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The Banach–Mazur game and domain theory

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Abstract. We prove that player α has a winning strategy in the Banach–Mazur game on a space X if and only if X is F-Y countably π-domain representable. We show that Choquet complete spaces are F-Y countably domain representable. We give an example of a space, which is F-Y countably domain representable, but which is not F-Y π-domain representable.

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1. Introduction. The famous Banach–Mazur game was invented by Mazur in 1935. For the history of game theory and facts about game theory, the reader is referred to the survey [12]. Let X be a topological space and $X = A \cup B$ be any given decomposition of X into two disjoint sets. The game $BM(X, A, B)$ is played as follows: Two players, named α and β, alternately choose open nonempty sets with $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \cdots$.

$$\begin{array}{lccc} \alpha & U_0 & & U_1 & & \\ & & & & \cdots \\ \beta & & V_0 & & V_1 \end{array}$$

Player α wins this game if $A \cap \bigcap_{n\in\omega} U_n \neq \emptyset$. Otherwise β wins.

We study a well-known modification of this game considered by Choquet in 1958, known as Banach–Mazur game or Choquet game. Player α and β alternately choose open nonempty sets with $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \cdots$. In the first round, player β starts by choosing a nonempty open set U_0.

$$\begin{array}{lccc} \beta & U_0 & & U_1 & & \\ & & & & \cdots \\ \alpha & & V_0 & & V_1 \end{array}$$

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Player α wins this play if $\bigcap_{n\in\omega} V_n \neq \emptyset$. Otherwise β wins. Denote this game by $BM(X)$. Every finite sequence of sets $(U_0,\dots,U_n)$, obtained by the first n steps in this game is called *partial play* of β. A *strategy* for player α in the game $BM(X)$ is a map s that assigns to each partial play $(U_0,\dots,U_n)$ of β a nonempty open set $V_n \subseteq U_n$. The strategy s is called a *winning strategy* for player α if player α always wins the play of the game using this strategy. The space X is called *weakly α-favorable* (see [13]) if X admits a winning strategy for player α in the game $BM(X)$. We say that a partial play $(W_0,\dots,W_k)$ is *stronger* than $(U_0,\dots,U_m)$ if $m \leq k$ and $U_0 = W_0,\dots,U_m = W_m$. Notice that if $(W_0,\dots,W_k)$ is stronger than $(U_0,\dots,U_m)$, then $s(W_0,\dots,W_k) \subseteq s(U_0,\dots,U_m)$, we denote this by $(U_0,\dots,U_m) \preceq (W_0,\dots,W_k)$. We denote a sequence $(U_0,\dots,U_k)$ by $\overrightarrow{U}(k)$.

The *strong Choquet* game is defined as follows:

β $U_0 \ni x_0$ $\quad$ $U_1 \ni x_1$

$\dots$

α $\quad$ V_0 $\quad$ V_1

Player β and α take turns in playing nonempty open subset, similar to the Banach–Mazur game. In the first round, player β starts by choosing a point x_0 and an open set U_0 containing x_0, then player α responds with an open set V_0 such that $x_0 \in V_0 \subseteq U_0$. In the n-th round, player β selects a point x_n and an open set U_n such that $x_n \in U_n \subseteq V_{n-1}$ and α responds with an open set V_n such that $x_n \in V_n \subseteq U_n$. Player α wins if $\bigcap_{n\in\omega} V_n \neq \emptyset$. Otherwise β wins. We say that a partial play $(W_0,x_0,\dots,W_k,x_k)$ is *stronger* than $(U_0,y_0,\dots,U_m,y_m)$ if $m \leq k$ and $U_0{=}W_0,\dots,U_m = W_m$ and $x_0 = y_0,\dots,x_m = y_m$. We denote this by $(U_0,y_0,\dots,U_m,y_m) \preceq (W_0,x_0,\dots,W_k, x_k)$. We denote a sequence $(W_0,x_0,\dots,W_k,x_k)$ by $(\overrightarrow{x} \circ \overrightarrow{W})(k)$. A topological space X is called *Choquet complete* if player α has a winning strategy in the strong Choquet game, and we then write $Ch(X)$.

For a topological space X, let $\tau(X)$ denote the topology on the set X and $\tau^*(X) = \tau(X)\backslash\{\emptyset\}$. A family $\mathcal{P}$ of open nonempty sets is called a *π-base* if for every open nonempty set U, there is $P \in \mathcal{P}$ such that $P \subseteq U$.

A *dcpo* (directed complete partial order) is a poset $(P,\sqsubseteq)$ in which every directed set has a supremum. If $p,q \in P$, then we say that "p is far below q" whenever for any directed set D with $q \sqsubseteq \sup(D)$, there is some $d \in D$ with $p \sqsubseteq d$. A *domain* is a dcpo in which every element q is the supremum of the directed set $\{p \in P :$ "p is far below q"$\}$. This notion has been introduced by D. Scott as a model for the λ-calculus, for more information see [1], [10]. Domain representable topological spaces were introduced by Bennett and Lutzer [2]. We say that a topological space is domain representable if it is homeomorphic to the space of maximal elements of some domain topologized with the Scott topology. In 2013, Fleissner and Yengulalp [3] introduced an equivalent definition of a *domain representable space* for T_1 topological spaces. We do not assume the antisymmetry condition on the relation $\ll$. As Önal and Vural suggested in [11], if we need an additional antisymmetric property, let us consider the equivalent relation E on the set Q defined by "pEq if and only if ($p \ll q$ and $q \ll p$) or $p = q$". We do not assume any separation axioms, if it is not explicitly stated.

We say that a topological space X is *F-Y* (*Fleissner–Yengulalp*) *countably domain representable* if there is a triple $(Q, \ll, B)$ such that

(D1) $B : Q \to \tau^*(X)$ and $\{B(q) : q \in Q\}$ is a base for $\tau(X)$,
(D2) $\ll$ is a transitive relation on Q,
(D3) for all $p, q \in Q$, $p \ll q$ implies $B(p) \supseteq B(q)$,
(D4) for all $x \in X$, the set $\{q \in Q : x \in B(q)\}$ is upward directed by $\ll$ (every pair of elements has an upper bound),
(D5_{ω_1}) if $D \subseteq Q$ and $(D, \ll)$ is countable and upward directed, then $\bigcap\{B(q) : q \in D\} \neq \emptyset$.

If the conditions (D1)–(D4) and the condition

(D5) if $D \subseteq Q$ and $(D, \ll)$ is upward directed, then $\bigcap\{B(q) : q \in D\} \neq \emptyset$

are satisfied, we say that the space X is *F-Y domain representable.*

In [4], Fleissner and Yengulalp introduced the notion of a π-*domain representable space*, as this is analogous to the notion of a domain representable space.

We say that a topological space X is *F-Y* (*Fleissner–Yengulalp*) *countably* π-*domain representable* if there is a triple $(Q, \ll, B)$ such that

(πD1) $B : Q \to \tau^*(X)$ and $\{B(q) : q \in Q\}$ is a π-base for $\tau(X)$,
(πD2) $\ll$ is a transitive relation on Q,
(πD3) for all $p, q \in Q$, $p \ll q$ implies $B(p) \supseteq B(q)$,
(πD4) if $q, p \in Q$ satisfy $B(q) \cap B(p) \neq \emptyset$, there exists $r \in Q$ satisfying $p, q \ll r$,
(πD5_{ω_1}) if $D \subseteq Q$ and $(D, \ll)$ is countable and upward directed, then $\bigcap\{B(q) : q \in D\} \neq \emptyset$.

If the conditions (πD1)–(πD4) and the condition

(πD5) if $D \subseteq Q$ and $(D, \ll)$ is upward directed, then $\bigcap\{B(q) : q \in D\} \neq \emptyset$

are satisfied, we say that the space X is *F-Y* π-*domain representable.*

2. π-domain representable spaces. In [5], Kenderov and Revalski have shown that the set $E = \{f \in C(X) : f$ attains its minimum in $X\}$ contains a G_δ dense subset of $C(X)$ is equivalent to the existence of a winning strategy for player α in the Banach–Mazur game. Oxtoby [9] showed that if X is a metrizable space, then player α has a winning strategy in $BM(X)$ if and only if X contains a dense completely metrizable subspace. Krawczyk and Kubiś [6] have characterized the existence of winning strategies for player α in the abstract Banach–Mazur game played with finitely generated structures instead of open sets. In [7], there has been presented a version of the Banach–Mazur game played on a partially ordered set. We give a characterization of the existence of a winning strategy for player α in the Banach–Mazur game using the notion "π-domain representable space" introduced by W. Fleissner and L. Yengulalp.

Theorem 1. *A topological space X is weakly α-favorable if and only if X is F-Y countably π-domain representable.*

Proof. If X is F-Y countably π-domain representable, then it is easy to show that X is weakly α-favorable.

Assume that X is weakly α-favorable. We shall show that X is F-Y countably π-domain representable. Let s be a winning strategy for player α in $BM(X)$. We consider a family Q consisting of all finite sequences $\left(\overrightarrow{U}_0(j_0),\ldots,\overrightarrow{U}_i(j_i)\right)$, where $\overrightarrow{U}_m(j_m)=(U_0^m,\ldots,U_{j_m}^m)$ is a partial play and $m\le i$, i.e.,

$$U_0^m\supseteq s(U_0^m)\supseteq U_1^m\supseteq s(U_0^m,U_1^m)\supseteq\ldots\supseteq U_{j_m}^m\supseteq s(U_0^m,\ldots,U_{j_m}^m)$$

and $s(\overrightarrow{U}_0(j_0))\supseteq\ldots\supseteq s(\overrightarrow{U}_i(j_i))$.

Let us define a relation $\ll$ on the family Q:

$$\begin{aligned}&\left(\overrightarrow{U}_0(j_0),\ldots,\overrightarrow{U}_i(j_i)\right)\ll\left(\overrightarrow{W}_0(l_0),\ldots,\overrightarrow{W}_k(l_k)\right)\text{ iff}\\ &s(\overrightarrow{U}_i(j_i))\supseteq s(\overrightarrow{W}_0(l_0))\\ &\quad\ \&\ i\le k\ \&\ \forall\, t\le i\ \exists\, r\le k\ \overrightarrow{U}_t(j_t)\preceq\overrightarrow{W}_r(l_r).\end{aligned}$$

Since $\preceq$ is transitive, $\ll$ is transitive.

Let us define a map $B:Q\to\tau^*(X)$ by the formula

$$B\left(\left(\overrightarrow{U}_0(j_0),\ldots,\overrightarrow{U}_i(j_i)\right)\right)=s(\overrightarrow{U}_i(j_i))$$

for $\left(\overrightarrow{U}_0(j_0),\ldots,\overrightarrow{U}_i(j_i)\right)\in Q$.

Since $\{s(V):V\in\tau^*(X)\}$ is a π-base, $\{B(q):q\in Q\}$ is a π-base for τ. It is easy to see that the map B satisfies the condition (πD3).

Towards item (πD4), let $p,q\in Q$ be such that $B(q)\cap B(p)\ne\emptyset$ and $p=\left(\overrightarrow{U}_0(j_0),\ldots,\overrightarrow{U}_i(j_i)\right)$, $q=\left(\overrightarrow{W}_0(l_0),\ldots,\overrightarrow{W}_k(l_k)\right)$. Since $V=B(p)\cap B(q)\subseteq s(\overrightarrow{U}_0(j_0))$ and s is a winning strategy, we find an element $\overrightarrow{U}'_0(j'_0)$ stronger than $\overrightarrow{U}_0(j_0)$ such that $s(\overrightarrow{U}'_0(j'_0))\subseteq V$. Step by step we find a partial play $\overrightarrow{U}'_t(j'_t)$ such that $\overrightarrow{U}_t(j_t)\preceq\overrightarrow{U}'_t(j'_t)$ and $s(\overrightarrow{U}'_t(j'_t))\subseteq s(\overrightarrow{U}'_{t-1}(j'_{t-1}))$ for $t\le i$. Since $s(\overrightarrow{U}'_i(j'_i))\subseteq s(\overrightarrow{W}_0(l_0))$, we find a partial play $\overrightarrow{W}'_0(l'_0)$ such that $\overrightarrow{W}_0(l_0)\preceq\overrightarrow{W}'_0(l'_0)$ and $s(\overrightarrow{W}'_0(l'_0))\subseteq s(\overrightarrow{U}'_i(j'_i))$. Similarly, as for the sequence p, for the sequence q, we define $\overrightarrow{W}'_t(l'_t)$ with $\overrightarrow{W}_t(l_t)\preceq\overrightarrow{W}'_t(l'_t)$ and $s(\overrightarrow{W}'_t(l'_t))\subseteq s(\overrightarrow{W}'_{t-1}(l'_{t-1}))$ for all $t\le k$.

Continuing in this way, we get an element $r=\left(\overrightarrow{U}'_0(j'_0),\ldots,\overrightarrow{U}'_i(j'_i),\overrightarrow{W}'_0(l'_0),\ldots,\overrightarrow{W}'_k(l'_k)\right)$ such that $p,q\ll r$ and $r\in Q$.

Next we show the condition (πD5_{ω_1}). Let $D\subseteq Q$ be a countable upward directed set and let $D=\{p_n:n\in\omega\}$. We define a chain $\{q_n:n\in\omega\}\subseteq D\subseteq Q$ such that $p_n\ll q_n$ for $n\in\omega$. By the condition (πD3), we get $\bigcap\{B(q_n):n\in\omega\}\subseteq\bigcap\{B(p):p\in D\}$. Each $q_n\in Q$ is of the form $q_n=\left(\overrightarrow{W}_0^n(l_0^n),\ldots,\overrightarrow{W}_{k_n}^n(l_{k_n}^n)\right)$.

Since $q_0\ll q_1$, there is $j_1\le k_1$ such that $\overrightarrow{W}_0^0(l_0^0)\preceq\overrightarrow{W}_{j_1}^1(l_{j_1}^1)$. We have

$$s(\overrightarrow{W}_0^0(l_0^0))\supseteq B(q_0)=s(\overrightarrow{W}_{k_0}^0(l_{k_0}^0))\supseteq s(\overrightarrow{W}_{j_1}^1(l_{j_1}^1))\supseteq B(q_1)=s(\overrightarrow{W}_{k_1}^1(l_{k_1}^1)).$$

Let $\overrightarrow{U}'_0(l^0_0) = \overrightarrow{W}^0_0(l^0_0)$ and $\overrightarrow{U}'_1(l^1_{j_1}) = \overrightarrow{W}^1_{j_1}(l^1_{j_1})$. Inductively, we can choose a sequence $\{s(\overrightarrow{U}'_n(l^n_{j_n})) : n \in \omega\}$ such that $\overrightarrow{U}'_n(l^n_{j_n}) \preceq \overrightarrow{U}'_{n+1}(l^{n+1}_{j_{n+1}})$ and

$$B(q_n) \supseteq s(\overrightarrow{U}'_{n+1}(l^{n+1}_{j_{n+1}})) \supseteq B(q_{n+1}).$$

Since s is a winning strategy for player α, we have

$$\emptyset \neq \bigcap\{s(\overrightarrow{U}'_n(l^n_{j_n})) : n \in \omega\} = \bigcap\{B(q_n) : n \in \omega\} \subseteq \bigcap\{B(p) : p \in D\}. \quad \square$$

We give an example of a space, which is F-Y countably domain representable, but which is not F-Y π-domain representable. Note that this space is F-Y countably π-domain representable and not F-Y domain representable.

Example 1. We consider the space

$$X = \sigma\big(\{0,1\}^{\omega_1}\big) = \big\{x \in \{0,1\}^{\omega_1} : |\mathrm{supp}\, x| \leq \omega\big\},$$

where $\mathrm{supp}\, x = \{\alpha \in \omega_1 : x(\alpha) = 1\}$ for $x \in \{0,1\}^{\omega_1}$, with the topology (ω_1-box topology) generated by the base

$$\mathcal{B} = \big\{\mathrm{pr}_A^{-1}(x) : A \in [\omega_1]^{\leq\omega}, x \in \{0,1\}^A\big\},$$

where $\mathrm{pr}_A : \sigma(\{0,1\}^{\omega_1}) \to \{0,1\}^A$ is a projection.

We shall define a triple $(Q, \ll, B)$. Let $Q = \mathcal{B}$, and the map $B : Q \to Q$ be the identity. Define a relation $\ll$ in the following way:

$$\mathrm{pr}_A^{-1}(x_A) \ll \mathrm{pr}_B^{-1}(x_B) \Leftrightarrow \mathrm{pr}_A^{-1}(x_A) \supseteq \mathrm{pr}_B^{-1}(x_B)$$

for any $\mathrm{pr}_A^{-1}(x_A), \mathrm{pr}_B^{-1}(x_B) \in \mathcal{B}$. It is easy to see that the relation $\ll$ is transitive and that it satisfies the condition (D3). Now, we prove the condition (D4). Let $x \in X$ and $\mathrm{pr}_{A_1}^{-1}(x_{A_1}), \mathrm{pr}_{A_2}^{-1}(x_{A_2}) \in \{\mathrm{pr}_A^{-1}(x_A) \in \mathcal{B} : x \in \mathrm{pr}_A^{-1}(x_A)\}$. Since $x \in \mathrm{pr}_{A_1}^{-1}(x_{A_1}) \cap \mathrm{pr}_{A_2}^{-1}(x_{A_2})$, we get $x_{A_1} \restriction A_2 = x_{A_2} \restriction A_1$. Set $A_3 = A_1 \cup A_2$ and let $x_{A_3} \in \{0,1\}^{A_3}$ be such that $x_{A_3} \restriction A_2 = x_{A_2}$ and $x_{A_3} \restriction A_1 = x_{A_1}$. We have $x_{A_3} \in \{0,1\}^{A_3}$ such that $x \in \mathrm{pr}_{A_3}^{-1}(x_{A_3}) \subseteq \mathrm{pr}_{A_1}^{-1}(x_{A_1}) \cap \mathrm{pr}_{A_2}^{-1}(x_{A_2})$. Hence $\mathrm{pr}_{A_1}^{-1}(x_{A_1}), \mathrm{pr}_{A_2}^{-1}(x_{A_2}) \ll \mathrm{pr}_{A_3}^{-1}(x_{A_3})$.

We prove the condition $(D5_{\omega_1})$. Let $D \subseteq \mathcal{B}$ be a countable upward directed family. We can construct a chain $\{\mathrm{pr}_{A_n}^{-1}(x_{A_n}) : n \in \omega\} \subseteq D$ such that for each set $\mathrm{pr}_A^{-1}(x_A) \in D$, there exists $n \in \omega$ such that $\mathrm{pr}_A^{-1}(x_A) \ll \mathrm{pr}_{A_n}^{-1}(x_{A_n})$.

Let $B = \bigcup\{A_n : n \in \omega\}$. Since $\{\mathrm{pr}_{A_n}^{-1}(x_{A_n}) : n \in \omega\}$ is a chain, there is $x_B \in \{0,1\}^B$ such that $x_B \restriction A_n = x_{A_n}$ for $n \in \omega$. Then

$$\bigcap\{\mathrm{pr}_{A_n}^{-1}(x_{A_n}) : n \in \omega\} = \mathrm{pr}_B^{-1}(x_B) \in \mathcal{B},$$

and $\mathrm{pr}_B^{-1}(x_B) \subseteq \bigcap D$. This completes the proof that the space $\sigma\big(\{0,1\}^{\omega_1}\big)$ is F-Y countably domain representable.

Now we show that $X = \sigma\big(\{0,1\}^{\omega_1}\big)$ is not F-Y π-domain representable. Suppose that there exists a triple $(Q, \ll, B)$ satisfying the conditions (πD1)–(πD5). The family $\mathcal{P} = \{B(q) : q \in Q\}$ is a π-base. By induction, we define a sequence $\{Q_\alpha : \alpha < \omega_1\}$ such that the following conditions are satisfied:

(1) $Q_\alpha \in [Q]^{\leq\omega}$ and Q_α is upward directed, for $\alpha < \omega_1$,

(2) $\bigcap\{B(q) : q \in Q_\alpha\} = \mathrm{pr}_{A_\alpha}^{-1}(x_{A_\alpha}) \in \mathcal{B}$ for some $A_\alpha \in [\omega_1]^{\leq\omega}$ and some $x_{A_\alpha} \in \{0,1\}^{A_\alpha}$, for $\alpha < \omega_1$,

(3) $Q_\alpha \subseteq Q_\beta$, for $\alpha < \beta < \omega_1$,
(4) if $\bigcap\{B(q) : q \in Q_\alpha\} = \mathrm{pr}^{-1}_{A_\alpha}(x_{A_\alpha})$ and $\bigcap\{B(q) : q \in Q_\beta\} = \mathrm{pr}^{-1}_{A_\beta}(x_{A_\beta})$ for some $A_\alpha, A_\beta \in [\omega_1]^{\leq\omega}$ and $x_{A_\alpha} \in \{0,1\}^{A_\alpha}$ and $x_{A_\beta} \in \{0,1\}^{A_\beta}$, then $\mathrm{supp}\, x_{A_\alpha} = \{\alpha \in A_\alpha : x(\alpha) = 1\} \subsetneq \{\alpha \in A_\beta : x(\alpha) = 1\} = \mathrm{supp}\, x_{A_\beta}$, for $\alpha < \beta < \omega_1$.

We define a set Q_0. Take any $r_0 \in Q$. There exist a set $A_0 \in [\omega_1]^{\leq\omega}$ and $x_{A_0} \in \{0,1\}^{A_0}$ such that $\mathrm{pr}^{-1}_{A_0}(x_{A_0}) \subseteq B(r_0)$. By conditions $(\pi D1), (\pi D3)$, $(\pi D4)$, there exists $r_1 \in Q$ such that $r_0 \ll r_1$ and $B(r_1) \subseteq \mathrm{pr}^{-1}_{A_0}(x_{A_0})$. Assume that we have defined $r_0 \ll \ldots \ll r_n$ and a chain $\{A_i : i \leq n\} \subseteq [\omega_1]^{\leq\omega}$ and $x_{A_i} \in \{0,1\}^{A_i}$ such that

$$\mathrm{pr}^{-1}_{A_{i-1}}(x_{A_{i-1}}) \supseteq B(r_i) \supseteq \mathrm{pr}^{-1}_{A_i}(x_{A_i}) \text{ for } i \leq n.$$

By conditions (πD1), (πD3), (πD4), there exists $r_{n+1} \in Q$ such that $r_n \ll r_{n+1}$ and $B(r_{n+1}) \subseteq \mathrm{pr}^{-1}_{A_n}(x_{A_n})$. There exist a set $A_{n+1} \in [\omega_1]^{\leq\omega}$ and $x_{A_{n+1}} \in \{0,1\}^{A_{n+1}}$ such that $\mathrm{pr}^{-1}_{A_{n+1}}(x_{A_{n+1}}) \subseteq B(r_{n+1})$. Let $Q_0 = \{r_n : n \in \omega\}$. Then $\bigcap\{B(q) : q \in Q_0\} = \bigcap\{\mathrm{pr}^{-1}_{A_n}(x_{A_n}) : n \in \omega\} = \mathrm{pr}^{-1}_A(x_A)$, where $A = \bigcup\{A_n : n \in \omega\}$ and $x_A \in \{0,1\}^A$ and $x_A \restriction A_n = x_{A_n}$ for all $n \in \omega$.

Assume that we have defined $\{Q_\alpha : \alpha < \beta\}$ which satisfies the conditions (1)–(4).

Let $\mathcal{R}_\beta = \bigcup\{Q_\alpha : \alpha < \beta\}$. The set $\mathcal{R}_\beta$ is upward directed by conditions (3), (1). Let $\mathcal{R}_\beta = \{p_n : n \in \omega\}$. By (2) and (3), we get $\bigcap\big\{B(p_n) : n \in \omega\big\} = \mathrm{pr}^{-1}_{A_\beta}(x_{A_\beta}) \in \mathcal{B}$ for some set $A_\beta \in [\omega_1]^{\leq\omega}$ and $x_{A_\beta} \in \{0,1\}^{A_\beta}$. There exist a set $A \in [\omega_1]^{\leq\omega}$ and $x_A \in \{0,1\}^A$ such that $\mathrm{pr}^{-1}_A(x_A) \subsetneq \mathrm{pr}^{-1}_{A_\beta}(x_{A_\beta})$ and $\mathrm{supp}\, x_{A_\beta} \subsetneq \mathrm{supp}\, x_A$. Since $\mathcal{P}$ is a π-base, we can find $r_\beta \in Q$ such that $B(r_\beta) \subseteq \mathrm{pr}^{-1}_A(x_A)$. Inductively, we can define a sequence $\{q_n : n \in \omega\} \subseteq Q$, a chain $\{A_n : n \in \omega\} \subseteq [\omega_1]^{\leq\omega}$, and a sequence $\{x_{A_n} \in \{0,1\}^{A_n} : n \in \omega\}$ such that $r_\beta, p_0 \ll q_0$, $q_{n-1}, p_n \ll q_n$, and

$$B(q_n) \supseteq \mathrm{pr}^{-1}_{A_n}(x_{A_n}) \supseteq B(q_{n+1}) \text{ for } n \in \omega.$$

Let $Q_\beta = \mathcal{R}_\beta \cup \{q_n : n \in \omega\}$. The set Q_β satisfies conditions (1)–(4), so we finish the induction. The set $\bigcup\{Q_\alpha : \alpha < \omega_1\}$ is upward directed.

By conditions (2), (3), we have

$$\begin{aligned}\bigcap\Big\{B(q) : q \in \bigcup\{Q_\alpha : \alpha < \omega_1\}\Big\} &= \bigcap\{\mathrm{pr}^{-1}_{A_\alpha}(x_{A_\alpha}) : \alpha < \omega_1\} = \\ &= \pi^{-1}_A(x_A), \text{ for } A = \bigcup\{A_\alpha : \alpha < \omega_1\} \text{ and } x_A \in \{0,1\}^A \\ &\quad \text{such that } x_A \restriction A_\alpha = x_{A_\alpha} \text{ for } \alpha < \omega_1,\end{aligned}$$

where $\pi_A : \{0,1\}^{\omega_1} \to \{0,1\}^A$ is the projection. By condition (4), we get $|\mathrm{supp}\, x_A| = \omega_1$. Hence $\pi^{-1}_A(x_A) \cap \sigma\big(\{0,1\}^{\omega_1}\big) = \emptyset$, a contradiction. □

Note that by the proof of [4, Proposition 8.3] it follows that if there exists a triple $(Q, \ll, B)$, which satisfies the conditions of the definition of F-Y countably π-domain representable and $|\bigcap\{B(q) : q \in D\}| = 1$ for every countable and upward directed set $D \subseteq Q$, then the space X is F-Y π-domain representable by this triple.

Theorem 2. *The Cartesian product of any family of F-Y countably π-domain representable spaces is F-Y countably π-domain representable.*

Proof. Let X be a product of a family $\{X_a : a \in A\}$ of F-Y countably π-domain representable spaces. Let $(Q_a, \ll_a, B_a)$ be a triple which satisfies conditions (πD1)–(πD4) and ($\pi \mathrm{D5}_{\omega_1}$) for the space X_a. Any basic nonempty open subset U in X is of the form $U = \prod\{U_a : a \in A\}$, where U_a is nonempty open subset of X_a and $U_a = X_a$ for all but a finite number of $a \in A$. We may assume that $0_a \in Q_a$ is the least element in Q_a and $B_a(0_a) = X_a$ for each $a \in A$. Put

$$Q = \left\{p \in \prod\{Q_a : a \in A\} : |\{a \in A : p(a) \neq 0_a\}| < \omega\right\}.$$

Define a relation $\ll$ on Q by the formula

$$p \ll q \iff p(a) \ll_a q(a) \text{ for all } a \in A,$$

where $p, q \in Q$. Let us define a map $B : Q \to \tau^*(X)$ by $B(p) = \prod\{B_a(p(a)) : a \in A\}$, where $p \in Q$. It is easy to check that $(Q, \ll, B)$ is a F-Y countably π-domain representing X. □

In a similar way, one can prove the above theorem also for F-Y countably domain representable, F-Y π-domain representable, and F-Y domain representable.

3. Domain representable spaces. In 2003, Martin [8] showed that if a space is domain representable, then player α has a winning strategy in the strong Choquet game. In 2015, Fleissner and Yengulalp [4] showed that it is sufficient that a space is F-Y countably domain representable. Now, we shall show that the property of being F-Y countably domain representable is necessary. For this purpose, we can use a triple $(Q, \ll, B)$ defined in [4, Proposition 8.3] or we can use a similar triple to the triple defined in the Theorem 1. Namely, if s is a winning strategy for player α, we consider a family Q consisting of all finite sequences $(\overrightarrow{x_0} \circ \overrightarrow{U}_0(j_0), \ldots, \overrightarrow{x_i} \circ \overrightarrow{U}_i(j_i))$, where $\overrightarrow{x_m} \circ \overrightarrow{U}_m(j_m) = (U_0^m, x_0^m, \ldots, U_{j_m}^m, x_{j_m}^m)$ is a partial play in the strong Choquet game for all $m \leq i$, i.e.,

$$U_0^m \supseteq s(U_0^m, x_0^m) \supseteq U_1^m \supseteq s(U_0^m, x_0^m, U_1^m, x_1^m) \supseteq \ldots \supseteq U_{j_m}^m \supseteq s(U_0^m, x_0^m, \ldots, U_{j_m}^m, x_{j_m}^m)$$

and $s(\overrightarrow{x_0} \circ \overrightarrow{U}_0(j_0)) \supseteq \ldots \supseteq s(\overrightarrow{x_i} \circ \overrightarrow{U}_i(j_i))$.

Let us define a relation $\ll$ on the family Q:

$$\left(\overrightarrow{x_0} \circ \overrightarrow{U}_0(j_0), \ldots, \overrightarrow{x_i} \circ \overrightarrow{U}_i(j_i)\right) \ll \left(\overrightarrow{y_0} \circ \overrightarrow{W}_0(l_0), \ldots, \overrightarrow{y_k} \circ \overrightarrow{W}_k(l_k)\right)$$
$$\text{iff } s\left(\overrightarrow{x_i} \circ \overrightarrow{U}_i(j_i)\right) \supseteq s\left(\overrightarrow{y_0} \circ \overrightarrow{W}_0(l_0)\right) \ \& \ i \leq k \ \&$$
$$\forall\, t \leq i\ \exists\, r \leq k\ \overrightarrow{x_t} \circ \overrightarrow{U}_t(j_t) \preceq \overrightarrow{y_r} \circ \overrightarrow{W}_r(l_r).$$

We define a map $B : Q \to \tau^*$ by the formula

$$B\left(\left(\overrightarrow{x_0} \circ \overrightarrow{U}_0(j_0), \ldots, \overrightarrow{x_i} \circ \overrightarrow{U}_i(j_i)\right)\right) = s\left(\overrightarrow{x_i} \circ \overrightarrow{U}_i(j_i)\right)$$

for each $\left(\overrightarrow{x_0} \circ \overrightarrow{U}_0(j_0), \ldots, \overrightarrow{x_i} \circ \overrightarrow{U}_i(j_i)\right) \in Q$.

As a consequence, we obtain:

Theorem 3. *A topological space X is Choquet complete if and only if it is F-Y countably domain representable.*

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