(T)$-function by giving 0 to v, that is not a PRDF. This is a contradiction with $\gamma_r(T) \equiv \gamma_R^P(T)$. Therefore, $\gamma_r(T') \equiv \gamma_R^P(T')$ and by induction on T', we have $T' \in \mathcal{F}$. We conclude that $T \in \mathcal{F}$ as it is obtained from T' by using Operation $\mathcal{O}_2$. Hence, we assume that the following fact holds.

Fact 1. If u is a strong support vertex of a tree T, then $\ell_u \leq 3$.

We root T at a leaf x_0 of a diametral path $x_0 x_1 \ldots x_d$ from x_0 to a leaf x_d farthest from x_0. Then Lemma 1 and Fact 1, implies that $3 \leq \deg(x_{d-1}) \leq 4$. Among all $\gamma_r(T)$-functions, let f be chosen so that the weight assigned to x_{d-1} is as large as possible. Since $\gamma_r(T) \equiv \gamma_R^P(T)$, it then follows that f is a $\gamma_R^P(T)$-function. We consider the following two cases:

Case 1. $\deg(x_{d-1}) = 4$.

Let $T' = T - x_d$. Clearly $f(x_{d-1}) = 2$ and $f|_{T'}$ is a PRDF on T'. Hence, $\gamma_r(T') \leq \gamma_R^P(T') \leq w(f|_{T'}) = \gamma_R^P(T) = \gamma_r(T)$. Assume that g is a $\gamma_r(T')$-function. Then we can extend g to a WRDF for T by assigning 0 to x_d, and so $\gamma_r(T) \leq \gamma_r(T')$. Consequently, $\gamma_r(T) = \gamma_r(T')$. Hence $\gamma_R^P(T) = \gamma_r(T) = \gamma_r(T') \leq \gamma_R^P(T')$ and so $\gamma_R^P(T) = \gamma_R^P(T')$. Therefore, $\gamma_r(T') = \gamma_R^P(T')$. If $\gamma_r(T')$ and $\gamma_R^P(T')$ are not

strongly equal, then every $\gamma_r(T')$-function which is not a PRDF can be extended to a $\gamma_r(T)$-function, by assigning 0 to x_d, that is not a PRDF. This leads to a contradiction with $\gamma_r(T) \equiv \gamma_R^P(T)$. Therefore $\gamma_r(T') \equiv \gamma_R^P(T')$ and by induction on T', $T' \in \mathcal{F}$ must be true. Therefore, $T \in \mathcal{F}$ must hold as it is derived from T' by applying Operation $\mathcal{O}_3$.

Case 2. $\deg(x_{d-1}) = 3$.

Then Lemma 2 implies that $f(x_{d-1}) = 2$. If $f(x_{d-2}) \neq 0$, then re-assigning to the vertices x_d and x_{d-1} the weight 1 produces a new $\gamma_r(T)$-function which is not a PRDF. This is a contradiction. Hence, $f(x_{d-2}) = 0$. If $u \neq x_{d-1}$ is a child strong support vertex of x_{d-2}, then we can assume that $f(u) = 2$ and so f is not a PRDF, which is a contradiction. On the other hand, Lemma 1 implies that x_{d-2} has not a weak support vertex as child. Thus every child $u \neq x_{d-1}$ of x_d is a leaf. Since $f(x_{d-2}) = 0$ holds, x_{d-2} is not a strong support vertex. Assume that x_{d-2} is a weak support vertex. Let $L(x_{d-2}) = \{x\}$ and $L(x_{d-1}) = \{x_d, z\}$. Without loss of generality, we may assume that $f(x) = 1$. Then re-assigning to the vertices x_d and z the weight 1 and to the vertex x_{d-1} the weight 0, we obtain a new $\gamma_r(T)$-function which is not a PRDF. This is a contradiction. Hence, we assume that $\deg(x_{d-2}) = 2$. Thus, Lemma 2 implies that $f(x_{d-1}) = 2$ and $f(x_d) = f(z) = f(x_{d-2}) = 0$. Since f is also a $\gamma_R^P(T)$-function, then $f(x_{d-3}) \neq 2$. Assume that $f(x_{d-3}) = 1$. Since f is a $\gamma_R^P(T)$-function, every vertex with weight 0 is adjacent to a vertex with weight 2, and then by giving to the vertex x_{d-3} the weight 0 and to the vertex x_{d-2} the weight 1, we obtain a new $\gamma_r(T)$-function, contradicting Lemma 2. Hence, $f(x_{d-3}) = 0$ and so $f(x_{d-4}) = 2$. Two subcases follow as follows:

Subcase 2.1 $\deg(x_{d-3}) \geq 3$.

First, we assume that x_{d-3} is a support vertex. Since $f(x_{d-3}) = 0$, it is not a strong support vertex so x_{d-3} is a weak support vertex. Let $L(x_{d-3}) = x'$. Without loss of generality, we can assume that $f(x') = 1$ since $f(x_{d-3}) = 0$. If x_{d-3} has a child u that is a strong support vertex, then we can assume that $f(u) = 2$, contradicting that f is a $\gamma_R^P(T)$-function of T. Thus, x_{d-3} does not have a strong support vertex as a child. On the other hand, Lemma 1 implies that x_{d-3} does not have a weak support vertex as a child. Hence, every child of x_{d-3}, except x', plays a role similar as x_{d-2}. Assume that $\deg(x_{d-3}) \geq 4$ and u is a child of x_{d-3} other than x' and x_{d-2}. Let $T' = T - T_{x_{d-2}}$. Since f is a $\gamma_R^P(T)$-function and $f(x_{d-2}) = 0$, hence $f|_{T'}$ is a PRDF on T', and so $\gamma_r(T') \leq \gamma_R^P(T') \leq \gamma_R^P(T) - 2 = \gamma_r(T) - 2$. Assume that f' is a $\gamma_r(T')$-function. Let $N(u) = \{x_{d-3}, x\}$ and $N(x) = \{u, y, z\}$. Then, we can assume that $f'(x) = 2$ and $f'(y) = f'(z) = f'(u) = 0$. If $f'(x_{d-3}) \neq 1$, then we can extend f' to a WRDF on T by assigning the weight 2 to x_{d-1} and the weight 0 to the three neighbors of it. Hence, $\gamma_r(T) \leq \gamma_r(T') + 2$. Next, assume that $f'(x_{d-3}) = 1$. Since $f'(u) = 0$, for every $w \in N(x_{d-3}) \setminus \{u\}$ such that $f'(w) = 0$, there exists at least one vertex $w' \in N(w) \setminus \{x_{d-3}\}$ such that $f'(w') \neq 0$. Hence, we can extend f' to a WRDF on T by assigning the weight 2 to x_{d-1} and the weight 0 to the three neighbors of it. Hence, $\gamma_r(T) \leq \gamma_r(T') + 2$. Consequently, in both cases, $\gamma_r(T') = \gamma_r(T) - 2$. Therefore, $\gamma_R^P(T') \leq \gamma_R^P(T) - 2 = \gamma_r(T) - 2 = \gamma_r(T') \leq \gamma_R^P(T')$ and so $\gamma_r(T') = \gamma_R^P(T')$. If $\gamma_r(T')$ and $\gamma_R^P(T')$ are not strongly equal, then every $\gamma_r(T')$-function which is not a PRDF is extended to a $\gamma_r(T)$-function by giving the weight 2 to x_{d-1} and the weight 0 to the three neighbors of it, that is not a PRDF. This leads to a contradiction with $\gamma_r(T) \equiv \gamma_R^P(T)$. Therefore, $\gamma_r(T') \equiv \gamma_R^P(T')$ and by induction on T', we have $T' \in \mathcal{F}$. We conclude that $T \in \mathcal{F}$, as it is obtained from T' by using Operation $\mathcal{O}_5$.

Thus, we can assume that $\deg(x_{d-3}) = 3$. If $\deg(x_{d-4}) = 2$, then clearly $f(x_{d-5}) = 0$ and so by re-assigning to the vertices x_{d-3} and x_{d-5} the value 1 and to the vertex x_{d-4} the value 0 we get a new $\gamma_r(T)$-function which is not a PRDF. This is a contradiction. Hence, we can assume that $\deg(x_{d-4}) \geq 3$. Let $T' = T - T_{x_{d-3}}$. Since f is a $\gamma_R^P(T)$-function and $f(x_{d-3}) = 0$, then $f|_{T'}$ is a PRDF on T' and so $\gamma_r(T') \leq \gamma_R^P(T') \leq \gamma_R^P(T) - 3 = \gamma_r(T) - 3$. Assume that f' is a $\gamma_r(T')$-function. If $f'(x_{d-4}) \neq 1$, then we can extend f' to a WRDF on T by assigning the weight 2 to x_{d-1}, the weight 1 to x' and the weight 0 to the remaining vertices of T, which is a WRDF of T of weight $\gamma_r(T') + 3$, and so $\gamma_r(T) \leq \gamma_r(T') + 3$. Next, assume that $f'(x_{d-4}) = 1$. If for every vertex $u \in N(x_{d-4})$ with $f'(u) = 0$, there exists at least one vertex $w \in N(u)$ with $f'(w) > 0$, then we can extend f' to a WRDF on T by assigning the weight 2 to x_{d-1}, the weight 1 to x' and the weight 0 to the remaining vertices of T,

which is a WRDF of T of weight $\gamma_r(T') + 3$, and so $\gamma_r(T) \leq \gamma_r(T') + 3$. We assume that there exists a vertex $u \in N(x_{d-4})$ with $f'(u) = 0$ such that for every $w \in N(u)$, we have $f'(w) = 0$. Then for every vertex $y \in N(x_{d-4}) - \{u\}$, we get $f'(y) \neq 0$, since f' is a $\gamma_r(T')$-function. Hence, there exists a vertex $y \in N(x_{d-4})$ such that $f'(y) \neq 0$, since $\deg(x_{d-4}) \geq 3$. Then by re-assigning to the vertices x_{d-4} and y the value 1, we get a new $\gamma_r(T)$-function g'. Consequently, $\gamma_r(T') = \gamma_r(T) - 3$. Hence, $\gamma_R^P(T') \leq \gamma_R^P(T) - 3 = \gamma_r(T) - 3 = \gamma_r(T') \leq \gamma_R^P(T')$ and so $\gamma_r(T') = \gamma_R^P(T')$.

Now assume that g' is a $\gamma_r(T')$-function. We first assume that $g'(x_{d-4}) = 0$. Then we can extend g' to a $\gamma_r(T)$-function by assigning the weight 2 to x_{d-1}, the weight 1 to x' and the weight 0 to the remaining vertices of T. This is a contradiction since every $\gamma_r(T)$-function assigns to the vertex x_{d-4} the value 2. Next, assume that $g'(x_{d-4}) = 1$. If for every vertex $u \in N(x_{d-4})$ with $g'(u) = 0$, there exists at least one vertex $w \in N(u)$ with $g'(w) > 0$, then we can extend g' to a $\gamma_r(T)$-function by assigning the weight 2 to x_{d-1}, the weight 1 to x' and the weight 0 to the remaining vertices of T. This is a contradiction since every $\gamma_r(T)$-function assigns to the vertex x_{d-4} the value 2. Hence, we can assume that every $\gamma_r(T')$-function assigns to the vertex x_{d-4} the value 2.

Also, if $\gamma_r(T')$ and $\gamma_R^P(T')$ are not strongly equal, then every $\gamma_r(T')$-function which is not a PRDF, can be extended to a $\gamma_r(T)$-function by assigning the weight 2 to x_{d-1}, the weight 1 to x' and the weight 0 to the remaining vertices of T, that is not a PRDF on tree T, which leads to a contradiction with $\gamma_r(T) \equiv \gamma_R^P(T)$. Therefore, $\gamma_r(T') \equiv \gamma_R^P(T')$ and so by induction on T', we have $T' \in \mathcal{F}$. We conclude that $T \in \mathcal{F}$ since it is obtained from T' by using Operation $\mathcal{O}_6$.

Now, assume that x_{d-3} is not a strong support vertex. Then, we observe that every child of x_{d-3} plays a similar role as x_{d-2}. Let $T' = T - T_{x_{d-2}}$. Since f is a $\gamma_R^P(T)$-function and $f(x_{d-2}) = 0$, then $f|_{T'}$ is a PRDF on T' and so $\gamma_r(T') \leq \gamma_R^P(T') \leq \gamma_R^P(T) - 2 = \gamma_r(T) - 2$. Assume that f' is a $\gamma_r(T')$-function. Let $N(u) = \{x_{d-3}, x\}$ and $N(x) = \{u, y, z\}$. Then, clearly we can assume that $f'(x) = 2$ and $f'(y) = f'(z) = f'(u) = 0$. If $f'(x_{d-3}) \neq 1$, then we can extend f' to a WRDF on T by assigning the weight 2 to x_{d-1} and the weight 0 to the three neighbors of it. Hence $\gamma_r(T) \equiv \gamma_r(T') + 2$. Next, assume that $f'(x_{d-3}) = 1$. Since $f'(u) = 0$, for every $w \in N(x_{d-3}) - \{u\}$ such that $f(w) = 0$, there exists at least one vertex $w' \in N(w) - \{x_{d-3}\}$ such that $f'(w') \neq 0$. Hence, we can extend f' to a WRDF on T by assigning the weight 2 to x_{d-1} and the weight 0 to the three neighbors of it. Hence, $\gamma_r(T) \leq \gamma_r(T') + 2$. Consequently, in both cases, $\gamma_r(T') = \gamma_r(T) - 2$. Therefore, $\gamma_R^P(T') \leq \gamma_R^P(T) - 2 = \gamma_r(T) - 2 = \gamma_r(T') \leq \gamma_R^P(T')$ and so $\gamma_r(T') = \gamma_R^P(T')$. Assume that g' is an arbitrary $\gamma_r(T')$-function. Then we can assume that $g'(u) = 0$. Hence, as before, it can be extended to a $\gamma_r(T)$-function g by assigning the weight 2 to x_{d-1} and the weight 0 to the three neighbors of it. Since $\gamma_r(T) \equiv \gamma_R^P(T)$, we deduce that g is a $\gamma_R^P(T)$-function and so g' is $\gamma_R^P(T')$-function. Therefore, $\gamma_r(T') \equiv \gamma_R^P(T')$ and via induction on T', $T' \in \mathcal{F}$ holds. Therefore, $T \in \mathcal{F}$ as it is derived via T' by applying Operation $\mathcal{O}_5$.

Subcase 2.2 $\deg(x_{d-3}) = 2$.

If $\deg(x_{d-4}) = 2$, then by re-assigning to the vertex x_{d-4} the value 0, to the vertex x_{d-3} the value 1 and to the vertex x_{d-5} the value $max\{f(x_{d-5}), 1\}$, we arrive at a new WRDF g with weight at most $\gamma_r(T)$ such that $g(x_{d-3}) \neq 0$. This is a contradiction. Thus, we may assume that $g(x_{d-4}) \geq 3$.

First, we assume that x_{d-4} is a support vertex. Let $T' = T - T_{x_{d-2}}$. Since $f(x_{d-2}) = 0$, hence $f|_{T'}$ is a PRDF for T' and so $\gamma_r(T') \leq \gamma_R^P(T') \leq \gamma_R^P(T) - 2 = \gamma_r(T) - 2$. Now, assume that g' be a $\gamma_r(T')$-function. Since x_{d-4} in tree T' is a strong support vertex, we can assume that $g'(x_{d-4}) = 2$. Then, we can extend g' to a WRDF on T by assigning the weight 2 to x_{d-1} and the weight 0 to the three neighbors of it. Hence, $\gamma_r(T) \leq \gamma_r(T') + 2$. Consequently, in both cases, $\gamma_r(T') = \gamma_r(T) - 2$. Therefore, $\gamma_R^P(T') \leq \gamma_R^P(T) - 2 = \gamma_r(T) - 2 = \gamma_r(T') \leq \gamma_R^P(T')$ and so $\gamma_r(T') = \gamma_R^P(T')$. Assume that g' is an arbitrary $\gamma_r(T')$-function. Then, we can assume that $g'(x_{d-4}) = 2$ and $g'(x_{d-3}) = 0$. Hence, as before, it can be extended to a $\gamma_r(T)$-function g by assigning the weight 2 to x_{d-1} and the weight 0 to the three neighbors of it. Since $\gamma_r(T) \equiv \gamma_R^P(T)$, we deduce that g is a $\gamma_R^P(T)$-function and so g' is $\gamma_R^P(T')$-function. Therefore, $\gamma_r(T') \equiv \gamma_R^P(T')$ and via induction on T', $T' \in \mathcal{F}$ holds true. Similarly, it is easy to see that $T \in \mathcal{F}$ holds. This can be seen as it is derived by T' applying Operation $\mathcal{O}_4$.

Next, suppose x_{d-4} is not a support vertex. By Lemma 1, the vertex x_{d-2} does not have a weak support vertex as a child. Also, by Lemma 2, the vertex x_{d-2} does not have a strong support vertex as a child with degree three. We assume that x_{d-4} has as a child, a strong support vertex v with $\ell_v = \deg(v) - 1 \geq 3$. Let $T' = T - T_v$. Then, clearly $f(v) = 2$ and $f|_{T'}$ is a PRDF for T'. So $\gamma_r(T') \leq \gamma_R^P(T') \leq \gamma_R^P(T) - 2 = \gamma_r(T) - 2$. Assume that g' is a $\gamma_r(T')$-function. g' can be extended to a WRDF on T by giving the weight 2 to v and the weight 0 to the all leaves neighbors of it. Hence, $\gamma_r(T) \leq \gamma_r(T') + 2$. Consequently, in both cases, $\gamma_r(T') = \gamma_r(T) - 2$. Hence, $\gamma_R^P(T') \leq \gamma_R^P(T) - 2 = \gamma_r(T) - 2 = \gamma_r(T') \leq \gamma_R^P(T')$ and so $\gamma_r(T') = \gamma_R^P(T')$. Now, assume that g' is an arbitrary $\gamma_r(T')$-function. It can then be extended to a $\gamma_r(T)$-function g by assigning the weight 2 to v and the weight 0 to the all leaves neighbors of it. Since $\gamma_r(T) \equiv \gamma_R^P(T)$, we deduce that g is a $\gamma_R^P(T)$-function and so g' is a $\gamma_R^P(T')$-function. Therefore, $\gamma_r(T') \equiv \gamma_R^P(T')$. Using induction on T', $T' \in \mathcal{F}$ holds true. Similarly, $T \in \mathcal{F}$. This is because T' and the Operation $\mathcal{O}_1$.

Next, we suppose that for any child support vertex v of x_{d-4}, we have $\ell_v < \deg(v) - 1$. Let $N(x_{d-1}) = \{x_d, y, x_{d-2}\}$. Assume that y_0 is a leaf vertex of a tree $T_{x_{d-4}}$, other than x_d and y, and at maximum distance from x_{d-4}. Let $y_0 y_1 \ldots y_t x_{d-4}$ be the shortest path from y_0 to x_{d-4}. Clearly, $t \leq 3$. As noted earlier, $t \notin \{0, 1\}$. We proceed depending on t.

We first suppose that $t = 2$. Then Lemma 1 implies that $\deg(y_1) \geq 3$ and so y_1 is a strong support vertex. Without loss of generality, we may assume that $f(y_1) = 2$ and so $f(y_2) \neq 0$, since f is a $\gamma_R^P(T)$-function and $f(x_{d-4}) = 2$. Hence, $\deg(y_2) \geq 3$. If y_2 is not a support vertex, then by Lemma 1 any child u of y_2 is a strong support vertex and so we can assume that $f(u) = 2$. Then by re-assigning to the vertex y_2 the value 0, we get a new WRDF with weight less than $\gamma_r(T)$. This is a contradiction. Assume that y_2 is a weak support vertex. Let $L(y_2) = \{y_2'\}$. Without loss of generality, we may assume that $f(y_2) = 1$ and $f(y_2') = 0$. Then by re-assigning to the vertex y_2 the value 0 and to the vertex y_2' the value 1, we produce a new $\gamma_r(T)$-function which is not a PRDF. This is a contradiction. Thus, we may assume that y_2 is a strong support vertex. If $\ell_{y_2} = 2$ and $L(y_2) = \{y_2', y_2''\}$, then we can assume that $f(y_2) = 2$ and $f(y_2') = f(y_2'') = 0$. Then by re-assigning to the vertex y_2 the value 0 and to the vertices y_2' and y_2'' the value 1, we get a new $\gamma_r(T)$-function which is not a PRDF. This is a contradiction. Hence, we may assume that $\ell_{y_2} \geq 3$ and so $f(y_2) = 2$. Then, Lemma 2 implies that $\deg(y_1) \geq 4$. Let $T' = T - T_{y_1}$. Then, clearly $f(y_1) = 2$ and $f|_{T'}$ is a PRDF for T' and so $\gamma_r(T') \leq \gamma_R^P(T') \leq \gamma_R^P(T) - 2 = \gamma_r(T) - 2$. Assume that g' is a $\gamma_r(T')$-function. Then, we can extend g' to a WRDF on T by assigning the weight 2 to y_1 and the weight 0 to the all leaves neighbors of it. Hence, $\gamma_r(T) \leq \gamma_r(T') + 2$. Consequently, in both cases, $\gamma_r(T') = \gamma_r(T) - 2$. Therefore, $\gamma_R^P(T') \leq \gamma_R^P(T) - 2 = \gamma_r(T) - 2 = \gamma_r(T') \leq \gamma_R^P(T')$ and so $\gamma_r(T') = \gamma_R^P(T')$. Now, assume that g' is an arbitrary $\gamma_r(T')$-function. Then it can be extended to a $\gamma_r(T)$-function g by assigning the weight 2 to y_1 and the weight 0 to all leaves neighbors of it. Since $\gamma_r(T) \equiv \gamma_R^P(T)$, we deduce that g is a $\gamma_R^P(T)$-function and so g' is a $\gamma_R^P(T')$-function. Therefore, $\gamma_r(T') \equiv \gamma_R^P(T')$. Using induction on T', $T' \in \mathcal{F}$ must be true. Therefore, we have $T \in \mathcal{F}$. This is due to T' and the Operation $\mathcal{O}_1$.

Suppose that $t = 3$. Since y_0 plays the same role as x_d, we may assume that $\deg(y_2) = \deg(y_3) = 2$ and y_1 is a strong support vertex of degree three. Let $T' = T - T_{x_{d-2}}$ and $L(y_1) = \{y_0, y_0'\}$. Then $f|_{T'}$ is a PRDF for T' and so $\gamma_r(T') \leq \gamma_R^P(T') \leq \gamma_R^P(T) - 2 = \gamma_r(T) - 2$. Now, assume that g' is a $\gamma_r(T')$-function. If $g'(x_{d-3}) \neq 1$, g' can be generalized to a WRDF on T by giving the weight 2 to x_d and the weight 0 to all neighbors of it. Hence, $\gamma_r(T) \leq \gamma_r(T') + 2$. Assume that $g'(x_{d-3}) = 1$. Then, we can assume that $g(y_1) = 2$, $g'(y_0) = g'(y_0') = g'(y_2) = 0$ and $g'(x_{d-4}) \neq 2$. If $g'(x_{d-4}) = 1$, then we can extend g' to a $\gamma_r(T)$-function g by re-assigning to the vertex x_{d-4} the value 2 and to x_{d-3} the value 0 and by assigning the weight 2 to x_{d-1} and the weight 0 to all neighbors of it, and so $\gamma_r(T) \leq \gamma_r(T') + 2$. Assume that $g'(x_{d-4}) = 0$. Then we can assume that $g'(y_3) = 1$. We can extend g' to a $\gamma_r(T)$-function g by re-assigning to the vertex x_{d-4} the value 2 and to the vertices x_{d-3} and y_3 the value 0 and by assigning the weight 2 to x_{d-1} and the weight 0 to all neighbors of it, and so $\gamma_r(T) \leq \gamma_r(T') + 2$. Consequently, in all cases, $\gamma_r(T') \leq \gamma_r(T) - 2$. Hence, $\gamma_R^P(T') \leq \gamma_R^P(T) - 2 = \gamma_r(T) - 2 = \gamma_r(T') \leq \gamma_R^P(T')$ and so $\gamma_r(T') = \gamma_R^P(T')$. Now, assume that g' be

an arbitrary $\gamma_r(T')$-function. Then, as before, it can be extended to a $\gamma_r(T)$-function g by re-assigning to the vertex x_{d-4} the value 2 and to the vertices x_{d-3} and y_3 the value 0 and by assigning the weight 2 to x_{d-1} and the weight 0 to all neighbors of it. Since $\gamma_r(T) \equiv \gamma_R^P(T)$, we deduce that g is a $\gamma_R^P(T)$-function and so g' is a $\gamma_R^P(T')$-function. Therefore, $\gamma_r(T') \equiv \gamma_R^P(T')$. Via induction on T', $T' \in \mathcal{F}$ holds true. Consequently, it can be seen that $T \in \mathcal{F}$. This is again due to T' as well as the Operation $\mathcal{O}_5$. The proof is then completed as desired. $\square$

3. Conclusions

Ian Stewart's work "Defend the Roman Empire!" [2] initially introduced the notion of Roman dominating functions. In recent years, many variations of Roman domination have been proposed due to different conditions on Roman domination, e.g., weak Roman domination [12] and perfect Roman domination [9]. Among the problems of Roman domination, the strong equality results have attracted great attention. In this paper, we studied the class of trees in which the perfect Roman domination number is strongly equal to the weak Roman domination number. This strongly equality means that each weak Roman dominatiche function of minimum weight is equivalent to a function that is deemed as perfect Roman dominating. We formulated some interesting conditions which we hope can be useful in stimulating relevant research.

References

1. Haynes, T.W.; Hedetniemi, S.T.; Slater, P.J. *Fundamentals of Domination in Graphs*; Marcel Dekker Inc.: New York, NY, USA, 1998.
2. Stewart, I. Defend the Roman Empire! *Sci. Am.* **1999**, *281*, 136–138. [CrossRef]
3. Cockayne, E.J.; Dreyer, P.A.; Hedetniemi, S.M.; Hedetniemi, S.T. Roman domination in graphs. *Discret. Math.* **2004**, *278*, 11–22. [CrossRef]
4. Alvarado, J.D.; Dantas, S.; Rautenbach, D. Strong equality of Roman and weak Roman domination in trees. *Discret. Appl. Math.* **2016**, *208*, 19–26. [CrossRef]
5. Bermudo, S.; Fernau, H.; Sigarreta, J.M. The differential and the Roman domination number of a graph. *Appl. Anal. Discret. Math.* **2014**, *8*, 155–171. [CrossRef]
6. Chellali, M.; Rad, N.J. Strong equality between the Roman domination and independent Roman domination numbers in trees. *Discuss. Math. Graph Theory* **2013**, *33*, 337–346. [CrossRef]
7. Rad, N.J.; Liu, C.H. Trees with strong equality between the Roman domination number and the unique response Roman domination number. *Australas. J. Comb.* **2012**, *54*, 133–140.
8. Shang, Y. Attack robustness and stability of generalized k-cores. *New J. Phys.* **2019**, *21*, 093013. [CrossRef]
9. Henning, M.A.; Klostermeyer, W.F.; MacGillivray, G. Perfect Roman domination in trees. *Discret. Appl. Math.* **2018**, *236*, 235–245. [CrossRef]
10. Darkooti, M.; Alhevaz, A.; Rahimi, S.; Rahbani, H. On perfect Roman domination number in trees: Complexity and bounds. *J. Comb. Optim.* **2019**, *38*, 712–720. [CrossRef]
11. Darkooti, M.; Alhevaz, A.; Rahimi, S.; Rahbani, H. Characterization of perfect Roman domination edge critical trees. *Le Matematiche* **2019**, *74*, 109–118.
12. Henning, M.A.; Hedetniemi, S.T. Defending the Roman empire–new strategy. *Discrete Math.* **2003**, *266*, 239–251. [CrossRef]
13. Valveny, M.; Pérez-Rosés, H.; Rodriguez-Velázquez, J.A. On the weak Roman domination number of lexicographic product graphs. *Discrete Appl. Math.* **2019**, *263*, 257–270. [CrossRef]

14. Hattingh, J.H.; Henning, M.A. Characterizations of trees with equal domination parameters. *J. Graph Theory* **2000**, *34*, 142–153. [CrossRef]
15. Haynes, T.W.; Henning, M.A.; Slater, P.J. Strong equality of domination parameters in trees. *Discret. Math.* **2003**, *260*, 77–87. [CrossRef]