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Complexity of Some Geometric Problems

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Abstract

We show that recognizing intersection graphs of convex sets has the same complexity as deciding truth in the existential first-order theory of the reals. Comparing this to similar results on the rectilinear crossing number and intersection graphs of line segments, we argue that there is a need to recognize this level of complexity as its own class.

1 Introduction

We show that determining whether a graph can be realized as an intersection graph of convex sets in the plane has the same complexity as deciding the truth of existential first-order sentences over the real numbers. This connection between geometry and logic is not uncommon: Kratochvíl and Matoušek [8], for example, showed that recognizing intersection graphs of line segments also has the same complexity as the first-order existential theory of the reals (this paper includes a new, slightly simplified proof of that result), and there are several other geometric problems that share the same complexity. We therefore suggest the introduction of a new¹ complexity class $\exists\mathbb{R}$, which captures the complexity of deciding the truth of the existential theory of the reals. The first combinatorial problem shown complete for $\exists\mathbb{R}$ was stretchability of pseudoline arrangements [10, 14, 13] (Mnëv shows

*Some of this work was done in the beautiful library at Oberwolfach during the seminar on Discrete Geometry in September 2008.

¹The complexity class is not entirely new, it has a name in the Blum-Shub-Smale model of computing over the reals, which is $\text{BP}(\mathbf{NP}_{\mathbb{R}}^0)$ [2]; however, it does not seem to play a major role in that model (as reflected by the complexity of the notation).

a much stronger result, his universality theorem, about the realizability of semi-algebraic sets through point-set configurations), and there have been several examples since. Very often, however, $\exists\mathbb{R}$ -completeness is not claimed explicitly; for example, in the case of the rectilinear crossing number, Bienstock gave a reduction from stretchability to the rectilinear crossing number problem. Since the problem can easily be shown to lie in $\exists\mathbb{R}$ (see Section 3) computing the rectilinear crossing number is $\exists\mathbb{R}$ -complete. So—in a sense—the complexity of the problem is known precisely, but it is not unusual to see the complexity question for the rectilinear crossing number listed as an open problem [11]. There is some good reason for that: we do not know how to capture $\exists\mathbb{R}$ well with respect to classical complexity classes: we know that it contains **NP** (this follows easily from the definition of $\exists\mathbb{R}$; alternatively, Shor gave a direct proof that stretchability is **NP**-hard [14]) and is itself contained in **PSPACE**, a remarkable improvement on Tarski’s original decision procedure for the theory of reals by Canny [3]. So, in a sense, we do *not* know the complexity of the rectilinear crossing number problem, since we can only position it between **NP** and **PSPACE**. However, we would like to argue that it is more productive to approach the situation in the same spirit as we do **NP**-completeness; **NP**-completeness of a problem does not exclude the possibility that the problem is in **P** or **EXP**-complete, but proving it **NP**-complete focuses that question on the real issue, namely not the particular problem, but the study of the structural aspects of **NP**-completeness as a hole. Something similar should be possible for $\exists\mathbb{R}$ -completeness. Knowing a problem is $\exists\mathbb{R}$ -complete does not tell us more than that it is **NP**-hard and in **PSPACE** in terms of classical complexity, but it does tell us where to start the attack: by studying the structure of $\exists\mathbb{R}$ -complete problems; so asking, like [11], whether the rectilinear crossing number can be decided in **NP** is really asking whether $\exists\mathbb{R}$ lies in **NP**. And that puts a different perspective on the problem. A solution will likely not come out of graph theory, or graph drawing, but out of a better understanding of real algebraic geometry and logic; what satisfiability is for **NP**, the existential theory of the reals is for $\exists\mathbb{R}$.

This note is only an initial attempt, classifying one new problem, intersection graphs of convex sets. We are working on a more exhaustive survey, that includes other problems.

2 Background

The existential theory of the reals is the set of true sentences of the form

$$(\exists x_1, \dots, x_n)[\varphi(x_1, \dots, x_n),$$

where φ is a quantifier-free Boolean formula (without negation) over the signature $(0, 1, +, *, <)$ interpreted over the universe of real numbers. It was first shown by Tarski that this theory is decidable; it is now known to be decidable in **PSPACE** by a result of Canny [3].

By disallowing negation, we restrict ourselves to strict inequalities, which is the version of the problem relevant to the examples presented in the current note; let us call the set of true sentences of this theory **STRICT INEQ**. With this we define the complexity class $\exists\mathbb{R}$ as the closure of **STRICT INEQ** under polynomial-time reductions. Note that $\mathbf{NP} \subseteq \exists\mathbb{R}$, since we can express satisfiability of a Boolean formula in $\exists\mathbb{R}$. For example, $(x \vee \neg y \vee z) \wedge (\neg x \vee y \vee z) \wedge (\neg x \vee \neg y \vee \neg z)$ is equivalent to

$$\begin{aligned} (\exists x, y, z)[& (-\varepsilon < x < 2) \wedge (-\varepsilon < y < 2) \wedge (-\varepsilon < z < 2) \\ & \wedge (x(1-y)z) + ((1-x)yz) + ((1-x)(1-y)(1-z)) < \varepsilon], \end{aligned}$$

if we choose $\varepsilon = 1/8(1 + 4m) = 1/104$ where m is the number of clauses, so $m = 3$ in the example. If the formula is satisfiable, then we assign a variable the value 0 if it is true and 1 otherwise, so that the sum becomes 0 which is less than ε ; in the example: $x = y = 0$ and $z = 1$ will do. For the reverse direction, assume that we have $x, y,$ and z satisfying the real formula. Each term of the sum is at least $-\varepsilon \cdot 2^2 = -4\varepsilon$; so the whole sum is at least $-4m\varepsilon \geq -12/104$, but then for the whole sum to be less than $1/104$ every term must be at most $1/104 + 12/104 = 1/8$. Now each term is the product of three factors, so at least one factor must be at most $(1/8)^{1/3} = 1/2$. Let the corresponding variable be true if the factor is of the form x and false if it is of the form $1 - x$. This yields a satisfying assignment of the original Boolean formula.

A *pseudoline* is a simple closed curve in the projective plane that is homeomorphic to a straight line. An *arrangement of pseudolines* is a collection of pseudolines so that each pair of pseudolines cross at most once (and do not touch). An arrangement is *simple* if no more than two pseudolines pass through a point. Two arrangements are *equivalent* if there is a homeomorphism of the projective plane turning one into the other. An arrangement of pseudolines is *simply stretchable* if it is equivalent to a simple arrangement of

straight lines. (So being simply stretchable means the original arrangement is simple and stretchable.)

Remark 2.1. If one wants to avoid the reference to the projective plane, one can define pseudolines in the plane as simple *x-monotone* curves, that is curves that cross every vertical line exactly once. If one takes this route, one needs to require that in an arrangement of pseudolines every pair of pseudolines crosses *exactly* once (as opposed to at most once).

Mnëv showed that STRICT INEQ reduces to SIMPLE STRETCHABILITY; since the reverse is also true, SIMPLE STRETCHABILITY is $\exists\mathbb{R}$ -complete. Shor later simplified the reduction [14]. From this it immediately follows that SIMPLE STRETCHABILITY is **NP**-hard, since $\exists\mathbb{R}$ -hardness implies **NP**-hardness as we saw above.

$\exists\mathbb{R}$ -hard problems typically require large representations; Goodman, Pollock and Sturmfels [6] showed that there are stretchable arrangements of n pseudolines whose coordinate representation requires 2^{cn} bits for some constant $c > 0$. (Equivalently, if we want to draw the endpoints on a grid, it must have size at least $2^{2^{c'n}}$ for some $c' > 0$.) Typically, reductions from an $\exists\mathbb{R}$ -hard problem A to another problem B are *geometric* in the sense that if we are given a geometric representation of B , we can derive a geometric representation of A which is of at most polynomial size in the bit-size of the original representation. For example, this is the case for Bienstock's reduction from simple stretchability to rectilinear crossing number. We can then conclude (as Bienstock did) that there are graphs for which any straight-line drawing with optimal rectilinear crossing number requires 2^{cn} bits of storage. All other reductions in this note are also geometric, so geometric representations of these problem will require exponential precision.

3 Rectilinear Crossing Number

The *rectilinear crossing number* of G , $\text{lin-cr}(G)$, is the smallest number of crossings in a straight-line drawing of G , that is, a drawing in which every edge is represented by a straight-line segment and at most two edges intersect in a point. The problem is **NP**-hard by Garey and Johnson's original proof that computing the crossing number is **NP**-hard [5] and it remains **NP**-hard even if the graph is cubic and a rotation system is given [7, 12]. Bienstock gave an easy and elegant reduction that shows that SIMPLE STRETCHABILITY reduces to deciding whether $\text{lin-cr}(G) \leq k$, even if G is restricted to be cubic [1].

Theorem 3.1 (Bienstock [1]). *Computing the rectilinear crossing number of a (cubic) graph is $\exists\mathbb{R}$ -complete. There are graphs for which the coordinates of the vertices in an lin-cr-optimal drawing of the graph require exponential precision (in the size of the graph).*

Proof. $\exists\mathbb{R}$ -hardness follows from Bienstock's reduction as does the claim about exponential precision, so we only have to show that determining whether $\text{lin-cr}(G) \leq k$ lies in $\exists\mathbb{R}$; the only, small, difficulty is that we do not know which edges of the graph cross, so we need to guess a subset of pairs of edges of size at most k using real numbers.

For three points $(x_i, y_i)_{i \in [3]}$ we define a predicate *left* which is true if no two of the points are on a common horizontal or vertical line and (x_3, y_3) lies to the left of the line passing through (x_1, y_1) and then (x_2, y_2) :

$$\begin{aligned} \text{left}(x_1, y_1, x_2, y_2, x_3, y_3) \\ := & \bigwedge_{1 \leq i < j \leq 3} ((x_i < x_j) \vee (x_i > x_j)) \wedge \bigwedge_{1 \leq i < j \leq 3} ((y_i < y_j) \vee (y_i > y_j)) \\ & \wedge ((x_2 - x_1)y_3 > x_1 + (y_2 - y_1)(x_3 - x_1)), \end{aligned}$$

and, similarly, *right*. With these we can define $\overline{\text{colinear}}$ to express that three points are not colinear and no two of them determine a horizontal or vertical line:

$$\begin{aligned} \overline{\text{colinear}}(x_1, y_1, x_2, y_2, x_3, y_3) \\ := \text{left}(x_1, y_1, x_2, y_2, x_3, y_3) \vee \text{right}(x_1, y_1, x_2, y_2, x_3, y_3). \end{aligned}$$

Also, we can define a predicate $\overline{\text{cross}}(x_1, y_1, x_2, y_2, x'_1, y'_1, x'_2, y'_2)$ that is true if and only if the two line segments determined by $(x_1, y_1), (x_2, y_2)$ and $(x'_1, y'_1), (x'_2, y'_2)$ do not have a point in common:

$$\begin{aligned} \overline{\text{cross}}(x_1, y_1, x_2, y_2, x'_1, y'_1, x'_2, y'_2) \\ := & (\text{left}(x_1, y_1, x_2, y_2, x'_1, y'_1) \wedge \text{left}(x_1, y_1, x_2, y_2, x'_2, y'_2)) \\ & \vee (\text{right}(x_1, y_1, x_2, y_2, x'_1, y'_1) \wedge \text{right}(x_1, y_1, x_2, y_2, x'_2, y'_2)) \\ & \vee (\text{left}(x'_1, y'_1, x'_2, y'_2, x_1, y_1) \wedge \text{left}(x'_1, y'_1, x'_2, y'_2, x_2, y_2)) \\ & \vee (\text{right}(x'_1, y'_1, x'_2, y'_2, x_1, y_1) \wedge \text{right}(x'_1, y'_1, x'_2, y'_2, x_2, y_2)). \end{aligned}$$

For a fixed k and $m = |E(G)|$, we can write a predicate $\text{atmost}_k(z_1, \dots, z_{m^2})$ which guarantees that at most k of the z_i are greater than zero:

$$\bigwedge_{i \in [m^2]} ((-1/2m^4 < z_i < 0) \vee (1 + 1/2m^2 < z_i)) \wedge \sum_{i \in [m^2]} z_i < k + 1.$$

Since $\text{lin-cr}(G) \leq \binom{m}{2}$, we can assume that $k \leq \binom{m}{2}$; so the sum of the negative z_i is at least $-1/2m^2$. If more than k of the z_i are positive, their sum is at least $k+1 + (k+1)/2m^2$, but then the total sum is at least $k+1$. On the other hand, given any subset of the z_i of size at most k , we can assign each z_i in the set $1 + 2/3m^2$ and every other z_i gets the value $-2/3m^4$, so that $\sum_{i \in [m^2]} z_i \leq k + 2/3 < k+1$, showing that any subset of the z_i can be realized by atmost_k .

With these predicates, we can express $\text{lin-cr}(G) \leq k$; to simplify the formula, suppose that $V(G) = [n]$, $E(G) = [m]$, and we have two functions $h, t : E \rightarrow V$ so that $h(e) = x$ and $t(e) = y$ if $e \in E$ is an edge between $x, y \in V$. We use $z_{(i-1)m+j} > 0$ to indicate that edges i and j are allowed to cross. Now $\text{lin-cr}(G) \leq k$ if and only if

$$\begin{aligned} & (\exists x_1, y_1, \dots, x_n, y_n, z_1, \dots, z_m) [\text{atmost}_k(z_1, \dots, z_m) \\ & \wedge \bigwedge_{i < j < k \in [n]} \overline{\text{colinear}}(x_i, y_i, x_j, y_j, x_k, y_k) \\ & \wedge \bigwedge_{\substack{i < j \in [m], \\ i, j \text{ non-adjacent}}} (z_{(i-1)m+j} > 0) \vee \overline{\text{cross}}(x_{h(i)}, y_{h(i)}, x_{t(i)}, y_{t(i)}, x_{h(j)}, y_{h(j)}, x_{t(j)}, y_{t(j)})]. \end{aligned}$$

□

4 Intersection Graphs of Segments

$G = (V, E)$ is an *intersection graph of line segments* if for each $v \in V$ there is a line segment ℓ_v in the plane so that $uv \in E$ if and only if ℓ_u and ℓ_v intersect.

Theorem 4.1 (Kratochvíl, Matoušek [8]). *Recognizing intersection graphs of line segments is $\exists\mathbb{R}$ -complete. There are graphs for which the coordinates of the endpoints of the line segments in any intersection representation of the graph require exponential precision (in the size of the graph).*

Remark 4.2. Kratochvíl and Pergel showed that the recognition of intersection graphs of line segments remains **NP**-hard if the graphs have girth at least k for any fixed k [9]. Can this be extended to $\exists\mathbb{R}$ -completeness?

We give a slightly simplified proof of Theorem 4.1; the argument will also be used in Theorem 5.1.

Lemma 4.3. *Suppose we have Jordan curves ℓ , $(\ell_i)_{i \in [n]}$, $(s_i^j)_{i \in [n-1], j \in [3]}$, and $(c_i)_{i \in [4n]}$ in the plane so that*

- (i) ℓ crosses ℓ_i , $i \in [n]$, and s_i^2 , $i \in [n-1]$,
- (ii) c_i crosses c_{i+1} (c_1 for $i = 4n$) exactly once, $i \in [4n]$,
- (iii) ℓ_i crosses c_{2i} and $c_{4n-2i+2}$, $i \in [n]$,
- (iv) both s_i^1 and s_i^3 cross s_i^2 , $i \in [n-1]$,
- (v) s_i^1 crosses c_{2i+1} and s_i^3 crosses $c_{4n-2i+1}$, $i \in [n-1]$,
- (vi) the only other crossings among these curves are between pairs of ℓ_i .

Then the curves ℓ_i cross ℓ either in order ℓ_1, \dots, ℓ_n or in the reverse of that order. The conclusion remains true if instead of (i) we only require that (i') ℓ crosses ℓ_i , $i \in [n]$, and (i'') s_i^2 , $i \in [n-1]$, may cross ℓ , but it does lie in the same connected component of $\mathbb{R}^2 - \cup_{i \in [4n]} c_i$ as ℓ .

We call the collection of curves $(s_i^j)_{i \in [n-1], j \in [3]}$, and $(c_i)_{i \in [4n]}$ and the way they cross each other and the curves ℓ and $(\ell_i)_{i \in [n]}$ the *ordering gadget* for ℓ with respect to $(\ell_i)_{i \in [n]}$. The intended drawing of the curves of the lemma is shown in Figure 1, but there are other drawings.

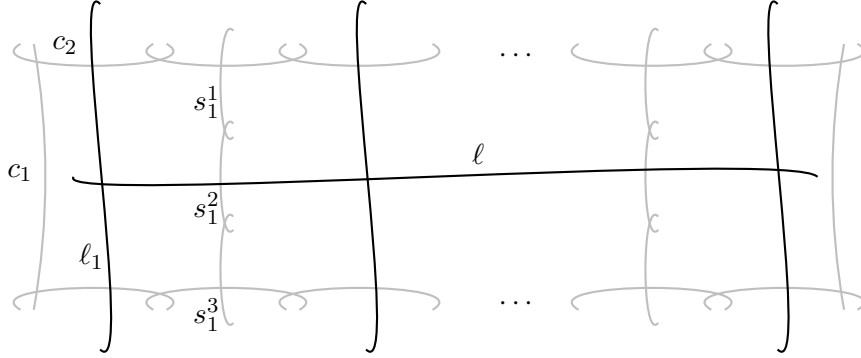


Figure 1: An ordering gadget

Proof. The set $\cup_{i \in [4n]} c_i$ contains a (unique) closed Jordan curve C . C separates the plane into two faces; without loss of generality (since we are dealing with curves), we may assume that ℓ lies in the inner face. Since ℓ crosses every s_i^2 , and these curves do not cross any c_j , all the s_i^2 also lie in the inner face of C (indeed, (i') is sufficient to draw this conclusion). Within each $S_i := c_{2i+1} \cup s_i^1 \cup s_i^2 \cup s_i^3 \cup c_{4n-2i+1}$, $i \in [n-1]$ choose a

Jordan arc s_i with endpoints on C . The s_i are chords of C that lie in the inner face of C (since s_i^2 does); moreover, the s_i do not intersect each other (since any two S_i are disjoint) or any of the ℓ_i (since S_i and ℓ_j are disjoint for all $i, j \in [n]$). Now the ends of the s_i and ℓ_i along C are in order $\ell_1, s_1, \ell_2, \dots, s_{n-1}, \ell_n, \ell_n, s_{n-1}, \dots, s_1, \ell_1$ (up to cyclic shifts). Since every ℓ_i has to cross ℓ and has to do so within C , it must do so in order ℓ_1, \dots, ℓ_n or its reverse. \square

Proof of Theorem 4.1. It is easy to see that the problem lies in $\exists\mathbb{R}$. To show $\exists\mathbb{R}$ -hardness, we reduce from SIMPLE STRETCHABILITY. Suppose we are given a simple arrangement \mathcal{A} of pseudolines. Remark 2.1 allows us to think of the arrangement as a set of simple, x -monotone curves.

Add a triangle T formed by three pairwise crossing curves so that all crossings of curves in \mathcal{A} lie within the region enclosed by T and one edge of T crosses all curves in \mathcal{A} (for example, choose a vertical line segment to the left of all crossings in \mathcal{A} that is long enough to cross every curve in \mathcal{A}). We can choose T so that we know the order of crossings of curves belonging to \mathcal{A} with the curves of T .

Cut off the pseudolines just beyond the boundary of T and for each curve ℓ (including the curves from T) add the ordering gadget for ℓ with respect to all remaining curves as constructed in Lemma 4.3. Also, require that curves c_{2i} and $c_{4n-2i+2}$ for ℓ cross the corresponding two c -curves of ℓ_i (see Figure 2). Let $G_{\mathcal{A}}$ be the resulting intersection graph of all curves.

In any (curvilinear) drawing of $G_{\mathcal{A}}$, the order of crossings along each curve from \mathcal{A} and T with curves from that set is as in the original arrangement or reversed by Lemma 4.3, since we added ordering gadgets for each of those curves. In particular, the crossings with T are first and last along each curve from \mathcal{A} , and therefore all crossings between curves of \mathcal{A} occur within the region enclosed by T . Now the crossings of \mathcal{A} with the edge of T that crosses all curves in \mathcal{A} occur either in the original order or in the reversed order. However, this means that the order of crossings along edges realizing \mathcal{A} must either all be in the original order, or all of them are reversed. Hence, if $G_{\mathcal{A}}$ can be realized as an intersection graph of straight-line segments, then \mathcal{A} is stretchable.

It is easy to see that if the original arrangement \mathcal{A} is stretchable, then so is the extended arrangement (the intended drawing of the ordering gadget is shown in Figure 1).

Finally, the reduction is geometric, so the claim about exponential precision follows. \square

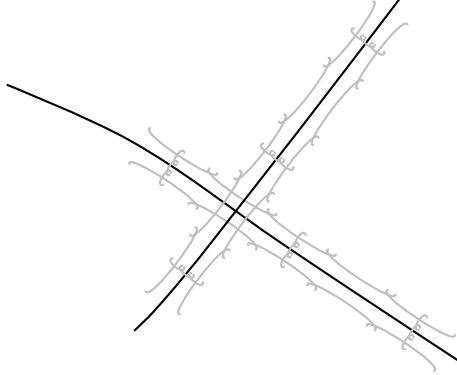


Figure 2: Two arrangement lines crossing, with gadgets.

5 Intersection Graphs of Convex Sets

$G = (V, E)$ is an *intersection graph of convex sets* if for every $v \in V$ there is a convex set C_v in the plane so that $uv \in E$ if and only if C_u and C_v intersect. We say two regions in the plane *intersect* if they share a common point. The problem is known to be in **PSPACE** and **NP-hard** [8].

Theorem 5.1. *Recognizing intersection graphs of convex sets is $\exists\mathbb{R}$ -complete. There are graphs for which any realization as intersection graphs of convex polygons requires exponential precision in writing down the coordinates of the vertices of the polygon.*

For the $\exists\mathbb{R}$ -hardness proof we carefully adapt the reduction from SIMPLE STRETCHABILITY to SEG we saw in Theorem 4.1, and we begin by restating Lemma 4.3 for convex sets.

Lemma 5.2. *Suppose we have convex sets L , $(L_i)_{i \in [n]}$, $(S_i^j)_{i \in [n-1], j \in [3]}$, and $(C_i)_{i \in [4n]}$ in the plane so that*

- (i) L intersects L_i , $i \in [n]$, and S_i^2 , $i \in [n-1]$,
- (ii) C_i intersects C_{i+1} (C_1 for $i = 4n$), $i \in [4n]$,
- (iii) L_i intersects C_{2i} and $C_{4n-2i+2}$, $i \in [n]$,
- (iv) both S_i^1 and S_i^3 intersect S_i^2 , $i \in [n-1]$,
- (v) S_i^1 intersects C_{2i+1} and S_i^3 intersects $C_{4n-2i+1}$, $i \in [n-1]$,

(vi) the only other intersections among these regions are between pairs of L_i .

Moreover, suppose we have Jordan curves ℓ in L and ℓ_i in L_i , $i \in [n]$ so that every ℓ_i crosses ℓ . Then the order of the intersections along ℓ is either ℓ_1, \dots, ℓ_n or the reverse of that order.

We call the collection of convex sets $(S_i^j)_{i \in [n], j \in [3]}$, and $(C_i)_{i \in [8n-4]}$ and the way they intersect each other and the sets L and $(L_i)_{i \in [n]}$ the *ordering gadget* for L with respect to $(\ell_i)_{i \in [n]}$. The intended drawing of the convex sets is similar to the one shown in Figure 1 with line segments replaced by convex sets.

Proof. Pick vertices $v_i \in C_i \cap C_{i+1}$, $i \in [4n]$, and $v_{4n} \in C_{4n} \cap C_1$, and let c_i be a straight-line segment in C_i connecting v_i to v_{i+1} (v_1 for $i = 4n$). Then the c_i form a cycle C without crossings (since any two non-adjacent segments of C belong to disjoint convex sets). Now we can extend ℓ_i in $L_i \cup C_{2i}$ so it connects to v_{2i-1} and in $L_i \cup C_{4n-2i+2}$ so it connects to $v_{4n-2i+1}$ without crossing C . Pick vertices $t_i^1 \in S_i^1 \cap S_i^2$ and $t_i^2 \in S_i^2 \cap S_i^3$. We can connect t_i^1 by a curve s_i^1 in $S_i^1 \cup C_{2i+1}$ to v_{2i} and t_i^2 by a curve s_i^3 in $S_i^3 \cup C_{4n-2i+1}$ to v_{4n-2i} without crossing any of the curves we have already constructed; finally, we can connect t_i^1 to t_i^2 within S_i^2 by a curve s_i^2 not crossing any other curve except, possibly, ℓ . Now extend the curves we have constructed slightly, so that shared endpoints become crossing points. The resulting curves fulfill Lemma 4.3 with conditions (i') and (ii') replacing (i): (ii') is true, since L intersects S_i^2 , $i \in [n-1]$ and none of these sets intersect C , so they must all lie on the same side of C . Now Lemma 4.3 allows us to conclude that ℓ is crossed by $(\ell_i)_{i \in [n]}$ in order ℓ_1, \dots, ℓ_n or the reverse of that order. \square

Proof of Theorem 5.1. To see that the problem lies in $\exists \mathbb{R}$ we first observe that every convex set can be replaced by a convex compact polygon with at most $O(n)$ vertices, where n is the number of convex sets: For each pair of convex sets that are required to intersect we pick a witness point in the intersection. Now replace every set with the convex hull of the witness points that lie within it (if there is no witness point in the set, we can simply eliminate that set, since it does not affect realizability); this does not change the intersection graph represented by the sets; moreover all sets are now compact polygons (with a total of at most $O(n^2)$ boundary vertices). In $\exists \mathbb{R}$, we can guess those vertices and verify that for any two convex sets that do not intersect there is a line so that the two convex sets are on opposite sides of the line. This suffices.

Suppose we are given a simple arrangement \mathcal{A} of pseudolines. As earlier, we think of the arrangement as a set of simple, x -monotone curves.

Let D be a disk which contains all the crossings of the pseudolines in its interior. Cut all the pseudolines at the boundary of D and let their order of intersection with the boundary be $A_1, \dots, A_n, A_1, \dots, A_n$. Add sets $(B_i)_{i \in [2n]}$, required to intersect cyclically: B_i with B_{i+1} and B_{2n} with B_1 , with no other intersections, and sets $(H_i)_{i \in [2n]}$, so that H_i intersects B_i , and A_i if $i \leq n$ and A_{i-n} otherwise. Now for each of the A -, B - and H -sets add the ordering gadget described in Lemma 5.2. Call the resulting intersection graph $G_{\mathcal{A}}$. (We will only make use of the ordering gadgets for $(A_i)_{i \in [n]}$, but we need to add them in such a way that they allow for the intersections with the other sets as well.)

If \mathcal{A} is stretchable, then the intersection graph we specified is realizable by convex sets (actually by line segments).

So suppose there is a drawing of convex sets realizing $G_{\mathcal{A}}$. Pick a vertex $u_i \in B_i \cap B_{i+1}$ for $i \in [2n-1]$, and $u_{2n} \in B_{2n} \cap B_1$, and let b_i be a straight-line segment between u_i and u_{i+1} (u_1 for $i = 2n$). Then the b_i form a cycle B (without crossings). None of the A_i intersect any of the B_j so all A_i must be on the same side of B . For each A_i , $i \in [n]$, pick a straight-line segment ℓ_i that starts in $A_i \cap H_i$ and ends in $A_i \cap H_{i+n}$. We claim that any two ℓ_i cross each other: each ℓ_i can be extended through H_i and H_{i+n} to connect to the cycle B . But then since two ℓ_i connect to alternating endpoints along B and both curves are on the same side of the cycle, the curves must cross; since the H_i do not intersect each other, that crossing must occur along the straight-line segments ℓ_i .

Now Lemma 5.2 implies that the order of crossings along each ℓ_i is either the original order or the reversed order; however, since the order of intersection with D is fixed by the cycle B , either all those orders are in the original order, or they are all reversed. But then, in either case, \mathcal{A} is stretchable.

The claim about exponential precision again follows because the reduction we gave is geometric. \square

Remark 5.3. As a simple application, we can give a new proof of a result in [4] which showed that RCC8 (the region-connection calculus, that is, topological inference) together with a predicate for convexity of regions has a consistency problem that is $\exists\mathbb{R}$ -complete. We will include details in the full version.

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