Periodicity of Circular Words

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We investigate some properties of circular words (or necklaces as mentioned in [6]). An ordinary word w is just a (finite or infinite) sequence of symbols (e.g. in the finite case: $w = w_1w_2 \dots w_n$). A set of symbols are called an alphabet and usually denoted by Σ. For a detailed introduction to combinatorics on words the reader can consult [3].

A circular word is obtained from a finite word w by joining it at the two extremes (i.e. the beginning and the end of w). We denote it by w_o and it can also be viewed as the set of all conjugates of w (i.e., $w_o = \{v \in \Sigma^* \mid v \text{ is a conjugate of } w\}$). It can easily be seen that if w is a primitive word, then the number of conjugates of w is $|w|$. Results about pattern avoidance of circular words can be found in [1, 2, 5]. Also [4] contains some applications to integer sequences.

For traditional (i.e., linear) words, the definition of a period is as follows. Let $v \in \Sigma^*$ be a word and $r = \frac{p}{q} \in \mathbb{Q}$ such that $0 < r < 1$ and $p = |v|$. Then $v^r = v_1v_2 \dots v_q$. A positive integer p is a *period* of w if there exist $v \in \Sigma^*$ such that $|v| = p \leq |w|$ and $w = v^r$ for some $r \in \mathbb{Q}$. We introduce new notions of *weak*- and *strong period*s of circular words and investigate their properties. The number $p > 0$ is a weak period of w_o if it is a period of some conjugate of w. Similarly, p is a strong period of w_o if it is a period of all conjugates of w_o.

A theorem of Fine and Wilf (see [3]) states that if p and q are both periods of a linear word w and $p + q - \gcd(p, q) \leq |w|$, then $\gcd(p, q)$ is also a period of w. This is not true for weak periods of circular words. Consider for example the word $(aabab)_o$ which has weak periods $2, 3, 4, 5$ and $2 + 3 - 1 \leq 5$, but $\gcd(2, 3) = 1$ is not a weak period of this word. Thus we investigate so called *paired period*s, that are in the form $(p, p + 1)$ such that p and $p + 1$ are both weak periods of w_o. Over the two letter alphabet, only words in the form $(ab)^*$ or $(ba)^*$ cannot have paired periods. The following list is a summary of some of our results:

- If w_o is a non empty circular word with strong period p, then p divides $|w|$. Corollary: If $|w|$ is a prime and there exists a strong period p of w_o such that $p < |w|$, then w is a unary word.
- Let $w \in \Sigma^*$ an arbitrary word. Then w_o has factors xx and yy with $|y| = |x| + 1$ iff $(|w| - |y|, |w| - |x|)$ is a paired period of w_o. A special case is when $xx = aa$ for some $a \in \Sigma$ and $yy = \varepsilon$ (the empty word). Then w has paired period $(|w| - 1, |w|)$.

Using our results we also state two conjectures about circular words obtained from the Thue-Morse and Fibonacci words:

- Circular words that are obtained from T_n (i.e., the nth Thue-Morse word) with $n \geq 5$ have only the paired periods that can be constructed from $\{\ell - 4, \ell - 3, \ell - 2, \ell - 1, \ell\}$, where $\ell = |T_n|$.
- Circular words that are obtained from F_n (i.e., the nth Fibonacci word) with $n \geq 4$ have only the paired periods that can be constructed from $\{k - 3, k - 2, k - 1, k\}$, where $k = |F_n|$.

References

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