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New multi-parameter Golay 2-complementary sequences and transforms

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Abstract. In this work, we develop a new unified approach to the so-called generalized Golay–Rudin–Shapiro (GRS) 2-complementary multi-parameter sequences. It is based on a new generalized iteration generating construction.

1. Introduction

Binary ± 1 -valued *Golay–Rudin–Shapiro* sequences (2-GRSS) associated with the cyclic group \mathbf{Z}_2^n were introduced independently by Golay [1, 2, 3] in 1949-1951, Shapiro [4,5] and Rudin [6] in 1951. M.J.E. Golay [2] introduced the general concept of "complementary pairs" of finite sequences all of whose entries are ± 1 . This was motivated by a highly non-trivial applications of infrared spectrometry. Then he gave an explicit construction for binary Golay complementary pairs of length 2^m and later [3] noted that the construction implies the existence of at least $2^m m! / 2$ binary Golay sequences of this length. They are known to exist for all lengths $N = 1^\alpha 10^\beta 26^\gamma$, where α, β, γ are integers and $\alpha, \beta, \gamma \geq 0$ (Turyan, [7]), but do not exist for any length N having a prime factor congruent to modulo 4 (Eliahou et al., [8]). In 1951, H. S. Shapiro [4, 5] introduced what became known, after 1963, as the "Rudin–Shapiro" polynomial pairs. Shapiro's work was entirely in pure mathematics. Budisin [9,10,11] using the work of Sivaswamy [12] gave a more general recursive construction for Golay complementary pairs and showed that the set of all binary Golay complementary pairs of length 2^m obtainable from it coincides with those given explicitly by Golay. For a survey of results on binary and non-binary Golay complementary pairs, see Byrnes [13] and Fan, Darnel [14], respectively. In 1999, Davis and Jedwab [15] gave an explicit description of a large class of Golay complementary sequences in terms of certain cosets of the first order Reed–Muller codes.

Discrete *Fourier–Golay–Rudin–Shapiro Transforms* (FGRST) in bases of different Golay–Rudin–Shapiro sequences can be used in many signal processing applications: multiresolution by discrete



orthogonal wavelet decomposition, digital audition, digital video broadcasting, communication systems (OFDM, MCDA), radar, and cryptographic systems.

To build the classical FGRST, the following actors are used: 1) the Abelian group \mathbf{Z}_2^n , 2) 2-point Fourier transform \mathcal{F}_2 , and 3) the complex field \mathbf{C} ; i.e., these transforms are associated with the triple $(\mathbf{Z}_2, \mathcal{F}_2, \mathbf{C})$. In this work, we develop a new unified approach to the so-called generalized complex-, $\mathbf{GF}(p)$ -, and Clifford-valued complementary sequences. The approach is associated not with the triple $(\mathbf{Z}_2, \mathcal{F}_2, \mathbf{C})$, but with $(\mathbf{Z}_2, \mathcal{CS}_2(\varphi, \alpha, \gamma), Alg)$ and $(\mathbf{Z}_2, \{\mathcal{CS}_2^1(\varphi_1, \alpha_1, \gamma_1), \dots, \mathcal{CS}_n^1(\varphi_n, \alpha_n, \gamma_n)\}, Alg)$, where $\mathcal{CS}_2^1(\varphi_1, \alpha_1, \gamma_1)$ and $\{\mathcal{CS}_2^1(\varphi_1, \alpha_1, \gamma_1), \mathcal{CS}_2^2(\varphi_2, \alpha_2, \gamma_2), \dots, \mathcal{CS}_n^1(\varphi_n, \alpha_n, \gamma_n)\}$ are a single transform or a set of arbitrary unitary (2×2) -transforms of type $\mathcal{CS}_2(\varphi, \alpha, \gamma) = \begin{bmatrix} e^{i\alpha} \cos \varphi & e^{i\gamma} \sin \varphi \\ e^{-i\gamma} \sin \varphi & -e^{-i\alpha} \cos \varphi \end{bmatrix}$, instead of $\mathcal{F}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ if $Alg = \mathbf{C}$.

The rest of the paper is organized as follows: in Section 2, the object of the study (*Golay–Rudin–Shapiro* binary sequences) is described. In Section 3, the iteration rule to construct the Golay matrix is introduced. In Section 4, the proposed method based on a new generalized iteration construction is explained to construct generalized multi-parameter Golay–Rudin–Shapiro sequences.

2. The object of the study

We begin with the description of the original Golay 2-complementary ± 1 -valued sequences.

Definition 1. Let $\text{com}^0(t) := (c_0, c_1, \dots, c_{N-1})$ and $\text{com}^1(t) := (s_0, s_1, \dots, s_{N-1})$, where $c_i, s_i \in \{\pm 1\}$. Both sequences $\text{com}^0(t), \text{com}^1(t)$ are called the (± 1) -valued complementary or Golay complementary pair over $\{\pm 1\}$, if $COR^0(\tau) + COR^1(\tau) = N\delta(\tau)$, or $\left(|COM^0(z)|^2 + |COM^1(z)|^2 \right)_{|z|=1} = N$, where $COR^0(\tau), COR^1(\tau)$ are the periodic correlation functions of $\text{com}^0(t), \text{com}^1(t)$ and $COM^0(z) = \mathcal{Z}\{\text{com}^0(t)\}, COM^1(z) = \mathcal{Z}\{\text{com}^1(t)\}$ are their \mathcal{Z} -transforms. Any sequence, which is a member of a Golay complementary pair, is called the Golay sequence and its \mathcal{Z} -transform $COM_k(z) = \mathcal{Z}\{\text{com}_k(t)\}$ is called the *Golay-Shapiro-Rudin polynomials*. We use two symbols $\mathbf{a}_n \in [0, 2^{n-1} - 1] = \mathbf{Z}_{2^n}$ and $\mathbf{t}_n \in [0, 2^{n-1} - 1] = \mathbf{Z}_{2^n}$ to enumerate Golay sequences and the discrete time, respectively. For integer $\mathbf{a}_n \in [0, 2^{n-1} - 1]$ and $\mathbf{t}_n \in [0, 2^{n-1} - 1]$ we shall use binary codes $\bar{\mathbf{a}}_n = (\alpha_1, \alpha_2, \dots, \alpha_n), \bar{\mathbf{t}}_n = (t_1, t_2, \dots, t_n)$, where $\alpha_i, t_i \in \{0, 1\}$. Let $\bar{\mathbf{a}}_n = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\bar{\mathbf{t}}_n = (t_1, t_2, \dots, t_n)$ be binary codes, then define

$$\mathbf{a}_n = |\bar{\mathbf{a}}_n| = |(\alpha_1, \alpha_2, \dots, \alpha_n)| = \sum_{i=1}^n \alpha_{n-i+1} 2^{i-1}, \quad \mathbf{t}_n = |\bar{\mathbf{t}}_n| = |(t_1, t_2, \dots, t_n)| = \sum_{i=1}^n t_{n-i+1} 2^{n-i}$$

as integers whose binary codes are $\bar{\mathbf{a}}_n = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\bar{\mathbf{t}}_n = (t_1, t_2, \dots, t_n)$, where α_n, t_1 are less significant bits (LSB) and α_1, t_n are most significant bits (MSB) of $\bar{\mathbf{a}}_n = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\bar{\mathbf{t}}_n = (t_1, t_2, \dots, t_n)$, respectively. Obviously,

$$\begin{array}{llll} \bar{\mathbf{a}}_1 = (\alpha_1) \in \mathbf{Z}_2, & \mathbf{a}_1 = \alpha_1 \in \mathbf{Z}_2, & \bar{\mathbf{t}}_1 = (t_1) \in \mathbf{Z}_2, & \mathbf{t}_1 = t_1 \in \mathbf{Z}_2, \\ \bar{\mathbf{a}}_2 = (\bar{\mathbf{a}}_1, \alpha_2) \in \mathbf{Z}_2 \times \mathbf{Z}_2 = \mathbf{Z}_2^2, & (\mathbf{a}_1, \alpha_2) \in \mathbf{Z}_2 \times \mathbf{Z}_2, & \bar{\mathbf{t}}_2 = (\bar{\mathbf{t}}_1, t_2) \in \mathbf{Z}_2 \times \mathbf{Z}_2 = \mathbf{Z}_2^2, & (\mathbf{t}_1, t_2) \in \mathbf{Z}_2 \times \mathbf{Z}_2, \\ \bar{\mathbf{a}}_3 = (\bar{\mathbf{a}}_2, \alpha_3) \in \mathbf{Z}_2^2 \times \mathbf{Z}_2 = \mathbf{Z}_2^3, & (\mathbf{a}_2, \alpha_3) \in \mathbf{Z}_2^2 \times \mathbf{Z}_2, & \bar{\mathbf{t}}_3 = (\bar{\mathbf{t}}_2, t_3) \in \mathbf{Z}_2^2 \times \mathbf{Z}_2 = \mathbf{Z}_2^3, & (\mathbf{t}_2, t_3) \in \mathbf{Z}_2^2 \times \mathbf{Z}_2, \\ \dots & \dots & \dots & \dots \\ \bar{\mathbf{a}}_n = (\bar{\mathbf{a}}_{n-1}, \alpha_n) \in \mathbf{Z}_2^{n-1} \times \mathbf{Z}_2 = \mathbf{Z}_2^n, & (\mathbf{a}_{n-1}, \alpha_n) \in \mathbf{Z}_2^{n-1} \times \mathbf{Z}_2, & \bar{\mathbf{t}}_n = (\bar{\mathbf{t}}_{n-1}, t_n) \in \mathbf{Z}_2^{n-1} \times \mathbf{Z}_2 = \mathbf{Z}_2^n, & (\mathbf{t}_{n-1}, t_n) \in \mathbf{Z}_2^{n-1} \times \mathbf{Z}_2, \end{array}$$

where $\mathbf{Z}_2^k = \{0, 1\}^k = \mathbf{Z}_2^k$ and $\mathbf{Z}_{2^k} = \{0, 1, 2, \dots, 2^k - 1\}$.

Let $\left\{ \text{com}_{(\bar{a}_n,0)}^{[n+1]}(\mathbf{t}_{n+1}), \text{com}_{(\bar{a}_n,1)}^{[n+1]}(\mathbf{t}_{n+1}) \right\}_{\mathbf{a}_n=0}^{2^n-1}$ be a set of 2^n pairs of complementary sequences of length 2^{n+1} .

Then the following matrix of depth $n+1$ has size $2^{n+1} \times 2^{n+1}$

$$\mathbf{G}_{2^{n+1}}^{[n+1]} = \bigoplus_{\mathbf{a}_{n+1}=0}^{2^{n+1}-1} \text{com}_{\mathbf{a}_{n+1}}^{[n+1]}(\mathbf{t}_{n+1}) = \bigoplus_{\mathbf{a}_n=0}^{2^n-1} \bigoplus_{\alpha_{n+1}=0}^1 \text{com}_{(\mathbf{a}_n, \alpha_{n+1})}^{[n+1]}(\mathbf{t}_{n+1}) = \bigoplus_{\mathbf{a}_n=0}^{2^n-1} \begin{bmatrix} \text{com}_{(\mathbf{a}_n,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(\mathbf{a}_n,1)}^{[n+1]}(\mathbf{t}_{n+1}) \end{bmatrix} =$$

$$= \bigoplus_{\mathbf{a}_{n-1}=0}^{2^{n-1}-1} \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1},0,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(\mathbf{a}_{n-1},0,1)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(\mathbf{a}_{n-1},1,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(\mathbf{a}_{n-1},1,1)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \dots \\ \text{com}_{(2^n-1,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(2^n-1,1)}^{[n+1]}(\mathbf{t}_{n+1}) \end{bmatrix} = \begin{bmatrix} \text{com}_{(0,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(0,1)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(1,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(1,1)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \dots \\ \text{com}_{(2^n-1,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(2^n-1,1)}^{[n+1]}(\mathbf{t}_{n+1}) \end{bmatrix} = \begin{bmatrix} \text{com}_{(0,0,\dots,0,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(0,0,\dots,0,1)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(0,0,\dots,1,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(0,0,\dots,1,1)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \dots \\ \text{com}_{(1,1,\dots,1,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(1,1,\dots,1,1)}^{[n+1]}(\mathbf{t}_{n+1}) \end{bmatrix} \tag{1}$$

and it is called the Golay matrix, where $\begin{bmatrix} \text{com}_{(\mathbf{a}_n,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(\mathbf{a}_n,1)}^{[n+1]}(\mathbf{t}_{n+1}) \end{bmatrix}$ are a pair of complementary sequences and

is the symbol of the vertical concatenation of $(2 \times 2^{n+1})$ -matrices $\begin{bmatrix} \text{com}_{(\mathbf{a}_n,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(\mathbf{a}_n,1)}^{[n+1]}(\mathbf{t}_{n+1}) \end{bmatrix}$. For example,

$$\mathbf{G}_{2^1}^{[1]} = \begin{bmatrix} \text{com}_0^{[1]}(\mathbf{t}_1) \\ \text{com}_1^{[1]}(\mathbf{t}_1) \end{bmatrix} = \bigoplus_{\mathbf{a}_1=0}^1 \text{com}_{\mathbf{a}_1}^{[1]}(\mathbf{t}_1), \quad \mathbf{G}_{2^2}^{[2]} = \left[\text{com}_{\mathbf{a}_2}^{[2]}(\mathbf{t}_2) \right]_{\mathbf{a}_2, \mathbf{t}_2=0}^3 = \bigoplus_{\mathbf{a}_1=0}^1 \begin{bmatrix} \text{com}_{(\mathbf{a}_1,0)}^{[2]}(\mathbf{t}_2) \\ \text{com}_{(\mathbf{a}_1,1)}^{[2]}(\mathbf{t}_2) \end{bmatrix} = \begin{bmatrix} \text{com}_{(0,0)}^{[2]}(\mathbf{t}_2) \\ \text{com}_{(0,1)}^{[2]}(\mathbf{t}_2) \\ \text{com}_{(1,0)}^{[2]}(\mathbf{t}_2) \\ \text{com}_{(1,1)}^{[2]}(\mathbf{t}_2) \end{bmatrix},$$

$$\mathbf{G}_{2^3}^{[3]} = \left[\text{com}_{\mathbf{a}_3}^{[3]}(\mathbf{t}_3) \right]_{\mathbf{a}_3, \mathbf{t}_3=0}^7 = \bigoplus_{\mathbf{a}_3=0}^7 \text{com}_{\mathbf{a}_3}^{[3]}(\mathbf{t}_3) = \bigoplus_{\mathbf{a}_2=0}^3 \begin{bmatrix} \text{com}_{(\mathbf{a}_2,0)}^{[3]}(\mathbf{t}_3) \\ \text{com}_{(\mathbf{a}_2,1)}^{[3]}(\mathbf{t}_3) \end{bmatrix} = \begin{bmatrix} \text{com}_{(0,0)}^{[3]}(\mathbf{t}_3) \\ \text{com}_{(0,1)}^{[3]}(\mathbf{t}_3) \\ \text{com}_{(1,0)}^{[3]}(\mathbf{t}_3) \\ \text{com}_{(1,1)}^{[3]}(\mathbf{t}_3) \\ \text{com}_{(2,0)}^{[3]}(\mathbf{t}_3) \\ \text{com}_{(2,1)}^{[3]}(\mathbf{t}_3) \\ \text{com}_{(3,0)}^{[3]}(\mathbf{t}_3) \\ \text{com}_{(3,1)}^{[3]}(\mathbf{t}_3) \end{bmatrix}.$$

3. Method. Iteration construction for original Golay sequences

The matrix $\mathbf{G}_{2^{n+1}}^{[n+1]}$ is constructed by the following iteration construction:

$$\mathbf{G}_{2^1}^{[1]} \xrightarrow{\mathcal{F}_2} \mathbf{G}_{2^2}^{[2]} \xrightarrow{\mathcal{F}_2} \dots \xrightarrow{\mathcal{F}_2} \mathbf{G}_{2^n}^{[n]} \xrightarrow{\mathcal{F}_2} \mathbf{G}_{2^{n+1}}^{[n+1]}. \tag{2}$$

The initial matrix $\mathbf{G}_{2^1}^{[1]}$ is formed by starting with the Fourier-Walsh (2×2) -matrix $\mathbf{G}_{2^1}^{[1]} = \mathcal{F}_2 = \begin{bmatrix} \text{com}_0^{[1]}(\mathbf{t}_1) \\ \text{com}_1^{[1]}(\mathbf{t}_1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and by the repeated application of the iteration construction to pairs of rows in the matrix. Let us suppose that we have the Golay matrix $\mathbf{G}_{2^n}^{[n]}$. We need to construct the next Golay matrix $\mathbf{G}_{2^{n+1}}^{[n+1]}$ using only $\mathbf{G}_{2^n}^{[n]}$ and $\mathcal{F}_2 = \mathbf{G}_{2^1}^{[1]}$. The Golay matrix $\mathbf{G}_{2^n}^{[n]}$ has a structure similar to (1):

$$\mathbf{G}_{2^n}^{[n]} = \begin{matrix} \boxed{2^n-1} \\ \boxed{} \\ \boxed{} \\ \boxed{} \\ \boxed{a_n=0} \end{matrix} \text{com}_{a_n}^{[n]}(\mathbf{t}_n) = \begin{matrix} \boxed{2^{n-1}-1} \\ \boxed{\phantom{2^{n-1}-1}} \\ \boxed{\phantom{2^{n-1}-1}} \\ \boxed{\phantom{2^{n-1}-1}} \\ \boxed{a_{n-1}=0} \end{matrix} \begin{bmatrix} \text{com}_{(a_{n-1},0)}^{[n]}(\mathbf{t}_{n+1}) \\ \text{com}_{(a_{n-1},1)}^{[n]}(\mathbf{t}_{n+1}) \end{bmatrix}. \quad (3)$$

To construct $\mathbf{G}_{2^{n+1}}^{[n+1]}$ from $\mathbf{G}_{2^n}^{[n]}$ we take each complementary pair of (3) in the form of $\begin{bmatrix} \text{com}_{(a_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \text{com}_{(a_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix}$

and construct shifted versa of their components

$$\begin{bmatrix} \mathbf{T}_{\mathbf{t}_n}^{2^n(0\oplus k)} \text{com}_{(a_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \mathbf{T}_{\mathbf{t}_n}^{2^n(1\oplus k)} \text{com}_{(a_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} = \text{diag}\left\{\mathbf{T}_{\mathbf{t}_n}^{2^n(0\oplus k)}, \mathbf{T}_{\mathbf{t}_n}^{2^n(1\oplus k)}\right\} \begin{bmatrix} \text{com}_{(a_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \text{com}_{(a_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} = \begin{bmatrix} \text{com}_{(a_{n-1},0)}^{[n]}(\mathbf{t}_n + 2^n(0\oplus k)) \\ \text{com}_{(a_{n-1},1)}^{[n]}(\mathbf{t}_n + 2^n(1\oplus k)) \end{bmatrix},$$

where $k=0,1$ and $\mathbf{T}_{\mathbf{t}_n}^{2^n s}$ is the shift operator on $2^n s$ positions in the time domain: $\mathbf{T}_{\mathbf{t}_n}^{2^n s} f(\mathbf{t}_n) := f(\mathbf{t}_n + 2^n s)$.

Now we construct the general building blocks for the Golay $(2^{n+1} \times 2^{n+1})$ -matrix $\mathbf{G}_{2^{n+1}}^{[n]}$:

$$\mathcal{F}_2 \cdot \begin{bmatrix} \mathbf{T}_{\mathbf{t}_n}^{2^n(0\oplus k)} \text{com}_{(a_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \mathbf{T}_{\mathbf{t}_n}^{2^n(1\oplus k)} \text{com}_{(a_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \left[\text{diag}\left\{\mathbf{T}_{\mathbf{t}_n}^{2^n(0\oplus k)}, \mathbf{T}_{\mathbf{t}_n}^{2^n(1\oplus k)}\right\} \right] \begin{bmatrix} \text{com}_{(a_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \text{com}_{(a_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} =$$

$$= \begin{bmatrix} \mathbf{T}_{\mathbf{t}_n}^{2^n(0\oplus k)} & \mathbf{T}_{\mathbf{t}_n}^{2^n(1\oplus k)} \\ \mathbf{T}_{\mathbf{t}_n}^{2^n(0\oplus k)} & -\mathbf{T}_{\mathbf{t}_n}^{2^n(1\oplus k)} \end{bmatrix} \begin{bmatrix} \text{com}_{(a_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \text{com}_{(a_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} = {}^{(k)}\mathcal{F}_2 \begin{bmatrix} \text{com}_{(a_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \text{com}_{(a_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix},$$

where

$${}^{(k)}\mathcal{F}_2 = \begin{bmatrix} \mathbf{T}_{\mathbf{t}_n}^{2^n(0\oplus k)} & \mathbf{T}_{\mathbf{t}_n}^{2^n(1\oplus k)} \\ \mathbf{T}_{\mathbf{t}_n}^{2^n(0\oplus k)} & -\mathbf{T}_{\mathbf{t}_n}^{2^n(1\oplus k)} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \left[\text{diag}\left\{\mathbf{T}_{\mathbf{t}_n}^{2^n(0\oplus k)}, \mathbf{T}_{\mathbf{t}_n}^{2^n(1\oplus k)}\right\} \right].$$

Using building blocks of $(2^n \times 2^n)$ -matrix $\mathbf{G}_{2^n}^{[n]}$, we construct the Golay $(2^{n+1} \times 2^{n+1})$ -matrix $\mathbf{G}_{2^{n+1}}^{[n+1]}$ according to the following iteration rule [16]:

$$\begin{bmatrix} \text{com}_{(a_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \text{com}_{(a_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} \begin{matrix} \nearrow \\ \\ \searrow \end{matrix} \begin{bmatrix} {}^{(0)}\mathcal{F} \begin{bmatrix} \text{com}_{(a_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \text{com}_{(a_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} \\ {}^{(1)}\mathcal{F} \begin{bmatrix} \text{com}_{(a_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \text{com}_{(a_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} \end{bmatrix} = \frac{\begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} & \mathbf{T}_{\mathbf{t}_n}^{2^n} \\ \mathbf{I}_{\mathbf{t}_n} & -\mathbf{T}_{\mathbf{t}_n}^{2^n} \end{bmatrix} \begin{bmatrix} \text{com}_{(a_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \text{com}_{(a_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix}}{\begin{bmatrix} \mathbf{T}_{\mathbf{t}_n}^{2^n} & \mathbf{I}_{\mathbf{t}_n} \\ \mathbf{T}_{\mathbf{t}_n}^{2^n} & -\mathbf{I}_{\mathbf{t}_n} \end{bmatrix} \begin{bmatrix} \text{com}_{(a_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \text{com}_{(a_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix}} = \quad (4)$$

$$= \frac{\begin{bmatrix} \text{com}_{(a_{n-1},0)}^{[n]}(\mathbf{t}_n) + \text{com}_{(a_{n-1},1)}^{[n]}(\mathbf{t}_n + 2^n) \\ \text{com}_{(a_{n-1},0)}^{[n]}(\mathbf{t}_n) - \text{com}_{(a_{n-1},1)}^{[n]}(\mathbf{t}_n + 2^n) \\ \text{com}_{(a_{n-1},1)}^{[n]}(\mathbf{t}_n) + \text{com}_{(a_{n-1},0)}^{[n]}(\mathbf{t}_n + 2^n) \\ -\text{com}_{(a_{n-1},1)}^{[n]}(\mathbf{t}_n) + \text{com}_{(a_{n-1},0)}^{[n]}(\mathbf{t}_n + 2^n) \end{bmatrix}}{\begin{bmatrix} \text{com}_{(a_{n-1},0,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(a_{n-1},0,1)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(a_{n-1},1,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(a_{n-1},1,1)}^{[n+1]}(\mathbf{t}_{n+1}) \end{bmatrix}},$$

where

$$\text{com}_{(a_{n-1},0,0)}^{[n+1]}(\mathbf{t}_{n+1}) = \text{com}_{(a_{n-1},0)}^{[n]}(\mathbf{t}_n) + \text{com}_{(a_{n-1},1)}^{[n]}(\mathbf{t}_n + 2^n),$$

$$\text{com}_{(a_{n-1},0,1)}^{[n+1]}(\mathbf{t}_{n+1}) = \text{com}_{(a_{n-1},0)}^{[n]}(\mathbf{t}_n) - \text{com}_{(a_{n-1},1)}^{[n]}(\mathbf{t}_n + 2^n),$$

$$\text{com}_{(\mathbf{a}_{n-1},0)}^{[n+1]}(\mathbf{t}_{n+1}) = \text{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n) + \text{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n + 2^n),$$

$$\text{com}_{(\mathbf{a}_{n-1},1)}^{[n+1]}(\mathbf{t}_{n+1}) = -\text{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n) + \text{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n + 2^n)$$

are complementary sequences of twice length, belonging to $\mathbf{G}_{2^{n+1}}^{[n+1]}$. Hence,

$$\begin{aligned} \mathbf{G}_{2^{n+1}}^{[n+1]} &= \begin{bmatrix} \oplus_{\mathbf{a}_{n+1}=0}^{2^{n+1}-1} \end{bmatrix} \left[\text{com}_{\mathbf{a}_{n+1}}^{[n+1]}(\mathbf{t}_{n+1}) \right] = \begin{bmatrix} \oplus_{\mathbf{a}_n=0}^{2^n-1} \end{bmatrix} \begin{bmatrix} \text{com}_{(\mathbf{a}_n,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(\mathbf{a}_n,1)}^{[n+1]}(\mathbf{t}_{n+1}) \end{bmatrix} = \begin{bmatrix} \oplus_{\mathbf{a}_{n-1}=0}^{2^{n-1}-1} \end{bmatrix} \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1},0,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(\mathbf{a}_{n-1},0,1)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(\mathbf{a}_{n-1},1,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(\mathbf{a}_{n-1},1,1)}^{[n+1]}(\mathbf{t}_{n+1}) \end{bmatrix} = \\ &= \begin{bmatrix} \oplus_{\mathbf{a}_{n-1}=0}^{2^{n-1}-1} \end{bmatrix} \begin{bmatrix} \mathbf{T}_{\mathbf{t}_n}^{2^n \cdot 0} & \mathbf{T}_{\mathbf{t}_n}^{2^n \cdot 1} \\ \mathbf{T}_{\mathbf{t}_n}^{2^n \cdot 0} & -\mathbf{T}_{\mathbf{t}_n}^{2^n \cdot 1} \end{bmatrix} \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \text{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} = \begin{bmatrix} \oplus_{\mathbf{a}_{n-1}=0}^{2^{n-1}-1} \end{bmatrix} \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n) + \text{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n + 2^n) \\ \text{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n) - \text{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n + 2^n) \\ \text{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n) + \text{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n + 2^n) \\ -\text{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n) + \text{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n + 2^n) \end{bmatrix} = \\ &= \begin{bmatrix} \oplus_{\mathbf{a}_{n-1}=0}^{2^{n-1}-1} \end{bmatrix} \begin{bmatrix} \mathbf{T}_{\mathbf{t}_n}^{2^n \cdot 1} & \mathbf{T}_{\mathbf{t}_n}^{2^n \cdot 0} \\ \mathbf{T}_{\mathbf{t}_n}^{2^n \cdot 1} & -\mathbf{T}_{\mathbf{t}_n}^{2^n \cdot 0} \end{bmatrix} \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \text{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} \\ &= \begin{bmatrix} \oplus_{\mathbf{a}_{n-1}=0}^{2^{n-1}-1} \end{bmatrix} \left(\begin{bmatrix} \oplus_{\alpha_n=0}^{2^n-1} \end{bmatrix} \begin{bmatrix} \mathbf{T}_{\mathbf{t}_n}^{2^n \cdot (0 \oplus \alpha_n)} \cdot \text{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n) + \mathbf{T}_{\mathbf{t}_n}^{2^n \cdot (1 \oplus \alpha_n)} \cdot \text{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n) \\ \mathbf{T}_{\mathbf{t}_n}^{2^n \cdot (0 \oplus \alpha_n)} \cdot \text{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n) - \mathbf{T}_{\mathbf{t}_n}^{2^n \cdot (1 \oplus \alpha_n)} \cdot \text{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} \right). \end{aligned} \tag{5}$$

This implies that

$$\text{com}_{(\mathbf{a}_{n+1})}^{[n+1]}(\mathbf{t}_{n+1}) = \text{com}_{(\mathbf{a}_{n-1}, \alpha_n, \alpha_{n+1})}^{[n+1]}(\mathbf{t}_n, t_{n+1}) = \sum_{t_{n+1}=0}^1 (-1)^{(\alpha_n \oplus t_{n+1}) \alpha_{n+1}} \text{com}_{(\mathbf{a}_{n-1}, \alpha_n \oplus t_{n+1})}^{[n]}(\mathbf{t}_n + 2^n \cdot t_{n+1}). \tag{6}$$

Hence,

$$\text{com}_{(\mathbf{a}_{n-1}, \alpha_n, \alpha_{n+1})}^{[n+1]}(\mathbf{t}_n, t_{n+1}) = (-1)^{(\alpha_n \oplus t_{n+1}) \alpha_{n+1}} \text{com}_{(\mathbf{a}_{n-1}, \alpha_n \oplus t_{n+1})}^{[n]}(\mathbf{t}_n) = (-1)^{\alpha_n \alpha_{n+1}} (-1)^{\alpha_{n+1} t_{n+1}} \text{com}_{(\mathbf{a}_{n-1}, \alpha_n \oplus t_{n+1})}^{[n]}(\mathbf{t}_n). \tag{7}$$

It is finally a recurrent relation between complementary sequences of $\mathbf{G}_{2^{n+1}}^{[n+1]}$ and $\mathbf{G}_{2^n}^{[n]}$.

Remark 1. Obviously,

$$\begin{aligned} \begin{bmatrix} \text{com}_{(\mathbf{a}_{n+1},0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(\mathbf{a}_{n+1},1)}^{[n+1]}(\mathbf{t}_{n+1}) \end{bmatrix} &= \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1},0,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(\mathbf{a}_{n-1},0,1)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(\mathbf{a}_{n-1},1,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(\mathbf{a}_{n-1},1,1)}^{[n+1]}(\mathbf{t}_{n+1}) \end{bmatrix} = \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \text{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} & \mathbf{T}_{\mathbf{t}_n}^{2^n} \\ \mathbf{I}_{\mathbf{t}_n} & -\mathbf{T}_{\mathbf{t}_n}^{2^n} \end{bmatrix} \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \text{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} = \\ &= \begin{bmatrix} \mathbf{T}_{\mathbf{t}_n}^{2^n} & \mathbf{I}_{\mathbf{t}_n} \\ \mathbf{T}_{\mathbf{t}_n}^{2^n} & -\mathbf{I}_{\mathbf{t}_n} \end{bmatrix} \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \text{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} = \\ &= \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \left(\mathbf{P}_2^0 \cdot \begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} & \\ & \mathbf{T}_{\mathbf{t}_n}^{2^n} \end{bmatrix} \cdot \mathbf{P}_2^0 \right) \cdot \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \text{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} \\ \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \left(\mathbf{P}_2^1 \cdot \begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} & \\ & \mathbf{T}_{\mathbf{t}_n}^{2^n} \end{bmatrix} \cdot \mathbf{P}_2^1 \right) \cdot \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \text{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \mathcal{F}_2 \cdot \left(\mathbf{P}_2^0 \cdot \begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} & \\ & \mathbf{T}_{\mathbf{t}_n}^{2^n} \end{bmatrix} \cdot \mathbf{P}_2^0 \right) \cdot \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \text{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} \\ \mathcal{F}_2 \cdot \left(\mathbf{P}_2^1 \cdot \begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} & \\ & \mathbf{T}_{\mathbf{t}_n}^{2^n} \end{bmatrix} \cdot \mathbf{P}_2^1 \right) \cdot \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \text{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} \end{bmatrix}, \end{aligned}$$

where $\mathbf{P}_2^0 = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$, $\mathbf{P}_2^1 = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$ are 2-cyclic shift operators.

Hence,

$$\begin{bmatrix} \text{com}_{(\mathbf{a}_n,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(\mathbf{a}_n,1)}^{[n+1]}(\mathbf{t}_{n+1}) \end{bmatrix} = \mathcal{F}_2 \cdot \left(\mathbf{P}_2^{\alpha_n} \cdot \begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} \\ \mathbf{T}_{\mathbf{t}_n}^{2^n} \end{bmatrix} \cdot \mathbf{P}_2^{\alpha_n} \right) \cdot \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \text{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} \quad (8)$$

and

$$\mathbf{G}_{2^{n+1}}^{[n+1]} = \begin{bmatrix} \text{com}_{(\mathbf{a}_n,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(\mathbf{a}_n,1)}^{[n+1]}(\mathbf{t}_{n+1}) \end{bmatrix} = \begin{bmatrix} \text{com}_{(\mathbf{a}_n,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(\mathbf{a}_n,1)}^{[n+1]}(\mathbf{t}_{n+1}) \end{bmatrix} = \mathcal{F}_2 \cdot \left(\mathbf{P}_2^{\alpha_n} \cdot \begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} \\ \mathbf{T}_{\mathbf{t}_n}^{2^n} \end{bmatrix} \cdot \mathbf{P}_2^{\alpha_n} \right) \cdot \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \text{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix}.$$

From (7) we obtain two expressions for $\text{com}_{(\mathbf{a}_n)}^{[n]}(\mathbf{t}_n)$:

$$\text{com}_{(\mathbf{a}_n)}^{[n]}(\mathbf{t}_n) = (-1)^{\sum_{i=1}^n (\alpha_{i-1} \oplus_{\frac{1}{2}} t_i) (\alpha_i \oplus_{\frac{1}{2}} t_{i+1})}, \quad (9)$$

$$\begin{aligned} \text{com}_{(\mathbf{a}_n)}^{[n]}(\mathbf{t}_n) &= (-1)^{\sum_{i=1}^{n-1} \alpha_i \alpha_{i+1}} \cdot (-1)^{\alpha_1 t_1 \oplus_{\frac{1}{2}} \alpha_2 t_2 \oplus_{\frac{1}{2}} \sum_{i=3}^n (\alpha_i \oplus_{\frac{1}{2}} \alpha_{i-2}) t_i} \cdot (-1)^{\sum_{i=1}^{n-1} t_i t_{i+1}} = \\ &= (-1)^{\sum_{i=1}^{n-1} \alpha_i \alpha_{i+1}} \cdot (-1)^{\langle \mathbf{a} | R | \mathbf{t} \rangle} \cdot (-1)^{\sum_{i=1}^{n-1} t_i t_{i+1}} = (-1)^{\sum_{i=1}^{n-1} \alpha_i \alpha_{i+1}} \cdot (-1)^{\langle \beta | \mathbf{t} \rangle} \cdot (-1)^{\sum_{i=1}^{n-1} t_i t_{i+1}}, \end{aligned} \quad (10)$$

where $\langle \mathbf{a} | R | \mathbf{t} \rangle = \alpha_1 t_1 \oplus_{\frac{1}{2}} \alpha_2 t_2 \oplus_{\frac{1}{2}} \sum_{i=3}^n (\alpha_i \oplus_{\frac{1}{2}} \alpha_{i-2}) t_i = \sum_{i=1}^n \beta_i t_i$. Here $\vec{\beta} = \vec{\alpha} R$, where $R = [\delta_{i,j} \oplus \delta_{i,j+2}]_{i,j=1}^n$ and $\alpha_0, t_{n+1} \equiv 0$.

Example 1. It is easy to construct $\mathbf{G}_{2^1}^{[1]}$, $\mathbf{G}_{2^2}^{[2]}$ and $\mathbf{G}_{2^3}^{[3]}$:

$$\begin{aligned} \mathbf{G}_{2^1}^{[1]} &\equiv \mathcal{F}_2 = \begin{bmatrix} \text{com}_{(0)}^{[1]}(\mathbf{t}_1) \\ \text{com}_{(1)}^{[1]}(\mathbf{t}_1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \text{com}_{(\alpha_1)}^{[1]}(t_1) \end{bmatrix} = \begin{bmatrix} (-1)^{\alpha_1 t_1} \end{bmatrix}, \\ \mathbf{G}_{2^2}^{[2]} &= \begin{bmatrix} \text{com}_{(0,0)}^{[2]}(\mathbf{t}_2) \\ \text{com}_{(0,1)}^{[2]}(\mathbf{t}_2) \\ \text{com}_{(1,0)}^{[2]}(\mathbf{t}_2) \\ \text{com}_{(1,1)}^{[2]}(\mathbf{t}_2) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \text{com}_{(\alpha_1, \alpha_2)}^{[2]}(t_1, t_2) \end{bmatrix} = \begin{bmatrix} (-1)^{(\tilde{\alpha}_0 \oplus_{\frac{1}{2}} t_1) (\alpha_1 \oplus_{\frac{1}{2}} t_2)} (-1)^{(\alpha_1 \oplus_{\frac{1}{2}} t_2) (\alpha_2 \oplus_{\frac{1}{2}} t_3)} \end{bmatrix} = \\ &= \text{diag} \left\{ (-1)^{\alpha_1 \alpha_2} \right\} \left[(-1)^{(\alpha_1 t_1 \oplus_{\frac{1}{2}} \alpha_2 t_2)} \right] \text{diag} \left\{ (-1)^{t_1 t_2} \right\}, \end{aligned}$$

where $\tilde{\alpha}_0, \tilde{t}_3 \equiv 0$;

$$\begin{aligned} \mathbf{G}_{2^3}^{[3]} &= \begin{bmatrix} \text{com}_{(0,0,0)}^{[3]}(\mathbf{t}_3) \\ \text{com}_{(0,0,1)}^{[3]}(\mathbf{t}_3) \\ \text{com}_{(0,1,0)}^{[3]}(\mathbf{t}_3) \\ \text{com}_{(0,1,1)}^{[3]}(\mathbf{t}_3) \\ \text{com}_{(1,0,0)}^{[3]}(\mathbf{t}_3) \\ \text{com}_{(1,0,1)}^{[3]}(\mathbf{t}_3) \\ \text{com}_{(1,1,0)}^{[3]}(\mathbf{t}_3) \\ \text{com}_{(1,1,1)}^{[3]}(\mathbf{t}_3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\ \hline 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 \\ \hline 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \\ \hline -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 \end{bmatrix} = \\ &= \begin{bmatrix} \text{com}_{(\alpha_1, \alpha_2, \alpha_3)}^{[3]}(t_1, t_2, t_3) \end{bmatrix} = \begin{bmatrix} (-1)^{(\alpha_0 \oplus_{\frac{1}{2}} t_1) (\alpha_1 \oplus_{\frac{1}{2}} t_2)} (-1)^{(\alpha_1 \oplus_{\frac{1}{2}} t_2) (\alpha_2 \oplus_{\frac{1}{2}} t_3)} (-1)^{(\alpha_2 \oplus_{\frac{1}{2}} t_3) (\alpha_3 \oplus_{\frac{1}{2}} t_4)} \end{bmatrix} = \\ &= \text{diag} \left\{ (-1)^{(\alpha_1 \alpha_2 \oplus_{\frac{1}{2}} \alpha_2 \alpha_3)} \right\} \cdot \begin{bmatrix} (-1)^{(\alpha_1 t_1 \oplus_{\frac{1}{2}} \alpha_2 t_2 \oplus_{\frac{1}{2}} (\alpha_3 \oplus_{\frac{1}{2}} \alpha_1) t_3)} \end{bmatrix} \cdot \text{diag} \left\{ (-1)^{(t_1 t_2 \oplus_{\frac{1}{2}} t_2 t_3)} \right\}, \end{aligned}$$

where $\tilde{\alpha}_0, \tilde{t}_4 \equiv 0$.

4. Generalization. Multi-parameter Golay sequences

In this section, we introduce the generalized Golay–Rudin–Shapiro sequences. They are represented by the following iteration construction

$$\begin{aligned} & \mathbf{G}_2^{[1]}[\mathcal{CS}_2(\varphi_1, \alpha_1, \gamma_1)] \xrightarrow{\mathcal{CS}_2(\varphi_2, \alpha_2, \gamma_2)} \mathbf{G}_2^{[2]}[\mathcal{CS}_2(\varphi_1, \alpha_1, \gamma_1), \mathcal{CS}_2(\varphi_2, \alpha_2, \gamma_2)] \xrightarrow{\mathcal{CS}_2(\varphi_3, \alpha_3, \gamma_3)} \dots \\ & \xrightarrow{\mathcal{CS}_2(\varphi_n, \alpha_n, \gamma_n)} \mathbf{G}_2^{[n]}[\mathcal{CS}_2(\varphi_1, \alpha_1, \gamma_1), \dots, \mathcal{CS}_2(\varphi_n, \alpha_n, \gamma_n)] \xrightarrow{\mathcal{CS}_2(\varphi_{n+1}, \alpha_{n+1}, \gamma_{n+1})} \mathbf{G}_2^{[n+1]}[\mathcal{CS}_2(\varphi_1, \alpha_1, \gamma_1), \dots, \mathcal{CS}_2(\varphi_n, \alpha_n, \gamma_n), \mathcal{CS}_2(\varphi_{n+1}, \alpha_{n+1}, \gamma_{n+1})], \end{aligned}$$

based on a sequence of unitary transforms:

$$\begin{aligned} \mathcal{CS}_2(\varphi_k, \alpha_k, \gamma_k) &= \left[(-1)^{\alpha t} \cdot \text{CS}_{\alpha, t}^{[k]}(\varphi_k, \alpha_k, \gamma_k) \right]_{\alpha, t=0}^1 = \left[(-1)^{\alpha t} \cdot \text{CS}_{\alpha, t}^{[k]} \right]_{\alpha, t=0}^1 = \begin{bmatrix} C_k & S_k \\ \bar{S}_k & -\bar{C}_k \end{bmatrix} = \\ &= \begin{bmatrix} C(\varphi_k, \alpha_k, \gamma_k) & S(\varphi_k, \alpha_k, \gamma_k) \\ \bar{S}(\varphi_k, \alpha_k, \gamma_k) & -\bar{C}(\varphi_k, \alpha_k, \gamma_k) \end{bmatrix} = \begin{bmatrix} e^{i\alpha_k} \cos \varphi_k & e^{i\gamma_k} \sin \varphi_k \\ e^{-i\gamma_k} \sin \varphi_k & -e^{-i\alpha_k} \cos \varphi_k \end{bmatrix}, \end{aligned} \tag{11}$$

$\forall k = 1, 2, \dots, n+1$, where

$$\begin{aligned} C_k &= C(\varphi_k, \alpha_k, \gamma_k) = e^{i\alpha_k} \cos \varphi_k, \text{ if } \alpha = 0, t = 0; & \bar{C}_k &= \bar{C}(\varphi_k, \alpha_k, \gamma_k) = e^{-i\alpha_k} \cos \varphi_k, \text{ if } \alpha = 0, t = 1; \\ S_k &= S(\varphi_k, \alpha_k, \gamma_k) = e^{i\gamma_k} \sin \varphi_k, \text{ if } \alpha = 1, t = 0; & \bar{S}_k &= \bar{S}(\varphi_k, \alpha_k, \gamma_k) = e^{-i\gamma_k} \sin \varphi_k, \text{ if } \alpha = 1, t = 1. \end{aligned}$$

For brevity, let $\theta_k := (\varphi_k, \alpha_k, \gamma_k)$, $\boldsymbol{\theta}_k := (\boldsymbol{\varphi}_k, \boldsymbol{\alpha}_k, \boldsymbol{\gamma}_k) := (\varphi_1, \alpha_1, \gamma_1; \varphi_2, \alpha_2, \gamma_2; \dots; \varphi_k, \alpha_k, \gamma_k)$. As in the previous case we assume that we have the Golay matrix $\mathbf{G}_2^{[n]}[\boldsymbol{\theta}_n] = \mathbf{G}_2^{[n]}[\boldsymbol{\varphi}_n, \boldsymbol{\alpha}_n, \boldsymbol{\gamma}_n]$. We need to construct the next Golay matrix $\mathbf{G}_2^{[n+1]}[\boldsymbol{\theta}_{n+1}]$ using only $\mathbf{G}_2^{[n]}[\boldsymbol{\theta}_n]$ and $\mathcal{CS}_2(\theta_{n+1})$. We use the following iteration construction

$$\begin{aligned} \mathbf{G}_2^{[n+1]}(\boldsymbol{\theta}_{n+1}) &= \mathbf{G}_2^{[n+1]}(\boldsymbol{\theta}_n, \theta_{n+1}) = \bigoplus_{\mathbf{a}_n=0}^{2^n-1} \begin{bmatrix} \text{com}_{(\mathbf{a}_n, 0)}^{[n+1]}(\mathbf{t}_{n+1} | \boldsymbol{\theta}_n, \theta_{n+1}) \\ \text{com}_{(\mathbf{a}_n, 1)}^{[n+1]}(\mathbf{t}_{n+1} | \boldsymbol{\theta}_n, \theta_{n+1}) \end{bmatrix} = \\ &= \bigoplus_{\mathbf{a}_{n-1}=0}^{2^{n-1}-1} \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1}, 0, 0)}^{[n+1]}(\mathbf{t}_{n+1} | \boldsymbol{\theta}_n, \theta_{n+1}) \\ \text{com}_{(\mathbf{a}_{n-1}, 0, 1)}^{[n+1]}(\mathbf{t}_{n+1} | \boldsymbol{\theta}_n, \theta_{n+1}) \\ \text{com}_{(\mathbf{a}_{n-1}, 1, 0)}^{[n+1]}(\mathbf{t}_{n+1} | \boldsymbol{\theta}_n, \theta_{n+1}) \\ \text{com}_{(\mathbf{a}_{n-1}, 1, 1)}^{[n+1]}(\mathbf{t}_{n+1} | \boldsymbol{\theta}_n, \theta_{n+1}) \end{bmatrix} = \bigoplus_{\mathbf{a}_{n-1}=0}^{2^{n-1}-1} \begin{bmatrix} C(\theta_{n+1})\mathbf{T}_{\mathbf{t}_n}^{2^n-0} & S(\theta_{n+1})\mathbf{T}_{\mathbf{t}_n}^{2^n-1} \\ \bar{S}(\theta_{n+1})\mathbf{T}_{\mathbf{t}_n}^{2^n-0} & -\bar{C}(\theta_{n+1})\mathbf{T}_{\mathbf{t}_n}^{2^n-1} \end{bmatrix} \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1}, 0)}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n) \\ \text{com}_{(\mathbf{a}_{n-1}, 1)}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n) \end{bmatrix} = \tag{12} \\ &= \bigoplus_{\mathbf{a}_{n-1}=0}^{2^{n-1}-1} \begin{bmatrix} \begin{bmatrix} C_{n+1} & S_{n+1} \\ \bar{S}_{n+1} & -\bar{C}_{n+1} \end{bmatrix} \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1}, 0)}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n) \\ \text{com}_{(\mathbf{a}_{n-1}, 1)}^{[n]}(\mathbf{t}_n + 2^n | \boldsymbol{\theta}_n) \end{bmatrix} \\ \begin{bmatrix} C_{n+1} & S_{n+1} \\ \bar{S}_{n+1} & -\bar{C}_{n+1} \end{bmatrix} \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1}, 0)}^{[n]}(\mathbf{t}_n + 2^n | \boldsymbol{\theta}_n) \\ \text{com}_{(\mathbf{a}_{n-1}, 1)}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n) \end{bmatrix} \end{bmatrix} = \\ &= \begin{bmatrix} \begin{bmatrix} C_{n+1} & S_{n+1} \\ \bar{S}_{n+1} & -\bar{C}_{n+1} \end{bmatrix} & & & & & \begin{bmatrix} \text{com}_{(\mathbf{0}, 0)}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n) \\ \text{com}_{(\mathbf{0}, 1)}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n) \\ \text{com}_{(\mathbf{0}, 0)}^{[n]}(\mathbf{t}_n + 2^n | \boldsymbol{\theta}_n) \\ \text{com}_{(\mathbf{0}, 1)}^{[n]}(\mathbf{t}_n + 2^n | \boldsymbol{\theta}_n) \\ \vdots \\ \text{com}_{(\mathbf{M}, 0)}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n) \\ \text{com}_{(\mathbf{M}, 1)}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n) \\ \text{com}_{(\mathbf{M}, 0)}^{[n]}(\mathbf{t}_n + 2^n | \boldsymbol{\theta}_n) \\ \text{com}_{(\mathbf{M}, 1)}^{[n]}(\mathbf{t}_n + 2^n | \boldsymbol{\theta}_n) \end{bmatrix} \end{bmatrix}, \end{aligned}$$

where $\mathbf{M} = 2^{n-1}$. From (12) we obtain

$$\mathbf{G}_{2^{n-1}}^{[n+1]}(\boldsymbol{\theta}_n, \theta_{n+1}) = \prod_{\mathbf{a}_{n-1}=0}^{2^{n-1}-1} \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1},0,0)}^{[n+1]}(\mathbf{t}_{n+1} | \boldsymbol{\theta}_n, \theta_{n+1}) \\ \text{com}_{(\mathbf{a}_{n-1},0,1)}^{[n+1]}(\mathbf{t}_{n+1} | \boldsymbol{\theta}_n, \theta_{n+1}) \\ \text{com}_{(\mathbf{a}_{n-1},1,0)}^{[n+1]}(\mathbf{t}_{n+1} | \boldsymbol{\theta}_n, \theta_{n+1}) \\ \text{com}_{(\mathbf{a}_{n-1},1,1)}^{[n+1]}(\mathbf{t}_{n+1} | \boldsymbol{\theta}_n, \theta_{n+1}) \end{bmatrix} = \prod_{\mathbf{a}_{n-1}=0}^{2^{n-1}-1} \begin{bmatrix} C(\theta_{n+1}) \cdot \text{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n) + S(\theta_{n+1}) \cdot \text{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n + 2^n | \boldsymbol{\theta}_n) \\ \bar{S}(\theta_{n+1}) \cdot \text{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n) - \bar{C}(\theta_{n+1}) \cdot \text{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n + 2^n | \boldsymbol{\theta}_n) \\ S(\theta_{n+1}) \cdot \text{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n) + C(\theta_{n+1}) \cdot \text{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n + 2^n | \boldsymbol{\theta}_n) \\ -\bar{C}(\theta_{n+1}) \cdot \text{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n) + \bar{S}(\theta_{n+1}) \cdot \text{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n + 2^n | \boldsymbol{\theta}_n) \end{bmatrix}. \tag{13}$$

Hence,

$$\begin{aligned} \text{com}_{(\mathbf{a}_{n+1})}^{[n+1]}(\mathbf{t}_n, t_{n+1} | \boldsymbol{\theta}_n, \theta_{n+1n+1}) &= \text{com}_{(\mathbf{a}_{n+1}, \alpha_n, \alpha_{n+1})}^{[n+1]}(\mathbf{t}_n, t_{n+1} | \boldsymbol{\theta}_n, \theta_{n+1}) = \\ &= \sum_{t_{n+1}=0}^1 (-1)^{(\alpha_n \oplus_2 t_{n+1}) \alpha_{n+1}} \cdot \text{CS}_{(\alpha_{n+1} \oplus_2 \alpha_n), t_{n+1}}^{[n+1]}(\theta_{n+1}) \cdot \text{com}_{(\mathbf{a}_{n+1}, \alpha_n \oplus_2 t_{n+1})}^{[n]}(\mathbf{t}_n + 2^n \cdot t_{n+1} | \boldsymbol{\theta}_n) \end{aligned}$$

and

$$\text{com}_{(\mathbf{a}_{n+1}, \alpha_n, \alpha_{n+1})}^{[n+1]}(\mathbf{t}_n, t_{n+1} | \boldsymbol{\theta}_n, \theta_{n+1}) = (-1)^{(\alpha_n \oplus_2 t_{n+1}) \alpha_{n+1}} \cdot \text{CS}_{(\alpha_{n+1} \oplus_2 \alpha_n), t_{n+1}}^{[n+1]}(\theta_{n+1}) \cdot \text{com}_{(\mathbf{a}_{n+1}, \alpha_n \oplus_2 t_{n+1})}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n). \tag{14}$$

This recurrent relation gives the following analytic expression for multi-parameter sequences

$$\text{com}_{\mathbf{a}_n}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n) = \prod_{i=1}^n (-1)^{(\alpha_{i-1} \oplus_2 t_i)(\alpha_i \oplus_2 t_{i+1})} \cdot \text{CS}_{(\alpha_{i-1} \oplus_2 \alpha_i), (t_i \oplus_2 t_{i+1})}^{[n+1]}(\theta_i). \tag{15}$$

Here are the particular cases:

1) if $\alpha_k = \gamma_k = 0$, then

$$\text{CS}_2(\theta_k) = \begin{bmatrix} C_k(\theta_k) & S_k(\theta_k) \\ \bar{S}_k(\theta_k) & -\bar{C}_k(\theta_k) \end{bmatrix} = \begin{bmatrix} \cos \varphi_k & \sin \varphi_k \\ \sin \varphi_k & -\cos \varphi_k \end{bmatrix} = [\text{CS}_{\alpha, t}(\varphi_k)]_{\alpha, t=0}^1, \quad \forall k = 1, 2, \dots, n,$$

and

$$\text{com}_{(\mathbf{a}_n)}^{[n]}(\mathbf{t}_n | \boldsymbol{\varphi}_n) = \prod_{i=1}^n (-1)^{(\alpha_{i-1} \oplus_2 t_i)(\alpha_i \oplus_2 t_{i+1})} \cdot \text{CS}_{(\alpha_{i-1} \oplus_2 \alpha_i), (t_i \oplus_2 t_{i+1})}^{[n+1]}(\varphi_i); \tag{16}$$

2) if $\alpha_k = \gamma_k = 0$, $\varphi_k = \pi / 4$, then

$$\text{CS}_2(\theta_k) = \begin{bmatrix} C_k(\theta_k) & S_k(\theta_k) \\ \bar{S}_k(\theta_k) & -\bar{C}_k(\theta_k) \end{bmatrix} = \begin{bmatrix} \cos(\pi / 4) & \sin(\pi / 4) \\ \sin(\pi / 4) & -\cos(\pi / 4) \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \forall k = 1, 2, \dots, n,$$

and

$$\text{com}_{(\mathbf{a}_n)}^{[n]}(\mathbf{t}_n) = \prod_{i=1}^n (-1)^{(\alpha_{i-1} \oplus_2 t_i)(\alpha_i \oplus_2 t_{i+1})}. \tag{17}$$

Remark 2. For further generalization on m -complementary sequences we rewrite iteration rule(12) as

$$\begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n) \\ \text{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n) \end{bmatrix} \begin{matrix} \nearrow \\ \searrow \end{matrix} \begin{bmatrix} C(\theta_{n+1}) \cdot \mathbf{I}_{\mathbf{t}_n} & S(\theta_{n+1}) \cdot \mathbf{T}_{\mathbf{t}_n}^{2^n} \\ \bar{S}(\theta_{n+1}) \cdot \mathbf{I}_{\mathbf{t}_n} & -\bar{C}(\theta_{n+1}) \cdot \mathbf{T}_{\mathbf{t}_n}^{2^n} \\ C(\theta_{n+1}) \cdot \mathbf{T}_{\mathbf{t}_n}^{2^n} & S(\theta_{n+1}) \cdot \mathbf{I}_{\mathbf{t}_n} \\ \bar{S}(\theta_{n+1}) \cdot \mathbf{T}_{\mathbf{t}_n}^{2^n} & -\bar{C}(\theta_{n+1}) \cdot \mathbf{I}_{\mathbf{t}_n} \end{bmatrix} \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n) \\ \text{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n) \\ \text{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n) \\ \text{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n) \end{bmatrix} =$$

$$\begin{aligned}
 &= \frac{\begin{bmatrix} C(\theta_{n+1}) & S(\theta_{n+1}) \\ \bar{S}(\theta_{n+1}) & -\bar{C}(\theta_{n+1}) \end{bmatrix} \cdot \left(\mathbf{P}_2^0 \cdot \begin{bmatrix} \mathbf{I}_{t_n} & \\ & \mathbf{T}_{t_n}^{2^n} \end{bmatrix} \cdot \mathbf{P}_2^0 \right) \cdot \begin{bmatrix} \text{com}_{(a_{n-1},0)}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n) \\ \text{com}_{(a_{n-1},1)}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n) \end{bmatrix}}{\begin{bmatrix} C(\theta_{n+1}) & S(\theta_{n+1}) \\ \bar{S}(\theta_{n+1}) & -\bar{C}(\theta_{n+1}) \end{bmatrix} \cdot \left(\mathbf{P}_2^1 \cdot \begin{bmatrix} \mathbf{I}_{t_n} & \\ & \mathbf{T}_{t_n}^{2^n} \end{bmatrix} \cdot \mathbf{P}_2^1 \right) \cdot \begin{bmatrix} \text{com}_{(a_{n-1},0)}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n) \\ \text{com}_{(a_{n-1},1)}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n) \end{bmatrix}} \\
 &= \frac{\mathcal{CS}_2(\theta_{n+1}) \cdot \left(\mathbf{P}_2^0 \cdot \begin{bmatrix} \mathbf{I}_{t_n} & \\ & \mathbf{T}_{t_n}^{2^n} \end{bmatrix} \cdot \mathbf{P}_2^0 \right) \cdot \begin{bmatrix} \text{com}_{(a_{n-1},0)}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n) \\ \text{com}_{(a_{n-1},1)}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n) \end{bmatrix}}{\mathcal{CS}_2(\theta_{n+1}) \cdot \left(\mathbf{P}_2^1 \cdot \begin{bmatrix} \mathbf{I}_{t_n} & \\ & \mathbf{T}_{t_n}^{2^n} \end{bmatrix} \cdot \mathbf{P}_2^1 \right) \cdot \begin{bmatrix} \text{com}_{(a_{n-1},0)}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n) \\ \text{com}_{(a_{n-1},1)}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n) \end{bmatrix}} \equiv \begin{bmatrix} \text{com}_{(a_{n-1},0,0)}^{[n+1]}(\mathbf{t}_{n+1} | \boldsymbol{\theta}_{n+1}) \\ \text{com}_{(a_{n-1},0,1)}^{[n+1]}(\mathbf{t}_{n+1} | \boldsymbol{\theta}_{n+1}) \\ \text{com}_{(a_{n-1},1,0)}^{[n+1]}(\mathbf{t}_{n+1} | \boldsymbol{\theta}_{n+1}) \\ \text{com}_{(a_{n-1},1,1)}^{[n+1]}(\mathbf{t}_{n+1} | \boldsymbol{\theta}_{n+1}) \end{bmatrix},
 \end{aligned}$$

where $\mathbf{P}_2^0 = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$, $\mathbf{P}_2^1 = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$ are 2-cyclic shift operators. Hence,

$$\begin{bmatrix} \text{com}_{(a_n,0)}^{[n]}(\mathbf{t}_{n+1} | \boldsymbol{\theta}_{n+1}) \\ \text{com}_{(a_n,1)}^{[n]}(\mathbf{t}_{n+1} | \boldsymbol{\theta}_{n+1}) \end{bmatrix} = \begin{bmatrix} \mathcal{CS}_2(\theta_{n+1}) \cdot \left(\mathbf{P}_2^{\alpha_n} \cdot \begin{bmatrix} \mathbf{I}_{t_n} & \\ & \mathbf{T}_{t_n}^{2^n} \end{bmatrix} \cdot \mathbf{P}_2^{\alpha_n} \right) \cdot \begin{bmatrix} \text{com}_{(a_{n-1},0)}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n) \\ \text{com}_{(a_{n-1},1)}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n) \end{bmatrix} \end{bmatrix} \tag{18}$$

and

$$\mathbf{G}_{2^{n+1}}^{[n+1]}(\boldsymbol{\theta}_{n+1}) = \bigoplus_{a_n=0}^{2^n-1} \begin{bmatrix} \text{com}_{(a_n,0)}^{[n]}(\mathbf{t}_{n+1} | \boldsymbol{\theta}_{n+1}) \\ \text{com}_{(a_n,1)}^{[n]}(\mathbf{t}_{n+1} | \boldsymbol{\theta}_{n+1}) \end{bmatrix} = \bigoplus_{a_n=0}^{2^n-1} \left[\mathcal{CS}_2(\theta_{n+1}) \cdot \left(\mathbf{P}_2^{\alpha_n} \cdot \begin{bmatrix} \mathbf{I}_{t_n} & \\ & \mathbf{T}_{t_n}^{2^n} \end{bmatrix} \cdot \mathbf{P}_2^{\alpha_n} \right) \cdot \begin{bmatrix} \text{com}_{(a_{n-1},0)}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n) \\ \text{com}_{(a_{n-1},1)}^{[n]}(\mathbf{t}_n | \boldsymbol{\theta}_n) \end{bmatrix} \right].$$

Example 3. It is easy to construct $\mathbf{G}_2^{[1]}[\boldsymbol{\theta}_1], \mathbf{G}_2^{[2]}[\boldsymbol{\theta}_2], \mathbf{G}_2^{[3]}[\boldsymbol{\theta}_3]$:

$$\begin{aligned}
 \mathbf{G}_2^{[1]}(\theta_1) &= \begin{bmatrix} \text{com}_{(0)}^{[1]}(\mathbf{t}_1 | \theta_1) \\ \text{com}_{(1)}^{[1]}(\mathbf{t}_1 | \theta_1) \end{bmatrix} = \begin{bmatrix} C_1(\theta_1) & S_1(\theta_1) \\ \bar{S}_1(\theta_1) & -\bar{C}_1(\theta_1) \end{bmatrix} = \begin{bmatrix} C_k(\varphi_k, \alpha_k) & S_k(\varphi_k, \gamma_k) \\ \bar{S}_k(\varphi_k, \gamma_k) & -\bar{C}_k(\varphi_k, \alpha_k) \end{bmatrix} = \\
 &= \begin{bmatrix} e^{i\alpha_k} \cos \varphi_k & e^{i\gamma_k} \sin \varphi_k \\ e^{-i\gamma_k} \sin \varphi_k & -e^{-i\alpha_k} \cos \varphi_k \end{bmatrix} = \left[(-1)^{\alpha} \cdot \text{CS}_{\alpha,t}^{[1]}(\theta_1) \right]_{\alpha,t=0}^1;
 \end{aligned}$$

$$\mathbf{G}_2^{[2]}(\theta_1, \theta_2) = \begin{bmatrix} \text{com}_{(0,0)}^{[2]}(\mathbf{t}_2 | \varphi_1, \alpha_1, \gamma_1; \varphi_2, \alpha_2, \gamma_2) \\ \text{com}_{(0,1)}^{[2]}(\mathbf{t}_2 | \varphi_1, \alpha_1, \gamma_1; \varphi_2, \alpha_2, \gamma_2) \\ \text{com}_{(1,0)}^{[2]}(\mathbf{t}_2 | \varphi_1, \alpha_1, \gamma_1; \varphi_2, \alpha_2, \gamma_2) \\ \text{com}_{(1,1)}^{[2]}(\mathbf{t}_2 | \varphi_1, \alpha_1, \gamma_1; \varphi_2, \alpha_2, \gamma_2) \end{bmatrix} =$$

$$= \left[(-1)^{(\alpha_1 \oplus_2 \alpha_2)(\alpha_2 \oplus_2 \tilde{\alpha}_3)} \cdot (-1)^{(\tilde{\alpha}_0 \oplus_2 \alpha_1)(\alpha_1 \oplus_2 \tilde{\alpha}_2)} \cdot \text{CS}_{(\alpha_1 \oplus_2 \alpha_2), (t_2 \oplus_2 \tilde{t}_3)}^{[2]}(\theta_2) \cdot \text{CS}_{(\tilde{\alpha}_0 \oplus_2 \alpha_1), (t_1 \oplus_2 \tilde{t}_2)}^{[1]}(\theta_1) \right] =$$

$$= \frac{\begin{bmatrix} C_2 & S_2 \\ \bar{S}_2 & -\bar{C}_2 \end{bmatrix} \begin{bmatrix} C_1 & S_1 \\ \bar{S}_1 & -\bar{C}_1 \end{bmatrix}}{\begin{bmatrix} C_2 & S_2 \\ \bar{S}_2 & -\bar{C}_2 \end{bmatrix} \begin{bmatrix} C_1 & S_1 \\ \bar{S}_1 & -\bar{C}_1 \end{bmatrix}} = \frac{\begin{bmatrix} C_2 C_1 & C_2 S_1 & S_2 \bar{S}_1 & -S_2 \bar{C}_1 \\ \bar{S}_2 C_1 & \bar{S}_2 S_1 & -\bar{C}_2 \bar{S}_1 & \bar{C}_2 \bar{C}_1 \\ S_2 \bar{S}_1 & -S_2 \bar{C}_1 & C_2 C_1 & C_2 S_1 \\ -\bar{C}_2 \bar{S}_1 & \bar{C}_2 \bar{C}_1 & \bar{S}_2 C_1 & \bar{S}_2 S_1 \end{bmatrix}}{\begin{bmatrix} C_2 C_1 & C_2 S_1 & S_2 \bar{S}_1 & -S_2 \bar{C}_1 \\ \bar{S}_2 C_1 & \bar{S}_2 S_1 & -\bar{C}_2 \bar{S}_1 & \bar{C}_2 \bar{C}_1 \\ S_2 \bar{S}_1 & -S_2 \bar{C}_1 & C_2 C_1 & C_2 S_1 \\ -\bar{C}_2 \bar{S}_1 & \bar{C}_2 \bar{C}_1 & \bar{S}_2 C_1 & \bar{S}_2 S_1 \end{bmatrix}} =$$

$$= \frac{\begin{bmatrix} e^{i\alpha_2} e^{i\alpha_1} \cos \varphi_2 \cos \varphi_1 & e^{i\alpha_2} e^{i\gamma_1} \cos \varphi_2 \sin \varphi_1 & e^{i\gamma_2} e^{-i\gamma_1} \sin \varphi_2 \sin \varphi_1 & -e^{i\gamma_2} e^{-i\alpha_1} \sin \varphi_2 \cos \varphi_1 \\ e^{-i\gamma_2} e^{i\alpha_1} \sin \varphi_2 \cos \varphi_1 & e^{-i\gamma_2} e^{i\gamma_1} \sin \varphi_2 \sin \varphi_1 & -e^{-i\alpha_2} e^{-i\gamma_1} \cos \varphi_2 \sin \varphi_1 & e^{-i\alpha_2} e^{-i\alpha_1} \cos \varphi_2 \cos \varphi_1 \\ e^{i\gamma_2} e^{-i\gamma_1} \sin \varphi_2 \sin \varphi_1 & -e^{i\gamma_2} e^{-i\alpha_1} \sin \varphi_2 \cos \varphi_1 & e^{i\alpha_2} e^{i\alpha_1} \cos \varphi_2 \cos \varphi_1 & e^{i\alpha_2} e^{i\gamma_1} \cos \varphi_2 \sin \varphi_1 \\ -e^{-i\alpha_2} e^{-i\gamma_1} \cos \varphi_2 \sin \varphi_1 & e^{-i\alpha_2} e^{-i\alpha_1} \cos \varphi_2 \cos \varphi_1 & e^{-i\gamma_2} e^{i\alpha_1} \sin \varphi_2 \cos \varphi_1 & e^{-i\gamma_2} e^{i\gamma_1} \sin \varphi_2 \sin \varphi_1 \end{bmatrix}}{\begin{bmatrix} e^{i\alpha_2} e^{i\alpha_1} \cos \varphi_2 \cos \varphi_1 & e^{i\alpha_2} e^{i\gamma_1} \cos \varphi_2 \sin \varphi_1 & e^{i\gamma_2} e^{-i\gamma_1} \sin \varphi_2 \sin \varphi_1 & -e^{i\gamma_2} e^{-i\alpha_1} \sin \varphi_2 \cos \varphi_1 \\ e^{-i\gamma_2} e^{i\alpha_1} \sin \varphi_2 \cos \varphi_1 & e^{-i\gamma_2} e^{i\gamma_1} \sin \varphi_2 \sin \varphi_1 & -e^{-i\alpha_2} e^{-i\gamma_1} \cos \varphi_2 \sin \varphi_1 & e^{-i\alpha_2} e^{-i\alpha_1} \cos \varphi_2 \cos \varphi_1 \\ e^{i\gamma_2} e^{-i\gamma_1} \sin \varphi_2 \sin \varphi_1 & -e^{i\gamma_2} e^{-i\alpha_1} \sin \varphi_2 \cos \varphi_1 & e^{i\alpha_2} e^{i\alpha_1} \cos \varphi_2 \cos \varphi_1 & e^{i\alpha_2} e^{i\gamma_1} \cos \varphi_2 \sin \varphi_1 \\ -e^{-i\alpha_2} e^{-i\gamma_1} \cos \varphi_2 \sin \varphi_1 & e^{-i\alpha_2} e^{-i\alpha_1} \cos \varphi_2 \cos \varphi_1 & e^{-i\gamma_2} e^{i\alpha_1} \sin \varphi_2 \cos \varphi_1 & e^{-i\gamma_2} e^{i\gamma_1} \sin \varphi_2 \sin \varphi_1 \end{bmatrix}},$$

where $\tilde{\alpha}_0, \tilde{t}_3 \equiv 0$;

$$\mathbf{G}_2^{[3]}(\theta_1, \theta_2, \theta_3) = \begin{bmatrix} \text{com}_{(0,0,0)}(\mathbf{t}_3 | \varphi_1, \alpha_1, \gamma_1; \varphi_2, \alpha_2, \gamma_2; \varphi_3, \alpha_3, \gamma_3) \\ \text{com}_{(0,0,1)}(\mathbf{t}_3 | \varphi_1, \alpha_1, \gamma_1; \varphi_2, \alpha_2, \gamma_2; \varphi_3, \alpha_3, \gamma_3) \\ \text{com}_{(0,1,0)}(\mathbf{t}_3 | \varphi_1, \alpha_1, \gamma_1; \varphi_2, \alpha_2, \gamma_2; \varphi_3, \alpha_3, \gamma_3) \\ \text{com}_{(0,1,1)}(\mathbf{t}_3 | \varphi_1, \alpha_1, \gamma_1; \varphi_2, \alpha_2, \gamma_2; \varphi_3, \alpha_3, \gamma_3) \\ \text{com}_{(1,0,0)}(\mathbf{t}_3 | \varphi_1, \alpha_1, \gamma_1; \varphi_2, \alpha_2, \gamma_2; \varphi_3, \alpha_3, \gamma_3) \\ \text{com}_{(1,0,1)}(\mathbf{t}_3 | \varphi_1, \alpha_1, \gamma_1; \varphi_2, \alpha_2, \gamma_2; \varphi_3, \alpha_3, \gamma_3) \\ \text{com}_{(1,1,0)}(\mathbf{t}_3 | \varphi_1, \alpha_1, \gamma_1; \varphi_2, \alpha_2, \gamma_2; \varphi_3, \alpha_3, \gamma_3) \\ \text{com}_{(1,1,1)}(\mathbf{t}_3 | \varphi_1, \alpha_1, \gamma_1; \varphi_2, \alpha_2, \gamma_2; \varphi_3, \alpha_3, \gamma_3) \end{bmatrix} = \begin{bmatrix} C_3 & S_3 & & & & & & & \\ \bar{S}_3 & -\bar{C}_3 & & & & & & & \\ & & C_3 & S_3 & & & & & \\ & & \bar{S}_3 & -\bar{C}_3 & & & & & \\ & & & & C_3 & S_3 & & & \\ & & & & \bar{S}_3 & -\bar{C}_3 & & & \\ & & & & & & C_3 & S_3 & \\ & & & & & & \bar{S}_3 & -\bar{C}_3 & \end{bmatrix} \cdot \begin{bmatrix} C_2 C_1 & C_2 S_1 & S_2 \bar{S}_1 & -S_2 \bar{C}_1 & & & & & \\ & & & & \bar{S}_2 C_1 & \bar{S}_2 S_1 & -\bar{C}_2 \bar{S}_1 & \bar{C}_2 \bar{C}_1 & \\ \bar{S}_2 C_1 & \bar{S}_2 S_1 & -\bar{C}_2 \bar{S}_1 & \bar{C}_2 \bar{C}_1 & C_2 C_1 & C_2 S_1 & S_2 \bar{S}_1 & -S_2 \bar{C}_1 & \\ S_2 \bar{S}_1 & -S_2 \bar{C}_1 & C_2 C_1 & C_2 S_1 & & & & & \\ & & & & -\bar{C}_2 \bar{S}_1 & \bar{C}_2 \bar{C}_1 & \bar{S}_2 C_1 & \bar{S}_2 S_1 & \\ & & & & S_2 \bar{S}_1 & -S_2 \bar{C}_1 & C_2 C_1 & C_2 S_1 & \\ -\bar{C}_2 \bar{S}_1 & \bar{C}_2 \bar{C}_1 & \bar{S}_2 C_1 & \bar{S}_2 S_1 & & & & & \end{bmatrix} = \begin{bmatrix} C_3 C_2 C_1 & C_3 C_2 S_1 & C_3 S_2 \bar{S}_1 & -C_3 S_2 \bar{C}_1 & S_3 \bar{S}_2 C_1 & S_3 \bar{S}_2 S_1 & -S_3 \bar{C}_2 \bar{S}_1 & S_3 \bar{C}_2 \bar{C}_1 \\ \bar{S}_3 C_2 C_1 & \bar{S}_3 C_2 S_1 & \bar{S}_3 S_2 \bar{S}_1 & -\bar{S}_3 S_2 \bar{C}_1 & -\bar{C}_3 \bar{S}_2 C_1 & -\bar{C}_3 \bar{S}_2 S_1 & \bar{C}_3 \bar{C}_2 \bar{S}_1 & -\bar{C}_3 \bar{C}_2 \bar{C}_1 \\ S_3 \bar{S}_2 C_1 & S_3 \bar{S}_2 S_1 & -S_3 \bar{C}_2 \bar{S}_1 & S_3 \bar{C}_2 \bar{C}_1 & C_3 C_2 C_1 & C_3 C_2 S_1 & C_3 S_2 \bar{S}_1 & -C_3 S_2 \bar{C}_1 \\ -\bar{C}_3 \bar{S}_2 C_1 & -\bar{C}_3 \bar{S}_2 S_1 & \bar{C}_3 \bar{C}_2 \bar{S}_1 & -\bar{C}_3 \bar{C}_2 \bar{C}_1 & \bar{S}_3 C_2 C_1 & \bar{S}_3 C_2 S_1 & \bar{S}_3 S_2 \bar{S}_1 & -\bar{S}_3 S_2 \bar{C}_1 \\ C_3 S_2 \bar{S}_1 & -C_3 S_2 \bar{C}_1 & C_3 C_2 C_1 & C_3 C_2 S_1 & -S_3 \bar{C}_2 \bar{S}_1 & S_3 \bar{C}_2 \bar{C}_1 & S_3 \bar{S}_2 C_1 & S_3 \bar{S}_2 S_1 \\ \bar{S}_3 S_2 \bar{S}_1 & -\bar{S}_3 S_2 \bar{C}_1 & \bar{S}_3 C_2 C_1 & \bar{S}_3 C_2 S_1 & \bar{C}_3 \bar{C}_2 \bar{S}_1 & -\bar{C}_3 \bar{C}_2 \bar{C}_1 & -\bar{C}_3 \bar{S}_2 C_1 & -\bar{C}_3 \bar{S}_2 S_1 \\ -S_3 \bar{C}_2 \bar{S}_1 & S_3 \bar{C}_2 \bar{C}_1 & S_3 \bar{S}_2 C_1 & S_3 \bar{S}_2 S_1 & C_3 S_2 \bar{S}_1 & -C_3 S_2 \bar{C}_1 & C_3 C_2 C_1 & C_3 C_2 S_1 \\ \bar{C}_3 \bar{C}_2 \bar{S}_1 & -\bar{C}_3 \bar{C}_2 \bar{C}_1 & -\bar{C}_3 \bar{S}_2 C_1 & -\bar{C}_3 \bar{S}_2 S_1 & \bar{S}_3 S_2 \bar{S}_1 & -\bar{S}_3 S_2 \bar{C}_1 & \bar{S}_3 C_2 C_1 & \bar{S}_3 C_2 S_1 \end{bmatrix}$$

Unitary transforms (11)

$$\mathcal{CS}_2(\varphi, \alpha, \gamma) = \left[(-1)^{\alpha t} \cdot \mathcal{CS}_{\alpha,t} \right]_{\alpha,t=0}^1 = \begin{bmatrix} C & S \\ \bar{S} & -\bar{C} \end{bmatrix} = \begin{bmatrix} e^{i\alpha} \cos \varphi & e^{i\gamma} \sin \varphi \\ e^{-i\gamma} \sin \varphi & -e^{-i\alpha} \cos \varphi \end{bmatrix}$$

form the group $SU(2, \mathbb{C})$ of all (2×2) -unitary matrices with complex entries and determinant equal to \pm one:

$$SU(2) = \left\{ \mathcal{CS}_2 \in \text{Mat}(2, \mathbb{C}) \mid (\mathcal{CS}_2^\dagger \cdot \mathcal{CS}_2 = \mathcal{CS}_2 \cdot \mathcal{CS}_2^\dagger = \mathbf{I}_2) \ \& \ (\det(\mathcal{CS}_2) = \pm 1) \right\}.$$

Let us introduce (2×2) -unitary matrices with \mathcal{Alg} -valued entries and determinant equal to \pm one:

$$\begin{aligned} \mathcal{CS}_2(\sigma, {}^1\varepsilon, {}^2\varepsilon) &= \begin{bmatrix} {}^1\varepsilon \cdot \sigma & {}^2\varepsilon \cdot \tilde{\sigma} \\ {}^2\bar{\varepsilon} \cdot \tilde{\sigma} & -{}^1\bar{\varepsilon} \cdot \sigma \end{bmatrix} = \begin{bmatrix} C_k(\sigma, {}^1\varepsilon) & S_k(\sigma, {}^2\varepsilon) \\ \bar{S}_k(\sigma, {}^2\varepsilon) & -\bar{C}_k(\sigma, {}^1\varepsilon) \end{bmatrix} = \left[(-1)^{\alpha t} \cdot \mathcal{CS}_{\alpha,t}(\sigma, {}^1\varepsilon, {}^2\varepsilon) \right]_{\alpha,t=0}^1 = \\ &= \left[(-1)^{\alpha t} \cdot \mathcal{CS}_{\alpha,t} \right]_{\alpha,t=0}^1 = \begin{bmatrix} C & S \\ \bar{S} & -\bar{C} \end{bmatrix}, \end{aligned}$$

where ${}^1\varepsilon, {}^2\varepsilon \in \mathcal{Alg}$, ${}^1\varepsilon \cdot {}^1\bar{\varepsilon} = |{}^1\varepsilon|^2 = 1$, ${}^2\varepsilon \cdot {}^2\bar{\varepsilon} = |{}^2\varepsilon|^2 = 1$, $\sigma, \tilde{\sigma} \in \mathcal{Alg}$, $\sigma^2 + \tilde{\sigma}^2 = 1$, and \mathcal{Alg} is an algebra (for example, Clifford algebras or finite Galois fields). In this case

$$\mathbf{G}_2^{[n]}(\boldsymbol{\sigma}_n, {}^1\boldsymbol{\varepsilon}_n, {}^2\boldsymbol{\varepsilon}_n) = \left[\text{com}_{u_n}^{[n]}(\mathbf{t}_n | \boldsymbol{\sigma}_n, {}^1\boldsymbol{\varepsilon}_n, {}^2\boldsymbol{\varepsilon}_n) \right] = \left[\prod_{i=1}^n (-1)^{(\alpha_{i-1} \oplus t_i)(\alpha_i \oplus t_{i+1})} \cdot \mathcal{CS}_{(\alpha_{i-1} \oplus \alpha_i), (t_i \oplus t_{i+1})}^{[n+1]}(\sigma_i, {}^1\varepsilon_i, {}^2\varepsilon_i) \right]$$

is the Fourier-Galois-Golay algebraic transform (FGGAT).

Example 4. Let $Alg = \mathbf{GF}(7)$, $\sigma = 2$, $\tilde{\sigma} = 5$, ${}^1\varepsilon, {}^2\varepsilon \equiv 1$. It is easy to check that $\sigma^2 + \tilde{\sigma}^2 = 2^2 + 5^2 = 1 \pmod{7}$. If $\mathcal{CS}_2(\sigma_1, {}^1\varepsilon_1, {}^2\varepsilon_1) = \mathcal{CS}_2(\sigma_2, {}^1\varepsilon_2, {}^2\varepsilon_2) = \begin{bmatrix} 2 & 5 \\ 5 & -2 \end{bmatrix}$, then

$$\mathbf{G}_{2^2}^{[2]} = \begin{bmatrix} \text{com}_{(0,0)}(\mathbf{t}_2) \\ \text{com}_{(0,1)}(\mathbf{t}_2) \\ \text{com}_{(1,0)}(\mathbf{t}_2) \\ \text{com}_{(1,1)}(\mathbf{t}_2) \end{bmatrix} = \begin{bmatrix} 2 & 5 & & \\ 5 & -2 & & \\ & & 2 & 5 \\ & & 5 & -2 \end{bmatrix} \begin{bmatrix} 2 & 5 & & \\ & & 5 & -2 \\ & & 2 & 5 \\ 5 & -2 & & \end{bmatrix} = \begin{bmatrix} 4 & 10 & & 4 & -10 \\ 10 & 4 & -10 & & 4 \\ 4 & -10 & & 4 & 10 \\ -10 & 4 & 10 & & 4 \end{bmatrix}$$

is the Fourier-Galois-Golay algebraic transform.

If $Alg = Clif$, where $Clif$ is the Clifford algebra, then $\mathbf{G}_{2^n}^{[n]}(\boldsymbol{\sigma}_n, {}^1\varepsilon_n, {}^2\varepsilon_n)$ is the Fourier-Clifford-Golay transform (FCGT), if $Alg = Ham$, where Ham is the quaternion Hamilton algebra, then $\mathbf{G}_{2^n}^{[n]}(\boldsymbol{\sigma}_n, {}^1\varepsilon_n, {}^2\varepsilon_n)$ is the Fourier-Hamilton-Golay transform (FHGT) and so on.

5. Conclusion

In this paper, we have shown a new unified approach to the so-called generalized complex- $\mathbf{GF}(p)$ -or Clifford-valued complementary sequences. The approach is based on a new iteration generating construction. This construction has a rich algebraic structure. It is associated not with the triple $(\mathbf{Z}_2^n, \mathcal{F}_2, \mathbf{C})$, but with

$$(\mathbf{Z}_2, \mathcal{CS}_2(\varphi, \alpha, \gamma), Alg) \text{ and } (\mathbf{Z}_2, \{\mathcal{CS}_2^1(\varphi_1, \alpha_1, \gamma_1), \mathcal{CS}_2^2(\varphi_2, \alpha_2, \gamma_2), \dots, \mathcal{CS}_n^1(\varphi_n, \alpha_n, \gamma_n)\}, Alg),$$

where $\mathcal{CS}_2^1(\varphi_1, \alpha_1, \gamma_1)$ and $\{\mathcal{CS}_2^1(\varphi_1, \alpha_1, \gamma_1), \mathcal{CS}_2^2(\varphi_2, \alpha_2, \gamma_2), \dots, \mathcal{CS}_n^1(\varphi_n, \alpha_n, \gamma_n)\}$ are a single transform or a

set of arbitrary unitary (2×2) -transforms of type $\mathcal{CS}_2(\varphi, \alpha, \gamma) = \begin{bmatrix} e^{i\alpha} \cos \varphi & e^{i\gamma} \sin \varphi \\ e^{-i\gamma} \sin \varphi & -e^{-i\alpha} \cos \varphi \end{bmatrix}$, if $Alg = \mathbf{C}$;

or orthogonal (2×2) -transforms of type $\mathcal{CS}_2(\sigma, {}^1\varepsilon, {}^2\varepsilon) = \begin{bmatrix} {}^1\varepsilon \cdot \sigma & {}^2\varepsilon \cdot \tilde{\sigma} \\ {}^2\bar{\varepsilon} \cdot \tilde{\sigma} & -{}^1\bar{\varepsilon} \cdot \sigma \end{bmatrix}$, if Alg is a finite algebra

(for example, a finite Galois field $\mathbf{GF}(q)$, or a finite Clifford algebra), where ${}^1\varepsilon, {}^2\varepsilon \in Alg$, ${}^1\varepsilon \cdot {}^1\bar{\varepsilon} = |{}^1\varepsilon|^2 = 1$, ${}^2\varepsilon \cdot {}^2\bar{\varepsilon} = |{}^2\varepsilon|^2 = 1$, $\sigma, \tilde{\sigma} \in Alg$, $\sigma^2 + \tilde{\sigma}^2 = 1$.

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