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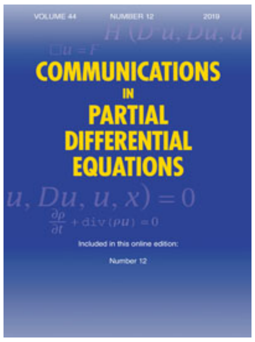

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Riesz continuity of the Atiyah–Singer Dirac operator under perturbations of local boundary conditions

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Dedicated to the memory of Alan G. R. McIntosh.

ABSTRACT
On a smooth complete Riemannian spin manifold with smooth compact boundary, we demonstrate that Atiyah–Singer Dirac operator $\not{D}_{\mathcal{B}}$ in L^2 depends Riesz continuously on L^∞ perturbations of local boundary conditions $\mathcal{B}$. The Lipschitz bound for the map $\mathcal{B} \to \not{D}_{\mathcal{B}}(1+\not{D}_{\mathcal{B}}^2)^{-\frac{1}{2}}$ depends on Lipschitz smoothness and ellipticity of $\mathcal{B}$ and bounds on Ricci curvature and its first derivatives as well as a lower bound on injectivity radius away from a compact neighbourhood of the boundary. More generally, we prove perturbation estimates for functional calculi of elliptic operators on manifolds with local boundary conditions.

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1. Introduction

The aim of this article and its companion [1] has been to prove perturbation estimates of quantities of the form

$$\left\| \frac{\widetilde{D}}{\sqrt{I+\widetilde{D}^2}} - \frac{D}{\sqrt{I+D^2}} \right\|_{L^2(\mathcal{M},\mathcal{V})\to L^2(\mathcal{M},\mathcal{V})},$$

where D and $\widetilde{D}$ are self-adjoint elliptic first-order partial differential operators, acting on sections of a vector bundle $\mathcal{V}$ over a smooth manifold $\mathcal{M}$. The symbol $f(\zeta) = \zeta(1+\zeta^2)^{-\frac{1}{2}}$ is a motivating example, yielding continuity results in the Riesz sense, but our methods apply equally well to more general holomorphic symbols around $\mathbb{R}$, which may be discontinuous at ∞. In [1], together with Alan McIntosh, we obtained results on complete manifolds $(\mathcal{M}, g)$ without boundary. In that case, the main example of operators D and $\widetilde{D}$ was the Atiyah–Singer Dirac operators on $\mathcal{M}$ with respect to two different metrics g and $\widetilde{g}$. The bound obtained was

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$$\left\|\frac{\widetilde{\mathrm{D}}}{\sqrt{\mathrm{I}+\widetilde{\mathrm{D}}^2}}-\frac{\mathrm{D}}{\sqrt{\mathrm{I}+\mathrm{D}^2}}\right\|_{\mathrm{L}^2(\mathcal{M},\mathcal{V})\to\mathrm{L}^2(\mathcal{M},\mathcal{V})}\lesssim \|\widetilde{\mathrm{g}}-\mathrm{g}\|_{\mathrm{L}^\infty(\mathcal{T}^{(2,0)}\mathcal{M})},$$

where the implicit constant depends on certain geometric quantities. Note that the two Dirac operators themselves depend also on the first derivatives of the metrics.

In the present paper, we consider the corresponding perturbation estimate on a manifold $\mathcal{M}$ (possibly noncompact) with smooth, compact boundary $\Sigma=\partial\mathcal{M}$. Our motivating example in this case is when both D and $\widetilde{\mathrm{D}}$ are the Atiyah–Singer Dirac operator, but with two different local boundary conditions, defined through two different subbundles $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ of $\mathcal{V}|_\Sigma$. For each boundary condition we assume self-adjointness and ellipticity so that the domains of D and $\widetilde{\mathrm{D}}$ are closed subspaces of $\mathrm{H}^1(\mathcal{V})$. The bound we obtain is

$$\left\|\frac{\widetilde{\mathrm{D}}}{\sqrt{\mathrm{I}+\widetilde{\mathrm{D}}^2}}-\frac{\mathrm{D}}{\sqrt{\mathrm{I}+\mathrm{D}^2}}\right\|_{\mathrm{L}^2(\mathcal{M},\mathcal{V})\to\mathrm{L}^2(\mathcal{M},\mathcal{V})}\lesssim \|\hat{\delta}(\widetilde{\mathcal{E}}_x,\mathcal{E}_x)\|_{\mathrm{L}^\infty(\Sigma)}, \tag{1.1}$$

where $\hat{\delta}(\mathcal{E}_x,\widetilde{\mathcal{E}}_x)=|\pi_{\mathcal{E}}(x)-\pi_{\widetilde{\mathcal{E}}}(x)|$ and $\pi_{\mathcal{E}}$ and $\pi_{\widetilde{\mathcal{E}}}$ respectively are the orthogonal projectors from $\mathcal{V}|_\Sigma$ to $\mathcal{E}$ and $\widetilde{\mathcal{E}}$. Again the implicit constant in the estimate depends on a number of geometric quantities which we list completely.

As described in the introduction of [1], an important application of these perturbation estimates is the study of spectral flow for unbounded self-adjoint operators. The study of the spectral flow was initiated by Atiyah and Singer in [2] and has important connections to particle physics. An analytic formulation of the spectral flow was given by Phillips in [3] and typically, the *gap metric*

$$\left\|\frac{\mathrm{i}+\widetilde{\mathrm{D}}}{\mathrm{i}-\widetilde{\mathrm{D}}}-\frac{\mathrm{i}+\mathrm{D}}{\mathrm{i}-\mathrm{D}}\right\|_{\mathrm{L}^2(\mathcal{M};\mathcal{V})\to\mathrm{L}^2(\mathcal{M};\mathcal{V})}$$

is used to understand the spectral flow for unbounded operators. The Riesz topology is a preferred alternative since the spectral flow in this topology better connects to topological and *K*-theoretic aspects of the spectral flow, which were observed in [2] for the case of bounded self-adjoint Fredholm operators. The main disadvantage is that it is typically harder to establish continuity in the Riesz topology. In particular we refer to the open problem pointed out by Lesch in the introduction of [4], namely whether a Dirac operator on a compact manifold with boundary depends Riesz continuously on pseudo-differential boundary conditions imposed on the operator.

The present article answers these questions to the positive, in the special case of local boundary conditions. Self-adjoint local boundary conditions are typically physical and a very large subclass of the so-called *Chiral* conditions are listed in [5] by Hijazi, Montiel and Roldàn as being self-adjoint boundary conditions. In particular, these exist in even dimensions or when the manifold is a space-like hypersurface in spacetime. The case of non-local boundary conditions defined by pseudo-differential projections appear to be beyond the scope of the methods used in the present paper but we anticipate they will be the object of further investigations in the future. The local nature of the boundary conditions enter the proof in a number of instances, but the most serious occurrence concern the so-called exponential off-diagonal estimates, which relies on the domains of the operators being preserved under multiplication by smooth, bounded functions. It is important to note that the right hand sides in the perturbation estimates that we obtain, namely $||\widetilde{g}-g||_{L^\infty(\mathcal{T}^{(2,0)}\mathcal{M})}$ and $||\hat{\delta}(\widetilde{\mathcal{E}}_x, \mathcal{E}_x)||_{L^\infty(\Sigma)}$, are supremum norms, which are smaller than estimates that can be obtained from operator theoretic arguments alone.

Like in [1], we use methods from operator theory and real harmonic analysis to obtain (1.1). For a self-adjoint operator, say D, the quadratic estimate

$$\int_0^\infty ||Q_t u||^2 \frac{dt}{t} \lesssim ||u||^2 \tag{1.2}$$

is immediate from the spectral theorem coupled with Fubini's theorem. Here $Q_t = tD(I + t^2D^2)^{-1}$ is a holomorphic approximation, adapted to the operator D, of the projection onto frequencies in a dyadic band around $1/t$. For the harmonic analyst, the estimate (1.2) yields continuity of a wavelet transform, adapted to D, and plays the same role in wavelet theory as Plancherel's theorem does in Fourier theory. We refer to [6] by Daubechies in the case Q_t is the projection onto scale t in the multiscale resolution. These ideas are also central in Littlewood–Paley theory.

Quadratic estimates like (1.2) are a flexible tool. They can be adapted to handle non-self-adjoint operators as well as non-commuting operators. Relevant to this article is the latter extension, where we want to estimate $f(\widetilde{D})-f(D)$ as in (1.1). By expressing these operators in terms of resolvents of $\widetilde{D}$ and D respectively via the Dunford functional calculus, such perturbation estimates can be obtained from quadratic estimates of the form

$$\int_0^\infty ||\widetilde{Q}_t A P_t u||^2 \frac{dt}{t} \lesssim ||u||^2. \tag{1.3}$$

Here $\widetilde{Q}_t$ is like Q_t above but for the operator $\widetilde{D}$, A typically is a bounded multiplication operator, and $P_t = (I + t^2D^2)^{-1}$ should be thought of as a holomorphic approximation, adapted to the operator D, to the projections onto frequencies smaller than $1/t$.

Just like in the non-self-adjoint case in (1.2), the estimates (1.3) are non-trivial and use the specific structure of the operators $\widetilde{D}$ and D. When these are differential operators, allowing non-smooth coefficients, we can use methods from harmonic analysis to handle (1.3) essentially as a Carleson embedding theorem. For operators with simpler structure than our Dirac operators, it is also possible to obtain higher order perturbation estimates. In this case the relevant quadratic estimates look like (1.6). For our Dirac operators, (1.3) more precisely amounts to the two estimates

$$\int_0^1 ||\widetilde{Q}_t A_1 \nabla (iI + D)^{-1} P_t u||^2 \frac{dt}{t} \lesssim ||A_1||_\infty^2 ||u||^2 \quad \text{and} \tag{1.4}$$

$$\int_0^1 ||t\widetilde{P}_t \operatorname{div} A_2 P_t u||^2 \frac{dt}{t} \lesssim ||A_2||_\infty^2 ||u||^2, \tag{1.5}$$

which need to be established for $u \in L^2(\mathcal{V})$, where A_1 and A_2 are L^∞ multipliers. Through a similarity transformation of $\widetilde{D}$, we can also assume that $\mathcal{D}(\widetilde{D}) = \mathcal{D}(D)$. Here $P_t = (I + t^2D^2)^{-1}$, $\widetilde{P}_t = (I + t^2\widetilde{D}^2)^{-1}$, $Q_t = tD(I + t^2D^2)^{-1}$, $\widetilde{Q}_t = t\widetilde{D}(I + t^2\widetilde{D}^2)^{-1}$.

At a first glance, trying to adapt the proofs in [1] for (1.4) and (1.5) to the case of manifolds with boundary seems to be a straightforward exercise. However, closer inspection reveals an interesting dichotomy. In [1], the estimate (1.5) was standard and well known to be equivalent to a certain measure being a Carleson measure, and the main new work was in establishing (1.4). Here the operator $A_1\nabla(\mathrm{iI}+\mathrm{D})^{-1}$ which is sandwiched between $\widetilde{\mathrm{Q}}_t$ and P_t, is not a multiplier but also incorporates a singular integral operator $\nabla(\mathrm{iI}+\mathrm{D})^{-1}$. To estimate, a Weitzenböck-type inequality for D is needed. Turning to a manifold with boundary, one sees that (1.4) follows as in [1], *mutatis mutandis*. Instead, the presence of boundary forces (1.5) to be a non-standard estimate, since new boundary terms appear in the absence of boundary conditions for the multiplier A_2. Indeed, in order for our estimates to be useful, we need to be able to allow for general A_2. More precisely, by Stokes' theorem

$$\int_{\mathcal{M}} \mathrm{g}\big(\widetilde{\mathrm{P}}_t t\mathrm{div} u, v\big)\, d\mu = \int_{\Sigma} \mathrm{g}\big(t\vec{\mathrm{n}}\cdot u, \widetilde{\mathrm{P}}_t v\big)\, d\sigma - \int_{\mathcal{M}} \big(u, t\nabla\widetilde{\mathrm{P}}_t v\big)\, d\mu.$$

The second term on the right hand side is bounded by $\|u\|_{\mathrm{L}^2}\|v\|_{\mathrm{L}^2}$ by the ellipticity and self-adjointness of $\widetilde{\mathrm{D}}$, but clearly the first term has no such bound. This means that in (1.5), the operators $\widetilde{\mathrm{P}}_t t\mathrm{div}$ are not even bounded, and standard estimates break down.

An important contribution of this paper lies in the new ideas needed to establish (1.5). Here, we observe that even though $\widetilde{\mathrm{P}}_t t\mathrm{div}$ is unbounded, the operator $\widetilde{\mathrm{P}}_t t\mathrm{div} A_2 \mathrm{P}_t$ as a whole is bounded by $\|A_2\|_{\mathrm{L}^\infty}$ (which is seen from Stokes' theorem and the ellipticity of D). Building on this observation, we prove (1.5) in Section 4.3 by adapting, in a non-trivial way, the standard harmonic analysis proof, usually referred to as a *local T(1)* argument. The inspiration for this analysis comes from [7] by Auscher, Axelsson, Hofmann and [8] by Axelsson, Keith, McIntosh. To be more precise, this allows us to reduce (1.5) for an arbitrary L^2 sections instead for certain test sections which vanish near the boundary Σ. For this special class of test sections, we are able to adapt the boundaryless estimates and (1.5) becomes standard.

The remainder of this article is organised as follows. In Section 2 we state in detail our main perturbation estimate in its general form, and show in Section 3 how it is applied to yield the motivating estimate for the Atiyah–Singer Dirac operator under perturbation of local boundary conditions. Then, Section 4 contains the proof of Theorem 2.1, as outlined above.

As aforementioned, this article is a sequel to the authors' joint paper [1] with Alan McIntosh. During our work on this project, McIntosh untimely passed away, leaving us in great sorrow. McIntosh's great heritage to mathematics include his widely celebrated unique blend of operator theory and harmonic analysis which has lead to breakthroughs like the proof of the Calderón conjecture on the L^2 boundedness of the Cauchy singular integral operator on Lipschitz curves, jointly with Coifman and Meyer in [9], and the proof of the Kato square root conjecture on the domain of the square root of elliptic second-order divergence form operators, jointly with Auscher, Hofmann, Lacey and Tchamitchian in [10].

The estimates in this article go back to the multilinear estimates pioneered by McIntosh in connection with [9]. There, expressions of the form

$$\int_0^\infty ||\mathrm{Q}_t A_1 \mathrm{P}_t A_2 \mathrm{P}_t A_3 \mathrm{P}_t \cdots A_k \mathrm{P}_t u||_{\mathrm{L}^2}^2 \frac{dt}{t} \tag{1.6}$$

were bounded by $||u||_{\mathrm{L}^2}^2$. Formally, the idea is to pass a derivative from Q_t, through the general L^∞ maps A_i, to the rightmost P_t, which becomes $\mathrm{Q}_t = t\mathrm{D}\mathrm{P}_t$, and conclude the desired estimate by (1.2). Concretely, this is achieved by harmonic analysis methods and Carleson measures. The power of this analysis is well known in real-variable harmonic analysis and, in fact, the necessary and much needed algebra of P_t and Q_t operators are in some circles of mathematicians referred to as *McIntoshery* (or in French *McIntosherie*).

In this article, we only employ the linear case $k=1$ of these multilinear estimates of McIntosh, leading to first-order perturbation estimates. Even though our work is yet another successful example of McIntoshery, we have nevertheless chosen to not add his name as an author. Both authors are former students of McIntosh, and we know he had as a firm principle for omitting his name from publications unless he clearly felt that he had contributed to the novelties of the article in a substantial way. Unfortunately, he could not join us this time.

2. Setup and statement of main theorem

2.1. Manifolds, bundles, and function spaces

Let $\mathcal{M}$ be a smooth manifold (possibly noncompact) with smooth boundary $\Sigma = \partial\mathcal{M}$. Throughout, we fix a smooth, Riemannian metric g on $\mathcal{M}$ and let ∇ denote the associated Levi-Civita connection. We assume that g is complete, by which we mean $(\mathcal{M}, \mathrm{g})$ is complete as a metric space. By $\mathring{\mathcal{M}}$, we denote the interior $\mathcal{M} \setminus \partial\mathcal{M}$. The induced volume measure is denoted by $d\mu$ on $\mathcal{M}$ and $d\sigma$ on Σ. Let $\vec{\mathrm{n}}$ be the unit outward normal vectorfield on Σ.

The tangent, cotangent bundles are denoted by $\mathrm{T}\mathcal{M}$ and $\mathrm{T}^*\mathcal{M}$ respectively, and the rank (p, q)-tensor bundle by $\mathcal{T}^{(p,q)}\mathcal{M}$.

For a smooth complex Riemannian bundle $(\mathcal{V}, \mathrm{h})$ on $\mathcal{M}$, let $\boldsymbol{\Gamma}(\mathcal{V})$ denote the set of measurable sections and $\mathrm{C}^{k,\alpha}(\mathcal{V})$ be the set of continuously k-differentiable sections with the k-th derivative being α-Hölder continuous up to the boundary. Note that when we write $\mathrm{C}^{k,\alpha}$, we do not assume $\mathrm{C}^{k,\alpha}$ with global control of the norm but rather, only $\mathrm{C}^{k,\alpha}$ regularity locally. We write $\mathrm{C}^k = \mathrm{C}^{k,0}$ and $\mathrm{C}^\infty(\mathcal{V}) = \cap_{k=1}^\infty \mathrm{C}^k(\mathcal{V})$. Moreover, define

$$\mathrm{C}_c^{k,\alpha}(\mathcal{V}) = \left\{u \in \mathrm{C}^{k,\alpha}(\mathcal{V}) : \mathrm{spt}\ u \subset \mathcal{M}\ \text{compact}\right\} \text{ and}$$
$$\mathrm{C}_{cc}^{k,\alpha}(\mathcal{V}) = \left\{u \in \mathrm{C}^{k,\alpha}(\mathcal{V}) : \mathrm{spt}\ u \subset \mathring{\mathcal{M}}\, \text{compact}\right\}.$$

Since Lipschitz maps will have special significance, we write $\mathrm{Lip}(\mathcal{V})$ to denote sections $\psi \in \mathrm{C}^{0,1}(\mathcal{V})$ with $||\nabla\psi||_{\mathrm{L}^\infty(\mathcal{V})} < \infty$.

For $1 \leq p < \infty$, denote the set of p-integrable measurable sections with respect to h and μ by $\mathrm{L}^p(\mathcal{V})$ with norm $||\xi||_p$. The space $\mathrm{L}^\infty(\mathcal{V})$ consist of $\xi \in \boldsymbol{\Gamma}(\mathcal{V})$ such that $|\xi| \leq C$ for some $C > 0$ almost-everywhere on $\mathcal{M}$. The norm $||\xi||_\infty$ is then the infimum over $C > 0$ such that this relation holds. The spaces $\mathrm{L}^p(\mathcal{V})$ are Banach spaces and $\mathrm{L}^2(\mathcal{V})$ is a Hilbert space with inner product $\langle\cdot,\cdot\rangle$. The latter space is what we shall be concerned with most in this paper and for simplicity of notation, we denote the norm $||\cdot||_2$ by $||\cdot||$. The restricted bundle $\mathcal{W} = \mathcal{V}|_\Sigma$ is a smooth, complex Riemannian bundle with metric $\mathrm{h}|_\Sigma$ and $\mathrm{L}^p(\mathcal{W})$ spaces are defined similarly on Σ with respect to the measure $d\sigma$.

Let ∇ be a connection on $\mathcal{V}$ that is compatible with h. Then, ∇ is a closeable operator in $\mathrm{L}^2(\mathcal{V})$ and we define the Sobolev spaces $\mathrm{H}^k(\mathcal{V})$ as the domain of the closure of the operator

$$\left(\nabla, \nabla^2, ..., \nabla^k\right) : \mathrm{L}^2 \cap \mathrm{C}^\infty(\mathcal{V}) \to \mathrm{L}^2 \cap \mathrm{C}^\infty\left(\oplus_{l=1}^k \mathcal{T}^{(l,0)}\mathcal{M} \otimes \mathcal{V}\right)$$

in L^2. Similarly, we obtain boundary Sobolev spaces $\mathrm{H}^k(\mathcal{V}|_\Sigma)$ from $\nabla|_\Sigma$. By compatibility, we have that

$$\langle \nabla u, v\rangle = \langle u, -\mathrm{tr}\nabla v\rangle$$

for $u \in \mathrm{L}^2_{\circ} \cap \mathrm{C}^\infty(\mathcal{V}), v \in \mathrm{L}^2 \cap \mathrm{C}^\infty(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$ and with either spt $u \subset \mathring{\mathcal{M}}$ compact or spt $v \subset \mathcal{M}$ compact. Thus, we obtain the divergence operator, defined as $\mathrm{div} = -\nabla_c{}^*$ as a densely-defined and closed operator with domain $\mathcal{D}(\mathrm{div})$ from the operator $\nabla_c : \mathrm{C}^\infty_{cc}(\mathcal{V}) \to \mathrm{C}^\infty_{cc}(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$.

2.2. Main theorem

In order to phrase the main theorem as in [1], we require some assumptions on the manifold. We say that $(\mathcal{M}, \mathrm{g}, \mu)$ has exponential volume growth if there exists $c_E \geq 1, \kappa, c > 0$ such that

$$0 < \mu(\mathrm{B}(x, tr)) \leq ct^\kappa \mathrm{e}^{c_E tr} \mu\left(\mathrm{B}(x, r)\right) < \infty, \tag{$\mathrm{E_{loc}}$}$$

for every $t \geq 1$ and g-balls $\mathrm{B}(x, r)$ of radius $r > 0$ at every $x \in \mathcal{M}$. The manifold $(\mathcal{M}, \mathrm{g})$ satisfies a *local Poincaré inequality* if there exists $c_P \geq 1$ such that for all $f \in \mathrm{H}^1(\mathcal{M})$,

$$||f - f_B||_{\mathrm{L}^2(B)} \leq c_P \mathrm{rad}(B)||f||_{\mathrm{H}^1(B)} \tag{$\mathrm{P_{loc}}$}$$

for all balls B in $\mathcal{M}$ such that the radius $\mathrm{rad}(B) \leq 1$.

We say that $(\mathcal{V}, \mathrm{h})$ satisfies *generalised bounded geometry*, or *GBG* for short, if there exist $\rho > 0$ and $C \geq 1$ such that, for each $x \in \mathcal{M}$, there exists a continuous local trivialisation $\psi_x : \mathrm{B}(x, \rho) \times \mathbb{C}^N \to \pi_{\mathcal{V}}^{-1}(\mathrm{B}(x, \rho))$ satisfying

$$C^{-1}|\psi_x^{-1}(y)u|_\delta \leq |u|_{\mathrm{h}(y)} \leq C|\psi_x^{-1}(y)u|_\delta,$$

for all $y \in \mathrm{B}(x, \rho)$, where δ denotes the usual inner product in $\mathbb{C}^N$ and $\psi_x^{-1}(y)u = \psi_x^{-1}(y, u)$ is the pullback of the vector $u \in \mathcal{V}_y$ to $\mathbb{C}^N$ via the local trivialisation ψ_x at $y \in \mathrm{B}(x, \rho)$. We call ρ the *GBG radius*. In typical application, the local trivialisations will be $\mathrm{C}^{0,1}$ or smooth.

Letting D and $\widetilde{\mathrm{D}}$ be first-order differential operators acting on a bundle $\mathcal{V}$ over $\mathcal{M}$ and $\mathscr{R} : \mathrm{H}^1(\mathcal{V}) \to \mathrm{H}^{\frac{1}{2}}(\mathcal{V}_\Sigma)$ the boundary trace map, we state the following assumptions adapted to our setting from [1]:

(A1) $\mathcal{M}$ and $\mathcal{V}$ are finite dimensional, quantified by $\dim \mathcal{M} < \infty$ and $\dim \mathcal{V} < \infty$,

(A2) $(\mathcal{M}, \mathrm{g})$ has exponential volume growth quantified by $c < \infty$, $c_E < \infty$ and $\kappa < \infty$ in ($\mathrm{E_{loc}}$),

(A3) a local Poincaré inequality ($\mathrm{P_{loc}}$) holds on $\mathcal{M}$ quantified by $c_P < \infty$, (A4) $\mathrm{T}^*\mathcal{M}$ has $\mathrm{C}^{0,1}$ GBG frames ν_j quantified by $\rho_{\mathrm{T}^*\mathcal{M}} > 0$ and $C_{\mathrm{T}^*\mathcal{M}} < \infty$, with $|\nabla \nu_j| < C_{G,\mathrm{T}^*\mathcal{M}}$ with $C_{G,\mathrm{T}^*\mathcal{M}} < \infty$,

(A5) $\mathcal{V}$ has $\mathrm{C}^{0,1}$ GBG frames e_j quantified by $\rho_{\mathcal{V}} > 0$ and $C_{\mathcal{V}} < \infty$, with $|\nabla e_j| < C_{G,\mathcal{V}}$ with $C_{G,\mathcal{V}} < \infty$,

(A6) D satisfies $|\mathrm{D} e_j| \leq C_{D,\mathcal{V}}$ with $C_{D,\mathcal{V}} < \infty$ almost-everywhere inside each GBG frame $\{e_j\}$,

(A7) We have $\eta\mathcal{D}(\mathrm{D}) \subset \mathcal{D}(\mathrm{D})$ for every bounded $\eta \in \mathrm{C}^\infty(\mathcal{M})$ with $\|\nabla\eta\|_\infty < \infty$, and $[\mathrm{D}, \eta]$ and $[\widetilde{\mathrm{D}}, \eta]$ are pointwise multiplication operators on almost-every fibre $\mathcal{V}_x$ with a constant $c_{\mathrm{D},\widetilde{\mathrm{D}}} > 0$ such that

$$|[\mathrm{D}, \eta]u(x)| \leq c_{\mathrm{D},\widetilde{\mathrm{D}}} |\nabla\eta(x)||u(x)| \tag{2.1}$$

for almost-every $x \in \mathcal{M}$ and the same estimate with D interchanged with $\widetilde{\mathrm{D}}$,

(A8) D and $\widetilde{\mathrm{D}}$ are self-adjoint operators which are essentially self-adjoint on their restriction to

$$\mathrm{C}_c^\infty(\mathcal{V}; \mathcal{B}) = \left\{u \in \mathrm{C}_c^\infty(\mathcal{V}) : \mathscr{R}u \in \mathcal{B}\right\},$$

where $\mathcal{B} = \mathrm{H}^{\frac{1}{2}}(\mathcal{E})$ with $\mathcal{E} \subset \mathcal{V}|_\Sigma$ a smooth subbundle of $\mathcal{V}|_\Sigma$, and both operators have domain $\mathcal{D}(\mathrm{D}) = \mathcal{D}(\widetilde{\mathrm{D}}) \subset \mathrm{H}^1(\mathcal{V})$ and with $\mathrm{C} \geq 1$ the smallest constant satisfying

$$\mathrm{C}^{-1}\|u\|_{\mathrm{D}} \leq \|u\|_{\mathrm{H}^1} \leq \mathrm{C}\|u\|_{\mathrm{D}} \quad \text{and} \quad \mathrm{C}^{-1}\|u\|_{\widetilde{\mathrm{D}}} \leq \|u\|_{\mathrm{H}^1} \leq \mathrm{C}\|u\|_{\widetilde{\mathrm{D}}} \tag{2.2}$$

for all $u \in \mathcal{D}(\mathrm{D}) = \mathcal{D}(\widetilde{\mathrm{D}})$ and where $\|\cdot\|_{\mathrm{D}} = \|\mathrm{D}\cdot\| + \|\cdot\|$, the operator norm, and

(A9) D satisfies the Riesz-Weitzenböck condition: $\mathcal{D}(\mathrm{D}^2) \subset \mathrm{H}^2(\mathcal{V})$ with

$$\|\nabla^2 u\| \leq c_W\left(\|\mathrm{D}^2 u\| + \|u\|\right) \tag{2.3}$$

for all $u \in \mathcal{D}(\mathrm{D}^2)$ with $c_W < \infty$.

The implicit constants in our perturbation estimates will be allowed to depend on

$$\begin{aligned}\mathrm{C}(\mathcal{M}, \mathcal{V}, \mathrm{D}, \widetilde{\mathrm{D}}) = \max\{&\dim\mathcal{M}, \dim\mathcal{V}, c, c_E, \kappa, c_P, \rho_{\mathrm{T}^*\mathcal{M}}, C_{\mathrm{T}^*\mathcal{M}}, C_{G,\mathrm{T}^*\mathcal{M}}, \\ &\rho_{\mathcal{V}}, C_{\mathcal{V}}, C_{G,\mathcal{V}}, c_{\mathrm{D}}, C_{\mathrm{D},\mathcal{V}}, \mathrm{C}, c_W\} < \infty.\end{aligned} \tag{2.4}$$

Our main theorem is the following.

Theorem 2.1. *Let $\mathcal{M}$ be a smooth manifold with smooth compact boundary $\Sigma = \partial\mathcal{M}$ and let g be a smooth metric on $\mathcal{M}$ such that $(\mathcal{M}, \mathrm{g})$ is complete as a metric space. Let $(\mathcal{V}, \mathrm{h}, \nabla)$ be a smooth vector bundle over $\mathcal{M}$ with smooth metric h and connection ∇ that are compatible.*

Let $\mathrm{D}, \widetilde{\mathrm{D}}$ be two first-order differential and assume the hypotheses (A1)-(A9) on $\mathcal{M}, \mathcal{V}, \mathrm{D}$ and $\widetilde{\mathrm{D}}$ and that

$$\widetilde{\mathrm{D}}\psi = \mathrm{D}\psi + A_1\nabla\psi + \operatorname{div} A_2\psi + A_3\psi, \tag{2.5}$$

holds in a distributional sense for $\psi \in \mathcal{D}(\mathrm{D}) = \mathcal{D}(\widetilde{\mathrm{D}})$, where

$$\begin{aligned} &A_1 \in \mathrm{L}^\infty(\mathcal{L}(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V}, \mathcal{V})), \\ &A_2 \in \mathrm{L}^\infty \cap \mathrm{Lip}(\mathcal{L}(\mathcal{V}, \mathrm{T}^*\mathcal{M} \otimes \mathcal{V})), \\ &A_3 \in \mathrm{L}^\infty(\mathcal{L}(\mathcal{V})), \end{aligned} \tag{2.6}$$

and let $||A||_\infty = ||A_1||_\infty + ||A_2||_\infty + ||A_3||_\infty$.

Then, for each $\omega \in (0, \pi/2)$ *and* $\sigma \in (0, \infty]$, *whenever* $f \in \mathrm{Hol}^\infty(\mathrm{S}^o_{\omega,\sigma})$, *we have the perturbation estimate*

$$||f(\widetilde{\mathrm{D}}) - f(\mathrm{D})||_{\mathrm{L}^2(\mathcal{V}) \to \mathrm{L}^2(\mathcal{V})} \lesssim ||f||_{\mathrm{L}^\infty(\mathrm{S}_{\omega,\sigma})} ||A||_\infty,$$

where the implicit constant depends on $\mathrm{C}(\mathcal{M}, \mathcal{V}, \mathrm{D}, \widetilde{\mathrm{D}})$.

Here $\mathrm{S}^o_{\omega,\sigma} := \{x + iy : y^2 < \tan^2 \omega x^2 + \sigma^2\}$, and we say that $f \in \mathrm{Hol}^\infty(\mathrm{S}^o_{\omega,\sigma})$ if it is holomorphic on $\mathrm{S}^o_{\omega,\sigma}$ and there exists $C > 0$ such that $|f(\zeta)| \leq C$. For a definition of functional calculi $f(\mathrm{D})$ and $f(\widetilde{\mathrm{D}})$ with symbols f bounded and holomorphic, see Section 2.3 in [1].

Remark 2.2. Self-adjointness of D and $\widetilde{\mathrm{D}}$ in Theorem 2.1 (A8) can be relaxed. Indeed, we only use self-adjointness to obtain the estimates (4.1) and (4.2). In the more general situation, that is, when the operator D or $\widetilde{\mathrm{D}}$ is only similar to a self-adjoint operator with similarity transform U, the constant $\frac{1}{2}||U||^2||U^{-1}||^2$ appears in place of $\frac{1}{2}$ in (4.1) and (4.2), and also enters in $\mathrm{C}(\mathcal{M}, \mathcal{V}, \mathrm{D}, \widetilde{\mathrm{D}})$.

We prove this theorem using real-variable harmonic analysis methods through the holomorphic bounded functional calculus in Section 4.

3. Application to the Atiyah–Singer Dirac operator

Throughout this section, in addition to assuming that $(\mathcal{M}, g)$ is a smooth and complete Riemannian manifold with compact boundary $\Sigma = \partial\mathcal{M}$, we assume that $\mathcal{M}$ is a *Spin manifold*.

Recall that the exterior algebra $\mathbf{\Omega}\mathcal{M} = \oplus_{p=0}^n \mathbf{\Omega}^p\mathcal{M}$ is a graded algebra, and it is vector-space isomorphic to the Clifford algebra which we denote by $\Delta\mathcal{M}$. Fix a spin structure $\mathrm{P}_{\mathrm{Spin}}(\mathcal{M})$ and let the associated Spin bundle be denoted by $\displaystyle{\not}\Delta\mathcal{M} = \mathrm{P}_{\mathrm{Spin}} \times_\eta {\not}\Delta\mathbb{R}^n$ corresponding to the standard complex representation $\eta : \Delta\mathbb{R}^n \to \mathcal{L}({\not}\Delta\mathbb{R}^n)$. Let $\cdot : \mathbf{\Gamma}(\Delta\mathcal{M}) \to \mathrm{End}({\not}\Delta\mathcal{M})$ denote Clifford multiplication on spinors.

Let ${\not}\mathrm{D}$ denote the Atiyah–Singer Dirac operator associated to ${\not}\Delta\mathcal{M}$, given locally in an orthonormal frame $\{e_k\}$ by the expression ${\not}\mathrm{D}\psi = e^k \cdot \nabla_{e_k}\psi$, where ∇ is the Spin connection. Denoting $\{{\not}e_\alpha\}$ to be an induced local orthonormal spin frame from $\{e_k\}$, the Spin connection takes the local expression $\nabla{\not}e_\alpha = \omega_E^2 \cdot {\not}e_\alpha$, where $\omega_E^2 = \frac{1}{2}\sum_{b<a} \omega_b^a \otimes e_b \cdot e_a$ is the lifting of the Levi-Civita connection 2-form to ${\not}\Delta\mathcal{M}$ and ω_b^a is the connection 1-form in $E = (e_1, \ldots, e_n)$. The symbol of this operator is $\mathrm{sym}_{{\not}\mathrm{D}}(\xi)\psi = \xi \cdot \psi$. We refer the reader to Lawson and Michelsohn [11] and Ginoux [12] for a more detailed exposition on spin structures, bundles and their associated operators.

To define ${\not}\mathrm{D}$ as a self-adjoint elliptic operator on $\mathrm{L}^2({\not}\Delta\mathcal{M})$ by imposing boundary conditions on $\mathcal{D}({\not}\mathrm{D})$ we will follow the framework developed by Bär and Ballmann [13] and specialised to Dirac-type operators in [14]. In particular, by a *local boundary condition* for $\not{D}$ we mean a space

$$\mathcal{B} = \mathrm{H}^{\frac{1}{2}}(\mathcal{E}) \quad \text{with} \quad \mathcal{E} \subset \not{S}\Sigma = \not{S}\mathcal{M}|_{\Sigma},$$

where $\mathcal{E}$ is a smooth subbundle. The operator $\not{D}$ with boundary condition $\mathcal{B}$, denoted $\not{D}_{\mathcal{B}}$, is the operator $\not{D}$ with domain

$$\mathcal{D}(\not{D}_{\mathcal{B}}) = \left\{\varphi \in \mathrm{L}^2(\not{S}\mathcal{M}) : \ \not{D}\varphi \in \mathrm{L}^2(\not{S}\mathcal{M}) \text{ and } \mathscr{R}\varphi \in \mathcal{B}\right\},$$

where $\mathscr{R}$ denotes the trace map. In particular, the choice $\mathcal{E} = 0$ yield $\not{D}_{\min}$ and $\not{D}_{\max} = \overline{\not{D}_{\mathrm{H}^{\frac{1}{2}}\not{S}\,\mathcal{E}}}$.

Two conditions we require of the local boundary condition $\mathcal{B}$ are as follows:

(i) Self-adjointness, which by Section 3.5 in [14] occurs if and only if $\mathrm{sym}_{\not{D}}(\vec{\mathrm{n}}^{\flat})$ maps the L^2 closure of $\mathcal{B}$ onto its orthogonal complement.

(ii) $\not{D}$-ellipticity, which is defined in terms of a self-adjoint boundary operator $\not{\partial}$ adapted to $\not{D}$ with principal symbol $\mathrm{sym}_{\not{\partial}}(\xi) = \mathrm{sym}_{\not{D}}(\vec{\mathrm{n}}^{\flat})^{-1} \circ \mathrm{sym}_{\not{D}}(\xi)$, and for which the operator

$$\pi_{\mathcal{B}} - \chi_{[0,\infty)}(\not{\partial}) : \mathrm{L}^2(\not{S}\Sigma) \to \mathrm{L}^2(\not{S}\Sigma)$$

is a Fredholm operator. Here, $\pi_{\mathcal{B}} : \mathrm{L}^2(\not{S}\Sigma) \to \mathcal{B}$ is projection induced from the fibrewise orthogonal projection $\pi_{\mathcal{E}} : \not{S}\Sigma \to \mathcal{E}$, and $\chi_{[0,\infty)}(\not{\partial})$ is the projection onto the positive spectrum of the operator $\not{\partial}$ (see Theorem 3.15 in [14]). This condition yields regularity up to the boundary, in the sense that $\not{D}u \in \mathrm{H}^{\mathrm{k}}_{\mathrm{loc}}(\not{S}\mathcal{M})$ if and only if $u \in \mathrm{H}^{\mathrm{k}+1}_{\mathrm{loc}}(\not{S}\mathcal{M})$ whenever $u \in \mathcal{D}(\not{D}_{\mathcal{B}})$. For a compact set $K \subset \mathcal{M}$, the constant C_{K} such that

$$C_K^{-1}||u||_{\not{D}^k_{\mathcal{B}},K} \leq ||u||_{\mathrm{H}^{\mathrm{k}},K} \leq C_K||u||_{\not{D}^k_{\mathcal{B}},K}$$

we call the *$\not{D}$-ellipticity constant of order k in K*. Here, $||u||^2_{T,K} = ||\chi_K Tu||^2 + ||\chi_K u||^2$. See Section 7.3-7.4 in [13] as well as Section 3.5 in [14].

We now state our perturbation result for the Atiyah–Singer Dirac operator $\not{D}_{\mathcal{B}}$ with a local boundary condition $\mathcal{B}$. For two local boundary conditions $\mathcal{B}$ and $\widetilde{\mathcal{B}}$, following Section 2 in Chapter IV in [15], we define the L^∞*-gap* between the subspaces $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ as

$$\hat{\delta}_\infty\big(\mathcal{B}, \widetilde{\mathcal{B}}\big) = ||\hat{\delta}\big(\mathcal{E}_x, \widetilde{\mathcal{E}}_x\big)||_{\mathrm{L}^\infty(\Sigma)} = \sup_{x\in\Sigma} |\pi_{\mathcal{E}}(x) - \pi_{\widetilde{\mathcal{E}}}(x)|,$$

where $\pi_{\mathcal{E}}$ and $\pi_{\widetilde{\mathcal{E}}}$ are the orthogonal projections from $\not{S}\Sigma$ to $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ respectively. We let $||\mathcal{B}||_{\mathrm{Lip}} = \sup_{x\in\Sigma} |\nabla \pi_E(x)|$, and similarly for $\widetilde{\mathcal{B}}$. For a set $Z \subset \mathcal{M}$ and $r > 0$, we write $Z_r = \{x \in \mathcal{M} : \rho_{\mathrm{g}}(x, Z) < r\}$, and $Z_r \sqcup Z_r$ to be the double of a neighbourhood Σ by pasting along Σ.

Theorem 3.1. *Let $(\mathcal{M}, \mathrm{g})$ be a smooth, Spin manifold with smooth, compact boundary $\Sigma = \partial\mathcal{M}$ that is complete as a metric space and suppose that there exists:*

(i) *a precompact open neighbourhood Z of Σ and $\kappa > 0$ such that* $\mathrm{inj}(\mathcal{M} \setminus Z, \mathrm{g}) > \kappa$,

(ii) *$C_R < \infty$ such that $|\mathrm{Ric}_{\mathrm{g}}| \leq C_R$ and $|\nabla \mathrm{Ric}_{\mathrm{g}}| \leq C_R$ on $\mathcal{M} \setminus Z$, and*

(iii) *a smooth metric g_Z on the double $Z_4 \sqcup Z_4$ obtained by pasting along Σ and $C_Z < \infty$ and $\kappa_Z > 0$ with $|\mathrm{Ric}_{\mathrm{g}_Z}| \leq C_Z$ and* $\mathrm{inj}(Z_2 \sqcup Z_2, \mathrm{g}_Z) \geq \kappa_Z$.

Fixing $C_B < \infty$, let $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ be two local self-adjoint $\not{D}$-elliptic boundary which satisfies:

iv. $||\mathcal{B}||_{\mathrm{Lip}} + ||\widetilde{\mathcal{B}}||_{\mathrm{Lip}} \leq C_B$, *and*
v. *$\not{D}$-ellipticity constants of orders 1 and 2 in a given compact neighbourhood K of the boundary.*

Then, for $\omega \in (0, \pi/2)$ and $\sigma > 0$, whenever we have $f \in \mathrm{Hol}^\infty(\mathrm{S}^{\mathrm{o}}_{\omega,\sigma})$, we have the perturbation estimate

$$||f(\not{D}_{\mathcal{B}}) - f(\not{D}_{\tilde{\mathcal{B}}})||_{\mathrm{L}^2 \to \mathrm{L}^2} \lesssim ||f||_\infty \hat{\delta}_\infty(\widetilde{\mathcal{B}}, \mathcal{B}),$$

where the implicit constant depends on $\dim \mathcal{M}$ and the constants appearing in (i)-(v).

Remark 3.2. The double of a smooth manifold with boundary by pasting along that boundary is again smooth (in terms of the differentiable structure). However, the canonical reflection of the metric may fail to be smooth across the boundary. The existence of a metric g_Z satisfying the assumed curvature bounds on $Z_2 \sqcup Z_2$ is always guaranteed, but we have included this in order to quantify the dependence of the constants in the perturbation estimate. See Section 3.1 for more details.

Example 3.3 (Boundary conditions in even dimensions). For $\mathcal{M}$ even dimensional, the Spin bundle splits $\not{\Delta}\mathcal{M} = \not{\Delta}^+\mathcal{M} \oplus^\perp \not{\Delta}^-\mathcal{M}$ (where $\not{\Delta}^\pm\mathcal{M}$ are the eigenspaces of $u \mapsto \vec{\mathrm{n}} \cdot u$) and

$$\not{D} = \begin{pmatrix} 0 & \not{D}^- \\ \not{D}^+ & 0 \end{pmatrix},$$

where $\not{D}^\pm : \not{\Delta}^\pm\mathcal{M} \to \not{\Delta}^\mp\mathcal{M}$. Again by even dimensionality, $\vec{\mathrm{n}} : \not{\Delta}^\pm\Sigma \to \not{\Delta}^\mp\Sigma$.

Let $\mathrm{B} \in \mathrm{End}(\not{\Delta}^+\Sigma)$ smooth and invertible, and define

$$\not{\Delta}_{\mathrm{B},x}\Sigma = \left\{(\psi, \vec{\mathrm{n}} \cdot \mathrm{B}\psi) : \psi \in \not{\Delta}^+_x\Sigma\right\} \text{ and } \not{\Delta}_{\mathrm{B}}\Sigma = \sqcup_{x \in \mathcal{M}} \not{\Delta}_{\mathrm{B},x}\Sigma,$$

which is a smooth sub-bundle of $\not{\Delta}\Sigma$. The boundary condition as considered by Gorokhovsky and Lesch in [16] is then given by $\mathcal{B}_{\mathrm{B}} = \mathrm{H}^{\frac{1}{2}}(\not{\Delta}_{\mathrm{B}}\Sigma)$.

When the boundary condition defining endomorphism B further satisfies $\mathrm{B}(x)^* = \mathrm{B}(x)$, then the boundary condition $\mathcal{B}_{\mathrm{B}}$ is $\not{D}$-elliptic and $\not{D}_B$ on $\mathrm{C}^\infty_c(\not{\Delta}\mathcal{M}; \mathcal{B}_{\mathrm{B}})$ is essentially self-adjoint. These facts are a consequence of Corollary 3.18 in [14], which guarantees $\not{D}$-ellipticity of the boundary condition $\mathcal{B}_{\mathrm{B}}$ since $\mathrm{sym}_{\not{\partial}}(\xi)$ interchanges $\not{\Delta}_{\mathrm{B}}\Sigma$ and $\not{\Delta}_{\mathrm{B}}^\perp\Sigma$ for $0 \neq \xi \in \mathrm{T}^*_x\Sigma$. The essential self-adjointness follows from invoking Theorem 3.11 in [14], since $\mathrm{sym}_{\not{D}}(\vec{\mathrm{n}})$ interchanges $\mathcal{B}_{\mathrm{B}}$ with its L^2-orthogonal complement $\mathcal{B}_{\mathrm{B}}^\perp = \{(\mathrm{B}\vec{\mathrm{n}} \cdot v, v) : v \in \mathrm{H}^{\frac{1}{2}}(\not{\Delta}^-\Sigma)\}$ in $\mathrm{H}^{\frac{1}{2}}(\not{\Delta}\Sigma)$.

Example 3.4. As noted in [5], Chiral conditions arise from an associated Chirality operator $G \in \mathrm{C}^\infty(\mathcal{L}(\not{\Delta}\mathcal{M}))$ satisfying: for all $X \in \mathrm{C}^\infty(\mathrm{T}\mathcal{M})$ and $\psi, \varphi \in \mathrm{C}^\infty(\not{\Delta}\mathcal{M})$,

$$G^2 = \mathrm{I}, \ \langle G\varphi, G\psi \rangle = \langle \varphi, \psi \rangle, \ \nabla_X(G\psi) = G\nabla X\psi, \ X \cdot G\varphi = -GX \cdot \varphi,$$

and the boundary condition is defined via the projector $\pi_G u = \frac{1}{2}(\mathrm{I} - \vec{\mathrm{n}} \cdot G)$. This is a self-adjoint local elliptic boundary condition which exists in any dimension (given the map G), and has been used in the study of asymptotically flat manifolds including black holes. See Section 5.2 in [5] for more details.

Proof of Theorem 3.1. Without loss of generality, we can assume that $\hat{\delta}_\infty(\mathcal{B}, \widetilde{\mathcal{B}}) \leq 1/2$, as the estimate is trivially true from the spectral theorem for $\hat{\delta}_\infty(\mathcal{B}, \widetilde{\mathcal{B}}) > 1/2$. Note that since the projectors $\pi_{\mathcal{E}}$ and $\pi_{\tilde{\mathcal{E}}}$ on $\not{\Delta}\Sigma$ to $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ respectively are orthogonal, $||2\pi_{\mathcal{E}} - \mathrm{I}||_\infty = 1$ and so we obtain:

(i) $||\pi_{\mathcal{E}} - \pi_{\tilde{\mathcal{E}}}||_\infty \leq \frac{1}{2||2\pi_{\mathcal{E}} - \mathrm{I}||_\infty}$, and
(ii) $||\nabla\pi_{\mathcal{E}}||_\infty + ||\nabla\pi_{\tilde{\mathcal{E}}}||_\infty \leq C_B$

We claim that there exists a $\mathrm{U} \in \mathrm{Lip}(\mathcal{L}(\not{\Delta}\mathcal{M}))$ with $||\mathrm{U} - \mathrm{I}||_\infty \leq \hat{\delta}_\infty(\mathcal{B}, \widetilde{\mathcal{B}}) \leq \frac{1}{2}$ and $||\nabla \mathrm{U}|| \lesssim C_B$ such that $\mathrm{U}\mathcal{B} = \widetilde{\mathcal{B}}$. To see this, set $\mathrm{U}_0 = \frac{1}{2}(\mathrm{I} + (2\pi_{\mathcal{E}} - \mathrm{I})(2\pi_{\tilde{\mathcal{E}}} - \mathrm{I}))$ and it is easy to see that $\pi_{\mathcal{E}} = \mathrm{U}_0^{-1}\pi_{\tilde{\mathcal{E}}}\mathrm{U}_0$. Fix $\epsilon > 0$ such that $[0, \epsilon) \times \Sigma \cong N^\epsilon$, where $N^\epsilon = \{x \in \mathcal{M} : \rho(x, \Sigma) < \epsilon\}$ and note that U_0 extends to a projection $\mathrm{U}'(x) = \mathrm{U}_0(x')$ for $x = (t, x') \in [0, \epsilon) \times \Sigma$. Then U is given by:

$$\mathrm{U}(x) = \begin{cases} \mathrm{I} & x \notin N^\epsilon, \\ \left(\mathrm{I} - \dfrac{\rho(x, \Sigma)}{\epsilon}\right)\mathrm{U}'(x) + \dfrac{\rho(x, \Sigma)}{\epsilon}\mathrm{I} & x \in N^\epsilon. \end{cases}$$

We verify the hypotheses (A1)–(A9) and invoke Theorem 2.1 with $\mathcal{V} = \not{\Delta}\mathcal{M}, \mathrm{D} = \not{D}_{\mathcal{B}}$ and $\widetilde{\mathrm{D}} = \mathrm{U}^{-1}\not{D}_{\tilde{\mathcal{B}}}\mathrm{U}$ to obtain the estimate

$$||f(\not{D}_{\mathcal{B}}) - f\big(\mathrm{U}^{-1}\not{D}_{\tilde{\mathcal{B}}}\mathrm{U}\big)||_{\mathrm{L}^2\to\mathrm{L}^2} \lesssim ||\mathrm{I} - \mathrm{U}||_\infty ||f||_\infty.$$

The passage from this to the required estimate follows from the fact that we have $||\mathrm{I} - \mathrm{U}||_\infty \leq 1/2$ by noting that $f(\mathrm{U}^{-1}\not{D}_{\tilde{\mathcal{B}}}\mathrm{U}) = \mathrm{U}^{-1}f(\not{D}_{\tilde{\mathcal{B}}})\mathrm{U}$ and that $||f(\not{D}_{\tilde{\mathcal{B}}}) - f(\mathrm{U}^{-1}\not{D}_{\tilde{\mathcal{B}}}\mathrm{U})||_{\mathrm{L}^2\to\mathrm{L}^2} \lesssim ||\mathrm{I} - \mathrm{U}||_\infty ||f||_\infty$.

The first hypothesis (A1) is immediate and (A2) and (A3) are a consequence of the fact that the curvature assumptions imply that $\mathrm{Ric}_g \geq -C_R$ (c.f. Theorem 5.6.4 and 5.6.5 in [17]).

The existence of GBG frames satisfying the required bounds in (A4), (A5), and (A6) follow from Proposition 3.6, which only depend on C_R, κ, C_Z and κ_Z. See Section 3.1.

Since we assume that $\mathcal{B}$ is a local boundary condition, we have that for every $\eta \in \mathrm{L}^\infty \cap \mathrm{Lip}(\mathcal{M})$, the domain inclusion $\eta\mathcal{D}(\not{D}_{\mathcal{B}}) \subset \mathcal{D}(\not{D}_{\mathcal{B}})$ holds. The commutator estimates follow from the fact that

$$[\not{D}, \eta]u = \mathrm{d}\eta \cdot u \quad \text{and} \quad \big[\mathrm{U}^{-1}\not{D}\mathrm{U}, \eta\big]u = \mathrm{U}^{-1}\mathrm{d}\eta \cdot \mathrm{U}u.$$

This shows (A7).

The hypothesis (A8) is a consequence of Proposition 3.8 and 3.9 since we assume that $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ are $\not{D}$-elliptic boundary conditions. Note that the constant arising from these propositions include the constant $C_{\mathrm{ell},K}$ in the ellipticity estimate

$$C_{\mathrm{ell},K}^{-1}||u||_{\not{D}_{\mathcal{B}},K} \leq ||u||_{\mathrm{H}^1,K} \leq C_{\mathrm{ell},K}||u||_{\not{D}_{\mathcal{B}},K}$$

whenever $u \in \mathcal{D}(\not{D}_{\mathcal{B}})$. The corresponding constant in the region $\mathcal{M} \setminus K$ depends on the geometric bounds (i)–(iii). In addition to these constants for $\not{D}_{\mathcal{B}}$, the corresponding estimate for the operator $\not{D}_{\tilde{\mathcal{B}}}$ includes the constant C_{B}. See Section 3.2 for details.

The remaining hypothesis is the Riesz-Weitzenböck hypothesis (A9). This is proved similar to Proposition 3.8, using the compact set K and $K_{\frac{1}{2}}$ near the boundary, along with the smooth cut-off f as they appear in the proof of this proposition. The estimate $\|\nabla^2(fu)\| \lesssim \|\not{D}_{\mathcal{B}}^2 u\| + \|u\|$ is obtained by arguing as in Proposition 3.18 in [1] via the cover provided by Lemma 3.7, and the remaining estimate $\|\nabla^2((1-f)u)\| \leq \widetilde{C}_{\mathrm{ell},K}(\|\not{D}_{\mathcal{B}}^2 u\| + \|u\|)$ is due to the boundary regularity result, Theorem 7.17 in [13]. Here, ellipticity constant $\widetilde{C}_{\mathrm{ell},K}$ is the constant

$$\widetilde{C}_{\mathrm{ell},K}^{-1}\|u\|_{\not{D}_{\mathcal{B}}^k,K} \leq \|u\|_{\mathrm{H}^k,K} \leq \widetilde{C}_{\mathrm{ell},K}\|u\|_{\not{D}_{\mathcal{B}}^k,K}$$

whenever $u \in \mathcal{D}(\not{D}_{\mathcal{B}}^k)$ for $k=1$, 2. The constant for the estimate in the region $\mathcal{M} \setminus K$ depend on the constants in (i)-(iii).

Lastly, the decomposition of the operator $\not{D}_{\tilde{\mathcal{B}}} - \not{D}_{\mathcal{B}} = A_1\nabla + \operatorname{div} A_2 + A_3$ distributionally proved in Proposition 3.12. See Section 3.3 for details. □

Throughout the remainder of this section, we assume the hypothesis of Theorem 3.1.

3.1. Geometric bounds in the presence of boundary

The way in which we prove Theorem 3.1 is via Theorem 2.1, which requires us to prove that under the geometric assumptions we make, the bundle $\not{\Delta}\mathcal{M}$ satisfies generalised bounded geometry and the first and second metric derivatives in each trivialisation are bounded.

We do this by considering the double of the manifold $\widetilde{\mathcal{M}} = \mathcal{M} \sqcup \mathcal{M}$, which is obtained by taking two copies of $\mathcal{M}$ and pasting along the boundary Σ to obtain a manifold without boundary. Since the boundary is smooth, this manifold is again smooth (in a differential topology sense, see Theorem 9.29 in [18]). By reflection, we obtain an extension $\mathrm{g}_{\mathrm{ext}}$ of the metric g to the whole of $\widetilde{\mathcal{M}}$. This metric is guaranteed to be continuous everywhere and smooth on $\widetilde{\mathcal{M}} \setminus \Sigma$, but in general, without imposing additional restrictions on the boundary, it will not be smooth. However, as we illustrate in the following lemma, we are able to construct a smooth metric sufficiently close to $\mathrm{g}_{\mathrm{ext}}$ that suffices to obtain the bounds we desire for $(\mathcal{M}, \mathrm{g})$.

Lemma 3.5. *There exists a smooth complete metric $\tilde{\mathrm{g}}$ on $\widetilde{\mathcal{M}}$ with $G \geq 1$ dependent on g_Z and g satisfying*

$$G^{-1}|u|_{\tilde{\mathrm{g}}} \leq |u|_{\mathrm{g}_{\mathrm{ext}}} \leq G|u|_{\tilde{\mathrm{g}}}$$

and for which there exists:

(i) $\tilde{\kappa} > 0$ *such that* $\mathrm{inj}(\widetilde{\mathcal{M}}, \tilde{\mathrm{g}}) > \tilde{\kappa}$,
(ii) $\widetilde{C}_R < \infty$ *such that* $|\mathrm{Ric}_{\tilde{\mathrm{g}}}| \leq \widetilde{C}_R$ *and* $|\nabla \mathrm{Ric}_{\tilde{\mathrm{g}}}| \leq \widetilde{C}_R$,
(iii) *a compact set $\mathcal{P}$ with $\mathcal{P} \neq \emptyset$ and $\Sigma \subset \mathcal{P}$ such that $\mathrm{g}_{\mathrm{ext}} = \tilde{\mathrm{g}}$ on $\widetilde{\mathcal{M}} \setminus \mathcal{P}$.*

The constants $\tilde{\kappa}, \widetilde{C}_R$ and depend on the original geometric bounds κ, C_R, κ_Z, C_Z.

Proof. Take Z from the hypothesis of Theorem 3.1 and let $\mathcal{P} = \overline{Z \sqcup Z}$. By hypothesis, since Z is precompact, we get that $\mathcal{P}$ is compact. As a consequence, if $\{x_n\}$ is a Cauchy sequence in $\mathcal{P}$, then it converges to some point and if $\{x_n\}$ is Cauchy in $\widetilde{\mathcal{M}} \setminus \mathring{\mathcal{P}}$, then it converges to some point in $\widetilde{\mathcal{M}} \setminus \mathring{\mathcal{P}}$ by the metric completeness of g. This establishes that g_{ext} is metric complete.

Next, let $\psi \in C^\infty(\widetilde{\mathcal{M}})$ be such that $\psi = 1$ on $\widetilde{\mathcal{M}} \setminus \mathring{\mathcal{P}}$ and $\psi = 0$ on $\mathcal{P}_{\frac{3}{2}} = \{x \in \widetilde{\mathcal{M}} : \rho_{g_{\text{ext}}}(x, \mathcal{P}) \leq \frac{3}{2}\}$. Since $\mathcal{P}_\epsilon$ is compact by construction, by the smoothness of the differentiable structure of $\widetilde{\mathcal{M}}$, there exists $G \geq 1$ such that g_{ext} and g_Z are G-close on $\mathcal{P}_2$. Define $\widetilde{g} = \psi g_{\text{ext}} + (1-\psi) g_Z$ and since $g_{\text{ext}} = \widetilde{g}$ away from $\mathcal{P}$, this shows that the quasi-isometry with constant G between g_{ext} and $\widetilde{g}$ and also establishes (iii).

Since g_Z satisfies a lower bound on injectivity radius on $Z_2 \sqcup Z_2$ as well as a Ricci curvature bound on this set, and since g satisfies similar bounds on Z, by construction of the metric $\widetilde{g}$, we obtain (i) and (ii) with the dependency as stated in the conclusion. □

Now, using this we can prove the main proposition that we require to prove the geometric bounds needed to prove Theorem 3.1.

Proposition 3.6. *There exist $r_H > 0$ and a constant $1 \leq C < \infty$ depending on κ, C_R, κ_Z and C_Z such that at each $x \in \mathcal{M}, \psi_x : B(x, r_H) \to \mathbb{R}^n$ corresponds to a coordinate system and inside that coordinate system with coordinate basis $\{\partial_j\}$ satisfying:*

$$C^{-1}|u|_{\psi_x^*\delta(y)} \leq |u|_{g(y)} \leq C|u|_{\psi_x^*\delta(y)}, \quad |\partial_k g_{ij}(y)| \leq C, \quad \text{and} \quad |\partial_k \partial_l g_{ij}(y)| \leq C,$$

for all $y \in B(x, r_H)$ and where δ is the Euclidean metric.

Proof. Utilising the metric $\widetilde{g}$ given by Lemma 3.5, we apply Theorem 1.2 in [19] to obtain $C^{2,\alpha}$-harmonic coordinates for the manifold $(\widetilde{\mathcal{M}}, \widetilde{g})$ with radius $\widetilde{r_H}$. We obtain the same conclusions for $(\mathcal{M}, \widetilde{g}|_{\mathcal{M}})$ as it is obtained via the subspace topology on $\widetilde{\mathcal{M}}$. The balls B_g and $B_{\widetilde{g}}$ are contained within the factor G given in the lemma, and away from the compact region $\mathcal{P}$ defined in the lemma, we have that $B_g = B_{\widetilde{g}}$. So, it suffices to set $r_H = \widetilde{r_H}/G$. On the region $\widetilde{\mathcal{M}} \setminus \mathcal{P}$, we have $C^{2,\alpha}$ control of the metric $\widetilde{g}$ and outside of this region, by compactness, we obtain control of as many derivatives of the metric as we like. By taking maximums of the constants appearing in the regions $\widetilde{\mathcal{M}} \setminus \mathcal{P}$ and $\mathcal{P}$, we obtain the constant C in the conclusion of this proposition. □

3.2. The domains of the operators

To invoke Theorem 2.1, we need to establish H^1 regularity for the operators $\not{D}_{\mathcal{B}}$ and $\not{D}_{\widetilde{\mathcal{B}}}$. To this end, we begin with the following covering lemma.

Lemma 3.7. *There exists $C_H < \infty$, $M > 0$ and a sequence of points x_i and a smooth partition of unity $\{\eta_i\}$ for $\mathcal{M}$ that is uniformly locally finite and subordinate to $\{B(x_i, r_H)\}$ satisfying:*

(i) $\sum_i |\nabla^j \eta_i| \leq C_H$ *for* $j = 0, \ldots, 3$, *and*
(ii) $1 \leq M \sum_i \eta_i^2$.

The $r_H > 0$ here is the harmonic radius guaranteed in Proposition 3.6.

Proof. Take the double of the manifold and the smooth metric given by Lemma 3.5. Then, by Lemma 1.1 in [19], on fixing $\rho > 0$ we find a sequence of points $x_i \in \widetilde{\mathcal{M}}$ such that (i) $\{\widetilde{B}(x_i, r)\}$ is a uniformly locally finite cover of $\widetilde{\mathcal{M}}$ for all $r \geq \rho$ and (ii) $\widetilde{B}(x_i, \rho/2) \cap \widetilde{B}(x_j, \rho/2) = \emptyset$ for all $i \neq j$. This relies purely on a measure counting argument since $\widetilde{\mathrm{g}}$ induces a measure satisfying exponential volume growth ($\mathrm{E}_{\mathrm{loc}}$) by the Ricci curvature lower bounds. Since $\widetilde{\mathrm{g}}$ is G-close to $\mathrm{g}_{\mathrm{ext}}$, the same is true for the metric $\mathrm{g}_{\mathrm{ext}}$, which is the metric guaranteed to be continuous obtained by reflection of g on $\mathcal{M}$ across Σ to the double $\widetilde{\mathcal{M}}$. Thus, a cover satisfying (i) and (ii) exists on $\widetilde{\mathcal{M}}$ replacing $\widetilde{\mathrm{g}}$ balls $\widetilde{B}$ with $\mathrm{g}_{\mathrm{ext}}$ balls B^{ext}.

Now, let r_{H} denote the radius obtained from Proposition 3.6, and set $\rho = r_H/16$. Let $\{x_i^{\mathcal{M}}\} \subset \mathring{\mathcal{M}}$ such that $\rho_{\mathrm{g}}(x_i^{\mathcal{M}}, \Sigma) > r_H/16$. Then $\{x_i^{\mathcal{M}}\} \subset \mathcal{M} \setminus Z'$, where $Z' = \{x \in \mathcal{M} : \rho_{\mathrm{g}}(x, \Sigma) \leq r_H/16\}$. Since Σ is compact, so is Z' and hence, there exists a finite number of points $\{x_j^{Z'}\}_{j=1}^{K}$ such that $Z' \subset \cup_{j=1}^{K} B(x_j^{Z'}, r_H/16)$. Then, the collection of points $\{\bar{x}_i\} = \{x_j^{\mathcal{M}}, x_k^{Z'}\}$ satisfies: $\mathcal{M} = \cup_i B(\bar{x}_i, r_H/16)$ with $\{B(\bar{x}_i, r_H/16)\}$ uniformly locally finite.

Inside each $B(\bar{x}_i, r_H/16)$ we have $\mathrm{C}^{2,\alpha}$ control of the metric, and therefore, the partition of unity $\{\eta_j\}$ with the gradient bound in the conclusion is obtained by proceeding as in the proof of Proposition 3.2 in [19]. □

With this lemma, we prove the following.

Proposition 3.8. *The embedding $\mathcal{D}(\not{D}_{\mathcal{B}}) \hookrightarrow \mathrm{H}^1(\mathcal{V})$ holds along with the ellipticity estimate $\|u\|_{\not{D}_{\mathcal{B}}} \simeq \|u\|_{\mathrm{H}^1}$ for all $u \in \mathcal{D}(\not{D}_{\mathcal{B}})$.*

Proof. Let K be a compact neighbourhood of Σ assumed in (v) of Theorem 3.1 and let $f : \mathcal{M} \to [0, 1]$ be smooth with $f = 1$ on $\mathcal{M} \setminus \mathring{K}$ and $f = 0$ on an open subset $\widetilde{K} \subset \mathring{K}$ with $\Sigma \subset \widetilde{K}$. Let $u \in \mathcal{D}(\not{D}_{\mathcal{B}})$ and we show that $\|\nabla(fu)\| + \|\nabla((1-f)u)\| \lesssim \|\not{D}_{\mathcal{B}}u\| + \|u\|$. Using the cover guaranteed by Lemma 3.7, we obtain that

$$\|\nabla(fu)\| \lesssim \|\not{D}_{\mathcal{B}}(fu)\| + \|fu\| \lesssim \|\not{D}_{\mathcal{B}}u\| + \|u\|,$$

where the first inequality is from running the exact same argument as Proposition 3.6 in [1] and the second inequality is from the fact that spt $\nabla f \subset K$ and hence bounded. For the remaining inequality, we note that since the boundary condition $\mathcal{B}$ is $\not{D}$-elliptic, Theorem 7.17 in [13] gives us that $u \in \mathrm{H}^{\mathrm{k}+1}_{\mathrm{loc}}(\not{\Delta}\mathcal{M}) \iff \not{D}_{\mathcal{B}}u \in \mathrm{H}^{\mathrm{k}}_{\mathrm{loc}}(\not{\Delta}\mathcal{M})$ whenever $u \in \mathcal{D}(\not{D}_{\mathcal{B}})$. Choosing $k = 0$, and the fact that spt $(1-f)u \subset K$, we get that

$$\|\nabla((1-f)u)\| \leq C_{\mathrm{ell},K}\big(\|\not{D}_{\mathcal{B}}((1-f)u)\| + \|(1-f)u\|\big) \lesssim \|\not{D}_{\mathcal{B}}u\| + \|u\|.$$

where $C_{\mathrm{ell},K} < \infty$ is a constant that depends on K.

The estimate $\|u\|_{\not{D}_{\mathcal{B}}} \lesssim \|u\|_{\mathrm{H}^1(\mathcal{V})}$ for $u \in \mathcal{D}(\not{D}_{\mathcal{B}})$ follows from the pointwise estimate $|\not{D}u| \lesssim |\nabla u|$ (c.f. Proposition 3.6 in [1]). □

Using this proposition, we prove the following.

Proposition 3.9. *The equality $\mathcal{D}(\not{D}_{\mathcal{B}}) = \mathcal{D}(\not{D}_{\tilde{\mathcal{B}}}\mathrm{U})$ holds.*

Proof. On fixing $\varphi \in \mathrm{C}^{\infty}_{c}(\not{\Delta}\mathcal{M})$, we compute at a point $x \in \mathcal{M}$ with a frame satisfying $\nabla_{e_i} e_j(x) = 0$:

$$\not{D}(\mathrm{U}\varphi) = e^i \cdot \nabla_{e_i}(\varphi^\alpha \mathrm{U} \not{e}_\alpha) = (e_i \varphi^\alpha) e^i \cdot \mathrm{U}\ \not{e}_\alpha + \varphi^\alpha \left(e_i \mathrm{U}^\beta_\alpha \right) \not{e}_\beta,$$

from which it follows directly that $|\not{D}(\mathrm{U}\varphi)|^2 \leq |\mathrm{U}|^2 |\nabla\varphi|^2 + |\nabla \mathrm{U}|^2 |\varphi|^2$. Now, for $\varphi \in \mathcal{D}(\not{D}_{\tilde{\mathcal{B}}})$, we have from Theorem 3.10 in [14] that there is a sequence $\varphi_n \in \mathrm{C}^\infty_c(\not{S}\mathcal{M}; \widetilde{\mathcal{B}})$ such that $\varphi_n \to \varphi$ in the graph norm of $\not{D}_{\tilde{\mathcal{B}}}$. Moreover, $\mathrm{U}\varphi_n \in \mathrm{C}^{0,1}_c(\not{S}\mathcal{M}, \mathcal{B}) \subset \mathcal{D}(\not{D}_{\mathcal{B}})$ and by Proposition 3.8, $||\nabla(\varphi_n - \varphi)|| \to 0$. Hence, combining this with our pointwise estimate and integrating, we obtain that

$$||\not{D}(\mathrm{U}\varphi_n - \mathrm{U}\varphi_m)|| \lesssim ||\mathrm{U}||_\infty ||\nabla(\varphi_n - \varphi_m)|| + ||\nabla \mathrm{U}||_\infty ||\varphi_n - \varphi_m|| \to 0$$

as $m, n \to \infty$. By the closedness of $\not{D}_{\mathcal{B}}$, we have that $\mathrm{U}\varphi \in \mathcal{D}(\not{D}_{\mathcal{B}})$. The reverse containment is obtained similarly. □

3.3. Decomposition of the difference of operators

A crucial assumption in Theorem 2.1 is to be able to write the difference of our operators $\not{D}_{\mathcal{B}}$ and $\mathrm{U}^{-1}\not{D}_{\tilde{\mathcal{B}}}\mathrm{U}$ as

$$\not{D}_{\mathcal{B}} - \mathrm{U}^{-1}\not{D}_{\tilde{\mathcal{B}}}\mathrm{U} = A_1 \nabla + \mathrm{div} A_2 + A_3$$

with $||A_i||_\infty$ controlled by $||\mathrm{U} - \mathrm{I}||_\infty$.

Our computations here are similar to those in Section 3 of [1], with the key observation being that the last term in Lemma 3.10 cannot be used as A_3, since it would yield only a bound $||A_3||_\infty \lesssim 1$ and not $||A_3||_\infty \lesssim ||\mathrm{U} - \mathrm{I}||_\infty$. Instead, we proceed via an application of the product rule for derivatives as in Lemma 3.11.

Throughout this subsection, unless otherwise stated, we fix an open set $\Omega \subset \mathring{\mathcal{M}}$ and let $\{e_i\}$ and $\{\not{e}_\alpha\}$ be orthonormal frames for $\mathrm{T}\mathcal{M}$ and $\not{S}\mathcal{M}$ respectively inside Ω.

Lemma 3.10. *For $\varphi \in \mathrm{C}^\infty(\not{S}\mathcal{M})$ we have the following pointwise equality almost-everywhere inside Ω:*

$$\left(\not{D} - \mathrm{U}^{-1}\ \not{D}\mathrm{U}\right)\varphi = X\nabla\varphi + Z^\Omega \varphi + \varphi^\alpha \left(\nabla_{e_j} \mathrm{U}^\beta_\alpha\right) \mathrm{U}^{-1} e^j \cdot \not{e}_\beta$$

with $X : \mathbf{\Gamma}(\mathrm{T}^*\mathcal{M} \otimes \not{S}\mathcal{M}) \to \mathbf{\Gamma}(\not{S}\mathcal{M})$ and $Z^\Omega : \mathbf{\Gamma}(\not{S}\Omega) \to \mathbf{\Gamma}(\not{S}\Omega)$ with almost-everywhere pointwise estimates

$$|X| \lesssim ||\mathrm{I} - \mathrm{U}||_\infty \quad \text{and} \quad |Z^\Omega| \lesssim ||\mathrm{I} - \mathrm{U}||_\infty,$$

where the implicit constants depends on the constants in Theorem 3.1.

Proof. A direction calculation yields that

$$\not{D}(\mathrm{U}\varphi) = \left(\nabla_{e_j}\varphi^\alpha\right) e^j \cdot \mathrm{U}\not{e}_\alpha + \varphi^\alpha \left(\nabla_{e_j} \mathrm{U}^\beta_\alpha\right) e^j \cdot \not{e}_\beta + \varphi^\alpha \mathrm{U}^\beta_\alpha e^j \cdot \nabla_{e_j} \not{e}_\beta.$$

Since the term $\nabla_{e_j}\ \not{e}_\beta = \omega^2_E(e_j) \cdot \not{e}_\beta$, multiplying this expression by U^{-1} on the left, and then subtracting it from the expression for $\not{D}\varphi$, we obtain that

$$\begin{aligned}\left(\not{D} - \mathrm{U}^{-1}\not{D}\mathrm{U}\right)\varphi &= \nabla_{e_j}\varphi^\alpha \left(e^j \cdot \not{e}_\alpha - \mathrm{U}^{-1} e^j \cdot \mathrm{U}\not{e}_\alpha\right) \\ &\quad + \left(e^j \cdot \omega^2_E(e_j) - \mathrm{U}^{-1} e^j \cdot \omega^2_E(e_j)\mathrm{U}\right) \cdot \varphi + \varphi^\alpha \left(\nabla_{e_j} \mathrm{U}^\beta_\alpha\right) \mathrm{U}^{-1} e^j \cdot \not{e}_\beta.\end{aligned}$$

To obtain a bound on the first expression to the right of this, we note that

$$e^j \cdot \not{e}_\alpha - \mathrm{U}^{-1} e^j \cdot \mathrm{U}\not{e}_\alpha = e^j \cdot (\mathrm{I}-\mathrm{U})\not{e}_\alpha + (\mathrm{I}-\mathrm{U}^{-1}) e^j \cdot \mathrm{U}\not{e}_\alpha,$$

and we can write

$$\nabla_{e_j} \varphi^\alpha e^j \cdot (\mathrm{I}-\mathrm{U})\not{e}_\alpha = X_1 \nabla \varphi - \varphi^\alpha e^j \cdot (\mathrm{I}-\mathrm{U}) \ \omega_E^2(e_j) \cdot \not{e}_\alpha,$$

where $X_1(\psi_k^\alpha e^k \otimes \not{e}_\alpha) = \psi_k^\alpha e^k \cdot (\mathrm{I}-\mathrm{U})\not{e}_\alpha$. Now, similarly, writing $X_2(\psi_k^\alpha e^k \otimes \not{e}_\alpha) = \psi_k^\alpha (\mathrm{I}-\mathrm{U}^{-1}) e^j \cdot \mathrm{U}\not{e}_\alpha$, we obtain that

$$\nabla_{e_j} \varphi^\alpha (\mathrm{I}-\mathrm{U}^{-1}) e^j \cdot \mathrm{U}\not{e}_\alpha = X_2 \nabla \varphi - \varphi^\alpha (\mathrm{I}-\mathrm{U}^{-1}) e^j \cdot \mathrm{U} \ \omega_E^2(e_j) \cdot \not{e}_\alpha.$$

Letting $X = X_1 + X_2$, we obtain that

$$\begin{aligned} \nabla_{e_j} \varphi^\alpha \left(e^j \cdot \not{e}_\alpha - \mathrm{U}^{-1} e^j \cdot \mathrm{U}\not{e}_\alpha\right) &= X \nabla \varphi \\ &\quad - e^j \cdot (\mathrm{I}-\mathrm{U}) \ \omega_E^2(e_j) \cdot \varphi - (\mathrm{I}-\mathrm{U}^{-1}) e^j \cdot \mathrm{U} \ \omega_E^2(e_j) \cdot \varphi. \end{aligned}$$

Now, note that

$$e^j \cdot \omega_E^2(e_j) - \mathrm{U}^{-1} e^j \cdot \ \omega_E^2(e_j)\mathrm{U} = e^j \cdot \omega_E^2(e_j)(\mathrm{I}-\mathrm{U}) + (\mathrm{I}-\mathrm{U}^{-1}) e^j \cdot \ \omega_E^2(e_j)\mathrm{U},$$

and on setting

$$\begin{aligned} Z^\Omega = {} & e^j \cdot \omega_E^2(e_j)(\mathrm{I}-\mathrm{U}) + (\mathrm{I}-\mathrm{U}^{-1}) e^j \cdot \omega_E^2(e_j)\mathrm{U} \\ & - e^j \cdot (\mathrm{I}-\mathrm{U}) \ \omega_E^2(e_j) - (\mathrm{I}-\mathrm{U}^{-1}) e^j \cdot \mathrm{U} \ \omega_E^2(e_j), \end{aligned}$$

we obtain the conclusion. □

This lemma illustrates that the main term to analyse is the last term $\varphi^\alpha (\nabla_{e_j} \mathrm{U}_\alpha^\beta) \mathrm{U}^{-1} e^j \cdot \not{e}_\beta$. This is the content of the following lemma.

Lemma 3.11. *For $\varphi \in \mathrm{C}^\infty(\not{\Delta}\mathcal{M})$, we have the following decomposition pointwise almost-everywhere inside Ω:*

$$\varphi^\alpha \left(\nabla_{e_j} \mathrm{U}_\alpha^\beta\right) \mathrm{U}^{-1} e^j \cdot \not{e}_\beta = L^\Omega \nabla \varphi + \operatorname{div} \ M^\Omega \varphi + N^\Omega \varphi.$$

The coefficients satisfy the estimates

$$\|L^\Omega\|_\infty + \|M^\Omega\|_\infty + \|N^\Omega\|_\infty \lesssim \|\mathrm{I}-\mathrm{U}\|_\infty \ \text{and} \ \|\nabla M^\Omega\|_\infty \lesssim 1,$$

where the implicit constants depend on the constants listed in Theorem 3.1.

Proof. First note that on letting $\varepsilon_\alpha^\beta = \delta_\alpha^\beta - \mathrm{U}_\alpha^\beta$, we have $\varphi^\alpha (\nabla_{e_j} \mathrm{U}_\alpha^\beta) \mathrm{U}^{-1} e^j \cdot \not{e}_\beta = -\varphi^\alpha (\nabla_{e_j} \varepsilon_\alpha^\beta) \mathrm{U}^{-1} e^j \cdot \not{e}_\beta$. Let $M^\Omega : \mathbf{\Gamma}(\not{\Delta}\Omega) \to \mathbf{\Gamma}(\mathrm{T}^*\Omega \otimes \not{\Delta}\Omega)$ written inside Ω as

$$M^\Omega \psi = \varphi^\alpha M_{\alpha,k}^\theta e^k \otimes \mathrm{U}^{-1} e^k \cdot \not{e}_\theta$$

with the coefficients to be determined later. Note that:

$$\begin{aligned} \nabla\left(M^\Omega \varphi\right) = {} & M_{\alpha,k}^\theta \left(\nabla_{e_j} \varphi^\alpha\right) e^j \otimes e^k \otimes \mathrm{U}^{-1} e^k \cdot \not{e}_\theta \\ & + \varphi^\alpha \nabla_{e_j} \left(M_{\alpha,k}^\theta\right) e^j \otimes e^k \otimes \mathrm{U}^{-1} e^k \cdot \not{e}_\theta + \varphi^\alpha M_{\alpha,k}^\theta e^j \otimes \nabla_{e_j} \left(e^k \otimes \mathrm{U}^{-1} e^k \cdot \not{e}_\theta\right). \end{aligned}$$

On taking the trace, and rearranging the equation,

$$\varphi^\alpha \nabla_{e_j}\left(M^\theta_{\alpha,j}\right)\mathrm{U}^{-1}e^j \cdot \not{e}_\theta = \mathrm{tr}_g \nabla\left(M^\Omega \varphi\right) - \left(\nabla_{e_j}\varphi^\alpha\right)M^\theta_{\alpha,j}\mathrm{U}^{-1}e^j \cdot \not{e}_\theta - \varphi^\alpha M^\theta_{\alpha,k}\mathrm{tr}\left(e^j \otimes \nabla_{e_j}\left(e^k \otimes \mathrm{U}^{-1}e^k \cdot \not{e}_\theta\right)\right).$$

So set $M^\theta_{\alpha,k} = \epsilon^\theta_\alpha$, which gives us an expression for $\varphi^\alpha(\nabla_{e_j}\epsilon^\beta_\alpha)\mathrm{U}^{-1}e^j \cdot \not{e}_\beta$.

It remains to show that the remaining terms in this expression can be decomposed to $L^\Omega\nabla\varphi + N\varphi$. Let $L^\Omega(e^j \otimes \not{e}_\alpha) = \mathrm{U}^{-1}e^j \cdot M^\theta_{\alpha,j}\not{e}_\theta$, then we have that

$$\mathrm{tr}_g\left(\varphi^\alpha \nabla_{e_j}\left(M^\theta_{\alpha,k}\right)e^j \otimes e^k \otimes \mathrm{U}^{-1}e^k \cdot \not{e}_\theta\right) = L^\Omega \nabla\varphi - \mathrm{U}^{-1}e^j \cdot M^\Omega \omega^2_E(e_j)\cdot\varphi.$$

Absorbing the error term in this computation along with the remaining term from the former expression, we can set

$$N^\Omega\varphi = -\varphi^\alpha\epsilon^\theta_\alpha \mathrm{tr}\left(e^j \otimes \nabla_{e_j}\left(e^k \otimes \mathrm{U}^{-1}e^k \cdot \not{e}_\theta\right)\right) - \mathrm{U}^{-1}e^j \cdot M^\Omega\omega^2_E(e_j)\cdot\varphi.$$

The estimates in the conclusion for $L^\Omega, M^\Omega, N^\Omega$ and ∇M^Ω follows from the definitions of these maps. □

Using these two lemmata, arguing in a similar way to Proposition 3.16 in [1], we obtain the following decomposition globally on $\mathring{\mathcal{M}}$.

Proposition 3.12. *We have that:*

$$\left(\not{D}_{\mathcal{B}} - \mathrm{U}^{-1}\not{D}_{\tilde{\mathcal{B}}}\mathrm{U}\right)\varphi = A_1\nabla\varphi + \mathrm{div}A_2\varphi + A_3\varphi$$

distributionally for all $\varphi \in \mathcal{D}(\not{D}_{\mathcal{B}})$ *where the coefficients* A_i *satisfy:*

$$A_1 \in \mathrm{L}^\infty(\mathcal{L}(\mathrm{T}^*\mathcal{M} \otimes \not{\Delta}\mathcal{M})),$$
$$A_2 \in \mathrm{L}^\infty \cap \mathrm{Lip}(\mathcal{L}(\not{\Delta}\mathcal{M}, \mathrm{T}^*\mathcal{M} \otimes \not{\Delta}\mathcal{M})),$$
$$A_3 \in \mathrm{L}^\infty(\mathcal{L}(\not{\Delta}\mathcal{M})),$$

with $||A_1||_\infty + ||A_2||_\infty + ||A_3||_\infty \lesssim ||\mathrm{I} - \mathrm{U}||_\infty$. *The implicit constants depend on the constants listed in Theorem 3.1.*

Proof. Following the proof of Proposition 3.16 in [1], it suffices to show that there exists a cover $\{B_j\}$ of balls with a fixed radius $r > 0$ with orthonormal frames $\{e_{j,l}\}$ inside B_j, and a Lipschitz partition of unity $\{\eta_j\}$ subordinate to $\{B_j\}$ satisfying: $|\nabla e_{j,l}| \leq C_1$ and $|\nabla\eta_j| \leq C_2$, where C_1 and C_2 are finite constants independent of j and l. The covering with the gradient bound on the partition of unity is given in Lemma 3.7 and the uniform control of $|\nabla e_{i,k}| \leq C_1$ is a consequence of the fact that each B_j corresponds to a ball in which we have $\mathrm{C}^{2,\alpha}$ uniform control of the metric. Then, as in Proposition 3.16 in [1], using Lemma 3.10 and Lemma 3.11, we set

$$A_1\varphi = X\varphi + \sum_j L^{B_j}\eta_j\varphi$$

$$A_2\varphi = \sum_j M^{B_j}\eta_j\varphi$$

$$A_3\varphi = \sum_j (N^{B_j} + Z^{B_j})\eta_j\varphi - \sum_j \mathrm{tr}\left(\nabla\eta_j \otimes \varphi\right).$$

It is readily verified that this yields the desired decomposition. □

4. Operator theory and harmonic analysis

Throughout this section, we assume the hypothesis of Theorem 2.1. Moreover, we assume that the reader is familiar with the holomorphic functional calculus via the Riesz-Dunford integral and how to estimate functional calculus of non-smooth operators with harmonic analysis. A brief description of this framework is included in Section 2.1 in [1], but [20] is a more detailed reference.

For $t > 0$, define the operators

$$\begin{aligned} \mathrm{R}_t &= \frac{1}{\mathrm{I} + \mathrm{i}t\mathrm{D}}, \quad \widetilde{\mathrm{R}}_t = \frac{1}{\mathrm{I} + \mathrm{i}t\widetilde{\mathrm{D}}}, \\ \mathrm{P}_t &= \frac{1}{\mathrm{I} + t^2\mathrm{D}^2}, \quad \widetilde{\mathrm{P}}_t = \frac{1}{\mathrm{I} + t^2\widetilde{\mathrm{D}}^2}, \\ \mathrm{Q}_t &= t\mathrm{D}\mathrm{P}_t, \quad \text{and} \quad \widetilde{\mathrm{Q}}_t = t\widetilde{\mathrm{D}}\widetilde{\mathrm{P}}_t. \end{aligned}$$

Due to self-adjointness, we have the bounds

$$\int_0^\infty ||\widetilde{\mathrm{Q}}_t u||^2 \frac{dt}{t} \leq \frac{1}{2}||u||^2 \quad \text{and} \quad \int_0^\infty ||\mathrm{Q}_t u||^2 \frac{dt}{t} \leq \frac{1}{2}||u||^2, \tag{4.1}$$

and

$$\sup_t ||\mathrm{R}_t||, \ \sup_t ||\widetilde{\mathrm{R}}_t||, \ \sup_t ||\mathrm{P}_t||, \ \sup_t ||\widetilde{\mathrm{P}}_t||, \ \sup_t ||\mathrm{Q}_t||, \ \sup_t ||\widetilde{\mathrm{Q}}_t|| \leq \frac{1}{2}. \tag{4.2}$$

Each of these operators are also self-adjoint.

We note the identities

$$\widetilde{\mathrm{R}}_t = \widetilde{\mathrm{P}}_t - \mathrm{i}\widetilde{\mathrm{Q}}_t \quad \text{and} \quad \mathrm{R}_t = \mathrm{P}_t - \mathrm{i}\mathrm{Q}_t, \tag{4.3}$$

as well as

$$\widetilde{\mathrm{R}}_t - \mathrm{R}_t = \widetilde{\mathrm{R}}_t[\mathrm{i}t(\mathrm{D} - \widetilde{\mathrm{D}})]\mathrm{R}_t \quad \text{and} \quad \widetilde{\mathrm{Q}}_t - \mathrm{Q}_t = -\widetilde{\mathrm{P}}_t[t(\widetilde{\mathrm{D}} - \mathrm{D})]\mathrm{P}_t - \widetilde{\mathrm{Q}}_t[t(\widetilde{\mathrm{D}} - \mathrm{D})]\mathrm{Q}_t. \tag{4.4}$$

Using the hypothesis that $\mathrm{D} - \widetilde{\mathrm{D}} = A_1\nabla + \operatorname{div} A_2 + A_3$,

$$\begin{aligned} &||(\widetilde{\mathrm{Q}}_t - \mathrm{Q}_t)f|| \\ &\leq ||\widetilde{\mathrm{P}}_t(tA_1\nabla)\mathrm{P}_t f|| + ||\widetilde{\mathrm{P}}_t(t \ \operatorname{div} A_2)\mathrm{P}_t f|| + ||\widetilde{\mathrm{P}}_t(tA_3)\mathrm{P}_t f|| \\ &\quad + ||\widetilde{\mathrm{Q}}_t(tA_1\nabla)\mathrm{Q}_t f|| + ||\widetilde{\mathrm{Q}}_t(t \ \operatorname{div} \ A_2)\mathrm{Q}_t f|| + ||\widetilde{\mathrm{Q}}_t(tA_3)\mathrm{Q}_t f||. \end{aligned} \tag{4.5}$$

4.1. Reduction to quadratic estimates

The goal of this subsection is to prove the following reduction of the main estimate in Theorem 2.1 to the two quadratic estimates appearing the hypothesis of the following proposition. It is these two quadratic estimates that allow us to access real-variable harmonic analysis methods. The proofs of these estimates are given in Sections 4.2 and 4.3 respectively.

Proposition 4.1. *Suppose that*

$$\int_0^1 ||\widetilde{\mathrm{Q}}_t A_1 \nabla (\mathrm{iI} + \mathrm{D})^{-1} \mathrm{P}_t u||^2 \frac{dt}{t} \le \mathrm{C}_1 ||A||_\infty^2 ||u||^2 \text{and}$$

$$\int_0^1 ||t\widetilde{\mathrm{P}}_t \ \operatorname{div} A_2 \mathrm{P}_t u||^2 \frac{dt}{t} \le \mathrm{C}_2 ||A||_\infty^2 ||u||^2$$

for all $u \in \mathrm{L}^2(\mathcal{V})$. *Then, for* $\omega \in (0, \pi/2)$ *and* $\sigma \in (0, \infty)$, *whenever* $f \in \mathrm{Hol}^\infty(\mathrm{S}^{\mathrm{o}}_{\omega,\sigma})$, *we obtain that*

$$||f(\widetilde{\mathrm{D}}) - f(\mathrm{D})|| \lesssim ||f||_\infty ||A||_\infty$$

where the implicit constant depends on $\mathrm{C}_1, \mathrm{C}_2$ *and* $\mathrm{C}(\mathcal{M}, \mathcal{V}, \mathrm{D}, \widetilde{\mathrm{D}})$.

First, we show that $f(\mathrm{D}) \sim f(\widetilde{\mathrm{D}})$ can be reduced to a quadratic estimate involving the difference of Q_t and $\widetilde{\mathrm{Q}}_t$. This is done via (4.5) and we estimate each of these terms using Proposition 4.5 and Proposition 4.7 in [1]. Unlike in the situation of [1] where the boundary was empty, we use the following trace lemma to control the estimate on the boundary. In what is to follow, $\mathscr{R} : \mathrm{H}^1(\mathcal{V}) \to \mathrm{H}^{\frac{1}{2}}(\mathcal{W})$ is the boundary trace map.

Proposition 4.2. *Let* $\widetilde{\mathrm{U}}_t$ *be one of* $\widetilde{\mathrm{R}}_t, \widetilde{\mathrm{P}}_t$ *or* $\widetilde{\mathrm{Q}}_t$ *and* U_t *be one of* $\mathrm{R}_t, \mathrm{P}_t, \mathrm{Q}_t$. *Then,*

$$\sup_{t>0} ||t\widetilde{\mathrm{U}}_t \ \operatorname{div} A_2 \mathrm{U}_t|| \lesssim ||A_2||_\infty.$$

Proof. Fix $u, v \in \mathrm{C}^\infty_{\mathrm{c}}(\mathcal{V}; \mathcal{B})$ and note that

$$\mathrm{h}(\operatorname{div} A_2 u, v) = \mathrm{h}(A_2 u, \nabla v) + \operatorname{div} W(u, v),$$

where $W(u, v) = (A_2)^j_{ik} u^i \delta_{jl} v^l dx^k$ inside an orthonormal frame, readily checked to be a well-defined covectorfield. By Stokes' theorem,

$$\langle \operatorname{div} A_2 u, v\rangle - \langle A_2 u, \nabla v\rangle = \int_\Sigma \mathrm{g}\big(W(u, v)|_\Sigma, \vec{\mathrm{n}}\big) d\sigma.$$

By Cauchy-Schwartz, compactness of Σ and smoothness of $\vec{\mathrm{n}}$, we obtain that

$$\left| \int_\Sigma \mathrm{g}\big(W(u, v)|_\Sigma, \vec{\mathrm{n}}\big) d\sigma \right| \lesssim ||A_2||_\infty ||\mathscr{R}u|| ||\mathscr{R}v||.$$

Next, note that whenever $\varphi \in \mathcal{D}(\mathrm{D})$ we have that $\varphi \in \mathcal{D}(\operatorname{div} A_2)$ and there exists a sequence $\varphi_n \in \mathrm{C}^\infty_{\mathrm{c}}(\mathcal{V}; \mathrm{B})$ such that $\varphi_n \to \varphi$ in $\mathcal{D}(\mathrm{D})$ by the essential self-adjointness of D. We prove that $\varphi_n \to \varphi$ in $\mathcal{D}(\operatorname{div} A_2)$. To prove this, note that $A_2 : \mathrm{C}^\infty(\mathcal{V}) \to \mathrm{C}^{0,1}(\mathrm{T}^*\mathcal{M} \otimes \mathcal{M})$ and fix a point $x \in \mathcal{M}$, choose an orthonormal frame $\{e_i\}$ for $\mathcal{V}$ and $\{dx^i\}$ for $\mathrm{T}^*\mathcal{M}$ with $\nabla e_i = \nabla dx^i = 0$ at x. For $\psi \in \mathrm{C}^\infty(\mathcal{V})$, $A_2 \psi = (A_2)^{jk}_i \psi^i dx^k \otimes e_j$, and

$$\operatorname{div} A_2 \psi = -\operatorname{tr} \nabla \Big((A_2)^{jk}_i \psi^i dx^k \otimes e_j \Big) = \sum_k \Big(\partial_k (A_2)^j_{ik} \Big) \psi^i + \sum_k (A_2)^j_{ik} \partial_k \psi^i) e_j.$$

Thus, $|\operatorname{div} A_2 \psi|^2 \lesssim ||\nabla A_2||_\infty^2 |\psi|^2 + ||A_2||_\infty^2 |\nabla \psi|^2$. Now, writing $\psi = \varphi_n - \varphi_m$, we obtain that

$$\|\mathrm{div}\, A_2(\varphi_n-\varphi_m)\|^2 \lesssim \|\nabla A_2\|_\infty^2 \|\varphi_n-\varphi_m\|^2 + \|A_2\|_\infty^2 \|\nabla(\varphi_n-\varphi_m)\|^2.$$

Since $\varphi_n \in \mathcal{D}(\mathrm{D})$, we have that $\|\nabla(\varphi_n-\varphi_m)\| \lesssim \|\mathrm{D}(\varphi_n-\varphi_m)\| + \|\varphi_n-\varphi_m\|$. Thus, we have that $\varphi_n \to \varphi$ and $\mathrm{div} A_2\varphi_n \to v$ and since $\mathrm{div} A_2$ is closed as A_2 is bounded, we obtain $\varphi \in \mathcal{D}(\mathrm{div} A_2)$ and $v = \mathrm{div} A_2 \varphi$.

Now, let $u, v \in \mathrm{L}^2(\mathcal{V})$. Since we assume that D is essentially self-adjoint on $\mathrm{C}_\mathrm{c}^\infty(\mathcal{V}; \mathcal{B})$, there exist sequences $u_n, v_m \in \mathrm{C}_\mathrm{c}^\infty(\mathcal{V}; \mathcal{B})$ such that $u_n \to \mathrm{U}_t u$ and $v_m \to \widetilde{\mathrm{U}}_t v$, with convergence in $\mathcal{D}(\mathrm{D}), \mathcal{D}(\nabla)$ and $\mathcal{D}(\mathrm{div} A_2)$ by what we have already established. Thus,

$$\begin{aligned}
|\langle t\widetilde{\mathrm{U}}_t\ \mathrm{div}\, A_2 \mathrm{U}_t u, v\rangle| &= |\lim_{m,n\to\infty} \langle t\ \mathrm{div}\, A_2 u_n, v_m\rangle| \\
&\leq \lim_{m,n\to\infty} |\langle tA_2 u_n, \nabla v_m\rangle| + \lim_{m,n\to\infty} \|A_2\|_\infty t \|\mathscr{R} u_n\| \|\mathscr{R} v_m\| \\
&\lesssim \lim_{m,n\to\infty} \|A_2\|_\infty \|u_n\| \big(\|t\widetilde{\mathrm{D}} v_m\| + t\|v_m\|\big) \\
&\quad + \|A_2\|_\infty t \|\mathscr{R}\mathrm{U}_t u\| \|\mathscr{R}\widetilde{\mathrm{U}}_t v\| \\
&\lesssim \|A_2\|_\infty \big(\|u\| + \sqrt{t}\|\mathscr{R}\mathrm{U}_t u\|\big)\|v\|,
\end{aligned}$$

where the last inequality follows from the standard boundary trace inequality. on $\sqrt{t}\|\mathscr{R}\widetilde{\mathrm{U}}_t v\|$ and from the uniform bounds on $\|t\nabla\widetilde{\mathrm{U}}_t v\| \lesssim \|t\widetilde{\mathrm{D}}\widetilde{\mathrm{U}}_t v\| + \|t\widetilde{\mathrm{U}}_t v\|$ and $t\|\widetilde{\mathrm{U}}_t v\|$. We obtain the conclusion by estimating $\|\mathscr{R}\mathrm{U}_t u\|$ similarly. □

As a consequence of this proposition and (4.5), we obtain

$$\sup_{t\in(0,1]} \|\widetilde{\mathrm{U}}_t - \mathrm{U}_t\| \lesssim \|A\|_\infty.$$

Using this, arguing exactly as in Section 4.2 in [1], we can reduce the required estimate in the conclusion of Proposition 4.1 to proving a quadratic estimate:

$$\int_0^1 \|\big(\widetilde{\mathrm{Q}}_t - \mathrm{Q}_t\big)u\|^2 \frac{dt}{t} \lesssim \|A\|_\infty^2 \|u\|^2$$

for all $u \in \mathrm{L}^2(\mathcal{V})$. From (4.5), we obtain that

$$\begin{aligned}
&\left(\int_0^1 \|\big(\widetilde{\mathrm{Q}}_t - \mathrm{Q}_t\big)u\|^2 \frac{dt}{t}\right)^{\frac{1}{2}} \\
&\qquad \leq \left(\int_0^1 \|\widetilde{\mathrm{P}}_t t A_1 \nabla \mathrm{P}_t u\|^2 \frac{dt}{t}\right)^{\frac{1}{2}} + \left(\int_0^1 \|\widetilde{\mathrm{P}}_t t \mathrm{div} A_2 \mathrm{P}_t u\|^2 \frac{dt}{t}\right)^{\frac{1}{2}} \\
&\qquad + \left(\int_0^1 \|\widetilde{\mathrm{P}}_t t A_3 \mathrm{P}_t u\|^2 \frac{dt}{t}\right)^{\frac{1}{2}} \\
&\qquad + \left(\int_0^1 \|\widetilde{\mathrm{Q}}_t t A_1 \nabla \mathrm{Q}_t u\|^2 \frac{dt}{t}\right)^{\frac{1}{2}} + \left(\int_0^1 \|\widetilde{\mathrm{Q}}_t t \mathrm{div} A_2 \mathrm{Q}_t u\|^2 \frac{dt}{t}\right)^{\frac{1}{2}} \\
&\qquad + \left(\|\widetilde{\mathrm{Q}}_t t A_3 \mathrm{Q}_t u\|^2 \frac{dt}{t}\right)^{\frac{1}{2}}.
\end{aligned} \tag{4.6}$$

Estimating as in Proposition 4.7 in [1], we bound the first, third and sixth term by $||A||_\infty^2||f||^2$. The second and forth terms are controlled by the hypothesis of Proposition 4.1. The only term that remains to be bounded is the penultimate term in this expression for which the estimate in Proposition 4.7 in [1] does not work. The way in which we estimate this term requires a slight excursion into interpolation theory.

Let $\mathrm{H}^1(\mathcal{V})$ denote the first-order Sobolev space on $\mathcal{V}$ and define

$$\mathrm{H}^s(\mathcal{V}) = \left[\mathrm{L}^2(\mathcal{V}), \mathrm{H}^1(\mathcal{V})\right]_{\theta=s},$$

for $s \in [0,1]$ where $[\cdot,\cdot]_\theta$ represents complex interpolation. Also, let

$$\mathrm{H}^s_0(\mathcal{V}) = \overline{\mathrm{C}^\infty_{cc}(\mathcal{V})}^{||\cdot||_{\mathrm{H}^s}}, \mathrm{H}^{-s}(\mathcal{V}) = \mathrm{H}^s_0(\mathcal{V})^*, \quad \text{and} \quad \mathrm{H}^s_{00}(\mathcal{V}) = \left[\mathrm{L}^2(\mathcal{V}), \mathrm{H}^1_0(\mathcal{V})\right]_{\theta=s}.$$

In order to gain an explicit expression for the norms in these interpolation scales, we connect these spaces to domains of operators. Let $\nabla_N = \overline{\nabla_2}$ and $\nabla_D = \overline{\nabla_0}$, where $\nabla_2 : \mathrm{C}^\infty \cap \mathrm{L}^2(\mathcal{V}) \to \mathrm{C}^\infty \cap \mathrm{L}^2(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$ and $\nabla_0 : \mathrm{C}^\infty_{cc}(\mathcal{V}) \to \mathrm{C}^\infty_{cc}(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$. The subscripts "$N$" and "$D$" are chosen for Neumann and Dirichlet respectively since $\mathrm{H}^1(\mathcal{V}) = \mathcal{D}(\nabla_N) = \mathcal{D}(\sqrt{\Delta_N})$ and $\mathrm{H}^1_0 = \mathcal{D}(\nabla_D) = \mathcal{D}(\sqrt{\Delta_D})$, where $\Delta_N = \nabla_N{}^*\nabla_N$ and $\Delta_D = \nabla_D{}^*\nabla_D$. Moreover, $||\cdot||_{\mathrm{H}^1} \simeq ||(\mathrm{I} + \sqrt{\Delta_N})\cdot||$ and $||\cdot||_{\mathrm{H}^1_0} \simeq ||(\mathrm{I} + \sqrt{\Delta_D})\cdot||$.

Consequently, by Theorem 6.6.9 in [21], we have that:

$$\mathrm{H}^s(\mathcal{V}) = \left[\mathrm{L}^2(\mathcal{V}), \mathrm{H}^1(\mathcal{V})\right]_{\theta=s} = \mathcal{D}\left(\left(\mathrm{I} + \sqrt{\Delta_N}\right)^s\right),$$
$$\mathrm{H}^s_{00}(\mathcal{V}) = \left[\mathrm{L}^2(\mathcal{V}), \mathrm{H}^1_0(\mathcal{V})\right]_{\theta=s} = \mathcal{D}\left(\left(\mathrm{I} + \sqrt{\Delta_D}\right)^s\right),$$

and in particular for $s \in [0,1]$,

$$||\cdot||_{\mathrm{H}^s} \simeq \left|\left|\left(\mathrm{I} + \sqrt{\Delta_N}\right)^s \cdot\right|\right| \quad \text{and} \quad ||\cdot||_{\mathrm{H}^{-s}} \simeq \left|\left|\left(\mathrm{I} + \sqrt{\Delta_N}\right)^{-s} \cdot\right|\right|.$$

Since the identity map embeds $\mathrm{H}^1_{00}(\mathcal{V}) \hookrightarrow \mathrm{H}^1(\mathcal{V})$ and $\mathrm{H}^0_{00}(\mathcal{V}) \hookrightarrow \mathrm{H}^0(\mathcal{V})$, we have by interpolation that

$$\mathcal{D}\left(\left(\mathrm{I} + \sqrt{\Delta_D}\right)^s\right) = \mathrm{H}^s_{00}(\mathcal{V}) \hookrightarrow \mathrm{H}^s(\mathcal{V}) = \mathcal{D}\left(\left(\mathrm{I} + \sqrt{\Delta_N}\right)^s\right)$$

for $s \in (0,1)$. Similarly, since $\mathcal{D}(\mathrm{D}) = \mathcal{D}(|\mathrm{D}|)$, where $|\mathrm{D}| = \sqrt{\mathrm{D}^2}$ and $||(\mathrm{I} + |\mathrm{D}|)u|| \simeq ||u|| + ||\mathrm{D}u||$, by the same Theorem 6.6.9 in [21],

$$\left[\mathrm{L}^2(\mathcal{V}), \mathcal{D}(\mathrm{D})\right]_{\theta=s} = \mathcal{D}\left(|\mathrm{D}|^s\right) = \mathcal{D}\left((\mathrm{I} + |\mathrm{D}|)^s\right).$$

The following key result is well known in the case of functions on the upper half space and smooth Euclidean domains by the work of Bergh and Löfström in [22] or Triebel in [23]. The following is a vector bundle version which, to our knowledge, does not seem to have been treated previously in the literature.

Lemma 4.3. *The equality* $\mathrm{H}^s(\mathcal{V}) = \mathrm{H}^s_0(\mathcal{V}) = \mathrm{H}^s_{00}(\mathcal{V})$ *holds whenever* $0 \leq s < 1/2$.

Proof. Now let $U_0 = \mathcal{M} \setminus Z$, where Z is a smooth precompact open neighbourhood of $\Sigma = \partial\mathcal{M}$ and (φ_j, ψ_j, U_j) trivialisations ψ_j inside charts $\varphi_j : U_j \to \mathbb{R}^n_+$ for $j = 1, \ldots, M$, so that $M = \cup_{j=0}^M U_j$. Let $\{\eta_j\}$ be a smooth partition of unity subordinate to $\{U_j\}$. We can choose η_j such that $|\nabla\eta_j| \leq C$ for some $C > 0$.

Define:

$$B_0 = \mathrm{L}^2(\mathcal{V}), \quad A_0 = \mathrm{L}^2(\mathcal{V}) \oplus \mathrm{L}^2\left(\mathbb{R}^n_+, \mathbb{C}^N\right)^M$$
$$B_1 = \mathrm{H}^1(\mathcal{V}), \quad A_1 = \mathrm{H}^1_0(\mathcal{V}) \oplus \mathrm{H}^1\left(\mathbb{R}^n_+, \mathbb{C}^N\right)^M$$
$$B_1^0 = \mathrm{H}^1_0(\mathcal{V}), \quad A_1^0 = \mathrm{H}^1_0(\mathcal{V}) \oplus \mathrm{H}^1_0\left(\mathbb{R}^n_+, \mathbb{C}^N\right)^M.$$

Now, define $S : B_0 \to A_0$ by

$$Su = \left(\eta_0, \psi_1(\eta_1 u) \circ \varphi_1^{-1}, ..., \psi_M(\eta_M u) \circ \psi_M^{-1}\right),$$

with j-th coordinate map extended to 0 outside of the support of η_j, and note S is an injection. Moreover, it is also a map $B_1 \mapsto A_1$ and $B_1^0 \mapsto A_1^0$. Also, define $R : A_0 \to B_0$ by

$$R(u_0, u_1, ..., u_M) = u_0 + \eta_1 \psi_1^{-1}(u_1 \circ \varphi_1) + ... + \eta_M \psi_M^{-1}(u_M \circ \varphi_M).$$

It is also easy to see that this is a map $A_1 \mapsto B_1$ and $A_1^0 \mapsto B_1^0$.

Now, note that $RS = \mathrm{I}$ on $\mathcal{L}(B_j, B_j)$ for $j = 0$, 1 and $\mathcal{L}(B_1^0, B_1^0)$. That is, R is a *retraction* and S is a *coretraction* associated to R. By Theorem (∗) in Section 1.2.4 of [23] we get that S is an isomorphic mapping from $\mathrm{H}^s(\mathcal{V}) \cong W$ for $s \in (0, 1)$ where W is a closed subspace of $\mathrm{H}^s_{00}(\mathcal{V}) \oplus \mathrm{H}^s(\mathbb{R}^n_+, \mathbb{C}^N)^M$. Similarly, we have that $\mathrm{H}^s_{00}(\mathcal{V}) \cong W_0$ with W_0 is a closed subspace of $\mathrm{H}^s_{00}(\mathcal{V}) \oplus \mathrm{H}^s_{00}(\mathbb{R}^n_+, \mathbb{C}^N)^M$. The subspace W is the range of SR restricted to $\mathrm{H}^s_{00}(\mathcal{V}) \oplus \mathrm{H}^s(\mathbb{R}^n_+, \mathbb{C}^N)^M$ and similarly W_0 is the range of SR restricted to $\mathrm{H}^s_{00}(\mathcal{V}) \oplus \mathrm{H}^s_{00}(\mathbb{R}^n_+, \mathbb{C}^N)^M$. But by Theorems 11.1 and 11.2 in [22], we obtain $\mathrm{H}^s_0(\mathbb{R}^n_+, \mathbb{C}^N) = \mathrm{H}^s_{00}(\mathbb{R}^n_+, \mathbb{C}^N) = \mathrm{H}^s(\mathbb{R}^n_+, \mathbb{C}^N)$ for $0 \leq s < 1/2$, and therefore, $W_0 = W$ for $0 \leq s < 1/2$. This shows that $\mathrm{H}^s(\mathcal{V}) = \mathrm{H}^s_{00}(\mathcal{V})$ for $0 \leq s < 1/2$.

To finish off the proof, note that $\|(\mathrm{I} + \sqrt{\Delta_N})u\| \lesssim \|(\mathrm{I} + \sqrt{\Delta_D})u\|$ so through interpolation we get $\|(\mathrm{I} + \sqrt{\Delta_N})^s u\| \lesssim \|(\mathrm{I} + \sqrt{\Delta_D})^s u\|$. Since $\mathrm{C}^\infty_{cc}(\mathcal{V})$ is dense in $\mathrm{H}^s_{00}(\mathcal{V}) = \mathcal{D}((\mathrm{I} + \sqrt{\Delta_D})^s)$, we have that $\mathrm{H}^s_{00}(\mathcal{V}) \hookrightarrow \mathrm{H}^s_0(\mathcal{V})$. But we have $\mathrm{H}^s_0(\mathcal{V}) \hookrightarrow \mathrm{H}^s(\mathcal{V})$ and since we have already proved $\mathrm{H}^s(\mathcal{V}) = \mathrm{H}^s_{00}(\mathcal{V})$ for $0 \leq s < 1/2$, we obtain the conclusion. □

With the aid of this lemma, we obtain the following.

Proposition 4.4. *The quadratic estimate*

$$\int_0^1 \|\widetilde{\mathrm{Q}}_t t \ \mathrm{div}\, A_2 \mathrm{Q}_t f\|^2 \frac{dt}{t} \lesssim \|f\|^2$$

holds for $f \in \mathrm{L}^2(\mathcal{V})$.

Proof. Fix $u \in \mathrm{L}^2(\mathcal{V})$ and estimate

$$\langle \widetilde{\mathrm{Q}}_t t \ \mathrm{div}\, A_2 \mathrm{Q}_t f, u\rangle = -\langle A_2 \mathrm{Q}_t f, t\nabla \widetilde{\mathrm{Q}}_t u\rangle + t\langle A_2 \mathscr{R} \mathrm{Q}_t f, \mathscr{R} \widetilde{\mathrm{Q}}_t u\rangle_{\mathrm{L}^2(\mathcal{W})}.$$

It is easy to see that

$$|\langle A_2 \mathrm{Q}_t f, t\nabla \widetilde{\mathrm{Q}}_t u\rangle| \lesssim \|A_2\|_\infty \|u\| \|\mathrm{Q}_t f\|,$$

so it remains to consider the boundary term. Note that

$$|t\langle A_2 \mathscr{R}Q_t f, \mathscr{R}\widetilde{Q}_t u\rangle_{L^2(\Sigma)}| \lesssim \|A_2\|_\infty t\|\mathscr{R}Q_t f\|_{L^2(\mathcal{W})}\|\mathscr{R}\widetilde{Q}_t u\|_{L^2(\mathcal{W})}.$$

By the standard boundary trace inequality, we obtain that $\sqrt{t}\|\mathscr{R}\widetilde{Q}_t u\|_{L^2(\mathcal{W})} \lesssim \|u\|$.

To bound $Q_t f$, let $\vec{N}$ be an extension of the normal vectorfield $\vec{n}$ on a compact neighbourhood around Σ. Then,

$$\begin{aligned} t\|\mathscr{R}Q_t f\|^2_{L^2(\mathcal{W})} &= t\int_{\mathcal{M}} \mathrm{div}(|Q_t f|^2 \vec{N})\,d\mu \\ &\lesssim t\int_{\mathcal{M}} \mathrm{Re}\ g(\nabla_{\vec{N}} Q_t f, Q_t f)\,d\mu + t\|Q_t f\|^2 \\ &\lesssim t|\langle \nabla_{\vec{N}} Q_t f, Q_t f\rangle| + t\|Q_t f\|^2. \end{aligned}$$

On fixing $0 < s < 1/2$, we note that

$$|\langle \nabla_{\vec{N}} Q_t f, Q_t f\rangle| \lesssim \|\nabla_{\vec{N}} Q_t f\|_{H^{-s}}\|Q_t f\|_{H^s}, \tag{4.7}$$

Now, note that $\nabla_{\vec{N}} : H^1(\mathcal{V}) \to L^2(\mathcal{V})$ and on defining $(\nabla_{\vec{N}} u)(v) = -\langle u, \nabla_{\vec{N}} v\rangle$ for $v \in C^\infty_c(\mathcal{V})$, we obtain that $\nabla_{\vec{N}} : L^2(\mathcal{V}) \to H^1_0(\mathcal{V})^* = H^{-1}(\mathcal{V})$ boundedly. By interpolation, we obtain that $\nabla_{\vec{N}} : [H^1(\mathcal{V}), L^2(\mathcal{V})]_{\theta=s} \to [L^2(\mathcal{V}), H^{-1}(\mathcal{V})]_{\theta=s}$ boundedly. Note, however, that

$$\left[H^1(\mathcal{V}), L^2(\mathcal{V})\right]_{\theta=s} = \left[L^2(\mathcal{V}), H^1(\mathcal{V})\right]_{\theta=1-s} = H^{1-s}(\mathcal{V}),$$

and that

$$\left[L^2(\mathcal{V}), H^{-1}(\mathcal{V})\right]_{\theta=s} = \left(\left[L^2(\mathcal{V}), H^1_0(\mathcal{V})\right]_{\theta=s}\right)^* = H^s_{00}(\mathcal{V})^* = H^s_0(\mathcal{V})^* = H^{-s}(\mathcal{V}),$$

where we have used that $L^2(\mathcal{V})$ is reflexive and Corollary 4.5.2 in [22] in the first equality and that $s < 1/2$ and Lemma 4.3 in the penultimate equality. On combining these facts, we obtain that

$$\|\nabla_{\vec{N}} Q_t f\|_{H^{-s}} \lesssim \|Q_t f\|_{H^{1-s}}.$$

Moreover, since $\mathcal{D}(|D|) \hookrightarrow H^1(\mathcal{V})$ and $\mathcal{D}(|D|^0) = L^2(\mathcal{V}) \hookrightarrow H^0(\mathcal{V}) = L^2(\mathcal{V})$, we have $\mathcal{D}(|D|^q) \hookrightarrow H^q(\mathcal{V})$ for $q \in [0,1]$ by interpolation and hence,

$$t^q\|Q_t f\|_{H^q} \lesssim \|t^q(I + |D|^q)Q_t f\| \le \|\psi_q(tD)f\| + \|Q_t f\|,$$

where $\psi_q(\zeta) = \zeta|\zeta|^q(1+\zeta^2)^{-1}$. Thus,

$$\begin{aligned} &t|\langle \nabla_{\vec{N}} Q_t f, Q_t f\rangle| \\ &\quad\lesssim \left(t^{1-s}\|Q_t f\|_{H^{1-s}}\right)\left(t^s\|Q_t f\|_{H^s}\right) \lesssim \|\psi_{1-s}(tD)f\|^2 + \|\psi_s(tD)f\|^2 + \|Q_t f\|^2, \end{aligned}$$

and therefore,

$$t\|\mathscr{R}Q_t f\|_{L^2(\mathcal{W})} \lesssim \|\psi_{1-s}(tD)f\|^2 + \|\psi_s(tD)f\|^2 + (1+t)\|Q_t f\|^2.$$

Noting that

$$\int_0^1 \|\psi_q(tD)f\|^2 \frac{dt}{t} \le C_q\|f\|^2$$

for $q \in [0,1)$ completes the proof. □

Remark 4.5. The equation (4.7) demonstrates the necessity of the interpolation methods since we can only conclude the desired quadratic estimates provided a derivative of order strictly less than 1 is applied to $\mathrm{Q}_t f$.

4.2. Harmonic analysis I

In this subsection, on drawing from the estimates in Section 5 in [1], we demonstrate how to handle the first quadratic estimate term

$$\int_0^1 \|\widetilde{\mathrm{Q}}_t A_1 \nabla (\mathrm{iI} + \mathrm{D})^{-1} \mathrm{P}_t f\|^2 \frac{dt}{t} \lesssim \|A\|_\infty^2 \|f\|^2$$

appearing in the hypothesis of Proposition 4.1. In order to avoid repetition, we encourage the reader to keep a copy of [1] handy to navigate through the remainder of this article.

The following is an itemisation of the notation that we will require from Section 5 of [1]:

- Dyadic cubes $\{Q_\alpha^k \subset \mathcal{M} : \alpha \in I_k, k \in \mathbb{N}\}$, with centres $z_\alpha^k \in Q_\alpha^k$, where $\cup_k Q_\alpha^k$ cover $\mathcal{M}$ almost everywhere, and when $\beta > \alpha$, $Q_\alpha^k \cap Q_\beta^l = \emptyset$ or $Q_\alpha^k \subset Q_\beta^l$. The cubes are of a fixed "length" $\delta \in (0, 1)$, and a δ^j cube contains an $a_0 \delta^j$ ball and has diameter at most $C_1 \delta^j$. The length of a cube Q is denoted $\ell(Q)$. The constant $\eta > 0$ is an exponent that measures smallness of the volume toward the edge of a cube with constant $C_2 > 0$. See Theorem 5.1 in [1].
- The scale is defined as $\mathrm{t_S} = \delta^J$ where $C_1 \delta^J \leq \rho/5$, with $\rho = \max\{\rho_{\mathrm{T}^*\mathcal{M}}, \rho_{\mathcal{V}}\}$, the maximum of the GBG radii of $\mathrm{T}^*\mathcal{M}$ and $\mathcal{V}$.
- The collection of dyadic cubes $\mathscr{Q}^j$, $\mathscr{Q} = \cup_{j \geq \mathrm{J}} \mathscr{Q}^j$, and $\mathscr{Q}_t$ for $t \leq \mathrm{t_S}$.
- The unique ancestor $\hat{Q} \in \mathscr{Q}^J$ for a dyadic cube $\mathscr{Q}$, the set of *GBG coordinates* $\mathscr{C}$, which for a cube $Q \in \mathscr{Q}^j$ is the GBG trivialisation pertaining to the unique GBG ball containing the cube in $\mathscr{Q}^J$ containing Q, and *dyadic GBG coordinates* $\mathscr{C}_\mathrm{J}$ which is the restriction of this GBG ball to the cube which contains it.
- The *cube integral* $\mathrm{B}(x_{\hat{Q}}, \rho) \times \mathscr{Q} \ni (x, Q) \mapsto (\int_Q \cdot)(x)$ defined on $\mathrm{L}^1_{\mathrm{loc}}(\mathcal{V})$ by

  $$\left(\int_Q u\right)(x) = \left(\int_Q u^i(y) d\mu(y)\right) e_i(x)$$

  where e_i is the GBG coordinates of Q, and *cube average* $u_Q = ⨍_Q u$ inside the GBG coordinate ball of Q and 0 outside it.

- For $t > 0$, the *dyadic averaging operator* $\mathbb{E}_t : \mathrm{L}^1_{loc}(\mathcal{V}) \to \mathrm{L}^1_{loc}(\mathcal{V})$ given by $\mathbb{E}_t(x) = (⨍_Q u)(x)$ where $x \ni Q$.
- For a $w = w^i e_i^{\mathbb{C}^N} \in \mathbb{C}^N$, the locally constant extension inside the GBG coordinates of Q are given by $\omega^c(x) = w^i e_i(x)$ and zero outside of this coordinate ball.
- Given a t-uniformly bounded family of operators $\mathbf{Q}_\mathrm{t}$, define the principal part $\gamma_t^{\mathbf{Q}}(x) : \mathbb{C}^N \cong \mathcal{V}_x \to \mathcal{V}_x$ of $\mathbf{Q}_t$ by by $\gamma_t^{\mathbf{Q}}(x) w = (\mathbf{Q}_t \omega^c)(x)$.

The following is a key lemma that is necessary in order to adapt the arguments of Section 5 of [1] to our manifold with boundary. It allows us to ensure that we can use a cut-off that restricts the estimates *away* from the boundary.

Lemma 4.6. *There exist constants $k_0, \widetilde{\eta}, \widetilde{C}_3 > 0$ such that for all cubes $Q \in \mathcal{Q}^k$ with $k > k_0$ and $\bar{Q} \cap \Sigma \neq \emptyset$, we have*

$$\mu\{x \in Q : \rho(x, \Sigma) \leq s\ell(Q)\} \leq \widetilde{C}_3 s^{\widetilde{\eta}} \mu(Q).$$

In particular, for every $Q \in \mathcal{Q}^k$ with $k > k_0$,

$$\mu\{x \in Q : \rho(x, \mathcal{M} \setminus (Q \setminus \Sigma)) \leq s\ell(Q)\} \leq \widetilde{C}_3 s^{\widetilde{\eta}} \mu(Q).$$

The constants $\widetilde{\eta}$ and $\widetilde{C}_3$ depends on η, a_0 and C_1 from Theorem 5.1 *in* [1].

Proof. Let $Z = \{x \in \mathcal{M} : \rho(x, \Sigma) \leq \varepsilon\}$ with $\varepsilon < 1$ chosen sufficiently small so that Z is a smooth compact submanifold of $\mathcal{M}$ with smooth boundary Σ. Let $\widetilde{Z}$ be the smooth compact manifold without boundary obtained by taking two copies of Z and identifying the boundaries, and extending the metric appropriately. This metric is C^0 and there exists a smooth C^∞ metric G-close to g for some $G \geq 1$. Consequently, without loss of generality, we assume that the metric extension is smooth. Let $k_\Sigma = \mathrm{inj}(\widetilde{Z}) > 0$.

By the compactness of $\widetilde{Z}$, we use Theorem 1.2 in [19] to obtain $C_\Sigma \geq 1$ such that for each $x \in \widetilde{Z}$, $(\psi_x, B(\frac{1}{2}k_\Sigma, x))$ is a coordinate chart with

$$C_\Sigma^{-1} |u|_{\psi_x^* \delta(y)} \leq |u|_{\mathrm{g}(y)} \leq C_\Sigma |u|_{\psi_x^* \delta(y)},$$

for each $y \in B(\frac{1}{2}k_\Sigma, x)$, and where δ is the Euclidean metric in that chart. In particular, since $Z \subset \widetilde{Z}$ and the topology of Z is the subspace topology inherited from $\widetilde{Z}$, we get that this holds for balls $B(x, r)$ in Z as well. From this, inside $(\psi_x, B(\frac{1}{2}k_\Sigma, x))$, on letting $\rho^*(x, y) = |\psi_x(x) - \psi_y(y)|$ and $\mathcal{L}^* = \psi_x^* \mathcal{L}$,

$$C_\Sigma^{-1} \ \rho^*(x, y) \leq \ \rho(x, y) \leq C_\Sigma \ \rho^*(x, y) \quad \text{and} \quad C_\Sigma^{-\frac{n}{2}} d\mathcal{L}^* \leq d\mu \leq C_\Sigma^{\frac{n}{2}} d\mathcal{L}^*. \tag{4.8}$$

Now, fix $k_0 > 0$ such that so that $C_1 \delta^{k_0} < \frac{1}{10} k_\Sigma$. Then, for all $k > k_0$, whenever $Q \in \mathcal{Q}^k$, we have that $Q \subset B(x_Q, \frac{1}{2}k_\Sigma)$, which corresponds to a coordinate system with control on the metric and measure as we have describe before.

Fix such a cube $Q \in \mathcal{Q}^k$ and define $Q_{\Sigma,s} = \{x \in Q : \rho(x, \Sigma) \leq s\ell(Q)\}$ and note that on using (4.8),

$$\psi_Q(Q_{\Sigma,s}) \subset E_{\Sigma,s} = \left\{x \in \psi_Q(Q) : \rho_{\mathbb{R}^n}\left(x, \mathbb{R}^{n-1} \cap \overline{\psi_Q(Q)} \leq C_\Sigma s \delta^k\right\}.$$

Similarly, we have that $\psi_Q(B(x_Q, C_1\delta^k)) \subset B_{\mathbb{R}^n}(\bar{x}_Q, C_\Sigma C_1 \delta^k) \subset \mathrm{Box}_{\mathbb{R}^n}(\bar{x}_Q, C_\Sigma C_1 \delta^k)$ where $\bar{x}_Q = \psi_Q(x_Q)$ and $\mathrm{Box}_{\mathbb{R}^n}(x, l)$ is a Euclidean box centred at x of length l. Then,

$$\begin{aligned}\mathcal{L}(E_{\Sigma,s}) &\leq \mathcal{L}^{n-1}\left(\mathbb{R}^n \cap \mathrm{Box}_{\mathbb{R}^n}\left(\bar{x}_Q, C_\Sigma C_1 \delta^k\right)\right) \times C_\Sigma s \delta^k \\ &\leq \left(C_\Sigma C_1 \delta^k\right)^{n-1} \times C_\Sigma s \delta^k = C_\Sigma^n C_1^{n-1} \delta^{nk} s.\end{aligned}$$

Similarly, we have that $\psi_Q\left(B\left(x_Q, a_0\delta^k\right)\right) \supset B_{\mathbb{R}^n}\left(\bar{x}_Q, C_\Sigma^{-1} a_0 \delta^k\right)$, and

$$\frac{\mu(Q_{\Sigma,s})}{\mu(Q)} \leq \frac{\mu(Q_{\Sigma,s})}{\mu(B(x_Q, a_0\delta^k)} \leq \frac{C_\Sigma^{\frac{n}{2}} \mathscr{L}(E_{\Sigma,s})}{C_\Sigma^{-\frac{n}{2}} \mathscr{L}(B_{\mathbb{R}^n}(\bar{x}_Q, C_\Sigma^{-1} a_0 \delta^k))}$$
$$\leq C_\Sigma^n \frac{C_\Sigma^n C_1^{n-1} \delta^{nk} s}{\omega_n (C_\Sigma^{-1} a_0 \delta^k)^n} = \frac{C_\Sigma^{3n} C_1^{n-1}}{\omega_n a_0^n} s,$$

where the first estimate follows from Theorem 5.1 (v) in [1], the second estimate from our previous calculation combined with (4.8), and where ω_n is the volume of the ball of unit radius in $\mathbb{R}^n$.

Set $\widetilde{\eta} = \max\{1, \eta\}$ and $\widetilde{C}_3 = \max\left\{C_3, \frac{C_\Sigma^{3n} C_1^{n-1}}{\omega_n a_0^n}\right\}$, and noting

$$\{x \in Q : \ \rho(x, \mathcal{M} \setminus (Q \setminus \Sigma)) \leq s\ell(Q)\} = \{x \in Q : \ \rho(x, \mathcal{M} \setminus Q) \leq s\ell(Q)\} \cup Q_{\Sigma,s},$$

completes the proof. $\square$

Proposition 4.7. *The quadratic estimate*

$$\int_0^1 ||\widetilde{\mathrm{Q}}_t A_1 \nabla (\mathrm{iI} + \mathrm{D})^{-1} \mathrm{P}_t u||^2 \frac{dt}{t} \lesssim ||A||_\infty^2 ||u||^2$$

holds for all $u \in \mathrm{L}^2(\mathcal{V})$, with the implicit constant depending on $\mathrm{C}(\mathcal{M}, \mathcal{V}, \mathrm{D}, \widetilde{\mathrm{D}})$.

Proof. We split the estimate as follows:

$$\begin{aligned}\int_0^1 ||\widetilde{\mathrm{Q}}_t A_1 \nabla (\mathrm{iI} + \mathrm{D})^{-1} \mathrm{P}_t u||^2 \frac{dt}{t} \lesssim & \int_0^1 \left|\left|\left(\widetilde{\mathrm{Q}}_t - \gamma_t \mathbb{E}_t\right) \mathrm{A}_1 \nabla (\mathrm{iI} + \mathrm{D})^{-1} \mathrm{P}_t u\right|\right|^2 \frac{dt}{t} \\ & + \int_0^1 ||\gamma_t \mathbb{E}_t \mathrm{A}_1 \nabla (\mathrm{iI} + \mathrm{D})^{-1} (\mathrm{I} - \mathrm{P}_t) u||^2 \frac{dt}{t} \\ & + \int_0^1 ||\gamma_t \mathbb{E}_t \mathrm{A}_1 \nabla (\mathrm{iI} + \mathrm{D})^{-1} u||^2 \frac{dt}{t}.\end{aligned}$$

Now, we note that the off-diagonal decay given in Lemma 5.9 in [1] is valid for our operator $\widetilde{\mathrm{Q}}_t A_1$ due to the local boundary conditions encoded in assumption (A7). Thus, we can apply Propositions 5.4, Lemma 5.8 and Proposition 5.12 in [1] to estimate the terms appearing in this decomposition. We give a brief description of how this is done.

The first term is estimated by using an argument similar to the proof of Proposition 5.4 and Theorem 2.4 in [1], with $\mathcal{W} = \mathrm{T}^*\mathcal{M} \otimes \mathcal{V}$. It suffices to note that since $||u||_{\mathrm{D}} \simeq ||u||_{\mathrm{H}^1}$ for $u \in \mathcal{D}(\mathrm{D})$, this argument can be run in verbatim. It simply remains to prove $||\nabla S u|| \lesssim ||u||_{\mathrm{H}^1}$ for $S = \nabla(\mathrm{iI} + \mathrm{D})^{-1}$. This argument is included in the proof of Theorem 2.4 in [1] on noting that the argument runs in verbatim due to assumption (A9).

For the middle term in the estimate, we use the argument in proving Proposition 5.10 in [1]. This argument is straightforward from establishing the *cancellation lemma*, Lemma 5.8 in [1]. To prove this lemma, we note that for each dyadic cube Q, and for each $u \in \mathcal{D}(\mathrm{D})$ with spt $u \subset Q \cap \mathring{\mathcal{M}}$, we have that

$$\left|\int_Q \mathrm{D}u \, d\mu\right| \lesssim \mu(Q)^{\frac{1}{2}} ||u|| \quad \text{and} \quad \left|\int_Q \nabla u \, d\mu\right| \lesssim \mu(Q)^{\frac{1}{2}} ||u||,$$

where the implicit constants depends on $\mathrm{C}(\mathcal{M}, \mathcal{V}, \mathrm{D}, \widetilde{\mathrm{D}})$. On coupling these estimates with Lemma 4.6, we obtain the statement of Lemma 5.8 in [1] in our present context.

The last term is obtained by a straightforward application of Proposition 5.12 in [1]. □

4.3. Harmonic analysis II

In this subsection, we prove the remaining estimate

$$\int_0^1 ||t\widetilde{\mathrm{P}}_t \mathrm{div} A_2 \mathrm{P}_t u||^2 \frac{dt}{t} \lesssim ||A||_\infty^2 ||u||^2$$

for all $u \in \mathrm{L}^2(\mathcal{V})$. It is in the proof of this estimate where the main novelty of the harmonic analysis in this article can be found. A key difficulty here is that the off-diagonal decay - and even L^2-boundedness - of $t\widetilde{\mathrm{P}}_t\mathrm{div}A_2$, which holds when $\mathcal{M}$ has no boundary, is not valid due to the fact that A_2 does not preserve boundary conditions. Despite this obstacle, on considering the operator $t\widetilde{\mathrm{P}}_t\mathrm{div}A_2\mathrm{P}_t$ instead as a whole, we are able to prove the required quadratic estimate. Our approach here is motivated by a similar argument in [7] by Auscher, Axelsson (Rosén) and Hofmann.

For the remainder of this subsection, let

$$\Theta_t = t\widetilde{\mathrm{P}}_t \mathrm{div} A_2 \mathrm{P}_t$$

and let γ_t denote the principal part of Θ_t we recall is $\gamma_t^\Theta(x)w = (\Theta_t\omega^c)(x)$, where ω^c is the constant section related to $w \in \mathcal{V}_x \cong \mathbb{C}^N$.

Lemma 4.8. *The operators Θ_t are uniformly bounded in $t > 0$ and have the off-diagonal decay estimate: there exists $C_\Theta > 0$ such that, for each $M > 0$, there exists a constant $C_{\Delta,M} > 0$ with*

$$||\chi E\Theta_t(\chi Fu)||_{\mathrm{L}^2(\mathcal{V})} \le C_{\Delta,M}||A||_\infty \left\langle \frac{\rho(E,F)}{t} \right\rangle^{-M} \exp\left(-C_\Theta \frac{\rho(E,F)}{t}\right) ||\chi Fu||_{\mathrm{L}^2(\mathcal{V})},$$

for every Borel set $E, F \subset \mathcal{M}, u \in \mathrm{L}^2(\mathcal{V})$, and where $\langle a \rangle = \max\{1, a\}$.

Proof. Uniform bounds for Θ_t were proved in Proposition 4.2. Building on this, we prove the off-diagonal estimates in the conclusion by reduction to corresponding such estimates for the resolvents R_t and $\widetilde{\mathrm{R}}_t$, which are immediate by replicating the argument of Lemma 5.3 in [9] in light of (A7).

Given $E, F \subset \mathcal{M}$ Borel with $\rho(E,F) > 0$, pick $\eta \in \mathrm{C}^\infty(\mathcal{M})$ such that $\eta(x) = 1$ when $\rho(x,E) < 1/3\ \rho(E,F)$ and $\eta(x) = 0$ when $\rho(x,F) < 1/3\ \rho(E,F)$ so that $||\nabla\eta||_\infty \lesssim 1/\rho(E,F)$. It suffices to prove the required estimates for $\widetilde{\mathrm{R}}_t t\mathrm{div}A_2\mathrm{R}_t$ since by replacing t by $-t$ in the estimates below and noting $\mathrm{P}_t = (\mathrm{R}_t + \mathrm{R}_{-t})/2$ and similarly $\widetilde{\mathrm{P}}_t = (\widetilde{\mathrm{R}}_t + \widetilde{\mathrm{R}}_{-t})/2$ yields the bound for Θ_t. Now, note that

$$||\chi_E\widetilde{\mathrm{R}}_t t\mathrm{div}A_2\mathrm{R}_t(\chi_F u)|| = ||\chi_E\left[\eta, \widetilde{\mathrm{R}}_t t\mathrm{div}A_2\mathrm{R}_t\right]\chi_F u||$$

and

$$\begin{aligned}&\left[\eta, \widetilde{\mathrm{R}}_t t\mathrm{div}A_2\mathrm{R}_t\right] \\ &\qquad = -\widetilde{\mathrm{R}}_t\left[\eta, \mathrm{i}t\widetilde{\mathrm{D}}\right]\widetilde{\mathrm{R}}_t t\mathrm{div}A_2\mathrm{R}_t + \widetilde{\mathrm{R}}_t[\eta, t\mathrm{div}]A_2\mathrm{R}_t - \left(\widetilde{\mathrm{R}}_t t\mathrm{div}A_2\mathrm{R}_t\right)[\eta, \mathrm{i}t\mathrm{D}]\widetilde{\mathrm{R}}_t.\end{aligned}$$

Since $[\eta, \widetilde{\mathrm{D}}], [\eta, \mathrm{div}], [\eta, \mathrm{D}]$ are multiplication operators whose L^∞ norm is bounded by $||\nabla\eta||_\infty$ and supported on

$$G = \left\{ x \in \mathcal{M} : \ \rho(x,E) \geq \frac{1}{3}\ \rho(E,F) \text{ and } \rho(x,F) \geq \frac{1}{3}\ \rho(E,F) \right\},$$

we obtain the conclusion from off-diagonal estimates for $\widetilde{\mathrm{R}}_t : \mathrm{L}^2(G;\mathcal{V}) \to \mathrm{L}^2(E;\mathcal{V})$ and $\mathrm{R}_t : \mathrm{L}^2(F;\mathcal{V}) \to \mathrm{L}^2(G;\mathcal{V})$, and from uniform bounds on $\widetilde{\mathrm{R}}_t t\mathrm{div} A_2 \mathrm{R}_t$ from Proposition 4.2. □

Next, we split the required estimate in the following way:

$$\int_0^1 ||\Theta_t u||^2 \frac{dt}{t} \leq \int_0^1 ||\Theta_t(\mathrm{I}-\mathrm{P}_t)u||^2 \frac{dt}{t} + \int_0^1 ||(\Theta_t - \gamma_t \mathbb{E}_t)\mathrm{P}_t u||^2 \frac{dt}{t} + \int_0^1 ||\gamma_t \mathbb{E}_t(\mathrm{P}_t - \mathrm{I})u||^2 \frac{dt}{t} + \int_0^1 ||\gamma_t \mathbb{E}_t u||^2 \frac{dt}{t} \tag{4.9}$$

The first three terms to the right of this expression can be handled relatively easily as the following lemma demonstrates.

Lemma 4.9. *We have that:*

$$\int_0^1 ||\Theta_t(\mathrm{I}-\mathrm{P}_t)u||^2 \frac{dt}{t} + \int_0^1 ||(\Theta_t - \gamma_t \mathbb{E}_t)\mathrm{P}_t u||^2 \frac{dt}{t} + \int_0^1 ||\gamma_t \mathbb{E}_t(\mathrm{P}_t - \mathrm{I})u||^2 \frac{dt}{t} \lesssim ||A||_\infty^2 ||u||^2.$$

Proof. For the first term, we estimate by noting that

$$\Theta_t(\mathrm{I}-\mathrm{P}_t) = \Theta_t t\mathrm{D}\mathrm{Q}_t = \left(t\widetilde{\mathrm{P}}_t \mathrm{div} A_2 \mathrm{Q}_t\right)\mathrm{Q}_t,$$

we obtain the required quadratic estimate using Proposition 4.2 to assert uniform bounds for $t\widetilde{\mathrm{P}}_t \mathrm{div} A_2 \mathrm{Q}_t$ and by noting that Q_t satisfies quadratic estimates (4.1). The two remaining estimates are handled via Propositions 5.4 and Proposition 5.10 in [1] with $S = \mathrm{I}$. The versions of these propositions in our current context can be obtained exactly the way described in the proof of Proposition 4.7. □

Thus, we have left with the last term in this expression, which we reduce to a Carleson measure estimate. That is, by Carleson's Theorem, the estimate of this term is obtained by proving that

$$d\nu(x,t) = |\gamma_t(x)|^2 \frac{d\mu(x)dt}{t}$$

is a Carleson measure. This is obtained if we prove for each cube $Q \in \mathcal{Q}$, and for Carleson regions $\mathrm{R}_Q = Q \times (0, \ell(Q))$,

$$\iint_{\mathrm{R}_Q} |\gamma_t(x)|^2 \frac{d\mu(x)dt}{t} \lesssim ||A||_\infty^2 \mu(Q). \tag{4.10}$$

The estimate we perform here is more intricate and involved than the Carleson measure estimate in Proposition 5.12 in [1], and we provide full details. First, observe the following important reduction.

Lemma 4.10. *Suppose that for every cube $Q \in \mathcal{Q}$ with $\ell(Q) \le \rho(Q, \Sigma)$ the Carleson estimate (4.10) holds. Then, (4.10) holds for every cube $Q \in \mathcal{Q}$.*

Proof. Fix $Q \in \mathcal{Q}^j$, with $j = \max\{k_0, J\}$ (with k_0 coming from Lemma 4.6), and define the two sets

$$\mathcal{A} = \left\{Q' \in \mathcal{Q} : Q' \subset Q \text{ and } \rho(Q', \Sigma) \ge \ell(Q')\right\},$$
$$\mathcal{B} = \left\{Q' \in \mathcal{Q} : Q' \subset Q \text{ and } \rho(Q', \Sigma) < \ell(Q')\right\}.$$

Now, consider the dyadic Whitney region $\mathcal{W}_{Q'} = Q' \times (\delta\ell(Q'), \ell(Q))$ so that

$$\mathrm{R}_Q = \left(\bigcup_{Q' \in \mathcal{A}} \mathcal{W}_{Q'}\right) \cup \left(\bigcup_{Q' \in \mathcal{B}} \mathcal{W}_{Q'}\right).$$

Note that $Q'' \subset Q'$ and $Q' \in \mathcal{A}$ implies that $Q'' \in \mathcal{A}$. Setting $\mathcal{A}_{\max}$ to be the maximal cubes in $\mathcal{A}$, we obtain that

$$\bigcup_{Q' \in \mathcal{A}} \mathcal{W}_{Q'} = \bigcup_{Q' \in \mathcal{A}_{\max}} \mathrm{R}_{Q'}.$$

On using the hypothesis, we obtain that

$$\sum_{Q' \in \mathcal{A}_{\max}} \iint_{\mathrm{R}_{Q'}} |\gamma_t|^2 \frac{d\mu dt}{t} \lesssim ||A||_\infty^2 \sum_{Q' \in \mathcal{A}_{\max}} \mu(Q') \lesssim ||A||_\infty^2 \mu(Q)$$

by the disjointedness of the cubes in $\mathcal{A}_{\max}$.

Next, note that from the off-diagonal decay of Θ_t, we obtain that $\Theta_t : \mathrm{L}^\infty(\mathcal{V}) \to \mathrm{L}^2_{\mathrm{loc}}(\mathcal{V})$, and reasoning as in Section 5.2 in [1], which comes from Corollary 5.3 in [8], we have that

$$\int_{Q'} |\gamma_t|^2 d\mu \lesssim \mu(Q')$$

and therefore,

$$\int_{Q'} |\gamma_t|^2 \frac{d\mu dt}{t} \lesssim \int_{\frac{\ell(Q')}{2}}^{\ell(Q')} \mu(Q') \frac{dt}{t} \lesssim \mu(Q').$$

Now, fix $k > j$ and note that $\delta^k \le \ell(Q)$ and for every cube $Q' \in \mathcal{B}_k = \mathcal{B} \cap \mathcal{Q}^k$, we have that $Q' \subset \{x \in Q : \ \rho(x, \Sigma) \le (C_1 + 1)\delta^k\}$. On invoking Lemma 4.6 with $s = \delta^k (C_1 + 1)\ell(Q)^{-1}$, we obtain that

$$\mu(Q') \lesssim \mu\{x \in Q : \ \rho(x, \Sigma) \le s\ell(Q)\} \lesssim \frac{\delta^{k\tilde{\eta}}}{\ell(Q)^{\tilde{\eta}}} \mu(Q) \lesssim \mu(Q),$$

where the second inequality follows from $\delta^k \le \ell(Q)$. Note now that if $Q' \in \mathcal{B}$ and $Q'' \subsetneq Q'$ then $\ell(Q'') \le \delta\ell(Q')$ and therefore,

$$\mathcal{W}_{Q'} = Q' \times \left(\delta\ell(Q'), \ell(Q')\right) \cap Q'' \times \left(\delta\ell(Q''), \ell(Q'')\right) = \mathcal{W}_{Q''} = \emptyset,$$

and therefore

$$\sum_{Q' \in \mathcal{B}} \iint_{\mathcal{W}_{Q'}} |\gamma_t|^2 \frac{d\mu dt}{t} \lesssim \sum_{k > j} \sum_{Q' \in \mathcal{B}_k} \iint_{\mathcal{W}_{Q'}} |\gamma_t|^2 \frac{d\mu dt}{t} \lesssim \mu(Q),$$

which completes the proof. □

We finally prove (4.10) for the remaining cubes Q bounded away from Σ.

Proposition 4.11. *Suppose that* $\rho(Q,\Sigma) \geq \ell(Q)$*. Then, the Carleson measure estimate (4.10) holds.*

Proof. Fix $w \in \mathbb{C}^N$, let $f_Q : \mathcal{M} \to [0,1]$ with $\operatorname{spt} f_Q$ compact, and $f_Q = 1$ on Q and 0 outside $\mathrm{B}(x_Q, 2\ell(Q))$ with $|\nabla f_Q| \lesssim \ell(Q)^{-1}$. Define $w_Q(x) = f_Q(x)w^c(x) = f_Q(x)w^i e_i(x)$ inside $B(x_{\hat{Q}}, \rho)$, the GBG trivialisation of Q. Note that, for $x \in Q$ and $t \leq \mathrm{t_S}, \mathbb{E}_t w_Q(x) = w^c$. Since the metric h is uniformly comparable to the trivial metric inside this trivialisation, and using the facts we have just mentioned,

$$\iint_{\mathrm{R}_Q} |\gamma_t|^2 \frac{d\mu dt}{t} \lesssim \sup_{|w|_\delta = 1} \iint_{\mathrm{R}_Q} |\gamma_t \mathbb{E}_t w_Q(x)|^2 \frac{d\mu dt}{t}.$$

We split

$$\begin{aligned} &\iint_{\mathrm{R}_Q} |\gamma_t \mathbb{E}_t w_Q(x)|^2 \frac{d\mu dt}{t} \\ &\leq \iint_{\mathrm{R}_Q} |(\gamma_t \mathbb{E}_t - \Theta_t) w_Q(x)|^2 \frac{d\mu dt}{t} + \iint_{\mathrm{R}_Q} |\Theta_t w_Q(x)|^2 \frac{d\mu dt}{t}. \end{aligned}$$

On following the exact same argument as in Proposition 5.11 in [1], noting that this proof only requires that Θ_t satisfies the off-diagonal estimates, we obtain that

$$\iint_{\mathrm{R}_Q} |(\gamma_t \mathbb{E}_t - \Theta_t) w_Q(x)|^2 \frac{d\mu dt}{t} \lesssim ||A||_\infty^2 \mu(Q).$$

For the remaining part, let

$$\Theta_t w_Q = t\widetilde{\mathrm{P}}_t \mathrm{div} A_2 (\mathrm{P}_t - \mathrm{I}) w_Q + t\widetilde{\mathrm{P}}_t \mathrm{div} A_2 w_Q.$$

We first obtain the required estimate on the second term. For that, observe $w_Q = 0$ near Σ and hence, $A_2 \omega_Q \in \mathcal{D}(\mathrm{div}_{\min})$. Using the identity $t\widetilde{\mathrm{P}}_t \mathrm{div}_{\min} = (\widetilde{\mathrm{Q}}_t + \mathrm{i}t\widetilde{\mathrm{P}}_t)(\nabla(\mathrm{iI} - \widetilde{\mathrm{D}})^{-1})^*$, we estimate

$$\int_0^{\ell(Q)} ||t\widetilde{\mathrm{P}}_t \ \mathrm{div}\ A_2 w_Q||^2 \ \frac{dt}{t} \lesssim ||(\nabla(\mathrm{iI} - \widetilde{\mathrm{D}})^{-1})^* A_2 w_Q||^2 \lesssim ||A||_\infty^2 \mu(Q).$$

To estimate the remaining term, we note that $t\widetilde{\mathrm{P}}_t \mathrm{div} A_2 (\mathrm{P}_t - \mathrm{I}) w_Q = -t\widetilde{\mathrm{P}}_t \,\mathrm{div}\, A_2 Q_t (t\mathrm{D} w_Q)$ and so by Proposition 4.2

$$||t\widetilde{\mathrm{P}}_t \,\mathrm{div}\, A_2 \big(\widetilde{\mathrm{P}}_t - \mathrm{I}\big) w_Q||^2 \lesssim t^2 ||A||_\infty^2 ||\mathrm{D} w_Q||^2 \lesssim t^2 ||A||_\infty^2 ||\nabla w_Q||^2 \lesssim t^2 ||A||_\infty^2 \frac{1}{\ell(Q)^2} \mu(Q).$$

Therefore,

$$\begin{aligned} \iint_{\mathrm{R}_Q} |t\widetilde{\mathrm{P}}_t \mathrm{div} A_2 \big(\widetilde{\mathrm{P}}_t - \mathrm{I}\big) w_Q|^2 \frac{d\mu dt}{t} &\leq \int_0^{\ell(Q)} ||t\widetilde{\mathrm{P}}_t \mathrm{div} A_2 (\mathrm{P}_t - \mathrm{I}) w_Q||^2 \frac{dt}{t} \\ &\lesssim ||A||_\infty^2 \mu(Q) \int_0^{\ell(Q)} \frac{t}{\ell(Q)^2} dt \lesssim ||A||_\infty^2 \mu(Q), \end{aligned}$$

which establishes the conclusion. □

Proof of Theorem 2.1. On combining the estimates in Section 4.3 and Proposition 4.7, the hypothesis of Proposition 4.1 is satisfied. This proves Theorem 2.1. □

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