A 3D Directional Shrinking Algorithm

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1. Introduction

A 3D binary picture [5] is a mapping that assigns the value of 0 or 1 to each point with integer coordinates in the 3D digital space denoted by $\mathbb{Z}^3$. Points having the value of 1 are called black points and form the objects of the picture, while 0's are called white ones and form the background, the holes, and the cavities of the picture.

The shrinking of binary pictures to similarly connected representations that have smaller foregrounds (i.e., fewer 1's) has found application as a fundamental preprocessing step in image processing [3]. Two forms of such shrinking have been emerged:

1. The picture is transformed to its topological kernel, where the shrunk picture is topologically equivalent to the original one;
2. objects (connected components) are shrunk to isolated points (i.e., single–point residues which may then be deleted).

The only 3D topology preserving shrinking algorithm has been proposed by Bertrand and Aktouf [2]. Their thinning algorithm can extract the topological kernel of an object if no end–point condition is applied. The strategy which is used for deleting 1's in parallel without altering the topology of the picture based upon subfields: the cubic grid $\mathbb{Z}^3$ is divided into 8 subfields which are successively activated in each iteration step. The parallel algorithm examines the $3 \times 3 \times 3$ neighborhood of the object points.

Two 3D shrinking algorithms belonging to the second type are known: Arcelli and Levialdi [1] proposed a parallel algorithm capable of transforming any finite object to an isolated 1 in a finite number of iteration step. Only the $2 \times 2 \times 2$ neighborhood of 1's are investigated and the object to be shrunk never leaves its circumscribing box. Hall and Küçük [4] developed the other algorithm which uses 2 subfields and examines the $3 \times 3 \times 3$ neighborhood of the object points.

In this work, a new 3D parallel shrinking algorithm is proposed for extracting the topological kernel of a binary picture. Our strategy which is used for preserving the topology is called directional or border sequential: Iteration steps are divided into 6 successive parallel subiterations, where only border 1's of a certain kind can be deleted in each subiteration. The algorithm examines the $3 \times 3 \times 3$ neighborhood of 1's and it is topology preserving for any $(26, 6)$ pictures.

2. Basic Notions and Results

Let p be a point in the 3D digital space $\mathbb{Z}^3$. Let us denote $N_j(p)$ (for $j = 6, 18, 26$) the set of points *j–adjacent* to a point p (see Fig. 8).

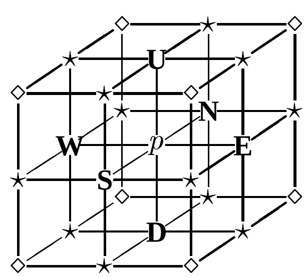

Figure 8: Frequently used adjacencies in $\mathbb{Z}^3$. The set $N_6(p)$ contains the central point p and points marked **U**, **N**, **E**, **S**, **W**, and **D**. The set of points $N_{18}(p)$ contains the set $N_6(p)$ and points marked "$\star$". The set of points $N_{26}(p)$ contains the set $N_{18}(p)$ and points marked "$\diamond$".

The sequence of distinct points $\langle x_0, x_1, \ldots, x_n \rangle$ is a *j–path* (for $j = 6, 18, 26$) of length n from point x_0 to point x_n in a non–empty set of points X if each point of the sequence is in X and x_i is j–adjacent to x_{i-1} for each $1 \leq i \leq n$. Note that a single point is a j–path of length 0. Two points are *j–connected* in the set X if there is a j–path in X between them. A set of points X is *j–connected* in the set of points $Y \supseteq X$ if any two points in X are j–connected in Y.

The *3D binary (m,n) digital picture* $\mathcal{P}$ is a quadruple $\mathcal{P} = (\mathbb{Z}^3, m, n, B)$ [5]. Each element of $\mathbb{Z}^3$ is called a *point* of $\mathcal{P}$. Each point in $B \subseteq \mathbb{Z}^3$ is called a *black point* and value 1 is assigned to it. Each point in $\mathbb{Z}^3 \setminus B$ is called a *white point* and value 0 is assigned to it. Adjacency m belongs to the black points and adjacency n belongs to the white points. A *black component* is a maximal m–connected set of points in B. A *white component* is a maximal n–connected set of points in $\mathbb{Z}^3 \setminus B$.

We are dealing with (26,6) pictures. It is assumed that any picture contains finitely many black points.

A black point p is said to be a *border point* if the set $N_6(p)$ contains at least one white point. A border point p is called a **U**–*border point* if the point marked by **U** in Fig. 8 is white. We can define **N**–, **E**–, **S**–, **W**–, and **D**–border points in the same way.

A black point is called a *simple point* if its deletion does not alter the topology of the picture. We make use the following result for (26,6) pictures:

CRITERION 1. [6]
Black point p is simple in picture $(\mathbb{Z}^3, 26, 6, B)$ if and only if all of the following three conditions hold:

1. *The set $(B \setminus \{p\}) \cap N_{26}(p)$ contains exactly one 26–component.*
2. *The set $(\mathbb{Z}^3 \setminus B) \cap N_6(p)$ is not empty and it is 6–connected in the set $(\mathbb{Z}^3 \setminus B) \cap N_{18}(p)$.*

Parallel reduction operations delete a set of black points and not only a single simple point (and each white point remains the same). We need to consider what is meant topology preservation when a number of black points are deleted simultaneously. The following sufficient conditions for parallel reduction operations of 3D (26,6) pictures are stated:

THEOREM 2. [7]
Let $\mathcal{T}$ be a parallel reduction operation. Let p be any black point in any picture $\mathcal{P} = (\mathbb{Z}^3, 26, 6, B)$ so that p is deleted by $\mathcal{T}$. Let $Q \subseteq (N_{18}(p) \setminus \{p\}) \cap B$ be any set of black points in picture $\mathcal{P}$. Operation $\mathcal{T}$ is topology preserving for (26,6) pictures if all of the following conditions hold:

1. *p is simple in the picture $(\mathbb{Z}^3, 26, 6, B \setminus Q)$.*
2. *No black component contained in a unit lattice cube can be deleted completely by operation $\mathcal{T}$.*

3. The New Shrinking Algorithm

The proposed directional 6–subiteration shrinking algorithm can be sketched by the following program:

Input: binary array X representing the picture $\mathcal{P} = (\mathbb{Z}^3, 26, 6, B)$;
Output: binary array Y representing the shrunk picture.

```
6_subiteration_shrinking_algorithm(X,Y)
  begin
    Y = X;
    repeat
      Y = deletion_from_U(Y);
      Y = deletion_from_D(Y);
      Y = deletion_from_N(Y);
      Y = deletion_from_S(Y);
      Y = deletion_from_E(Y);
      Y = deletion_from_W(Y);
    until no points are deleted;
  end.
```

Our algorithm terminates when there are no more black points to be deleted. Since all considered input pictures are finite, the shrinking algorithm will terminate.

Deletable points in a subiteration are given by a set of $3 \times 3 \times 3$ matching templates. A black point is deletable if at least one template in the set of templates matches it. Templates are described by three kinds of elements, "●" (black), "○" (white), and "." ("don't care"), where "don't care" matches either black or white point in a given picture.

The first subiteration (`deletion_from_U`) assigned to the deletion direction **U** can delete certain **U**–border points; the second subiteration associated with the deletion direction **D** attempt to delete **D**–border points, and so on.

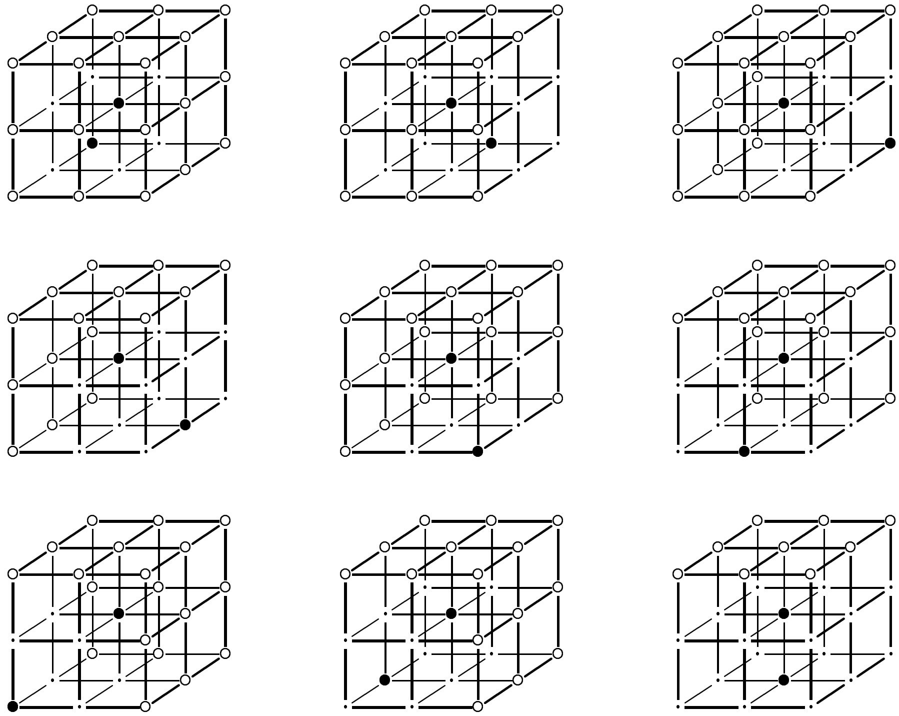

Figure 9: The set of templates assigned to the first subiteration of the proposed algorithm. These templates can delete certain **U**–border points. Notations: every position marked "•" matches a black point; every position marked "◦" matches a white point; every "·" ("don't care") matches either a black or a white point.

The set of templates presented in Fig. 9 is assigned to the first subiteration. The deletable points of the other five subiterations can be obtained by proper rotations and/or reflections of the templates in Fig. 9. Note that choosing another order of the deletion directions yields another algorithm, but it does not alter the topological correctness.

Note that the templates of our algorithm can be regarded as a "careful" characterization of simple points. This Boolean characterization makes easy implementation possible.

The topological correctness is stated by the following theorem:

THEOREM 3.
Each subiteration of the proposed 3D shrinking algorithm is a topology preserving reduction for $(26, 6)$ *pictures. (Therefore, the entire algorithm is topology preserving, too.)*

Theorem 3. can be proved easily by using Criterion 1. and Theorem 2.

The proposed algorithm is capable of transforming each simply–connected object (i.e., object without holes and cavities [5]) to an isolated point. The shrunk multiply–connected object (i.e., object containing holes or cavities) is a closed "thin" curve or a closed "thin" surface.

References

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