MANY-PARAMETER M-COMPLEMENTARY GOLAY SEQUENCES AND TRANSFORMS

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Abstract

In this paper, we develop the family of Golay–Rudin–Shapiro (GRS) m-complementary many-parameter sequences and many-parameter Golay transforms. The approach is based on a new generalized iteration generating construction, associated with n unitary many-parameter transforms and n arbitrary groups of given fixed order. We are going to use multi-parameter Golay transform in Intelligent-OFDM-TCS instead of discrete Fourier transform in order to find out optimal values of parameters optimized PARP, BER, SER, anti-eavesdropping and anti-jamming effects.

Keywords: complementary sequences, many-parameter orthogonal transforms, fast algorithms, OFDM systems.

Introduction

Binary ± 1-valued *Golay – Rudin – Shapiro* sequences (2-GRSS) associated with the cyclic group $\mathbf{Z}_2^n$ were introduced independently by Golay [1, 2, 3] in 1949-1951, Shapiro [4, 5] and Rudin [6] in 1951. M.J.E.Golay [2] introduced the general concept of "complementary pairs" of finite sequences all of whose entries are ± 1. For building the classical FGRST in bases of classical 2-GRSS the following actors are used: 1) Abelian group $\mathbf{Z}_2$, 2) 2-point Fourier transform $\mathcal{F}_2$, and 3) complex field $\mathbf{C}$, *i.e.,* these transforms are associated with the triple $(\mathbf{Z}_2, \mathcal{F}_2, \mathbf{C})$.

In previous papers [7, 8], we have shown a new unified approach to the $\mathbf{GF}(p)$ -, or Clifford-valued complementary sequences and Golay transforms. It was associated not with the triple $(\mathbf{Z}_2, \mathcal{F}_2, \mathbf{C})$, but with triples

$$(\mathbf{Z}_2 , \{\mathcal{CS}_2^1(\varphi_1, \alpha_1, \gamma_1), \mathcal{CS}_2^2(\varphi_2, \alpha_2, \gamma_2),...,$$

$$\mathcal{CS}_n^1(\varphi_n, \alpha_n, \gamma_n)\}, \mathcal{A}lg)$$

and $\left(\mathbf{Z}_2, \mathcal{CS}_2(\varphi, \alpha, \gamma), \mathcal{A}lg\right)$, where $\{\mathcal{CS}_2^1(\varphi_1, \alpha_1, \gamma_1),$ $\mathcal{CS}_2^2(\varphi_2, \alpha_2, \gamma_2),...,\mathcal{CS}_n^1(\varphi_n, \alpha_n, \gamma_n)\}$ is a set of arbitrary unitary(2×2) -transforms of type

$$\mathcal{CS}_2(\varphi_k, \alpha_k, \gamma_k) = \begin{bmatrix} e^{i\alpha_k}\cos\varphi_k & e^{i\gamma_k}\sin\varphi_k \\ e^{-i\gamma_k}\sin\varphi_k & -e^{-i\alpha_k}\cos\varphi_k \end{bmatrix},$$

$$k = 1,...,n,$$

and $\mathcal{CS}_2(\varphi, \alpha, \gamma)$ is a single transform, $\mathcal{A}lg$ is an algebra (for example, Clifford algebra).

In this work, we develop a new unified approach to the so-called generalized multi-parameter $m-$ complementary sequences. This construction has a rich algebraic structure. It is associated not with the triple $(\mathbf{Z}_2, \mathcal{F}_2, \mathbf{C})$, but with

1) $\left(\mathbf{Z}_m, \mathbf{U}_m, \mathcal{A}lg\right)$, 2) $\left(\mathbf{Z}_m, \{\mathbf{U}_m^1, \mathbf{U}_m^2,..., \mathbf{U}_m^n\}, \mathcal{A}lg\right)$,

3) $\left(\mathbf{Gr}_m, \{\mathbf{U}_m^1, \mathbf{U}_m^2,..., \mathbf{U}_m^n\}, \mathcal{A}lg\right)$,

4) $\left(\{\mathbf{Gr}_m^1, \mathbf{Gr}_m^2,..., \mathbf{Gr}_m^n\}, \{\mathbf{U}_m^1, \mathbf{U}_m^2,..., \mathbf{U}_m^n\}, \mathcal{A}lg\right)$.

where $\{\mathbf{Gr}_m^1, \mathbf{Gr}_m^2,..., \mathbf{Gr}_m^n\}$ is a set of arbitrary finite groups of given order *m* Here $\{\mathbf{U}_m^1, \mathbf{U}_m^2,..., \mathbf{U}_m^n\}$ is a set of arbitrary unitary (*m×m*) – transforms represented in the many-parameter Jacobi-Euler form [9 – 10]:

$$\mathbf{U}_m^1 = \mathbf{U}_m^1(\varphi_0^1, \varphi_1^1,..., \varphi_q^1) = \mathbf{U}_m^1(\boldsymbol{\varphi}_q^1) = \prod_{r=1}^{m-1}\prod_{s=r+1}^{m}\mathbf{J}(\varphi_{r,s}^1),$$

$$\mathbf{U}_m^2 = \mathbf{U}_m^2(\varphi_0^2, \varphi_1^2,..., \varphi_q^2) = \mathbf{U}_m^2(\boldsymbol{\varphi}_q^2) = \prod_{r=1}^{m-1}\prod_{s=r+1}^{m}\mathbf{J}(\varphi_{r,s}^2),$$

...,

$$\mathbf{U}_m^n = \mathbf{U}_m^n(\varphi_0^n, \varphi_1^n,..., \varphi_q^n) = \mathbf{U}_m^n(\boldsymbol{\varphi}_q^n) = \prod_{r=1}^{m-1}\prod_{s=r+1}^{m}\mathbf{J}(\varphi_{r,s}^n),$$

where

$$\mathbf{J}(\varphi_{r,s}) = \begin{array}{c} \\ \\ r \\ \\ s \\ \\ \end{array}\begin{pmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & c(\varphi_{r,s}) & \cdots & s(\varphi_{r,s}) & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & s(\varphi_{r,s}) & \cdots & -c(\varphi_{r,s}) & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix},$$

is the Jacobi orthonormal rotation with reflection, $\boldsymbol{\varphi}_q^1 = (\varphi_0^1, \varphi_1^1,..., \varphi_q^1),..., \boldsymbol{\varphi}_q^n = (\varphi_0^n, \varphi_1^n,..., \varphi_q^n)$ are the Jacobi parameters, $q = C_m^2 = m(m-1)/2$, $c(\varphi_{r,s}) = \cos(\varphi_{r,s})$, $s(\varphi_{r,s}) = \sin(\varphi_{r,s})$.

The rest of the paper is organized as follows: in Section 2, the object of the study (*Golay – Rudin – Shapiro m*-ary sequences) is described. In Section 3 we propose method based on new generalized iteration rule with *n* unitary (*m×m*)-transforms $\mathbf{U}_m^1, \mathbf{U}_m^2,..., \mathbf{U}_m^n$ and single group $\mathbf{Z}_m$. Then we generalize the previously method on *n* unitary (*m×m*)-transforms $\mathbf{U}_m^1, \mathbf{U}_m^2,..., \mathbf{U}_m^n$ and on *n* finite groups $\{\mathbf{Gr}_m^1, \mathbf{Gr}_m^2,..., \mathbf{Gr}_m^n\}$. In Section 5 we derive fast algorithms for binary Golay transforms.

The object of the study.
New iteration construction for original Golay sequences

We begin by describing the original Golay *m*-complementary sequences.

Definition 1. *A generalization of the Golay complementary pair, known as the Golay m-Complementary m-element Set (m-GCS) of complex-valued sequences* [11]

$$m\text{-GCS} = \begin{cases} \text{com}_0(t) := \big(c_0(0), \quad c_0(1), \ ..., \ c_0(m-1)\big), \\ \text{com}_1(t) := \big(c_1(0), \quad c_1(1), \ ..., \ c_1(m-1)\big), \\ ...., \\ \text{com}_{m-1}(t) := \big(c_{m-1}(0), c_{m-1}(1),...,c_{m-1}(m-1)\big) \end{cases}$$

is defined by $\sum_{k=0}^{m-1} COR_k(\tau) = m \cdot \delta(\tau), \ \sum_{k=0}^{m-1} |COM_1(z)|^2 = m,$

where $\left\{ COR_k(\tau) \right\}_{k=0}^{m-1}$ are the periodic autocorrelation functions of $\left\{ \text{com}_k(t) \right\}_{k=1}^{m}$ and $COM_k(z) = \mathcal{Z}\left\{ \text{com}_k(t) \right\}$ are their $\mathcal{Z}$ − transforms.

We use two symbols $\alpha_n \in [0, m^{n-1}-1] = \mathbf{Z}_{m^n}$ and $t_n \in [0, m^{n-1}-1] = \mathbf{Z}_{m^n}$ for numeration of Golay sequences and discrete time, respectively. For integer $\alpha_n \in [0, m^{n-1}-1] = \mathbf{Z}_{m^n}$ and $t_n \in [0, m^{n-1}-1] = \mathbf{Z}_{m^n}$ we shall use m-arycodes $\vec{a}_n = (\alpha_1, \alpha_2, ..., \alpha_n), \vec{t}_n = (t_1, t_2, ..., t_n),$ where $\alpha_i t_i \in \{0, 1, ..., m-1\} = \mathbf{Z}_m, i = 1, 2, ..., n.$

Let $\vec{a}_n = (\alpha_1, \alpha_2, ..., \alpha_n)$ *and* $\vec{t}_n = (t_1, t_2, ..., t_n)$ *be m-ary codes, then define*

$$\alpha_n = |\vec{a}_n| = \sum_{i=1}^{n} \alpha_{n-i+1} m^{i-1}, \text{ and } t_n = |\vec{t}_n| = \sum_{i=1}^{n} t_{n-i+1} m^{n-i}$$

as integers whose m-ary codes are $\vec{a}_n = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $\vec{t}_n = (t_1, t_2, ..., t_n),$ where α_n, t_1 are less significant bits (LSB) and α_1, t_n are most significant bits (MSB) of $\vec{a}_n = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $\vec{t}_n = (t_1, t_2, ..., t_n),$ respectively. Obviously,

$$\vec{a}_1 = (\alpha_1) \in \mathbf{Z}_m, \qquad\qquad \mathbf{\alpha}_1 = \alpha_1 \in \mathbf{Z}_m,$$
$$\vec{a}_2 = (\vec{a}_1, \alpha_2) \in \mathbf{Z}_m \times \mathbf{Z}_m = \mathbf{Z}_m^2, \qquad (\mathbf{\alpha}_1, \alpha_2) \in \mathbf{Z}_m \times \mathbf{Z}_m,$$
$$\vec{a}_3 = (\vec{a}_2, \alpha_3) \in \mathbf{Z}_m^2 \times \mathbf{Z}_m = \mathbf{Z}_m^3, \qquad (\mathbf{\alpha}_2, \alpha_3) \in \mathbf{Z}_{m^2} \times \mathbf{Z}_m,$$
$$..., \qquad ....,$$
$$\vec{a}_n = (\vec{a}_{n-1}, \alpha_n) \in \mathbf{Z}_m^{n-1} \times \mathbf{Z}_m = \mathbf{Z}_m^n, \quad (\mathbf{\alpha}_{n-1}, \alpha_n) \in \mathbf{Z}_{m^{n-1}}^1 \times \mathbf{Z}_m;$$
$$\vec{t}_1 = (t_1) \in \mathbf{Z}_m, \qquad\qquad t_1 = t_1 \in \mathbf{Z}_m,$$
$$\vec{t}_2 = (\vec{t}_1, t_2) \in \mathbf{Z}_m \times \mathbf{Z}_m = \mathbf{Z}_m^2, \qquad (t_1, t_2) \in \mathbf{Z}_m \times \mathbf{Z}_m,$$
$$\vec{t}_3 = (\vec{t}_2, t_3) \in \mathbf{Z}_m^2 \times \mathbf{Z}_m = \mathbf{Z}_m^3, \qquad (t_2, t_3) \in \mathbf{Z}_{m^2} \times \mathbf{Z}_m,$$
$$..., \qquad ....,$$
$$\vec{t}_n = (\vec{t}_{n-1}, t_n) \in \mathbf{Z}_m^{n-1} \times \mathbf{Z}_m = \mathbf{Z}_m^n, \qquad (\mathbf{t}_{n-1}, t_n) \in \mathbf{Z}_{m^{n-1}} \times \mathbf{Z}_m.$$

Let $\left\{ \text{com}_{\mathbf{a}_{n+1}}^{[n+1]}(\mathbf{t}_{n+1}) \right\}$ be m^{n+1}-element set of m complementary sequences (of length m^{n+1}), where $\alpha_{n+1}, t_{n+1} = 0, 1, .., m^{n+1}-1$ They form rows of a $(m^{n+1} \times m^{n+1})$-matrix $\mathbf{G}_{m^{n+1}}^{[n+1]} = \left[\text{com}_{\mathbf{a}_{n+1}}^{[n+1]}(\mathbf{t}_{n+1}) \right]_{\mathbf{a}_{n+1}, \mathbf{t}_{n+1}=0}^{m^{n+1}-1},$ that is called *the m-Golay matrix*. Here index [*n*+1] shows that Golay matrix have been obtained on the *n*+1 iteration step. We are going to group these rows (sequences) as

$$\mathbf{G}_{m^{n+1}}^{[n+1]} = \boxplus_{\mathbf{a}_{n+1}=0}^{m^{n+1}-1} \text{com}_{(\mathbf{a}_{n+1})}^{[n+1]}(\mathbf{t}_{n+1}) = \boxplus_{\mathbf{a}_n=0}^{m^n-1} \left(\boxplus_{\alpha_{n+1}=0}^{m-1} \text{com}_{(\mathbf{a}_n, \alpha_{n+1})}^{[n+1]}(\mathbf{t}_{n+1}) \right) = \boxplus_{\mathbf{a}_n=0}^{m^n-1} \begin{bmatrix} \text{com}_{(\mathbf{a}_n,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(\mathbf{a}_n,1)}^{[n+1]}(\mathbf{t}_{n+1}) \\ ... \\ \text{com}_{(\mathbf{a}_n,m-1)}^{[n+1]}(\mathbf{t}_{n+1}) \end{bmatrix}. \qquad (1)$$

Let us to select the more fine structure of the *m*-Golay matrix:

$$\mathbf{G}_{m^{n+1}}^{[n+1]} = \boxplus_{\mathbf{a}_{n+1}=0}^{m^{n+1}-1} \text{com}_{(\mathbf{a}_{n+1})}^{[n+1]}(\mathbf{t}_{n+1}) = \boxplus_{\mathbf{a}_n=0}^{m^n-1} \begin{bmatrix} \text{com}_{(\mathbf{a}_n,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(\mathbf{a}_n,1)}^{[n+1]}(\mathbf{t}_{n+1}) \\ .... \\ \text{com}_{(\mathbf{a}_n,m-1)}^{[n+1]}(\mathbf{t}_{n+1}) \end{bmatrix} = \boxplus_{\mathbf{a}_{n-1}=0}^{m^{n-1}-1} \left(\boxplus_{\alpha_n=0}^{m-1} \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1},\alpha_n,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(\mathbf{a}_{n-1},\alpha_n,1)}^{[n+1]}(\mathbf{t}_{n+1}) \\ .... \\ \text{com}_{(\mathbf{a}_{n-1},\alpha_n m-1)}^{[n+1]}(\mathbf{t}_{n+1}) \end{bmatrix} \right) = \boxplus_{\mathbf{a}_{n-1}=0}^{m^{n-1}-1} \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1},0,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(\mathbf{a}_{n-1},0,1)}^{[n+1]}(\mathbf{t}_{n+1}) \\ ... \\ \text{com}_{(\mathbf{a}_{n-1},0,m-1)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \hline \text{com}_{(\mathbf{a}_{n-1},1,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(\mathbf{a}_{n-1},1,1)}^{[n+1]}(\mathbf{t}_{n+1}) \\ ... \\ \text{com}_{(\mathbf{a}_{n-1},1,m-1)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \vdots \\ \vdots \\ \hline \text{com}_{(\mathbf{a}_{n-1},m-1,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(\mathbf{a}_{n-1},m-11)}^{[n+1]}(\mathbf{t}_{n+1}) \\ ... \\ \text{com}_{(\mathbf{a}_{n-1},m-1m-1)}^{[n+1]}(\mathbf{t}_{n+1}) \end{bmatrix}. \qquad (2)$$

Example 1. For *n* = 1 and *n* = 2 we have, respectively,

$$\mathbf{G}_{3^1}^{[1]} = \left[\mathrm{com}_{\alpha_1}^{[1]}(\mathbf{t}_1)\right]_{\alpha_1,\mathbf{t}_1=0}^{2} = \bigboxplus_{\alpha_1=0}^{2} \mathrm{com}_{\alpha_1}^{[1]}(\mathbf{t}_1) = \begin{bmatrix} \mathrm{com}_{(0)}^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_{(1)}^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_{(2)}^{[1]}(\mathbf{t}_1) \end{bmatrix},$$

$$\mathbf{G}_{3^2}^{[2]} = \bigboxplus_{\alpha_1=0}^{2} \begin{bmatrix} \mathrm{com}_{(\alpha_1,0)}^{[2]}(\mathbf{t}_2) \\ \mathrm{com}_{(\alpha_1,1)}^{[2]}(\mathbf{t}_2) \\ \mathrm{com}_{(\alpha_1,2)}^{[2]}(\mathbf{t}_2) \end{bmatrix} = \begin{bmatrix} \mathrm{com}_{(0,0)}^{[2]}(\mathbf{t}_2) \\ \mathrm{com}_{(0,1)}^{[2]}(\mathbf{t}_2) \\ \mathrm{com}_{(0,2)}^{[2]}(\mathbf{t}_2) \\ \hline \mathrm{com}_{(1,0)}^{[2]}(\mathbf{t}_2) \\ \mathrm{com}_{(1,1)}^{[2]}(\mathbf{t}_2) \\ \mathrm{com}_{(1,2)}^{[2]}(\mathbf{t}_2) \\ \hline \mathrm{com}_{(2,0)}^{[2]}(\mathbf{t}_2) \\ \mathrm{com}_{(2,1)}^{[2]}(\mathbf{t}_2) \\ \mathrm{com}_{(2,2)}(\mathbf{t}_2) \end{bmatrix}. \ \square$$

The matrix $\mathbf{G}_{m^{n+1}}^{[n+1]}$ is constructed by an iteration construction. The initial matrix $\mathbf{G}_{m^1}^{[1]}$ is formed by starting with an arbitrary unitary $(m\times m)$-matrix (in many-parameter form or not)

$$\mathbf{U}_m = \left[A_\alpha(t)\right] := \mathbf{G}_{m^1}^{[1]} = \begin{bmatrix} \mathrm{com}_0^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_1^{[1]}(\mathbf{t}_1) \\ \dots\dots \\ \mathrm{com}_{m-1}^{[1]}(\mathbf{t}_1) \end{bmatrix} =$$

$$= \begin{bmatrix} A_0(0) & A_0(1) & A_0(2) & \dots & A_0(m-1) \\ A_1(0) & A_1(1) & A_1(2) & \dots & A_1(m-1) \\ A_2(0) & A_2(1) & A_2(2) & \dots & A_2(m-1) \\ \dots & \dots & \dots & \dots & \dots \\ A_{m-1}(0) & A_{m-1}(1) & A_{m-1}(2) & \dots & A_{m-1}(m-1) \end{bmatrix},$$

where $A_\alpha(t) \in \mathcal{A}lg$,

$\mathrm{com}_\alpha^{[1]}(t) = \left(A_\alpha(0),\ A_\alpha(1),\dots,\ A_\alpha(m-1)\right)$.

Example 2. The initial matrix $\mathbf{G}_{m^1}^{[1]}$ can be the Fourier transform on Abelian group $\mathbf{Z}_m$:

$$\mathbf{G}_{m^1}^{[1]} = \begin{bmatrix} \mathrm{com}_0^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_1^{[1]}(\mathbf{t}_1) \\ \mathrm{com}_2^{[1]}(\mathbf{t}_1) \\ \dots\dots \\ \mathrm{com}_{m-1}^{[1]}(\mathbf{t}_1) \end{bmatrix} = \tag{3}$$

$$= \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \varepsilon^{1\cdot1} & \varepsilon^{1\cdot2} & \dots & \varepsilon^{1\cdot(m-1)} \\ 1 & \varepsilon^{2\cdot1} & \varepsilon^{2\cdot2} & \dots & \varepsilon^{2\cdot(m-1)} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \varepsilon^{(m-1)\cdot1} & \varepsilon^{(m-1)\cdot2} & \dots & \varepsilon^{(m-1)\cdot(m-1)} \end{bmatrix},$$

where $\varepsilon_m = \sqrt[m]{1} \in \mathcal{A}lg$, $\mathrm{com}_k^{[1]}(\mathbf{t}) = \left(1,\varepsilon^{k\cdot1},\varepsilon^{k\cdot2},\dots,\varepsilon^{k\cdot(m-1)}\right)$, $(k=0,1,\dots,m-1)$ are characters $\mathbf{Z}_m$. $\square$

It is easy to check that

$$\left(\left|\mathrm{COM}_0(z)\right|^2 + \left|\mathrm{COM}_1(z)\right|^2 + \dots + \left|\mathrm{COM}_{m-1}(z)\right|^2\right)_{|z|=1} = m.$$

Indeed,

$$\sum_{k=1}^{m-1} \left|\mathrm{COM}_k(z)\right|^2 = \sum_{k=1}^{m-1} \mathrm{COM}_k(z)\overline{\mathrm{COM}}^k(\overline{z}) =$$

$$= \sum_{k=1}^{m-1}\left(\sum_{t=0}^{m-1} a_k(t)z^t\right)\left(\sum_{s=0}^{m-1} \overline{a}_k(s)\overline{z}^s\right) =$$

$$= \sum_{s=0}^{m-1}\sum_{t=0}^{m-1}\left(\sum_{k=0}^{m-1} a_k(t)\overline{a}_k(s)\right)z^t\overline{z}^s = \sum_{s=0}^{m-1}\sum_{t=0}^{m-1}\delta_{t-s}z^t\overline{z}^s = \sum_{t=0}^{m-1}\left|z\right|^{2t},$$

since $\sum_{k=0}^{m-1} a_k(t)\overline{a}_k(s) = \delta_{t-s}$ is true for an arbitrary unitary (orthogonal) matrix. Hence,

$$\left.\left(\sum_{k=1}^{m-1}\left|\mathrm{COM}_k(z)\right|^2\right)\right|_{|z|=1} = \left(\sum_{t=0}^{m-1}\left|z\right|^{2t}\right)_{|z|=1} = m$$

and initial sequences in the form of rows of an unitary matrix (in particular case, in the form of characters $\mathrm{com}_k(\mathbf{t}_1) = (1,\varepsilon^{k\cdot1},\varepsilon^{k\cdot2},\dots,\varepsilon^{k\cdot(m-1)})$ of cyclic group $\mathbf{Z}_m$) are the Golay *m*-complementary sequences.

Methods

The matrix $\mathbf{G}_{m^{n+1}}^{[n+1]}$ is constructed by an iteration construction

$$\mathbf{G}_{m^1}^{[1]}(\mathbf{U}_m) \xrightarrow{\mathbf{U}_m^2} \mathbf{G}_{m^2}^{[2]}(\mathbf{U}_m^1,\mathbf{U}_m^2) \xrightarrow{\mathbf{U}_m^3} \dots \xrightarrow{\mathbf{U}_m^{n+1}} \mathbf{G}_{m^{n+1}}^{[n+1]}(\mathbf{U}_m^1,\dots,\mathbf{U}_m^n,\mathbf{U}_m^{n+1}), \tag{4}$$

where

$$\mathcal{U}_{n+1} := \left\{\mathbf{U}_m^1,\dots,\mathbf{U}_m^n,\mathbf{U}_m^{n+1}\right\} = \left\{\mathcal{U}_n,\mathbf{U}_m^{n+1}\right\},$$
$$\mathcal{U}_n := \left\{\mathbf{U}_m^1,\dots,\mathbf{U}_m^n\right\}.$$

Here $\mathbf{U}_m^s(\boldsymbol{\varphi}_q^s) = \left[A_\alpha^s(t\,|\,\boldsymbol{\varphi}_q^s)\right]_{\alpha,t=0}^{m-1} \in SU(\mathcal{A}lg,m)$ $(s=1,2,\dots,n)$ are a sequence of unitary many-parameter $(m\times m)$-transforms, belonging to the special unitary group $SU(\mathcal{A}lg,m)$, where $s=1,2,\dots,n+1$ and $A_\alpha^s(t\,|\,\boldsymbol{\varphi}_q^s)$ are $\mathcal{A}lg$-valued many-parameter sequences.

Let us assume that we have *m*-Golay matrix $\mathbf{G}_{m^n}^{[n]}(\mathbf{U}_1,\dots,\mathbf{U}_n) = \mathbf{G}_{m^n}^{[n]}(\mathcal{U}_n)$ (depending on *n* previous transforms $\mathbf{U}_m^1,\dots,\mathbf{U}_m^n$). We need to construct the next *m*-Golay matrix $\mathbf{G}_{m^{n+1}}^{[n+1]}(\mathbf{U}_m^1,\dots,\mathbf{U}_m^{n+1}) = \mathbf{G}_{m^{n+1}}^{[n+1]}(\mathcal{U}_{n+1})$ using only $\mathbf{G}_{m^n}^{[n]}(\mathbf{U}_m^1,\dots,\mathbf{U}_m^n)$ and $\mathbf{U}_m^{n+1}$. We are going to use for *m*-Golay matrix $\mathbf{G}_{m^n}^{[n]}(\mathcal{U}_n)$ the same structure as in (1):

$$\mathbf{G}_{m^n}^{[n]}(\mathcal{U}_n) = \bigboxplus_{\alpha_n=0}^{m^n-1} \mathrm{com}_{(\alpha_n)}^{[n]}(\mathbf{t}_n\,|\,\mathcal{U}_n) =$$

$$= \bigboxplus_{\alpha_n=0}^{m^n-1} \begin{bmatrix} \mathrm{com}_{(\alpha_{n-1},0)}^{[n]}(\mathbf{t}_n\,|\,\mathcal{U}_n) \\ \mathrm{com}_{(\alpha_{n-1},1)}^{[n]}(\mathbf{t}_n\,|\,\mathcal{U}_n) \\ \dots\dots\dots \\ \mathrm{com}_{(\alpha_{n-1},m-1)}^{[n]}(\mathbf{t}_n\,|\,\mathcal{U}_n) \end{bmatrix}. \tag{5}$$

For constructing $\mathbf{G}_{m^{n+1}}^{[n+1]}(\mathcal{U}_{n+1})$ from $\mathbf{G}_{m^n}^{[n]}(\mathcal{U}_n)$ we take each complementary set in the form

$$m\text{-GCS}^{[n]}\left(\mathcal{U}_n\right) = \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n \mid \mathcal{U}_n) \\ \text{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n \mid \mathcal{U}_n) \\ \dots\dots\dots\dots \\ \text{com}_{(\mathbf{a}_{n-1},m-1)}^{[n]}(\mathbf{t}_n \mid \mathcal{U}_n) \end{bmatrix}$$

and construct *m* shifted versa of their components

$$m\text{-GCS}^{[n]}\left(\mathcal{U}_n\right) \to \begin{array}{l} \nearrow \quad m\text{-GCS}_{\alpha_n=0}^{[n]}\left(\mathcal{U}_{n+1}\right), \\ \quad m\text{-GCS}_{\alpha_n=1}^{[n]}\left(\mathcal{U}_{n+1}\right), \\ \quad \dots\dots \\ \searrow \quad m\text{-GCS}_{\alpha_n=m-1}^{[n]}\left(\mathcal{U}_{n+1}\right), \end{array}$$

where

$$m\text{-GCS}_{\alpha_n}^{[n]}\left(\mathcal{U}_{n+1}\right) = \mathbf{U}_m^{n+1}\left(\mathbf{P}_m^{\alpha_n}\begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} & & & \\ & \mathbf{T}_{\mathbf{t}_n}^{1\cdot m^n} & & \\ & & \ddots & \\ & & & \mathbf{T}_{\mathbf{t}_n}^{(m-1)\cdot m^n} \end{bmatrix}\tilde{\mathbf{P}}_m^{\alpha_n}\right) \times$$

$$\times \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n \mid \mathcal{U}_n) \\ \text{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n \mid \mathcal{U}_n) \\ \dots\dots\dots\dots \\ \text{com}_{(\mathbf{a}_{n-1},m-1)}^{[n]}(\mathbf{t}_n \mid \mathcal{U}_n) \end{bmatrix} \equiv \begin{bmatrix} \text{com}_{(\mathbf{a}_n,0)}^{[n+1]}(\mathbf{t}_{n+1} \mid \mathcal{U}_{n+1}) \\ \text{com}_{(\mathbf{a}_n,1)}^{[n+1]}(\mathbf{t}_{n+1} \mid \mathcal{U}_{n+1}) \\ \dots\dots\dots\dots \\ \text{com}_{(\mathbf{a}_n,m-1)}^{[n+1]}(\mathbf{t}_{n+1} \mid \mathcal{U}_{n+1}) \end{bmatrix}. \quad (6)$$

Here $\alpha_n = 0, 1, \dots, m-1$, $\mathbf{P}_m^{\alpha_n}$ is the cyclic permutation operator on α_n positions (modulo *m*), $\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m}^n s}$ is the shift operator on $m^n s$ positions $\mathbf{T}_{\mathbf{t}_n}^{m^n s} f(\mathbf{t}_n) := f(\mathbf{t}_n + m^n \text{ s})$, $\tilde{\mathbf{P}}_m$ is transposed matrix of $\mathbf{P}_m$.

According to (1) we obtain

$$\mathbf{G}_{m^{n+1}}^{[n+1]}(\mathcal{U}_{n+1}) = \bigboxplus_{\alpha_n=0}^{m^n-1}\begin{bmatrix} \text{com}_{(\mathbf{a}_n,0)}^{[n+1]}(\mathbf{t}_{n+1} \mid \mathcal{U}_{n+1}) \\ \text{com}_{(\mathbf{a}_n,1)}^{[n+1]}(\mathbf{t}_{n+1} \mid \mathcal{U}_{n+1}) \\ \dots\dots\dots\dots \\ \text{com}_{(\mathbf{a}_n,m-1)}^{[n+1]}(\mathbf{t}_{n+1} \mid \mathcal{U}_{n+1}) \end{bmatrix} =$$

$$= \bigboxplus_{\alpha_n=0}^{m^n-1}\mathbf{U}_m^{n+1}\cdot\left(\mathbf{P}_m^{\alpha_n}\cdot\begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} & & & \\ & \mathbf{T}_{\mathbf{t}_n}^{1\cdot m^n} & & \\ & & \ddots & \\ & & & \mathbf{T}_{\mathbf{t}_n}^{(m-1)\cdot m^n} \end{bmatrix}\cdot\tilde{\mathbf{P}}_m^{\alpha_n}\right) \times \quad (7)$$

$$\times \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n \mid \mathcal{U}_n) \\ \text{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n \mid \mathcal{U}_n) \\ \dots\dots\dots\dots \\ \text{com}_{(\mathbf{a}_{n-1},m-1)}^{[n]}(\mathbf{t}_n \mid \mathcal{U}_n) \end{bmatrix},$$

and, consequently,

$$\text{com}_{(\mathbf{a}_{n-1},\alpha_n,\alpha_{n+1})}^{[n+1]}(\mathbf{t}_{n+1} \mid \mathcal{U}_{n+1}) =$$

$$= \sum_{\beta_n=0}^{m-1} a_{\alpha_{n+1}}^{n+1}(\beta_n)\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m}^n(\beta_n\oplus\alpha_n)}\text{com}_{(\mathbf{a}_{n-1},\beta_n)}^{[n]}(\mathbf{t}_n \mid \mathcal{U}_n).$$

Since $\mathbf{t}_{n+1} = (\mathbf{t}_n, t_{n+1})$, then believing $t_{n+1} = \alpha_n \underset{m}{\oplus} \beta_n$, we obtain:

$$\text{com}_{(\mathbf{a}_{n-1},\alpha_n,\alpha_{n+1})}^{[n+1]}(\mathbf{t}_{n+1} \mid \mathcal{U}_{n+1}) = \text{com}_{(\mathbf{a}_{n-1},\alpha_n,\alpha_{n+1})}^{[n+1]}(\mathbf{t}_n, t_{n+1} \mid \mathcal{U}_{n+1}) =$$

$$= \sum_{t_{n+1}=0}^{m-1} A_{\alpha_{n+1}}^{n+1}(\alpha_n \underset{m}{\oplus} t_{n+1})\mathbf{T}_{\mathbf{t}_n}^{m^n t_{n+1}}\text{com}_{(\mathbf{a}_{n-1},\alpha_n\underset{m}{\oplus}t_{n+1})}^{[n]}(\mathbf{t}_n \mid \mathcal{U}_n) = \quad (8)$$

$$= \sum_{t_{n+1}=0}^{m-1} A_{\alpha_{n+1}}^{n+1}(\alpha_n \underset{m}{\oplus} t_{n+1})\mathbf{T}_{\mathbf{t}_n}^{m^n t_{n+1}}\text{com}_{(\mathbf{a}_{n-1},\alpha_n\underset{m}{\oplus}t_{n+1})}^{[n]}(\mathbf{t}_n + m^n t_{n+1} \mid \mathcal{U}_n).$$

So,

$$\text{com}_{(\mathbf{a}_{n-1},\alpha_n,\alpha_{n+1})}^{[n+1]}(\mathbf{t}_n, t_{n+1} \mid \mathcal{U}_{n+1}) =$$

$$= A_{\alpha_{n+1}}^{n+1}(\alpha_n \underset{m}{\oplus} t_{n+1})\cdot\text{com}_{(\mathbf{a}_{n-1},\alpha_n\underset{m}{\oplus}t_{n+1})}^{[n]}(\mathbf{t}_n \mid \mathcal{U}_n). \quad (9)$$

It is finally recurrent relation between *m*-complementary sequences of $\mathbf{G}_{m^{n+1}}^{[n+1]}[\mathcal{U}_{n+1}]$ and $\mathbf{G}_{m^n}^{[n]}[\mathcal{U}_n]$. From (9) we obtain expression for $\text{com}_{\bar{\mathbf{a}}_{n+1}}^{[n+1]}(\mathbf{t}_{n+1} \mid \mathcal{U}_{n+1})$:

$$\text{com}_{(\mathbf{a}_{n+1})}^{[n+1]}(\mathbf{t}_{n+1}) = \prod_{s=1}^{n} A_{\alpha_{s+1}\underset{m}{\oplus}t_{s+2}}^{s+1}(\alpha_s \underset{m}{\oplus} t_{s+1}), \quad \alpha_0, t_{n+2} \equiv 0. (10)$$

In particular, for matrices in the form of the Fourier transform $\mathbf{U}_m^1 = \mathbf{U}_m^2 = \dots = \mathbf{U}_m^n = [\varepsilon_m^{\alpha t}]$ we have

$$\text{com}_{(\mathbf{a}_{n-1},\alpha_n,\alpha_{n+1})}^{[n+1]}(\mathbf{t}_{n+1}) = \text{com}_{(\mathbf{a}_{n-1},\alpha_n,\alpha_{n+1})}^{[n+1]}(\mathbf{t}_n, t_{n+1}) =$$

$$= \varepsilon_m^{\sum_{s=1}^{n}(\alpha_s\underset{m}{\oplus}t_{s+1})\left(\alpha_{s+1}\underset{m}{\oplus}t_{s+2}\right)}. \quad (11)$$

Where $\alpha_0, t_{n+2} \equiv 0$. New sequences in (9) are orthogonal and *m*-complementary sequences.

Generalizations

In this section, we introduce generalized *m*-complementary sequences. It is based on using new permutation matrices $\mathbf{P}_m^{\alpha_n}$ in (7). The mappings g: $\mathbf{X}\to\mathbf{X}$ of a set $\mathbf{X}$ into (or onto) itself are of particular importance. They form the following set $\mathbf{X^X}$: $= \{g|g: \mathbf{X}\to\mathbf{X} \}$.

Definition 2. *One-to-one map from a set* $\mathbf{X}$ *to itself* g: $\mathbf{X}\to\mathbf{X}$, $x' = g(x) = g\circ x$ *is called a transformation of the set X.*

If $\mathbf{X}$ is finite and consists of *m* elements (for example, $\mathbf{X} = \{0, 1, 2, \dots, m\}$) then a transformation of the set X is called a *permutation*. As is well known, the set of all permutations of $\mathbf{X}$ forms a group $S_m = Sum\{\mathbf{X}\}$ in which the product $\sigma\pi$ of a pair of permutations σ, π is defined by $(\sigma\pi)\circ x$: $= \sigma\circ(\pi\circ x)$.

If $\mathbf{X}$ contains more than two elements, S_m is not commutative. Any subgroup of S_m is called a *permutation group* on $\mathbf{X}$, or a *group of permutations* of $\mathbf{X}$. We shall say that the permutations in $Sym(X)$ *act* or *operate* on the elements of $\mathbf{X}$.

Definition 3. *A homomorphism of a group on a set* h: $\mathbf{Gr}\to Sym\{\mathbf{X}\}$ *is called a permutation representation (or realization) of* .

The image $h(\mathbf{Gr})\subset Sym\{\mathbf{X}\}$ is a permutation group and the elements of are represented as permutations of . A permutation representation is equivalent to an action of on the set : To specify an action, we need to define for element $g\in\mathbf{Gr}$ the corresponding permutation $h(g)$ of , that is, $h(g)\circ x$ for any $x\in\mathbf{X}$. We are going to write $h(g)\circ x$

in the short form $g \circ x$ and to call the group of transformations of . The pair $\langle \rangle$ is called a space with transformation group the elements $x \in \mathbf{X}$ are called points of the space .

Definition 4. *If is a permutation group of degree , then the permutation representation of is the linear permutation representation of* : $\mathbf{P}: \mathbf{Gr} \rightarrow \mathrm{GL}_m(\mathcal{A}lg)$ *which maps to the corresponding permutation matrix* $\mathbf{P}(g)$, .

$$\mathbf{P}(0) = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}, \mathbf{P}(1) = \begin{bmatrix} & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ 1 & & & & \end{bmatrix}, \mathbf{P}(2) = \begin{bmatrix} & & 1 & & \\ & & & 1 & \\ & & & & \ddots \\ 1 & & & & \\ & 1 & & & \end{bmatrix}, ..., \mathbf{P}(m-1) = \begin{bmatrix} & & & & 1 \\ 1 & & & & \\ & \ddots & & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}.$$

In particular, for $m = 2$ and $m = 3$ we have

$$\mathbf{P}(0) = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \mathbf{P}(1) = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix};$$

$$\mathbf{P}(0) = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \mathbf{P}(1) = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}, \mathbf{P}(2) = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}. \ \square$$

That is, acts on by permuting the standard basis vectors $\{e_n\}_{n \in \mathbf{X}} \in \mathcal{A}lg^m$ such that

$$\mathbf{P}(g)e_n = e_{g \circ n} = e_{n'} \in \{e_n\}_{n \in \mathbf{X}},$$

where $\mathbf{P}(g)'s$ are the operators in $\mathcal{A}lg^m$ which define the above mentioned *linear* representation.

Example 3. Let

$$\mathbf{X} = [0, 1, ..., m-1], \ \mathbf{Gr} = \mathbf{Z}_m = \left\langle \{0, 1, ..., m-1\}, \underset{m}{\oplus} \right\rangle$$

be the cyclic group of order *m*. Then

In expression (7) was used *linear* permutation representation $\mathbf{P}(g)$ of only one group . However, we can use others finite groups of given order *m*. Let $\mathbf{Gr} = \mathbf{Gr}_m = \{g_\alpha\}_{\alpha=0}^{m-1}$ be a group of given order *m* and $\{\mathbf{P}(g_\alpha)\}_{\alpha=0}^{m-1}$. Then

$$\mathbf{G}_{m^{n+1}}^{[n+1]}(\mathcal{U}_{n+1}; \mathbf{Gr}_m) = \underset{\alpha_n=0}{\overset{m^n-1}{\boxplus}} \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_n, 0)}^{[n]}(\mathbf{t}_n | \mathcal{U}_{n+1}; \mathbf{Gr}_m) \\ \mathrm{com}_{(\boldsymbol{\alpha}_n, 1)}^{[n]}(\mathbf{t}_n | \mathcal{U}_{n+1}; \mathbf{Gr}_m) \\ \cdots \cdots \cdots \cdots \cdots \\ \mathrm{com}_{(\boldsymbol{\alpha}_n, m-1)}^{[n]}(\mathbf{t}_n | \mathcal{U}_{n+1}; \mathbf{Gr}_m) \end{bmatrix} =$$

$$= \underset{\boldsymbol{\alpha}_n=0}{\overset{m^n-1}{\boxplus}} \mathbf{U}_m^{n+1} \cdot \left(\mathbf{P}_m(g_{\alpha_n}) \cdot \begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} & & & \\ & \mathbf{T}_{\mathbf{t}_n}^{1 \cdot m^n} & & \\ & & \ddots & \\ & & & \mathbf{T}_{\mathbf{t}_n}^{(m-1) \cdot m^n} \end{bmatrix} \cdot \tilde{\mathbf{P}}_m(g_{\alpha_n}) \right) \cdot \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1}, 0)}^{[n]}(\mathbf{t}_n | \mathcal{U}_n; \mathbf{Gr}_m) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1}, 1)}^{[n]}(\mathbf{t}_n | \mathcal{U}_n; \mathbf{Gr}_m) \\ \cdots \cdots \cdots \cdots \cdots \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1}, m-1)}^{[n]}(\mathbf{t}_n | \mathcal{U}_n; \mathbf{Gr}_m) \end{bmatrix}$$

$$(12)$$

is the Golay matrix associated with triple $\left(\mathbf{Gr}_m, \{\mathbf{U}_m^1, \mathbf{U}_m^2, ..., \mathbf{U}_m^{n+1}\}, \mathcal{A}lg \right)$.

Example 4. For $m = 4$ we have two groups: $\mathbf{Z}_4 = \{0, 1, 2, 3\}$ and $\mathbf{Z}_2 \times \mathbf{Z}_2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. For both groups we have the following permutation representations:

$$\mathbf{P}(0) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \mathbf{P}(1) = \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{bmatrix}, \mathbf{P}(2) = \begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix}, \mathbf{P}(3) = \begin{bmatrix} & & & 1 \\ 1 & & & \\ & 1 & & \\ & & 1 & \end{bmatrix},$$

$$\mathbf{P}(0, 0) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \mathbf{P}(0, 1) = \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}, \mathbf{P}(1, 0) = \begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix}, \mathbf{P}(1, 1) = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}.$$

Hence, we can construct two different set of Golay matrices associated with two triples

1) $\left(\mathbf{Z}_4, \{\mathbf{U}_m^1, \mathbf{U}_m^2, ..., \mathbf{U}_m^{n+1}\}, \mathcal{A}lg \right)$,

2) $\left(\mathbf{Z}_2 \times \mathbf{Z}_2, \{\mathbf{U}_m^1, \mathbf{U}_m^2, ..., \mathbf{U}_m^{n+1}\}, \mathcal{A}lg \right)$,

respectively.   □

Let $\mathcal{G}_{n+1} := \{\mathbf{Gr}_m^1, \mathbf{Gr}_m^2, ..., \mathbf{Gr}_m^n, \mathbf{Gr}_m^{n+1}\} = \{\mathcal{G}_m^n, \mathbf{Gr}_m^{n+1}\}$ be a set of arbitrary groups of given order m: $\mathbf{Gr}_m^1 = \{g_{\alpha_1}^1\}_{\alpha_1=0}^{m-1}, ..., \mathbf{Gr}_m^{n+1} = \{g_{\alpha_{n+1}}^1\}_{\alpha_{n+1}=0}^{m-1}$. Then we

can use on each kth iteration permutation representations $\{\mathbf{P}_m^k(g_{\alpha_k})\}_{\alpha_k=0}^{m-1}$ for $\mathbf{Gr}_m^k$. In this case, we obtain the following Golay transform

$$\mathbf{G}_{m^{n+1}}^{[n+1]}(\mathcal{U}_{n+1};\mathcal{G}_{n+1}) = \boxplus_{\mathbf{a}_n=0}^{m^n-1} \begin{bmatrix} \mathrm{com}_{(\mathbf{a}_n,0)}^{[n]}(\mathbf{t}_n \mid \mathcal{U}_{n+1};\mathcal{G}_{n+1}) \\ \mathrm{com}_{(\mathbf{a}_n,1)}^{[n]}(\mathbf{t}_n \mid \mathcal{U}_{n+1};\mathcal{G}_{n+1}) \\ .... \\ \mathrm{com}_{(\mathbf{a}_n,m-1)}^{[n]}(\mathbf{t}_n \mid \mathcal{U}_{n+1};\mathcal{G}_{n+1}) \end{bmatrix} =$$

$$= \boxplus_{\mathbf{a}_n=0}^{m^n-1} \mathbf{U}_m^{n+1} \cdot \left(\mathbf{P}_m^{n+1}(g_{\alpha_n}) \cdot \begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} & & & \\ & \mathbf{T}_{\mathbf{t}_n}^{1 \cdot m^n} & & \\ & & \ddots & \\ & & & \mathbf{T}_{\mathbf{t}_n}^{(m-1)\cdot m^n} \end{bmatrix} \cdot \tilde{\mathbf{P}}_m^{n+1}(g_{\alpha_n}) \right) \cdot \begin{bmatrix} \mathrm{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n \mid \mathcal{U}_n;\mathcal{G}_n) \\ \mathrm{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n \mid \mathcal{U}_n;\mathcal{G}_n) \\ .... \\ \mathrm{com}_{(\mathbf{a}_{n-1},m-1)}^{[n]}(\mathbf{t}_n \mid \mathcal{U}_n;\mathcal{G}_n) \end{bmatrix}.$$

(13)

It is associated with triple

$$\left(\{\mathbf{Gr}_m^1, \mathbf{Gr}_m^2, ..., \mathbf{Gr}_m^{n+1}\}, \{\mathbf{U}_m^1, \mathbf{U}_m^2, ..., \mathbf{U}_m^{n+1}\}, \mathcal{A}lg\right).$$

Fast Golay transforms

Let us consider expressions (8) and (9) for $m=2$ (*i.e.*, expressions (6) and (7) from our work [7]):

$$\mathrm{com}_{(\mathbf{a}_{n+1})}^{[n+1]}(\mathbf{t}_{n+1}) = \mathrm{com}_{(\mathbf{a}_{n-1},\alpha_n,\alpha_{n+1})}^{[n+1]}(\mathbf{t}_n,t_{n+1}) =$$
$$= \sum_{t_{n+1}=0}^{1} (-1)^{\left(\alpha_n \underset{2}{\oplus} t_{n+1}\right)\alpha_{n+1}} \mathrm{com}_{(\mathbf{a}_{n-1},\alpha_n \underset{2}{\oplus} t_{n+1})}^{[n]}\left(\mathbf{t}_n + 2^n \cdot t_{n+1}\right),$$

(14)

$$\mathrm{com}_{(\mathbf{a}_{n-1},\alpha_n,\alpha_{n+1})}^{[n+1]}(\mathbf{t}_n,t_{n+1}) = (-1)^{\left(\alpha_n \underset{2}{\oplus} t_{n+1}\right)\alpha_{n+1}} \mathrm{com}_{(\mathbf{a}_{n-1},\alpha_n \underset{2}{\oplus} t_{n+1})}^{[n]} \times$$
$$\times \left(\mathbf{t}_n\right) = (-1)^{\alpha_n \alpha_{n+1}}(-1)^{\alpha_{n+1} t_{n+1}} \mathrm{com}_{(\mathbf{a}_{n-1},\alpha_n \underset{2}{\oplus} t_{n+1})}^{[n]}\left(\mathbf{t}_n\right)$$

(15)

and find matrix representations of these expressions. We introduce the following σ-parametrized $(2^n \times 2^n)$-matrix:

$$^\sigma\mathbf{G}_{2^n}^{[n]} := \boxplus_{\mathbf{a}_n=0}^{2^n-1} P_2^\sigma \begin{bmatrix} \mathrm{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \mathrm{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} = \begin{cases} ^0\mathbf{G}_{2^n}^{[n]} = \boxplus_{\mathbf{a}_n=0}^{2^n-1} \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \begin{bmatrix} \mathrm{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \mathrm{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix}, & \sigma = 0, \\[4mm] ^1\mathbf{G}_{2^n}^{[n]} = \boxplus_{\mathbf{a}_n=0}^{2^n-1} \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \begin{bmatrix} \mathrm{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \mathrm{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix}, & \sigma = 1; \end{cases}$$

$$= \begin{cases} ^0\mathbf{G}_{2^n}^{[n]} = \boxplus_{\mathbf{a}_n=0}^{2^n-1} P_2^0 \begin{bmatrix} \mathrm{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \mathrm{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix}, & \sigma = 0 \\[4mm] ^1\mathbf{G}_{2^n}^{[n]} = \boxplus_{\mathbf{a}_n=0}^{2^n-1} P_2^1 \begin{bmatrix} \mathrm{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \mathrm{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix}, & \sigma = 1 \end{cases} = \begin{cases} ^0\mathbf{G}_{2^n}^{[n]} = \boxplus_{\mathbf{a}_n=0}^{2^n-1} \begin{bmatrix} \mathrm{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \mathrm{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix}, & \sigma = 0 \\[4mm] ^1\mathbf{G}_{2^n}^{[n]} = \boxplus_{\mathbf{a}_n=0}^{2^n-1} \begin{bmatrix} \mathrm{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n) \\ \mathrm{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n) \end{bmatrix}, & \sigma = 1, \end{cases}$$

and construct the direct sum of introduced matrices

$$\tilde{\mathbf{G}}_{2^{n+1}}^{[n+1]} = \bigoplus_{\sigma=0}^{1} {}^{(\sigma)}\mathbf{G}_{2^n}^{[n]} = \begin{bmatrix} {}^{(0)}\mathbf{G}_{2^n}^{[n]} \\ \hline {}^{(1)}\mathbf{G}_{2^n}^{[n]} \end{bmatrix} = \begin{bmatrix} \left(\mathbf{I}_{2^{n-1}} \otimes \mathbf{P}_2^0\right)\mathbf{G}_{2^n}^{[n]} \\ \hline \left(\mathbf{I}_{2^{n-1}} \otimes \mathbf{P}_2^1\right)\mathbf{G}_{2^n}^{[n]} \end{bmatrix} =$$

$$= \begin{bmatrix} \boxplus_{\mathbf{a}_{n-1}=0}^{2^{n-1}-1} \begin{bmatrix} \mathrm{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \mathrm{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} \\ \hline \boxplus_{\mathbf{a}_{n-1}=0}^{2^{n-1}-1} \begin{bmatrix} \mathrm{com}_{(\mathbf{a}_{n-1},1)}^{[n]}(\mathbf{t}_n + 2^n) \\ \mathrm{com}_{(\mathbf{a}_{n-1},0)}^{[n]}(\mathbf{t}_n + 2^n) \end{bmatrix} \end{bmatrix}.$$

(16)

From (16) we see that $\tilde{\mathbf{G}}_{2^{n+1}}^{[n+1]}$ represents $\mathrm{com}_{(\mathbf{a}_{n-1},\alpha_n \underset{2}{\oplus} t_{n+1})}^{[n]}\left(\mathbf{t}_n + 2^n \cdot t_{n+1}\right)$ in (14). It is easy to see, that

$$\tilde{\mathbf{G}}_{2^{n+1}}^{[n+1]} = \begin{bmatrix} \left[\mathbf{I}_{2^{n-1}} \otimes \mathbf{P}_2^0\right] \\ \hline \left[\mathbf{I}_{2^{n-1}} \otimes \mathbf{P}_2^1\right] \end{bmatrix} \times \begin{bmatrix} \mathbf{G}_{2^n}^{[n]} \\ \hline \mathbf{G}_{2^n}^{[n]} \end{bmatrix} = \left[\delta_{\alpha_{n+1}}^{(2)}(t_{n+1})\left[\mathbf{I}_{2^{n-1}} \otimes \mathbf{P}_2^{t_{n+1}}\right]\right] \times \left[\mathbf{I}_2 \otimes \mathbf{G}_{2^n}^{[n]}\right] = \mathbf{P}_2^{\{t_{n+1}\}} \cdot \left[\mathbf{I}_2 \otimes \mathbf{G}_{2^n}^{[n]}\right],$$

where

$$\mathbf{P}_{2^{n+1}}^{\{t_{n+1}\}} := \left[\delta_{\alpha_{n+1}}^{(2)}(t_{n+1}) \left[\mathbf{I}_{2^{n-1}} \otimes \mathbf{P}_2^{t_{n+1}} \right] \right] = \left[\begin{array}{c} \left[\mathbf{I}_{2^{n-1}} \otimes \mathbf{P}_2^0 \right] \\ \hline \left[\mathbf{I}_{2^{n-1}} \otimes \mathbf{P}_2^1 \right] \end{array} \right]$$

is the permutation matrix with controlling digit $\{t_{n+1}\}$. According to (15) the Golay matrix $\mathbf{G}_{2^{n+1}}^{[n+1]}$ is the product of three matrices

$$\mathbf{G}_{2^{n+1}}^{[n+1]} = \Delta\left\{ (-1)^{\alpha_n \alpha_{n+1}} \right\} \left[\delta_{\mathbf{a}_n, \mathbf{t}_n}^{(2^n)} (-1)^{\alpha_{n+1} t_{n+1}} \right] \tilde{\mathbf{G}}_{2^{n+1}}^{[n+1]} = \Delta\left\{ (-1)^{\alpha_n \alpha_{n+1}} \right\} \left[\delta_{\mathbf{a}_n, \mathbf{t}_n}^{(2^n)} (-1)^{\alpha_{n+1} t_{n+1}} \right],$$

$$\left[\delta_{\alpha_{n+1}}^{(2)}(t_{n+1}) \left[\mathbf{I}_{2^{n-1}} \otimes \mathbf{P}_2^{t_{n+1}} \right] \right] \left[\mathbf{I}_2 \otimes \mathbf{G}_{2^n}^{[n]} \right] = \Delta\left\{ (-1)^{\alpha_n \alpha_{n+1}} \right\} \left[\delta_{\mathbf{a}_n, \mathbf{t}_n}^{(2^n)} (-1)^{\alpha_{n+1} t_{n+1}} \right] \mathbf{P}_{2^{n+1}}^{\{t_{n+1}\}} \left[\mathbf{I}_2 \otimes \mathbf{G}_{2^n}^{[n]} \right].$$
(17)

Where $\Delta\{(-1)^{\alpha_n \alpha_{n+1}}\} = \mathbf{diag}\{(-1)^{\alpha_n \alpha_{n+1}}\}$ is diagonal matrix, and $\left[\delta_{\mathbf{a}_n, \mathbf{t}_n}^{(2^n)} (-1)^{\alpha_{n+1} t_{n+1}} \right]$ has the following structure

$$\left[\delta_{\mathbf{a}_n, \mathbf{t}_n}^{(2^n)} (-1)^{\alpha_{n+1} t_{n+1}} \right] = \left[\mathbf{I}_{2^n} \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix} \middle| \mathbf{I}_{2^n} \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right] = \left[\mathbf{I}_{2^n} \middle| \mathbf{I}_{2^n} \right] \hat{\otimes} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} =$$

$$= \left[\left[\delta_{\mathbf{a}_n, \mathbf{t}_n}^{(2^n)} \right] \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix} \middle| \left[\delta_{\mathbf{a}_n, \mathbf{t}_n}^{(2^n)} \right] \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right] = \begin{cases} \alpha_{n+1}=0 \\ \alpha_{n+1}=1 \\ \alpha_{n+1}=0 \\ \alpha_{n+1}=1 \\ \alpha_{n+1}=0 \\ \alpha_{n+1}=1 \\ \alpha_{n+1}=0 \\ \alpha_{n+1}=1 \end{cases} \begin{array}{c} t_{n+1}=0 \quad \vdots \quad t_{n+1}=1 \\ \left[\begin{array}{cc|cc} 1 & & 1 & \\ 1 & & -1 & \\ \hline & 1 & & 1 \\ & 1 & & -1 \\ \hline & \vdots & & \vdots \\ & \vdots & & \vdots \\ \hline & & 1 & & 1 \\ & & 1 & & -1 \end{array} \right] := \mathbb{N}_{2^{n+1}}. \end{array}$$
(18)

Here $\hat{\otimes}$ is new tensor product:

$$\left[\mathbf{I}_{2^n} \middle| \mathbf{I}_{2^n} \right] \hat{\otimes} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} := \left[\mathbf{I}_{2^n} \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix} \middle| \mathbf{I}_{2^n} \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right].$$
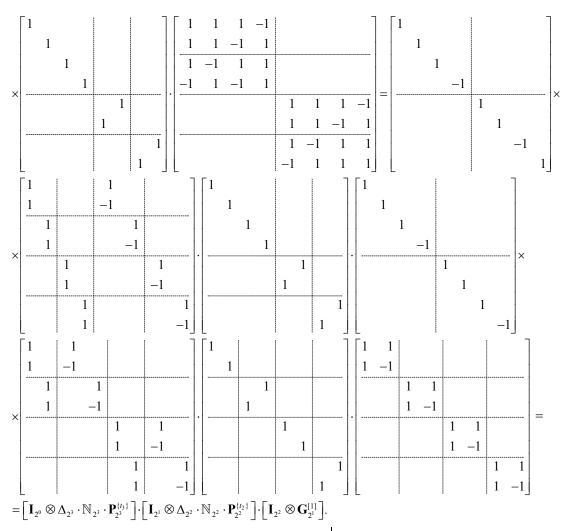
From recurrent relation (17) we obtain

$$\mathbf{G}_{2^n}^{[n]} = \left(\prod_{k=2}^n \left[\mathbf{I}_{2^{n-k}} \otimes \Delta_{2^k} \cdot \mathbb{N}_{2^k} \cdot \mathbf{P}_{2^k}^{\{t_k\}} \right] \right) \cdot \left[\mathbf{I}_{2^{n-k+1}} \otimes \mathbf{G}_{2^1}^{[1]} \right] = \prod_{k=2}^n \left\{ \mathbf{I}_{2^{n-k}} \otimes \left[\Delta\left\{ (-1)^{\alpha_{k-1}\alpha_k} \right\} \cdot \delta_{\mathbf{a}_{k-1}, \mathbf{t}_{k-1}}^{(2^{k-1})} (-1)^{\alpha_k t_k} \right] \cdot \right.$$

$$\left. \cdot \left[\delta_{\alpha_k}^{(2)}(t_k) \left[\mathbf{I}_{2^{k-2}} \otimes \mathbf{P}_2^{t_k} \right] \right] \right\} \cdot \left[\mathbf{I}_{2^{n-k+1}} \otimes \mathbf{G}_{2^1}^{[1]} \right].$$
(19)

This expression represents the fast algorithm for the Golay transform.

Example 5.

$$\mathbf{G}_{2^2}^{[2]} = \begin{bmatrix} \mathrm{com}_{(0,0)}^{[2]}(\mathbf{t}_2) \\ \mathrm{com}_{(0,1)}^{[2]}(\mathbf{t}_2) \\ \mathrm{com}_{(1,0)}^{[2]}(\mathbf{t}_2) \\ \mathrm{com}_{(1,1)}^{[2]}(\mathbf{t}_2) \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ \hline 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{array} \right] = \left[\begin{array}{c|c} 1 & \\ & 1 \\ \hline & \\ & & \end{array} \begin{array}{c} \\ \\ 1 \\ -1 \end{array} \right] \left[\begin{array}{cc|cc} 1 & 1 & \\ 1 & -1 & \\ \hline & & 1 & 1 \\ & & 1 & -1 \end{array} \right] \left[\begin{array}{c|c} 1 & \\ & 1 \\ \hline & 1 \\ 1 & \end{array} \right] \left[\begin{array}{cc|cc} 1 & 1 & \\ 1 & -1 & \\ \hline & & 1 & 1 \\ & & 1 & -1 \end{array} \right] =$$

$$= \left[\mathbf{I}_{2^0} \otimes \Delta_{2^2} \cdot \mathbb{N}_{2^2} \cdot \mathbf{P}_{2^2}^{\{t_2\}} \right] \cdot \left[\mathbf{I}_{2^1} \otimes \mathbf{G}_{2^1}^{[1]} \right].$$

$$\mathbf{G}_{2^3}^{[3]} = \begin{bmatrix} \mathrm{com}_{(0,0,0)}^{[3]}(\mathbf{t}_3) \\ \mathrm{com}_{(0,0,1)}^{[3]}(\mathbf{t}_3) \\ \mathrm{com}_{(0,1,0)}^{[3]}(\mathbf{t}_3) \\ \mathrm{com}_{(0,1,1)}^{[3]}(\mathbf{t}_3) \\ \mathrm{com}_{(1,0,0)}^{[3]}(\mathbf{t}_3) \\ \mathrm{com}_{(1,0,1)}^{[3]}(\mathbf{t}_3) \\ \mathrm{com}_{(1,1,0)}^{[3]}(\mathbf{t}_3) \\ \mathrm{com}_{(1,1,1)}^{[3]}(\mathbf{t}_3) \end{bmatrix} = \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\ \hline 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 \\ \hline 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \\ \hline -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 \end{array} \right] = \left[\begin{array}{cccc|cccc} 1 & & & & & & & \\ & 1 & & & & & & \\ \hline & & 1 & & & & & \\ & & & -1 & & & & \\ \hline & & & & 1 & & & \\ & & & & & 1 & & \\ \hline & & & & & & -1 & \\ & & & & & & & 1 \end{array} \right] \cdot \left[\begin{array}{cccc|cccc} 1 & & & & 1 & & & \\ 1 & & & & -1 & & & \\ \hline & 1 & & & & 1 & & \\ & 1 & & & & -1 & & \\ \hline & & 1 & & & & 1 & \\ & & 1 & & & & -1 & \\ \hline & & & 1 & & & & 1 \\ & & & 1 & & & & -1 \end{array} \right] \times$$

$$\times \begin{bmatrix} 1 \\ & 1 \\ && 1 \\ &&& 1 \\ &&&& 1 \\ &&&&& 1 \\ &&&&&& 1 \\ &&&&&&& 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ &&&& 1 & 1 & 1 & -1 \\ &&&& 1 & 1 & -1 & 1 \\ &&&& 1 & -1 & 1 & 1 \\ &&&& -1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ & 1 \\ && 1 \\ &&& -1 \\ &&&& 1 \\ &&&&& 1 \\ &&&&&& -1 \\ &&&&&&& 1 \end{bmatrix} \times$$

$$\times \begin{bmatrix} 1 && 1 \\ 1 && -1 \\ && 1 && 1 \\ && 1 && -1 \\ &&&& 1 && 1 \\ &&&& 1 && -1 \\ &&&&&& 1 && 1 \\ &&&&&& 1 && -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ && 1 \\ & 1 \\ &&& 1 \\ &&&& 1 \\ &&&&&& 1 \\ &&&&& 1 \\ &&&&&&& 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ & 1 \\ && -1 \\ &&& 1 \\ &&&& 1 \\ &&&&& 1 \\ &&&&&& 1 \\ &&&&&&& -1 \end{bmatrix} \times$$

$$\times \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ && 1 & 1 \\ && 1 & -1 \\ &&&& 1 & 1 \\ &&&& 1 & -1 \\ &&&&&& 1 & 1 \\ &&&&&& 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ & 1 \\ &&& 1 \\ && 1 \\ &&&& 1 \\ &&&&&& 1 \\ &&&&& 1 \\ &&&&&&& 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ && 1 & 1 \\ && 1 & -1 \\ &&&& 1 & 1 \\ &&&& 1 & -1 \\ &&&&&& 1 & 1 \\ &&&&&& 1 & -1 \end{bmatrix} =$$

$$= \left[\mathbf{I}_{2^0} \otimes \Delta_{2^3} \cdot \mathbb{N}_{2^3} \cdot \mathbf{P}_{2^3}^{\{t_3\}} \right] \cdot \left[\mathbf{I}_{2^1} \otimes \Delta_{2^2} \cdot \mathbb{N}_{2^2} \cdot \mathbf{P}_{2^2}^{\{t_2\}} \right] \cdot \left[\mathbf{I}_{2^2} \otimes \mathbf{G}_{2^1}^{[1]} \right]. \qquad \square$$

Conclusion and future researches

In this paper, we have shown a new unified approach to the so-called generalized multi-parameter *m*–complementary sequences. The approach is based on a new iteration generating construction. This construction has a rich algebraic structure. It is associated not with the triple $(\mathbf{Z}_2, \mathcal{F}_2, \mathbf{C})$, but with

1) $(\mathbf{Z}_m, \mathbf{U}_m, \mathcal{A}lg)$,

2) $\left(\mathbf{Z}_m, \left\{ \mathbf{U}_m^1, \mathbf{U}_m^2, ..., \mathbf{U}_m^n \right\}, \mathcal{A}lg \right)$,

3) $\left(\mathbf{Gr}_m, \left\{ \mathbf{U}_m^1, \mathbf{U}_m^2, ..., \mathbf{U}_m^n \right\}, \mathcal{A}lg \right)$ or with

4) $\left(\left\{ \mathbf{Gr}_m^1, \mathbf{Gr}_m^2, ..., \mathbf{Gr}_m^n \right\}, \left\{ \mathbf{U}_m^1, \mathbf{U}_m^2, ..., \mathbf{U}_m^n \right\}, \mathcal{A}lg \right)$,

where $\left\{ \mathbf{U}_m^1, \mathbf{U}_m^2, ..., \mathbf{U}_m^n \right\}$ is a set of arbitrary unitary $(m \times m)$-transforms and $\left\{ \mathbf{Gr}_m^1, \mathbf{Gr}_m^2, ..., \mathbf{Gr}_m^n \right\}$ is a set of arbitrary groups of given order *m*. Furthermore, we have derived demonstrated fast algorithms for Golay transforms.

We are going to use generalized multi-parameter *m*-complementary sequences as subcarriers of Intelligent OFDM telecommunication system. Most of the data transmission systems nowadays use orthogonal frequency division multiplexing telecommunication system (OFDM-TCS) based on the discrete Fourier transform (DFT) $\mathcal{F}_N$. The conventional OFDM will be denoted by the symbol $\mathcal{F}_N$-**OFDM**. Conventional OFDM-TCS makes use of signal orthogonality of the multiple sub-carriers $e^{j2\pi kn/N}$ (complex exponential harmonics). Sub-carriers $\left\{ \mathbf{subc}_k(n) \right\}_{k=0}^{N-1} = \left\{ e^{j2\pi kn/N} \right\}_{k=0}^{N-1}$ form matrix of DFT $\mathcal{F}_N = \left[\mathbf{subc}_k(n) \right]_{k,n=0}^{N-1} \equiv \left[e^{j2\pi kn/N} \right]_{k,n=0}^{N-1}$.

At the time, the idea of using the fast algorithm of different orthogonal transforms $\mathbf{U}_N = \left[\mathbf{subc}_k(n) \right]_{k,n=0}^{N-1}$ for a software-based implementation of the OFDM's modulator and demodulator, transformed this technique from an attractive, but difficult to implement idea, into an incredibly successful story of the data transmission. OFDM-TCS, based on arbitrary orthogonal (unitary) transform $\mathbf{U}_N$ will be denoted as $\mathbf{U}_N$-**OFDM**. The idea which links $\mathcal{F}_N$-**OFDM** and $\mathbf{U}_N$-**OFDM** is that, in the same manner that the complex exponentials $\left\{ e^{j2\pi kn/N} \right\}_{k=0}^{N-1}$ are orthogonal to each-other, the members of a family of $\mathbf{U}_N$-sub-carriers $\left\{ \mathbf{subc}_k(n) \right\}_{k=0}^{N-1}$ (rows of the matrix $\mathcal{U}_N$) will satisfy the same property. The $\mathbf{U}_N$-**OFDM** reshapes the multi-carrier transmission concept, by using carriers $\left\{ \mathbf{subc}_k(n) \right\}_{k=0}^{N-1}$ in-

stead of OFDM's complex exponentials $\left\{e^{j2\pi kn/N}\right\}_{k=0}^{N-1}$. In this paper, we propose a simple and effective anti-eavesdropping and anti-jamming Intelligent OFDM system, based on MPTs. In our Intelligent-OFDM-TCS we are going to use multi-parameter Golay transform $\mathbf{G}_{2n}(\varphi_1, \varphi_2, \ldots, \varphi_q)$ at the place of DFT $\mathcal{F}_N$. We are going to study of Intell- $\mathbf{G}_{2n}(\varphi_1, \varphi_2, \ldots, \varphi_q)$-OFDM-TCS to find out optimal values of parameters optimized PARP, BER, SER, anti-eavesdropping and anti-jamming effects.

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