

# Constructing Graphs with Given Eigenvalues and Angles

Dragan Stevanović

**Introduction.** The basic problem of spectral graph theory is how to construct all graphs with given eigenvalues. This problem is very difficult and for now there is no better method than to construct all graphs with given number of vertices and select those which have the given eigenvalues. If we also consider angles, beside eigenvalues, this problem becomes tractable. There are several known algorithms in the literature for the construction of graphs of special kind given its eigenvalues and angles. Cvetková gave such algorithm for trees [1], and for tree-like cubic graphs [4].

Previously, Cvetković [2] gave the method that uses eigenvalues and angles only to construct the graph which is the supergraph of all graphs with given eigenvalues and angles. Such supergraph is the *quasi-graph* in general case. If we also know the eigenvalues and angles of graph's complement, we can construct the *fuzzy image* of a graph, which enhances upon the quasi-graph. In the case of trees, that supergraph is the *quasi-bridge graph*, whose construction is much simpler than that of quasi-graph and fuzzy image.

In [3] Cvetković gave the lower bound on distance between vertices based on eigenvalues and angles of graph. In Section we give new lower bound, similar to this one and show that two are independent of each other.

Based on the lower bound on distance and the supergraph of all graphs with given eigenvalues and angles, in Section we give the branch & bound algorithm to construct all graphs with given eigenvalues and angles.

*Notions* Let  $G$  be the graph on  $n$  vertices with adjacency matrix  $A$ . Let the vectors  $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$  constitute the standard orthonormal basis for  $R^n$ . Then  $A$  has spectral decomposition  $A = \mu_1 P_1 + \mu_2 P_2 + \dots + \mu_m P_m$ , where  $\mu_1 > \mu_2 > \dots > \mu_m$ , and  $P_i$  represents the orthogonal projection of  $R^n$  onto  $\mathcal{E}(\mu_i)$  (moreover  $P_i^2 = P_i = P_i^T$ ,  $i = 1, \dots, m$ ; and  $P_i P_j = O$ ,  $i \neq j$ ). The nonnegative quantities  $\alpha_{ij} = \cos \beta_{ij}$ , where  $\beta_{ij}$  is the angle between  $\mathcal{E}(\mu_i)$  and  $\mathbf{e}_j$ , are called *angles* of  $G$ . Since  $P_i$  represents orthogonal projection of  $R^n$  onto  $\mathcal{E}(\mu_i)$  we have  $\alpha_{ij} = \|P_i \mathbf{e}_j\|$ . The sequence  $\alpha_{ij}$  ( $j = 1, 2, \dots, n$ ) is the  *$i$ th eigenvalue angle sequence*,  $\alpha_{ij}$  ( $i = 1, 2, \dots, m$ ) is the  *$j$ th vertex angle sequence*. The *angle matrix*  $\mathcal{A}$  of  $G$  is defined to be the matrix  $\mathcal{A} = \|\alpha_{ij}\|_{m,n}$  provided its columns (i.e. the vertex angles sequences) are ordered lexicographically. The angle matrix is a graph invariant.

**Lower bounds on distances.** Let  $d(j, k)$  be the distance between vertices  $j$  and  $k$  in  $G$ . Cvetkově [3] gave the following lower bound on  $d(j, k)$ .

**Theorem 2 (Cvetković [3])** *If  $g = \min \{s : \sum_{i=1}^m |\mu_i^s| \alpha_{ij} \alpha_{ik} \geq 1\}$ , then  $d(j, k) \geq g$ .*

The following theorem exploits the similar idea.

**Theorem 3** *If  $g = \min \{s : \sum_{i=1}^m |\mu_i^{s+2}| \alpha_{ij} \alpha_{ik} \geq d_j + d_k + \delta_{s-1} - s\}$ , where  $\delta_{s-1}$  is the sum of  $s-1$  smallest degrees of vertices other than  $j$  and  $k$ , then  $d(j, k) \geq g$ .*

**Example.** For the tree shown in Fig. 2 from Theorem 2 we have  $d(u, v) \geq 2$ , while from Theorem 3 follows  $d(u, v) \geq 3$ . On the other hand, from Theorem 2 follows  $d(w, v) \geq 3$ , while Theorem 3 gives only that  $d(w, v) \geq 1$ . This shows that lower bounds given in these lemmas are independent of each other. In order to get better lower bound, one must then take the greater of the values given by these theorems.  $\square$

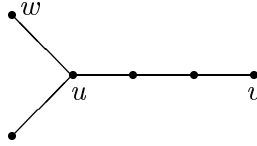


Figure 2: Graph from the Example of Section 2.

**Quasi-bridge graphs.** Cvetković in [2] gave the necessary condition for two vertices  $u$  and  $v$  to be joined by a bridge, and called it the *bridge condition*. He defined the *quasi-bridge graph*  $QB(G)$  of the graph  $G$  as the graph with the same vertices as  $G$ , with two vertices adjacent if and only if they fulfill the bridge condition. If  $G$  is a tree, then we obviously have that  $G$  is a spanning tree of  $QB(G)$ .

The bridge condition is not sufficient for the existence of the bridge. On the other hand, there are trees for which the equality  $QB(G) = G$  holds. Such examples are the stars  $S_n$  and the double stars  $DS_{m,n}$  with  $m \neq n$ . From the statistical testing of random trees, the number of quasi-bridges of trees appears to be linearly bounded in the number of vertices. We believe that this fact holds for all trees.

**Conjecture 4** *If  $e(G)$  denotes the number of edges of graph  $G$ , then for trees hold  $e(QB(G)) = O(e(G))$ .*

**Quasi-graphs and fuzzy images.** Cvetković in [2] also gave the necessary condition for two vertices  $u$  and  $v$  to be adjacent, and called it the *edge condition*. He defined the *quasi-graph*  $Q(G)$  of the graph  $G$  as the graph with the same vertices as  $G$ , with two vertices adjacent if and only if they fulfill the edge condition. Obviously, any graph is spanning subgraph of its quasi-graph.

Since the edge condition in  $\overline{G}$  is a necessary condition for non-adjacency in  $G$ , it follows that any two distinct vertices of  $G$  are adjacent either in  $Q(G)$  or in  $Q(\overline{G})$ . If they are adjacent in one and not adjacent in the other, then their status coincides with that in  $Q(G)$ . Cvetković [2] defined the *fuzzy image*  $FI(G)$  as the graph with the same vertex set as  $G$  and two kinds of edges, solid and fuzzy. Vertices  $u$  and  $v$  of  $FI(G)$  are

- 1) non-adjacent if they are non-adjacent in  $Q(G)$  and adjacent in  $Q(\overline{G})$ ;
- 2) joined by a solid edge if they are adjacent in  $Q(G)$  and non-adjacent in  $Q(\overline{G})$ ;
- 3) joined by a fuzzy edge if they are adjacent in both  $Q(G)$  and  $Q(\overline{G})$ .

For the construction of  $FI(G)$  one must know the eigenvalues and angles of  $\overline{G}$ . If  $G$  is regular and both  $G$  and  $\overline{G}$  are connected then this information is already known from the eigenvalues and angles of  $G$  [5].

Except for small values of  $n$  (up to 4), it is very difficult to use the edge condition practically, so that the problem of its implementation still remains. Since the coefficients of the characteristic polynomials are integers, we can use the following much weaker theorem.

**Theorem 5** *Let  $G$  be a graph with  $n$  vertices, and let  $wv$  be an edge of  $G$ . Then  $P_{G-u}(n)P_{G-v}(n)$  is quadratic residue modulus  $P_G(n)$  for every  $n \in \mathbb{Z}$ .*

**The Constructing Algorithm.** The proposed algorithm is of branch & bound type. Before the algorithm enters the main loop, it determines the graph  $G^*$  that is the supergraph of all graphs with given eigenvalues and angles. Let  $G$  denote the fictitious graph with given eigenvalues and angles, with vertex set  $V(G)$  and edge set  $E(G)$ . In the case of trees we have  $G^* = QB(G)$ , in the case of regular connected graphs with connected complements  $G^* = FI(G)$ , while in other cases  $G^* = Q(G)$ . Then for each pair  $(u, v)$  of vertices, algorithm determines the lower bound  $d(u, v)$  on distance in  $G$  between vertices  $u$  and  $v$ , based on Lemmas 2 and 3.

At level 0 of the algorithm we arbitrarily choose the vertex  $u_1$ , and pass to the level 1, where it enters the main loop. When we come at level  $i$  ( $i \geq 1$ ) we have already constructed the subgraph  $G$  induced by vertices  $v_1, v_2, \dots, v_i$ . This fact gives us the possibility to use the Interlacing theorem to check if we are on the good way with construction. Then we choose the vertex  $u_{i+1}$  from the set of remaining vertices, select its neighbors from the set  $\{v_1, v_2, \dots, v_i\}$ , and pass to the next level, until  $G$  is constructed in whole. The vertex  $u_{i+1}$  is chosen such that the total number of neighborhoods of  $u_{i+1}$  that must be tried out is minimized.

## References

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