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Abstract

In this paper, we consider the valuation of a CDS (credit default swap) contract when the reference asset is assumed to follow a regime-switching model with the volatility allowed to jump among different states. Our motivation originates from empirical evidence demonstrating the existence of regime-switching in real markets. The default probability is analytically derived first, based on which a closed-form formula for the CDS price is obtained so that it can be easily implemented for practical purposes. Finally, numerical experiments are carried out to show quantitatively some properties of the CDS price under the regime-switching model.

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Analytically pricing credit default swaps under a regime switching model

Wenting Chen ^{*} Xinjiang He [†] Xinzi Qiu [‡]

Abstract

In this paper, we consider the valuation of a CDS (credit default swap) contract when the reference asset is assumed to follow a regime switching model with the volatility allowed to jump among different states. Our motivation originates from empirical evidence demonstrating the existence of regime switching in real markets. The default probability is analytically derived first, based on which a closed-form formula for the CDS price is derived so that it can be easily implemented for practical purposes. Finally, numerical experiments are carried out to show quantitatively some properties of the CDS price under the regime-switching model.

AMS(MOS) subject classification.

Keywords. CDS, default probability, regime switching, closed-form analytical solution.

1 Introduction

Nowadays, credit derivatives are becoming increasingly important because they can make the risk tradable so that the credit risk can be effectively managed. Among these, the

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CDS (credit default swap) is one of the most important and basic types. In a typical CDS contract, there are actually three parties involved, two of which are the so-called CDS buyer and seller and they enter into the CDS contract against the default of the reference asset belonging to the third party. In specific, the CDS buyer needs to make periodic payments to the seller, and in exchange, its seller has the obligation to compensate the buyer in case of a default event occurs. In other words, a CDS contract can transfer the credit risk from its buyer to the seller who is willing to undertake such risk. With the growing volume of CDSs trading in financial markets, a large amount of research interests have been put into the accurate determination of the CDS price.

One of the most important factors that has significant impacts on the accurate determination of the CDS price is the choice of the default model. In the literature, two main kinds of models, namely, the reduced-form models and the structural models, are widely adopted. The formers are introduced in [1, 14], and are then further developed by Lando [15], Madan & Unal [16] and a number of other authors. These models are very popular as they are mathematically appealing with the probability of default being able to be extracted from the historical data and the CDS price straightforwardly determined. However, it should be noted that one of the main disadvantages of these models is the failure in capturing the wide range of default correlations between different companies.

Structural models, as another alternative, use the evolution of the reference asset to determine the time when the default would occur, providing both intuitive economic interpretations and endogenous explanations of credit defaults. The very first model in this category is established by Merton [17], who made an assumption that a default event will be triggered if the value of the reference asset drops below a certain level. Unfortunately, despite the popularity of Merton's approach, the assumption that the reference asset price follows a geometric Brownian motion is inappropriate because it will lead to a significantly smaller default probability [18]. Consequently, a number of modifications have emerged. For example, a pure jump process is considered in [6] with the reference asset price modeled

by a Poisson process, whereas in [19], the asset price is assumed to follow a jump-diffusion process. Recently, He & Chen [4, 12] adopted a generalized mixed fractional Brownian motion and a multiscale stochastic volatility model to reflect different mechanics of the asset returns and derived corresponding closed-form pricing formulae for the CDS contract.

It should be pointed out that most of the models mentioned above are unable to capture the changing beliefs of investors towards the states of real markets, which prompts the development of the regime-switching model [11]. This particular model was introduced by Hamilton [10], who assumed that the volatility of the underlying price follows a Markov chain so that it can vary according to different states. This so-called regime-switching model becomes increasingly popular among researchers and market practitioners because a lot of empirical evidence has already suggested that the dynamics of the underlying price are better captured by allowing volatility to switch between different states [2, 5, 8, 9, 13]. In this paper, we consider the valuation of the CDS under the regime-switching model and derive a closed-form analytical formula for the CDS price. Through numerical experiments, the influence of introducing regime switching into the geometric Brownian motion is shown, and the impacts of different parameters on the CDS price are quantitatively discussed as well.

The rest of the paper is organized as follows. In Section 2, the default probability is analytically derived, based on which the closed-form solution for the CDS price is obtained under the regime-switching model. In Section 3, numerical examples and discussions are presented, followed by some concluding remarks in the last section.

2 Closed-form pricing formula

In this section, a closed-form analytical solution for the price of the CDS under the regime-switching model is presented. We shall first derive the default probability of the reference asset, based on which the CDS price is determined by analyzing cash flows.

2.1 Default probability under the regime switching model

One of the most important factors in the CDS contract is the default probability of the reference asset, which represents the likelihood of a default of the reference asset taking place within a certain period. In fact, this is also a key step in the derivation of the price of the CDS contract. In this subsection, we shall consider the derivation of this important factor.

We assume that the reference asset S_t follows a regime-switching model under the risk-neutral measure \mathbb{Q} . In this model, the volatility is allowed to jump among different states following a Markov chain ¹. In specific, S_t satisfies

$$\frac{dS_t}{S_t} = rdt + \sigma_{X_t}dW_t. \quad (2.1)$$

Here, W_t is a standard Brownian motion independent of the Markov chain X_t , which is defined as

$$X_t = \begin{cases} (1, 0)', & \text{when the economy is believed to be in State 1,} \\ (0, 1)', & \text{when the economy is believed to be in State 2,} \end{cases}$$

with v' denoting the transpose of the vector v . The transition probability between the two states of the Poisson process is

$$P(t_{ij} > t) = e^{-\lambda_{ij}t}, \quad \text{for } i, j = 1, 2, i \neq j.$$

Here, λ_{ij} is the transition rate from State i to j , and t_{ij} is the time spent in State i before transferring to State j . σ_{X_t} is the volatility controlled by the Markov chain X_t . It is equal to σ_1 and σ_2 when the economy is in State 1 and State 2, respectively. We remark that if the two transition rates are both equal to zero, the regime-switching model will degenerate

¹For the illustration purpose, we will only discuss the two-state Markov chain and the extension to the case of arbitrary but finite states should be quite straightforward.

to the standard Black-Scholes model.

Let D be the default level, which means that the default of the reference asset will be triggered if its value is less than or equal to D at T . This implies that the default probability of the reference asset is equivalent to the value of $P^{\mathbb{Q}}(S_T \leq D)$, where $P^{\mathbb{Q}}$ is the probability under the measure \mathbb{Q} . If we make the transformation of $y_T = \ln(S_T)$, it is straightforward that

$$P^{\mathbb{Q}}(S_T \leq D) = P^{\mathbb{Q}}(y_T \leq \ln D) = \int_{-\infty}^{\ln D} p(y) dy, \quad (2.2)$$

where $p(y)$ is the probability density function of y_T . According to the relationship between the distribution function and the characteristic function, we obtain

$$P^{\mathbb{Q}}(y_T \leq \ln D) = \frac{1}{2} - \frac{1}{\pi} \int_0^{+\infty} \text{Real} \left[\frac{e^{j\phi \ln D} f(\phi; T, 0, y_0, X_0)}{j\phi} \right] d\phi, \quad (2.3)$$

with j being the imaginary unit. Also, $f(\phi; T, s, y_s, X_s)$ is denoted as the characteristic function of y_T at the current time $s (< T)$. In fact, (2.3) implies that once the particular characteristic function is successfully derived, the analytical formula for the no-default probability can be worked out straightforwardly, as will be presented in the following theorem.

Theorem 1 *If the reference asset price S_t follows (2.1), then the characteristic function of the log-price y_T is equal to*

$$f(\phi; T, s, y_s, X_s) = e^{rj\phi(T-s) + j\phi y_s} \langle e^M X_s, I \rangle, \quad (2.4)$$

where $I = (1, 1)'$, $\langle \cdot, \cdot \rangle$ represents the inner product of two vectors, and

$$M = \begin{bmatrix} -\frac{1}{2}(j\phi + \phi^2)\sigma_1^2(T-s) - \lambda_{12}(T-s) & \lambda_{21}(T-s) \\ \lambda_{12}(T-s) & -\frac{1}{2}(j\phi + \phi^2)\sigma_2^2(T-s) - \lambda_{21}(T-s) \end{bmatrix}. \quad (2.5)$$

Proof.

Let $\mathbb{F}_t^Y = \{\mathcal{F}_u^Y, s \leq u \leq t\}$ and $\mathbb{F}_t^X = \{\mathcal{F}_u^X, s \leq u \leq t\}$ be the natural filtrations generated by the Brownian motion and the Markov chain, respectively, from the current time s to time t . According to the definition of the characteristic function, we have

$$f(\phi; T, s, y_s, X_s) = E^Q[e^{j\phi y_T} | \mathbb{F}_s^Y, \mathbb{F}_s^X], \quad (2.6)$$

where Y is the log of the asset price and X is the Markov chain. It should be pointed out that the introduction of the Markov chain has added additional difficulty in calculating explicitly the right hand side of (2.6). However, with the division of the task into two steps, we still have managed to find out the explicit form of f . The first step is to introduce the conditional characteristic function of y_t , which is specified as

$$g(\phi; T, s, y_s) = E^Q[e^{j\phi y_T} | \mathbb{F}_s^Y, \mathbb{F}_T^X], \quad (2.7)$$

with all the information of the Markov chain X_t in the time period $t \in [s, T]$ being given. Then, the characteristic function $f(\phi; t, y_0, X_0)$ can be worked out in the second step by taking the expectation of the conditional characteristic function with respect to the Markov chain, i.e.,

$$f(\phi; T, s, y_s, X_s) = E^Q[g(\phi; T, s, y_s) | \mathbb{F}_s^X], \quad (2.8)$$

with the help of the tower rule for the expectation. In the following, we shall work out the two stages in details.

In the first step, by using the rule of the risk-neutral pricing theory, the PDE system governing the conditional characteristic function $g(\phi; T, s, y_s)$ can be derived as

$$\begin{cases} \frac{\partial g}{\partial s} + \frac{1}{2}\sigma_s^2 \frac{\partial^2 g}{\partial y^2} + (r - \frac{1}{2}\sigma_s^2) \frac{\partial g}{\partial y} = 0, \\ g|_{s=T} = e^{j\phi y_T}, \end{cases} \quad (2.9)$$

where σ_s is a time-dependent parameter and is equal to $\langle \sigma_v, X_s \rangle$, with $\sigma_v = (\sigma_1, \sigma_2)'$. To seek a solution of (2.9), we assume that $g(\phi; T, s, y_s)$ is in the form of

$$g(\phi; T, s, y_s) = e^{C(\phi; T, s) + j\phi y_s}. \quad (2.10)$$

The substitution of (2.10) into (2.9) yields the following ODE (ordinary differential equation) system

$$\begin{cases} \frac{\partial C}{\partial s} - \frac{1}{2}\sigma_s^2\phi^2 + j\phi(r - \frac{1}{2}\sigma_s^2) = 0, \\ C|_{s=T} = 0. \end{cases} \quad (2.11)$$

Clearly, $C(\phi; T, s)$ can be derived by direct integration as

$$\begin{aligned} C(\phi; T, s) &= rj\phi(T-s) - \frac{1}{2}(j\phi + \phi^2) \int_s^T \sigma_t^2 dt \\ &= rj\phi(T-s) - \frac{1}{2}(j\phi + \phi^2) \int_s^T \langle \sigma_v^2, X_t \rangle dt, \end{aligned} \quad (2.12)$$

After the derivation of $g(\phi; T, s, y_s)$, we now turn to the second step to determine f . By substituting (2.12) into (2.8), we obtain

$$\begin{aligned} f(\phi; T, s, y_s, X_s) &= E^Q[e^{rj\phi(T-s) - \frac{1}{2}(j\phi + \phi^2) \int_s^T \langle \sigma_v^2, X_t \rangle dt + j\phi y_s} | \mathbb{F}_s^X] \\ &= e^{rj\phi(T-s) + j\phi y_s} E^Q[e^{-\frac{1}{2}(j\phi + \phi^2) \int_s^T \langle \sigma_v^2, X_t \rangle dt} | \mathbb{F}_s^X]. \end{aligned} \quad (2.13)$$

By using a similar approach as adopted [3], the only unknown term contained in (2.13) can be worked out explicitly as

$$E^Q[e^{-\frac{1}{2}(j\phi + \phi^2) \int_s^T \langle \sigma_v^2, X_t \rangle dt} | X_s] = \langle e^M X_s, I \rangle. \quad (2.14)$$

For simplicity, the proof is left in the appendix for interested readers. This has completed the proof of this theorem.

The derived analytical expression of the default probability has now paved the way to calculate the CDS price, the details of which will be illustrated in the following subsection.

2.2 Valuation of the CDS contract

With the analytical expression of the default probability available, we shall, in this subsection, derive closed-form analytical expression for the CDS contract. Unlike most financial derivatives, the price of the CDS contract refers to the regular amount that the buyer needs to pay to the seller, and is often measured by a percentage of the notional value of the reference asset.

In order to determine the CDS price, it is necessary to analyze the cash flows between the buyer and the seller. We now denote c as the CDS price and M as the notional value of the reference asset. If we further assume that an amount of cM should be made by the buyer to the seller at a series of discrete times $t_i, i = 0, 1, 2, \dots, N$ with $0 = t_0 < t_1 < t_2 < \dots < t_N = T$ regularly, then the present value of the buyer's cash flow P_1 can be expressed as

$$P_1 = \sum_{i=0}^N cM e^{-rt_i},$$

where r is the risk-free interest rate. On the other hand, the payment from the seller will only come into effect if the default occurs. In this case, the seller needs to pay the compensation fee of $(1 - L)M$ at time T , where L represents the recovery rate. Therefore, the present value of the seller's cash flow P_2 is

$$P_2 = e^{-rT} M(1 - L)P^Q(S_T \leq D). \quad (2.15)$$

As a swap contract, it should be fair to both parties when it is initiated, which implies that the initial value of the contract should be zero. In other words, we have $P_1 = P_2$,

implying that

$$\sum_{i=0}^N cM e^{-rt_i} = e^{-rT} M(1-L)P^Q(S_T \leq D). \quad (2.16)$$

The substitution of (2.3) into (2.16) further yields

$$c = \frac{e^{-rT}(1-L) \left\{ \frac{1}{2} - \frac{1}{\pi} \int_0^{+\infty} \text{Real} \left[\frac{e^{j\phi \ln D} f(\phi; T, 0, y_0, X_0)}{j\phi} \right] d\phi \right\}}{\sum_{i=0}^N e^{-rt_i}}, \quad (2.17)$$

where the explicit expression of f is provided in Theorem 1.

With the availability of the closed-form analytical solution², it becomes easier to conduct some quantitative analysis. This will be the main issue in the next section.

3 Numerical examples and discussions

As shown in the previous section, the analytical expression for the price of the CDS is derived rigorously. Therefore, there is no need to further address the accuracy of our solution and present any calculated results. However, from the view point that a comparison with results determined by other approach may give readers a sense of verification of the newly established formula, several numerical examples are still provided in this section. Furthermore, with the help of the formula, we also analyze quantitatively the impacts of different parameters on the CDS price.

In the following, unless otherwise stated, values of parameters are listed as follows. The current state is assumed to be State 1, and the risk-free interest rate r is equal to 5%. The default level D and the current underlying price S_0 are equal to 80 and 90, respectively. The two transition rates, λ_{12} and λ_{21} , take the value of 10 and 20 respectively. The time to expiry T is set to be 5 (years), and the number of payments is 20. The volatility of

²It should be remarked that the extension to the case where the default can occur at any time is not trivial, and no closed-form analytical solution can be found.

the regime switching model for State 1 is $\sigma_1 = 0.1$ for both cases, whereas the volatility of State 2 (σ_2) takes the value of 0.05 and 0.2 for Case 1 and Case 2, respectively. It should be pointed out that for comparison purposes, the volatility for the standard Brownian motion σ takes the same value as σ_1 .

Before we study various properties of the CDS price under the regime-switching model, it is necessary for us to first verify our newly derived formula to ensure that there are no algebraic errors. We compare the default probabilities calculated from our formula with those obtained through the Monte-Carlo simulation. Here, 200,000 sample paths are adopted, and the results produced by the Monte-Carlo simulation with such a large number of sample paths could be viewed as "benchmark" solutions. From Fig 1(a), it is clear that our results agree well with the benchmark solutions with the maximum relative error being less than 0.4%, as shown in Fig 1(b). This demonstrates the validation of our formula. On the other hand, one could also observe from Fig 1(a) that the default probability is a monotonic decreasing function of the reference asset price. This is indeed financially meaningful, as an increase in the price of the reference asset will add some degrees of difficulty for the price to fall below the default level, resulting in a smaller probability.

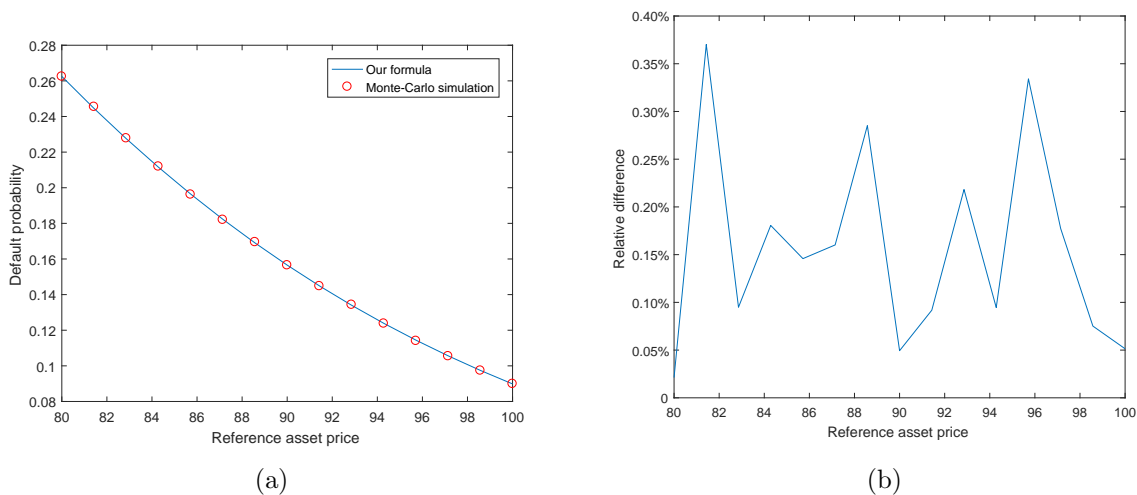


Figure 1: Our formula vs Monte-Carlo simulation.

With confidence in the newly derived formula, we now turn to examine the quantitative

effects of various parameters on the price of the CDS contract. Depicted in Figure 2 is the comparison of the default probability under the standard Brownian motion and the regime switching model. It is interesting to notice that the default probability exhibits a similar pattern under both models. That is, such a probability increases to a certain level and then decreases as the time to expiry increases. One could also observe from this figure that the default probability of the standard Brownian motion is higher but lower than that of Case 1 and Case 2, respectively. This is also reasonable, because a lower volatility will result in a lower default probability, and the overall volatility level of the standard Brownian motion is higher but lower than that of Case 1 and Case 2, respectively.

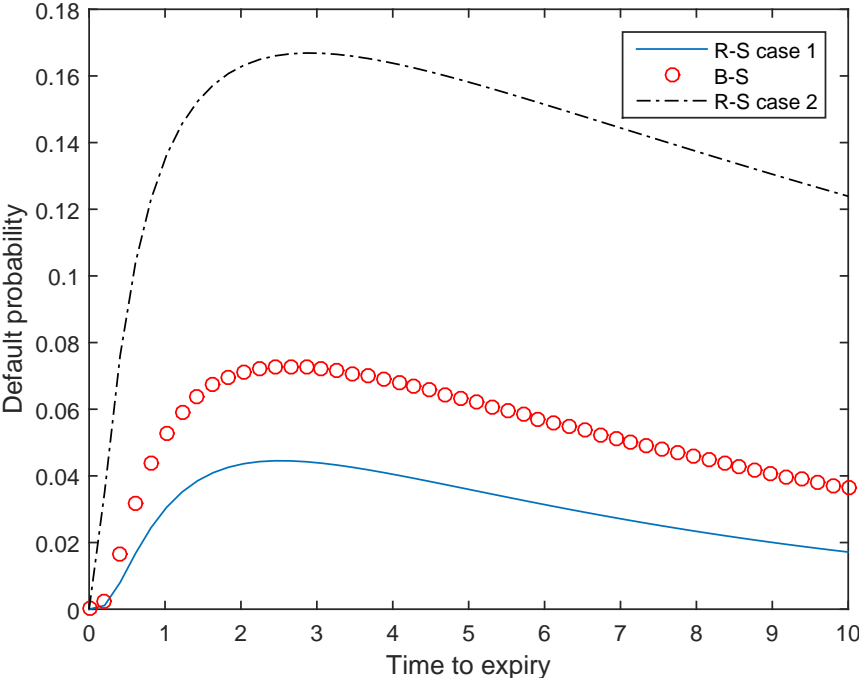


Figure 2: Comparison of default probabilities under different models.

Now, we shall investigate the impact of the transition rates on the CDS price. We introduce a scale parameter z into the transition rates, i.e., $\lambda_{12} = 10 * z$ and $\lambda_{21} = 20 * z$, and the CDS price as a function of z is shown in Figure 3. From this figure, it is clear that our model degenerates to the standard Brownian motion if the two transition rates are equal to zero. This agrees with what we have mentioned previously. On the other hand,

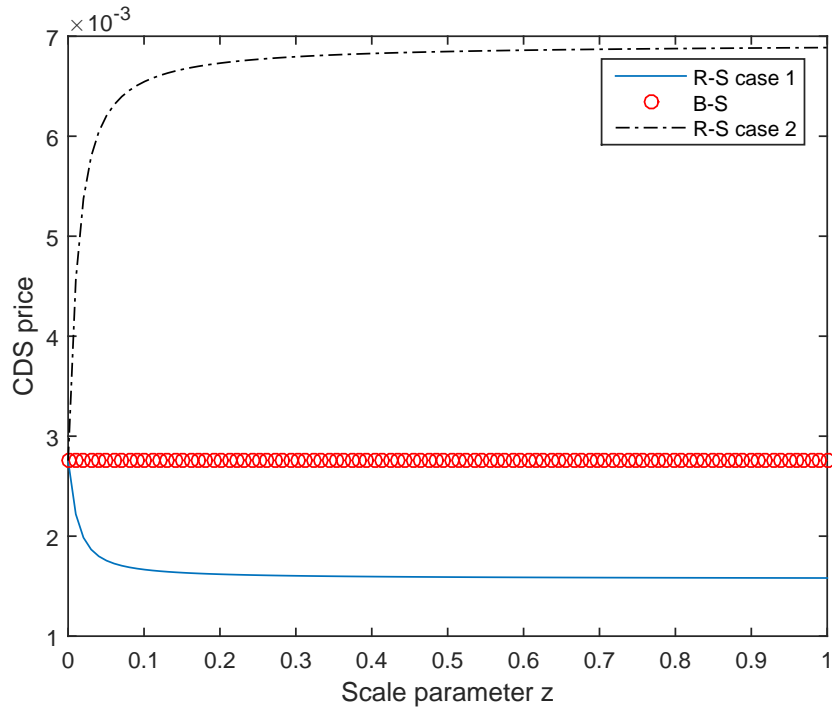


Figure 3: CDS prices under different models with respect to a scale parameter z .

one could also observe that the CDS price is a decreasing but increasing function of the transition rates for Case 1 and Case 2, respectively. This can be explained by a financially meaningful argument as follows. In Case 1, the volatility of the current state is larger than that of the other state. In this situation, increasing the transition rate implies that the probability of the volatility switching to the lower value becomes larger. This will result in a smaller CDS price because a lower volatility corresponds to lower risk.

In Fig 4, we show the CDS price with different times to expiry under both the B-S and R-S models. From this figure, it is clear that the CDS price increases first and then decreases as the time is gradually away from the expiry. This is quite similar to what the default probability does, as shown in Fig 2, and is not surprising at all because the default probability and the CDS price are positively correlated, as shown in (2.17).

In Fig 5, the CDS price is plotted against the number of payments. One could observe that the CDS price under both models is monotonically decreasing with respect to the number of payments. Financially, if the buyer of the CDS contract pays more often,

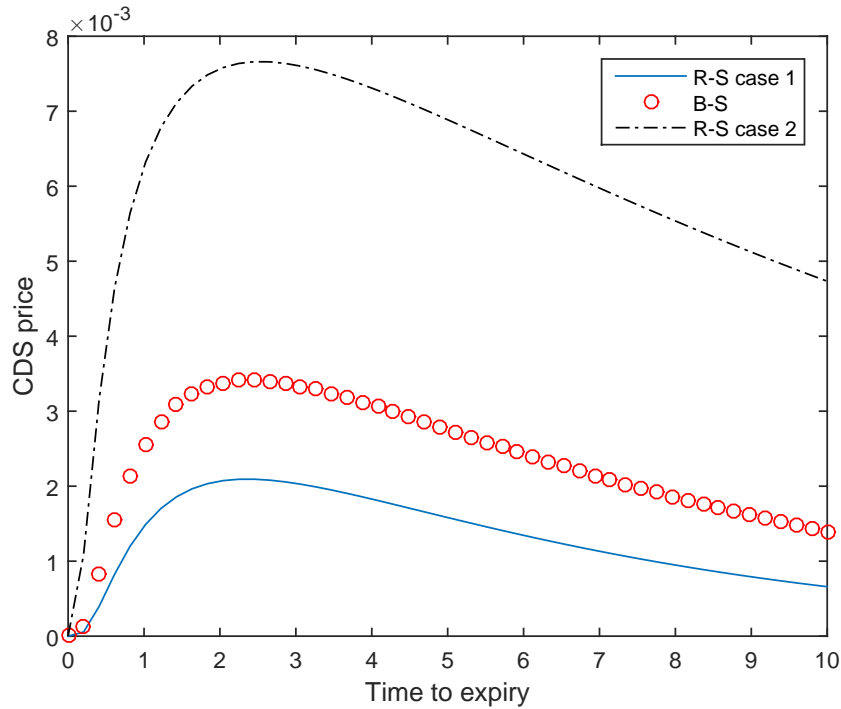


Figure 4: CDS prices under different models with respect to the time to expiry.

he/she should certainly pay less at one time, provided that the compensation from the seller remains unchanged. It can also be witnessed from this figure that the gap between the CDS price under the two models is narrowed down when the number of payments is increased.

4 Conclusion

In this paper, the pricing of the CDS is investigated under a regime switching model, which nests the standard Brownian motion as a special case. By analyzing the cash flows of the buyer and seller of the CDS contract, a closed-form pricing formula for this contract is derived by making use of the analytical expression of the default probability. After the newly derived formula being numerically verified with the Monte-Carlo simulation, some quantitative analyses are conducted, demonstrating the impacts of different parameters on the CDS prices.

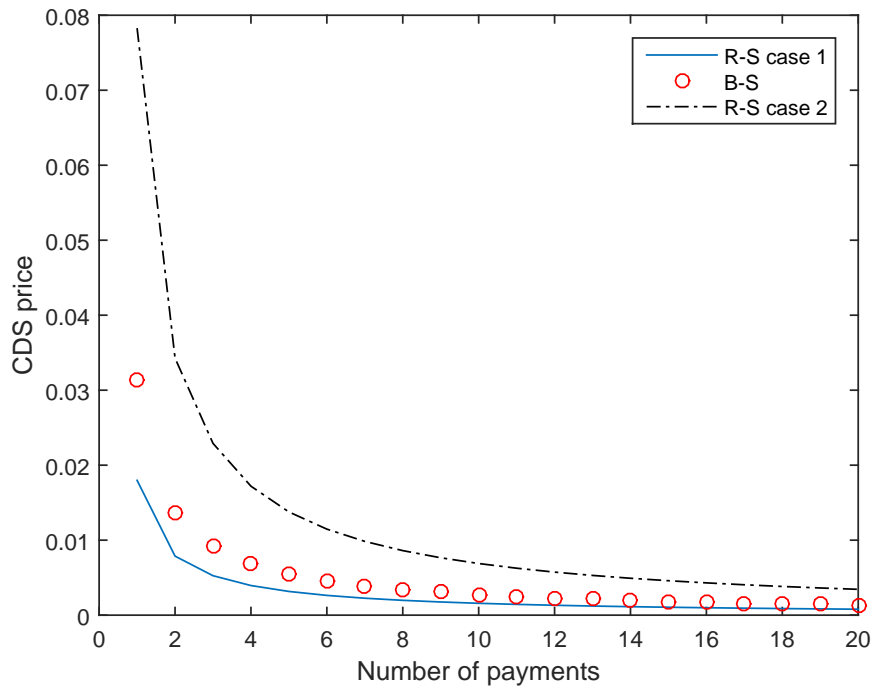


Figure 5: CDS prices under different models with respect to the number of payments.

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Appendix

In this appendix, we provide the proof for (2.14). According to the results in [7], the Markov chain X_t can be expressed as

$$X_t = X_0 + \int_0^t A' X_s ds + M_t, \quad (\text{A-1})$$

where M_t is a martingale with respect to the filtration generated by X_t , and A is a matrix defined as

$$A = \begin{bmatrix} -\lambda_{12} & \lambda_{12} \\ \lambda_{21} & -\lambda_{21} \end{bmatrix}. \quad (\text{A-2})$$

If we define

$$Z_T = X_T e^{\int_s^T \langle \sigma_v^2, X_t \rangle dt}, \quad (\text{A-3})$$

it is not difficult to show that the total derivative of Z_T is

$$dZ_T = e^{\int_s^T \langle \sigma_v^2, X_t \rangle dt} [(A' X_T) dT + dM_T] + \langle \sigma_v^2, X_T \rangle e^{\int_s^T \langle \sigma_v^2, X_t \rangle dt},$$

with the substitution of the expression of X_T . Also, from the identity

$$\langle \sigma_v^2, X_T \rangle = \text{diag}[\sigma_v^2] X_T,$$

where $\text{diag}[v]$ is a diagonal matrix with the vector v as the main elements on the main diagonal, we can further obtain

$$dZ_T = (A' + \text{diag}[\sigma_v^2]) X_T e^{\int_s^T \langle \sigma_v^2, X_t \rangle dt} + dM_T e^{\int_s^T \langle \sigma_v^2, X_t \rangle dt}. \quad (\text{A-4})$$

Integrating on both sides of the above equation yields

$$Z_T - Z_s = \int_s^T (A' + \text{diag}[\sigma_v^2]) X_u e^{\int_s^u \langle \sigma_v^2, X_t \rangle dt} du + \int_s^T e^{\int_s^u \langle \sigma_v^2, X_t \rangle dt} dM_u,$$

the second integral on the right hand side of which is a martingale. Therefore, given $Z_s = X_s$, we can arrive at

$$E[Z_T] = X_s + \int_s^T (A' + \text{diag}[\sigma_v^2]) E[Z_u] du, \quad (\text{A-5})$$

by taking the expectation. If we denote $H_T = E[Z_T]$, and take the derivative on both sides of (A-5) with respect to T , we find that H_T satisfies the following ODE system

$$\begin{cases} \frac{dH_T}{dT} = (A' + \text{diag}[\sigma_v^2])H_T, \\ H_s = X_s, \end{cases} \quad (\text{A-6})$$

the result of which can be derived as

$$E[Z_T] = H_T = e^{\int_s^T (A' + \text{diag}[\sigma_v^2]) dt} X_s. \quad (\text{A-7})$$

On the other hand, if we further assume that the probability of X_T being $(1, 0)'$ is π , we can calculate $E[Z_T]$ from the definition of Z_T as

$$\begin{aligned} E[Z_T] &= \begin{bmatrix} E(e^{\int_s^T \langle \sigma_v^2, X_t \rangle dt}) \\ 0 \end{bmatrix} \pi + \begin{bmatrix} 0 \\ E(e^{\int_s^T \langle \sigma_v^2, X_t \rangle dt}) \end{bmatrix} (1 - \pi) \\ &= \begin{bmatrix} \pi E(e^{\int_s^T \langle \sigma_v^2, X_t \rangle dt}) \\ (1 - \pi) E(e^{\int_s^T \langle \sigma_v^2, X_t \rangle dt}) \end{bmatrix}. \end{aligned} \quad (\text{A-8})$$

This clearly shows that $E(e^{\int_s^T \langle \sigma_v^2, X_t \rangle dt})$ is actually the sum of each element in $E[Z_T]$, and thus we obtain

$$E(e^{\int_s^T \langle \sigma_v^2, X_t \rangle dt}) = \langle e^{\int_s^T (A' + \text{diag}[\sigma_v^2]) dt} X_s, I \rangle. \quad (\text{A-9})$$

This has completed the proof.