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Equilibrium states on higher-rank Toeplitz non-commutative solenoids

Abstract

We consider a family of higher-dimensional non-commutative tori, which are twisted analogues of the algebras of continuous functions on ordinary tori and their Toeplitz extensions. Just as solenoids are inverse limits of tori, our Toeplitz non-commutative solenoids are direct limits of the Toeplitz extensions of non-commutative tori. We consider natural dynamics on these Toeplitz algebras, and we compute the equilibrium states for these dynamics. We find a large simplex of equilibrium states at each positive inverse temperature, parametrized by the probability measures on an (ordinary) solenoid.

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EQUILIBRIUM STATES ON HIGHER-RANK TOEPLITZ NONCOMMUTATIVE SOLENOIDS

ZAHRA AFSAR, ASTRID AN HUEF, IAIN RAEBURN, AND AIDAN SIMS

ABSTRACT. We consider a family of higher-dimensional noncommutative tori, which are twisted analogues of the algebras of continuous functions on ordinary tori, and their Toeplitz extensions. Just as solenoids are inverse limits of tori, our Toeplitz noncommutative solenoids are direct limits of the Toeplitz extensions of noncommutative tori. We consider natural dynamics on these Toeplitz algebras, and compute the equilibrium states for these dynamics. We find a large simplex of equilibrium states at each positive inverse temperature, parametrised by the probability measures on an (ordinary) solenoid.

1. INTRODUCTION

Classical solenoids are inverse limits of tori. There are noncommutative analogues of tori, which are the twisted group algebras $C^*(\mathbb{Z}^n, \sigma)$ of the abelian group \mathbb{Z}^n . For $n = 2$, these are the rotation algebras A_θ generated by two unitaries U, V satisfying the commutation relation $UV = e^{2\pi i\theta}VU$. When θ is irrational, these are simple C^* -algebras, and have been extensively studied (see, for example, [10, Chapter VI]). For $\theta = 0$, we recover the commutative algebra $C(\mathbb{T}^2)$, and hence the A_θ are also known as “noncommutative tori.” In [24], Latrémolière and Packer studied a family of noncommutative solenoids that are direct limits of noncommutative tori. (The connection is that the commutative algebra of continuous functions on a solenoid is the direct limit of the algebras of continuous functions on the approximating tori.)

Following surprising results about phase transitions for the KMS states of the Toeplitz algebras of the $ax + b$ -semigroup of the natural numbers [21, 19], many authors have studied the KMS structure of Toeplitz extensions in other settings. Typically, these Toeplitz extensions exhibit more interesting KMS structure. This recent work has covered Toeplitz algebras of directed graphs and their higher-rank analogues [15, 16, 7, 13, 8] (after earlier work in [11]), Toeplitz algebras arising in number theory [9], the Nica-Toeplitz extensions of Cuntz-Pimsner algebras [19, 17, 18, 1, 4], and Toeplitz algebras associated to self-similar actions [22, 23]. In [6], Brownlowe, Hawkins and Sims described Toeplitz extensions of the noncommutative solenoids from [24], and considered a natural dynamics on this extension. They showed that for each inverse temperature $\beta > 0$, the KMS_β states are parametrised by the probability measures on a commutative solenoid which is the inverse limit of 1-dimensional tori [6, Theorem 6.6].

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Here we consider a family of higher-rank noncommutative solenoids and their Toeplitz extensions. As for the algebras of higher-rank graphs [16], there is an obvious gauge action of a torus \mathbb{T}^d on these algebras, but to get a dynamics one has to choose an embedding of \mathbb{R} in the torus. We fix $r \in [0, \infty)^d$, giving an embedding $t \mapsto e^{itr}$ of \mathbb{R} in \mathbb{T}^d , and compose with the gauge action to get a dynamics α^r .

The building blocks in [6] are Toeplitz noncommutative tori in which one generator U is unitary, the other V is an isometry, the relation is still given by $UV = e^{2\pi i\theta}VU$, and the dynamics fixes U . Here we fix $d, k \in \mathbb{N}$. Our blocks B_θ are Toeplitz noncommutative tori generated by a unitary representation U of \mathbb{Z}^d and a Nica-covariant isometric representation V of \mathbb{N}^k , and the commutation relation is given by $U_n V_p = e^{2\pi i p^T \theta^n} V_p U_n$ for a fixed $k \times d$ matrix θ with entries in $[0, \infty)$. Then the dynamics α^r is given by a vector $r \in (0, \infty)^k$; it fixes the unitaries U_n , and multiplies V_p by $e^{itp^T r}$.

We begin by describing the direct system of Toeplitz noncommutative tori whose limit is the Toeplitz noncommutative solenoid of the title. Everything is defined in terms of presentations of the blocks: building the connecting maps is in particular quite complicated, and requires us to be careful with the notation, which we try to keep consistent throughout the paper. We then discuss the dynamics, which is again defined using actions on the individual blocks. Then, remarkably, we have a presentation of the direct limit which allows us to state our main result as Theorem 2.7. This gives a satisfyingly explicit description of the KMS_β states in terms of measures on a commutative solenoid of the form $\varprojlim \mathbb{T}^d$. This concrete description is new even in the case $k = d = 1$ studied in [6].

The first step in the proof of our theorem is an analysis of the KMS states of a building block B_θ , which we do in §3. The description in Proposition 3.7 looks rather like the descriptions of KMS states on graph algebras in [15, Theorem 3.1] and [16, Theorem 6.1], and on algebras associated to local homeomorphisms in [2, Theorem 5.1]: we find a subinvariance relation which identifies the measures on the torus associated to KMS states, and then describe the solutions of that relation in terms of a concrete simplex of measures.

In the next section (§4), we show how the subinvariance relations for the building blocks combine to give one continuously parametrised subinvariance relation for the direct limit (Theorem 4.1). We then describe the solutions to this new subinvariance relation in Theorem 5.1, which is the key technical result in the paper. This solution is very concrete, involving a formula which is reminiscent of a multi-variable Laplace transform, and is much more direct than the *ad hoc* approach used in [6].

In the last section, we give a concrete description of the isomorphism $\mu \mapsto \psi_\mu$ of the simplex $P(\varprojlim \mathbb{T}^d)$ of probability measures on the solenoid onto the simplex of KMS_β states on the Toeplitz noncommutative torus. Then by evaluating these KMS states on generators, we arrive at the explicit values described in Theorem 2.7.

2. TOEPLITZ NONCOMMUTATIVE SOLENOIDS

We define a Toeplitz noncommutative solenoid as the direct limit of a sequence of blocks, which we call Toeplitz noncommutative tori. So we begin by looking at

these blocks. In the course of this section we will introduce notation which will be used throughout the paper.

First we fix positive integers d and k . We write A^T for the transpose of a matrix A . We view elements of \mathbb{R}^k as column vectors, and write the inner product of $n, p \in \mathbb{R}^k$ in matrix notation as $p^T n$. We use similar conventions for \mathbb{R}^d .

The pair $(\mathbb{Z}^k, \mathbb{N}^k)$ is a quasi-lattice ordered group in the sense of Nica [25]. Indeed, for every $p, q \in \mathbb{N}^k$, the element $p \vee q$ defined pointwise by

$$(p \vee q)_j = \max\{p_j, q_j\} \quad \text{for } 1 \leq j \leq k$$

is a least upper bound for p and q , so it is lattice-ordered. An isometric representation $V : \mathbb{N}^k \rightarrow B(H)$ is *Nica-covariant* if it satisfies

$$V_p V_p^* V_q V_q^* = V_{p \vee q} V_{p \vee q}^* \quad \text{for all } p, q \in \mathbb{N}^k,$$

or equivalently [20, (1.4)] if

$$V_p^* V_q = V_{(p \vee q) - p} V_{(p \vee q) - q}^* \quad \text{for all } p, q \in \mathbb{N}^k.$$

For $\theta \in M_{k,d}([0, \infty))$, we consider the universal C^* -algebra B_θ generated by a unitary representation U of \mathbb{Z}^d and a Nica-covariant isometric representation V of \mathbb{N}^k such that

$$(2.1) \quad U_n V_p = e^{2\pi i p^T \theta n} V_p U_n \quad \text{for } p, q \in \mathbb{N}^k \text{ and } n \in \mathbb{Z}^d.$$

We then have also

$$(2.2) \quad U_n V_p^* = (V_p U_{-n})^* = (e^{-2\pi i p^T \theta(-n)} U_{-n} V_p)^* = e^{-2\pi i p^T \theta n} V_p^* U_n.$$

Direct calculation shows that for $p, q, p', q' \in \mathbb{N}^k$ and $n, n' \in \mathbb{Z}^d$, we have

$$\begin{aligned} V_p U_n V_q^* V_{p'} U_{n'} V_{q'}^* &= V_p U_n V_{(q \vee p') - q} V_{(q \vee p') - p'}^* U_{n'} V_{q'}^* \\ &= e^{2\pi i ((q \vee p') - q)^T \theta n + ((q \vee p') - p')^T \theta n'} V_{p + (q \vee p') - q} U_{n + n'} V_{q' + (q \vee p') - p'}^*, \end{aligned}$$

and we deduce that

$$B_\theta = \overline{\text{span}}\{V_p U_n V_q^* : n \in \mathbb{Z}^d \text{ and } p, q \in \mathbb{N}^k\}.$$

We call B_θ a *Toeplitz noncommutative torus*.

Now we move on to noncommutative solenoids. First we need some more conventions. We write \mathbb{S}^d for the compact quotient space $\mathbb{R}^d / \mathbb{Z}^d$, and view functions $f \in C(\mathbb{S}^d)$ as \mathbb{Z}^d -periodic continuous functions $f : \mathbb{R}^d \rightarrow \mathbb{C}$. We write $M(\mathbb{S}^d)$ for the set of positive measures on \mathbb{S}^d , and view measures $\mu \in M(\mathbb{S}^d)$ as positive functionals $f \mapsto \int_0^1 f d\mu$ on $C(\mathbb{S}^d)$. Then $\|\mu\| := \mu(\mathbb{S}^d)$ is the norm of the corresponding functional, and $P(\mathbb{S}^d) := \{\mu \in M(\mathbb{S}^d) : \|\mu\| = 1\}$ is the set of probability measures.

We consider three sequences of matrices $\{\theta_m\} \subset M_{k,d}([0, \infty))$, $\{D_m\} \subset M_k(\mathbb{N})$, and $\{E_m\} \subset M_d(\mathbb{N})$ such that: each D_m is diagonal with entries larger than 1; each E_m has $\det E_m > 1$; and

$$(2.3) \quad D_m \theta_{m+1} E_m = \theta_m \quad \text{for } m \geq 1.$$

We choose a sequence $\{r^m\} = \{(r_j^m)\}$ of vectors in $(0, \infty)^k$ satisfying

$$(2.4) \quad r^{m+1} = D_m^{-1} r^m \quad \text{for } m \geq 1$$

Notice that both sequences are determined by the first terms $\theta_1 \in M_{k,d}([0, \infty))$ and $r^1 \in [0, \infty)^k$.

Example 2.1. We fix $N \geq 2$, and set $d = k = 1$, $D_m = E_m = N$ for $m \geq 2$, $\theta_1 \in (0, \infty)$ and $\theta_m = N^{-2(m-1)}\theta_1$. Taking the equivalence classes of the θ_m in $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ yields an example of the set-up of [6] except that we are insisting that $\theta_m = N^2\theta_{m+1}$ as real numbers, not just as elements of \mathbb{S} . This has the consequence that $\theta_m \rightarrow 0$ as $m \rightarrow \infty$, which need not happen in the situation of [6]; but see Remark 2.2 below.

Remark 2.2. Our hypothesis that $D_m\theta_{m+1}E_m = \theta_m$ exactly, and not just modulo \mathbb{Z}^d , seems to be crucial in our arguments. Specifically, to assemble the sequences of KMS states that we will construct on the approximating subalgebras B_m into a KMS state on B_∞ , we will need to show that the associated probability measures ν_m (see Proposition 3.7(a)) intertwine through the maps induced by the E_m^T . We do this in Lemma 6.2, and we indicate there the step in the first displayed calculation where it is critical that $D_m\theta_{m+1}E_m = \theta_m$ exactly. This prompted us to review carefully the arguments of [6] and we believe that those arguments also require that $N^2\theta_{m+1} = \theta_m$ exactly. Specifically, the calculation at the end of the proof of [6, Theorem 6.9] implicitly treats θ_j as an element of \mathbb{R} (there are many solutions to $N^k\gamma = \theta_j$ in \mathbb{S}). Similarly the formulas in [6, Section 8] that involve setting $r_j = \beta/(N^j\theta_j)$ only make sense if θ_j is an element of \mathbb{R} . In particular, in the final displayed calculation in the proof of [6, Lemma 8.1], it is critical that $N^2\theta_{j+1} = \theta_j$ exactly.

For each m there is a Toeplitz noncommutative torus $B_m := B_{\theta_m}$ with generators $U_{m,n}$ and $V_{m,p}$ such that: $U : n \mapsto U_{m,n}$ is a unitary representation of \mathbb{Z}^d , $V : p \mapsto V_{m,p}$ is a Nica-covariant isometric representation of \mathbb{N}^k , and the pair U, V satisfy the commutation relation (2.1) for the matrix θ_m .

Next we use the matrices D_m and E_m to build homomorphisms from B_m to B_{m+1} .

Proposition 2.3. *Suppose that m is a positive integer. Then there is a homomorphism $\pi_m : B_m \rightarrow B_{m+1}$ such that $\pi_m(U_{m,n}) = U_{m+1, E_m n}$ and $\pi_m(V_{m,p}) = V_{m+1, D_m p}$.*

Proof. We define $U : \mathbb{Z}^d \rightarrow B_{m+1}$ by $U_n = U_{m+1, E_m n}$ and $V : \mathbb{N}^k \rightarrow B_{m+1}$ by $V_p = V_{m+1, D_m p}$. Then since D_m and E_m have entries in \mathbb{N} , U is a unitary representation of \mathbb{Z}^d and V is an isometric representation of \mathbb{N}^k .

We claim that V is Nica-covariant. To see this, we take $p, q \in \mathbb{N}^k$. Then Nica covariance of $p \mapsto V_{m+1, p}$ implies that

$$(2.5) \quad \begin{aligned} V_p V_p^* V_q V_q^* &= V_{m+1, D_m p} V_{m+1, D_m p}^* V_{m+1, D_m q} V_{m+1, D_m q}^* \\ &= V_{m+1, (D_m p) \vee (D_m q)} V_{m+1, (D_m p) \vee (D_m q)}^*. \end{aligned}$$

Now recall that D_m is diagonal¹, with diagonal entries $d_{m,j}$, say. Then for $1 \leq j \leq k$ we have

$$((D_m p) \vee (D_m q))_j = \max\{(D_m p)_j, (D_m q)_j\} = \max\{d_{m,j} p_j, d_{m,j} q_j\}$$

¹This is crucial here. For example, consider

$$D = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then $De_1 = e_1$, $De_2 = e_1 + e_2$, $e_1 \vee e_2 = e_1 + e_2$, and $D(e_1 \vee e_2) = 2e_1 + e_2$ is not the same as $(De_1) \vee (De_2) = e_1 + e_2$.

$$\begin{aligned}
&= d_{m,j} \max\{p_j, q_j\} = d_{m,j}(p \vee q)_j \\
&= (D_m(p \vee q))_j.
\end{aligned}$$

Thus

$$V_{m+1, (D_m p) \vee (D_m q)} = V_{m+1, D_m(p \vee q)} = V_{p \vee q},$$

and (2.5) says that V is Nica covariant.

We next claim that U and V satisfy the commutation relation (2.1). We take $n \in \mathbb{Z}^d$ and $p \in \mathbb{N}^k$, and compute using the commutation relation in B_{m+1} :

$$\begin{aligned}
U_n V_p &= U_{m+1, E_m n} V_{m+1, D_m p} \\
&= e^{2\pi i (D_m p)^T \theta_{m_1} E_m n} V_{m+1, D_m p} U_{m+1, E_m n} \\
&= e^{2\pi i p^T (D_m \theta_{m_1} E_m)^n} V_{m+1, D_m p} U_{m+1, E_m n} \\
&= e^{2\pi i p^T \theta_{m_1} n} V_p U_n \quad \text{using (2.3)}.
\end{aligned}$$

Now the universal property of B_m gives the desired homomorphism π_m . \square

Remark 2.4. Although we don't think we use this anywhere, the homomorphisms π_m are in fact injective. One way to see this is to use the Nica covariance of $n \mapsto V_{m,n}$ to get a homomorphism $\pi_{V_m} : \mathcal{T}(\mathbb{N}^k) \rightarrow B_{\theta^m}$, and interpret (2.1) as saying that (π_{V_m}, U_m) is a covariant representation of a dynamical system $(\mathcal{T}(\mathbb{N}^k), \mathbb{Z}^d, \gamma^m)$ in the algebra B_{θ^m} . Then B_{θ^m} has the universal property which characterises the crossed product $\mathcal{T}(\mathbb{N}^k) \rtimes_{\gamma^m} \mathbb{Z}^d$, and we can deduce from the equivariant uniqueness theorem for the crossed product (for example, [26, Corollary 4.3]) that the representation

$$\pi_{D_m, E_m} := \pi_{V_{m+1} \circ D_m} \rtimes (U_{m+1} \circ E_m)$$

of $\mathcal{T}^k(\mathbb{N}^k) \rtimes_{\gamma^m} \mathbb{Z}^d$ in $\mathcal{T}^k(\mathbb{N}^k) \rtimes_{\gamma^{m+1}} \mathbb{Z}^d$ is faithful.

We now define our *higher-rank Toeplitz noncommutative solenoid* to be the direct limit

$$(2.6) \quad B_\infty := \varinjlim_{m \in \mathbb{N}} (B_m, \pi_m).$$

We write $\pi_{m,\infty}$ for the canonical homomorphism of B_m into B_∞ . To ease notation we also write $U_{m,n}$ for the image $\pi_{m,\infty}(U_{m,n})$ in B_∞ .

Now we use the vectors $r^m \in (0, \infty)^k$ from our set-up to define the dynamics we propose to study.

Proposition 2.5. *There is a dynamics $\alpha : \mathbb{R} \rightarrow \text{Aut } B_\infty$ such that*

$$(2.7) \quad \alpha_t(V_{m,p} U_{m,n} V_{m,q}^*) = e^{it(p-q)^T r^m} V_{m,p} U_{m,n} V_{m,q}^*.$$

Proof. Since U_m and $V'_m : p \mapsto e^{itp^T r^m} V_{m,p}$ satisfy the same relations in B_m as U_m and V_m , there is a dynamics $\alpha^{r^m} : \mathbb{R} \rightarrow \text{Aut } B_m$ such that

$$\alpha^{r^m}(V_{m,p} U_{m,n} V_{m,q}^*) = e^{it(p-q)^T r^m} V_{m,p} U_{m,n} V_{m,q}^*.$$

We claim that $\pi_m \circ \alpha_t^{r^m} = \alpha_t^{r^{m+1}} \circ \pi_m$. To see this, we compute on generators. First, for $n \in \mathbb{Z}^d$ we have

$$\begin{aligned}
\alpha_t^{r^{m+1}}(\pi_m(U_{m,n})) &= \alpha_t^{r^{m+1}}(U_{m+1, E_m n}) = U_{m+1, E_m n} \\
&= \pi_m(U_{m,n}) = \pi_m(\alpha_t^{r^m}(U_{m,n})).
\end{aligned}$$

Second, for $p \in \mathbb{N}^k$, and using the relation (2.4) at the crucial step to pass from r^{m+1} to r^m , we have

$$\begin{aligned} \alpha_t^{r^{m+1}}(\pi_m(V_{m,p})) &= \alpha_t^{r^{m+1}}(V_{m+1,D_m p}) = e^{it(D_m p)^T r^{m+1}} V_{m+1,D_m p} \\ &= e^{itp^T D_m r^{m+1}} \pi_m(V_{m,p}) = e^{itp^T r^m} \pi_m(V_{m,p}) \\ &= \pi_m(\alpha_t^{r^m}(V_{m,p})). \end{aligned}$$

Now the universal property of the direct limit implies that for each $t \in \mathbb{R}$, there is an automorphism α_t of B_∞ such that $\alpha_t \circ \pi_{m,\infty} = \pi_{m,\infty} \circ \alpha_t^{r^m}$. The formula (2.7) (which implicitly involves the homomorphisms $\pi_{m,\infty}$) implies that $t \mapsto \alpha_t$ is a strongly continuous action of \mathbb{R} on B_∞ . \square

Our goal is to describe the KMS states of the dynamical system (B_∞, α) . But first we pause to establish some conventions about probability measures on inverse limits.

Remark 2.6. All measures in this paper are positive Borel measures. We view probability measures on a compact space X as states on the C^* -algebra $C(X)$ of continuous functions. We write $P(X)$ for the set of probability measures on X .

When $\{h_m : X_{m+1} \rightarrow X_m : m \in \mathbb{N}\}$ is an inverse system of compact spaces with each h_m surjective, the inverse limit $\varprojlim (X_m, h_m)$ is also a compact space. We write $h_{m,\infty}$ for the canonical map of $X_\infty := \varprojlim (X_m, h_m)$ onto X_m , so that we have $h_{m,\infty} = h_m \circ h_{m+1,\infty}$ for all $m \in \mathbb{N}$. The maps $h_{m,\infty}$ induce maps $h_{m,\infty*}$ on measures: if μ is a probability measure on X_∞ , then $\mu_m := h_{m,\infty*}(\mu)$ is the measure on X_m such that

$$\int_{X_m} f d\mu_m = \int_{X_\infty} (f \circ h_{m,\infty}) d\mu \quad \text{for } f \in C(X_m).$$

Conversely, because each h_m is surjective, for any sequence of probability measures $\{\mu_m \in P(X_m) : m \in \mathbb{N}\}$ such that $\mu_m = h_{m*}(\mu_{m+1})$ for all m there is a probability measure $\mu \in P(X_\infty)$ such that $\mu_m = h_{m,\infty*}(\mu)$ for all m (see [5, Lemma 6.1], for example). Thus the simplices $P(\varprojlim X_m)$ and $\varprojlim P(X_m)$ are canonically isomorphic.

To state our main result, we need to observe that, because the entries in the E_m are integers, multiplication by E_m^T on \mathbb{R}^d maps \mathbb{Z}^d into \mathbb{Z}^d and hence induces a homomorphism E_m^T of $\mathbb{S}^d = \mathbb{R}^d / \mathbb{Z}^d$ onto itself. We show that the KMS states are parametrised by the probability measures on the inverse limit $\varprojlim (\mathbb{S}^d, E_m^T)$, which is an ordinary solenoid. We write $E_{m,\infty}^T$ for the projection of $\varprojlim (\mathbb{S}^d, E_m^T)$ on the m th copy of \mathbb{S}^d , so that we have

$$E_{m,\infty}^T = E_m^T \circ E_{m+1,\infty}^T \quad \text{for } m \in \mathbb{N}.$$

The main theorem of this paper is the following; we prove it at the end of the paper.

Theorem 2.7. *Suppose that $\mu \in P(\varprojlim (\mathbb{S}^d, E_m^T))$ and $\beta > 0$. Let $\{\mu_m\}$ be the corresponding sequence of probability measures on \mathbb{S}^d . For $m \in \mathbb{N}$ and $n \in \mathbb{N}^d$, we define the moment $M_{m,n}(\mu)$ to be the number*

$$M_{m,n}(\mu) = \int_{\mathbb{S}^d} e^{2\pi i x^T n} d\mu_m(x) = \int_{\varprojlim (\mathbb{S}^d, E_m^T)} e^{2\pi i E_{m,\infty}^T(x)^T n} d\mu(x).$$

Then there is a KMS_β state ψ_μ on (B_∞, α) such that

$$(2.8) \quad \psi_\mu(V_{m,p}U_{m,n}V_{m,q}^*) = \delta_{p,q} e^{-\beta p^T r^m} \prod_{j=1}^k \frac{\beta r_j^m}{\beta r_j^m - 2\pi i(\theta_m^T n)_j} M_{m,n}(\mu).$$

The map $\mu \mapsto \psi_\mu$ is an affine homeomorphism of $P(\varprojlim(\mathbb{S}^d, E_m^T))$ onto the simplex $KMS_\beta(B_\infty, \alpha)$ of KMS_β states.

Remark 2.8. As a reality check, we take $p = q = 0$ and $n = 0$. Then $V_{m,p}U_{m,n}V_{m,q}^*$ is the identity $1_{B_m} = 1_{B_\infty}$, and our formula collapses to $\psi_\mu(1) = 1$.

Remark 2.9. It is interesting to set $d = k = 1$ and compare the formula (2.8) with the formula (6.4) in Theorem 6.9 of [6], which on the face of it looks different. The point is that the integral on the right-hand side of [6, (6.4)] is with respect to the subinvariant measure associated to the probability measure μ , which in our notation would be ν_{μ_m} . There is no specific description for this measure in [6]: they get an isomorphism of the simplex $P(\varprojlim \mathbb{S})$ onto the simplex of subinvariant measures by specifying it on the extreme points (see [6, Lemma 8.2]). We reconcile the two approaches in Remark 5.4.

3. EQUILIBRIUM STATES ON A TOEPLITZ NONCOMMUTATIVE TORUS

In this section, we fix $\theta \in M_{k,d}([0, \infty))$, and investigate the KMS states on the Toeplitz noncommutative torus B_θ .

For $n \in \mathbb{Z}^d$, we write g_n for the character on \mathbb{S}^d given by $g_n(x) = e^{2\pi i x^T n}$, and $\iota : C(\mathbb{S}^d) \rightarrow C^*(\mathbb{Z}^d) \subset B_\theta$ for the isomorphism such that $\iota(g_n) = U_n$. Then we have

$$B_\theta = \overline{\text{span}}\{V_p \iota(f) V_q^* : f \in C(\mathbb{S}^d), p, q \in \mathbb{N}^k\}.$$

For $y \in \mathbb{R}^d$ we define $R_y : \mathbb{S}^d \rightarrow \mathbb{S}^d$ by $R_y(x) = x + y$. Later, we will also write R_y^* for the automorphism of $C(\mathbb{S}^d)$ given by $R_y^* f = f \circ R_y$, and R_{y^*} for the dual map on measures defined by

$$\int_{\mathbb{S}^d} f dR_{y^*}(\mu) = \int_{\mathbb{S}^d} R_y^*(f) d\mu = \int_{\mathbb{S}^d} f \circ R_y d\mu.$$

The assignment $y \mapsto R_y^*$ is a strongly continuous action R of \mathbb{R}^d on $C(\mathbb{S}^d)$, and each R_{y^*} is norm-preserving.

Lemma 3.1. For $f \in C(\mathbb{S}^d)$ and $p \in \mathbb{N}^k$ we have

$$(3.1) \quad V_p \iota(f) = \iota(f \circ R_{-\theta^T p}) V_p \quad \text{and} \quad V_p^* \iota(f) = \iota(f \circ R_{\theta^T p}) V_p^*.$$

Proof. Since $C(\mathbb{S}^d) = \overline{\text{span}}\{g_n : x \mapsto e^{2\pi i x^T n} : n \in \mathbb{Z}^d\}$, it suffices to check (3.1) for $f = g_n$. Let $n \in \mathbb{Z}^d$. Then (2.1) gives

$$V_p \iota(g_n) = V_p U_n = e^{-2\pi i p^T \theta n} U_n V_p = e^{-2\pi i p^T \theta n} \iota(g_n) V_p.$$

Since

$$e^{-2\pi i p^T \theta n} g_n(x) = e^{-2\pi i p^T \theta n} e^{2\pi i x^T n} = g_n(x - \theta^T p) = (g_n \circ R_{-\theta^T p})(x),$$

the first equality follows. The second follows from a similar computation using (2.2). \square

Remark 3.2. The minus sign in the first identity in (3.1) is crucial. As a reality check, notice that the signs in the two formulas have to be different, because $V_p^*V_p = 1$ means the $\pm\theta^T p$ have to cancel. As a corollary, note that $V_pV_p^*$, which is a proper projection, commutes with the $\iota(f)$. (To see that $V_pV_p^* \neq 1$, we can use the specific representation of B_θ constructed in the proof of Proposition 3.7(b).)

We now fix $r \in (0, \infty)^k$. The universal property of B_θ gives a dynamics $\alpha^r : \mathbb{R} \rightarrow \text{Aut } B_\theta$ such that

$$(3.2) \quad \alpha_t^r(U_n) = U_n \quad \text{and} \quad \alpha_t^r(V_p) = e^{itp^T r} V_p \quad \text{for } n \in \mathbb{Z}^d, p \in \mathbb{N}^k, t \in \mathbb{R}.$$

Then $\alpha_t^r(V_p U_n V_q^*) = e^{it(p-q)^T r} V_p U_n V_q^*$, and hence

$$\{V_p U_n V_q^* : n \in \mathbb{Z}^d, p, q \in \mathbb{N}^k\}$$

is a set of α^r -analytic elements spanning an α^r -invariant dense subset of B_θ .

To describe the KMS_β states of (B_θ, α^r) , it was tempting to apply [3, Theorem 6.1] to the Toeplitz algebra of the commuting homeomorphisms $h_j : x \mapsto x + \theta_j$ associated to the rows θ_j of θ . That result is in several ways more general than we need, but has an unfortunate hypothesis of rational independence on the set $\{r_j\}$ which we prefer to avoid.

Proposition 3.3. *Suppose that $\beta > 0$ and ϕ is a KMS_β state of (B_θ, α^r) . Then ϕ is a KMS_β state of (B_θ, α^r) if and only if*

$$(3.3) \quad \phi(V_p U_n V_q^*) = \delta_{p,q} e^{-\beta p^T r} \phi(U_n) \quad \text{for } n \in \mathbb{Z}^d \text{ and } p, q \in \mathbb{N}^k.$$

To prove Proposition 3.3 we need two lemmas. The arguments are based on the proofs of Lemmas 5.2 and 5.3 in [16].

Lemma 3.4. *Suppose that $\beta > 0$ and ϕ is a KMS_β state of (B_θ, α^r) . If $p, q \in \mathbb{N}^k$ satisfy $p^T r = q^T r$, then*

- (a) $\phi(V_p U_n V_p^*) = \phi(V_q U_n V_q^*)$ for $n \in \mathbb{Z}^d$; and
- (b) $|\phi(V_p \iota(f) V_q^*)| \leq \phi(V_p \iota(f) V_p^*)$ for positive $f \in C(\mathbb{S}^d)$.

Proof. For (a), since V_q is an isometry, we have

$$\phi(V_p U_n V_p^*) = \phi(V_p U_n (V_q^* V_q) V_p^*) = \phi((V_p U_n V_q^*) (V_q V_p^*)),$$

and since $p^T r = q^T r$ the KMS condition gives

$$\phi(V_p U_n V_p^*) = e^{-\beta(p-q)^T r} \phi(V_q V_p^* (V_p U_n V_q^*)) = \phi(V_q U_n V_q^*).$$

For (b), we take a positive function f in $C(\mathbb{S}^d)$. By linearity and continuity, part (a) implies that $\phi(V_q \iota(f) V_q^*) = \phi(V_p \iota(f) V_p^*)$. Using the Cauchy–Schwarz inequality at the second step, we calculate:

$$\begin{aligned} |\phi(V_p \iota(f) V_q^*)|^2 &= |\phi((V_p \iota(\sqrt{f})) (V_q \iota(\sqrt{f}))^*)|^2 \\ &\leq \phi(V_p \iota(f) V_p^*) \phi(V_q \iota(f) V_q^*) \\ &= \phi(V_p \iota(f) V_p^*)^2. \end{aligned}$$

Since both sides are the squares of non-negative numbers, we can take square roots, and we retrieve (b). \square

Lemma 3.5. *Suppose that $\beta > 0$ and ϕ is a KMS_β state of (B_θ, α^r) . Suppose that $p, q \in \mathbb{N}^k$ satisfy $p^T r = q^T r$ and that $f \in C(\mathbb{S}^d)$. Write $P := (p \vee q) - p$. Then*

$$(3.4) \quad \phi(V_p \iota(f) V_q^*) = \phi(V_{p+lP} \iota(f \circ R_{l\theta^T P}) V_{q+lP}^*) \quad \text{for all } l \in \mathbb{N}.$$

If $p \neq q$, then $\phi(V_p \iota(f) V_q^*) = 0$.

Proof. We prove (3.4) by induction on l . The base case $l = 0$ is trivial. Now suppose that (3.4) holds for $l \geq 0$. The inductive hypothesis gives

$$\begin{aligned} \phi(V_p \iota(f) V_q^*) &= \phi(V_{p+lP} \iota(f \circ R_{l\theta^T P}) V_{q+lP}^*) \\ &= \phi(V_{p+lP} \iota(f \circ R_{l\theta^T P}) V_{q+lP}^* V_{q+lP} V_{q+lP}^*). \end{aligned}$$

Since the dynamics α^r fixes the element $V_{q+lP} V_{q+lP}^*$, the KMS condition implies that

$$\phi(V_p \iota(f) V_q^*) = \phi(V_{q+lP} V_{q+lP}^* V_{p+lP} \iota(f \circ R_{l\theta^T P}) V_{q+lP}^*),$$

and Nica covariance gives

$$\begin{aligned} \phi(V_p \iota(f) V_q^*) &= \phi(V_{q+lP} V_{((q+lP) \vee (p+lP)) - (q+lP)}^* V_{((q+lP) \vee (p+lP)) - (p+lP)} \iota(f \circ R_{l\theta^T P}) V_{q+lP}^*). \end{aligned}$$

For $c \in \mathbb{N}^k$ we have $(p+c) \vee (q+c) = (p \vee q) + c$. Thus

$$\begin{aligned} \phi(V_p \iota(f) V_q^*) &= \phi(V_{q+lP} V_{(p \vee q) - q} V_{(p \vee q) - p}^* \iota(f \circ R_{l\theta^T P}) V_{q+lP}^*) \\ &= \phi(V_{(p \vee q) + lP} V_P^* \iota(f \circ R_{l\theta^T P}) V_{q+lP}^*) \\ &= \phi(V_{(p \vee q) + lP} \iota(f \circ R_{l\theta^T P} \circ R_{\theta^T P}) V_{q+(l+1)P}^*) \quad \text{by Lemma 3.1} \\ &= \phi(V_{p+(l+1)P} \iota(f \circ R_{(l+1)\theta^T P}) V_{q+(l+1)P}^*) \end{aligned}$$

because $(p \vee q) + lP = p + (l+1)P$. This completes the inductive step, and hence the proof of (3.4).

Now suppose that $p \neq q$. Then at least one of P and $(p \vee q) - q$ is nonzero. We argue the case where $P \neq 0$, and the other case follows by taking adjoints. For $l \in \mathbb{N}$ we have

$$\begin{aligned} |\phi(V_p \iota(f) V_q^*)| &= |\phi(V_{p+lP} \iota(f \circ R_{l\theta^T P}) V_{q+lP}^*)| \\ &\leq \phi(V_{p+lP} \iota(f \circ R_{l\theta^T P}) V_{p+lP}^*) \quad \text{by Lemma 3.4(b)} \\ &= e^{-\beta(p+lP)^T r} \phi(V_{p+lP}^* V_{p+lP} \iota(f \circ R_{l\theta^T P})) \\ &= e^{-\beta(p+lP)^T r} \phi(\iota(f \circ R_{l\theta^T P})) \\ &\leq e^{-\beta(p+lP)^T r} \|f\|_\infty. \end{aligned}$$

Since $P > 0$ and $r \in (0, \infty)^k$, we have $(p+lP)^T r \rightarrow \infty$ as $l \rightarrow \infty$, and hence $e^{-\beta(p+lP)^T r} \|f\|_\infty \rightarrow 0$ as $l \rightarrow \infty$. Thus $\phi(V_p \iota(f) V_q^*) = 0$. \square

Proof of Proposition 3.3. First suppose that ϕ is a KMS_β state for (B_θ, α^r) . For $n \in \mathbb{Z}^d$ and $p, q \in \mathbb{N}^k$, two applications of the KMS condition give

$$(3.5) \quad \phi(V_p U_n V_q^*) = e^{-\beta p^T r} \phi(U_n V_q^* V_p) = e^{-\beta(p-q)^T r} \phi(V_p U_n V_q^*).$$

It follows immediately that if $(p - q)^T r \neq 0$, then $\phi(V_p U_n V_q^*) = 0$. If $(p - q)^T r = 0$ but $p \neq q$, then Lemma 3.5 gives $\phi(V_p U_n V_q^*) = 0$. This combined with the first equality in (3.5) gives

$$\phi(V_p U_n V_q^*) = \delta_{p,q} e^{-\beta p^T r} \phi(U_n V_q^* V_p) = \delta_{p,q} e^{-\beta p^T r} \phi(U_n)$$

because V_p is an isometry. This is the desired formula (3.3).

Now suppose that ϕ is a state satisfying (3.3). Since the $V_p U_n V_q^*$ are analytic elements spanning a dense α^r -invariant subspace of B_θ , it suffices to fix $p, q, b, c \in \mathbb{N}^k$ and $n, n' \in \mathbb{Z}^d$, and show that

$$(3.6) \quad \phi(V_p U_n V_q^* V_b U_{n'} V_c^*) = e^{-\beta(p-q)^T r} \phi(V_b U_{n'} V_c^* V_p U_n V_q^*).$$

Let $P := (q \vee b) - b$ and $Q := (q \vee b) - q$. Then $P, Q \in \mathbb{N}^k$ are the unique elements such that $P \wedge Q = 0$ and $P + b = Q + q$, and Nica covariance says that $V_q^* V_b = V_Q V_P^*$. Now we calculate, using first the identities (2.1) and (2.2), and then (at the last step) the assumption (3.3):

$$(3.7) \quad \begin{aligned} \phi(V_p U_n V_q^* V_b U_{n'} V_c^*) &= \phi(V_p U_n V_Q V_P^* U_{n'} V_c^*) \\ &= e^{2\pi i Q^T \theta n} \phi(V_p (V_Q U_n) V_P^* U_{n'} V_c^*) \\ &= e^{2\pi i (Q^T \theta n + P^T \theta n')} \phi(V_{Q+p} U_{n+n'} V_{P+c}^*) \\ &= \delta_{Q+p, P+c} e^{-\beta(Q+p)^T r} e^{2\pi i (Q^T \theta n + P^T \theta n')} \phi(U_{n+n'}). \end{aligned}$$

Similarly, let $M := (c \vee p) - p$ and $N := (c \vee p) - c$. Then $M, N \in \mathbb{N}^k$ are the unique elements such that $M \wedge N = 0$ and $M + p = N + c$, and the right-hand side of (3.6) is

$$(3.8) \quad \begin{aligned} e^{-\beta(p-q)^T r} \phi(V_b U_{n'} V_N V_M^* U_n V_q^*) \\ &= e^{-\beta(p-q)^T r} e^{2\pi i (N^T \theta n' + M^T \theta n)} \phi(V_{b+N} U_{n+n'} V_{q+M}^*) \\ &= \delta_{N+b, M+q} e^{-\beta(p-q+b+N)^T r} e^{2\pi i (N^T \theta n' + M^T \theta n)} \phi(U_{n+n'}). \end{aligned}$$

To see that (3.7) is equal to (3.8), we first show that the two Kronecker deltas have the same value. For this, observe that by definition of M, N, P, Q , we have

$$(P + b) + (N + c) = (Q + q) + (M + p),$$

and consequently $(N + b) - (M + q) = (Q + p) - (P + c)$. Thus $\delta_{Q+p, P+c} = 1$ if and only if $\delta_{N+b, M+q} = 1$. So it now suffices to prove that (3.7) equals (3.8) when $Q + p = P + c$ and $N + b = M + q$.

We first claim that $M = Q$ and $N = P$. By assumption, we have $M + q = N + b$, and we have $P + b = Q + q$ by definition of P, Q . Subtracting these equations, we obtain $M - Q = N - P$, and rearranging gives $M - N = Q - P$. Since $P \wedge Q = 0$ and $M \wedge N = 0$, we deduce that $Q = (Q - P) \vee 0 = (M - N) \vee 0 = M$, and then $P = N$ too, as claimed.

We now have

$$e^{2\pi i (Q^T \theta n + P^T \theta n')} = e^{2\pi i (M^T \theta n + N^T \theta n')},$$

and so it remains to check that

$$e^{-\beta(p-q+b+N)^T r} = e^{-\beta(p+Q)^T r}.$$

For this, we apply $N = P$, from above, at the second equality and $b + P = q + Q$, by definition of Q, P , at the third to get

$$\begin{aligned} (p - q) + (b + N) &= p + (b + N - q) = p + (b + P - q) \\ &= p + (q + Q - q) = p + Q, \end{aligned}$$

which gives the result. Thus ϕ is a KMS_β state. \square

Lemma 3.6. *Write θ_j for the j th row of θ . Then the series*

$$(3.9) \quad \sum_{p \in \mathbb{N}^k} e^{-\beta p^T r} R_{\theta^T p^*}$$

*converges in the operator norm of $B(C(\mathbb{S}^d))$ to an inverse for $\prod_{j=1}^k (\text{id} - e^{-\beta r_j} R_{\theta_j^T *})$.*

Proof. We first need to understand the sum (3.9), which we want to calculate as an iterated sum. So we interpret (3.9) as a $B(C(\mathbb{S}^d))$ -valued integral over \mathbb{N}^k with respect to counting measure σ (for which all functions on \mathbb{N}^k are measurable). Since each $R_{\theta^T p}$ is norm-preserving, we have

$$\|e^{-\beta p^T r} R_{\theta^T p^*}\| = e^{-\beta p^T r} = \prod_{j=1}^k e^{-\beta p_j r_j}.$$

By Tonelli's theorem, we have

$$\begin{aligned} \sum_{p \in \mathbb{N}^k} \|e^{-\beta p^T r} R_{\theta^T p^*}\| &= \sum_{p_k=0}^{\infty} \cdots \sum_{p_1=0}^{\infty} \left(\prod_{j=1}^k e^{-\beta p_j r_j} \right) \\ &= \sum_{p_k=0}^{\infty} \cdots \sum_{p_2=0}^{\infty} \left(\prod_{j=2}^k e^{-\beta p_j r_j} \right) \left(\sum_{p_1=0}^{\infty} e^{-\beta p_1 r_1} \right) \\ &= \sum_{p_k=0}^{\infty} \cdots \sum_{p_2=0}^{\infty} \left(\prod_{j=2}^k e^{-\beta p_j r_j} \right) (1 - e^{-\beta r_1})^{-1}. \end{aligned}$$

Repeating this $k - 1$ more times gives

$$\sum_{p \in \mathbb{N}^k} \|e^{-\beta p^T r} R_{\theta^T p^*}\| = \prod_{j=1}^k (1 - e^{-\beta r_j})^{-1}.$$

Thus the function $p \mapsto e^{-\beta p^T r} R_{\theta^T p^*}$ is integrable with respect to σ , and Fubini's theorem for functions with values in a Banach space (for example, [12, Theorem II.16.3]) implies that

$$\begin{aligned} \sum_{p \in \mathbb{N}^k} e^{-\beta p^T r} R_{\theta^T p^*} &= \sum_{p_k=0}^{\infty} \cdots \sum_{p_1=0}^{\infty} \left(\prod_{j=1}^k e^{-\beta p_j r_j} R_{p_j \theta_j^T *} \right) \\ &= \prod_{j=1}^k \left(\sum_{p_j=0}^{\infty} (e^{-\beta r_j} R_{\theta_j^T *})^{p_j} \right). \end{aligned}$$

Writing the infinite sum as a limit of partial sums shows that

$$(3.10) \quad \sum_{p_j=0}^{\infty} (e^{-\beta r_j} R_{\theta_j^*})^{p_j} (\text{id} - e^{-\beta r_j} R_{\theta_j^*}) = \text{id}.$$

To simplify the product

$$\left(\prod_{j=1}^k \left(\sum_{p_j=0}^{\infty} (e^{-\beta r_j} R_{\theta_j^*})^{p_j} \right) \right) \left(\prod_{j=1}^k (\text{id} - e^{-\beta r_j} R_{\theta_j^*}) \right),$$

we write the left-hand product from $j = k$ to $j = 1$, and the right-hand one from $j = 1$ to $j = k$. Now k applications of (3.10) show that the product telescopes to the identity id of $B(C(\mathbb{S}^d))$. \square

The next proposition is an analogue of [16, Theorem 6.1] and [3, Theorem 6.1].

Proposition 3.7. *Fix $\beta \in (0, \infty)$.*

- (a) *Suppose that ϕ is a KMS_{β} state for (B_{θ}, α^r) , and let $\nu \in P(\mathbb{S}^d)$ be the measure such that*

$$\phi(\iota(f)) = \int_{\mathbb{S}^d} f d\nu \quad \text{for } f \in C(\mathbb{S}^d).$$

Suppose that $F \subset \mathbb{N}^k$ is a finite set such that $p \neq q \in F$ implies $p \wedge q = 0$. Then the measure ν satisfies the subinvariance relation

$$(3.11) \quad \prod_{p \in F} (\text{id} - e^{-\beta p^T r} R_{\theta^* p}) (\nu) \geq 0.$$

- (b) *Define $y_{\beta} := \sum_{p \in \mathbb{N}^k} e^{-\beta p^T r}$, and suppose that κ is a positive measure on \mathbb{S}^d with total mass y_{β}^{-1} . Write θ_j for the j th row of θ . Then*

$$\nu = \nu_{\kappa} := \prod_{j=1}^k (\text{id} - e^{-\beta r_j} R_{\theta_j^*})^{-1} (\kappa)$$

is a subinvariant probability measure, and there is a KMS_{β} state ϕ_{ν} of (B_{θ}, α^r) such that

$$(3.12) \quad \phi_{\nu}(V_p \iota(f) V_q^*) = \delta_{p,q} e^{-\beta p^T r} \int_{\mathbb{S}^d} f d\nu \quad \text{for } p, q \in \mathbb{N}^k \text{ and } f \in C(\mathbb{S}^d).$$

- (c) *The map $\kappa \mapsto \phi_{\nu_{\kappa}}$ is an affine isomorphism of the simplex*

$$\Sigma_{\beta,r} = \{ \text{positive measures } \kappa : \|\kappa\| = y_{\beta}^{-1} \}$$

onto the simplex of KMS_{β} states of (B_{θ}, α^r) .

Proof. (a) We take a positive function $f \in C(\mathbb{S}^d)_+$ and compute

$$(3.13) \quad \int_{\mathbb{S}^d} f d \left(\prod_{p \in F} (\text{id} - e^{-\beta p^T r} R_{\theta^* p}) (\nu) \right) = \int_{\mathbb{S}^d} f \circ \left(\prod_{p \in F} (\text{id} - e^{-\beta p^T r} R_{\theta^* p}) \right) d\nu \\ = \int_{\mathbb{S}^d} \sum_{S \subset F} (-1)^{|S|} \left(\prod_{p \in S} e^{-\beta p^T r} \right) \left(f \circ \prod_{p \in S} R_{\theta^* p} \right) d\nu.$$

We write $p_S := \sum_{p \in S} p$, and observe that $\prod_{p \in S} e^{-\beta p^T r} = e^{-\beta p_S^T r}$ and $\prod_{p \in S} R_{\theta^T p} = R_{\theta^T p_S}$. Thus

$$\begin{aligned}
(3.13) &= \int_{\mathbb{S}^d} \sum_{S \subset F} (-1)^{|S|} e^{-\beta p_S^T r} (f \circ R_{\theta^T p_S}) d\nu \\
&= \phi \left(\sum_{S \subset F} (-1)^{|S|} e^{-\beta p_S^T r} \iota(f \circ R_{\theta^T p_S}) V_{p_S}^* V_{p_S} \right) \quad \text{since } V_{p_S}^* V_{p_S} = 1 \\
&= \phi \left(\sum_{S \subset F} (-1)^{|S|} V_{p_S} \iota(f \circ R_{\theta^T p_S}) V_{p_S}^* \right) \quad \text{by the KMS condition} \\
&= \phi \left(\sum_{S \subset F} (-1)^{|S|} V_{p_S} V_{p_S}^* \iota(f) \right) \quad \text{by (3.1)}.
\end{aligned}$$

Because the set F has the property that $p \wedge q = 0$ for $p \neq q \in F$, Nica covariance gives $V_p V_p^* V_q V_q^* = V_{p \vee q} V_{p \vee q}^* = V_{p+q} V_{p+q}^*$ for $p \neq q \in F$. Thus for each $S \subset F$, we have $V_{p_S} V_{p_S}^* = \prod_{p \in S} V_p V_p^*$, and

$$\sum_{S \subset F} (-1)^{|S|} V_{p_S} V_{p_S}^* = \prod_{p \in F} (1 - V_p V_p^*).$$

The latter product is a projection, and it is fixed by the action α . Hence another application of the KMS condition gives

$$\begin{aligned}
\int_{\mathbb{S}^d} f d \left(\prod_{p \in F} (\text{id} - e^{-\beta p^T r} R_{\theta^T p}) (\nu) \right) &= \phi \left(\prod_{p \in F} (1 - V_p V_p^*) \iota(f) \right) \\
&= \phi \left(\left(\prod_{p \in F} (1 - V_p V_p^*) \right)^2 \iota(f) \right) \\
&= \phi \left(\prod_{p \in F} (1 - V_p V_p^*) \iota(f) \prod_{p \in F} (1 - V_p V_p^*) \right).
\end{aligned}$$

This last term is positive because the argument of ϕ is a positive element of B_θ , and this proves (a).

(b) We have

$$\prod_{j=1}^k (\text{id} - e^{-\beta r_j} R_{\theta_j}) (\nu) = \kappa \geq 0,$$

so ν is subinvariant. By Lemma 3.6 we have

$$\begin{aligned}
(3.14) \quad \int_{\mathbb{S}^d} 1 d\nu &= \int_{\mathbb{S}^d} 1 d \left(\sum_{p \in \mathbb{N}^k} e^{-\beta p^T r} R_{\theta^T p} (\kappa) \right) \\
&= \sum_{p \in \mathbb{N}^k} e^{-\beta p^T r} \int_{\mathbb{S}^d} 1 \circ R_{\theta^T p} d\kappa \\
&= \sum_{p \in \mathbb{N}^k} e^{-\beta p^T r} \|\kappa\| = y_\beta \|\kappa\| = 1,
\end{aligned}$$

and hence ν is a probability measure.

We will build a KMS_β state using a representation of B_θ on $\ell^2(\mathbb{N}^k) \otimes L^2(\mathbb{S}^d, \kappa)$. Recall that we write g_n for the trigonometric polynomial $g_n(x) = e^{2\pi i x^T n}$. Then the formula $W_n f := g_n f$ defines a unitary representation W of \mathbb{Z}^d on $L^2(\mathbb{S}^d, \kappa)$.

Write $\{\delta_p : p \in \mathbb{N}^k\}$ for the orthonormal basis of point masses for $\ell^2(\mathbb{N}^k)$, and let D_n be the bounded operator such that $D_n \delta_p := e^{2\pi i p^T \theta n} \delta_p$. Then D is a unitary representation of \mathbb{Z}^d on $\ell^2(\mathbb{N}^k)$, and hence $D \otimes W$ is a unitary representation of \mathbb{Z}^d on $\ell^2(\mathbb{N}^k) \otimes L^2(\mathbb{S}^d, \kappa)$.

Let T be the usual Toeplitz representation of \mathbb{N}^k by isometries on $\ell^2(\mathbb{N}^k)$. Then we have

$$(T_p \otimes 1)(D_n \otimes W_n)(\delta_q \otimes f) = e^{2\pi i q^T \theta n} (\delta_{p+q} \otimes W_n f),$$

and

$$\begin{aligned} (D_n \otimes W_n)(T_p \otimes 1)(\delta_q \otimes f) &= e^{2\pi i (p+q)^T \theta n} (\delta_{p+q} \otimes W_n f) \\ &= e^{2\pi i p^T \theta n} (T_p \otimes 1)(D_n \otimes W_n). \end{aligned}$$

Hence the universal property of B_θ gives a representation

$$\pi : B_\theta \rightarrow B(\ell^2(\mathbb{N}^k) \otimes L^2(\mathbb{S}^d, \kappa))$$

such that $\pi(U_n) = (D_n \otimes W_n)$ and $\pi(V_p) = T_p \otimes 1$.

Since $\sum_{p \in \mathbb{N}^k} e^{-\beta p^T r}$ is convergent, there is a positive linear functional $\phi_\nu : B_\theta \rightarrow \mathbb{C}$ such that

$$\phi_\nu(a) := \sum_{p \in \mathbb{N}^k} e^{-\beta p^T r} (\pi(a)(\delta_p \otimes 1) | \delta_p \otimes 1).$$

Then (3.14) implies that $\phi_\nu(1) = 1$, and ϕ_ν is a state. To see that ϕ_ν is a KMS_β state, we take $p, q \in \mathbb{N}^k$, $n \in \mathbb{Z}^d$, and calculate:

$$\begin{aligned} (3.15) \quad \phi_\nu(V_p \iota(g_n) V_q^*) &= \phi_\nu(V_p U_n V_q^*) \\ &= \sum_{b \in \mathbb{N}^k} e^{-\beta b^T r} ((D_n \otimes W_n)(T_q^* \delta_b \otimes 1) | T_p^* \delta_b \otimes 1) \\ &= \sum_{b \geq p \vee q} e^{-\beta b^T r} (e^{2\pi i (b-q)^T \theta n} \delta_{b-q} \otimes g_n | \delta_{b-p} \otimes 1) \\ &= \delta_{p,q} \sum_{b \geq p} e^{-\beta b^T r} e^{2\pi i (b-p)^T \theta n} (g_n | 1) \\ &= \delta_{p,q} \left(\sum_{b \in \mathbb{N}^k} e^{-\beta (b+p)^T r} e^{2\pi i b^T \theta n} \right) \int_{\mathbb{S}^d} g_n d\kappa. \end{aligned}$$

In particular,

$$(3.16) \quad \phi_\nu(\iota(g_n)) = \phi_\nu(U_n) = \left(\sum_{b \in \mathbb{N}^k} e^{-\beta b^T r} e^{2\pi i b^T \theta n} \right) \int_{\mathbb{S}^d} g_n d\kappa.$$

Thus

$$\phi_\nu(V_p U_n V_q^*) = \delta_{p,q} e^{-\beta p^T r} \phi_\nu(U_n),$$

and ϕ_ν is a KMS_β state by Proposition 3.3.

From (3.15), we have

$$\begin{aligned} \phi_\nu(U_n) &= \sum_{b \in \mathbb{N}^k} e^{-\beta b^T r} \int_{\mathbb{S}^d} e^{2\pi i (x+b^T \theta)^T n} d\kappa(x) \\ &= \sum_{b \in \mathbb{N}^k} e^{-\beta b^T r} \int_{\mathbb{S}^d} g_n \circ R_{\theta^T b} d\kappa \end{aligned}$$

$$= \int_{\mathbb{S}^d} g_n d\left(\sum_{b \in \mathbb{N}^k} e^{-\beta b^T r} R_{\theta^T b^*}(\kappa)\right),$$

which by Lemma 3.6 is $\int_{\mathbb{S}^d} g_n d\nu$. Thus

$$\phi_\nu(V_p \iota(g_n) V_q^*) = \delta_{p,q} e^{-\beta p^T r} \phi_\nu(\iota(g_n)) = \delta_{p,q} e^{-\beta p^T r} \int_{\mathbb{S}^d} g_n d\nu.$$

Since $C(\mathbb{S}^d) = \overline{\text{span}}\{g_n : n \in \mathbb{Z}^d\}$, (3.12) follows from (3.16) and the linearity and continuity of ϕ_ν .

(c) We first observe that both maps $\kappa \mapsto \nu_\kappa$ and $\nu \mapsto \phi_\nu$ are affine, and hence so is the composition. To see that the composition is surjective, we take a KMS_β state ϕ , restrict it to the range of ι to get a measure ν , and take

$$\kappa = \prod_{j=1}^k (\text{id} - e^{-\beta r_j} R_{\theta_{j^*}})(\nu).$$

Then the formula (3.3) implies that ϕ and ϕ_{ν_κ} agree on the elements $V_p \iota(f) V_q^*$, and hence by linearity and continuity on all of B_θ . Thus $\phi = \phi_{\nu_\kappa}$. The procedure which sends ϕ to κ is weak* continuous and inverts $\kappa \mapsto \phi_{\nu_\kappa}$. Thus it is a continuous bijection of one compact Hausdorff space onto another, and is therefore a homeomorphism. Thus so is the inverse $\kappa \mapsto \phi_{\nu_\kappa}$. \square

4. THE SUBINVARIANCE RELATION FOR THE DIRECT LIMIT

We now return to the set-up in which the dynamics α on the direct limit B_∞ is given by a sequence $\{r^m\}$.

Suppose that ϕ is a KMS_β state of (B_∞, α) and ν_m are the measures on \mathbb{S}^d that implement the restrictions of $\phi \circ \pi_{m,\infty}$ to $C(\mathbb{S}^d) \subset B_m$. Since the embeddings π_m are all unital, so are the $\pi_{m,\infty}$. Thus for each m , the restriction $\phi \circ \pi_{m,\infty}$ is a KMS_1 state of (B_m, α^{r^m}) , and hence is given by a probability measure ν_m which satisfies the subinvariance relations for $\theta = \theta_m$ in (3.11) parametrised by subsets F of $\{1, \dots, k\}$. But here, since $\phi \circ \pi_{m,\infty} = \phi \circ \pi_{m+l,\infty} \circ \pi_{m,m+l}$ for $l \in \mathbb{N}$, the measure ν_m satisfies a sequence of subinvariance relations parametrised by l as well as F . Our first main result says that these can be combined into one master subinvariance relation with real parameters $s \in [0, \infty)^k$.

We now describe our continuously parametrised subinvariance relation. For $k = 1$ this follows from [6, Definition 6.7 and Theorem 6.9].

Theorem 4.1. *Suppose that ϕ is a KMS_β state on (B_∞, α) and $m \in \mathbb{N}$. We write ι_m for the inclusion of $C(\mathbb{S}^d)$ in B_m , and then*

$$\iota_m(C(\mathbb{S}^d)) = \overline{\text{span}}\{U_{m,n} : n \in \mathbb{N}^d\}.$$

Let ν_m be the probability measure on \mathbb{S}^d such that

$$(4.1) \quad \phi \circ \pi_{m,\infty}(\iota_m(f)) = \int_{\mathbb{S}^d} f d\nu_m \quad \text{for } f \in C(\mathbb{S}^d).$$

Write $\theta_{m,j}$ for the j th row of the matrix θ_m . Then for every $s \in [0, \infty)^k$, we have

$$(4.2) \quad \prod_{j=1}^k (\text{id} - e^{-\beta s_j r_j^m} R_{s_j \theta_{m,j}^*})(\nu_m) \geq 0.$$

We prove Theorem 4.1 at the end of this section. We first need two preliminary results.

The homomorphism $\pi_m : B_m \rightarrow B_{m+1}$ maps $\iota_m(C(\mathbb{S}^d))$ into $\iota_{m+1}(C(\mathbb{S}^d))$. When we view $\iota_m(C(\mathbb{S}^d))$ as $\overline{\text{span}}\{U_{m,n}\}$, the homomorphism π_m is characterised by

$$\pi_m(U_{m,n}) = U_{m+1,E_m n};$$

when we view $\iota_m(C(\mathbb{S}^d))$ as $\{\iota_m(f) : f \in C(\mathbb{S}^d)\}$, π_m is induced by the covering map $E_m^T : \mathbb{S}^d \rightarrow \mathbb{S}^d$, and hence we have $\pi_m(\iota_m(f)) = \iota_{m+1}(f \circ E_m^T)$. In particular, $\pi_m|_{C(\mathbb{S}^d)}$ is $(E_m^T)^* : f \mapsto f \circ E_m^T$. The corresponding map on measures is given by E_{m*}^T :

$$\int_{\mathbb{S}^d} f d\pi_m(\nu) = \int_{\mathbb{S}^d} f dE_{m*}^T(\nu) = \int_{\mathbb{S}^d} (f \circ E_m^T) d\nu.$$

Lemma 4.2. *Suppose that ϕ is a KMS $_\beta$ state on (B_∞, α) . For $m \in \mathbb{N}$, let ν_m be the probability measure on \mathbb{S}^d satisfying (4.1). Then for every finite subset F of \mathbb{N}^k such that $p \wedge q = 0$ for all $p \neq q \in F$, we have*

$$\prod_{p \in F} (\text{id} - e^{-\beta(D_m^{-1}p)^T r^m} R_{\theta_m^T D_m^{-1} p^*})(\nu_m) \geq 0.$$

Proof. We apply Proposition 3.7(a) to the state $\phi \circ \pi_{m+1, \infty}$ of $(B_{m+1}, \alpha^{r^{m+1}})$. We deduce that

$$(4.3) \quad \prod_{p \in F} (\text{id} - e^{-\beta p^T r^{m+1}} R_{\theta_{m+1}^T p^*})(\nu_{m+1}) \geq 0.$$

To convert this to a statement about ν_m , we want to apply E_{m*}^T to the left-hand side. We first observe that

$$(4.4) \quad \begin{aligned} E_m^T \circ R_{\theta_{m+1}^T p^*}(x) &= E_m^T x - E_m^T \theta_{m+1}^T p \\ &= E_m^T x - \theta_m^T D_m^{-1} p \quad \text{using (2.3)} \\ &= R_{\theta_m^T D_m^{-1} p^*} \circ E_m^T(x). \end{aligned}$$

Since E_{m*}^T preserves positivity and $h \mapsto h_*$ is covariant with respect to composition, (4.3) implies that

$$\begin{aligned} 0 &\leq E_{m*}^T \left(\prod_{p \in F} (\text{id} - e^{-\beta p^T r^{m+1}} R_{\theta_{m+1}^T p^*})(\nu_{m+1}) \right) \\ &= \left(\prod_{p \in F} (\text{id} - e^{-\beta p^T r^{m+1}} R_{\theta_m^T D_m^{-1} p^*}) \right) \circ E_{m*}^T(\nu_{m+1}) \quad \text{using (4.4)} \\ &= \prod_{p \in F} (\text{id} - e^{-\beta p^T (D_m)^{-1} r^m} R_{\theta_m^T D_m^{-1} p^*})(\nu_m) \quad \text{using (2.4)} \\ &= \prod_{p \in F} (\text{id} - e^{-\beta (D_m^{-1} p)^T r^m} R_{\theta_m^T D_m^{-1} p^*})(\nu_m). \quad \square \end{aligned}$$

For a positive integer l , we can apply the argument of Lemma 4.2 to the embedding $\pi_{m,m+l}$ of B_m in B_{m+l} . This amounts to replacing the matrix D_m with $D_{m,m+l} := D_{m+l-1} D_{m+l-2} \cdots D_{m+1} D_m$, E_m with a similarly defined $E_{m,m+l}$, θ_{m+1} with θ_{m+l} , and r^{m+1} with r^{m+l} . We obtain:

Corollary 4.3. *Suppose that ϕ is a KMS_β state on (B_∞, α) , and ν_m is the probability measure satisfying (4.1). Then for every positive integer l and for every finite subset F of \mathbb{N}^k such that $p \wedge q = 0$ for all $p \neq q \in F$, we have*

$$\prod_{p \in F} (\text{id} - e^{-\beta(D_{m,m+l}^{-1})^T r^m} R_{\theta_m^T D_{m,m+l}^{-1} p^*})(\nu_m) \geq 0.$$

Proof of Theorem 4.1. For each $l \geq 0$ and $p \in \mathbb{N}^k$, we can apply Corollary 4.3 to the finite subset $F_p := \{p_j e_j : 1 \leq j \leq k\}$ of \mathbb{N}^k . This gives us the subinvariance relation

$$(4.5) \quad \prod_{j=1}^k (\text{id} - e^{-\beta(D_{m,m+l}^{-1} p_j e_j)^T r^m} R_{\theta_m^T D_{m,m+l}^{-1} p_j e_j^*})(\nu_m) \geq 0.$$

Each factor in the left-hand side L of (4.5) has the form $\text{id} - e^{-s} R_{v^*}$. Since $(e^{-s} R_{v^*})(e^{-t} R_{w^*}) = e^{-(s+t)} R_{(v+w)^*}$, the product $(\text{id} - e^{-s} R_{v^*})(\text{id} - e^{-t} R_{w^*})$ of two such terms collapses to

$$\text{id} - e^{-s} R_{v^*} - e^{-t} R_{w^*} + e^{-(s+t)} R_{(v+w)^*}.$$

Thus we can expand

$$L = \text{id} + \sum_{\emptyset \neq G \subset \{1, \dots, k\}} (-1)^{|G|} e^{-\beta(D_{m,m+l}^{-1} p_G)^T r^m} R_{\theta_m^T D_{m,m+l}^{-1} p_G^*}(\nu_m),$$

where $p_G := \sum_{j \in G} p_j e_j$.

For each fixed $f \in C(\mathbb{S}^d)$ and $\nu \in P(\mathbb{S}^d)$, the function $s \mapsto \int R_s(f) d\nu$ on \mathbb{R}^k is continuous, being the composition of the norm-continuous map $s \mapsto R_s(f)$ and the bounded functional given by integration against ν . We now consider a positive function f in $C(\mathbb{S}^d)$: we write $f \in C(\mathbb{S}^d)_+$. For $s \in [0, \infty)^k$ and $G \subset \{1, \dots, k\}$, we write $s_G = \sum_{j \in G} s_j e_j$. Then

$$g_G : s \mapsto \int f d(e^{-\beta s_G^T r^m} R_{\theta_m^T s_G^*})(\nu_m)$$

is continuous, and so is the linear combination

$$L(s) := \int f d\left(\sum_{G \subset \{1, \dots, k\}} (-1)^{|G|} e^{-\beta s_G^T r^m} R_{\theta_m^T s_G^*}\right)(\nu_m).$$

The subinvariance relation (4.5) says that $L(s) \geq 0$ for all s of the form $D_{m,m+l} p$ for $l \geq 0$ and $p \in \mathbb{N}^k$.

Since each of the matrices D_m is diagonal with entries $d_{m,j}$, say, at least 2, we have

$$D_{m,m+l}^{-1} p_j e_j = \left(\prod_{n=0}^{l-1} d_{m+n,j}^{-1}\right) p_j e_j.$$

Since $d_{n,j} \geq 2$ for all n and j , the rational numbers of the form $(\prod_{n=0}^{l-1} d_{m+n,j}^{-1}) p_j$ are dense in $[0, \infty)$. Thus the vectors s for which $L(s) \geq 0$ form a dense subset of $[0, \infty)^k$, and the continuity of L implies that $L(s) \geq 0$ for all $s \in [0, \infty)^k$. A measure ν which has $\int f d\nu \geq 0$ for all $f \in C(\mathbb{S}^d)_+$ is a positive measure, and this is what we had to prove. \square

5. THE SOLUTION OF THE SUBINVARIANCE RELATION

We now describe the solutions to the subinvariance relation (4.2). We observe that the formula on the right of (5.1) below is the Laplace transform of a periodic function, and as such is given by an integral over a finite rectangle. This observation motivated our calculations, but in the end we found it easier to work with the trigonometric polynomials $x \mapsto e^{2\pi i n x}$.

Theorem 5.1. *Let $\theta \in M_{k,d}(\mathbb{S}^d)$, $\beta \in (0, \infty)$ and $r = (r_j) \in (0, \infty)^k$. Denote the j th row of θ by θ_j .*

(a) *For each $\mu \in M(\mathbb{S}^d)$, there is a nonnegative measure $\nu_\mu \in M(\mathbb{S}^d)$ such that*

$$(5.1) \quad \int_{\mathbb{S}^d} f d\nu_\mu = \int_{[0,\infty)^k} e^{-\beta w^T r} \int_{\mathbb{S}^d} f(x + \theta^T w) d\mu(x) dw \quad \text{for } f \in C(\mathbb{S}^d),$$

and ν_μ has total mass $\|\mu\| \prod_{j=1}^k (\beta r_j)^{-1}$. The measure $\nu = \nu_\mu$ satisfies the subinvariance relation

$$(5.2) \quad \prod_{j=1}^k (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^T *})(\nu) \geq 0 \quad \text{for } s \in [0, \infty)^k.$$

(b) *For each ν satisfying the subinvariance relation (5.2), there is a measure $\mu_\nu \in M(\mathbb{S}^d)$ such that*

$$(5.3) \quad \int_{\mathbb{S}^d} f d\mu_\nu = \lim_{s_k \rightarrow 0^+} \cdots \lim_{s_1 \rightarrow 0^+} \frac{1}{s_k \cdots s_1} \int_{\mathbb{S}^d} f d(\prod_{j=1}^k (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^T *}))(\nu)$$

for $f \in C(\mathbb{S}^d)$, and μ_ν has total mass $\|\nu\| \prod_{j=1}^k (\beta r_j)$.

(c) *The map $\mu \mapsto \nu_\mu$ is an affine homeomorphism of $M(\mathbb{S}^d)$ onto the simplex of measures satisfying the subinvariance relation (5.2), and the inverse takes ν to μ_ν .*

Remark 5.2. When we apply Theorem 5.1, the measure μ comes from a KMS state, and hence is a probability measure. The corresponding solutions of the subinvariance relation (5.2) are the elements of the simplex

$$\Sigma_{\beta,r} := \left\{ \kappa \in M(\mathbb{S}^d) : \kappa(\mathbb{S}^d) = \prod_{j=1}^k (\beta r_j)^{-1} \right\}.$$

This is the analogue for our situation of the simplex Σ_β appearing in [3, Theorem 6.1]: because the local homeomorphisms $h_i : x \mapsto x + \theta_i$ of \mathbb{S}^d are homeomorphisms, the function f_β of [3] is the constant function $x \mapsto \prod_{j=1}^k (\beta r_j)$, which we denote by $f_{\beta,r}$ (see Proposition 4.3 in [3]).

Remark 5.3. A measure ν that satisfies the subinvariance relation (5.2) also satisfies the analogous relation involving $\prod_{j \in J} (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^T *})$ for any subset J of $\{1, \dots, k\}$. To see this, observe that for any vector $y \in [0, \infty)^d$, R_y is an isometric positivity-preserving linear operator on $C(\mathbb{S}^d)$. Hence so are R_{y^*} and $e^{-\beta s_j r_j} R_{y^*}$. Since the numbers $-\beta r_j$ are negative, the series $\sum_{n=0}^{\infty} e^{-\beta s_j r_j n} R_{y^*}^n$ converges in norm in the Banach space of bounded linear operators on $M(\mathbb{S}^d)$ to an inverse for $\text{id} - e^{-\beta s_j r_j} R_{y^*}$. Hence applying this inverse allows us to remove factors from the subinvariance relation without losing positivity.

Remark 5.4 (Reality check). We reassure ourselves that the description of subinvariant measures in Theorem 5.1 is consistent with the description in [6, Theorem 7.1]. There $d = k = 1$, and they describe the simplex of subinvariant probability measures by specifying the extreme points of the simplex.

We recall that the matrices $D_m \in M_1(\mathbb{N}) = \mathbb{N}$ and $E_m \in M_1(\mathbb{N})$ are all the same integer $N \geq 2$, and the sequence θ_m then satisfies $N^2\theta_{m+1} = \theta_m$. In terms of our generators, the dynamics $\alpha : \mathbb{R} \rightarrow \text{Aut } B_m$ in [6] is given by

$$\alpha_t(V_{m,p}U_{m,n}V_{m,q}^*) = e^{it(p-q)N^{-m}}V_{m,p}U_{m,n}V_{m,q}^*$$

(see [6, Proposition 6.3]), which is our α^{r^m} with $r^m = N^{-m}$. We are interested in KMS_β states, so the subinvariant probability measures for (B_m, α) are those in the set denoted Ω_{sub}^r for $r = \beta N^{-m}\theta_m^{-1} = \beta r^m\theta_m^{-1}$ (see [6, Notation 6.8]²).

Since the calculation in [6] is about extreme points, we start with a point mass $\delta_y \in P(\varprojlim \mathbb{S})$. Then $(E_m^T)_*\delta_y$ is the point mass $\mu_m = \delta_{y_m}$, where y_m is obtained by realising y as a sequence $\{y_m\}$ satisfying $Ny_{m+1} = y_m$. Then the measure ν_{μ_m} in Theorem 5.1(a) is given by

$$\begin{aligned} \int f d\nu_{\mu_m} &= \int_0^\infty e^{-\beta wr^m} \int_0^1 f(x + \theta_m w) d\mu_m(x) dw \\ &= \int_0^\infty e^{-\beta wr^m} f(y_m + \theta_m w) dw. \end{aligned}$$

For $f(x) = e^{2\pi i n x}$, we get

$$\int f d\nu_{\mu_m} = e^{2\pi i n y_m} \int_0^\infty e^{-\beta wr^m} e^{2\pi i n \theta_m w} dw,$$

and a change of variables gives

$$\begin{aligned} \int f d\nu_{\mu_m} &= e^{2\pi i n y_m} \int_0^\infty e^{-\beta \theta_m^{-1} v r^m} e^{2\pi i n v \theta_m^{-1}} dv \\ &= e^{2\pi i n y_m} \theta_m^{-1} \int_0^\infty e^{-(\beta r^m \theta_m^{-1})v} e^{2\pi i n v} dv. \end{aligned}$$

Now we recognise the integral as the Laplace transform of the periodic function $x \mapsto e^{2\pi i n x}$, and hence

$$(5.4) \quad \int f d\nu_{\mu_m} = e^{2\pi i n y_m} \theta_m^{-1} \frac{1}{1 - e^{-\beta r^m \theta_m^{-1}}} \int_0^1 e^{-(\beta r^m \theta_m^{-1})v} e^{2\pi i n v} dv.$$

In the notation of [6], we set $r := \beta r^m \theta_m^{-1}$, and rewrite (5.4) as

$$\begin{aligned} \int f d\nu_{\mu_m} &= e^{2\pi i n y_m} \beta^{-1} N^m \int_0^1 \frac{r}{1 - e^{-r}} e^{-rv} e^{2\pi i n v} dv \\ &= \beta^{-1} N^m \int_0^1 e^{2\pi i n v} d(R_{y_m})_*(m_r)(v). \end{aligned}$$

This shows that the measure ν_{μ_m} is a multiple of the measure $(R_{y_m})_*(m_r)$ appearing in [6, Theorem 7.1]. We are off by the scalar $\beta^{-1} N^m$ because that theorem

²The displayed equation there is meant to say this, as opposed to $r = \beta N^{-m}\theta_m$, which is the way we first read it.

is about the simplex of subinvariant *probability* measures, and the measures ν_μ in Theorem 5.1 have total mass $(\beta r^m)^{-1} = \beta^{-1} N^m$.

For the proof of Theorem 5.1(a), we need the following lemma, which is known to probabilists as the *inclusion-exclusion principle*. We couldn't find a good reference for this measure-theoretic version, but fortunately it is relatively easy to prove by induction on the number k of subsets.

Lemma 5.5. *Suppose that λ is a finite measure on a space X and $\{S_j : 1 \leq j \leq k\}$ is a finite collection of measurable subsets of X . For each subset G of $\{1, \dots, k\}$, we set $S_G := \bigcap_{j \in G} S_j$. Then*

$$\lambda\left(\bigcup_{j=1}^k S_j\right) = \sum_{\emptyset \neq G \subset \{1, \dots, k\}} (-1)^{|G|-1} \lambda(S_G).$$

Proof of Theorem 5.1(a). We first claim that there is a positive functional I on $C(\mathbb{S}^d)$ such that $I(f)$ is given by the right-hand side of (5.1). Indeed, the estimate

$$\left| \int_{[0, \infty)^k} e^{-\beta w^T r} \int_{\mathbb{S}^d} f(x + \theta^T w) d\mu(x) dw \right| \leq \int_{[0, \infty)^k} e^{-\beta w^T r} \int_{\mathbb{S}^d} \|f\|_\infty d\mu(x) dw,$$

shows that the right-hand side of (5.1) determines a bounded function $I : C(\mathbb{S}^d) \rightarrow \mathbb{C}$. This function I is linear because the integral is linear, and $f \geq 0$ implies $I(f) \geq 0$ because all the integrands in (5.1) are non-negative. Thus there is a finite nonnegative measure ν_μ satisfying (5.1). The norm of the integral is given by the total mass of the measure ν_μ , which is

$$\int_{\mathbb{S}^d} 1 d\nu_\mu = \int_{[0, \infty)^k} e^{-\beta w^T r} \|\mu\| dw.$$

To compute the exact value of the integral, observe that

$$e^{-\beta w^T r} = e^{-\beta \sum_{j=1}^k w_j r_j} = \prod_{j=1}^k e^{-\beta w_j r_j}.$$

Thus

$$\|\nu_\mu\| = \|\mu\| \int_{[0, \infty)^k} \prod_{j=1}^k e^{-\beta w_j r_j} dw = \|\mu\| \prod_{j=1}^k \int_0^\infty e^{-\beta w_j r_j} dw_j = \|\mu\| \prod_{j=1}^k (\beta r_j)^{-1}.$$

This proves the assertions in the first sentence of part (a).

For the subinvariance relation, we fix $f \in C(\mathbb{S}^d)_+$, and aim to prove that

$$\int_{\mathbb{S}^d} f d\left(\prod_{j=1}^k (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^*}) (\nu_\mu)\right) \geq 0.$$

As in the proof of Theorem 4.1, we write

$$\prod_{j=1}^k (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^*}) = \text{id} + \sum_{\emptyset \neq G \subset \{1 \leq j \leq k\}} (-1)^{|G|} e^{-\beta s_G^T r} R_{\theta^T s_G^*},$$

with $s_G := \sum_{j \in G} s_j e_j$. For $j \leq k$ we define $S_j = \{v \in [0, \infty) : v_j \geq s_j\}$, and $S_G := \bigcap_{j \in G} S_j$. Then

$$\begin{aligned}
 (5.5) \quad \int_{\mathbb{S}^d} f d(e^{-\beta s_G^T r} R_{\theta^T s_G^*})(\nu_\mu) &= \int_{\mathbb{S}^d} e^{-\beta s_G^T r} (f \circ R_{\theta^T s_G}) d\nu_\mu \\
 &= \int_{[0, \infty)^k} e^{-\beta w^T r} e^{-\beta s_G^T r} \int_{\mathbb{S}^d} f(x + \theta^T w + \theta^T s_G) d\mu(x) dw \\
 &= \int_{S_G} e^{-\beta v^T r} \int_{\mathbb{S}^d} f(x + \theta^T v) d\mu(x) dv.
 \end{aligned}$$

Since f is fixed, we can define a measure m on $[0, \infty)^k$ by

$$\int_{[0, \infty)^k} g dm = \int_{[0, \infty)^k} g(v) e^{-\beta v^T r} \int_{\mathbb{S}^d} f(x + \theta^T v) d\mu(x) dv.$$

Now (5.5) says that

$$\int_{\mathbb{S}^d} f d(e^{-\beta s_G^T r} R_{\theta^T s_G^*})(\nu_\mu) = m(S_G).$$

Thus

$$\int_{\mathbb{S}^d} f d\left(\prod_{j=1}^k (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^*})\right)(\nu_\mu) = m([0, \infty)^k) + \sum_{\emptyset \neq G \subset \{1 \leq j \leq k\}} (-1)^{|G|} m(S_G).$$

By the inclusion-exclusion principle, this is

$$m([0, \infty)^k) - m(\bigcup_{j=1}^k S_j) = m(\prod_{j=1}^k [0, s_j]) \geq 0. \quad \square$$

We now move towards a proof of part (b), and for that the first problem is to prove that the iterated limit in (5.3) exists. We will work with l satisfying $1 \leq l \leq k$, and show by induction on l that the iterated limit

$$\lim_{s_l \rightarrow 0^+} \cdots \lim_{s_1 \rightarrow 0^+}$$

exists. We will be doing some calculus, so we often assume that our test functions f belong to the dense subalgebra $C^\infty(\mathbb{S}^d)$ of $C(\mathbb{S}^d)$ consisting of smooth functions all of whose derivatives are also periodic.

We begin by establishing that, even after dividing by the numbers which are going to 0, the norms of the measures remain uniformly bounded.

Lemma 5.6. *Suppose that ν is a finite positive measure on \mathbb{S}^d satisfying the subinvariance relation (5.2). Then for each $s \in (0, \infty)^k$,*

$$\lambda_s := \frac{1}{s_k s_{k-1} \cdots s_1} \prod_{j=1}^k (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^*})(\nu)$$

is a positive measure with total mass

$$(5.6) \quad \|\lambda_s\| \leq \left(\prod_{j=1}^k (\beta r_j) \right) \|\nu\|.$$

Proof. The subinvariance relation implies that the measure is positive. For the estimate on the total mass of λ_s , we deal with the variables s_i separately. So for $1 \leq l \leq k$, we set

$$\sigma_l := \prod_{j=l}^k (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^T *})(\nu),$$

which by Remark 5.3 are all positive measures. We have

$$\begin{aligned} \int_{\mathbb{S}^d} 1 d\left(\prod_{j=1}^k (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^T *})(\nu)\right) &= \int_{\mathbb{S}^d} 1 d\sigma_1 \\ &= \int_{\mathbb{S}^d} 1 d((\text{id} - e^{-\beta s_1 r_1} R_{s_1 \theta_1^T *})(\sigma_2)) \\ &= \int_{\mathbb{S}^d} 1 \circ (\text{id} - e^{-\beta s_1 r_1} R_{s_1 \theta_1^T}) d\sigma_2 \\ &= \int_{\mathbb{S}^d} (1 - e^{-\beta s_1 r_1}) d\sigma_2. \end{aligned}$$

So for all $s_1 > 0$,

$$(5.7) \quad \frac{1}{s_1} \int_{\mathbb{S}^d} 1 d\left(\prod_{j=1}^k (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^T *})(\nu)\right) = \int_{\mathbb{S}^d} \frac{1 - e^{-\beta s_1 r_1}}{s_1} d\sigma_2.$$

The integrand here is

$$\frac{1 - e^{-\beta s_1 r_1}}{s_1} = \frac{f(0) - f(s_1)}{s_1} \quad \text{for } f(s_1) := e^{-\beta s_1 r_1}.$$

Hence for each fixed $s_1 > 0$, the mean value theorem implies that there exists $c \in (0, s_1)$ such that

$$\frac{1 - e^{-\beta s_1 r_1}}{s_1} = -f'(c) = -(-\beta r_1 e^{-\beta c r_1}),$$

which is a positive number less than βr_1 . Thus (5.7) is at most $\beta r_1 \|\sigma_2\|$.

Now we repeat this argument, first to see that

$$\frac{1}{s_2} \int_{\mathbb{S}^d} 1 d(\text{id} - e^{-\beta s_2 r_2} R_{s_2 \theta_2^T *})(\sigma_3)$$

has mass at most $\beta r_2 \|\sigma_3\|$. After $k - 2$ more steps, we arrive at the estimate (5.6). \square

Lemma 5.7. *Suppose that $1 \leq j \leq k$ and that $\lambda \in M(\mathbb{S}^d)$ satisfies*

$$(\text{id} - e^{-\beta s r_j} R_{s \theta_j^T *})(\lambda) \geq 0 \quad \text{for all } s > 0.$$

Then for all $f \in C^\infty(\mathbb{S}^d)$, we have

$$(5.8) \quad \lim_{s \rightarrow 0^+} \left(\frac{1}{s} \int_{\mathbb{S}^d} f d(\text{id} - e^{-\beta s r_j} R_{s \theta_j^T *})(\lambda) \right) = \beta r_j \int_{\mathbb{S}^d} f d\lambda - \int_{\mathbb{S}^d} \theta_j^T(\nabla f) d\lambda.$$

Proof. Let $g : \mathbb{S}^d \times \mathbb{R} \rightarrow \mathbb{C}$ be the function $g(x, s) = e^{-\beta sr_j} f(x + s\theta_j^T)$. The term on the left of (5.8) can be rewritten as

$$\frac{1}{s} \int_{\mathbb{S}^d} (f(x) - e^{-\beta sr_j} f(x + s\theta_j^T)) d\lambda(x) = \frac{1}{s} \int_{\mathbb{S}^d} (g(x, 0) - g(x, s)) d\lambda(x).$$

So we want to show that the function G defined by $G(s) := \int_{\mathbb{S}^d} g(x, s) d\lambda(x)$ is differentiable at 0 with $-G'(0)$ equal to the right-hand side of (5.8).

We compute

$$\frac{\partial g}{\partial s}(x, s) = -\beta r_j e^{-\beta sr_j} f(x + s\theta_j^T) + e^{-\beta sr_j} \theta_j^T \nabla f(x + s\theta_j^T).$$

The Cauchy-Schwarz inequality for the inner product $\theta_j^T(\nabla f) = (\theta_j | \nabla f)$ then gives

$$(5.9) \quad \left| \frac{\partial g}{\partial s}(x, s) \right| \leq \beta r_j \|f\|_\infty + \|\theta_j^T\|_2 \|\nabla f(x + s\theta_j^T)\|_2.$$

The right-hand side is uniformly bounded on \mathbb{S}^d , and hence there is an integrable function on \mathbb{S}^d that dominates the right-hand side for all $s \in [0, 1]$, say. Thus we can differentiate under the integral sign, using Theorem 2.27 of [14], for example. We deduce that G is differentiable on $[0, 1]$ with derivative

$$G'(s) = \int_{\mathbb{S}^d} (-\beta r_j e^{-\beta sr_j} f(x + s\theta_j^T) + e^{-\beta sr_j} \theta_j^T \nabla f(x + s\theta_j^T)) d\lambda(x).$$

Taking $s = 0$ gives the negative of the right-hand side of (5.8), as required. \square

Our next step is the inductive argument, which is quite a complicated one. As a point of notation, for each tuple $I = \{i_1, \dots, i_m\}$ with entries in $\{1, 2, \dots, k\}$, and for $f \in C^\infty(\mathbb{S}^d)$, we write $|I| := m$ and $D_I f$ for the partial derivative

$$D_I f := \frac{\partial^m f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_m}}.$$

Lemma 5.8. *Suppose that ν is a positive measure on \mathbb{S}^d satisfying the subinvariance relation (5.2). Let $1 \leq l \leq k$.*

(a) *The iterated limit*

$$\lim_{s_l \rightarrow 0^+} \cdots \lim_{s_1 \rightarrow 0^+} \frac{1}{s_l \cdots s_1} \int_{\mathbb{S}^d} f d(\prod_{j=1}^l (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^T *})) (\nu)$$

exists for all $f \in C(\mathbb{S}^d)$.

(b) *Write*

$$\Sigma_l := \bigcup_{n=0}^l \{1, \dots, k\}^n.$$

Then there are real scalars $\{K_I^l : I \in \Sigma_l\}$ such that $K_\emptyset^l = \prod_{j=1}^l (\beta r_j)$, and: for every $f \in C^\infty(\mathbb{S}^d)$ and for every measure ν on \mathbb{S}^d satisfying the subinvariance relation (5.2), the limit in (a) is

$$(5.10) \quad \int_{\mathbb{S}^d} \left(\sum_{I \in \Sigma_l} K_I^l D_I f \right) d\nu.$$

Proof. We prove by induction on l that the limit in (a) exists for every $f \in C^\infty(\mathbb{S}^d)$, and that there exist the scalars K_I^l . Then, since we know from Lemma 5.6 that the measures λ_s are norm-bounded by $(\prod_{j=1}^k(\beta r_j))\|\nu\|$ and $C^\infty(\mathbb{S}^d)$ is norm dense in $C(\mathbb{S}^d)$, we get convergence in (a) also for $f \in C(\mathbb{S}^d)$.

When $l = 1$, the index set Σ_1 consists of the empty set \emptyset and the one-point sets $\{j\}$. Lemma 5.7 implies that $K_\emptyset^1 = \beta r_1$ and $K_{\{j\}}^1 = \theta_1^T e_j = \theta_{1j}$.

We fix l between 1 and $k - 1$, and suppose as our inductive hypothesis that for every measure λ such that

$$(5.11) \quad \prod_{j=1}^l (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^T *})(\lambda) \geq 0 \quad \text{for all } s \in [0, \infty)^k,$$

we have such scalars $\{K_I^l\}$ parametrised by $I \in \Sigma_l$. We now have to start with a measure κ that satisfies

$$(5.12) \quad \prod_{j=1}^{l+1} (\text{id} - e^{-\beta s_j r_j} R_{\theta_j^T *})(\kappa) \geq 0 \quad \text{for all } s \in [0, \infty)^k,$$

and find suitable scalars K_I^{l+1} .

We define

$$\lambda := (\text{id} - e^{-\beta s_{l+1} r_{l+1}} R_{s_{l+1} \theta_{l+1}^T *})(\kappa).$$

Remark 5.3 reassures us that λ is another positive measure, and (5.12) implies that it satisfies (5.11). The induction hypothesis gives

$$\begin{aligned} L(s_{l+1}) &:= \lim_{s_l \rightarrow 0^+} \cdots \lim_{s_1 \rightarrow 0^+} \frac{1}{s_{l+1} \cdots s_1} \int_{\mathbb{S}^d} f d \left(\prod_{j=1}^{l+1} (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^T *}) \right) (\kappa) \\ &= \frac{1}{s_{l+1}} \left(\int_{\mathbb{S}^d} \left(\sum_{I \in \Sigma_l} K_I^l D_I f \right) d\lambda \right). \end{aligned}$$

Lemma 5.7 implies that

$$\begin{aligned} \lim_{s_{l+1} \rightarrow 0^+} L(s_{l+1}) &= \beta r_{l+1} \int_{\mathbb{S}^d} \left(\sum_{I \in \Sigma_l} K_I^l D_I f \right) d\lambda - \int_{\mathbb{S}^d} \theta_{l+1}^T \nabla \left(\sum_{I \in \Sigma_l} K_I^l D_I f \right) d\lambda \\ &= \beta r_{l+1} \int_{\mathbb{S}^d} \left(\sum_{I \in \Sigma_l} K_I^l D_I f \right) d\lambda - \int_{\mathbb{S}^d} \left(\sum_{I \in \Sigma_l} \sum_{i=1}^d K_I^l \theta_{l+1, i} \frac{\partial D_I f}{\partial x_i} \right) d\lambda(x). \end{aligned}$$

To finish off the inductive step, we set $K_\emptyset^{l+1} = \beta r_{l+1} K_\emptyset^l$, and for $I' = (I, i_{|I|+1})$ we set

$$K_{I'}^{l+1} = \begin{cases} K_I^l \theta_{(l+1)i_{l+1}} & \text{if } |I| = l \\ \beta r_{l+1} K_{I'}^l - K_I^l \theta_{l+1, i_{|I|+1}} & \text{if } |I| < l. \end{cases}$$

This completes the inductive step, and hence the proof. \square

Proof of Theorem 5.1(b). Lemma 5.8 shows that the limit exists for all $f \in C(\mathbb{S}^d)$, and for $f \in C^\infty(\mathbb{S}^d)$ gives us a formula for the limit. The limit is linear in f , positive when f is, and is bounded by $\|f\|_\infty (\prod_{j=1}^k(\beta r_j))\|\nu\|$. Thus it is given by a finite positive measure μ_ν . Since the total mass of the measure is integration against the constant function 1, and since 1 is smooth, the total mass is given by (5.10).

But since all derivatives of 1 are zero, the only nonzero terms are the ones on which $I = \emptyset$. Now the formula for K_\emptyset^k implies that $\|\mu_\nu\| = (\prod_{j=1}^k (\beta r_j)) \|\nu\|$. \square

We now work towards the proof of Theorem 5.1(c). To prove that $N : \mu \mapsto \nu_\mu$ is a bijection of the measures arising from KMS_β states onto the subinvariant measures, we prove that N is one-to-one and that $M : \nu \mapsto \mu_\nu$ satisfies $N \circ M(\nu) = \nu$ for all subinvariant measures ν . We then have $N \circ (M \circ N)(\mu) = (N \circ M) \circ N(\mu) = N(\mu)$, and injectivity of N implies $(M \circ N)(\mu) = \mu$. Thus Theorem 5.1(c) follows from the following proposition.

Proposition 5.9. *Suppose that ν is a measure on \mathbb{S}^d satisfying the subinvariance relation (5.2) and with total mass $\prod_{j=1}^k (\beta r_j)^{-1}$. Then $\nu = \nu_{\mu_\nu}$.*

Suppose that ν is a subinvariant measure and $f \in C^\infty(\mathbb{S}^d)$. We need to show that the functional defined by integrating against ν_{μ_ν} , which is defined in parts (a) and (b) of the theorem as

$$(5.13) \quad \int_{[0, \infty)^k} e^{-\beta w^T r} \lim_{s \rightarrow 0^+} \frac{1}{s_k \cdots s_1} \int_{\mathbb{S}^d} f(x + \theta^T w) d\left(\prod_{j=1}^k (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^T *})\right)(\nu)(x) dw,$$

is in fact implemented by ν . We will do this by peeling off the iterated limit one variable at a time. For this, the next lemma is crucial.

Lemma 5.10. *Consider a positive measure λ on \mathbb{S}^d , $b \in (0, \infty)$ and $v \in \mathbb{S}^d$. For $f \in C^\infty(\mathbb{S}^d)$, we have*

$$\begin{aligned} \int_0^\infty e^{-bt} \lim_{s \rightarrow 0^+} \frac{1}{s} \int_{\mathbb{S}^d} f(y + tv) d((\text{id} - e^{-bs} R_{sv*})\lambda)(y) dt \\ = \lim_{s \rightarrow 0^+} \int_0^\infty \frac{e^{-bt}}{s} \int_{\mathbb{S}^d} f(y + tv) d((\text{id} - e^{-bs} R_{sv*})\lambda)(y) dt. \end{aligned}$$

Proof. For $s > 0$, we have

$$\frac{1}{s} \int_{\mathbb{S}^d} f(y + tv) d((\text{id} - e^{-bs} R_{sv*})\lambda)(y) = \frac{1}{s} \int_{\mathbb{S}^d} (f(y + tv) - e^{-bs} f(y + tv + sv)) d\lambda(y).$$

We write this last integrand as

$$\begin{aligned} K(y, s, t) &= s^{-1} (f(y + tv) - e^{-bs} f(y + tv + sv)) \\ &= s^{-1} (f(y + tv) - e^{-bs} f(y + tv) + e^{-bs} f(y + tv) - e^{-bs} f(y + tv + sv)) \\ &= \frac{1 - e^{-bs}}{s} f(y + tv) + e^{-bs} \frac{f(y + tv) - f(y + tv + sv)}{s}. \end{aligned}$$

We estimate the first summand using the mean value theorem on e^{-bs} , and the second summand using the same theorem on f , to find

$$|K(y, s, t)| \leq b \|f\|_\infty + \|v^T(\nabla f)\|_\infty.$$

Thus we have

$$\left| \frac{e^{-bt}}{s} \int_{\mathbb{S}^d} K(y, s, t) d\lambda(y) \right| \leq e^{-bt} (b \|f\|_\infty + \|v^T(\nabla f)\|_\infty) \|\lambda\|.$$

Now the result follows from the dominated convergence theorem for Lebesgue measure on $[0, \infty)$ (modulo the trick of observing that it suffices to work with sequences $s_n \rightarrow 0+$ — see the proof of [14, Theorem 2.27]). \square

Proof of Proposition 5.9. As in the proof of Theorem 5.1(b), Lemma 5.8 implies that there is a positive measure η on \mathbb{S}^d such that for $g \in C^\infty(\mathbb{S}^d)$, we have

$$\int_{\mathbb{S}^d} g d\eta = \lim_{s_k, \dots, s_2 \rightarrow 0^+} \frac{1}{s_k \cdots s_2} \int_{\mathbb{S}^d} g d\left(\prod_{j=2}^k (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^T})\right)(\nu).$$

Since the operators $\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^T}$ commute with each other,

$$(5.14) \quad \int_{\mathbb{S}^d} g d(\text{id} - e^{-\beta s_1 r_1} R_{s_1 \theta_1^T})(\eta) \\ = \lim_{s_k, \dots, s_2 \rightarrow 0^+} \frac{1}{s_k \cdots s_2} \int_{\mathbb{S}^d} g d\left(\prod_{j=1}^k (\text{id} - e^{-\beta s_j r_j} R_{s_j \theta_j^T})\right)(\nu).$$

Now we need some complicated notation to implement the peeling process. First of all, we fix $f \in C^\infty(\mathbb{S}^d)$. In an attempt to avoid an overdose of subscripts, we write $s = (s_1, \hat{s})$, $w = (w_1, \hat{w})$ and $r = (r_1, \hat{r})$. We also write $\hat{\theta}$ for the $k-1 \times d$ matrix obtained from θ by deleting the first row: thus θ^T has block form $(\theta_1^T \hat{\theta}^T)$. With the new notation, (5.13) becomes

$$\int_{[0, \infty)^{k-1}} e^{-\beta \hat{w}^T \hat{r}} \int_0^\infty e^{-\beta r_1 w_1} \times \\ \times \lim_{s_1 \rightarrow 0^+} \frac{1}{s_1} \int_{\mathbb{S}^d} f(x + \hat{\theta}^T \hat{w} + w_1 \theta_1^T) d(\text{id} - e^{-\beta r_1 w_1} R_{s_1 \theta_1^T})(\eta)(x) dw_1 d\hat{w}.$$

Now we can apply Lemma 5.10 to the inside integrals, which gives

$$(5.15) \quad \int_{\mathbb{S}^d} f d\nu_{\mu_\nu} = \int_{[0, \infty)^{k-1}} e^{-\beta \hat{w}^T \hat{r}} \lim_{s_1 \rightarrow 0^+} \frac{1}{s_1} \int_0^\infty e^{-\beta r_1 w_1} \times \\ \times \int_{\mathbb{S}^d} f(x + \hat{\theta}^T \hat{w} + w_1 \theta_1^T) d(\text{id} - e^{-\beta r_1 w_1} R_{s_1 \theta_1^T})(\eta)(x) dw_1 d\hat{w}.$$

We now consider the function g on $(0, \infty)$ defined by

$$g(s_1) := \frac{1}{s_1} \int_0^\infty e^{-\beta r_1 w_1} \int_{\mathbb{S}^d} f(x + \hat{\theta}^T \hat{w} + w_1 \theta_1^T) d(\text{id} - e^{-\beta r_1 w_1} R_{s_1 \theta_1^T})(\eta)(x) dw_1.$$

We aim to prove that $g(s_1) \rightarrow \int_{\mathbb{S}^d} f(x + \hat{\theta}^T \hat{w}) d\eta(x)$ as $s_1 \rightarrow 0^+$. To this end, we compute:

$$g(s_1) = \frac{1}{s_1} \int_0^\infty e^{-\beta r_1 w_1} \int_{\mathbb{S}^d} f(x + \hat{\theta}^T \hat{w} + w_1 \theta_1^T) d\eta(x) dw_1 \\ - \frac{1}{s_1} \int_0^\infty e^{-\beta r_1 (w_1 + s_1)} \int_{\mathbb{S}^d} f(x + \hat{\theta}^T \hat{w} + (w_1 + s_1) \theta_1^T) d\eta(x) dw_1.$$

Changing the variable in the second integral to get an integral over $[s_1, \infty)$ gives

$$g(s_1) = \frac{1}{s_1} \int_0^{s_1} e^{-\beta r_1 w_1} \int_{\mathbb{S}^d} f(x + \hat{\theta}^T \hat{w} + w_1 \theta_1^T) d\eta(x) dw_1.$$

Now we have

$$\begin{aligned} g(s_1) &= \int_{\mathbb{S}^d} f(x + \hat{\theta}^T \hat{w}) d\eta(x) \\ &= \frac{1}{s_1} \int_0^{s_1} e^{-\beta r_1 w_1} \int_{\mathbb{S}^d} f(x + \hat{\theta}^T \hat{w} + w_1 \theta_1^T) d\eta(x) dw_1 - \int_{\mathbb{S}^d} f(x + \hat{\theta}^T \hat{w}) d\eta(x) \\ &= \frac{1}{s_1} \int_0^{s_1} \int_{\mathbb{S}^d} (e^{-\beta r_1 w_1} f(x + \hat{\theta}^T \hat{w} + w_1 \theta_1^T) - f(x + \hat{\theta}^T \hat{w})) d\eta(x) dw_1. \end{aligned}$$

Since $y \mapsto e^{-\beta y^T r} f(x + \hat{\theta}^T \hat{w} + \theta^T y)$ is uniformly continuous, there exists δ such that

$$0 \leq w_1 < \delta \implies |e^{-\beta r_1 w_1} f(x + \hat{\theta}^T \hat{w} + w_1 \theta_1^T) - f(x + \hat{\theta}^T \hat{w})| < \frac{\epsilon}{\|\eta\|}.$$

So for $0 \leq s_1 < \delta$ we have

$$\left| g(s_1) - \int_{\mathbb{S}^d} f(x + \hat{\theta}^T \hat{w}) d\eta(x) \right| \leq \frac{1}{s_1} \int_0^{s_1} \int_{\mathbb{S}^d} \frac{\epsilon}{\|\eta\|} d\eta(x) dw_1 = \epsilon.$$

Thus $g(s_1) \rightarrow \int_{\mathbb{S}^d} f(x + \hat{\theta}^T \hat{w}) d\eta(x)$, as we wanted.

Putting the formula for $\lim_{s_1 \rightarrow 0^+} g(s_1)$ in (5.15) gives

$$\int_{\mathbb{S}^d} f d\nu_{\mu_\nu} = \int_{[0, \infty)^{k-1}} e^{-\beta \hat{w}^T \hat{r}} \int_{\mathbb{S}^d} f(x + \hat{\theta}^T \hat{w}) d\eta(x) d\hat{w},$$

which is the right-hand side of (5.13) with one $\lim_{s \rightarrow 0^+}$ and one \int_0^∞ removed. Repeating the argument $k-1$ times gives

$$\int_{\mathbb{S}^d} f d\nu_{\mu_\nu} = \int_{\mathbb{S}^d} f d\nu,$$

as required. \square

As described before Proposition 5.9, this completes the proof of Theorem 5.1.

6. A PARAMETRISATION OF THE EQUILIBRIUM STATES

We are now ready to describe the KMS states of our system. At the end of the section, we will use the following theorem to prove our main result.

Theorem 6.1. *Consider our standard set-up, and suppose that $\beta > 0$.*

- (a) *Suppose that $\mu \in P(\varprojlim(\mathbb{S}^d, E_m^T))$. Define measures $\mu_m \in P(\mathbb{S}^d)$ by $\mu_m = E_{m, \infty^*}^T(\mu)$ and take ν_{μ_m} to be the subinvariant measure on \mathbb{S}^d obtained by applying Theorem 5.1 to the measure μ_m . Then there is a KMS_β state ψ_μ of (B_∞, α) such that*

$$(6.1) \quad \psi_\mu(V_{m,p} U_{m,n} V_{m,q}^*) = \delta_{p,q} e^{-\beta p^T r^m} \prod_{j=1}^k (\beta r_j^m) \int_{\mathbb{S}^d} e^{2\pi i x^T n} d\nu_{\mu_m}(x).$$

- (b) *The map $\mu \mapsto \psi_\mu$ is an affine homeomorphism of $P(\varprojlim(\mathbb{S}^d, E_m^T))$ onto $\text{KMS}_\beta(B_\infty, \alpha)$.*

To prove the theorem, we first build some maps between the spaces of subinvariant measures. We will make use of Theorem 5.1, but the measures described there are not all normalised. To ensure we are dealing with probability measures, we introduce the numbers

$$c_m := \prod_{j=1}^k (\beta r_j^m) = \beta^k \left(\prod_{j=1}^k r_j^m \right) \quad \text{and} \quad d_m := \det D_m.$$

Because D_m is diagonal, Equation 2.4 shows that the two sets of numbers are related by $d_m c_{m+1} = c_m$.

In particular, the functions f_{β, r^m} from Remark 5.2 have constant value c_m , and so $f_{\beta, r^{m+1}} = d_m^{-1} f_{\beta, r^m}$. Thus with $\Sigma_{\beta, r}$ from Proposition 3.7(c), we can define $\sigma_m : \Sigma_{\beta, r^{m+1}} \rightarrow \Sigma_{\beta, r^m}$ by

$$\sigma_m(\nu) = d_m^{-1} E_{m*}^T(\nu).$$

Lemma 6.2. *Suppose that $\mu \in P(\varprojlim(\mathbb{S}^d, E_m^T))$, and define $\mu_m := E_{m, \infty*}^T(\mu)$ for $m \geq 1$. Then the measures ν_{μ_m} given by Theorem 5.1 satisfy $\sigma_m(\nu_{\mu_{m+1}}) = \nu_{\mu_m}$.*

Proof. We take $f \in C(\mathbb{S}^d)$, and compute using (5.1):

$$\begin{aligned} \int_{\mathbb{S}^d} f d\sigma_m(\nu_{\mu_{m+1}}) &= d_m^{-1} \int_{\mathbb{S}^d} f \circ E_m^T d\nu_{\mu_{m+1}} \\ &= d_m^{-1} \int_{[0, \infty)^k} e^{-\beta w^T r^{m+1}} \int_{\mathbb{S}^d} (f \circ E_m^T)(x + \theta_{m+1}^T w) d\mu_{m+1}(x) dw \\ &= d_m^{-1} \int_{[0, \infty)^k} e^{-\beta w^T r^{m+1}} \int_{\mathbb{S}^d} f(E_m^T x + E_m^T \theta_{m+1}^T w) d\mu_{m+1}(x) dw \\ &= d_m^{-1} \int_{[0, \infty)^k} e^{-\beta w^T D_m^{-1} r^m} \int_{\mathbb{S}^d} f(E_m^T x + \theta_m^T D_m^{-1} w) d\mu_{m+1}(x) dw, \end{aligned}$$

where at the last step we used³ both (2.4) and (2.3). Now substituting $v = D_m^{-1} w$ in the outside integral gives

$$(6.2) \quad \int_{\mathbb{S}^d} f d\sigma_m(\nu_{\mu_{m+1}}) = \int_{[0, \infty)^k} e^{-\beta v^T r^{m+1}} \int_{\mathbb{S}^d} f(E_m^T x + \theta_m^T v) d\mu_{m+1}(x) dv.$$

We write $s := \theta_m^T v$ and consider the translation automorphism τ_s of $C(\mathbb{S}^d)$ defined by $\tau_s(f)(x) = f(x + s)$. Then the inside integral on the right of (6.2) is

$$\begin{aligned} \int_{\mathbb{S}^d} f(E_m^T x + \theta_m^T v) d\mu_{m+1}(x) &= \int_{\mathbb{S}^d} (\tau_s(f) \circ E_m^T) d\mu_{m+1} \\ &= \int_{\mathbb{S}^d} \tau_s(f) d(E_m^T)_*(\mu_{m+1}) \\ &= \int_{\mathbb{S}^d} f(x + \theta_m^T v) d\mu_m(x). \end{aligned}$$

Putting this back into the double integral in (6.2) gives the right-hand side of (5.1) for the measure μ_m , and we deduce from (5.1) that

$$\int_{\mathbb{S}^d} f d\sigma_m(\nu_{\mu_{m+1}}) = \int_{\mathbb{S}^d} f d\nu_{\mu_m} \quad \text{for all } f \in C(\mathbb{S}^d),$$

³This also uses that $D_m \theta_{m+1} E_m = \theta_m$ on the nail, i.e. as opposed to modulo \mathbb{Z} . Otherwise the difference would appear in the last formula multiplied by the real variable w .

as required. \square

Proof of Theorem 6.1(a). Since the maps

$$E_{m,\infty}^T : \varprojlim \mathbb{S}^d \rightarrow \mathbb{S}^d$$

are surjective, each μ_m is a probability measure on \mathbb{S}^d . Thus we deduce from Theorem 5.1 that $\nu_m := (\prod_{j=1}^k (\beta r_j^m)) \nu_{\mu_m}$ is a probability measure satisfying the subinvariance relation (5.2). Thus Proposition 3.7(b) gives a KMS state ψ_m of (B_m, α^{r^m}) such that $\psi_m(U_{m,n}) = \int_{\mathbb{S}^d} e^{2\pi i x^T n} d\nu_m(x)$. We now need to check that $\psi_{m+1} \circ \pi_m = \psi_m$ so that we can deduce from [6, Proposition 3.1] that the ψ_m combine to give a KMS_β state of (B_∞, α) .

Since we are viewing measures as functionals on $C(\mathbb{S}^d)$, the map E_{m*}^T on $M(\mathbb{S}^d)$ is induced by the continuous function $E_m^T : x \mapsto E_m^T x$ on \mathbb{S}^d . Then for $f \in C(\mathbb{S}^d)$

$$(6.3) \quad \int_{\mathbb{S}^d} f \circ E_m^T(x) d\nu_{m+1}(x) = \int_{\mathbb{S}^d} f(x) dE_{m*}^T(\nu_{m+1})(x).$$

For the functions $g_n \in C(\mathbb{S}^d)$ given by $g_n(x) = e^{2\pi i n^T x}$ (so that $\iota_m(g_n) = U_{m,n} \in B_m$), we have

$$g_n \circ E_m^T(x) = e^{2\pi i (E_m^T x)^T n} = e^{2\pi i x^T E_m n} = g_{E_m n}(x).$$

Substituting this on the left-hand side of (6.3) gives

$$(6.4) \quad \int_{\mathbb{S}^d} g_{E_m n}(x) d\nu_{m+1}(x) = \int_{\mathbb{S}^d} g_n(x) dE_{m*}^T(\nu_{m+1})(x).$$

Using again $d_m = \det D_m$ and $c_m = \prod_{j=1}^k (\beta r_j^m)$ and the relation $d_m c_{m+1} = c_m$, Lemma 6.2 gives

$$(6.5) \quad \begin{aligned} E_{m*}^T(\nu_{m+1}) &= \sigma_m(d_m \nu_{m+1}) = d_m c_{m+1} \sigma_m(\nu_{\mu_{m+1}}) \\ &= d_m c_{m+1} \nu_{\mu_m} = c_m \nu_{\mu_m} = \nu_m. \end{aligned}$$

Using (6.4) at the third step, and (6.5) at the fourth step, we now calculate:

$$\begin{aligned} \psi_{m+1}(\pi_m(U_{n,m})) &= \psi_{m+1}(U_{m+1, E_m n}) = \int_{\mathbb{S}^d} g_{E_m n} d\nu_{m+1} \\ &= \int_{\mathbb{S}^d} g_n dE_{m*}^T(\nu_{m+1}) = \int_{\mathbb{S}^d} g_n d\nu_m = \psi_m(U_{m,n}). \end{aligned}$$

Thus the states ψ_m give an element (ψ_m) of the inverse limit $\varprojlim \text{KMS}_\beta(B_m, \alpha^{r^m})$, and surjectivity of the isomorphism in [6, Proposition 3.1] gives a KMS_β state ψ_μ of (B_∞, α) such that $\psi_m = \psi_\mu \circ \pi_{m,\infty}$ for $m \geq 1$. \square

Remark 6.3. We observe that the KMS_β state of Theorem 6.1(a) is given on $B_\infty = \overline{\text{span}}\{V_{m,p} U_{m,n} V_{m,q}^*\}$ by

$$\psi_\mu(V_{m,p} U_{m,n} V_{m,q}^*) = \delta_{p,q} e^{-\beta p^T r^m} \int_{\mathbb{S}^d} g_n d(\nu_{E_{m,\infty}^T(\mu)}).$$

Proof of Theorem 6.1(b). We first prove that every KMS_β state has the form ϕ_μ . So suppose that ϕ is a KMS_β state of (B_∞, α) . Then for each $m \geq 1$, $\phi \circ \pi_{m,\infty}$ is a KMS_β state of (B_m, α^{r^m}) , and hence there are probability measures ν_m such that

$$\phi \circ \pi_{m,\infty}(f) = \int_{\mathbb{S}^d} f d\nu_m \quad \text{for all } f \in C(\mathbb{S}^d)$$

and $E_{m*}^T(\nu_{m+1}) = \nu_m$ for all $m \geq 1$. Theorem 4.1 implies that each ν_m satisfies the corresponding subinvariance relation. More specifically, we write $P_m^{\text{sub}}(\mathbb{S}^d)$ for the set of probability measures satisfying (4.2), and then we have $\nu_m \in P_m^{\text{sub}}(\mathbb{S}^d)$.

Once more using $d_m = \det D_m$ and $c_m = \prod_{j=1}^k (\beta r_j^m)$, the construction of Theorem 5.1(a) gives a function $\mu \mapsto c_m \nu_\mu$ from $P(\mathbb{S}^d)$ to the simplex $P_m^{\text{sub}}(\mathbb{S}^d)$. Lemma 6.2 gives commutative diagrams

$$\begin{array}{ccccc} \mu_m & & P(\mathbb{S}^d) & \xleftarrow{E_{m*}^T} & P(\mathbb{S}^d) & & \mu_{m+1} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ c_m \nu_{\mu_m} & & P_m^{\text{sub}}(\mathbb{S}^d) & \xleftarrow{E_{m*}^T} & P_{m+1}^{\text{sub}}(\mathbb{S}^d) & & c_{m+1} \nu_{\mu_{m+1}}, \end{array}$$

and Theorem 5.1 implies that the vertical arrows are bijections, with inverse given by $\nu \mapsto \mu_\nu$. A simple set-theoretic argument then implies that we also have commutative diagrams

$$\begin{array}{ccccc} c_m^{-1} \mu_{\nu_m} & & P(\mathbb{S}^d) & \xleftarrow{E_{m*}^T} & P(\mathbb{S}^d) & & c_{m+1}^{-1} \mu_{\nu_{m+1}} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \nu_m & & P_m^{\text{sub}}(\mathbb{S}^d) & \xleftarrow{E_{m*}^T} & P_{m+1}^{\text{sub}}(\mathbb{S}^d) & & \nu_{m+1}. \end{array}$$

Thus the sequence $(\mu_m) := (c_m^{-1} \mu_{\nu_m})$ belongs to the inverse limit $\varprojlim (P(\mathbb{S}^d), E_{m*}^T)$, and hence is given by a probability measure $\mu \in P(\varprojlim(\mathbb{S}^d, E_m^T))$. We want to show that $\phi = \psi_\mu$. Since both are states, it suffices to check that they agree on elements $V_{m,p} U_{m,n} V_{m,q}^*$. Since ϕ is a KMS_β state and the measure ν_m implements ϕ on $C(\mathbb{S}^d) = \overline{\text{span}}\{U_{m,n}\}$, we have

$$\phi(V_{m,p} U_{m,n} V_{m,q}^*) = \delta_{p,q} e^{-\beta p^T r^m} \int_{\mathbb{S}^d} e^{2\pi i x^T n} d\nu_m(x).$$

Since $\nu = \nu_{\mu_\nu}$ for all ν and $\mu_{\nu_m} = c_m \mu_m$, we have $\nu_m = c_m \nu_{\mu_m}$ and

$$\phi(V_{m,p} U_{m,n} V_{m,q}^*) = \delta_{p,q} e^{-\beta p^T r^m} c_m \int_{\mathbb{S}^d} e^{2\pi i x^T n} d\nu_{\mu_m}(x).$$

This is precisely the formula for $\psi_\mu(V_{m,p} U_{m,n} V_{m,q}^*)$ in (6.1). Thus $\phi = \psi_\mu$.

Since each ψ_μ is a state, it follows from the formula (6.1) that $\mu \mapsto \psi_\mu$ is affine and weak* continuous from $M(\mathbb{S}^d) = C(\mathbb{S}^d)^*$ to the state space of B_∞ . The formula (6.1) also implies that $\mu \mapsto \psi_\mu$ is injective, and since we have just shown that it is surjective, we deduce that it is a homeomorphism of the compact space $P(\varprojlim(\mathbb{S}^d, E_n^T))$ onto the simplex of KMS_β states of (B_∞, α) . \square

Proof of Theorem 2.7. According to (6.1) in Theorem 6.1, we have to compute

$$\int_{[0,\infty)^k} e^{2\pi i x^T n} d\nu_{\mu_m}(x),$$

which by Theorem 5.1 is

$$\begin{aligned}
(6.6) \quad & \int_{[0,\infty)^k} e^{-\beta w^T r^m} \int_{\mathbb{S}^d} e^{2\pi i(x+\theta_m^T w)^T n} d\mu_m(x) dw \\
&= \int_{[0,\infty)^k} e^{-\beta w^T r^m} e^{2\pi i w^T \theta_m n} \int_{\mathbb{S}^d} e^{2\pi i x^T n} d\mu_m(x) dw \\
&= \int_{[0,\infty)^k} e^{-\beta w^T r^m} e^{2\pi i w^T \theta_m n} M_{m,n}(\mu) dw.
\end{aligned}$$

We can rewrite the integrand as

$$e^{-\beta w^T r^m} e^{2\pi i w^T \theta_m n} = e^{\sum_{j=1}^k w_j(-\beta r_j^m + 2\pi i(\theta_m n)_j)} = \prod_{j=1}^k e^{w_j(-\beta r_j^m + 2\pi i(\theta_m n)_j)}.$$

When we view $\int_{[0,\infty)^k} dw$ as an iterated integral, we find that

$$(6.6) = \prod_{j=1}^k \left(\int_0^\infty e^{w_j(-\beta r_j^m + 2\pi i(\theta_m n)_j)} M_{m,n}(\mu) dw_j \right).$$

Since $\beta > 0$ and each $r_j^m > 0$, we have

$$|e^{w_j(-\beta r_j^m + 2\pi i(\theta_m n)_j)}| = e^{w_j(-\beta r_j^m)} \rightarrow 0 \quad \text{as } w_j \rightarrow \infty.$$

Thus

$$(6.6) = \prod_{j=1}^k \frac{e^{w_j(-\beta r_j^m + 2\pi i(\theta_m n)_j)}}{-\beta r_j^m + 2\pi i(\theta_m n)_j} M_{m,n}(\mu) \Big|_0^\infty = \prod_{j=1}^k \frac{1}{\beta r_j^m - 2\pi i(\theta_m n)_j} M_{m,n}(\mu),$$

and the result follows from (6.1). \square

REFERENCES

- [1] Z. Afsar, N. Brownlowe, N.S. Larsen and N. Stammeier, Equilibrium states on right LCM semigroup C^* -algebras, *Int. Math. Res. Not. IMRN*, to appear; doi:10.1093/imrn/rnx162.
- [2] Z. Afsar, A. an Huef and I. Raeburn, KMS states on C^* -algebras associated to local homeomorphisms, *Internat. J. Math.* **25** (2014), article no. 1450066, 28 pp.
- [3] Z. Afsar, A. an Huef and I. Raeburn, KMS states on C^* -algebras associated to a family of $*$ -commuting local homeomorphisms, *J. Math. Anal. Appl.* **464** (2018), 965–1009.
- [4] Z. Afsar, N.S. Larsen and S. Neshveyev, KMS states on Nica-Toeplitz algebras, preprint, arXiv:1807.05822.
- [5] L.W. Baggett, N.S. Larsen, J.A. Packer, I. Raeburn and A. Ramsay, Direct limits, multiresolution analyses, and wavelets, *J. Funct. Anal.* **258** (2010), 2714–2738.
- [6] N. Brownlowe, M. Hawkins and A. Sims, The Toeplitz noncommutative solenoid and its Kubo-Martin-Schwinger states, *Ergod. Theory Dyn. Syst.*, to appear; doi:10.1017/etds.2017.7.
- [7] J. Christensen, Symmetries of the KMS simplex, *Comm. Math. Phys.*, to appear; doi:10.1007/s00220-018-3250-5.
- [8] J. Christensen, KMS states on the Toeplitz algebras of higher-rank graphs, preprint, arXiv:1805.09010.
- [9] J. Cuntz, C. Deninger and M. Laca, C^* -algebras of Toeplitz type associated with algebraic number fields, *Math. Ann.* **355** (2013), 1383–1423.
- [10] K.R. Davidson, *C^* -Algebras by Example*, Fields Inst. Monographs, vol. 6, Amer. Math. Soc., Providence, 1996.
- [11] R. Exel and M. Laca, Partial dynamical systems and the KMS condition, *Comm. Math. Phys.* **232** (2003), 223–277.

- [12] J.M.G. Fell and R.S. Doran, *Representations of *-Algebras, Locally Compact Groups, and Banach *-Algebraic Bundles*, Vol. 1, Academic Press, San Diego, 1988.
- [13] J. Fletcher, A. an Huef and I. Raeburn, A program for finding all KMS states on the Toeplitz algebra of a higher-rank graph, *J. Operator Theory*, to appear; arXiv:1801.03189.
- [14] G.B. Folland, *Real Analysis: Modern Techniques and their Applications*, second ed., Wiley, New York, 1999.
- [15] A. an Huef, M. Laca, I. Raeburn and A. Sims, KMS states on the C^* -algebras of finite graphs, *J. Math. Anal. Appl.* **405** (2013), 388–399.
- [16] A. an Huef, M. Laca, I. Raeburn and A. Sims, KMS states on C^* -algebras associated to higher-rank graphs, *J. Funct. Anal.* **266** (2014), 265–283.
- [17] E.T.A. Kakariadis, On Nica-Pimsner algebras of C^* -dynamical systems over \mathbb{Z}_+^n , *Int. Math. Res. Not. IMRN*, **2017**, 1013–1065.
- [18] E.T.A. Kakariadis, Equilibrium states and entropy theory for Nica-Pimsner algebras, preprint, arXiv:1806.02443.
- [19] M. Laca and S. Neshveyev, Type III₁ equilibrium states of the Toeplitz algebra of the affine semigroup over the natural numbers, *J. Funct. Anal.* **261** (2011), 169–187.
- [20] M. Laca and I. Raeburn, Semigroup crossed products and the Toeplitz algebras of non-abelian groups, *J. Funct. Anal.* **139** (1996), 415–440.
- [21] M. Laca and I. Raeburn, Phase transition on the Toeplitz algebra of the affine semigroup over the natural numbers, *Adv. Math.* **225** (2010), 643–688.
- [22] M. Laca, I. Raeburn, J. Ramagge and M.F. Whittaker, Equilibrium states on the Cuntz-Pimsner algebras of self-similar actions, *J. Funct. Anal.* **266** (2014), 6619–6661.
- [23] M. Laca, I. Raeburn, J. Ramagge and M.F. Whittaker, Equilibrium states on operator algebras associated to self-similar actions of groupoids on graphs, *Adv. Math.* **331** (2018), 268–325.
- [24] F. Latrémolière and J.A. Packer, Noncommutative solenoids, preprint, arXiv:1110.6227.
- [25] A. Nica, C^* -algebras generated by isometries and Wiener-Hopf operators, *J. Operator Theory* **27** (1992), 17–52.
- [26] I. Raeburn, On crossed products by coactions and their representation theory, *Proc. London Math. Soc.* **64** (1992), 625–652.

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