Analysis of permutation routing algorithms[2]

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In this paper we analyze some permutation routing algorithms for different kind of mesh architectures. We give lower bounds for the number of steps of arbitrary on–line or off–line algorithms on rectangular meshes with buses . Finally we give lower and upper bounds for the expected number of steps of the basic greedy algorithm on linear array without bus.

One possible alternative of the traditional single processor computer is the multiprocessor parallel computer. The effectiveness of such a computer highly depends on how fast the necessary data can be sent to the appropriate location. The architecture or an efficient routing algorithm can significantly decrease the developing costs of such a computer. Therefore researchers turned their attention to either the hardware architecture or the analysis of the different algorithms.

During the analysis different architectures used to be considered. One of the simplest architecture is the linear array. In this case each processor is connected with the two neighboring processors.

In some cases using buses is practical for data communication to increase the efficiency of the computer. The bus is connected to all processors, but only one processor may use the bus at a given step.

In this paper we deal with the permutation routing problem: each processor should send a message to another one. We deal with the classical variant, where each processor can receive at most one message. Suppose that the neighboring processors are connected with a full–duplex line, which enables them to exchange a message chosen from the message queue at each step. We say that the problem is solved, if each message arrived at its destination. In the simplest cases we assume that the processors have enough memory to store the waiting messages. The efficiency of the algorithm is measured by the number of steps required to solve the problem. In each steps only the neighboring processors can send messages to each other, or exactly one processor can use the buses to which it is connected.

The efficiency of the algorithms can be measured two ways: we can investigate the number of steps in the worst–case or we can analyse the expected number of steps for solving a problem. Formally we can give the above mentioned efficiency by the following: let $\pi = (i_1, i_2, \dots, i_n)$ an example, in which the processor j send a message to the processor i_j ($j = 1, 2, \dots, n$). Denote Π_n the set of all permutations of n. If $S_A(n, \pi)$ denotes the number of steps of algorithm A for π in case of n processors, then the algorithm requires

$$W_A(n) = \max_{\pi \in \Pi_n} S_A(n, \pi)$$

steps in the worst–case.

Lower bound for two–dimensional bused meshes. Cheung and Lau proved that for an $n \times n$ bused mesh each algorithm A requires at least $0.691n$ steps. In this paper we investigate, whether this bound can be improved for a $m \times n$ rectangular mesh.

Theorem 0.1 *For each algorithm A on an $m \times n$ $(m \geq n)$ rectangular mesh with row and column buses*

$$W_A(m, n) \geq \max\left(\frac{2}{3}m, \frac{7n + 3m}{8} - \frac{\sqrt{(3m - n)^2 + 16n^2}}{8}\right)$$

It is easy to see that if $m = n$ then the above bound gives the Cheung és Lau [2] result for square meshes.

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Average case analysis for one dimensional routing problem. Consider now the one dimensional problem without bus. Denote n the number of processors, and let A be the basic greedy algorithm, which solves the problem such a way, that each processor sends its message to its neighbor in the direction of the destination processor. If the message arrives at its destination, then the processor stores the data, but it still remains able to receive and send further messages.

Then the number of steps required to solve the problem equals to the maximal distance of the messages from their destination. Formally we can say that if $\Pi = \pi(1), \pi(2), ..., \pi(n)$ defines the problem, then the number of steps required by algorithm A is

$$L_A^{\Pi}(n) = \max_{i=1,...,n} \{|i - \pi(i)|\}.$$

In the following we would like to determine the average–case behaviour of algorithm A. To do this we suppose that the algorithm gets each permutation of the elements $1, ..., n$ with equal probability. Denote $S_j(n)$ the number of permutations of elements $1, ..., n$ for which

$$\max_{i=1,...,n} \{|i - \pi(i)|\} = j.$$

Then the expected number of steps for A is

$$E_A(n) = \frac{\sum_{j=1}^{n-1} j \cdot S_j(n)}{n!}.$$

Because the values of $S_j(n)$ are not known exactly for each j, or they can be given by only a very complicated formula (see [8]), so our aim is to give a useful lower bound for this expected value. In the following we give the theorem for the expected number of steps of algorithm A.

Theorem 0.2 $E_A(n) \geq n + 2 - 2\sqrt{n+1}$.

From the above theorem the following surprising fact follows:

$$\lim_{n \to \infty} \frac{E_A(n)}{n} = 1,$$

i.e. the asymptotic average–case behaviour of algorithm A is not better than its asymptotic worst–case behaviour.

References

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