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A Random Walk in a Random Environment

Edwin Andrew Sanchez

A RANDOM WALK IN A RANDOM ENVIRONMENT - SANCHEZ

THE UNIVERSITY OF NEW MEXICO
ALBUQUERQUE, NEW MEXICO 87106

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NAME AND ADDRESS	DATE

Edwin Andrew Sanchez
Candidate

Mathematics and Statistics
Department

This dissertation is approved, and it is acceptable in quality and form for publication on microfilm:

Approved by the Dissertation Committee:

Robert F. Coghlan, Chairperson

Reuben Hersh

Richard J. Griego

Accepted:

Bernard Spolsky
Dean, Graduate School

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Date

A RANDOM WALK IN A RANDOM ENVIRONMENT

BY

EDWIN ANDREW SANCHEZ

B.S., NEW MEXICO HIGHLANDS UNIVERSITY, 1971

M.S., NEW MEXICO HIGHLANDS UNIVERSITY, 1972

DISSERTATION

Submitted in Partial Fulfillment of the
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ABSTRACT OF DISSERTATION

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Edwin Andrew Sanchez, Ph.D.
Department of Mathematics and Statistics
The University of New Mexico, 1978

In this dissertation we consider a model of a random walk, $\{Z_n\}$, on $R^{(1)}$ where the distribution of $\{Z_n\}$ is dependent on a stochastic process $\{Y_n\}$ and is given by $g_y(z)$ where $Y_{n-1} = y$. Conditions on the environmental process $\{Y_n\}$ are given for which transience or recurrence of the random walk $\{Z_n\}$ can be determined. The probability of absorption and mean time problems are solved when $\{Y_n\}$ is a finite Markov chain and $\{Z_n\}$ is a classical random walk on the integer lattice.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

The theory of Random Walks has been the subject of numerous articles and books, beginning with Levy's Theorie de l'addition des variables aleatoires (1937) and continuing through Spitzer's Principles of Random Walk. In all of these works it is assumed that as the process evolves the transition probabilities are fixed and that, for each realization of the process, these transition probabilities are identical. Then Solomon in Random Walks in a Random Environment (1975), allowed the random walk, at the beginning of each realization, to choose according to some stochastic process, the transition probabilities that would govern the evolution of the random walk for that particular realization. Once again, though, his model required that the transition probabilities remain fixed for that realization of the process.

In this paper we will formulate a model for a Random Walk in a Random Environment (RWRE) in which the transition probabilities are chosen randomly at each step in time, as the walk evolves. This model follows in the vein of the many recent papers in the areas of random evolutions and random environments, such as the work done by Griego and Hersh [8] on Random Evolutions, W. Smith and Wilkinson [16] on Branching Processes in Random Environments (BPRE), Athreya and Karlin [1] on BPRE's, W. Smith [15] on BPRE's and Torrez [20] on Birth and Death Process in a Random Environment (BDRE). We will first formulate our model

for RWRE's, then we will show how it is similar to the above models, and how Solomon's model differs from all the other models for random evolutions.

The model we will study is then as follows: The RWRE will be a two component stochastic sequence (Y_n, Z_n) where the $\{Y_n\}$ component will be the environmental or control process and the $\{Z_n\}$ component will be the random walk. It should be noted that, although $\{Z_n\}$, as we will define it, will not be a random walk in the classical sense, we will still refer to it as a random walk. We will then, as in the case of classical random walks, be interested in answering the following questions about the random walk $\{Z_n\}$: Is it recurrent or transient; Can we solve the probability of absorption problem, and the problem of the expected duration of the walk, and if so, what are the forms of the solutions?

To answer these questions we will first have to specify the structure of the control process $\{Y_n\}$ and then specify how the random walk $\{Z_n\}$ evolves given that we are in a particular environment y. The control process $\{Y_n\}$ will be defined on (Ω, F). In its most general setting $\{Y_n\}$ will be a stationary ergodic process. To be able to derive more succinct results concerning the $\{Z_n\}$ process, we will further restrict the stochastic nature of $\{Y_n\}$, requiring that it be a positive φ-recurrent Markov chain, or as in the case of chapter three, that it be a finite recurrent Markov chain.

The manner in which the evolution of the random walk $\{Z_n\}$ is controlled by the environmental process $\{Y_n\}$ is as follows: To each state $y \in \Omega$ we associate with it the transition probabilities $g_y(B)$,

where g_y is a probability distribution on $R^{(k)}$, the space on which $\{Z_n\}$ evolves, and

$$g_y(B) = P(Z_{n+1} = B + z \mid (Y_0,Z_0),(Y_1,Z_1),\ldots,(Y_n,Z_n) = (y,z))$$

$$\text{for every } y \in \Omega,\ z \in R^{(k)},\ \text{ and } n = 0,1,2,\ldots \text{ and} \tag{1}$$

$$\int_{R^{(k)}} g_y(z)dz = 1 \ \forall\ y \in \Omega .$$

So by the above we see that within a particular environment the transition probabilities are homogeneous. In the special case that $\{Z_n\}$ is on $R^{(1)}$ and is allowed to only move to neighboring integer valued states, we have that

$$\begin{aligned} p_y &= P(Z_{n+1} = k + 1 \mid (Y_0,Z_0),(Y_1,Z_1),\ldots,(Y_n,Z_n) = (y,k)) \\ q_y &= P(Z_{n+1} = k + 1 \mid (Y_0,Z_0),(Y_1,Z_1),\ldots,(Y_n,Z_n) = (y,k)) \end{aligned} \tag{2}$$

with

$$p_y + q_y = 1 \ \forall\ y \in \Omega \quad \text{and} \quad \begin{cases} p_y \neq 0 \\ q_y \neq 0 \end{cases} \text{for any } y \in \Omega .$$

Note that in this model, if $\{Y_n\}$ is Markovian, then (Y_n,Z_n) is a Markov chain. Even in this case, however, the Z component is not Markovian in general.

The above model differs from that of Torrez, in that in his model the process $\{Z_n\}$ was a Birth and Death Chain. We will consider generalized random walks, as in (1) above, which generalize the Birth and Death assumptions, but in all over models the law of evolution of the Z component will be spatially homogeneous. Smith & Wilkinson and Athreya & Karlin considered the case where $\{Z_n\}$ was a Branching Process.

Smith and Wilkinson then made $\{Y_n\}$ an i.i.d. sequence [16] and later assumed $\{Y_n\}$ was a Markov Chain [17], whereas Athreya and Karlin only required that $\{Y_n\}$ be a stationary ergodic process [1]. In their work on BPRE's the control of $\{Y_n\}$ on $\{Z_n\}$ can be expressed in terms of probability generating functions and the analysis then depends on special properties of generating functions. While we will deal with the same types of questions, we will require different techniques, especially those based on spatial homogenity combined with renewal theory.

Solomon's model differs from the above models in one major aspect; all of the above models can be described by a bivariate process (Y_n, Z_n) where $\{Y_n\}$ is a non-trivial sequence of random variables, whereas in Solomon's model the process $\{Y_n\}$ is such that $Y_n = Y_1$, $n = 1,2,3,..$ Solomon then allows Y_1 to determine a random perturbation of the homogeneous environment of a classical random walk, so the probability of a step to the right from state n becomes $P_{n,y}$ in the y^{th} environment. Thus these environments are not spatially homogeneous in general. The resulting Z process does not fit into our context of RWRE's nor into Torrez's BDRE's. However, it is because of the similarities between our model and the models of Smith and Wilkinson, Athreya and Karlin, and Torrez, that we will refer to our model as a Random Walk in a Random Environment, despite the fact that Solomon used this terminology to describe his model, which differs as discussed above from the previous works in Random Environments.

In ~~this~~ chapter II we will discuss the problem of recurrence and transience of $\{Z_n\}$ given that $\{Y_n\}$ satisfies certain conditions.

In the case where $\{Y_n\}$ is a stationary ergodic sequence, we say that $\{Z_n\}$ is recurrent if, for every neighborhood of the origin $A \subset R^k$ and for every $z \in R^{(k)}$, we have that $P_z(Z_n \in A \text{ i.o.}) = 1$, and that it is transient if, for every bounded $A \subset R^{(k)}$ and for every $z \in R^{(k)}$, we have that $P_z(Z_n \in A \text{ i.o.}) = 0$, where P_z is the probability measure generated by starting the Z process at z, and having Y evolve according to its stationary ergodic law.

In the case where $\{Y_n\}$ is a φ-recurrent Markov Chain, we will say that $\{Z_n\}$ is recurrent if, for every neighborhood of the origin $A \subset R^{(k)}$ and for every $z \in R^{(k)}$ and for every $y \in \Omega$, we have that $P_{y,z}(Z_n \in A \text{ i.o.}) = 1$ and that it is transient if, for every bounded $A \subset R^{(k)}$ and for every $z \in R^{(k)}$ and for every $y \in \Omega$, $P_{y,z}(Z_n \in A \text{ i.o.}) = 0$. Where $P_{y,z}$ is the probability measure generated by letting the (Y,Z) process start at (y,z). It should be noted that, when $\{Y_n\}$ is a positive φ-recurrent Markov Chain with invariant probability distribution π, we have that $P_{\pi,z}(Z_n \in A \text{ i.o.}) = 1$ where A is as above if and only if $P_{y,z}(Z_n \in A \text{ i.o.}) = 1$ for every $y \in \Omega$ and that $P_{\pi,z}(Z_n \in A \text{ i.o.}) = 0$ if and only if $P_{y,z}(Z_n \in A \text{ i.o.}) = 0$ for every $y \in \Omega$. In the classical case a random walk is always either transient or recurrent; in our model $\{Z_n\}$ may be neither. The example below will illustrate the difficulty in determining recurrence properties for the RWRE $\{Z_n\}$.

Example 1. Let $\{Z_n\}$ be a symmetric simple random walk on the integers and let $\{Y_n\}$ be a symmetric simple random walk on the two dimensional integer lattice. Both of the above are recurrent random walks. Then let

$\{Z_n\}$ be independent of $\{Y_n\}$: in this case the control process in fact has no effect on the random walk. Then (Y_n, Z_n) is a symmetric three dimensional random walk and, therefore, it is not recurrent. Thus recurrence of $\{Z_n\}$ cannot be identified with recurrence of the chain (Y_n, Z_n) even when $\{Y_n\}$ is Markovian. In chapter two we will develop conditions on the control process under which the recurrence or transience of $\{Z_n\}$ can be determined.

In chapter three we will concern ourselves with the question of the probability of absorption and the mean time to absorption when $\{Y_n\}$ is a Markov Chain on an irreducible, finite state space. We will solve the problem when we have both a left and a right barrier. In the case where $\{Y_n\}$ only has two states and we only have a left barrier, we will compare the probability of absorption for the RWRE to the classical random walk where $p = p_1\pi_1 + p_2\pi_2$, and $\pi = (\pi_1, \pi_2)$ is the invariant starting distribution for Y .

In chapter four we will consider certain examples and examine the behavior of the solutions to the absorption problem when we make small perturbations in the control process or in the transition probabilities.

$[Z_n]$ [illegible]

from the [illegible]

three di[illegible]

resour[illegible]

$[X_n Z_n]$ [illegible]

condition [illegible]

ranges of [illegible]

It i[illegible]

probabilit[illegible]

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problem wi[illegible]

$[T_n]$ only [illegible]

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is shown [illegible]

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determinatio[illegible]

CHAPTER II

RECURRENCE AND TRANSIENCE

In this chapter we will give conditions on the control process $\{Y_n\}$ under which we can state conditions which will determine recurrence and transience for the random walk process $\{Z_n\}$. In some cases it will be possible to show the dichotomy that $\{Z_n\}$ is either transient or recurrent. When $\{Y_n\}$ is a Markov Chain (Ω,F) will denote the state space of $\{Y_n\}$ and as before we will let π denote its stationary distribution. (Throughout this chapter $\{Y_n\}$ will be such that π, the invariant measure, does exist and is finite, hence P_π exists.) When $\{Y_n\}$ is ergodic, we let π denote the distribution of Y_0.

We will first show that, when $\{Y_n\}$ is a stationary ergodic process, $\{Z_n\}$ is defined on $\mathbb{R}$ and satisfies certain integrability conditions, then $E[Z_1] \neq 0$ implies that $\{Z_n\}$ is transient and Z_n converges to $\operatorname{sgn}[E(Z_1)] \cdot \infty$ a.s. P_π. In the second section we will show that, when $\{Y_n\}$ is a positive φ-recurrent Markov Chain which has a C-set, $\{Z_n\}$ is defined on $\mathbb{R}$ and its evolution again satisfies certain integrability conditions, we then have

1) $E[Z_1] = 0 \iff \{Z\}$ is recurrent

2) $E[Z_1] \neq 0 \iff \{Z_n\}$ is transient

and then $Z_n \to \operatorname{sgn}[E(Z_1)] \cdot \infty$ a.s.

Section 2.1.

In this section $\{Y_n\}$ is a stationary ergodic sequence on (Ω,F,P) with shift operator T. From the discussion on page 5 in Chapter 1, it follows that $W_n = Z_{n+1} - Z_n$ is a stationary sequence of random variables. By proposition 6.32 (Brieman), if C is an invariant event for $\{W_n\}$, then C is also a tail event. We then consider the following measurable function of $\{Y_n\}$: $P(C|\{Y_n\})$. For a fixed sequence $\{Y_n\}_0$ the W_n's are an independent sequence.

We then know by the Kolmogorov Zero-One Law that

$$P(C|\{Y\}) = 0 \text{ or } 1 . \tag{1}$$

Note that for some sequence $\{Y_n\}_1$ we could have that $P(C|\{Y_n\}_1)$ equals 0 while for another sequence $\{Y_n\}_2$ we may have that $P(C|\{Y_n\}_2)$ equals 1. So while $P(C|\{Y_n\})$ may only assume one of two values, it is not clear that it is necessarily a constant function. We would like to show that in fact $P(C|\{Y_n\})$ is a constant function, but we will first prove that it is an invariant function of $\{Y_n\}$.

To accomplish this we will use proposition 6.5 (Brieman) to extend $\{Y_n,W_n\}$ to be a double-ended stationary sequence. We will then have that for $\mathcal{B} = \tilde{f}(Y_0,Y_{\pm 1},Y_{\pm 2},Y_{\pm 3},\ldots)$ and $C \in \tilde{f}(W_0,W_{\pm 1},W_{\pm 2},W_{\pm 3},\ldots)$ where C is invariant, that

$$TC = C \text{ and } T\mathcal{B} = \mathcal{B} . \tag{2}$$

By the definition of conditional probability we know that $P(C|\mathcal{B})$ is a random variable which satisfies:

$$P(AC) = \int_A P(C|\mathcal{B})dP \quad \forall \quad A \in \mathcal{B} .$$

By the stationarity of $\{Y_n\}$ we have that

$$\int_A P(C|\mathcal{B})dP = \int_{TA} TP(C|\mathcal{B})dP$$

and

$$P(AC) = P(TA \cap TC) .$$

If we then let

$$A' = TA$$

we have that

$$P(A' \cap TC) = \int_{A'} TP(C|\mathcal{B})dP \quad \forall \quad A' \in T\mathcal{B} .$$

Hence by Radon-Nikodyn we have that

$$TP(C|\mathcal{B}) = P(TC|T\mathcal{B}) \quad \text{a.s. } P . \tag{3}$$

So from (2) and (3) we get that

$$TP(C|\mathcal{B}) = P(TC|T\mathcal{B}) = P(C|\mathcal{B})$$

Which implies that $P(C|\{Y_n\})$ is an invariant function of $\{Y_n\}$, hence it is a.s. constant.

Since $P(C|\{Y_n\}$ is an a.s. constant function of $\{Y_n\}$, that would imply that

$$P(C|\{Y_n\}) = P(C) \quad \text{a.s. } P$$

and in fact $P(C) = 0$ or 1 . Then using definition 6.30 (Brieman) we know, since every invariant set C has probability zero or one, that $\{W_n\}$ must be ergodic. The above argument, together with the ergodic

theorem, proves the following theorem.

Theorem 2.1. If $\{Y_n\}$ is a stationary ergodic process and if $\{Z_n\}$ is defined on $\mathbb{R}$, then $W_n = Z_{n+1} - Z_n$ is a stationary ergodic process and, if $E(W_1)$ exists, then $\frac{Z_n}{n} \to E[W_1]$ a.s.

According to the model specified in Chapter one, we have, given that we are at a point $Y_1 = y$ in the control process, that the distribution of w_1 is given by a distribution function $g_y(z)$. Hence

$$E[W_1] = \int_\Omega \int_{-\infty}^{\infty} x \, g_y(z)dz \, \pi(dy) .$$

We will refer to the next two conditions as conditions * and throughout the rest of the Chapter we will assume that conditions * are satisfied, unless specified otherwise.

1) $f(y) = \int_{-\infty}^{\infty} x \, g_y(z)dz$ exists for π-a.e. y

*

2) $\int f(y) \, \pi(dy)$ exists .

The above two conditions insure that $E[w_1]$ does exist. We will use the notation $E[w_1] = \pi f$. We then have the following corollary

Corollary 2.2. If the conditions of Theorem 2.1 are in effect and if condition * also holds and if in addition $\pi f \neq 0$, then $\{Z_n\}$ is transient and $Z_n \to \text{sgn}(\pi f) \cdot \infty$ a.s.

It should be noted at this point that as in the case of classical random walks we have shown that $\pi f \neq 0$ implies transience. Then the question that arises is, if $\pi f = 0$, does that imply Z_n

is recurrent? In the situation we have above the answer is not known. From the corollary it is clear that if we have recurrence then $\pi f = 0$.

Section 2.2.

In this section we will give conditions on the control process which will insure that we do get a dichotomy for recurrence and transience. To do this we will need to treat the two cases separately. Since conditions for transience will follow immediately from the previous section, we will deal with it first. Then we will look at the conditions for recurrence in two parts. First when $\{Y_n\}$ has a positive atom, and second, assuming only that $\{Y_n\}$ has a C-set, using a result of Nummelin [12].

The first result for transience is an immediate consequence of Theorem 2.1 and is given below.

Theorem 2.3. If $\{Y_n\}$ is a positive φ-recurrent Markov Chain on an arbitrary state space with stationary initial distribution π . If $g_y(z)$ satisfies conditions * then $\{W_n\}$ is a stationary ergodic process and $\frac{Z_n}{n} \to \pi f$ a.s. P_μ for every initial distribution μ on the Y chain.

Proof. The φ-recurrence of $\{Y_n\}$ implies that Ω is indecomposable. The positive φ-recurrence implies that π exists and is finite and unique. Then by Theorem 7.16 (Brieman) we know that $\{Y_n\}$ started with the distribution π is a stationary ergodic process. Hence Theorem 2.1 then gives us the above result for P_π. The result for P_μ follows

from Proposition 4.3 and Theorem 5.1 of Orey [13]. This result immediately gives us the following corollary.

Corollary 2.4. If $\{Y_n\}$ is a positive φ-recurrent Markov Chain on an arbitrary state space and if $g_y(z)$ satisfies conditions * and if $\pi f = E[W_1] \neq 0$ then Z_n is transient and $Z_n \to (\text{sgn } \pi f) \cdot \infty$ a.s. P_μ for every initial distribution μ on the Y chain.

We will now consider the problem: Under what conditions does $E[W_1] = 0$ imply that Z_n is recurrent. To do this we will first assume that $\{Y_n\}$ has a positive atom. Then we will look at the process Z_{τ_n} where τ_n is the n^{th} return time to the atom of the control process.

Recall that for A to be a positive atom of $\{Y_n\}$, it requires that $\pi(A) > 0$ and $P(y_0,x) = P(y_1,x)$ for every y_0 and y_1 elements of A. Note that in the discrete case a point with positive mass is an atom. We will now prove the preliminary results concerning recurrence. First we will consider the following.

Define τ_n to be the n^{th} return time to a positive atom A. If $\{Y_n\}$ is a positive φ-recurrent Markov Chain, we know that $E[\tau_1] < \infty$. We then prove the following lemma.

Lemma 2.4. If $\{Y_n\}$ is a positive φ-recurrent Markov Chain with a positive atom A and $h: \Omega \to \mathbb{R}$ with h being π-integrable then

$$\pi(A)E[V_1] = \pi h \quad \text{where} \quad V_1 = \sum_{k=1}^{\tau_1} h(Y_k) .$$

Proof. From the proof of Theorem 2.3 we know that $\{Y_n\}$ is a stationary ergodic sequence, hence $\{h(Y_n)\}$ is an ergodic sequence, and

$$\frac{1}{n}\sum_{k=1}^{\tau_n} h(Y_k) \to \pi h \quad \text{a.s.} \quad P_\pi \ . \tag{4}$$

Since $\tau_n \to \infty$ and $\tau_n < \infty$ a.s. P_π, (4) implies

$$\frac{1}{\tau_n}\sum_{k=1}^{\tau_n} h(Y_k) \to \pi h \quad \text{a.s.} \quad P_\pi \tag{5}$$

and

$$\frac{1}{\tau_n}\sum_{k=1}^{\tau_1} h(Y_k) \to 0 \quad \text{a.s.} \quad P_\pi \ . \tag{6}$$

Putting (5) and (6) together we see that

$$\frac{1}{\tau_n}\sum_{k=\tau_1+1}^{\tau_n} h(Y_k) \to \pi f \quad \text{a.s.} \quad P_\pi \ .$$

If we then set

$$V_k = \sum_{\tau_k+1}^{\tau_{k+1}} h(Y_k)$$

we see that the V_k's are an i.i.d. sequence of random variables. Then by the strong law of large numbers

$$\frac{1}{n}\sum_{1}^{n} V_k \to E(V_1) \quad \text{a.s.} \quad P_\pi$$

and

$$\frac{\tau_n}{n} \to E[\tau_1] \quad \text{a.s.} \quad P_\pi$$

with

$$E[\tau_1] = \frac{1}{\pi(A)}$$

$$\Rightarrow \frac{n}{\tau_n} \to \pi(A) \quad \text{a.s.} \quad P_\pi \ .$$

So we have that

$$\frac{1}{\tau_n} \sum_{k=\tau_1+1}^{\tau_n} h(Y_k) = \frac{n}{\tau_n} \cdot \frac{n-1}{n} \cdot \frac{1}{n-1} \sum_{k=1}^{n-1} V_k .$$

Hence

$$\frac{1}{\tau_n} \sum_{k=\tau_1+1}^{\tau_n} h(Y_k) \to \pi(A) E[V_1] . \tag{7}$$

Equations (6) and (7) then give us the desired result

$$\pi(A)\ E[V_1] = \pi h .$$

So we then have the following lemma.

Lemma 2.5. If $\{Y_n\}$ is as in Lemma 2.4 and if $g_y(z)$ satisfies conditions * and if $E[W_1] = 0$, then $E[Z_{\tau_1}] = 0$.

Proof. Let $h(y) = f(y) = \int_{-\infty}^{\infty} z\, g_y(z) dz$, hence $\pi h = \pi f = E[W_1] = 0$ (where f is the function defined by *), and $V_1 = \sum_{k=1}^{\tau_1} h(Y_k)$.

Now

$$E[Z_{\tau_1}] = E[\sum_{k=1}^{\tau_1} W_k]$$

$$= E[\sum_{k=1}^{\tau_1} E\{W_k | Y_k\}]$$

$$= E[\sum_{k=1}^{n} h(Y_k)] = E[V_1]$$

but

$$E[V_1] = \frac{\pi h}{\pi(A)} = 0$$

$$\Rightarrow E[Z_{\tau_1}] = 0 .$$

We then prove the following theorem.

Theorem 2.6. If $\{Y_n\}$ is a positive φ-recurrent Markov Chain with a positive atom A and if $\{Z_n\}$ is defined on $\mathbb{R}$ and $g_y(z)$ satisfies conditions * and if $E[W_1] = 0$ then Z_n is recurrent.

Proof. Let τ_n be the time of the n^{th} return to A , the positive atom. Then by Lemma 2.5 we get that $E[Z_{\tau_1}] = 0$. We then define

$$X_k = Z_{\tau_k} - Z_{\tau_{k-1}}$$

which is a sequence of independent identically distributed random variables. Note that

$$E[X_1] = E[Z_{\tau_k} - Z_{\tau_{k-1}}] = 0 .$$

If we then apply Theorem 3.38 (Brieman), we have that

$$Z_{\tau_n} = \sum_{k=1}^{n} (Z_{\tau_k} - Z_{\tau_{k-1}}) = \sum_{k=1}^{n} X_k$$

is recurrent. This implies that $\{Z_n\}$ is recurrent, i.e. if

$$P[Z_{\tau_n} \in \mathcal{B} \text{ i.o.}] = 1$$

then

$$P[Z_n \in \mathcal{B} \text{ i.o.}] = 1 .$$

Now if $\{Y_n\}$ does not have a positive atom then $\{X_k\}$ as defined in the proof is not an i.i.d. sequence of random variables. So we cannot apply Theorem 3.38 (Brieman). We will then use Nummelin's method to extend the original process to a new one which does contain a positive atom, and then use the proof as above.

Take $\{Y_n\}$ to be a positive φ-recurrent Markov Chain, which has a C-set where we define a C-set as in Orey [13].

<u>Definition</u>. Relative to the σ-finite measure φ on (Ω,F) a set $C \in F$ is called a C-set if $\varphi(C) > 0$ and there exists a positive integer k such that

$$\inf_{(x,y) \in C \times C} p^k(x,y) > 0$$

where $p^k(x,y)$ is the k-step transition probability density, which is the Radon-Nikodym derivative of the φ-absolutely continuous part of

$$P^k(x,\cdot) .$$

We will assume for now that k equals 1 . Then using Nummelin's method we form a new chain $\{Y_n^*\}$ on $\Omega \times \{0,1\}$ which has a positive atom and such that the marginal of $\{Y_n^*\}$ on Ω is the chain $\{Y_n\}$. (cf. Theorem 2.3 of Newmelin [12]) We define the random walk associated with $(y,i) \in \Omega \times \{0,1\}$ by

$$P^*[Z_{n+1} = B+z \mid (Y_0^*,Z_0),(Y_1^*,Z_1),\ldots,(Y_n^*,Z_n) = (y,i,z)] = g_y(B)$$

for every $y \in \Omega$, $z \in R^{(k)}$, and $B \subset R^{(k)}$, where $g_{y_0}(z)$ was given by the original process and satisfied

$$g_{y_0}(B) = P[Z_{n+1} = B+z \mid (Y_0,Z_0),(Y_1,Z_1),\ldots,(Y_n,Z_n) = (y_0,z)] .$$

Thus the Z component of the RWRE $[Y_n^*,Z_n]$ has the same distribution as the Z component of $[Y_n,Z_n]$.

If π^* is the invariant probability for $\{Y_n^*\}$, then by proposition 2.2 of Nummelin we have that the Ω marginal of π^* is the invariant probability π for $\{Y_n\}$. Furthermore

$$E_{\pi^*}[W_1^*] = E_{\pi}[W_1] .$$

We then have the following lemma.

Lemma 2.7. If $\{Y_n\}$ is a positive recurrent Markov Chain with a C-set such that $k = 1$ and Z_n is defined on $\mathbb{R}$ and $g_y(z)$ satisfies conditions * and if $E[W_1] = 0$, then Z_n is recurrent.

Proof. Since we have a C-set with $k = 1$ we know that $\{Y_n{*}\}$ is a positive φ-recurrent Markov Chain with an atom and $E[W_1{*}] = 0$ and $W_n = W_n{*}$ so $Z_n = Z_n{*}$. Hence by Theorem 2.6 we get that $Z_n{*}$ is recurrent which immediately implies that Z_n is recurrent.

Now suppose that in the above k is strictly greater than 1. We then use Lemma 1.5, ii) in Nummelin, to form $\{Y_{nk}\}$ a positive φ-recurrent Markov Chain. So we have the following theorem.

Theorem 2.8. If $\{Y_n\}$ is a positive φ-recurrent Markov Chain with a C-set and if $\{Z_n\}$ is defined on $R^{(1)}$ and $g_y(z)$ satisfies conditions * with $E[W_1] = 0$ then $\{Z_n\}$ is recurrent.

Proof. Let k_0 be the value of k in the definition of C-set. We then look at $\{Y_0, Y_{k_0}, Y_{2k_0}, Y_{3k_0}, \ldots\}$, a positive φ-recurrent Markov Chain with a C-set with $k = 1$. We then know that $\hat{W}_1$, the one-step transition probability for the new random walk is given by $\hat{W}_1 = \sum_{k=1}^{k_0} W_k$.

$$\Rightarrow E_\pi[\hat{W}_1] = E_\pi[\sum_{k=1}^{k_0} W_k] = \sum_{k=1}^{k_0} E_\pi[W_k] = 0$$

So $\{Y_{nk_0}\}$ meets the conditions of Lemma 2.7, hence we get that Z_{nk_0} is recurrent. This implies that Z_n is recurrent.

So for a positive φ-recurrent Markov Chain which has a C-set and for which $g_y(z)$ satisfies conditions * , we then have the following dichotomy in force:

1) $E[W_1] = 0 \iff Z_n$ is recurrent,

2) $E[W_1] \neq 0 \iff Z_n$ is transient and then $Z_n \to \text{sgn}(\pi f)\cdot\infty$ a.s. P_π.

We now consider the case where $\{Z_n\}$ is defined on the two-dimensional integer lattice, $I^{(2)}$. We know from Gihman and Skorohod's The Theory of Stochastic Process I, that if $\{X_n\}$ is a classical random walk on $I^{(2)}$, and if $E[X_1] = 0$ and $E[|X_1|^2] < \infty$ then $\{X_n\}$ is recurrent.

Theorem 2.9. If $\{Y_n\}$ is a positive φ-recurrent Markov Chain with a positive atom and if $\{Z_n\}$ is defined on the two-dimensional integer lattice and $g_y(z)$ meets the equivalent conditions to conditions * when Z takes its values in $R^{(2)}$, and if $E[W_1] = 0$ and $E[|W_1|^2] = E[{W_1^{(1)}}^2] + E[{W_1^{(2)}}^2] < \infty$ where $W_1^{(i)}$, $i = 1,2$, are the components of the W_1 vector, and if one of the two conditions below is met:

i) $E_\pi[\tau_1] < \infty$ (equivalently $E_{\pi_A}[\tau_1^2] < \infty$) and there is a bounded B in $I^{(2)}$ such that $g_y(B) = 1$ for every y in the complement of A, where π_A is the invariant distribution on A for the process $\{Y_{\tau_n}\}$. (i.e. the process restricted to A)

or

ii) there exists $r,s > 1$ with $\frac{1}{r} + \frac{1}{s} = 1$ and there exists a function h on $\{1,2,3,\ldots,\}$ to $(0,\infty)$ such that

a) $\sum_{i=1}^{2} \sum_{k=-\infty}^{\infty} |k|^{2r}\, g_\pi^{(i)}(k) < \infty$,

b) $\sum_{n=1}^{\infty} \frac{1}{h(n)} < \infty$, and

c) $E_{\pi}[h(\tau_A)^{2s-1}] < \infty$.

Then Z_n is recurrent.

Proof. It is easily seen that for this situation, Lemma 2.5 can be extended. So we have $E[W_1] = 0$ implies that $E[Z_{\tau_1}] = 0$. Then if we look at (Y_n, W_n) which is a φ-recurrent Markov Chain, we see that since there is no feedback in the model that the marginal on its stationary initial distribution, $\pi(y,w)$, of Y must be π . So that the initial distribution of Y is π, hence the marginal of W must be g_{π} . We will then use the notation

$$g_{\pi} = \begin{pmatrix} g_{\pi}^{(1)} \\ g^{2}_{(\pi)} \end{pmatrix}$$

and, note that $(A, R^{(2)})$ is a positive atom of (Y_n, W_n). If we then define $\tilde{h}(Y, W_n) = W_n$, we have that $\pi[|\tilde{h}|^2] = E[|W_1|^2] = E[W_1^{(1)^2}] + E[W_1^{(2)^2}]$ and that $\pi\tilde{h} = 0$. Then we note that for $i = 1,2$,

$$\int_{\Omega-A}\int_{R^{(2)}} |W_1^{(i)}|^{2r} \pi(dy,dw) \leq \int_{\Omega}\int_{R^2} |W_1^{(i)}|^{2r} \pi(dy,dw)$$

$$= \int_{R^2} |W_1^{(i)}|^{2r} g_{\pi}(dw) = \sum_{k=-\infty}^{\infty} |k|^{2r} g_{\pi}^{(i)}(k) .$$

So by Theorem 4.2, Cogburn [4], applied to each component of W_1, we have the $E_{\pi_A}[|Z_{\tau_1}|^2] = E_{\pi_A}[|\sum_{k=1}^{\tau_1} W_k|^2] < \infty$ if either condition i) or ii) holds.

Therefore, if we again let $X_k = Z_{\tau_k} - Z_{\tau_{k-1}}$, we get that $\{X_k\}$ is

an i.i.d. sequence of integer-valued random vectors defined on $I^{(2)}$ with $E[X_1] = 0$ and $E[|X_1|^2] < \infty$. So by theorem 3, page 124 of Gihman and Skorohod [6] we have that $Z_{\tau_n} = S_n = \sum_{k=1}^{n} X_k$ is a recurrent random walk, hence Z_n is recurrent.

It should be noted that if $E[W_1] \neq 0$, then one of the two components has expectation not equal to zero, and by Theorem 2.6, that component would be transient, hence Z_n would be transient. Using the techniques of Theorem 2.8 we get the following corollary to Theorem 2.9.

Corollary 2.10. If $\{Y_n\}$ is a positive φ-recurrent Markov Chain with a C-set and if $\{Z_n\}$ is as in Theorem 2.9 and all other conditions of Theorem 2.9 are met, then Z_n is recurrent.

CHAPTER III

PROBABILITY OF ABSORPTION AND MEAN TIME TO ABSORPTION

Section 3.1.

In this chapter we will consider bichains (Y_n, Z_n) such that the control process $\{Y_n\}$ is a finite recurrent Markov Chain and such that Z given Y is a random walk with its transition probabilities defined by:

$$p_i = P(Z_{n+1} = k + 1 \mid (Y_0, Z_0),\ (Y_1, Z_1), \ldots, (Y_n, Z_n) = (i,k))$$
$$q_i = P(Z_{n+1} = k - 1 \mid (Y_0, Z_0),\ (Y_1, Z_1), \ldots, (Y_n, Z_n) = (i,k)) \tag{1}$$

for all integer values k. We will assume throughout this chapter that

$$p_i \neq 0 \text{ and } q_i \neq 0 \text{ for any } i = 1,2,\ldots,n \text{ and}$$

$$p_i \neq p_j \text{ for } i \neq j \text{ and } p_i + q_i = 1 \text{ for all } i = 1,2,\ldots,n .$$

We will be interested in solving the absorption problem, given that we have an absorbing barrier at $(Y,Z) = (\cdot, 0)$, and that the bichain is started at (y,z) with $z > 0$. The methods used will also allow that we have a right barrier at $(\cdot, a)$, in this situation (Y,Z) will be started at (y,z) where $0 < z < a$. In both of these cases we will consider the problem of the expected duration of the walk, when the probability of absorption is equal to 1.

We will let the transition matrix of the control process $\{Y_n\}$ be denoted by

$$B = \begin{bmatrix} \pi_{11} & \pi_{12} & \cdots & \pi_{1n} \\ \pi_{21} & \pi_{22} & & \pi_{2n} \\ \vdots & & & \vdots \\ \pi_{n1} & & & \pi_{nn} \end{bmatrix} \tag{3}$$

where as usual

$$\pi_{ij} = P(Y_{n+1} = j \mid Y_n = i) \quad . \tag{4}$$

Then since $\{Y_n\}$ is a finite recurrent Markov Chain, it has an invariant probability distribution π, which we will denote by a $1 \times n$ vector, $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$. We can think of π_k as the proportion of time that $\{Y_n\}$ in the long run will spend in stake k. Hence if we let $\vec{p} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix}$ and $p = \pi \cdot \vec{p}$ then p can be viewed as the average probability of taking a step to the right.

From the results of Chapter 2 we know that, for the one barrier case, the probability of absorption is less than one if and only if $P > \frac{1}{2}$ (i.e. if the process is recurrent or drifts to the left, then the probability of absorption is one). So, when we only have a barrier at $(\cdot,0)$, we will assume $P > \frac{1}{2}$ but, unless specified otherwise, we will assume $0 < P < 1$ for the two barrier problems.

We now want to consider the probability of absorption at the barrier $(\cdot,0)$, given we are at some point (i,k) where $0 < k < a$. We will set

$$f(i,k) = P(\text{absorption} \mid (Y_0, Z_0) = (i,k)). \tag{5}$$

Then since the probability of absorption from (i,k) is equal to the sum of the products of the probability of going to state (j,ℓ) in one step, times the probability of being absorbed from (j,ℓ), we have that for every $i = 1,2,3,\ldots,n$ and $k = 1,2,3,\ldots,a-1$ that

$$f(i,k) = p_i \sum_{j=1}^{n} \pi_{ij} f(j,k+1) + q_i \sum_{j=1}^{n} \pi_{ij} f(j,k-1) . \tag{6}$$

We will now solve by a constructive process. Since there is no feedback in the system, it is reasonable to attempt a separation of variables and, if one looks at the solution to this problem for the classical random walk, it is known that $f(k) = a\, r^k$ is the solution. Hence we will attempt a solution of the form

$$f(i,k) = C(i)\, r^k . \tag{7}$$

Then plugging the above into (6) we get

$$C(i)\, r^k = p_i \sum_{j=1}^{n} \pi_{ij} C(j)\, r^{k+1} + q_i \sum_{j=1}^{n} \pi_{ij} C(j)\, r^{k-1} \tag{8}$$

or

$$C(i)\, r^k = (p_i\, r^{k+1} + q_i\, r^{k-1}) \sum_j \pi_{ij} C(j) .$$

If we then divide both sides by r^{k-1} we get

$$C(i)\, r = (p_i\, r^2 + q_i) \sum_j \pi_{ij} C(j) \quad \forall\ i = 1,2,\ldots,n . \tag{9}$$

If we then let

$$\gamma_i = p_i\, r^2 + q_i , \quad i = 1,2,\ldots,n$$

and

$$\Gamma = (\gamma_{ij})_{n\times n} \quad \text{where} \quad \begin{cases} \gamma_{ii} = \gamma_i & i = 1,2,\ldots,n \\ \gamma_{ij} = 0 & \text{for } i \neq j \end{cases}$$

and

$$C = \begin{pmatrix} C(1) \\ C(2) \\ \vdots \\ C(n) \end{pmatrix}$$

Then we can write (9) in matrix notation as

$$\Gamma BC = rc \tag{10}$$

or in equivalent form

$$(\Gamma B - rI)\, c = 0 \ .$$

So values of r and c , which satisfy the above, exist if and only if

$$\det(\Gamma B - rI) = 0 \ . \tag{11}$$

It should be remembered that Γ is a function of r , hence we are not merely looking for eigenvectors and eigenvalues of ΓB . Since $r = 0$ is a trivial solution of (6), we will assume $r \neq 0$. We will then let

$$A_r = \Gamma B - rI \ .$$

So

$$A_r = \begin{bmatrix} \gamma_1\pi_{11}-r & \gamma_1\pi_{12} & \cdots & \gamma_1\pi_{1n} \\ \gamma_2\pi_{21} & \gamma_2\pi_{22}-r & \cdots & \gamma_2\pi_{2n} \\ \vdots & \vdots & & \vdots \\ \gamma_n\pi_{n1} & \gamma_n\pi_{n2} & \cdots & \gamma_n\pi_{nn}-r \end{bmatrix} . \tag{11}$$

It is easy to see that the $\det A_r$ is a polynomial in r of degree $2n$. Now the coefficient of r^{2n} is then

$$a_{2n} = (\prod_{i=1}^{n} p_i) \det B$$

and the constant term

$$a_0 = (\prod_{i=1}^{n} q_i) \det B .$$

So if $\det B \neq 0$, then we see that $a_{2n} \neq 0$ and $a_0 \neq 0$. Hence $\det A_r$ is a polynomial of degree $2n$, which does not have $r = 0$ as a root. We will denote the solutions by $r_1, r_2, \ldots, r_{2n}$. Then for each r_ℓ, $\ell = 1,2,\ldots,2n$, we can find

$$C_\ell = \begin{pmatrix} C_\ell(1) \\ C_\ell(2) \\ \vdots \\ C_\ell(n) \end{pmatrix} .$$

We then note that, since all the coefficients of the polynomial, $\det A_r$, are real, the roots will appear in conjugate pairs. If r_1 and r_2 are conjugate roots, it is easily seen that $C_1 = \bar{C}_2$. We next observe that $\mathrm{Re}(c(i)r^k)$ and $\mathrm{Im}(c(i)r^k)$ are independent solutions for $f(i,k)$. The fact that they are solutions is trivial and we will show below that they are independent. If we assume that they are linearly dependent, then there is a real α such that

$$\mathrm{Re}(c(i)r^k) = \alpha \, \mathrm{Im}(c(i)r^k) \tag{12}$$

holds for all values of i and k. Hence it must hold for $k = 0$. So we get

$$\mathrm{Re}(c(i)) = \alpha \, \mathrm{Im}(c(i)) . \tag{13}$$

It is easy to see that det A_i is a polynomial in λ of degree $2n$, and the coefficient of λ^{2n} is then

$$a_{2n} = \left(\prod_{j=1}^{n} q_j\right) \det S$$

and the constant term

$$a_0 = \left(\prod_{j=1}^{n} q_j\right) \det T \; .$$

So if $\det S \neq 0$, then we see that $a_{2n} \neq 0$ and $a_0 \neq 0$. Hence det A_i is a polynomial of degree $2n$ which does not have $\lambda = 0$ as a root. We will denote the solutions by $\{\lambda_1, \ldots, \lambda_{2n}\}$. Then for each λ_k, $k = 1,2,\ldots,2n$, we can find

$$C_k = \begin{pmatrix} c_k(1) \\ c_k(2) \\ \vdots \\ c_k(2n) \end{pmatrix}$$

We then note that, since all the coefficients of the polynomial det A_i are real, the roots will appear in conjugate pairs. If λ_j and λ_k are conjugate roots, it is easily seen that $C_j = \bar{C}_k$. We next observe that [illegible] and [illegible] are independent solutions for (3.10). The fact that they are solutions is obvious and we will show below that they are independent. If we assume that they are linearly dependent, then there is a real α such that

[illegible] (3.11)

holds for all values of i and [illegible]. Hence it must hold for [illegible].

So we get

[illegible] (3.12)

If we then expand (12), we get

$$Re(c(i))Re(r^k) - Im(c(i))Im(r^k) = \alpha\ Re(c(i))Im(r^k) + \alpha\ Im(c(i))Re(r^k)\ .$$

From (13) we know that

$$Re(c(i))Re(r^k) = \alpha\ Im(c(i))Re(r^k),$$

so we have that

$$-Im(c(i))Im(r^k) = \alpha^2\ Im(c(i))Im(r^k)$$

or $\alpha^2 = -1$. So there is no real α which will satisfy (12). Hence for any conjugate pair of roots r_1 and r_2 we have four solutions, $Re(c_1(i){r_1}^k)$, $Im(c_1(i){r_1}^k)$, $Re(c_2(i){r_2}^k)$ and $Im(c_2(i){r_2}^k)$. If we then assume $Imr_1 < 0$ and $Imr_2 > 0$, we will then use the two following independent solutions: $Re(c_1(i){r_1}^k)$ and $Im(c_2(i){r_2}^k)$.

We will then use the following sets of indices to distinguish the three distinct types of roots and solutions they generate:

$$E = \{\ell \mid r_\ell \text{ real root}\}$$

$$F = \{\ell \mid r_\ell \text{ complex and } Imr_\ell > 0\}$$

$$G = \{\ell \mid r_\ell \text{ complex and } Imr_\ell < 0\}\ .$$

It should be noted that E, F, and G are mutually disjoint and their union is $\{1,2,3,\ldots,2n\}$. We can then prove the following theorem.

<u>Theorem 3.1</u>. If $\det B \neq 0$ and the $2n$ roots of $\det A_r$ are distinct then the values of r_ℓ and the eigenvectors C_ℓ associated with them give us $2n$ linearly independent solutions and hence the general solution to the probability of absorption problem is

$$f^*(i,k) = \sum_{\ell \in E} c_\ell(i) r_\ell^k + \sum_{\ell \in F} \operatorname{Im}(c_\ell(i) r_\ell^{\,k}) + \sum_{\ell \in G} \operatorname{Re}(c_\ell(i) r_\ell^{\,k}) .$$

Proof. From Section 1.1 of Hildebrand [9] we know that if we have 2n linearly independent solutions for $f(i,k)$ then the general solution is a linear combination of these 2n solutions. Hence $f^*(i,k)$ which is a linear combination of these 2n independent solutions must then be the general solution. Note that since $c_\ell(i)$ is the component of an eigenvector then any multiple, $a_\ell c_\ell(i)$, is also the component of an eigenvector $a_\ell c_\ell$, hence we shall let c_ℓ absorb the constant.

We will now consider the following example which shows that the 2n roots need not all be distinct. Let

$$B = \begin{bmatrix} .6 & .4 \\ .4 & .6 \end{bmatrix}$$

then the invariant distribution $\pi = (.5,.5)$. If we let $\vec{p} = \begin{bmatrix} .2 \\ .8 \end{bmatrix}$ then $\pi\vec{p} = .5$ and equation 6 then yields

$$A_r = \begin{bmatrix} (.2r^2 + .8)(.6) - r & (.2r^2 + .8)(.4) \\ (.8r^2 + .2)(.4) & (.8r^2 + .2)(.6) - r \end{bmatrix} .$$

We then wish to evaluate $\det A_r$ and find values of r such that $\det A_r = 0$. If we add the second column to the first we get

$$\det A_r = \begin{vmatrix} .2r^2 + .8 - r & .08r^2 + .32 \\ .8r^2 + .2 - r & .48r^2 + .12 - r \end{vmatrix}$$

$$= \frac{1}{100^2} \begin{vmatrix} (r-1)(2r-8) & 8r^2 + 32 \\ (r-1)(8r-2) & 48r^2 + 12 - 100r \end{vmatrix}$$

adding the second row to the first we get

$$= \frac{(r-1)}{100^2} \begin{vmatrix} 10r - 10 & 56r^2 - 100r + 44 \\ 8r - 2 & 48r^2 + 12 - 100r \end{vmatrix}$$

$$= \frac{(r-1)^2}{100^2} \begin{vmatrix} 10 & 56r - 44 \\ 8r - 2 & 48r^2 - 100r + 12 \end{vmatrix}$$

So we see that $r = 1$ is a double root of $\det A_r = 0$. Hence we now know that the $2n$ roots need not be distinct. If they had been distinct then Theorem 3.1 would have been in effect, but since they are not, we know from the proof of Theorem 3.1 that some other solutions are needed.

In the case of classical random walks we know that, when $p = q = \frac{1}{2}$ we get $f(k) = a^k$, also to be a solution. This then suggests that, if r is a double root to $\det A_r$, then we try the solution $f(i,k) = k\, c(i)r^k$. But an attempt to plug this into (6) leads to problems. We need to modify the above attempt, but first we will consider the following result concerning a double root.

<u>Theorem 3.2</u>. If we have a double root r_0 and if the rank of A_{r_0} is $n - 2$ then there exist two linearly independent eigenvectors c both of which satisfy $\Gamma BC = r_0 c$. Furthermore the general solution is

[illegible]

Adding the second row to the first we get

[illegible]

So we see that $r = 1$ is a double root of $\det A_r = 0$. [illegible] the know that the [illegible] roots need not be distinct. If they had been distinct then Theorem 3.1 would have been in effect, but since this is not, we know from the proof of Theorem 3.1 that some other solutions are needed.

In the case of classical vector [illegible] we know that, when [illegible] we get [illegible] ought to be a solution. This line suggests that, if r is a double root of $\det A_r$, then we try the solution [illegible]. But an attempt to plug this into [illegible] leads to [illegible]. We need to modify the above attempt, and first we will consider the following result concerning a double root.

Theorem 3.3. If we have a double root r_1 and if the rank of [illegible] $n - 1$ then there exist two linearly independent eigenvectors [illegible] of which satisfy [illegible]. Furthermore the general solution is

$$f(i,k) = \sum_{\ell \in E} C_\ell(i) r_\ell^{\,k} + \sum_{\ell \in F} \mathrm{Im}(C_\ell(i) r_\ell^{\,k}) + \sum_{\ell \in G} \mathrm{Re}(C_\ell(i) r_\ell^{\,k})$$

where E, F, and G are as before and ℓ runs through the 2n linearly independent solutions generated by the pairs $\{C_i, r_i\}$ where r_i is a root of $\det A_r = 0$ and C_i runs through all eigenvectors of ΓB for each r_i .

Proof. From elementary matrix theory we know that two linearly independent eigenvectors exist when the rank of A_{r_0} is $n - 2$. Their independence assures us of two linearly independent solutions for $f(i,k)$ associated with r_0 the double root.

So double roots will cause a problem iff $\mathrm{rank}(\Gamma B - rI) = n - 1$. We will then assume for the rest of this section that $\mathrm{rank}\, A_r = n - 1$. If r is a known solution for (11) we try the solution

$$f(i,k) = [k\, C(i) + \alpha(i)]\, r^k \tag{14}$$

$$[k\, C(i) + \alpha(i)] r^k = p_i \sum_{j=1}^{n} \pi_{ij} [(k+1)C(j) + \alpha(j)] r^{k+1}$$

$$+ q_i \sum_{j=1}^{n} \pi_{ij} [(k-1)C(j) + \alpha(j)] r^{k-1} \tag{15}$$

$$\forall\ i = 1,2,\ldots,n \quad k \in I$$

dividing through by r^{k-1} and collecting terms we get

$$k\, C(i) r + \alpha(i) r = k(p_i r^2 + q_i) \sum_j \pi_{ij}\, C(j) + (p_i r^2 - q_i) \sum_j \pi_{ij}\, C(j)$$

$$+ (p_i r^2 + q_i) \sum_j \pi_{ij}\, \alpha(j) \ . \tag{16}$$

If we let $k = 0$ in (16) we get

$$\alpha(i)r = (p_i r^2 - q_i) \sum_j \pi_{ij} C(j) + (p_i r^2 + q_i) \sum_j \pi_{ij} \alpha(j) \qquad \forall\, i = 1,2,3,\ldots,n \;. \tag{17}$$

If we then subtract (17) from (16) we get

$$\begin{aligned} k\, C(i)r &= k(p_i r^2 + q_i) \sum_j \pi_{ij} C(j) \\ \Rightarrow \quad C(i)r &= (p_i r^2 + q_i) \sum_j \pi_{ij} C(j) \;. \end{aligned} \tag{18}$$

Since r is a root of $\det A_r = 0$ then we know that there exists C which satisfies (18). Hence to show that (14) is a solution we only need to show that there are α_n's which satisfy (17).

We will let

$$\alpha = \begin{pmatrix} \alpha(1) \\ \alpha(2) \\ \vdots \\ \alpha(n) \end{pmatrix}$$

and

$$\tilde{\Gamma} = (\tilde{\gamma}_{ij})_{n\times n} \quad \text{where} \quad \begin{cases} \tilde{\gamma}_{ii} = q_i - p_i r^2 & i = 1,2,\ldots,n \\ \tilde{\gamma}_{ij} = 0 & \text{for } i \neq j \;. \end{cases}$$

To simplify notation we will use C_i and α_i to denote $C(i)$ and $\alpha(i)$ respectively. We can then rewrite (17) as follows.

$$\gamma_i \sum_j \pi_{ij}\alpha_j - \alpha_i r = \tilde{\gamma}_i \sum_j \pi_{ij} C_j \qquad \forall\; i \;,$$

which we can then rewrite in matrix notation as

$$(\Gamma B - rI)\alpha = \tilde{\Gamma} Bc \;.$$

So we need to show that $\tilde{\Gamma}BC$ is in the range of $\Gamma B - rI$. To do this we will consider the corollary to Theorem 1.3 in Moore's Elements of Linear Algebra and Matrix Theory, which tells us that the system $A_r\alpha = \tilde{\Gamma}BC$ is solvable if and only if the ranks of A_r and its augmented matrix are equal. Since it is assumed that rank A_r equals $n - 1$, we need to show that the rank of its augmented matrix, which is given below, is also $n - 1$.

$$[A_r : \tilde{\Gamma}Bc] = \begin{bmatrix} \gamma_1\pi_{11}-r & \gamma_1\pi_{12} & \cdots & \gamma_1\pi_{1n} & \tilde{\gamma}_1 \sum_j \pi_{1j}C_j \\ \gamma_2\pi_{21} & \gamma_2\pi_{22}-r & \cdots & \gamma_2\pi_{2n} & \tilde{\gamma}_2 \sum_j \pi_{2j}C_j \\ \vdots & \vdots & & & \vdots \\ \gamma_n\pi_{n1} & \gamma_n\pi_{n2} & \cdots & \gamma_n\pi_{nn}-r & \tilde{\gamma}_n \sum_j \pi_{2j}C_j \end{bmatrix} \qquad (19)$$

We then consider λ the non-zero left eigenvector of ΓB, which we know exists. We can assume without loss of generality that the first component of λ does not equal zero, $\lambda_1 \neq 0$. Then in (19), if we take λ_i times the i^{th} row and add it to λ_1 times the first row, we will have a new matrix which has the same rank as the augmented matrix. But since λ is the left eigenvector the new vector $\lambda[A_r : \tilde{\Gamma}BC]$ is

$$\left[0 \quad 0 \quad \cdots \quad 0 \quad \sum_i \tilde{\gamma}_i\pi_i \sum_j \pi_{ij}C_j\right]. \qquad (20)$$

Since the rank of the augmented matrix is at least as large as the rank of the original matrix, we only need show that the rank of the augmented matrix is not n . So from (20) we only need to show that

[illegible]

then we will consider the conditions on Theorem 1.3 in Bodewig's [illegible]

N Linear Algebra and Matrix Theory, which tells us that the system

[illegible]

and matrix are equal. Hence it is assumed that each λ_i [illegible]

we need to show that the rank of the augmented matrix, which is given below, is also

$$[\lambda_i, [M]] = [\text{illegible}] \qquad (19)$$

On the smallest λ, the nonzero left eigenvalue of $[M]$, which we have assumed. We can assume without loss of generality that the first component of λ does not equal zero, $\lambda_1 \neq 0$. Then in (19), if we take λ_i times the i^{th} row and add it to λ_1 times the first row, we will have a new matrix which has the same rank as the augmented matrix. But since λ is the left eigenvector, the new vector $[\lambda_i][M]$ is

$$[0 \quad 0 \quad \ldots \quad \text{[illegible]}] \qquad (20)$$

Since the rank of the augmented matrix is at least as large as the rank of the original matrix, we only need show that the rank of the augmented matrix is the [illegible] of (20) we [illegible]

$$\sum_i \pi_i \tilde{\gamma}_i \sum_j \pi_{ij} C_j = 0 \quad .$$

Now if $\lambda_i C_i = 0 \;\forall\; i$, we have the following lemma.

<u>Lemma 3.3</u>. If the rank $A_r = n - 1$, and $\lambda_i C_i = 0 \;\forall\; i = 1,2,\ldots,n$ then the rank of the augmented matrix is $n - 1$.

<u>Proof</u>. From the above discussion we know that we only need to show that

$$\sum_i \lambda_i \tilde{\gamma}_i \sum_j \pi_{ij} C_j = \lambda \tilde{\Gamma} Bc = 0$$

but since Γ^{-1} exists and $BC = r\Gamma^{-1}c$

$$\Rightarrow \lambda\tilde{\Gamma}BC = r\, \lambda\tilde{\Gamma}\Gamma^{-1}c = r \sum_i \lambda_i \frac{\tilde{\gamma}_i}{\gamma_i} C_i = r \sum_i \pi_i C_i \frac{\tilde{\gamma}_i}{\gamma_i} \quad ,$$

but $\lambda_i C_i = 0 \;\forall\; i \implies \lambda\tilde{\Gamma}BC = 0$, hence from (20) we have that the rank of the augmented matrix is $n - 1$.

We will now assume that there is a value i_0 for which $\lambda_{i_0} C_{i_0} \neq 0$, and we will assume without loss of generality that $i_0 = 1$. We will then multiply the j^{th} column of A_r by C_j and add it to the first column times C_1 for $j = 1,2,\ldots,n$. Then we will multiply the i^{th} row of the resulting matrix by λ_i and add it to the first row times λ_1 for $i = 1,2,\ldots,n$. The resulting matrix R_r then has rank $n - 1$. Hence

$$R_r = \begin{bmatrix} \lambda\Gamma BC - \lambda cr & [\lambda(\Gamma B-rI)]_2 & \cdots & [\lambda(\Gamma B-rI)]_n \\ [(\Gamma B-rI)c]_2 & \gamma_2\pi_{22}-r & \cdots & \gamma_2\pi_{2n} \\ \vdots & \vdots & & \vdots \\ [(\Gamma B-rI)c]_n & \gamma_n\pi_{n2} & \cdots & \gamma_n\pi_{nn}-r \end{bmatrix} \quad (21)$$

(where $[\beta]_i$ is the i^{th} component of the vector $\beta]$. Note that for $r = r_0$ each term in the first row and in the first column is equal to 0 . Since rank $R_{r_0} = n - 1$ then letting $\tilde{R}_r$ be the submatrix obtained from R_r by deleting the first row and column, we have

$$\text{rank } \tilde{R}_{r_0} = \text{rank} \begin{bmatrix} \gamma_2\pi_{22}-r_0 & \gamma_2\pi_{23} & \cdots & \gamma_2\pi_{2n} \\ \gamma_3\pi_{32} & \gamma_3\pi_{33}-r_0 & \cdots & \gamma_3\pi_{3n} \\ \vdots & \vdots & & \vdots \\ \gamma_n\pi_{n2} & \gamma_n\pi_{n3} & \cdots & \gamma_n\pi_{nn}-r_0 \end{bmatrix} = n - 1, \quad (22)$$

hence $\det \tilde{R}_{r_0} \neq 0$. If we let

$$P = (p_{ij})_{n \times n} \text{ where } \begin{cases} p_{ii} = p_i & i = 1,2,\ldots,n \\ p_{ij} = 0 & i \neq j \end{cases}$$

and

$$Q = (q_{ij})_{n \times n} \text{ where } \begin{cases} q_{ii} = q_i & i = 1,2,\ldots,n \\ q_{ij} = 0 & i \neq j \end{cases} ,$$

we can then prove the following lemma.

Lemma 3.4. If r_0 is a double root and if rank $R_{r_0} = n - 1$ and if λ and c, the eigenvectors associated with r_0, satisfy $\lambda_1 \neq 0$ and $c_1 \neq 0$ then rank $\tilde{R}_{r_0} = n - 1$ and $\lambda \Gamma Bc - \lambda cr$ which equals $\lambda PBc\, r^2 - \lambda cr + \lambda QBc$ is a perfect square and its double root is

$$r = r_0 = \sqrt{\frac{\lambda QBc}{\lambda PBc}} .$$

Proof. From above we know that $\det \tilde{R}_{r_0} \neq 0$ and that $\tilde{R}_{r_0}$ is an $(n-1) \times (n-1)$ matrix, hence its rank must be $n - 1$. Next we wish to consider $\frac{d}{dr} \det R_r$. The derivative of the determinant of the matrix R_r is the sum of the determinants of n matrices, each the same as R_r except that in the first, the elements of the first row have been replaced by their derivatives with respect to r, in the second the elements of the second row have been replaced by their derivatives with respect to r, etc. Hence

$$\frac{d}{dr} \det R_r = \sum_{i=1}^{n} \det R_r^{(i)} .$$

Where $R_r^{(i)}$ is the same as R_r except that the elements of the i^{th} row have been replaced by their derivatives. Note that for $i > 1$, $\det R_r^{(i)}$ can be evaluated by expanding by the first row. But the $[\det R_r^{(i)}]_{r=r_0} = 0$ because each element of the first row equals zero for $r = r_0$. Hence

$$\left[\frac{d}{dr} \det R_r\right]_{r=r_0} = \det R_r^{(1)}{}_{r=r_0} .$$

To evaluate $[\det R_r^{(1)}]_{r = r_0}$ we will expand by the first column. Then for every i greater than 1, LEMMA 2.4. If x_0 is a double root [illegible] and if y^0 and z^0 are right and left nullvectors with [illegible] respectively, $y_1^0 \neq 0$ and $z_1^0 \neq 0$ then rank $B_0 = n + 1$ and [illegible] $\det B_0$ equals [illegible] is a contour [illegible] the double root [illegible]

$$[illegible]$$

Proof. From above we know that $\det A_0 \neq 0$ and that B_0 is a $(n+1) \times (n+1)$ matrix, hence its rank must be $n + 1$. Next we wish to consider $\frac{d}{dx}\det B_0$. The derivative of the determinant of the matrix B_0 is the sum of the determinants of n matrices, each the same as B_0 except that in the first, the elements of the first row have been replaced by their derivatives with respect to x, in the second the elements of the second row have been replaced by their derivatives with respect to x, etc. Hence

$$\frac{d}{dx}\det B_0 = \sum_{i=1}^{n} \det B_0^{(i)}$$

where $B_0^{(i)}$ is the same as B_0 except that the elements of the ith row have been replaced by their derivatives. Note that for $i \neq 1$, $\det B_0^{(i)}$ can be evaluated by expanding by the first row. But then $\det B_0^{(i)}\big|_{x=x_0} = 0$ because each element of the first row equals zero for $x = x_0$. Hence

$$\frac{d}{dx}\det B_0\Big|_{x=x_0} = \det B_0^{(1)}\Big|_{x=x_0}$$

To evaluate $\det B_0^{(1)}\big|_{x=x_0}$ we will expand by the first column. Then for every i greater than 1

$$[\gamma_i \sum_j \pi_{ij} c_j - c_i r]_{r=r_0} .$$

Hence

$$\left[\det R_r^{(1)}\right]_{r=r_0} = \left[\frac{d}{dr} \left(\sum_i \sum_j \pi_{ij} \gamma_i \lambda_i c_j - \sum_i c_i \pi_i r\right]_{r=r_0} \left[\det \tilde{R}_r \; r=r_0\right].$$

Hence we have that

$$\left[\frac{d}{dr} \det R_r\right]_{r=r_0} = \left[\frac{d}{dr}\left(\sum_i \sum_j \pi_{ij} \gamma_i \lambda_i c_j - \sum_j c_i \lambda_i r\right)\right]_{r=r_0} \left[\det \tilde{R}_r\right]_{r=r_0} .$$

Next note that $\det R_r$ is an $2n^{th}$ degree polynomial with a double root at r_0 we know that

$$\det R_r = (r - r_0)^2 g(r)$$

$$\Rightarrow \frac{d}{dr} \det R_r = 2(r - r_0(g(r) + (r - r_0)^2 g'(r)$$

$$\Rightarrow \left[\frac{d}{dr} \det R_r\right]_{r=r_0} = 0 .$$

But then, since $[\det \tilde{R}_r]_{r=r_0} \neq 0$, we then have that

$$\left[\frac{d}{dr}\left(\sum_i \sum_j \pi_{ij} \gamma_i \lambda_i c_j - \sum_i c_i \lambda_i r\right)\right]_{r=r_0} = 0 .$$

This together with the fact that

$$\left(\sum_i \sum_j \pi_{ij} \gamma_i \lambda_i c_j - \sum_i c_i \lambda_i r\right)_{r=r_0} = 0$$

imply that

$$\sum_i \sum_j \pi_{ij} \gamma_i \lambda_i c_j - \sum_i c_i \lambda_i r , \tag{23}$$

is a perfect square and since

$$\Gamma = r^2 P + Q \tag{24}$$

we can write (23) as

$$\lambda \Gamma Bc - r\lambda c = \lambda PBc\, r^2 - \lambda cr + \lambda QBc\ , \tag{25}$$

hence

$$r = \sqrt{\frac{\lambda QBc}{\lambda PBc}}\ .$$

We are now in the position to show that the conclusion of Lemma 3.3 holds even when $\lambda_i c_i \neq 0 \ \forall i$.

Lemma 3.5. If r_0 is a double root of $\det A_r$, with λ and c as its corresponding eigenvectors, where rank $A_{r_0} = n - 1$ and there is a value i_0 for which $\lambda_{i_0} c_{i_0} \neq 0$, then the rank of the augmented matrix, (19), is $n - 1$.

Proof. Without loss of generality we can assume $i_0 = 1$. Then as in the proof of Lemma 3.3 we only need to show that $\lambda \tilde{\Gamma} Bc = 0$.

$$\lambda \Gamma Bc = \lambda(Q - r^2 P)Bc = \lambda QBc - r^2 \lambda PBc \tag{26}$$

but from Lemma 3.4 we know that

$$r = \sqrt{\frac{\lambda QBc}{\lambda PBc}} \quad \text{or} \quad r^2 = \frac{\lambda QBc}{\lambda PBc}$$

so (26) becomes

$$\lambda QBc - \frac{\lambda QBc}{\lambda PBc} \cdot \lambda PBc = \lambda QBc - \lambda QBc = 0\ ,$$

hence the rank of the augmented matrix is $n - 1$.

Lemma 3.6. If r_0 is a double root of $\det A_r$, where rank $A_{r_0} = n-1$, then the system $A_{r_0}\alpha = \tilde{\Gamma}Bc$ is solvable.

Proof. From Lemmas 3.3 and 3.5, we know that the rank of the augmented matrix is $n - 1$, hence from the corollary to Theorem 1.3 in Moore's text, we know that the above system is solvable. We then have the following theorem.

Theorem 3.7. If $\det B \neq 0$ and r_0 is a double root of $\det A_r$, where rank $A_{r_0} = n - 1$, then the general solution to the probability of absorption problem is

$$f(i,k) = (c_0(i)k + \alpha_i)r_0^{\,k} + \sum_{\ell\in E} c_\ell(i)r_\ell^{\,k} + \sum_{\ell\in F} \mathrm{Im}(c_\ell(i)r_\ell^{\,k})$$

$$+ \sum_{\ell\in G} \mathrm{Re}(c_\ell(i)r_\ell^{\,k}) ,$$

where E, F, and G are as defined before Theorem 3.1 .

It should be noted that when $P = \frac{1}{2}$, $r = 1$ is a double root.

Theorem 3.8. If $P = \frac{1}{2}$ then $r_0 = 1$ is a double root and rank $A_r = n-1$.

Proof. If $r_0 = 1$ is a root trivially, associated with it are $\lambda = \pi$ and $c = J = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$.

Hence

$$\lambda PBc = \pi PJ = \frac{1}{2}$$

and

$$\lambda QBc = \pi QJ = \frac{1}{2}$$

and

$$\lambda c = \pi J = 1$$

Lemma 4.2. If q_0 is a [illegible] root of [illegible] and q_0 [illegible], then [illegible]
then the process X_n is [illegible].

Proof. From Lemmas 4.2 and 4.3, we know that the roots of the equation
[illegible] in Section 3
(iii), we know that the above series is convergent. We thus have the
following theorem.

Theorem 4.3. If [illegible] and q_0 is a double root of [illegible] and A_{q_0} [illegible]
rank $A_{q_0} = k - 1$, then the general solution to the probability of extinction problem is

$$[illegible]$$

$$[illegible]$$

where X, Y, and Q are as defined before Theorem 4.1.

It should be noted that when [illegible] $q_0 = 1$ is a double root.

Theorem 4.4. If [illegible] then $q_0 = 1$ is a double root and rank A_{q_0} [illegible]

Proof. If $q_0 = 1$ is a root [illegible] associated with it are [illegible]
and [illegible]

Hence

$$[illegible]$$

and

$$[illegible]$$

and

$$[illegible]$$

so

$$\pi PBJ\, r^2 - \pi Jr + \pi QBJ = \frac{1}{2} r^2 - r + \frac{1}{2} = \frac{1}{2} (r-1)^2 .$$

Hence, it is a perfect square. So $r - 1$ is a double root.

For a root, r, of multiplicity $\ell > 2$, if such a root exists, it is conjectured that if rank $A_r = n - 1$ then the following are solutions to the probability of absorption problem:

$$f_1(i,k) = c(i)r^k ,$$

$$f_2(i,k) = (c(i)k + \alpha(i))r^k ,$$

$$f_2(i,k) = (\frac{c(i)}{2} k^2 + \alpha(i)k + \beta(i))\, r^k ,$$

$$\vdots \qquad\qquad \vdots$$

$$f_\ell(i,k) = \left(\frac{c(i)}{\ell!} k^\ell + \frac{\alpha(i)}{(\ell-1)!} k^{\ell-1} + \ldots + w_i\right) r^k .$$

Section 3.2.

In this section we will restrict our attention to the case where the control process only has two states. We will then study the behavior of $\det A_r$ as a function of r and attempt to make some statements about the values of r for which $\det A_r = 0$. We will assume throughout this section that $\det B \neq 0$. The reason for this restriction is, that for a two-state control process, $\det B = 0$ implies that the chain $\{Y_n\}$ consists of i.i.d. random variables and that $\{Z_n\}$ is an ordinary random walk in the average environment.

From equation 11, section 3.1, we see that $\det A_r = \det[\Gamma B - rI]$ or in this case

$$\det A_r = \begin{vmatrix} (r^2 p_1 + q_1)\pi_{11} - r & (r^2 p_1 q_1)\pi_{12} \\ (r^2 p_2 + q_2)\pi_{21} & (r^2 p_2 + q_2)\pi_{22} - 1 \end{vmatrix}$$

$$= \begin{vmatrix} r^2 p_1 + q_1 - r & (r^2 p_1 + q_1)\pi_{12} \\ r^2 p_2 + q_2 - r & (r^2 p_2 + q_2)\pi_{22} - r \end{vmatrix}. \qquad (27)$$

We then note that

$$r^2 p_i + q_i - r = r^2 p_i + 1 - p_i - r = p_i(r^2-1) - (r-1)$$

$$= p_i(r+1)(r-1) - (r-1) = (r-1)(p_i r - q_i)$$

so (27) above becomes

$$(r-1) \begin{vmatrix} p_1 r - q_1 & (r^2 p_1 + q_1)\pi_{12} \\ p_2 r - q_2 & (r^2 p_2 + q_2)\pi_{22} - r \end{vmatrix}.$$

From this and from Section 3.1 we see that $r = 1$ is a solution. To locate the other zeros we will evaluate

$$\det \tilde{A}_r = \begin{vmatrix} p_1 r - q_1 & (r^2 p_1 + q_1)\pi_{12} \\ p_2 r - q_2 & (r^2 p_2 + q_2)\pi_{22} - r \end{vmatrix} \qquad (28)$$

From equation (2), Section 3.5, we see that [illegible] in this case

$$\det [\text{illegible}] = \begin{vmatrix} [\text{illegible}] & [\text{illegible}] \\ [\text{illegible}] & [\text{illegible}] \end{vmatrix}$$

$$(27) \qquad = \begin{vmatrix} [\text{illegible}] & [\text{illegible}] \\ [\text{illegible}] & [\text{illegible}] \end{vmatrix}$$

We then note that

$$[\text{illegible}] = \ldots + \rho_1 (q^2-1) + (q-1)$$

$$= \rho_1 (q+1)(q-1) + (q-1) = (q-1)(\rho_1 q + \rho_1)$$

so (27) above becomes

$$(q-1) \begin{vmatrix} [\text{illegible}] & [\text{illegible}] \\ [\text{illegible}] & [\text{illegible}] \end{vmatrix}$$

From this and from Section 3.1 we see that [illegible] = 1 is a solution. We denote the above matrix we will evaluate

$$(28) \qquad \det [\text{illegible}] = \begin{vmatrix} [\text{illegible}] & [\text{illegible}] \\ [\text{illegible}] & [\text{illegible}] \end{vmatrix}$$

for the following set of values; $\{-1, 0, \frac{q_1}{p_1}, \frac{q_2}{p_2}, 1, \frac{q}{p}\}$.

For $r = -1$, (28) becomes

$$\begin{vmatrix} -p_1 - q_1 & \pi_{12} \\ -p_2 - q_2 & \pi_{22} + 1 \end{vmatrix} = \begin{vmatrix} -1 & \pi_{12} \\ -1 & \pi_{22} + 1 \end{vmatrix}$$

$$= -\pi_{22} - 1 + \pi_{12} = -\pi_{22} - 1 + 1 - \pi_{11} = -(\pi_{22}+\pi_{11}) < 0 . \tag{29}$$

Hence $[\det \tilde{A}_r]_{r=-1} < 0$.

For $r = 0$ (28) becomes

$$\begin{vmatrix} -q_1 & q_1\pi_{12} \\ -q_2 & q_2\pi_{22} \end{vmatrix} = q_k q_2[-\pi_{22} + \pi_{12}] .$$

$$= q_1 q_2[-\pi_{22} + 1 - \pi_{11}] = q_1 q_2[1 - \pi_{11} - \pi_{22}] . \tag{30}$$

If we then note that

$$\det B = \begin{vmatrix} \pi_{11} & 1 - \pi_{11} \\ 1 - \pi_{22} & \pi_{22} \end{vmatrix} = \pi_{11}\pi_{22} - (1 - \pi_{11})(1 - \pi_{22})$$

$$= \pi_{11}\pi_{22} - 1 + \pi_{11} + \pi_{22} - \pi_{11}\pi_{22} = \pi_{11} + \pi_{22} - 1 .$$

We can rewrite (30) as

$$[\det A_r]_{r=0} = -(\det B) q_1 q_2 . \tag{31}$$

Next for $r = \frac{q_1}{p_1}$ (28) becomes

$$\begin{vmatrix} 0 & \left(\frac{q_1^2}{p_1} + q_1\right)\pi_{12} \\ p_2 \cdot \frac{q_1}{p_1} - q_2 & \left(\frac{q_1^2}{p_1^2} p_2 + q_2\right)\pi_{22} - \frac{q_1}{p_1} \end{vmatrix}$$

$$= -\left(p_2 \frac{q_1}{p_1} - q_2\right)\left(\frac{q_1^2}{p_1} + q_1\right)\pi_{12}$$

$$= \frac{1}{p_1^2}(p_1 q_2 - p_2 q_1)(q_1^2 + q_1 p_1)\pi_{12} \quad ,$$

then we note that $p_1q_2 - p_2q_1 = p_1(1 - p_2) - p_2(1 - p_1) = p_1 - p_1p_2 - p_2 - p_2p_1 = p_1 - p_2$ and $q_1^2 + q_1p_1 = q_1^2 + q_1(1 - q_1) = q_1^2 + q_1 - q_1^2 = q_1$, so we have that

$$[\det \tilde{A}_r]_{r = \frac{q_1}{p_1}} = \frac{q_1}{p_1^2}(p_1 - p_2)\pi_{12} \quad . \tag{32}$$

For $r = \frac{q_2}{p_2}$, (28) becomes

$$\begin{vmatrix} p_1 \frac{q_2}{p_2} - q_1 & \frac{q_2^2}{p_2^2} p_1 + q_1 \cdot \pi_{12} \\ 0 & \frac{q_2^2}{p_2} + q_2 \quad \pi_{22} - \frac{q_2}{p_2} \end{vmatrix}$$

$$= \left(p_1 \frac{q_2}{p_2} - q_1\right)\left[\left(\frac{q_2^2}{p_2} + q_2\right)\pi_{22} - \frac{q_2}{p_2}\right]$$

[illegible] becomes

[illegible]

Then we note that [illegible]

[illegible] and [illegible] so we have that

(see 4.2) [illegible]

And all (15) becomes

[illegible]

$$= \frac{1}{p_2^{\,2}} (p_1q_2 - p_2q_1)\left[(q_2^{\,2} + q_2p_2)\pi_{22} - q_2\right] ,$$

which by arguments similar to the above we get

$$\left[\det \tilde{A}_r\right]_{r=\frac{q_2}{p_2}} = \frac{q_2=}{p_2^{\,2}} [\pi_{22} - 1] . \tag{33}$$

For $r = 1$, $\det \tilde{A}_r$ becomes

$$\begin{vmatrix} p_1 - q_1 & \pi_{12} \\ p_2 - q_2 & \pi_{22} - 1 \end{vmatrix} .$$

Multiplying the above by $\frac{1}{\pi_1}\begin{vmatrix} \pi_1 & \pi_2 \\ 0 & 1 \end{vmatrix} = 1$ we get

$$\frac{1}{\pi_1} \begin{vmatrix} p - q & \pi_1\pi_{12} + \pi_2\pi_{22} - \pi_2 \\ p_2 - q_2 & \pi_{22} - 1 \end{vmatrix}$$

$$= \frac{1}{\pi_1} \begin{vmatrix} p - q & \pi_2 - \pi_2 \\ p_2 - q_2 & \pi_{22} - 1 \end{vmatrix} = \frac{1}{\pi_1} (p - q)(\pi_{22} - 1) .$$

So
$$[\det \tilde{A}_r]_{r=1} = \frac{1}{\pi_1} (p - q)(\pi_{22} - 1) . \tag{34}$$

We will assume without loss of generality that $p_1 > p_2$ throughout this section. Then from (28) we observe that the leading coefficient of the 3rd degree polynomial $\det \tilde{A}_r$ is

$$p_1p_2(\pi_{22} - \pi_{12}) = p_1p_2(\pi_{22} - 1 + \pi_{11}) = p_1p_2 \det B .$$

We will then summarize the above results in the following table.

Table 3.1

$\det \tilde{A}_r$

	$P > \frac{1}{2}$		$P = \frac{1}{2}$		$P < \frac{1}{2}$	
	det B>0	det B<0	det B>0	det B<0	det B>0	det B<0
$r \downarrow -\infty$	-	+	-	+	-	+
$r = -1$	-	-	-	-	-	-
$r = 0$	-	+	-	+	-	+
$r = \frac{q_1}{p_1}$	+	+	+	+	+	+
$r = 1$	-	-	0	0	+	+
$r = \frac{q_2}{p_2}$	-	-	-	-	-	-
$r \uparrow \infty$	+	-	+	-	+	-

The reason for putting 1 before $\frac{q_2}{p_2}$ in the above is because for $P = \frac{1}{2}$ and $P < \frac{1}{2}$, $1 < \frac{q_2}{p_2}$ and in some cases for $P > \frac{1}{2}$ this is also true. From the above we can then see that there are always three real and distinct roots for $\det \tilde{A}_r = 0$.

We will now consider the one barrier problem, say a barrier at $(\cdot,0)$ with (Y_n,Z_n) started at $(i,3)$ where $z > 0$, which requires $P > \frac{1}{2}$ to be nontrivial. From the above table we see that there are two non-zero roots, r_1 and r_2 such that $r_1 < r_2$ and their magnitudes are less than 1. If we let $r^* = \max(|r_1|,|r_2|)$, then for large values of k, $f(i,k)$ will behave like $c^*(i)\, r^{*k}$. Hence since

$f(k) = c(\frac{q}{p})^k$ is the probability of absorption for the average environment, a natural point of interest is the relationship of r^* to $\frac{q}{p}$.

Now consider the boundary conditions for the one barrier problem: $f(i,0) = 1$ and $\lim_{k \uparrow \infty} f(i,k) = 0$ for $i = 1,2$. We then note that $f(i,k+1) < f(i,k)$ for every $k \geq 0$. The fact that this inequality must hold is shown below. If we assume that $f(i,k+1) = f(i,k)$ for some $k = a$ and $i = 1$, and if we then set up a barrier at $(\cdot,a)$. We have that $f(1,a+1) = f(1,a) = 1$ and $f(2,a)$ must also be 1 since it is on the boundary. We then have that

$$f(1,a+1) = p_i \sum_{j=1}^{2} \pi_{ij} f(j,a+2) + q_i \sum_{j=1}^{2} \pi_{ij} f(j,a) .$$

But $f(1,a+1) = f(1,a) = f(2,a) = 1$, hence $1 = p_i \sum_{j=1}^{2} \pi_{ij} f(j,a+2)$ $+ q_i \sum \pi_{ij}(1)$ which implies that $1 = p_i \sum_{j=1}^{2} \pi_{ij} f(j,a+2) + q_i$. So $\sum_{j=1}^{2} \pi_{ij} f(j,a+2)$ must equal 1. Then since $f(j,k) \leq 1$ for all j and k, $f(j,a+2) = 1$ for $j = 1,2$. Then since $f(1,a+1)$ $= f(1,a+2) = 1$, we can show as above that $f(i,a+3) = 1$ for $i = 1,2$. Hence $f(i,k) = 1$ for all (i,k). But then $\lim_{k \uparrow \infty} f(i,k) \neq 0$, so $f(i,k+1) < f(i,k)$ for all i and k. Hence r^* must be positive and must then be equal to r_2.

So from Table 1 we know that to compare r^* and $\frac{q}{p}$, we only need to determine $\operatorname{sgn}(\det A_{\frac{q}{p}})$. We will first take (27) and multiply by

$$\frac{1}{\pi_1}\begin{vmatrix} \pi_1 & \pi_2 \\ 0 & 1 \end{vmatrix} = 1 \text{ to get}$$

$$\det \tilde{A}_r = \begin{vmatrix} pr - q & \pi_1(r_2p_1+q_1)\pi_{12} + \pi_2(r^2p_2+q_2)\pi_{22} - \pi_2 r \\ p_2r-q_2 & (r^2p_2 + q_2)\pi_{22} - r \end{vmatrix} .$$

Plugging $r = \frac{q}{p}$ into the above and observing that the (1,1) component is zero, the above becomes

$$-\left(\frac{p_2q}{p} - q_2\right)\left[\pi_1\pi_{12}\left(\frac{p_1q^2}{p^2} + q_1\right) + \pi_2\pi_{22}\left(\frac{p_2q^2}{p^2} + q_2\right) - \pi_2\frac{q}{p}\right.$$
$$= \frac{1}{p^3}\left(q_2p - p_2q\right)\left[\pi_1\pi_{12}(p_1q^2 + q_1p^2) + \pi_2\pi_{22}(p_2q^2 + q_2p^2) - \pi_2qp\right] . \qquad (35)$$

We then note that

$$q_2p - p_2q = (1-p_2)p - p_2(1-p) = p-p_2p-p_2+p_2p = p-p_2$$

and that

$$\pi_2 = \pi_1\pi_{12} + \pi_2\pi_{22}$$

hence

$$\pi_2qp = \pi_1\pi_{12}(p-p^2) + \pi_2\pi_{22}(p-p^2)$$

and that

$$p_iq^2+q_ip^2 = p_i(1-2p+p^2) + (1-p_i)p^2 = p_i-2pp_i+p_ip^2+p^2-p_ip^2 = p_i-2pp_i+p^2 .$$

So (35) becomes

$$\frac{1}{p^3}(p-p_2)\left[\pi_1\pi_{12}(p_1-2pp_1+p^2-p+p^2) + \pi_2\pi_{22}(p_2-2pp_2+p^2-p+p^2)\right]. \qquad (36)$$

Then note that

$$p_i - 2pp_i + 2p^2 - p = p_i(1-2p) - p(1-2p) = (p_i-p)(1-2p)$$

so (36) becomes

$$\frac{1}{p^3}(p-p_2)(1-2p)\left[\pi_1\pi_{12}(p_1-p) + \pi_2\pi_{22}(p_2-p)\right] . \tag{37}$$

We then note that

$$p_1-p = p_1 - \pi_1 p_1 - \pi_2 p_2 = \pi_2(p_1-p_2)$$

and

$$p_2-p = p_2 - \pi_1 p_1 - \pi_2 p_2 = \pi_1(p_2-p_1) .$$

So (37) becomes

$$\frac{1}{p^3}(p-p_2)(1-2p)\pi_1\pi_2\left[\pi_{12}(p_1-p_2) + \pi_{22}(p_2-p_1)\right]$$

$$\frac{1}{p^3}(p-p_2)(1-2p)\pi_1\pi_2(p_2-p_1)(-\pi_{12}+\pi_{22}) \tag{38}$$

$$\frac{1}{p^3}(p-p_2)(1-2p)\pi_1\pi_2(p_2-p_1) \det B .$$

Then we know that

$$p - p_2 > 0, \quad 1 - 2p < 0, \quad \text{and} \quad p_2 - p_1 < 0 .$$

So the sign of (38) is the same as sign det B , hence from Table 3.1 we know that

$$\frac{q}{p} > r_2 \quad \text{if} \quad \det B < 0$$

$$\frac{q}{p} < r_2 \quad \text{if} \quad \det B > 0$$

and

$$\frac{q}{p} = r_2 \quad \text{if} \quad \det B = 0 .$$

So we have the following theorem.

Theorem 3.9. If $\{Y_n\}$ is a two-state recurrent Markov Chain such that $p > \frac{1}{2}$, then there are four distinct real values of r for the probability of absorption problem, $-1 < r_1 < r_2 < 1 = r_3$ and $|r_4| > 1$. Moreover, $r^2 > |r_1|$, so that in the one barrier problem the solution is of the form $f(i,k) = c_1(i)r_1{}^k + c_2(i)r_2{}^k$ and for large k is asymptotically $f(i,k) \sim c_2(i)r_2{}^k$, and

$$r_2 > \frac{q}{p} \quad \text{if} \quad \det B > 0 \quad ,$$

$$r_2 = \frac{q}{p} \quad \text{if} \quad \det B = 0 \quad ,$$

$$r_2 < \frac{q}{p} \quad \text{if} \quad \det B < 0 \; .$$

Section 3.3.

In this section we will discuss the problem of the expected duration of the walk given that the probability of absorption is 1 . In the case where we have two barriers, and we start the walk between the two barriers, the assumption that $p_i \neq 1$ and $q_i \neq 1$ for any i, assures us that the probability of absorption is 1 . In the case where we only have a barrier on the left at $(\cdot,0)$, if $p < \frac{1}{2}$ then the probability of absorption is 1 .

We will first consider the case where $p \neq \frac{1}{2}$. This will give solutions for both the one and two barrier problems. Then for the two barrier problem, we will need to consider the case where $p = \frac{1}{2}$, separately.

The difference equation which we need to solve it then as follows:

$$D(i,k) = p_i \sum_j \pi_{ij} D(j,k+1) + q_i \sum_j \pi_{ij} D(j,k-1) + 1 \qquad (39)$$

[illegible] [illegible] [illegible] is a [illegible] [illegible] [illegible] that
$p < \frac{1}{2}$. Then there are four distinct real values of [illegible] and the proba-
bility of absorption becomes, [illegible] [illegible] [illegible] and [illegible]
Moreover, [illegible] it is [illegible] the [illegible] [illegible] [illegible]
is of the form [illegible] and for large [illegible]
approximation [illegible]

[illegible] [illegible] and [illegible]

[illegible] if and [illegible]

$\gamma_2 < \frac{1}{2}$ if and [illegible]

Section 3.3

In this section we will discuss the proof of the expected dura-
tion of the walk given that the probability [illegible] of the
base state we have two barriers, and we start the walk between the two
barriers, the assumption that [illegible] and [illegible] for the two [illegible] ensures
us that the probability of absorption at [illegible]. On the other hand we only
have a barrier on the left at [illegible], if $p < q$ then the probability of
absorption is 1.

We will first consider the case where $p \neq q$. This will give solu-
tions for both the one and two barrier problems. Then for the two barr-
ier problem, we will need to consider the case where $p = q$, separately.

The difference equation which we need to solve in order to obtain

[illegible]

Note that the difference equation associated with the probability of absorption problem is the homogeneous form of the above equation. Since we have already solved the homogeneous problem, we only need to find a particular solution to (39). Then the generalized solution to (39) is the particular solution plus the general solution to the homogeneous equation.

As in the previous section we will solve by a constructive method. We will try the solution

$$D(i,k) = \sigma_i k + w_i \quad .$$

Plugging into (39) we get

$$\sigma_i k + w_i = p_i \sum_j \pi_{ij}[\sigma_j(k+1)+w_j] + q_i \sum_j \pi_{ij}[\sigma_p(k-1)+w_j] + 1$$

$$\sigma_i k + w_i = k \sum_j \pi_{ij}\sigma_j + (p_i - q_i) \sum_j \pi_{ij}\sigma_j + \sum_j \pi_{ij} w_j + 1 \quad . \tag{40}$$

If we let $k = 0$, then we get

$$w_i = (p_i - q_i) \sum_j \pi_{ij}\sigma_j + \sum_j \pi_{ij} w_j + 1 \quad . \tag{41}$$

And subtracting (41) from (40) we see that

$$\sigma_i k = k \sum_j \pi_{ij}\sigma_j$$

or

$$\sigma_i = \sum_j \pi_{ij}\sigma_j \; .$$

Then the stochastic nature of B, immediately yields $\sigma_i = \sigma$, for every i, where σ is a non-zero constant. Hence (41) becomes

$$w_i = \sigma(p_i - q_i) + \sum_j \pi_{ij} w_j + 1$$

$$\Rightarrow \sum_j \pi_{ij} w_j - w_i = \sigma(q_i - p_i) + 1 .$$

If we let $w = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$ and $J = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$

$$\Rightarrow \quad Bw - w = \sigma(Q - P)J + J \quad . \tag{42}$$

If we multiply through by π , the invariant probability distribution, we get

$$\pi Bw - \pi w = \sigma\pi(Q - P)J + \pi J \quad , \tag{43}$$

but

$$\pi B = \pi \quad \text{and} \quad \pi(a - P) = q - p$$

so (43) becomes

$$0 = \sigma(q - p) + 1$$

and if $p \neq \frac{1}{2}$, which we will assume holds unless specified otherwise, then $q - p \neq 0$. Hence $\sigma = \frac{1}{p - q}$. Then from elementary matrix theory, we know that there is a vector w , which satisfies

$$(B - I)w = \frac{1}{p - q}(Q - P)J + J \tag{44}$$

if and only if $\pi\left[\frac{1}{p - q} (Q - P)J + J\right] = 0$. But σ was picked so as to satisfy this condition, hence there is an w to satisfy (44). So we have the following theorem.

Theorem 3.10. If $p \neq \frac{1}{2}$, then a particular solution for (20), the expected duration of the walk is $D(i,k) = \frac{k}{q - p} + w_i$ where the w_i's satisfy

$$\sum \pi_{ij} w_j - w_i = \frac{q_i - p_i}{q - p} + 1 = \frac{2(p - p_i)}{1 - 2p_i} .$$

The general solution is

$$D(i,k) = \frac{k}{q - p} + w_i + f(i,k) ,$$

where $f(i,k)$ is the general solution to the homogeneous cos, which was given in Section 3.1. Note that the form of $f(i,k)$ depends on the nature of the roots of $\det A_r$.

We now wish to consider the special case where $p = \frac{1}{2}$. From Theorem 3.8 we know that $r_0 = 1$ is a double root. We will now show that when $p = \frac{1}{2}$, then a particular solution to (39) is

$$D(i,k) = \sigma k^2 + 2\sigma\mu_i k + \beta_i . \tag{45}$$

Plugging into (39) we get

$$\sigma k^2 + 2\sigma\mu_i k + \beta_i = p_i \sum_j \pi_{ij}[\sigma(k+1)^2 + 2\sigma\mu_j(k+1) + \beta_j]$$

$$+ q_i \sum_j \pi_{ij}[\sigma(k-1)^2 + 2\sigma\mu_j(k-1) + \beta_j] + 1 \tag{46}$$

$$= \sigma k^2 + 2\sigma k(p_i - q_i) + \sigma + 2\sigma k \sum_j \pi_{ij}\mu_j + 2\sigma(p_i - q_i) \sum_j \pi_{ij}\mu_j + \sum_j \pi_{ij}\beta_j + 1 .$$

Since true for all k, letting $k = 0$ we get

$$\beta_i = \sigma + 2\sigma(p_i - q_i) \sum_j \pi_{ij}\mu_{ij} + \sum_j \pi_{ij}\beta_j + 1 . \tag{47}$$

Similarly we have that

$$2\sigma\mu_i k = 2\sigma k(p_i - q_i) + 2\sigma k \sum_j \pi_{ij}\mu_j$$

or

$$\mu_i = (p_i - q_i) + \sum_j \pi_{ij}\mu_j \tag{48}$$

$$\Rightarrow \sum \pi_{ij}\mu_j - \mu_i = q_i - p_i \quad \forall\ i = 1,2,\ldots,n$$

or in matrix notation

$$(B - I)\mu = (Q - P)J \tag{49}$$

but note then that this is merely equation (19) in Section 3.1 . (i.e., $\Gamma = I$, $Bc = J$ and $(Q - P) = \tilde{\Gamma}$) .

So we know that there is a μ which satisfies the above. Therefore, if we can show that there are values for the β_i's, which satisfy (47), then we will have shown that (45) is a particular solution to the expected duration of the walk.

We will then rewrite (47) as

$$\sum_j \pi_{ij}\beta_j - \beta_i = 2\sigma(q_i - p_i) - (\sigma + 1), \quad i = 1,2,\ldots,n \tag{50}$$

or in matrix notation

$$(B - I)\beta = 2\sigma(Q - P)B\mu - (\sigma + 1)J\ . \tag{51}$$

Again we only need show that

$$\pi[2\sigma(Q - P)B\mu - (\sigma + 1)J] = 0$$

this is equivalent to

$$2\sigma\pi(Q - P)B\mu - (\sigma + 1) = 0$$

which is equivalent to

$$w_{ik} = \sum \eta_{ij} w_{jk} \quad \text{for } i < q_i$$

or

$$A_{ij}(x) = k_{ij} + \sum_j r_{ij} w_j \qquad (48)$$

$$\sum w_{ij} r_{jk} = q_i - q_k, \quad i,k = 1,2,\ldots,n$$

or in matrix notation

$$(R - I)w = (q - k)I \qquad (49)$$

but note that this is merely equation (19) in section 3.3 (i.e., $r = R$, $Rb = d$ and $(s - T) = q$).

So we know that there is a w_i which satisfies the above. Therefore, if we can show that there are values for the q_i's which satisfy (47), then we will have shown that (48) is a particular solution to the expected duration of the walk.

We will then rewrite only as

$$\sum r_{ij} q_j - q_i = \text{const}_i = w_i(r - (n + 1)), \quad i = 1,2,\ldots,n \qquad (50)$$

or in matrix notation

$$(R - I)q = \Lambda(q - I)w_i - (q + 1)I \qquad (51)$$

Again we only need show that

$$R(w)(q - I)w_i - (q + 1)I = 0$$

this is equivalent to

$$R(w)(q - I)w_i - (q + 1) = 0$$

which is equivalent to

$$\sigma = \begin{cases} -1 & \text{if } 2\pi(Q-P)B\mu = 0 \\ \dfrac{1}{2\pi(Q-P)B\mu - 1} & \text{if } 2\pi(Q-P)B\mu \neq 0 \end{cases} . \qquad (52)$$

Then, since π, Q, P, B, and μ are known to us, we can define σ to assure that values of β_i's exist to satisfy (47). So we have the following theorem.

<u>Theorem 3.11</u>. If $p = \frac{1}{2}$, then a particular solution to the mean time to absorption problem for the two barrier problem is

$$D(i,k) = \sigma k^2 + 2\sigma\mu_i k + \beta_i$$

where σ, μ_i and β_i are as above.

CHAPTER IV

NUMERICAL EXAMPLES

In this chapter we will consider numerical examples of RWRE's . First we will consider four examples in which $\vec{p}$ and π will remain fixed but B will vary. We will calculate the probability of absorption for the two barrier problem in all four examples and compare them to one another as well as to the probability of absorption for the classical random walk where $p = \pi \cdot \vec{p}$.

The second set of examples will be where $\vec{p}$ and B remain fixed but we look at the two barrier problem for two different sets of barriers:

$$\{(\cdot,0), (\cdot,15)\} \text{ and } \{(\cdot,0), (\cdot,25)\} .$$

In these examples p will equal $\frac{1}{2}$, and we will solve the probability of absorption problem as well as the expected duration of the walk for both sets of barriers. We will compare these results to the classical case.

The third example will be of a three state control process in which we get complex roots. For this example we will solve the one barrier probability of absorption problem.

The next example will be of a five state control process in which we will solve the two barrier probability of absorption problem.

The last example will be of a RWRE in which the random walk is a generalized random walk and can take one or two steps to the left or

CHAPTER IV

NUMERICAL EXAMPLES

In this chapter we will consider numerical examples of Chapter 3. First we will consider four examples in which p and r will remain fixed but N will vary. We will calculate the probability of absorption for the two barrier problem in all four examples and compare them to one another as well as to the probability of absorption for the classical random walk where $p = r = \frac{1}{2}$.

The second set of examples will be where p and N remain fixed and we look at the two barrier problem for two different sets of barriers

$$\{\{1,2\}, \{-1,-2\}\} \text{ and } \{\{1,2\}, \{-2,-3\}\} .$$

In these examples r will equal $\frac{1}{2}$, and we will solve the probability of absorption problem as well as the expected duration of the walk for both sets of barriers. We will compare these results to the classical case.

The third example will be of a three state Markov process in which we get complex roots. For this example we will solve the one barrier probability of absorption problem.

The next example will be of a five state Markov process in which we will solve the two barrier probability of absorption problem.

The last example will be of a case in which the random walk is a generalized random walk and we find the expected duration of the walk.

right. That is

$$p_2^{(i)} = P(Z_{n+1} = 2 \mid (Y_n, Z_n) = (i,0))$$

$$p_1^{(i)} = P(Z_{n+1} = 1 \mid (Y_n, Z_n) = (i,0))$$

$$q_1^{(i)} = P(Z_{n+1} = -1 \mid (Y_n, Z_n) = (i,0))$$

$$q_2^{(i)} = P(Z_{n+1} = -2 \mid (Y_n, Z_n) = (i,0)) ,$$

where $p_2^{(i)}$, $p_1^{(i)}$, $q_1^{(i)}$, and $q_2^{(i)}$ are all greater than zero for $i = 1,2$, and $p_2^{(i)} + p_1^{(i)} + q_1^{(i)} + q_2^{(i)} = 1$ for $i = 1,2$.

Before proceeding with the examples a few comments need to be made about the method used to obtain the roots r_ℓ and their corresponding vectors c_2 . One method that could be used is to treat $\det A_r$ as a function of r and then to use Horner's method or a combination of Newton's method with a bisection method to track down the zeros. Another method would be to expand $\det A_r$ into a polynomial of degree $2n$ and then solve by a Newton Raphson bisection method. It should be noted that due to the magnitude of the coefficients that an attempt to track down the zeros by use of Newton's method alone will not always converge.

A much more convenient method is to note that we can rewrite $(\Gamma B - I)c = 0$ as $r^2 PBc - rIc + QBc = 0$ and then we can write a companion matrix for the above in the form

$$r \begin{bmatrix} -I & PB \\ I & 0 \end{bmatrix} \begin{bmatrix} c \\ rc \end{bmatrix} = \begin{bmatrix} -QB & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} c \\ rc \end{bmatrix} . \qquad (53)$$

We then observe that if B^{-1} exists, (i.e. det $B \neq 0$) then,

$\begin{bmatrix} -I & PB \\ I & 0 \end{bmatrix}^{-1}$ exists and we can rewrite the above as

$$r\begin{pmatrix} r \\ rc \end{pmatrix} = \begin{bmatrix} -I & PB \\ I & 0 \end{bmatrix}^{-1} \begin{bmatrix} -QB & 0 \\ 0 & I \end{bmatrix} \begin{pmatrix} c \\ rc \end{pmatrix} .$$

Then we will solve the above by using Matlab, a computer package developed by Cleve Moler. If B is singular then there are other numerical methods which can be used bo solve (53). Since all the examples in this chapter are such that det $B \neq 0$, we have used Matlab to find the solutions to (53).

In the first four examples $\vec{p}$ and π will be fixed as below,

$$\vec{p} = [.9,.4] \quad \pi = [\tfrac{1}{2}, \tfrac{1}{2}] .$$

Hence $\pi\vec{p} = .65 = p$. We will then solve the probability of absorption problem for the following transition matrices.

$$B_1 = \begin{bmatrix} .9 & .1 \\ .1 & .9 \end{bmatrix} \qquad B_2 = \begin{bmatrix} .51 & .49 \\ .49 & .51 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} .49 & .51 \\ .51 & .49 \end{bmatrix} \quad \text{and} \quad B_4 = \begin{bmatrix} .1 & .9 \\ .9 & .1 \end{bmatrix} .$$

The following table then gives the values of r for each of the four examples.

Then we will solve the above by using Matlab, a computer package developed by Cleve Moler. If A is singular then there are other numerical methods which can be used to solve (53). Since all the examples in this chapter are such that $\det A \neq 0$, we have used Matlab to find the solutions to (53).

In the first four examples $\bar{\pi}$ and r will be fixed as below.

$$\bar{\pi} = [illegible], \quad r = [illegible]$$

Hence [illegible]. We will then solve the probability of absorption problem for the following transition matrices.

$P_1 =$ [illegible], $P_2 =$ [illegible]

$P_3 =$ [illegible] and $P_4 =$ [illegible]

The following table then gives the values of r for each of the four examples.

Table 4.1

Roots

Example	r_1	r_2	r_3	r_4
1	.0960040608	.8020821583	1	2.16441378
2	.0033939113	.5423998672	1	90.53753955
3	-.0034638897	.5345963563	1	-90.00335469
4	-.2384528295	.4263696772	1	-1.63930574

It is of interest to note that the stationary solution is $\frac{q}{r} = .538461539$. In fact, if $\det B = 0$ then we would have $r_1 = 0$ and $r_2 = \frac{q}{p}$.

Tables 4.2, 4.3, 4.4, and 4.5 then give the solutions for the probability of absorption problem when we have barriers at $(\cdot,0)$ and $(\cdot,15)$.

It should be noted for example one, that since $\det B_1 \gg 0$, $f(k)$ the solution for the classical random walk should be smaller than $f(1,k)$ and $f(2,k)$ for sufficiently large k. In Table 4.2 we see that this is true for $k \geq 3$. For B_2, which is very close to zero, we see in Table 4.3 that $f(1,k) < f(k) < f(2,k)$ as would be expected. In Table 4.4 for B_3, which again is close to zero, we see that $f(1,k) < f(k) < f(2,k)$. But for example four, since $\text{Det } B_4 \ll 0$, we have that $f(k)$ is larger than $f(1,k)$ and $f(2,k)$ for $k \geq 2$.

Table 4.2

k	f(1,k)	f(2,k)	f(k)
0	1	1	1
1	.3555	.8580	.5384
2	.2388	.6900	.2899
3	.1835	.5501	.1560
4	.1430	.4374	.0840
5	.1107	.3470	.0452
6	.0850	.2744	.0243
7	.0643	.2162	.0130
8	.0477	.1695	.0070
9	.0344	.1319	.0037
10	.0238	.1017	.0020
11	.0154	.0771	.0010
12	.0088	.0566	.0055
13	.0039	.0388	.0002
14	.0007	.0212	.0001
15	0	0	0

Table 4.3

k	f(1,k)	f(2,k)	f(k)
0	1	1	1
1	.3617	.7179	.5384
2	.1955	.3899	.2899
3	.1060	.2114	.1560
4	.0574	.1146	.0840
5	.0311	.0621	.0452
6	.0168	.0337	.0243
7	.0091	.0182	.0130
8	.0049	.0098	.0070
9	.0026	.0053	.0037
10	.0014	.0028	.0020
11	6.92×10^{-4}	.0015	.0010
12	3.28×10^{-4}	7.56×10^{-4}	.0005
13	1.31×10^{-4}	3.63×10^{-4}	.0002
14	2.45×10^{-5}	1.49×10^{-4}	.0001
15	0	0	0

Table 4.4

k	f(1,k)	f(2,k)	f(k)
0	1	1	1
1	.3600	.7141	.5384
2	.1930	.3811	.2899
3	.1032	.2037	.1560
4	.0551	.1089	.0840
5	.0294	.0582	.0452
6	.0157	.0310	.0243
7	.0083	.0166	.0130
8	.0044	.0088	.0070
9	.0023	.0047	.0037
10	.0012	.0025	.0020
11	6.05×10^{-4}	.0013	.0010
12	2.844×10^{-4}	6.43×10^{-4}	.0005
13	1.13×10^{-4}	3.05×10^{-4}	.0002
14	2.11×10^{-5}	1.24×10^{-4}	.0001
15	0	0	0

Table 4.5

k	f(1,k)	f(2,k)	f(k)
0	1	1	1
1	.2979	.6658	.5384
2	.1576	.2268	.2899
3	.0599	.1103	.1560
4	.0273	.0438	.0840
5	.0112	.0194	.0452
6	.0049	.0081	.0243
7	.0021	.0035	.0130
8	8.8×10^{-4}	.0015	.0070
9	3.72×10^{-4}	6.31×10^{-4}	.0037
10	1.57×10^{-4}	2.67×10^{-4}	.0020
11	6.53×10^{-5}	1.12×10^{-4}	.0010
12	2.58×10^{-5}	4.61×10^{-5}	.0005
13	9.62×10^{-6}	1.76×10^{-5}	.0002
14	1.68×10^{-6}	6.25×10^{-6}	.0001
15	0	0	0

In the next set of examples we will let

$$B = \begin{bmatrix} \frac{6}{7} & \frac{1}{7} \\ \frac{1}{7} & \frac{6}{7} \end{bmatrix} \quad \text{and} \quad \vec{P} = \begin{bmatrix} \frac{3}{7} \\ \frac{4}{7} \end{bmatrix} .$$

We then have that the roots are $r_1 = \frac{2}{5}$, $r_2 = 1$, $r_3 = 1$, and $r_4 = \frac{5}{2}$. We must then find a solution to

$$(B - I)\alpha = (Q - P)Bc \quad .$$

It is easy to show that $\alpha = \begin{pmatrix} 0 \\ c_2(1) \end{pmatrix}$ will satisfy the above. Then using Matlab we solve the probability of absorption problem and mean time to absorption problem with barriers at $(\cdot,0)$ and $(\cdot,15)$. These results are given in Table 4.6. Tables 4.7 and 4.8 contain the results for barriers at $(\cdot,0)$ and $(\cdot,25)$. In Tables 4.6 and 4.8 it should be noted that since det $B > 0$ the mean to absorption is greater for the classical random walk than for the RWRE.

Table 4.6

k	f(1,k)	f(2,k)	f(k)
0	1	1	1
1	.94937	.9100	.93333
2	.88975	.83463	.86667
3	.82652	.76510	.80000
4	.76186	.69792	.73333
5	.69661	.63167	.66667
6	.63114	.56580	.60000
7	.56556	.50008	.53333
8	.49992	.43444	.46667
9	.43420	.36886	.40000
10	.36833	.30339	.3333
11	.30208	.23814	.26667
12	.23490	.17348	.20000
13	.16537	.11025	.1333
14	.09000	.05063	.06667
15	0	0	0

k	D(1,k)	D(2,k)	D(k)
0	0	0	0
1	10.50	16.85	14
2	21.05	28.86	26
3	30.54	37.83	36
4	38.50	44.51	44
5	44.77	49.17	50
6	49.28	51.95	54
7	51.98	52.88	56
8	52.88	51.98	56
9	51.95	49.28	54
10	49.17	44.77	50
11	44.51	38.50	44
12	37.83	30.54	36
13	28.86	21.05	26
14	16.85	10.50	14
15	0	0	0

Table 4.7

k	f(1,k)	f(2,k)	f(k)
0	1	1	1
1	.96943	.94566	.96
2	.93343	.90015	.92
3	.89526	.85817	.88
4	.85622	.81761	.84
5	.81683	.77761	.80
6	.77730	.73784	.76
7	.73771	.69815	.72
8	.69810	.65851	.68
9	.65849	.61887	.64
10	.61887	.57925	.60
11	.57924	.53962	.56
12	.53962	.50000	.52
13	.50000	.46038	.48
14	.46038	.42076	.44
15	.42075	.38113	.40
16	.38113	.34151	.36
17	.34149	.30190	.32
18	.30185	.26229	.28
19	.26216	.22270	.24
20	.22239	.18317	.20
21	.18239	.14378	.16
22	.14183	.10474	.12
23	.09985	.06657	.08
24	.05434	.03057	.04
25	0	0	0

Table 4.8

k	D(1,k)	D(2,k)	D(k)
0	0	0	0
1	17.50	29.30	24
2	36.30	51.72	46
3	54.52	70.31	66
4	71.43	86.28	84
5	86.72	100.10	100
6	100.28	111.99	114
7	112.06	122.00	126
8	122.03	130.18	136
9	130.19	136.54	144
10	136.54	141.08	150
11	141.08	143.80	154
12	143.80	144.71	156
13	144.71	143.80	156
14	143.80	141.08	154
15	141.08	136.54	150
16	136.54	130.19	144
17	130.18	122.03	136
18	122.00	112.06	126
19	111.99	100.28	114
20	100.10	86.72	100
21	86.28	71.43	84
22	70.31	54.52	66
23	51.72	36.30	46
24	29.30	17.50	24
25	0	0	0

In the next example we let

$$B = \begin{bmatrix} .1 & .2 & .7 \\ .6 & .1 & .3 \\ .4 & .5 & . \end{bmatrix} \quad \text{and} \quad \vec{P} = \begin{bmatrix} .6 \\ .3 \\ .7 \end{bmatrix}$$

The roots are then

$r_1 = -.1425 + .1238i$, $r_4 = 1$,

$r_2 = -.1425 - .1238i$, $r_5 = -4.3431 + 2.1148i$,

$r_3 = .8018$, $r_6 = -4.3431 - 2.1148i$.

We then solve the one barrier probability of absorption problem and find that the solutions are:

$$f(1,k) = \mathrm{Re}(.0588\ r_1{}^k) + \mathrm{Im}(.0269\ r_2{}^k) + .9731\ r_3{}^k ,$$

$$f(2,k) = \mathrm{Re}((-.0498 - .0722i)r_1{}^k) + \mathrm{Im}((-.023 - .033i)r_2{}^k) + 1.095\ r_3{}^k$$

$$f(3,k) = \mathrm{Re}((-.025 + .045i)r_1{}^k) + \mathrm{Im}((-.012 + .0205i)r_2{}^k) + .966\ r_3{}^k .$$

Numerical results are given in Table 4.9. It should be noted from the table that for $k \geq 8$, $f(k)$ is greater than $f(i,k)$, $i = 1,2,3$.

Table 4.9

k	f(1,k)	f(2,k)	f(3,k)	f(k)
0	1	1	1	1
1	.7837	.8813	.7698	.8112
2	.6237	.7065	.6218	.6580
3	.5201	.5640	.4984	.5338
4	.4022	.4528	.3995	.4330
5	.3225	.3629	.3204	.3512
6	.2586	.2910	.2569	.2849
7	.2074	.2333	.2060	.2311
8	.1663	.1871	.1652	.1875
9	.1333	.1500	.1324	.1521
10	.1069	.1203	.1062	.1234
11	.0857	.0965	.0852	.1001
12	.0687	.0773	.0683	.0812
13	.0551	.0620	.0549	.0658
14	.0442	.0497	.0439	.0534
15	.0354	.0399	.0352	.0433
20	.0117	.0132	.0117	.0152
25	.0039	.0044	.0039	.0053
30	.0013	.0015	.0013	.0019
35	4.28×10^{-4}	4.81×10^{-4}	4.25×10^{-4}	6.59×10^{-4}
40	1.42×10^{-4}	1.59×10^{-5}	1.41×10^{-4}	2.32×10^{-4}
45	4.70×10^{-5}	5.29×10^{-5}	4.67×10^{-5}	8.13×10^{-5}
50	1.56×10^{-5}	1.75×10^{-5}	1.55×10^{-5}	2.86×10^{-5}

In the next example we let

$$B = \begin{bmatrix} .4 & .2 & .1 & .1 & .2 \\ .2 & .6 & .1 & .05 & .05 \\ .1 & .1 & .5 & .2 & .1 \\ .1 & .05 & .2 & .45 & .2 \\ .2 & .05 & .1 & .2 & .45 \end{bmatrix} \quad \text{and} \quad \vec{P} = \begin{bmatrix} .9 \\ .8 \\ .6 \\ .4 \\ .3 \end{bmatrix} .$$

The roots are then

$$\begin{aligned} r_1 &= .02383 , & r_6 &= 1 , \\ r_2 &= .11066 , & r_7 &= 2.65647 , \\ r_3 &= .14418 , & r_8 &= 3.9035 , \\ r_4 &= .20900 , & r_9 &= 7.3083 , \\ r_5 &= .74335 , & r_{10} &= 14.4784 . \end{aligned}$$

It should be noted, that five of the roots have magnitude less than 1 , as would be required if we wanted to solve the one barrier problem. Table 4.10 then gives the numerical results for the two barrier problem with barriers at $(\cdot,0)$ and $(\cdot,15)$.

Table 4.10

k	f(1,k)	f(2,k)	f(3,k)	f(4,k)	f(5,k)	f(k)
0	1	1	1	1	1	1
1	.5158	.5171	.7118	.8284	.8689	.6659
2	.3695	.3448	.5141	.6287	.6548	.4432
3	.2710	.2484	.3767	.4671	.4862	.2947
4	.1988	.1814	.2768	.3449	.3593	.1957
5	.1452	.1321	.2031	.2539	.2646	.1297
6	.1053	.0956	.1484	.1861	.1941	.0857
7	.0757	.0685	.1077	.1358	.1417	.0564
8	.0537	.0484	.0775	.0984	.1027	.0368
9	.0374	.0334	.0550	.0705	.0738	.0238
10	.0252	.0223	.0383	.0499	.0523	.0151
11	.0161	.0140	.0259	.0345	.0363	.0093
12	.0094	.0080	.0165	.0230	.0243	.0054
13	.0044	.0037	.0095	.0142	.0154	.0029
14	7.95×10^{-4}	.0011	.0040	.0072	.0082	.0011
15	0	0	0	0	0	0

The last numerical example is of a generalized random walk in a random environment. We will let

$$B = \begin{bmatrix} .8 & .2 \\ .6 & .4 \end{bmatrix} \quad \text{and}$$

$p_2^{(1)} = .25$, $p_1^{(1)} = .4$, $q_1^{(1)} = .25$, $q_2^{(1)} = .1$,

$p_2^{(2)} = .2$, $p_1^{(2)} = .2$, $q_1^{(2)} = .5$, and $q_2^{(2)} = .1$.

If we then use the notation,

$$P_2 = \begin{bmatrix} .25 & 0 \\ 0 & .2 \end{bmatrix}, \quad P_1 = \begin{bmatrix} .4 & 0 \\ 0 & .2 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} .25 & 0 \\ 0 & .5 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} .1 & 0 \\ 0 & .1 \end{bmatrix},$$

we find that the values of r and c which will solve the difference equation associated with the probability of absorption problem must also satisfy the following:

$$r^4 P_2 Bc + r^3 P_1 Bc - r^2 Ic + rQ_1 Bc + Q_2 Bc = 0 .$$

We can then form a companion matrix for the above and solve as before. We then find that

$$r_1 = -.1966 , \qquad r_5 = 1 ,$$

$$r_2 = -.1027 , \qquad r_6 = -3.005 ,$$

$$r_3 = .1910 , \qquad r_7 = 4.301 ,$$

$$r_4 = .7280 , \qquad r_8 = -5.515 .$$

To solve the two barrier problem, we must set up four barriers. These are then: $(\cdot,0)$, $(\cdot,1)$, $(\cdot,21)$, and $(\cdot,22)$. The boundary conditions are then: $f(1,0) = 1$, $f(2,0) = 1$, $f(1,1) = 1$, $f(2,1) = 1$, $f(1,21) = 0$, $f(2,21)$, $= 0$, $f(1,22) = 0$, and $f(2,22) = 0$. So we have eight equations in eight unknowns. The numerical results are given in Table 4.11.

Table 4.11

k	f(1,k)	f(2,k)
0	1	1
1	1	1
2	.6371	.7775
3	.4741	.5755
4	.3413	.4204
5	.2485	.3060
6	.1804	.2224
7	.1309	.1616
8	.0949	.1172
9	.0687	.0849
10	.0496	.0614
11	.0357	.0442
12	.0256	.0318
13	.0182	.0228
14	.0128	.0162
15	.0089	.0113
16	.0061	.0078
17	.0040	.0053
18	.0025	.0034
19	.0014	.0021
20	.0007	.0011
21	0	0
22	0	0

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