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Acknowledgements

Enumerating Alternating Permutations
with One Alternating Descent

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ABSTRACT   This paper introduces a new statistic for alternating permutations, called an alternating descent. Specifically this paper focuses on alternating permutations with one alternating descent. We then enumerate these permutations by decomposing them into four sets.

INTRODUCTION

Permutations as combinatorial objects will be the basis for this paper. Two of their most basic attributes are ascents and descents. We will examine in this paper a statistic for alternating permutations. Alternating permutations have been studied by Stanley[2] and Dulucq and Simion[1]. This statistic is based on the concept of a descent, called *alternating descents*. We will focus our attention, however, solely on the alternating permutations containing one alternating descent.

In the first section we define this statistic and its notation. In Section 2, we introduce the decomposition of these permutations into four subsets that will then be enumerated in Sections 3 through 6.

1. ALTERNATING DESCENTS

DEFINITION 1.1.

A *permutation* is a specific ordering of a set of elements. Since we consider finite sets this definition coincides with the definition of a permutation as a one-to-one and onto map.

A permutation w may be expressed as a word where $w = w(1) \, w(2) \cdots w(n)$.

DEFINITION 1.2.

An *ascent* in a permutation is a position i such that $w(i) < w(i+1)$. A *descent* in a permutation is a position i such that $w(i) > w(i+1)$.

DEFINITION 1.3.

An *alternating permutation* is a permutation with ascents in all odd positions and descents in all even positions.

EXAMPLE 1.4.

The alternating permutation of S_1 is {1}.
The alternating permutation of S_2 is {12}.
The alternating permutations of S_3 are {132, 231}.
The alternating permutations of S_4 are {1324, 1423, 2314, 2413, 3412}.

Permutations can also be displayed visually. In these graphs, one can see the relative differences between consecutive values. For alternating permutations, what is immediately apparent is that they zig-zag. These graphs can be formed by plotting for each i $(i, w(i))$, then connecting each plotted point.

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EXAMPLE 1.5.

Consider $w=362415$; w can be expressed visually with each point representing a term in w:
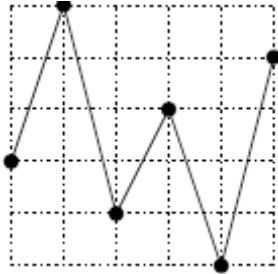
If we care only about ascents and descents, and not the relative differences between consecutive values, we can also capture this zig-zag property without using a graph. We will use this form in later analysis.

EXAMPLE 1.6.

If w is the alternating permutation $w=24153867$ then w may be written as $w=\begin{smallmatrix} & 4 & & 5 & & 8 & & 7 \\ 2 & & 1 & & 3 & & 6 & \end{smallmatrix}$.

Counting the number of descents in an alternating permutation is uninteresting because we will always know exactly how many there are: if the alternating permutation $w \in S_n$, then there are $\left\lceil \frac{n-1}{2} \right\rceil$ ascents and $\left\lfloor \frac{n-1}{2} \right\rfloor$ descents. So we will examine a different statistic.

DEFINITION 1.7.

Given the alternating permutation w, read the odd positions of w from right to left and then the even positions from left to right. This mapping of w of size n will be called w', where its size is also n. Equivalently, if the alternating permutation is in the zig-zag form, read the letters starting from the bottom right corner and continuing clockwise all the way around to the top right. The number of alternating descents of w, denoted $altdes(w)$, is defined as $altdes(w)=des(w')$, the number of descents of w'.

We can also find w' and $altdes(w)$ for non-alternating permutations w, but here will focus only on the alternating permutations.

EXAMPLE 1.8.

For $w=24153867$ from Example 1.6, we have $w'=63124587$. There are three descents in w', so $altdes(w)=2$.

EXAMPLE 1.9.

There are no alternating permutations with 1 alternating descent in either S_1 or S_2 because there are no descents in either $(1)'=1$ or $(12)'=12$.

EXAMPLE 1.10.

The alternating permutation of S_3 with one alternating descent is 132 where $(132)'=213$.

EXAMPLE 1.11.

The alternating permutations of S_4 with one alternating descent are 1324, 2413, and 3412 where $(1324)'=2134$, $(2413)'=1243$, and $(1342)'=1342$.

2. DECOMPOSING $E_{N,1}$

Let $\varepsilon_{n,1}$ be the set of alternating permutations of size n with 1 alternating descent. We will show that the set of alternating permutations of n letters with one alternating descent, $\varepsilon_{n,1}$, can be decomposed as a union of the following pairwise disjoint sets:

$$\mathcal{E}_{n,1} = A_n \sqcup B_n \sqcup C_n \sqcup D_n .$$

Given that $|S|$ is the the number of elements of S, we have:

$$|A_n| = |\mathcal{E}_{n-1,1}|$$

$$|B_n| = |B_{n-1}| + |C_{n-1}| + |D_{n-1}|$$

$$|C_n| = |B_{n-2}| + |C_{n-2}| + |D_{n-2}|$$

$$|D_n| = 1 \text{ for } n > 3.$$

3. SET A_n

The set A_n is the set of all alternating permutations with one alternating descent ending in n if n is even and ending in 1 if n is odd. The set A_n will be acquired directly from the permutations w in $\varepsilon_{n-1,1}$ via the map $\psi(w)$, defined as follows:

- If n is even, then obtain $\psi(w)$ by inserting n as the last letter of each permutation in $\varepsilon_{n-1,1}$.
- If n is odd, then obtain $\psi(w)$ by inserting 1 as the last letter of each permutation in $\varepsilon_{n-1,1}$, shifting every other letter's value up by one.

EXAMPLE 3.1.

For $w = 24351$, we have $n = 6$, and so $\psi(w) = 243516$.

EXAMPLE 3.2.

For $w = 143526$, we have $n = 7$, and so $\psi(w) = 2546371$.

PROPOSITION 3.3.

For any $w \in \varepsilon_{n-1,1}$, we have $\psi(w) \in \varepsilon_{n,1}$.

Proof. When n is even, there is a descent in the final position of w, that is $w(n-2) > w(n-1)$. Adding n after the final position of w, we must surely have $w(n-1) < n$. This also guarantees that $w'(n-1) < n$, which means there is no new alternating descent. Since we begin with an alternating permutation with one alternating descent and only insert n in the final position, we preserve its alternating structure and its alternating descent. Therefore $\psi(w)$ is in $\varepsilon_{n,1}$.

When n is odd, there is an ascent in the final position of w, that is $w(n-2) < w(n-1)$. So when 1 is inserted after the final position of w and every other letter in w is shifted up by one, it is certain that $w(n-1) > 1$. Since we begin with an alternating permutation with one alternating descent and only add 1 to the final position and shift every other letter up by one, we preserve its alternating structure and its alternating descent. Therefore $\psi(w)$ is in $\varepsilon_{n,1}$.

It is easy to check that the map $\psi(w)$ is one-to-one

and onto, therefore the cardinality of A_n is equal to the cardinality of $\varepsilon_{n-1,1}$. If we assume that w does not lie in A_n, then w has the alternating descent in the top row if n is even and in the bottom row if n is odd.

4. SET B_n

The set B will be acquired from the permutations in $\varepsilon_{n-1,1} \setminus A_{n-1}$. This means that the permutations in B will come from those in $\varepsilon_{n-1,1}$ that do not end in n if $n-1$ is even or 1 if $n-1$ is odd.

EXAMPLE 4.1

$\varepsilon_{3,1}$ contains only the permutation 132. $\varepsilon_{4,1}$ contains the permutations 1324, 3412, and 2413. Because $\psi(132) = 1324$, the permutations of $\varepsilon_{4,1} \setminus A_4$ are 2413 and 3412.

Before we can show how set B is generated, we must first understand the process of complementing a permutation.

DEFINITION 4.2.

Consider the permutation $w = w(1)w(1)w(2)\dots w(n)$. To complement a permutation, w, replace each $w(i)$ with $n+1-w(i)$. This operation will be expressed as $\overline{w}$.

EXAMPLE 4.3.

For $w = 1365247$, we have $\overline{w} = 7523641$.

EXAMPLE 4.4.

For the alternating permutation $w = \begin{smallmatrix} & 4 & & 6 & & 7 & \\ 2 & & 1 & & 3 & & 5 \end{smallmatrix}$, we have $\overline{w} = \begin{smallmatrix} 6 & & 7 & & 5 & & 3 \\ & 4 & & 2 & & 1 & \end{smallmatrix}$.

Note in the second example how complementing an alternating permutation preserves its zig-zag quality.

The set B will be generated from the permutations $w \in \varepsilon_{n-1,1} \setminus A_{n-1}$, acquired by the map, $\zeta(w)$, defined as follows:

- Consider the permutation, w, in its zig-zag form. If n is even, then to obtain $\zeta(w)$, first find $\overline{w}$. Then shift each letter in the top row of $\overline{w}$ that is to the left of the alternating descent, including the position of alternating descent, to the left by one position.

This includes moving the top leftmost letter of $\overline{w}$ to the bottom leftmost position. Then insert in the remaining slot of $\overline{w}$, which will be immediately to the right of the descent, the letter n.

- Consider the permutation, w, in its zig-zag from. If n is odd, then to obtain $\zeta(w)$, first find $\overline{w}$. Then shift each letter in the top row of $\overline{w}$ to the left by one entry. This includes moving the top leftmost letter of $\overline{w}$ to the bottom leftmost position. Finally, insert in the remaining slot of $\overline{w}$, which will be the top rightmost position, the letter n.

EXAMPLE 4.5

For $w = \begin{smallmatrix} & 3 & & 6 & & 7 & \\ 2 & & 1 & & 5 & & 4 \end{smallmatrix}$, we have $n=8$. First find $\overline{w}$, which is $\begin{smallmatrix} 6 & & 7 & & 3 & & 4 \\ & 5 & & 2 & & 1 & \end{smallmatrix}$. Then move the top leftmost letter to the bottom leftmost position and shift each letter in the top row of $\overline{w}$ that is left of the alternating descent to the left by one position, so $\overline{w}$ becomes $\begin{smallmatrix} & 7 & & & 3 & & 4 \\ 6 & & 5 & & 2 & & 1 \end{smallmatrix}$. Finally, insert n in the empty remaining top position, so we have $\begin{smallmatrix} & 7 & & 8 & & 3 & & 4 \\ 6 & & 5 & & 2 & & 1 \end{smallmatrix}$. So $\zeta(w) = 67582314$.

EXAMPLE 4.6

For $w = \begin{smallmatrix} & 5 & & 7 & & 8 & & 6 \\ 4 & & 3 & & 2 & & 1 \end{smallmatrix}$, we have $n=9$. First we find $\overline{w}$, which is $\begin{smallmatrix} 5 & & 6 & & 7 & & 8 \\ & 4 & & 2 & & 1 & & 3 \end{smallmatrix}$. Then we move the letter in the top leftmost position of $\overline{w}$ to the bottom leftmost position and shift each letter of the top row of $\overline{w}$ to the left by one position, which gives $\begin{smallmatrix} & 6 & & 7 & & 8 & \\ 5 & & 4 & & 2 & & 1 & & 3 \end{smallmatrix}$. Finally, we insert n in the empty remaining top position, yielding $\begin{smallmatrix} & 6 & & 7 & & 8 & & 9 & \\ 5 & & 4 & & 2 & & 1 & & 3 \end{smallmatrix}$. So $\zeta(w) = 564728193$.

PROPOSITION 4.7.

For any $w \in \varepsilon_{n-1,1} \setminus A_{n-1}$, we have $\zeta(w) \in \varepsilon_{n,1}$.

Proof. Consider w in its zig-zag form. When n is even, w has an alternating descent in its bottom row. By complementing the permutation, this alternating descent now comes in the top row. Therefore when we shift every letter to the left of the alternating descent to the left by one, including moving the top leftmost letter to the bottom leftmost position, we add no new alternating descent. Inserting the letter n into this permutation does not add a descent, but increases the

value of the descent. The alternating structure of the permutation is preserved because we know the value of n must be greater than every other letter. Thus we have $\zeta(w) \in \varepsilon_{n,1}$.

Consider w in its zig-zag form. When n is odd, w has an alternating descent in its top row. By complementing the permutation, this alternating descent now comes in the bottom row. Therefore when we shift each letter of the top row over by one position and shift the top leftmost letter to the bottom row, no new alternating descent is added. The alternating structure is preserved because the alternating descent occurs in the bottom row of $\overline{w}$, the top row must be in ascending order, which means that there will be an ascent in the first position of $\zeta(w)$. Additionally, when we insert n into the top rightmost position, $n > \overline{w}(n-3)$ and $n > \overline{w}(n-1)$ must be true. Thus we have $\zeta(w) \in \varepsilon_{n,1}$.

5. SET C_n

Similar to set B, the permutations of set C will be generated from the permutations in $\varepsilon_{n-2,1} \setminus A_{n-2}$. The set C will generated from the permutations $w \in \varepsilon_{n-2,1} \setminus A_{n-2}$, acquired by the map, $\sigma(w)$, defined as follows:

- Consider the permutation, w, in its zig-zag form. If n is odd, then to obtain $\sigma(w)$, examine the bottom row of w. Insert $n-i-1$ in the position immediately right of the descent, where i is the number of letters that are to the right of the descent on the bottom row. Preserving the value of the letters in the bottom row, increase the values of the letters of the top row by at least one, making sure not to acquire a new letter in the top row that has the same value as one in the bottom row, and also inserting ascending letters in the remaining top position.
- Consider the permutation, w, in its zig-zag form. If n is even, then to obtain $\sigma(w)$, first insert the letter 1 to the right of the bottom rightmost letter of w. Then insert $n-1$ to the right of the top rightmost letter. Next, increase each letter in the

permutation by one except the letter $n-2$ which we increase by two, not including the two letters we have just inserted.

EXAMPLE 5.1.

For $w = \begin{smallmatrix} & 4 & & 5 & \\ 1 & & 3 & & 2 \end{smallmatrix}$, we have $n=7$. First we find that $i=2$, and $n-i-1=4$ because there are two letters right of the descent. Next we insert this letter, yielding $\begin{smallmatrix} & 4 & & 5 & \\ 1 & & 4 & & 3 & & 2 \end{smallmatrix}$. We then increase the letters in the top row and insert the letter n, or in this case 7, to find $\zeta(w) = \begin{smallmatrix} & 5 & & 6 & & 7 & \\ 1 & & 4 & & 3 & & 2 \end{smallmatrix}$.

EXAMPLE 5.2.

For $w = \begin{smallmatrix} & 4 & & 2 & \\ 3 & & 1 & \end{smallmatrix}$, we have $n=6$. First we insert the letter 1 to the right of the bottom rightmost letter, which yields $\begin{smallmatrix} & 4 & & 2 & \\ 3 & & 1 & & 1 \end{smallmatrix}$. We then insert the letter $n-1$, or in this case 5, to the right of the top rightmost letter, which yields $\begin{smallmatrix} & 4 & & 2 & & 5 \\ 3 & & 1 & & 1 \end{smallmatrix}$. Next we increase all the letters by one, except the letter $n-2$, which we increase by two, and those that we have just inserted, to find $\zeta(w) = \begin{smallmatrix} & 6 & & 3 & & 5 \\ 4 & & 2 & & 1 \end{smallmatrix}$.

PROPOSITION 5.3.

For any $w \in \varepsilon_{n-2,1} \setminus A_{n-2}$, we have $\zeta(w) \in \varepsilon_{n,1}$.

The proof is omitted here as it is too technical to include, but it uses the same methods as the proof of Propositions 3.3 and 4.7.

6. SET D_n

DEFINITION 6.1.

The alternating permutation of size n with no alternating descents has $w'(i) = i$. We will call this permutation d_n, with $n \in \mathbb{N}$ equal to its size.

EXAMPLE 6.2.

For $n=7$, we have $d_7 = 4536271$ where $d_7' = 1234567$.

The permutation in set D_n will be generated from d_n. This permutation will be generated by the map $\delta(d_n)$ defined as follows:

- For all permutations of size n, to obtain $\delta(d_n)$,

increase the letter $n-1$ by one and decrease the letter n by one.

EXAMPLE 6.3.

For $n=6$, we have $d_6 = \begin{smallmatrix} & 4 & & 5 & & 6 \\ 3 & & 2 & & 1 \end{smallmatrix}$, so $\delta(d_6) = \begin{smallmatrix} & 4 & & 6 & & 5 \\ 3 & & 2 & & 1 \end{smallmatrix}$.

PROPOSITION 6.4.

For any d_n with $n > 3$, we have $\delta(d_n) \in \varepsilon_{n,1}$.

This proposition can be easily verified, so the proof here is omitted.

7. CONCLUSION

Given that the sets A_n, B_n, C_n, and D_n are defined as in Sections 2 through 5, we arrive at the following conclusion.

THEOREM 7.1.

The set $\varepsilon_{n,1}$ can be decomposed as the union of four pairwise disjoint sets:

$$\mathcal{E}_{n,1} = A_n \sqcup B_n \sqcup C_n \sqcup D_n .$$

The proof of this theorem is omitted here due its length and technicality.

If we let $|\varepsilon_{n,1}| = e_{n,1}$, the number of alternating permutations with one alternating descent can be found by the recurrence relation defined as follows [3]:

$$e_{n,1} = e_{n-1,1} + F(n) - 1$$

where $F(n)$ is the function giving the Fibonacci numbers.

Further, if we let $F(n) - 1 = \int n$, we can express this recursively as [4]:

$$\int_n = \int_{n-1} + \int_{n-2} + 1$$

From Examples 1.9, 1.10 and 1.11, we know that $|\varepsilon_{1,1}| = |\varepsilon_{2,1}| = 0$, $|\varepsilon_{3,1}| = 1$ and $|\varepsilon_{4,1}| = 3$, therefore we will begin this sequence by enumerating $|\varepsilon_{3,1}|$. If we let $e_{3,1} = 1$, $e_{4,1} = 3$, $\int_1 = 0$, and $\int_2 = 0$, we acquire the sequence:

$$1\ 3\ 7\ 14\ 26\ 46\ 79\ \dots$$

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Matthew Francis, Brittany Fitzpatrick, and Kate Dovenberg take a quick break from presenting their research at the 10th Annual Natural Sciences, Mathematics, and Technology Showcase.