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**OPTIMAL ROW-COLUMN DESIGNS FOR CORRELATED ERRORS
AND
NESTED ROW-COLUMN DESIGNS FOR UNCORRELATED ERRORS**

by
Nizam Uddin

A dissertation submitted to the faculty of
Old Dominion University in partial fulfillment of the
requirements for the degree of

DOCTOR OF PHILOSOPHY
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May, 1989

Approved by :

~~John P. Morgan~~ (Director)

ABSTRACT

Optimal row-column designs for correlated errors and nested row-column designs for uncorrelated errors

Nizam Uddin
Old Dominion University, 1989
Director: Dr. John P. Morgan

In this dissertation the design problems are considered in the row-column setting for second order autonormal errors when the treatment effects are estimated by generalized least squares, and in the nested row-column setting for uncorrelated errors when the treatment effects are estimated by ordinary least squares. In the former case, universal optimality conditions are derived separately for designs in the plane and on the torus using more general linear models than those considered elsewhere in the literature. Examples of universally optimum planar designs are given, and a method is developed for the construction

of optimum and near optimum designs, that produces several infinite series of universally optimum designs on the torus and near optimum designs in the plane. Efficiencies are calculated for planar versions of the torus designs, which are found to be highly efficient with respect to some commonly used optimality criterion. In the nested row-column setting, several methods of construction of balanced and partially balanced incomplete block designs with nested rows and columns are developed, from which many infinite series of designs are obtained. In particular, 149 balanced incomplete block designs with nested rows and columns are listed (80 appear to be new) for the number of treatments, $v \leq 101$, a prime power.

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TABLE OF CONTENTS

| | page |
|---|------|
| LIST OF TABLES | v |
| CHAPTER | |
| 1. INTRODUCTION..... | 1 |
| 2. OPTIMAL TWO-DIMENSIONAL DESIGNS FOR CORRELATED ERRORS | 11 |
| 2.1 Introduction | 11 |
| 2.2 Optimality Conditions | 12 |
| 2.3 Construction of Designs With v Blocks | 19 |
| 3. OPTIMAL TORUS DESIGNS FOR CORRELATED ERRORS | 30 |
| 3.1 Introduction | 30 |
| 3.2 Designs With Second Order Balance | 33 |

| CHAPTER | page |
|---|------|
| 3.3 Two Generalizations | 39 |
| 3.4 Designs For Non Prime Power v | 49 |
| 3.5 Efficiency Calculations | 56 |
| 4. BALANCED INCOMPLETE BLOCK DESIGNS | |
| WITH NESTED ROWS AND COLUMNS | 64 |
| 4.1 Introduction | 64 |
| 4.2 Construction of <i>BIBRC</i> Designs | 68 |
| 4.3 Designs With Fewer Replicates | 77 |
| 5. PARTIALLY BALANCED NESTED | |
| ROW-COLUMN DESIGNS | 88 |
| 5.1 Introduction | 88 |
| 5.2 Rectangular and Latin Square Type Designs With | |
| Nested Rows and Columns | 90 |
| 5.3 Pseudocyclic Designs With Nested Rows and Columns ... | 95 |
| 6. CONCLUDING REMARKS | 103 |
| 7. REFERENCES | 115 |

LIST OF TABLES

| TABLE | Page |
|---|------|
| 2.1 k and m satisfying conditions of Theorems 2.5-2.7 | 28 |
| 3.1 r values satisfying (3.9) | 43 |
| 3.2 Partitions of Z_v -0 and possible design sizes | 53 |
| 3.3 Correlations obtained from (3.11) | 57 |
| 3.4 Two designs for $v = 5$ relative to universal optimality | 59 |
| 3.5 6×8 for $v = 8$ relative to universal optimality | 60 |
| 3.6 Comparison of 6×6 designs for $v = 9$ | 61 |
| 3.7 Comparison of 10×10 designs for $v = 10$ | 61 |
| 3.8 Two designs for $v = 13$ relative to hypothetical optimum | 63 |
| 4.1 Constructed <i>BIBRCs</i> for $v \leq 101$ | 83 |

Chapter 1

INTRODUCTION

Statistical design for experimental studies to compare a number of factors or factor combinations, called treatments, refers to an allocation of the treatments to experimental units (subunits of the experimental material such that two different units may receive different treatments). Some examples of treatments and experimental units respectively are fertilizers and plots in field experiments, drugs and patients in clinical experiments, diets and animals in animal feeding experiments, and teaching methods and groups of students in educational experiments. The aim of designing such an experiment is to compare the treatments as efficiently as possible on the basis of responses obtained from the experimental units. The basis of the theory of statistical design (henceforth referred to as design only) considered here is the estimation of the parameter vector τ in the fixed effects additive linear model

$$Y = X\tau + Z\beta + \varepsilon, \quad \text{cov}(\varepsilon) = \Sigma. \quad (1.1)$$

Here τ is the vector of parameters of interest (treatment effects), β is a vector of parameters that should be included in the model to eliminate heterogeneity (if any) in the experimental material (block effects, which may be zero), X is the treatment/plot incidence matrix that determines the allocation of treatments to experimental units, and Z is the block/plot incidence matrix. The problem is to choose a design, in a well defined class of designs, that satisfies statistical criteria set by the experimenter; such a choice is called an optimum (in some sense) design. So the design matrix X is chosen in such a way that $\hat{\tau}$, the estimator of the parameter vector τ , satisfies some well defined statistical properties such as unbiasedness, minimization of the average variance of all elementary treatment contrasts, minimization of the maximum variance of elementary treatment contrasts, or other properties to be discussed.

The information matrix (which henceforth will be called the C -matrix) for the estimation of treatment parameters plays an important role in the theory of optimal design. The C -matrix for the estimation of τ under generalized least squares for the model (1.1) is

$$C = X'\Sigma^{-1}X - X'\Sigma^{-1}Z(Z'\Sigma^{-1}Z)^{-1}Z'\Sigma^{-1}X.$$

Kiefer (1975) introduced universal optimality using symmetric convex decreasing functionals on the class of C -matrices. Let C_d be the information matrix of the estimator $\hat{\tau}$ for a design d under the model (1.1) and ϕ be a convex function

mapping C_d to the real line. Then a design d^* is called universally optimum relative to a class of designs, say χ , if d^* minimizes $\phi(C_d)$ over χ for every ϕ which is convex and invariant under each permutation of rows and (the same on) columns, and has the property that $\phi(bC_d) \leq \phi(C_d)$ for all $b > 1$. If C_d has zero row and column sums for all $d \in \chi$, then by Proposition 1 of Kiefer (1975) sufficient conditions for a design $d^* \in \chi$ to be universally optimum are :

- (a) C_{d^*} is completely symmetric in the sense that all diagonal elements are equal and all off-diagonal elements are equal, and
- (b) $\text{trace}(C_{d^*}) \geq \text{trace}(C_d)$ for all $d \in \chi$.

The three well known optimality criteria A -, D -, and E - optimality (defined in Chapter 3) are contained in universal optimality.

Recent research has paid much attention to the analysis and construction of designs when the yields from experimental units are correlated. Comprehensive studies on the analysis of field trials when the yields from the neighboring plots are correlated are given by Bartlett (1978) and Wilkinson et al. (1983). Since Bartlett's (1978) paper the investigation of optimum designs for spatially correlated error models has become one of the fast growing fields of research in experimental design. Under the assumption of known Σ of some particular types, optimality of complete and incomplete block designs using generalized least squares analysis were investigated by a number of authors, see, for example,

Kiefer & Wynn (1984), Gill & Shukla (1985a), and Kunert (1987). Optimum two dimensional designs for correlated errors are only beginning to be investigated, see Kiefer & Wynn (1981), Martin (1982, 1986), and Gill & Shukla (1985b). Two dimensional designs but with only one dimensional correlations are considered by Kunert (1984, 1985); repeated measurements designs, change-over designs, and residual effects designs are some of the important examples in this case.

Of particular interest here are the results for two-dimensional correlations, as they are a primary topic of this dissertation. Martin (1986) investigated the optimality of two dimensional designs under seven different covariance structures, but using a simple model with treatment effects only. In his paper, Martin (1986) enumerated small designs (3×3 's, 4×4 's and 5×5 's), compared their efficiencies using generalized and ordinary least squares analysis, and made recommendations concerning design properties leading to efficient designs. Martin (1986) discusses designs with only one rectangular array. Gill & Shukla (1985b) have considered b rectangular arrays but again for a model with treatment effects only. A non-stationary second order autonormal process has been assumed in Gill & Shukla's (1985b) model 2, with an additional assumption that the errors in different blocks are uncorrelated. The model with treatment effects only would be appropriate if the blocks are homogeneous, but such an assumption is very unlikely to be satisfied in practice. Hence in Chapter 2 of this dissertation,

the design problem for a more general model including block effects has been considered, allowing the experimenter to eliminate heterogeneity due to blocks. Following a brief introduction in section 2.1, conditions for universally optimum designs are established in section 2.2. Construction of optimum and near optimum designs, using the method of differences, are considered in section 2.3 under certain assumptions. Two examples of universally optimum designs are given.

The results of Chapter 2 and investigation of the papers mentioned above show that universally optimum two dimensional planar designs of a reasonable size are hardly possible for a variety of correlation structures partly because of the absence of equal number of neighbors of a treatment in the corner, in the end rows and columns, and in the interior plots of a design. Martin (1986) has several recommendations concerning design properties that will give reasonable efficiency and balance across a range of correlation structures. These are, in fact, in accord with the exact properties Martin (1982) has obtained for designs on the torus, and it is the torus approach that will be taken in Chapter 3 of this dissertation. A full description of the model and underlying correlation structure for the errors will be given there. Following a brief introduction and optimality results in section 3.1, construction of these designs, which like those of Chapter 2 fall under the general heading of "neighbor designs", are given in sections 3.2 and 3.3, where several infinite series of universally optimum torus designs for

v , a prime power, are obtained. For v , not a prime power, some designs are given which do not satisfy exact balance properties but are reasonably balanced in rows and columns combined and in diagonals, and/or collectively in rows, columns and diagonals. These nearly balanced designs have the property that no treatment is neighbored by itself in any direction, insuring A -optimality under least squares analysis for many correlation structures (see Kiefer & Wynn (1981), Martin (1986)). Section 3.4 deals with the construction of these nearly optimal designs, where for all $v \leq 30$ not covered by the previous constructions, results of a computer search are reported. In section 3.5, A -, E -, D -, and R - efficiencies are calculated for planar versions of some torus designs.

As already indicated, the investigation of optimal designs for spatial correlations leads to the area of neighbor designs, that take into account which treatments occur next to which other treatments and how often. Neighbor designs balanced for nearest neighbors have drawn much attention over the last decade because of their high efficiency for correlation models as well as their applications in polycross experimentation. Optimal designs for most correlation structures require that the design be balanced for first nearest neighbors, see, for example, Kiefer & Wynn (1981, 1984), Martin (1982,1986), Kunert (1983, 1984, 1985, 1987), Gill & Shukla (1985a, 1985b), Ipinoyomi (1986), and Morgan & Chakravarti (1988). For the purposes of polycross experimentation, direc-

tionally (concerned with ordered pairs) and non-directionally (concerned with unordered pairs) balanced nearest neighbor designs were considered by, among others, Wright (1962, 1965), Freeman (1967, 1969, 1979a, 1979b, 1981), Dénes & Keedwell (1974), Oleson (1976), Bailey (1984), Street (1986), Morgan (1988a, 1988b, 1988c, 1989), and Afsarinejad & Seeger (1988). Among others, Freeman (1979a), Morgan (1988a, 1988b, 1989) and Afsarinejad & Seeger (1988) considered bordered design to improve the precision of the estimates. While discussing Barlett's (1978) paper, Dyke (1978) also advocated the use of border plots but expressed reservations for their use in two dimensional designs. The designs of Chapter 2 fall into the class of neighbor designs, and those in Chapter 3 can be used as bordered designs preserving neighbor properties.

The Papadakis method for making use of information from nearest neighbors (see Papadakis (1937)), use of directional and non-directional nearest neighbor balance (possibly in each of 8 directions), use of borders around the plots of an experiment, and use of generalized least squares with an appropriate covariance structure, are nothing but ways to improve the precision of the estimates of treatments when the yields from neighboring plots are correlated. Another suggested approach is : use standard least squares analysis, ignoring correlations, but with elimination of heterogeneity in two dimensions within each block. The detailed studies of Pearce (1978) and Kempton & Howes (1981) demonstrate that the

row-column designs of the lattice type with block sizes 5, 6, and 7 are as effective as the covariance adjustment in achieving the local control. Pearce (1978) states that “row-column designs were best of all when used well” and “balanced row-column designs proved quite the most effective of the methods tried”, while Kempton & Howes (1981) found that “a standard row-column design performs at least as well”. Jarret & Hall (1982) argue that the designs in b separated rectangular arrays with the elimination of heterogeneity due to rows and columns within each array could be used as an alternative to neighbor designs, especially when the array size is small.

The above discussion clearly suggests that nested row-column designs are reasonable competitors to nearest neighbor designs and methods. These designs may be considered as robust (to some degree of efficiency) against correlation. In addition nested row-column designs are useful in other situations, e.g. in field trials where the two directions represent two possible trends on the ground. Hence Chapters 4 and 5 of this dissertation are devoted to block designs with nested rows and columns, primarily as an useful alternative to designs of Chapters 2 and 3.

Block designs with nested rows and columns are designs for v treatments in b blocks of size $k = pq$, where each block is further grouped into p rows and q columns. These designs were introduced by Singh & Dey (1979) for the

elimination of heterogeneity in two directions within each block. Analysis of data obtained from such designs is given in Singh & Dey (1979); see also Dey (1986) and John (1987). The constructions of members of two subclasses of these designs, called balanced and partially balanced incomplete block designs with nested rows and columns, have been considered by Street(1981), Agrawal & Prasad (1982a, 1982b, 1983), Ipinyomi & John (1985), and Cheng (1986). New constructions in these subclasses are considered in Chapters 4 and 5.

Balanced incomplete block designs with nested rows and columns, *BIBRCs*, constructed by Agrawal & Prasad (1982a) have the strong property that each of the row, column and block component designs is a balanced incomplete block design. Chapter 4 presents a new method for the construction of balanced incomplete block designs with nested rows and columns, based on the method of differences, that takes advantage of the fact that if $p = q$ a sufficient condition for balance is that the rows and columns together give a balanced incomplete block design and the blocks give a balanced incomplete block design. Following a general construction in section 4.2, many infinite series of these designs are obtained in section 4.3. These designs have the additional property that the block size does not necessarily vary as v varies. This allows the experimenter to use small block sizes for large v , a recommended criteria for row-column designs to be an alternative to neighbor designs.

In general, nested row-column designs with perfect balance require a fairly large number of replications, because of combinatorial relationships among the parameters of the design. In addition, the methods of construction available so far (including those given in Chapter 4) are largely limited to prime power numbers of treatments. Hence there is a need to consider designs for composite numbers and with fewer replicates, possibly compromising the perfect balance property. This is the topic of Chapter 5. In section 5.2 several methods for construction of rectangular and Latin square type nested row-column designs with fewer restrictions and comparatively smaller number of replications are presented. Constructions of these designs based on the pseudocyclic association scheme are given in section 5.3.

Finally examples are used to explain various constructions in all chapters of this dissertation. For the purposes of constructions of proposed designs some properties of finite fields are established in various chapters. The dissertation concludes with a closing discussion in Chapter 6.

Chapter 2

Optimal two-dimensional designs for correlated errors

2.1. INTRODUCTION

The design problems for correlated errors resulting from rectangular arrays of experimental units have recently been investigated by a number of authors. The optimality of Latin squares for a nearest neighbor covariance structure under least squares analysis has been considered by Kiefer & Wynn (1981). Among others Martin (1982), Gill & Shukla (1985b) and Kunert (1988) have addressed the optimal design problem in a model with treatment effects only under generalized least squares. Martin (1986) has considered both least squares and generalized least squares under various covariance structures, but again for a simpler model with treatment effects only. For most error covariance structures considered so far, universal optimality arguments require more than one rectangular array

(block), in which case it would be desirable to include block effects in the model to eliminate heterogeneity due to blocks. In this chapter optimal designs for a model with treatment and block effects are investigated. Errors are assumed to follow the second order autonormal process with the errors from different blocks being uncorrelated. Gill & Shukla (1985b) have assumed this error structure for their model 2 with treatment effects only.

Under generalized least squares, conditions for a design to be universally optimum in the class of sets of v equireplicate $v \times v$ blocks for v treatments are derived in section 2.2, where an example of a universally optimum design is given. Then it is shown that these designs are at least 99% efficient in the class of all sets of $v \times v$ blocks (allowing unequal replication). In section 2.3, a method is given that produces highly efficient designs (to be called near optimal designs) and eventually gives the universally optimum designs of section 2.2. The concept of near optimal designs won't be clear until the end of section 2.2.

2.2. OPTIMALITY CONDITIONS

Let there be b separated blocks of treatments, each arranged into p rows and q columns. With y_{klm} the yield from row l , column m of square k , our model is

$$y_{klm} = \mu_k + \tau_{[klm]} + e_{klm}$$

μ_k being the fixed effect of block k , $\tau_{[klm]}$ the fixed effect of the treatment applied

to plot klm , and e_{klm} a random error term. In matrix notation this is

$$Y = (I_b \otimes 1_{pq})\mu + X\tau + e$$

where $X^T = (X_1^T, \dots, X_b^T)$, X_k is the plot/treatment incidence matrix for block k , I_b is the identity matrix of order b , 1_{pq} is a $pq \times 1$ column vector of 1's, and \otimes stands for the kronecker product of matrices. It is assumed that yields from different blocks are uncorrelated so $\text{var}(e) = I_b \otimes W$ where W is a known variance matrix of order pq . The C -matrix for estimation of treatment contrasts is

$$C = \sum_{k=1}^b X_k^T [W^{-1} - \frac{1}{w} W^{-1} 11^T W^{-1}] X_k$$

where $w = 1^T W^{-1} 1$.

We will attempt to use the method of Kiefer (1975) : the universally optimal design is that which assigns the treatments to the plots in such a way that $\text{tr}(C)$ is maximized and C is made completely symmetric. If $\lambda_1, \lambda_2, \dots, \lambda_{v-1}$ are the non-zero eigenvalues of C , the term “universal optimality” means that $\Phi_p(C) = \left[\sum_{i=1}^{v-1} \lambda_i^{-p} / (v-1) \right]^{1/p}$ is minimized for $0 < p < \infty$ over all admissible allocations of treatments.

The correlation matrix to be considered is that of the symmetric second order autonormal planar process :

$$\sigma^2 W^{-1} = I_{pq} - \beta_1 (I_p \otimes N_q + N_p \otimes I_q) - \beta_2 N_p \otimes N_q \quad (2.1)$$

where N_t of order t satisfies

$$(N_t)_{ij} = \begin{cases} 1 & \text{if } (i - j) = \pm 1 \\ 0 & \text{otherwise,} \end{cases}$$

and $\beta_1 \geq 0$, $\beta_2 \geq 0$, $\beta_1 + \beta_2 \leq 1/4$. This is the nonstationary planar version of the second order autonormal torus process discussed by Martin (1982). Alternatively one could obtain a stationary planar process by using a finite portion of the infinite torus process, but there is no simple parametric representation of the resulting W^{-1} , and that process may well be less realistic. Nonstationarity in the proposed model shows up in the variances of the edge plots, which for the unbordered designs to be discussed here is the more plausible behaviour.

Gill & Shukla (1985b, pp. 2188-2189) consider a coarser version of the model, using the same W but taking $\mu_k \equiv 0$. They do not find any optimal designs.

To investigate C define

c_i^k = the number of times the treatment i occurs on one of the four corner plots of block k ,

e_i^k = the number of times the treatment i occurs on one of the $2(p + q - 4)$ edge plots of block k ,

m_i^k = the number of times the treatment i occurs on one of the $(p - 2)(q - 2)$ interior ("middle") plots of block k .

As defined here, corner, edge, and interior plots form a partition of the pq plots

of a block, $\sum_{i=1}^v (c_i^k + e_i^k + m_i^k) = pq$. Also define

N_{ij}^{RC} = the number of times treatments i and j occur as row or column neighbors in the b blocks,

N_{ij}^D = the number of times treatments i and j occur as diagonal neighbors in the b blocks,

r_i^k = the number of replicates of treatment i in square k .

Straightforward counting gives

$$(C)_{ii} = r_i - 2\beta_1 N_{ii}^{RC} - 2\beta_2 N_{ii}^D - \frac{1}{w} \sum_{k=1}^b (y_i^k)^2 \quad (2.2)$$

and

$$(C)_{ij} = -\beta_1 N_{ij}^{RC} - \beta_2 N_{ij}^D - \frac{1}{w} \sum_{k=1}^b y_i^k y_j^k \quad (2.3)$$

where $y_i^k = (c_i^k + e_i^k + m_i^k) - (2c_i^k + 3e_i^k + 4m_i^k)\beta_1 - (c_i^k + 2e_i^k + 4m_i^k)\beta_2$ and $r_i = \sum_{k=1}^b r_i^k$. Since $\beta_1, \beta_2, w > 0$ it is clear from (2.2) that $\text{tr}(C) = \sum_{i=1}^v (C)_{ii}$ is maximized if $N_{ii}^{RC} = 0 = N_{ii}^D$ for every i , and $\sum_{i=1}^v \sum_{k=1}^b (y_i^k)^2$ is minimized. Hence there should be no like neighbors, and since $\sum_{i=1}^v \sum_{k=1}^b y_i^k$ is a constant, the y_i^k 's should be as equal as possible, their mean value being

$$\bar{y} = [pq - 2(2pq - p - q)\beta_1 - 4(p - 1)(q - 1)\beta_2]/v.$$

Now specialize to the case $p = q = v$. In appendix I it is shown that for equireplicate designs (r_i^k 's all equal) and $v \geq 4$, the y_i^k 's are as equal as possible when $(c_i^k, e_i^k, m_i^k) = (1, 2, v - 3)$ or $(0, 4, v - 4)$ for each i, k . (This does not

necessarily hold if the r_i^k 's are allowed to vary - see appendix II - a possibility that will be treated later). Note that this condition implies that each corner of a block has a different treatment.

The next task is to obtain complete symmetry of C , i.e., make $(C)_{ii}$ constant in i and $(C)_{ij}$ constant in $i \neq j$. From (2.2) and (2.3) the conditions are N_{ij}^{RC} equal in $i \neq j$, N_{ij}^D equal in $i \neq j$, $\sum_{k=1}^b (y_i^k)^2$ equal in i , and $\sum_{k=1}^b y_i^k y_j^k$ equal in $i \neq j$. The conditions for maximization of trace give

$$y_i^k = v - 4(v-1)\beta_1 - 4(v-2)\beta_2 - \beta_2 c_i^k \text{ where } c_i^k = 0 \text{ or } 1.$$

So the last two conditions hold if and only if each treatment is in an equal number of corners, and every pair of treatments occurs in an equal number of blocks for which they are both in a corner. This says that the corner design (i.e. the design given by taking the 4 corner plots of each $v \times v$ square as a one-dimensional block) is a balance incomplete block design (*BIBD*).

THEOREM 2.1. *A design for v treatments in $v \times v$ blocks is universally optimum in the class of equireplicate blocks for the second order autonormal process if*

$$N_{ii}^{RC} = N_{ii}^D = 0 \text{ for all } i,$$

$$N_{ij}^{RC} \text{ are equal for all } i \neq j,$$

$$N_{ij}^D \text{ are equal for all } i \neq j,$$

$$(c_i^k, e_i^k, m_i^k) = (1, 2, v-3) \text{ or } (0, 4, v-4) \text{ for every } i, k, \text{ and}$$

the corner design is a BIBD.

Simply stated, Theorem 2.1 says to use a design balanced for combined row and column neighbors and balanced for diagonal neighbors, which also achieves certain counts for treatment occurrences on the corner, edge, and interior plots.

Example 2.1. An example of Theorem 2.1 design is the following set of 4×4 squares for 4 treatments. In fact, it can be shown that this design is universally optimum in the wider class of designs without equal r_i^k 's.

| | | | |
|---------|---------|---------|---------|
| 1 3 1 2 | 2 4 2 3 | 3 1 3 4 | 4 2 4 1 |
| 4 2 4 3 | 1 3 1 4 | 2 4 2 1 | 3 1 3 2 |
| 1 3 1 2 | 2 4 2 3 | 3 1 3 4 | 4 2 4 1 |
| 4 2 4 3 | 1 3 1 4 | 2 4 2 1 | 3 1 3 2 |

In section 2.3 a general method of construction will be given from which designs with $b = v$ satisfying Theorem 2.1 except for the *BIBD* corner condition can be obtained (the construction will also be used later in Chapter 3 as a step in the construction of universally optimal torus designs). It is the corner conditions that are especially restrictive here in the sense of requiring large b (compare the similar problem with end conditions encountered by Kunert (1987) for one-way designs), so that their relaxation will be desirable if it can be done with a small loss of efficiency. It will next be shown that a Theorem 2.1 design in v blocks, but with the relaxed condition that the corner design is binary and equireplicate, has Φ_p -efficiency ≥ 0.99 within the class of $v \times v$ squares. Although the number of replicates, v^2 , is large, it is comparable to that required for optimal and

high efficiency designs for one dimensional designs with correlated errors (see Kunert(1987) and Morgan & Chakravarti (1988), for example).

For these designs, $b = p = q = v$, $r_i = v^2$, $N_{ii}^{RC} = N_{ii}^D = 0$, $N_{ij}^{RC} = 4v$, $N_{ij}^D = 4(v-1)$, $y_i^k = v - 4(v-1)\beta_1 - 4(v-2)\beta_2 - \beta_2 c_i^k$, $c_i^k \in \{0, 1\}$, $\delta_{ii} = \sum_{k=1}^b c_i^k = \sum_{k=1}^b (c_i^k)^2 = 4$, and $\delta_{ij} = \sum_{k=1}^b c_i^k c_j^k \leq 4$. Also writing $\Delta = (\delta_{ij})$ and using J for a $v \times v$ matrix of 1's, (2.2) and (2.3) give

$$C = (v^2 + 4v\beta_1 + 4(v-1)\beta_2)I - [v + 4\beta_1 + \frac{4}{v}\beta_2(v-1 - \frac{4}{w}\beta_2)]J - \frac{\beta_2^2}{w}\Delta.$$

Now the C -matrix for the corner design under ordinary least squares is

$C^* = 4I - \frac{1}{4}\Delta$. Write $\xi = v^2 + 4v\beta_1 + 4(v-1)\beta_2 - \frac{16}{w}\beta_2^2$ and $\xi^* = \frac{4}{w}\beta_2^2$. Then

$$C - \xi(I - \frac{1}{v}J) = \xi^*C^*.$$

If λ_i is a nonzero eigenvalue of C , then $\lambda_i - \xi$ is an eigenvalue of the left-hand matrix, which since $\xi^* > 0$ and C^* is non-negative definite, satisfies

$$\lambda_i - \xi \geq 0, \text{ i.e. } \lambda_i \geq \xi \Rightarrow \Phi_p(C) \leq \xi^{-1}.$$

To obtain a bound on the efficiency, an optimal value of Φ_p must be calculated. A hypothetical universally optimum design would have

$C_{ii}^{opt} = v^2 - \frac{1}{w} \sum_{k=1}^v (\bar{y})^2 = (v^3 - w)/v$ and hence a common non-zero eigenvalue of $\frac{v}{v-1}C_{ii}^{opt} = (v^3 - w)/(v-1)$. Thus the Φ_p -efficiency of the proposed design is at least $\{(v-1)/(v^3 - w)\}/\xi^{-1} = \{1 + 16\beta_2^2/(w\xi)\}^{-1}$.

THEOREM 2.2. *A design for v treatments in v $v \times v$ blocks satisfying the conditions of Theorem 2.1 except that the BIBD condition is replaced by the corner design being binary and equireplicate, satisfies*

$$\Phi_p - \text{efficiency} \geq \{1 + 16\beta_2^2/(w\xi)\}^{-1}$$

for the second order autonormal model (2.1). It is easily verified that Φ_p -eff ≥ 0.99 for $v \geq 4$ and $0 \leq p \leq \infty$.

In fact, simple calculus shows that for fixed v the worst case is at $\beta_1 = 0$, $\beta_2 = 1/4$, for which

$$1 + 16\beta_2^2/(w\xi) = 1 + \frac{1}{v(v-1)(2v+3)}$$

and the 0.99 bound follows.

2.3. CONSTRUCTION OF DESIGNS WITH v BLOCKS

The method of differences has been used by a number of authors to construct two-dimensional neighbor designs (see, e.g. Bailey (1984), Street (1986), Morgan (1988a), and Ipinyomi & Freeman (1988)), and here will be used on the Galois fields. First a general construction for neighbor designs in $p \times p$ blocks is described in Theorem 2.3. The method is then used to obtain Theorem 2.2 designs. To begin, let $G = GF_v$ be a finite field of order v , and $a = (a_1, a_2, \dots, a_p)$, $b =$

(b_1, b_2, \dots, b_p) be vectors of elements of G . Also write $a^* = (a_1 - a_2, a_2 - a_3, \dots, a_{p-1} - a_p)$ for the vector of forward neighbor differences of a , and let $B(a, b)$ be the array with (l, m) element $a_l + b_m$.

THEOREM 2.3. *Suppose there exist p_1 -vectors a and b on an abelian group G such that*

- (i) $\pm a^* \cap \pm b^* = \emptyset$,
- (ii) $\pm a^* \cup \pm b^*$ is each nonzero group element $4(p_1 - 1)/(v - 1)$ times,
- (iii) $B(\pm a^*, \pm b^*)$ is each nonzero group element $4(p_1 - 1)^2/(v - 1)$ times.

Then $B(a, b)$ is a balanced neighbor difference array, in that the v arrays $B(a, b) + g, g \in G$, are together balanced for combined first row and column neighbors, and for first diagonal neighbors. Furthermore, no treatment neighbors itself.

PROOF: The combined row and column neighbor differences are p_1 copies of $\pm a^* \cup \pm b^*$, which by (ii) are balanced. The diagonal neighbor differences are $B(\pm a^*, \pm b^*)$, which by (iii) are balanced. By (i) – (iii) none of the differences is zero. \square

Note that the Theorem 2.3 designs have non-directional neighbor balance, and that balance is for rows and columns combined, and for diagonal directions combined. Hence they would be appropriate for roughly square plots in which

the error process is assumed symmetric. Neighbors in each of rows, columns, and the two diagonal directions can be balanced by using another v arrays given by a 90° rotation of the v arrays of the theorem. Rotating each of the resulting $2v$ arrays 180° gives a directional design, balanced for neighbors in each of the 8 directions.

The initial problem here is in finding the sets a^* and b^* , from which a and b can be “reconstructed” (though if one wished to consider models with blocking factors consideration of neighbor differences alone would not be sufficient). It is clear from (ii) of Theorem 2.3 that $4|(v-1)$. When discussing a^* , b^* we can WLOG take $p = (v+3)/4$; larger p will give multiples of these two sets. (In fact, a^* and b^* need not even be of the same size when we consider torus constructions in Chapter 3 : just each an integer multiple of $(v-1)/4$.) Hence the problem becomes: partition the nonzero elements of G into equal-sized subsets S_1 and S_2 satisfying

$$g \in S_i \Rightarrow -g \in S_i, \quad (2.4)$$

$$\cup S_i = G - 0, \quad (2.5)$$

$$B(S_1, S_2) \text{ contains each} \\ \text{nonzero group element } (v-1)/4 \text{ times.} \quad (2.6)$$

THEOREM 2.4. *Let S_1 and S_2 be sets satisfying (2.4) and (2.5). They satisfy (2.6) if and only if they are initial blocks for a BIBD with $2v$ blocks of size $(v-1)/2$.*

PROOF: The group table for G , after deletion of the zero row and column, can be broken into four subtables: $B(S_1, S_1)$, $B(S_1, S_2)$, $B(S_2, S_1)$, and $B(S_2, S_2)$. Then (2.6) is satisfied iff $B(S_1, S_1)$ and $B(S_2, S_2)$ together have each nonzero group element with equal frequency. By (2.4) $S_i = S_{i1} \cup S_{i2}$ where $S_{i2} = -S_{i1}$, so that $B(S_i, S_i)$ can be broken into the four subtables $B(-S_{i1}, S_{i2})$, $B(S_{i1}, -S_{i2})$, $B(S_{i1}, -S_{i1})$, and $B(S_{i2}, -S_{i2})$. Deleting the $(v-1)/4$ zeros in each of the last two tables leaves all the symmetric differences for the set S_i . Hence S_1 and S_2 satisfy (2.6) if and only if all the symmetric differences within S_1 and S_2 are together each nonzero group element $(v-3)/2$ times, and the theorem is proved. \square

COROLLARY 2.1. *Let v be a prime power of the form $4t+1$. Then sets S_1 and S_2 satisfying (2.4)-(2.6) exist on GF_v .*

PROOF: Let S_1 be the set of quadratic residues on GF_v , and S_2 the quadratic non-residues. It is well known that S_1 and S_2 generate the required BIBD (e.g. Raghavarao, 1971). The result follows since -1 is quadratic. \square

Using this corollary, we can take $\pm a^*$ and $\pm b^*$ of Theorem 2.3 as four copies

of S_1 and S_2 respectively to obtain designs satisfying Theorem 2.2. So this sets the problem : find an ordering a of the elements of GF_v such that $\pm a^*$ is 4 copies of the quadratic residues. Such an ordering will be called a *quadratic neighbor difference ordering*, or *QND*. Then taking $b = xa$ where x is a primitive element of GF_v , $B(a, b)$ will be a Latin square which is an initial block for a Theorem 2.2 design. Furthermore, (2.2), (2.3) and the argument leading to Theorem 2.1 make it evident that $B(a, b)$ is universally optimum for the first order model (set $\beta_2 = 0$) among all $v \times v$ squares.

Example 2.2. As an example of this technique, for $v = 5$ take $a = (0, 1, 2, 3, 4)$ and $b = (0, 2, 4, 1, 3)$. The initial block is

$$B(a, b) = \begin{array}{ccccc} & 0 & 2 & 4 & 1 & 3 \\ & 1 & 3 & 0 & 2 & 4 \\ 2 & 4 & 1 & 3 & 0 & \\ 3 & 0 & 2 & 4 & 1 & \\ 4 & 1 & 3 & 0 & 2 & \end{array}$$

from which the other four blocks are

$$\begin{array}{cccc} 1 & 3 & 0 & 2 & 4 & 2 & 4 & 1 & 3 & 0 & 3 & 0 & 2 & 4 & 1 & 4 & 1 & 3 & 0 & 2 \\ 2 & 4 & 1 & 3 & 0 & 3 & 0 & 2 & 4 & 1 & 4 & 1 & 3 & 0 & 2 & 0 & 2 & 4 & 1 & 3 \\ 3 & 0 & 2 & 4 & 1 & 4 & 1 & 3 & 0 & 2 & 0 & 2 & 4 & 1 & 3 & 1 & 3 & 0 & 2 & 4 \\ 4 & 1 & 3 & 0 & 2 & 0 & 2 & 4 & 1 & 3 & 1 & 3 & 0 & 2 & 4 & 2 & 4 & 1 & 3 & 0 \\ 0 & 2 & 4 & 1 & 3 & 1 & 3 & 0 & 2 & 4 & 2 & 4 & 1 & 3 & 0 & 3 & 0 & 2 & 4 & 1 \end{array}$$

Note that the corner design for these squares is a *BIBD*, so Theorem 2.1 is satisfied.

For brevity and ease of notation we will use the Legendre symbol $\chi(y)$ for $y \in GF_v$, defined by

$$\chi(y) = \begin{cases} 1 & \text{if } y \text{ is a quadratic residue} \\ -1 & \text{if } y \text{ is a quadratic non-residue} \\ 0 & \text{if } y = 0. \end{cases}$$

To construct a *QND* on GF_v , $v = 4t + 1$, first consider the following cyclic arrangement of the $v - 1$ non-zero elements :

$$c = (x^m, x^{m+k}, x^{2m}, x^{2m+k}, \dots, x^{m(v-1)/2}, x^{m(v-1)/2+k})$$

where, so that the elements of c are all distinct, k is odd and the greatest common divisor of m and $v - 1$ is $(m, v - 1) = 2$. Again, x is any primitive element of the field. The $2(v - 1)$ symmetric neighbor differences are

$$\begin{aligned} &\pm x^m(1 - x^k), \pm x^{2m}(1 - x^{k-m}), \pm x^{2m}(1 - x^k), \pm x^{3m}(1 - x^{k-m}), \\ &\pm x^{3m}(1 - x^k), \dots, \pm x^{(v-1)m/2}(1 - x^k), \pm x^m(1 - x^{k-m}) \end{aligned}$$

which can be grouped as

$$\pm(1 - x^k)(x^m, x^{2m}, \dots, x^{(v-1)m/2}), \pm(1 - x^{k-m})(x^m, x^{2m}, \dots, x^{(v-1)m/2}).$$

Since $(m, v - 1) = 2$, these are all quadratic residues 4 times if and only if $\chi(1 - x^k) = \chi(1 - x^{k-m}) = 1$. The construction method is to break the cycle c and adjoin the missing element 0 in such a fashion that the set of neighbor differences is preserved. The cases $v = 8t + 1$ and $8t + 5$ are treated separately.

THEOREM 2.5. *Let $v = 8t + 5$ be a prime power. If there exists an integer m such that $(m, v - 1) = 2$ and $x^m - 2$ is a quadratic residue, then there exists a QND of GF_v .*

PROOF: Break c between x^m and x^{m+k} , and join 0 to x^m . It will be shown that under the conditions of the theorem, k and m can be chosen so that the result is a QND. The lost differences of c are $\pm(x^m - x^{m+k}) = \pm x^m(1 - x^k)$, and the gained differences from adding 0 are $\pm x^m$, so the neighbor differences of c are preserved if and only if $1 - x^k = -1$, i.e. $x^k = 2$, determining k (for $v = 8t + 5$, 2 is always a quadratic non-residue). With k so chosen, $\chi(1 - x^k) = \chi(-1) = \chi(x^{4t+2}) = 1$ and $\chi(1 - x^{k-m}) = \chi(1 - 2x^{-m}) = \chi(x^m - 2) = 1$, showing that the differences from c are the quadratic residues 4 times each, completing the proof. \square

Though not very strict, the conditions of Theorem 2.5 do not always hold, in particular failing for $v = 13$. We will, however, develop a simple sufficient condition, and show that they hold for all $13 < v < 500$.

Let $\varphi(v - 1)$ be the number of positive integers less than and coprime to $v - 1$ (φ is Euler's φ -function). The number of $m \pmod{(v - 1)}$ satisfying $m \equiv 2 \pmod{4}$ is $(v - 1)/4 = 2t + 1$, of which $\varphi(v - 1)/2$ satisfy $(m, v - 1) = 2$, and t satisfy $\chi(x^m - 2) = 1$ (see appendix III). So if $\varphi(v - 1) > 2(t + 1)$, Theorem 2.5 is satisfied. For $v < 500$ this condition fails only for $v = 13, 61$, and 421 , and

for the latter two numbers Theorem 2.5 is satisfied anyway. A single pair k, m is given in Table 2.1 for each $13 < v < 100$. A QND for $v = 13$ is $(0, 1, 11, 2, 12, 3, 4, 5, 9, 6, 10, 7, 8)$.

Theorem 2.5 does not work for $v = 8t + 1$ because in this case 2 is a quadratic residue. Theorem 2.5 approaches the problem by breaking the cycle c in two places.

THEOREM 2.6. *Let $v = 8t + 1$ be a prime power. If there exists an integer m satisfying $(m, v - 1) = 2$, $\chi(x^m - 2^{-1}) = 1$, and $\chi(x^m + 2^{-1}) = -1$, then there exists a QND of GF_v .*

PROOF: Break c between x^m and x^{m+k} , and again between $x^{(l-1)m+k}$ and x^{lm} , where l and k are to be determined. Join these two pieces by making x^{lm} and x^{m+k} adjacent, and attach 0 to x^m . The lost neighbor differences from c are $\pm x^m(1 - x^k)$ and $\pm x^{lm}(1 - x^{k-m})$, and those gained are $\pm x^m(x^k - x^{(l-1)m})$ and $\pm x^m$.

Let k be given by $x^k = x^m + 2^{-1}$. By the conditions of the theorem k is odd, $\chi(1 - x^k) = \chi(2^{-1} - x^m) = \chi(x^m - 2^{-1}) = 1$, and $\chi(1 - x^{k-m}) = \chi(2^{-1}x^{-m}) = 1$, so the neighbor differences from c are the quadratic residues 4 times each.

It remains to show that with appropriate choice of l the neighbor differences of c are preserved. Let l be given by $x^{-(l-1)m} = 1 - x^{k-m}$. Multiplying by

$\pm x^{lm}$ gives $\pm x^m = \pm x^{lm}(1 - x^{k-m})$. Also $-x^{(l-1)m} = -(1 - x^{k-m})^{-1} = -(2^{-1}x^{-m})^{-1} = 2x^m = 1 - 2x^k \Rightarrow 1 - x^k = x^k - x^{(l-1)m}$. \square

The conditions of Theorem 2.6 are more restrictive than those of Theorem 2.5, but again the number of available m grows with v so that a solution is almost always found. For $v < 500$ Theorem 2.6 fails only for $v = 9$ and 17 . A final result covers these failed cases.

THEOREM 2.7. *Let $v = 8t + 1$ be a prime power for which $x^k = 1 + x^{2t}$ is a quadratic non-residue. If there exists an integer m such that $(m, v - 1) = 2$ and $\chi(x^m - x^k) = 1$, then there exists a QND of GF_v .*

PROOF: Break the cycle c between x^m and x^{m+k} , and between x^{lm} and x^{lm+k} , where l is chosen so that $(l - 1)m \equiv 2t \pmod{(v - 1)}$. Join the resulting two sections into a v -vector by making x^m and x^{lm} adjacent to the missing element 0 . The lost differences from c are $\pm x^m(1 - x^k) = \pm x^{2t+m} = \pm x^{lm}$ and $\pm x^{lm}(1 - x^k) = \pm x^{4t+m} = \pm x^m$, which are the gained differences. \square

Theorems 2.5-2.7 with the examples for $v = 5$ and 13 give the desired set of Latin squares for all prime power $v \equiv 1 \pmod{4}$, $v < 500$. We illustrate these techniques by applying Theorem 2.7 for $v = 17$. With $x = 3$, $k = 5$, and $m = 2$ the cycle c is

$$c = (9, 11, 13, 14, 15, 7, 16, 12, 8, 6, 4, 3, 2, 10, 1, 5).$$

Any l such that $2(l-1) \equiv 4 \pmod{16}$ can be used ; take $l = 3$. Breaking the cycle between 9 and 11, and between 15 and 7, then joining 9 and 15 to zero gives

$$a = (11, 13, 14, 15, 0, 9, 5, 1, 10, 2, 3, 4, 6, 8, 12, 16, 7).$$

Table 2.1. k and m satisfying conditions of Theorems 2.5-7.

| v | Theorem | k | m | Primitive root or polynomial |
|-----|---------|-----|-----|---------------------------------|
| 9 | 2.7 | 1 | 6 | $x^2 + x + 2$ |
| 17 | 2.7 | 5 | 2 | 3 |
| 25 | 2.6 | 1 | 14 | $x^2 + x + 2$ |
| 29 | 2.5 | 1 | 6 | 2 |
| 37 | 2.5 | 1 | 14 | 2 |
| 41 | 2.6 | 9 | 6 | 6 |
| 49 | 2.6 | 19 | 34 | $x^2 + x + 3$ |
| 53 | 2.5 | 1 | 6 | 2 |
| 61 | 2.5 | 1 | 14 | 2 |
| 73 | 2.6 | 19 | 2 | 5 |
| 81 | 2.6 | 65 | 2 | $x^3 + x$ |
| 89 | 2.6 | 19 | 2 | 3 |
| 97 | 2.6 | 29 | 2 | 5 |

In summary, the procedures of Theorems 2.5-2.7 simply break a first neighbor balanced cyclic arrangement of the nonzero elements of G into two or more pieces, then adjoin these pieces including zero in such a fashion that the lost differences are gained. Note that multiplying one of the designs constructed here by x^1, x^2, \dots, x^{v-1} produces $v(v-1)$ $v \times v$ blocks satisfying Theorem 2.1. If the six differences from the corner design are equally divided among quadratic residues and non-residues, just use x^1, x^3, \dots, x^{v-2} .

We close this chapter with a note that we do not know if the partitioning of Theorem 2.4 can be had for non-prime powers. Enumeration shows that it is not possible for $v = 21$ and 33.

Chapter 3

Optimal torus designs for correlated errors

3.1. INTRODUCTION

In Chapter 2, two-dimensional designs were considered for the second order autonormal planar error process. Universally optimum and near optimum designs there require a large number of replications for the treatments. Also the optimum designs considered elsewhere for various covariance structures are not free from this drawback. Primarily the conditions on corner plots and plots in end rows and end columns make attainment of complete symmetry in the C -matrix with a small number of replications impossible. The corners, end rows, and end columns can be removed by taking blocks on the torus. Consequently universally optimal designs can be constructed for comparatively smaller number of replications (see Martin (1982)). Besides optimality arguments with reasonable design sizes the

torus approach has other advantages. Torus designs are expected to be highly efficient for correlated errors in the plane (compare section 3.5 below), and these designs can be used as bordered designs (see Freeman (1979a), Morgan (1989)) in the plane preserving all balanced properties of the torus design. For these reasons the torus approach is taken in this chapter. While Martin (1982) has considered single torus designs only, here more than one torus is considered and corresponding block effects are included in the model.

Let y_{ijk} be the observation in row i and column j of the $p \times q$ torus lattice k ($=1,2,\dots,t$) with arbitrary initial cell $(1,1,k)$. The model considered here is

$$y_{ijk} = \mu_k + \tau_{[ijk]} + e_{ijk} \quad (3.1)$$

where the errors follow the symmetric second order autonormal torus process:

$$\sigma^2 \text{var}^{-1}(e) = I_t \otimes [I_{pq} - \alpha(I_p \otimes C_q + C_p \otimes I_q) - \gamma(C_p \otimes C_q)] \quad (3.2)$$

Here C_q is the $q \times q$ matrix with

$$(C_q)_{ij} = \begin{cases} 1 & \text{if } i - j \equiv \pm 1 \pmod{q} \\ 0 & \text{otherwise,} \end{cases}$$

and α , γ , and σ^2 are constants. Then it can be shown that if α , $\gamma > 0$, a design is universally optimum for estimation of treatment contrasts if

- (i) every treatment has each other treatment as first neighbor equally often in rows and columns combined,

(3.3)

- (ii) every treatment has each other treatment as first neighbor equally often in diagonals,
- (iii) no treatment immediately neighbors itself in rows, columns, or diagonals,
- (iv) the set of t toruses is equireplicate, with each pair of distinct treatments appearing together in an equal number of toruses.

Note that (iv) says that the toruses form a balanced block design as defined by Kiefer(1975). If interest focuses on the treatment means $\mu + \tau_i$ in the simpler model $\mu_k = \mu$ for all k , (iv) is relaxed to equireplication in the t toruses combined. In general, equireplication in each of the toruses is sufficient for (iv) to hold. In the terminology of Martin (1982), (i) and (ii) say the design has second order neighbor balance.

The optimality result above can be extended to higher order torus processes after defining neighbor relationships as follows. Say two points on an $p \times q$ torus lattice are $[\frac{i(i+1)}{2} + j]$ -th order neighbors if one can be reached from the other by moving $\pm i$ rows and $\pm j$ columns, or $\pm i$ columns and $\pm j$ rows, $j \leq i \leq \max[\text{int}(p/2), \text{int}(q/2)]$. Then an order w variance balanced design is balanced for d -th order neighbors for each $d = 1, 2, \dots, w$, (Martin, 1982). Universal optimality will then depend on the signs of the parameters in the order w torus lattice process.

The primary concern here is to construct designs for the second order case,

and second order neighbors will be referred to by the more natural “first diagonal neighbors”. However some higher order results are also obtained. Section 3.2 deals with the construction of universally optimum torus designs for v , a prime power of the form $4t + 1$. Two generalizations of the section 3.2 results are given in section 3.3. In section 3.4 designs are obtained for non-prime powers which as closely as possible meet the optimality conditions, and hence should be near-optimal. Some numerical comparisons for planar versions of torus designs are made in section 3.5.

3.2. DESIGNS WITH SECOND ORDER BALANCE

As mentioned in Chapter 2 the general construction described in Theorem 2.3 can be used as a step in the construction of torus designs. In this chapter $p_1 = (v + 3)/4$ is taken in Theorem 2.3 to obtain minimal size designs. Using corollary 2.1, we now construct a and b of minimal size by

$$a = (1, x^2, x^4, \dots, x^{(v-1)/2}) \text{ and } b = (x, x^3, x^5, \dots, x^{(v+1)/2}) \quad (3.4)$$

where x is any primitive element of GF_v . Using the above a and b and applying Theorem 2.3 we have

COROLLARY 3.1. *Let v be a prime power of the form $4t + 1$. Then there exist $v(v + 3)/4 \times (v + 3)/4$ squares which are balanced for first row and column neighbors combined, are balanced for first diagonal neighbors, and have no like first neighbors.*

Since none of the designs in this chapter have like first neighbors, this fact will be omitted in succeeding results.

Example 3.1. The corollary 3.1 design for $v = 9$. The powers of x in additive form are $x = (1, 0)$, $x^2 = (2, 1)$, $x^3 = (2, 2)$, $x^4 = (0, 2)$, $x^5 = (2, 0)$, $x^6 = (1, 2)$, $x^7 = (1, 1)$, $x^8 = (0, 1)$.

Adding mod(3,3), then writing i for x^i , gives these 9 squares :

$$\begin{array}{lll}
 \begin{array}{ccc} 7 & 5 & 2 \\ 8 & 1 & 7 \\ 6 & 2 & 3 \end{array} & B + (1, 0) = \begin{array}{ccc} 2 & 0 & 8 \\ 7 & 5 & 2 \\ 3 & 8 & 4 \end{array} & B + (2, 0) = \begin{array}{ccc} 8 & 1 & 7 \\ 2 & 0 & 8 \\ 4 & 7 & 6 \end{array} \\
 \\
 \begin{array}{ccc} 6 & 2 & 3 \\ 4 & 7 & 6 \\ 1 & 3 & 5 \end{array} & B + (1, 1) = \begin{array}{ccc} 3 & 8 & 4 \\ 6 & 2 & 3 \\ 5 & 4 & 0 \end{array} & B + (2, 1) = \begin{array}{ccc} 4 & 7 & 6 \\ 3 & 8 & 4 \\ 0 & 6 & 1 \end{array} \\
 \\
 \begin{array}{ccc} 1 & 3 & 5 \\ 0 & 6 & 1 \\ 7 & 5 & 2 \end{array} & B + (1, 2) = \begin{array}{ccc} 5 & 4 & 0 \\ 1 & 3 & 5 \\ 2 & 0 & 8 \end{array} & B + (2, 2) = \begin{array}{ccc} 0 & 6 & 1 \\ 5 & 4 & 0 \\ 8 & 1 & 7 \end{array}
 \end{array}$$

In this section methods of adjoining the v arrays of Theorem 2.3 designs with $p_1 = (v + 3)/4$ are considered, the procedures for which will lead us to torus designs and what we shall call pseudo-torus designs. Suppose B_i and B_j are two

$p_1 \times p_1$ components such that the last column of B_i is the first column of B_j . If we form a $p_1 \times (2p_1 - 1)$ array by merging B_i and B_j at this common end column, this repeated set of column neighbors is lost, while the diagonal and row neighbors are unaffected. We seek to adjoin the v arrays in this fashion so that a balanced set of neighbors is lost. Our first result is for the integers mod (v) .

THEOREM 3.1. *Suppose a and b are p_1 -vectors on a cyclic group G satisfying Theorem 2.3, and that $(b_{p_1} - b_1, v) = 1$. Then there exists a $(p_1 - 1) \times v(p_1 - 1)$ torus or pseudo-torus design balanced for combined first row and column neighbors, and balanced for first diagonal neighbors.*

PROOF: Let $w_2 = b_2 - b_1$ and write $B_i = B(a, b) + w_2(i - 1)$ for $i = 1, 2, \dots, v$. Then the last column of B_i is the first of B_{i+1} and we can merge the v B_i 's into a single cylindrical $p_1 \times v(p_1 - 1)$ array, losing the first neighbors in $a + g$, $g \in G$. To maintain the combined row and column balance of the Theorem 2.1 design, we need to also lose the set of first neighbors in $b + g$, $g \in G$. Let $w_1 = a_{p_1} - a_1$.

case (i). Suppose $w_1 = 0$. Then the first and last rows of our $p_1 \times v(p_1 - 1)$ cylinder design are identical. Connecting the ends of our cylinder by merging these two rows gives a $(p_1 - 1) \times v(p_1 - 1)$ torus design with the required properties.

case (ii). If $w_1 \neq 0$ then the first and last rows of our cylinder are not identical. However the last row must be a cyclic permutation of the first, so that

with an appropriate “twisting” of our cylinder, the merging used in case (i) can be achieved. Because this twisting distorts the torus lattice, we will call these pseudo-torus designs. \square

We see that the designs falling into case (i) of Theorem 3.1 are universally optimum for the second order symmetric torus lattice process with $\alpha > 0$ and $\gamma > 0$. If the designs are to be used in the plane, the same neighbors are lost whether a torus or a pseudo-torus design is used, and the distinction is unimportant. If we wish to preserve neighbor balance in the plane by bordering, this too can be done for either case. Theorem 3.1 and all succeeding torus and pseudo-torus results could be equivalently stated in terms of fully bordered planar designs; we take the torus approach for optimality arguments and for the cohesion and simplicity afforded the constructions.

Example 3.2. On Z_5 take $a = (0, 1, 2, 3)$ and $b = (0, 3)$ where $S_1 = \{1, 4\}$ and $S_2 = \{2, 3\}$. Merging the developed $R(a, b)$'s as in Theorem 3.1 gives this 4×5 Youden design, which as a pseudo-torus design has distinct pairs of treatments as first neighbors 4 times in rows and columns combined, and 4 times in diagonals. For clarity borders have been included, but will henceforth be omitted.

| | | | | | | | | |
|---|----|----|----|----|----|----|----|---|
| 1 | | 4 | 2 | 0 | 3 | 1 | | 4 |
| | .. | .. | .. | .. | .. | .. | .. | |
| 2 | : | 0 | 3 | 1 | 4 | 2 | : | 0 |
| 3 | : | 1 | 4 | 2 | 0 | 3 | : | 1 |
| 4 | : | 2 | 0 | 3 | 1 | 4 | : | 2 |
| 0 | : | 3 | 1 | 4 | 2 | 0 | : | 3 |
| | .. | .. | .. | .. | .. | .. | .. | |
| 1 | | 4 | 2 | 0 | 3 | 1 | | 4 |

In view of corollary 3.1 we have the following:

COROLLARY 3.2. *Let v be a prime of the form $4t + 1$. Then there is a $(v - 1)/4 \times v(v - 1)/4$ torus or pseudo-torus design balanced for first row and column neighbors combined, and balanced for first diagonal neighbors.*

Theorem 3.2 extends the method of Theorem 3.1 for non-cyclic groups.

THEOREM 3.2. *Let $v = v_1 v_2$, $1 < v_1 < v$. Let a and b be p_1 -vectors on an abelian group G of order v satisfying Theorem 2.1. Suppose $w_1 = a_{p_1} - a_1$ and $w_2 = b_{p_1} - b_1$ are such that w_2 generates a subgroup G_0 of order v_2 , and $G_j = G_0 + jw_1$ are the cosets of G_0 , $j = 1, 2, \dots, v_1$. Then there exists a $v_1(p_1 - 1) \times v_2(p_1 - 1)$ pseudo-torus design which is balanced for first row and column neighbors combined, and is balanced for first diagonal neighbors.*

PROOF: Let $B_{i,1} = B(a, b) + (i - 1)w_2$, $i = 1, 2, \dots, v_2$, and form the $p_1 \times v_2(p_1 - 1)$ cylindrical array B_1 by adjoining the $B_{i,1}$'s at their common end columns. Now let $B_j = B_1 + (j - 1)w_1$, $j = 1, 2, \dots, v_1$; these v_1 cylinders

are the v components of the Theorem 2.1 design adjoined so that the column neighbors given by first neighbors in $a + g$, $g \in G$ have been lost. If the first row of B_1 is denoted by r_1 , the last row of B_{v_2} is $r_1 + w_1 v_1$, which must be a cyclic permutation of r_1 . A corresponding set of row neighbors is lost by merging the B_j 's into a single array via their common end rows. \square

It should be noted that Theorem 3.2 designs are true torus designs whenever w_1 is of order v_1 . In particular, this will be the case when $v = q^2$ for prime q and $G = GF_v$.

COROLLARY 3.3. *Let v be a squared prime of the form $4t + 1$. Then there exists a $v^{1/2}(v - 1)/4 \times v^{1/2}(v - 1)/4$ torus design balanced for first row and column neighbors combined, and balanced for first diagonal neighbors.*

PROOF: For a and b given in (3.4), $w_1 = -2$ and $w_2 = -2x$ clearly satisfy Theorem 3.2. \square

Example 3.3. For $v = 9$ we construct the 6×6 design of corollary 3.3 by combining the arrays of example 3.1. Here $G = GF_{3^2}$ with $a = (1, x^2, x^4)$ and $b = xa$. Identifying x^i with i ($i = 1, 2, \dots, 8$) we obtain

| | | | | | |
|---|---|---|---|---|---|
| 7 | 5 | 2 | 0 | 8 | 1 |
| 8 | 1 | 7 | 5 | 2 | 0 |
| 6 | 2 | 3 | 8 | 4 | 7 |
| 4 | 7 | 6 | 2 | 3 | 8 |
| 1 | 3 | 5 | 4 | 0 | 6 |
| 0 | 6 | 1 | 3 | 5 | 4 |

This is a knight's move torus design with second order neighbor balance. Rows and columns together are a *PBIB*(2) of Latin square type.

COROLLARY 3.4. *Let $v = q^n$ be an odd prime power of the form $4t + 1$, $n > 2$. Then there exist $q^{n-2} \times q(v-1)/4 \times q(v-1)/4$ toruses together balanced for first row and column neighbors combined, and for first diagonal neighbors.*

PROOF: Let a and b be as in (3.4) for the additive group G of GF_v and form the $q(v-1)/4 \times q(v-1)/4$ torus T by adjoining q^2 of the v Theorem 2.3 components as in the proof of Theorem 3.2. Let $A = \{c_1 + c_2x : c_1, c_2 \in Z_q\}$ be the additive subgroup generated by w_1 and w_2 . Our q^{n-2} arrays are $T + y$ where y takes on one value in each of the q^{n-2} cosets of G/A . \square

3.3. TWO GENERALIZATIONS

In section 3.2, single torus constructions were obtained for v , a prime power of the form $4t + 1$, by using a partition of the nonzero elements of G into two equal sized subsets. Designs in several toruses for v not necessarily of the form $4t + 1$ can be obtained by partitioning the nonzero elements of G into more than

two subsets. Let $v = 2mt + 1$ and suppose we can find subsets S_1, S_2, \dots, S_m of G , $|S_i| = 2t$, satisfying

$$g \in S_i \Rightarrow -g \in S_i, \quad (3.5)$$

$$\cup S_i = G - 0, \text{ and} \quad (3.6)$$

$$B(S_i, S_{i+1}) \text{ for } i = 1, 2, \dots, m \text{ are each} \\ \text{nonzero element of } G \text{ exactly } 2t \text{ times,} \quad (3.7)$$

where we write $S_{m+1} = S_1$. Then construct $m(t+1)$ -vectors a_1, a_2, \dots, a_m such that $\pm a_i^* = S_i$. We can then generalize our previous results by applying the techniques of Theorems 3.1 or 3.2 or Corollary 3.4, successively taking a, b as each pair a_i, a_{i+1} ($a_{m+1} = a_1$).

THEOREM 3.3. *Let $v = 2tm + 1$ be a prime or prime power. Sets S_1, S_2, \dots, S_m satisfying (3.5), (3.6), and (3.7) exist on $G = GF_v$.*

PROOF: Let x be a primitive element of G and $S_1 = (1, x^m, x^{2m}, \dots, x^{(2t-1)m})$ and $S_i = x^{i-1}S_1$. S_1 is closed under multiplication and $x^{tm} = -1$ is in S_1 , so $g \in S_i \Rightarrow -g \in S_i$. By inspection of $B(S_1, S_2)$ one can see that its entries are those of $B_1 =$

$$(x^0, x^m, x^{2m}, \dots, x^{(2t-1)m}) \otimes (1+x, 1+x^{m+1}, 1+x^{2m+1}, \dots, 1+x^{(2t-1)m+1})$$

where \otimes is the Kronecker product. Defining $B_i = x^{i-1}B_1$, it follows easily that

B_i contains the entries of $B(S_i, S_{i+1})$, and B_1, B_2, \dots, B_m collectively contain the elements of

$(x^0, x^1, x^2, \dots, x^{2tm-1}) \otimes (1+x, 1+x^{m+1}, 1+x^{2m+1}, \dots, 1+x^{(2t-1)m+1})$, that is, each nonzero element of G $2t$ times. \square

Let h be given by $(1-x^m)^{-1} = x^h$. Then with $a_1 = (x^h, x^{h+m}, \dots, x^{h+tm})$, $a_i = x^{(i-1)m} a_1$ satisfy $\pm a_i^* = S_i$ of Theorem 3.3. Combining these results yields the following corollaries.

COROLLARY 3.5. *Let v be a prime power of the form $2tm + 1$, $m > 1$. There exist vm $(t+1) \times (t+1)$ squares that are together balanced for combined first row and column neighbors, and for first diagonal neighbors.*

COROLLARY 3.6. *Let v be a prime of the form $2tm + 1$, $m > 1$. Then there exist m $t \times vt$ toruses that are together balanced for combined first row and column neighbors, and for first diagonal neighbors.*

COROLLARY 3.7. *Let $v = q^n$ be an odd prime power of the form $2tm + 1$, $m, n > 1$. Then there exists mq^{n-2} $qt \times qt$ toruses that are together balanced for combined first row and column neighbors, and for first diagonal neighbors.*

PROOF: The desired q^{n-2} $qt \times qt$ arrays will arise from each $B(a_i, a_{i+1})$ by applying the technique in the proof of corollary 3.4. \square

The number of separate arrays in corollaries 3.5 - 3.7 can be halved when $m = 2$ since only one of the two pairs S_1, S_2 and S_2, S_1 need be used. More generally, the number of arrays can be halved if the number of times each S_i is used can be reduced from 2 to 1. This requires that the number of S_i 's be even, so write $v = 2tm' + 1 = 4tm + 1$ where $m' = 2m$ is even. The general problem is to find subsets S_1, S_2, \dots, S_{2m} of G , $|S_i| = 2t$, satisfying (3.5), (3.6), and

$$R(S_{2i-1}, S_{2i}) \text{ for } i = 1, 2, \dots, m \text{ are each} \\ \text{nonzero element of } G \text{ exactly } t \text{ times.} \quad (3.8)$$

We then apply our methods to m pairs a_i, b_i such that $\pm a_i^* = S_{2i-1}$ and $\pm b_i^* = S_{2i}$, $i = 1, 2, \dots, m$. Under certain conditions the sets of Theorem 3.3 can be so partitioned.

THEOREM 3.4. *For $v = 4tm + 1 = 2tm' + 1$ a prime power, the sets S_1, S_2, \dots, S_{2m} of Theorem 3.3, in some order, satisfy (3.8) if for some integer r*

$$\{1 \pm x^{2r-1}, 1 \pm x^{2m+2r-1}, 1 \pm x^{4m+2r-1} \dots, 1 \pm x^{2(t-1)m+2r-1}\} \quad (3.9)$$

is composed of t quadratic residues and t non-residues.

PROOF: We take $S_1 = \{1, x^{2m}x^{4m}, \dots, x^{2(2t-1)m}\}$, $S_2 = x^{2r-1}S_1$, and $S_{2i-1} = x^{2(i-1)}S_1$, $S_{2i} = x^{2(i-1)}S_2$ for $i = 1, 2, \dots, m$. Using the relation

$x^{2tm} = -1$, $B(S_1, S_2)$ can be written as $(x^0, x^{2m}, x^{4m}, \dots, x^{2(2t-1)m}) \otimes$
 $(1 \pm x^{2r-1}, 1 \pm x^{2m+2r-1}, 1 \pm x^{4m+2r-1} \dots, 1 \pm x^{2(t-1)m+2r-1})$.
 $B(S_{2i-1}, S_{2i}) = x^{2(i-1)} B(S_1, S_2)$ implies that the m tables together contain
 $(x^0, x^2, x^m, \dots, x^{4tm-2}) \otimes$
 $(1 \pm x^{2r-1}, 1 \pm x^{2m+2r-1}, 1 \pm x^{4m+2r-1} \dots, 1 \pm x^{2(t-1)m+2r-1})$. The left-hand
vector containing all the quadratic residues gives the result. \square

Now write $x^h = (1 - x^{2m})^{-1}$. Then for $a_1 = (x^h, x^{h+2m}, \dots, x^{2tm})$ and
 $b_1 = x^{2r-1} a_1$, $a_i = x^{2(i-1)} a_1$ and $b_i = x^{2(i-1)} b_1$ satisfy $\pm a_i^* = S_{2i-1}$ and
 $\pm b_i^* = S_{2i}$ in the proof of Theorem 3.4. Hence we can halve the number of arrays
in corollaries 3.5- 3.7 whenever (3.9) holds. This is not always possible (see Table
3.1 below).

Table 3.1. r values satisfying (3.9)

| v | t | m | r | primitive root or polynomial |
|-----|-----|-----|------------|---------------------------------|
| 17 | 2 | 2 | 2 | 3 |
| 25 | 2 | 3 | 1, 2, 3 | $x^2 + x + 2$ |
| | 3 | 2 | non-exist. | |
| 37 | 3 | 3 | 2 | 2 |
| 41 | 2 | 5 | 3 | 11 |
| | 5 | 2 | non-exist. | |
| 49 | 6 | 2 | 1, 2 | $x^2 + 2x + 5$ |
| | 4 | 3 | 2 | |
| | 3 | 4 | 1, 2, 3, 4 | |
| | 2 | 6 | 2, 5 | |

The following example shows that even for GF_v , the approach of (3.8) is indeed distinct from that of (3.9), and not simply a method for halving those designs.

Example 3.4. For $v = 25$ put $t = 3$ and $m = 2$. One can check that (3.9) is not satisfied. The following 4 sets satisfy (3.5), (3.6), and (3.8): $S_1 = (1, x^4, x^8, x^{12}, x^{16}, x^{20})$, $S_2 = x^2 S_1$, $S_3 = x S_1$, and $S_4 = x^3 S_1$. But $B(S_2, S_3)$ and $B(S_1, S_4)$ together give each quadratic residue 4 times and each quadratic non-residue twice, so these sets do not satisfy (3.7). With these sets we get two 15×15 toruses (compare corollary 3.9 below), one of which is

| | | | | | | | | | | | | | | |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 0 | 23 | 4 | 10 | 8 | 22 | 16 | 13 | 0 | 4 | 12 | 10 | 22 | 15 | 16 |
| 5 | 17 | 21 | 18 | 7 | 19 | 14 | 9 | 5 | 21 | 2 | 18 | 19 | 6 | 14 |
| 19 | 6 | 14 | 5 | 17 | 21 | 18 | 7 | 19 | 14 | 9 | 5 | 21 | 2 | 18 |
| 1 | 4 | 3 | 20 | 22 | 24 | 11 | 0 | 1 | 3 | 10 | 20 | 24 | 16 | 11 |
| 12 | 21 | 8 | 15 | 19 | 13 | 23 | 5 | 12 | 8 | 18 | 15 | 13 | 14 | 23 |
| 13 | 14 | 23 | 12 | 21 | 8 | 15 | 19 | 13 | 23 | 5 | 12 | 8 | 18 | 15 |
| 7 | 3 | 6 | 9 | 24 | 17 | 2 | 1 | 7 | 6 | 20 | 9 | 17 | 11 | 2 |
| 10 | 8 | 22 | 16 | 13 | 0 | 4 | 12 | 10 | 22 | 15 | 16 | 0 | 23 | 4 |
| 0 | 23 | 4 | 10 | 8 | 22 | 16 | 13 | 0 | 4 | 12 | 10 | 22 | 15 | 16 |
| 19 | 6 | 14 | 5 | 17 | 21 | 18 | 7 | 19 | 14 | 9 | 5 | 21 | 2 | 18 |
| 20 | 22 | 24 | 11 | 0 | 1 | 3 | 10 | 20 | 24 | 16 | 11 | 1 | 4 | 3 |
| 1 | 4 | 3 | 20 | 22 | 24 | 11 | 0 | 1 | 3 | 10 | 20 | 24 | 16 | 11 |
| 13 | 14 | 23 | 12 | 21 | 8 | 15 | 19 | 13 | 23 | 5 | 12 | 8 | 18 | 15 |
| 9 | 24 | 17 | 2 | 1 | 7 | 6 | 20 | 9 | 17 | 11 | 2 | 7 | 3 | 6 |
| 7 | 3 | 6 | 9 | 24 | 17 | 2 | 1 | 7 | 6 | 20 | 9 | 17 | 11 | 2 |

Entries (except 0) are expressed as powers of primitive x . The second torus is obtained by multiplying by x . The following are immediate.

COROLLARY 3.8. *Let v be a prime power of the form $4mt + 1$, and suppose (3.9) holds. Then there exist $vm = (t + 1) \times (t + 1)$ squares that are together balanced for combined first row and column neighbors, and for first diagonal neighbors.*

COROLLARY 3.9. *Let v be a prime of the form $4tm + 1$, and suppose (3.9) holds. Then there exist $m = t \times vt$ toruses that are together balanced for combined first row and column neighbors, and for first diagonal neighbors.*

A small complication arises when connecting the developed $B(a_i, b_i)$ in two directions. We first give the result for even powers of primes. But first we state

LEMMA 3.1. *The nonzero elements of GF_q in GF_{q^n} are all quadratic residues if n is even.*

COROLLARY 3.10. *Let $v = q^n$ be an odd prime power of the form $4tm + 1$, where $n > 1$ is even, and suppose (3.9) holds. Then there exists $mq^{n-2} = qt \times qt$ toruses that are together balanced for combined first row and column neighbors, and for first diagonal neighbors.*

PROOF: It is sufficient to show that the method of corollary 3.4 can be used to form $q^{n-2} = qt \times qt$ toruses from the developed $B(a_1, b_1)$'s. Hence we must show

that $w_1 = x^{h+2tm} - x^h = -2x^h$ and $w_2 = x^{2r-1}w_1$ generate distinct additive subgroups. Now -2 generates the subfield GF_q , which for even n is composed only of quadratic residues and 0, so h and $h + 2r - 1$ having different parities establishes the result. \square

The elements of GF_q in GF_{q^n} are powers of x^h where $h = (q^n - 1)/(q - 1)$. So corollary 3.10 holds for odd $n > 1$ iff $2r - 1$ is not an odd multiple of h . Of course the a_i 's and b_i 's given in the proof are only one of many possibilities, and it appears that the corollary will hold for all odd $n > 1$ as well.

Again, we know from section 3.2 that (3.9) always holds when $m = 1$. Next we show that (3.9) holds when $t = 1$, that is, that $1 \pm x^{2r-1}$ is one quadratic residue and one non-residue for some integer r . Multiplying by $y = x^{1-2r}$ this becomes, equivalently, $y \pm 1$ is one quadratic residue and one quadratic non-residue. The proof of the following lemma is in appendix IV.

LEMMA 3.2. *Let $v = 4m + 1 = q^n$ be a power of the odd prime q . There exists a quadratic non-residue $y \in GF_v$ such that one of $\{y - 1, y + 1\}$ is a quadratic residue, and the other a quadratic non-residue. If $n > 1$, this y may be chosen so that is not in the subfield GF_v .*

COROLLARY 3.11. *Let $v = 4m + 1 = q^n$ where q is an odd prime and $n > 1$. Then there exist $q^{n-2}(v - 1)/4$ $q \times q$ toruses that are together balanced for combined first row and column neighbors, and for first diagonal neighbors.*

PROOF: This is just corollary 3.10 with $t = 1$ and $x^{2r-1} = y^{-1}$ where y is given by Lemma 3.2. That y , and hence y^{-1} , is not in the subfield GF_q , removes the corollary 3.10 restriction that n be even. \square

Note that these designs have only $(v - 1)/4$ replicates.

Proper choice of y for these designs can often give balance of neighbors at higher orders. For the corollary 3.11 designs we may take $a_1 = (0, 1)$, $b_1 = (0, y)$, and $(a_i, b_i) = x^{2(i-1)}(a_1, b_1)$, where $y = x^{2r-1}$ is given by lemma 3.2. The first diagonal differences in $B(a_1, b_1)$ are $\pm(y + 1)$ and $\pm(y - 1)$, showing how first diagonal balance is obtained. The differences for looking at neighbors separated by i rows or i columns are

$$\pm ix^{2(u-1)} \text{ and } \pm iyx^{2(u-1)} \quad u = 1, 2, \dots, m$$

which is a balanced set since y is a quadratic non-residue. Hence the corollary 3.11 designs are balanced for combined i -th row and column neighbors, $i = 1, 2, \dots, (q - 1)/2$. For neighbors separated by $\pm i$ rows and $\pm j$ columns, the

differences are

$$x^{2(u-1)}\{\pm(jy+i), \pm(jy-i)\} = jx^{2(u-1)}\{\pm(y+ij^{-1}), \pm(y-ij^{-1})\}$$

$$u = 1, 2, \dots, m$$

Hence we have the following corollary

COROLLARY 3.12. *If there exists a quadratic non-residue $y \in GF_q$ such that $\{y+i, y-i\}$ contains exactly one quadratic residue and one quadratic non-residue for each $i = 1, 2, \dots, q-1$, then the design of corollary 3.11 is balanced for neighbors of all orders.*

Example 3.5. A balanced lattice for 25 treatments that is also balanced for torus neighbors of all orders. This is the corollary 3.12 design with $y^{-1} = x^{2r-1} = x$, using the primitive polynomial $x^2 + x + 2$. Again i is written for x^i

| | | | | | | | | | | | | | | |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 11 | 16 | 15 | 18 | 2 | 13 | 18 | 17 | 20 | 4 | 15 | 20 | 19 | 22 | 6 |
| 4 | 23 | 14 | 6 | 3 | 6 | 1 | 16 | 8 | 5 | 8 | 3 | 18 | 10 | 7 |
| 8 | 21 | 17 | 24 | 22 | 10 | 23 | 19 | 2 | 24 | 12 | 1 | 21 | 4 | 2 |
| 19 | 7 | 1 | 0 | 13 | 21 | 9 | 3 | 0 | 15 | 23 | 11 | 5 | 0 | 17 |
| 9 | 20 | 10 | 12 | 5 | 11 | 22 | 12 | 14 | 7 | 13 | 24 | 14 | 16 | 9 |
| | | | | | | | | | | | | | | |
| 17 | 22 | 21 | 24 | 8 | 19 | 24 | 23 | 2 | 10 | 21 | 2 | 1 | 4 | 12 |
| 10 | 5 | 20 | 12 | 9 | 12 | 7 | 22 | 14 | 11 | 14 | 9 | 24 | 16 | 13 |
| 14 | 3 | 23 | 6 | 4 | 16 | 5 | 1 | 8 | 6 | 18 | 7 | 3 | 10 | 8 |
| 1 | 13 | 7 | 0 | 19 | 3 | 15 | 9 | 0 | 21 | 5 | 17 | 11 | 2 | 23 |
| 15 | 2 | 16 | 18 | 11 | 17 | 4 | 18 | 20 | 13 | 19 | 6 | 20 | 22 | 15 |

The condition of corollary 3.12 is achievable for $n = 2$ and $q=3,5,7$, and 11 but not for $q=13,17$, or 19. The corollary 3.11 designs will be balanced to

sixth order if additionally $\{(y \pm 2), (y \pm 2^{-1})\}$ are two quadratic residues and two quadratic non-residues. This can be done for $n = 2$ and $q=13, 17$, and 19 ; no other cases have been checked, but it is suspected that this will hold for all q .

3.4 DESIGNS FOR NON PRIME POWER v

In this section designs will be constructed using cyclic groups for small v . In most cases we will not be able to attain the perfect balance of sections 3.2 and 3.3 : the approach here is to keep the range in neighbor counts small, still allowing no like first neighbors. We concentrate on the method used in (2.4)-(2.6), but will not demand that S_1 and S_2 are equal sized subsets, and will relax (2.6). The effect of the former will depend on the method of merging the component arrays; the latter relaxes the demand of exact diagonal neighbor balance.

Consider first the case of $v = 4t + 3$. Let S_1 and S_2 be a partition of $Z_v - 0$, $|S_1| = 2t$, $|S_2| = 2t+2$, satisfying (2.4)-(2.5) and

$$B(S_1, S_2) \text{ contains the nonzero elements of } Z_v \text{ with} \\ \text{frequencies } f_1 < f_2 < \dots < f_s. \quad (3.10)$$

Let a, b be such that $\pm a^* = S_1$ and $\pm b^* = S_2$, and write $w_1 = a_{t+1} - a_1$ and $w_2 = b_{t+2} - b_1$. Then in the $v \times (t+1) \times (t+2)$ arrays $\{B(a, b) + g : g \in Z_v\}$ each pair of distinct treatments occurs as first neighbors $t+1$ or $t+2$ times in rows and columns combined, and f_1, f_2, \dots, f_s times in diagonals. If we choose b

such that $(w_2, v) = 1$ then we can merge the $B(a, b)$'s via common end columns into a $(t + 1) \times (4t + 3)(t + 1)$ cylinder design with first neighbors balanced for rows and columns combined. Alternatively, as a torus or pseudo-torus design, the dimensions are $t \times (4t + 3)(t + 1)$, with combined first row/column neighbor counts of t and $t + 1$. In either case the diagonal first neighbors are the same as in the $B(a, b)$'s. If $(w_1, v) = 1$ we can get a $t(4t + 3) \times (t + 1)$ pseudo-torus design with the same counts; of the three, the cylinder design is to be preferred in planar applications. If w_1 generates a subgroup G_1 of order v_1 and w_2 is such that $G_1 + iw_2$ for $i = 1, 2, \dots, v_2 (v = v_1 v_2)$ are the cosets of G_1 in Z_v , then we can obtain a $v_1 t \times v_2 (t + 1)$ torus or pseudo-torus design again with the same neighbor counts. Of course the size of a (b) can be increased so that $\pm a^*$ ($\pm b^*$) is multiple copies of S_1 (S_2), multiplying the number of rows (columns) of the design; this may be useful for some of the small treatment numbers (see Table 3.2), but can further spread the row/column neighbor counts depending on the method of adjoining the $B(a, b)$'s.

For $v = 4t + 1$ (we are interested in the case when v is not a prime or prime power) the procedures are the same except that $|S_1| = |S_2| = 2t$. Hence a pseudo-torus design will be balanced for combined first row and column neighbors, while the cylinder row/column neighbor counts will be t and $t + 1$.

When v is even the order 2 element requires that a larger a or b be used if

the row/column neighbor counts are to be kept reasonably balanced. For $v = 4t$ partition $Z_v = 0$ as S_1, S_2 , $|S_1| = 2t - 1$, $|S_2| = 2t$ satisfying (2.4) and (3.10). Now find a and b such that $\pm a^* =$ two copies of S_1 and $\pm b^* = S_2$. Then in the $2t \times (t + 1)$ array $B(a, b)$

- (i) the symmetric row differences are $2t$ copies of S_2
- (ii) the symmetric column differences are $2(t + 1)$ copies of S_1
- (iii) the symmetric diagonal differences are $B(\pm a^*, \pm b^*) =$ two copies of $B(S_1, S_2)$.

Since each column of $B(a, b)$ gives two copies of S_1 , the $2t \times 4t^2$ cylinder designs will be balanced for combined row and column neighbors. Any torus or pseudo-torus design will have row/column neighbor counts of $2t - 1$ and $2t$.

For $v = 4t + 2$ we need S_1, S_2 satisfying (2.4) and (3.10) with $|S_1| = 2t - 1$, $|S_2| = 2t$, and a, b such that $\pm a^* =$ two copies of S_1 and $\pm b^* = S_2$. Then in the $2(t + 1) \times (t + 1)$ array $B(a, b)$

- (iv) the symmetric row differences are $2(t + 1)$ copies of S_2
- (v) the symmetric column differences are $2(t + 1)$ copies of S_1
- (vi) the symmetric diagonal differences are two copies of $B(S_1, S_2)$.

So torus or pseudo-torus designs will have combined row and column neighbor counts of $2t$ and $2t + 1$; for cylinder designs the counts are $2t + 1$ and $2(t + 1)$, or $2t$ and $2(t + 1)$, as the arrays are adjoined via rows or columns, respectively.

Interesting here is that the row and column neighbor counts will be balanced if $\pm b^*$ is two copies of S_2 and the $B(a, b)$'s are adjoined by rows.

All sets S_1, S_2 on Z_v giving the smallest value of $\sum n_i f_i^2$, where n_i is the number of elements of Z_v occurring with frequency f_i in $B(S_1, S_2)$, are given in Table 3.2 for $v \leq 30$, along with possible design sizes. On the torus for the second order autonormal process this method gives the MS-optimal design within this class. In all of the cases here at least one of the partitions also minimizes $f_s - f_1$, the value of which is listed for each partition.

Just two sets S_1, S_2 have been chosen here so as to obtain designs in single arrays. It is not, however, always the case that for given S_1 and S_2 , a and b can be found so that w_1 and w_2 are of appropriate orders to generate a single array design. When v is even, if the number of odd elements in S_2 is a multiple of 4, then for the constructions given above both w_1 and w_2 must be even, and thus together generate a subgroup of order no greater than $v/2$. In general if the subgroup G_3 generated by w_1 and w_2 is of order v_3 , then the initial torus found by merging $B(a, b)$'s according to w_1 and w_2 may be developed into v/v_3 toruses by addition of elements of distinct cosets of G_3 . Alternatively a different partition (with larger $\sum n_i f_i^2$) could be used; these are also listed (where necessary) in Table 3.2.

Table 3.2. Partitions of $Z_v - 0$ and Possible Design Sizes

| v | S_1 | $f_s - f_1$ | single array design sizes |
|-----|------------------------------------|-------------|--|
| 7 | 1, 6 | 1 | 2×14 ζ 1×14 2×7 |
| 8 | 1, 4, 7 | 1 | 4×16 ζ 3×16 6×8 12×4 24×2 |
| 10 | 1, 2, 5, 8, 9 or 1, 4, 5, 6, 9 | 3 | 5×20 10×10 25×4 50×2 |
| 11 | 1, 3, 8, 10 | 2 | 3×33 ζ 2×33 22×3 |
| 12 | 3, 4, 6, 8, 9* | 1 | 6×36 ζ 5×36 10×18 15×12 |
| | 3, 5, 6, 7, 9* | 2 | 20×9 30×6 60×3 |
| 14 | 1, 2, 5, 7, 9, 12, 13 | 2 | 7×42 14×21 49×6 98×3 |
| 15 | 1, 2, 5, 10, 13, 14 | 2 | 4×60 ζ 3×60 9×20 |
| | 1, 3, 4, 11, 12, 14 | 2 | 15×12 45×4 |
| 16 | 1, 2, 4, 8, 12, 14, 15 | 2 | 8×64 ζ 7×64 14×32 |
| | | | 28×16 56×8 112×4 |
| | 1, 2, 6, 7, 9, 11, 12, 16, 17* | 2 | |
| 18 | 1, 2, 4, 5, 9, 13, 14, 16, 17* | 2 | 9×72 18×36 27×24 |
| | 1, 2, 4, 8, 9, 10, 14, 16, 17 | 2 | 54×12 81×8 162×4 |
| 19 | 1, 2, 6, 8, 11, 13, 17, 18 | 2 | 5×95 ζ 4×95 76×5 |
| 20 | 1, 2, 3, 7, 10, 13, 17, 18, 19* | 2 | 10×100 ζ 9×100 18×50 36×25 |
| | 1, 2, 4, 9, 10, 11, 16, 18, 19* | 3 | 45×20 90×10 180×5 |
| 21 | 1, 2, 3, 5, 10, 11, 16, 18, 19, 20 | 2 | 5×105 15×35 |
| 22 | 1, 2, 3, 5, 10, 11, 12, 17, 19 | | |
| | 20, 21* | 2 | 11×110 22×55 |
| | 1, 3, 4, 5, 8, 11, 14, 17, 18 | | |
| | 19, 21* | 2 | 121×10 242×5 |
| | 1, 2, 4, 6, 7, 11, 15, 16, 18 | | |
| | 20, 21* | 3 | |
| 23 | 1, 2, 3, 7, 9, 14, 16, 20, 21, 22 | 3 | 6×138 ζ 5×138 115×6 |
| 24 | 1, 2, 4, 5, 10, 12, 14, 19, 20 | | |
| | 22, 23* | 2 | 12×144 ζ 11×144 22×72 |
| | 1, 2, 3, 7, 10, 12, 14, 17, 21 | | |
| | 22, 23* | 3 | 33×48 44×36 66×24 |
| | | | 88×18 132×12 264×6 |

Table 3.2. (continued)

| v | S_1 | $f_s - f_1$ | single array design sizes |
|-----|-----------------------------------|-------------|---|
| 26 | 1, 2, 3, 4, 8, 10, 13, 16, 18, 22 | | |
| | 23, 24, 25* | 3 | 13×156 26×78 |
| | 1, 2, 3, 5, 6, 11, 13, 15, 20, 21 | | |
| | 23, 24, 25* | 3 | 169×12 338×6 |
| | 1, 2, 4, 5, 6, 11, 13, 15, 20, 21 | | |
| 27 | 22, 24, 25* | 3 | |
| | 1, 3, 4, 6, 9, 10, 17, 18, 21 | | |
| | 23, 24, 26 | 3 | 7×189 ς 6×189 18×63 |
| | | | 54×21 162×7 |
| | 1, 2, 7, 10, 11, 12, 14, 16, 17 | | |
| 28 | 18, 21, 26, 27* | 3 | 14×196 ς 13×196 26×98 |
| | 1, 3, 4, 5, 8, 13, 14, 15, 20, 23 | | |
| | 24, 25, 27* | 3 | 52×49 91×28 182×14 364×7 |
| | 1, 2, 3, 5, 9, 10, 12, 15, 18, 20 | | |
| | 21, 25, 27, 28, 29 | 3 | 15×210 30×105 45×70 75×42 |
| 30 | 1, 2, 3, 6, 8, 9, 11, 15, 19, 21 | | |
| | 22, 24, 27, 28, 29 | 4 | 90×35 150×21 225×14 450×7 |

* w_1 and w_2 must both be even; * does not minimize $\sum n_i f_i^2$; ς = cylinder design (all others are toruses).

Designs with size marked ς in Table 3.2 are cylinder designs; such designs are given only when they result in exact row/column neighbor balance. Two comments concerning variants on the design sizes are worthy of mention. First, any torus design or cylinder design can be divided into sections with all neighbor counts preserved by bordering. For instance, a 4×16 cylinder design for 8 treatments could be layed out as two 4×8 side-bordered arrays. Secondly, as mentioned above, larger designs can be obtained by choosing a and/or b such

that their differences give replicates of the required sets, though care must be given to the row and columns counts if this is done. Hence, for instance, a 7×16 torus design for 8 treatments is a possibility.

Given the sets S_1 and S_2 construction depends only on appropriate choice of a and b . For the second order autonormal process on the torus this amounts to obtaining w_1 and w_2 of the desired orders. Practically speaking for planar applications, one can be guided by the general recommendations of Martin (1986) for long-term correlation structures: efficient designs should have as few like third order neighbors as possible (we already have no like first and second order neighbors), and to keep $\text{var}(\hat{\tau}_i - \hat{\tau}_j)$ as constant as possible balance neighbors to as high an order as possible. Designs here could also be adapted for the process c_2 of Martin (1986), for which efficiency requires that third order like neighbors occur as frequently as possible; our method would then require that $\pm a^*$ and/or $\pm b^*$ be multiple copies of S_1 and S_2 . These designs will not be efficient for short term correlations, which require a large number of like diagonal neighbors; for this situation see Morgan (1989).

Let $a^{2*} = (a_1 - a_3, a_2 - a_4, \dots, a_{p-2} - a_p, a_{p-1} + a_1 - a_p - a_2)$; a^{2*} is the set of second forward neighbor differences for cyclic merging of a . Similarly defining b^{2*} , the symmetric differences for higher order torus neighbors are

$\pm a^{2*}, \pm b^{2*}$ third order

$B(\pm a^{2*}, \pm b^*), B(\pm a^*, \pm b^{2*})$ fourth order

$B(\pm a^{2*}, \pm b^{2*})$ fifth order

and so on upon defining a^{3*} etc. Frequencies of occurrence and numbers of zeros in these lists can be used to guide the choice of a and b as one is concerned with balance and efficiency, respectively.

3.5. EFFICIENCY CALCULATIONS

In this section numerical comparisons are used to investigate the behavior in the plane of some of the constructed designs. The model is as given in section 3.1, but with planar correlations

$$\text{cov}(e_{ij}, e_{i'j'}) =$$

$$\frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\cos(g\theta_1) \cos(h\theta_2) d\theta_1 d\theta_2}{1 - 2\alpha\cos(\theta_1) - 2\alpha\cos(\theta_2) - 4\gamma\cos(\theta_1)\cos(\theta_2)} \quad (3.11)$$

where $|i - i'| = g$, $|j - j'| = h$, in this section only $k = 1$ array is considered, and $|\alpha| + |\gamma| < \frac{1}{4}$ (see Moran(1973)). This is different than the non-stationary planar process considered by Gill and Shukla (1985b), which they defined in terms of $\text{var}^{-1}(e)$, being in fact the stationary version of the torus process mentioned in section 2.2.

Let $\rho_{gh} = \rho_{hg}$ be the correlation for plots separated by g rows and h columns. In the calculations below we take $\rho_{10} = .1, .2, \dots, .5$ and $\rho_{11} \cong \rho_{10}^{\sqrt{2}}$. These may be roughly appropriate for square plots, but are of course somewhat arbitrary : manipulation of α and γ can produce a wide range of behaviors in the correlations. The exact values of α and γ used, along with the first few correlations, are given in the following table.

Table 3.3. Correlations obtained from (3.11)

| α | γ | ρ_{10} | ρ_{11} | ρ_{20} | ρ_{12} |
|----------|----------|-------------|-------------|-------------|-------------|
| .0881 | .0192 | .100 | .039 | .012 | .007 |
| .1485 | .0298 | .200 | .103 | .048 | .034 |
| .1890 | .0284 | .300 | .181 | .109 | .084 |
| .21635 | .02084 | .400 | .274 | .194 | .160 |
| .23422 | .011822 | .500 | .379 | .300 | .263 |

Note that these correlations decay more slowly than the process C_2 considered by Martin (1986), for which $\rho_{gh} = \rho_{10}^{(g^2+h^2)^{1/2}}$.

The C matrix for estimation of treatment contrasts is

$$C = X'(\Sigma^{-1} - \frac{1}{1'\Sigma^{-1}1}\Sigma^{-1}J\Sigma^{-1})X$$

where X is the plot/treatment incidence matrix, Σ the correlation matrix for e , and J and 1 are a matrix and a vector of 1's, respectively. Then

$$\text{tr}(C) \leq \text{tr}(\Sigma^{-1}) - \frac{1'\Sigma^{-1}1}{v} + \sum_{s \neq s'} \sum_{s'' \neq s'} \sigma^{ss''} I(\sigma^{ss''} > 0) = \frac{v-1}{\lambda^*}, \text{ say.}$$

Here $\Sigma^{-1} = (\sigma^{ss'})$, and $I(\sigma^{ss'} > 0) = 1$ if $\sigma^{ss'} > 0$, or 0 otherwise. Let $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{v-1}$ be the nonzero eigenvalues of C^- . A universally optimum design arrived at by the method of Kiefer (1975) would have $\lambda_1 = \lambda_2 = \dots = \lambda_{v-1} = \lambda^*$, providing a standard against which to evaluate the proposed designs; we will use the A ($= \sum \lambda_i$), E ($= \lambda_{v-1}$), and D ($= \prod \lambda_i$) criteria. We will also use $R = \lambda_{v-1}/\lambda_1$ as a simple measure of the dispersion in the design. This is just the ratio of the largest and smallest variances of estimated treatment contrasts, which is 1 for the hypothetical universally optimum design. With this definition, the A -efficiency is defined as $A\text{-eff} = (A\text{-value for hypothetical optimal design}) / (A\text{-value of the proposed design})$. When comparing a proposed design with some other design, the numerator of $A\text{-eff}$ should be read as the A -value of that other design. The E -, D -, and R -efficiencies are defined in a similar fashion.

We first consider the design of example 3.2 for 5 varieties. This design may be extended row-wise by successively adding 1 (mod 5) to the last row; the 5×5 thus obtained is design $D5.185$ of Martin (1986). Values of A -, E -, D -, and R -eff for the 4×5 and 5×5 each appear in Table 3.4. Note in particular that all of the A -efficiencies are greater than 99%, and that the loss is small even in terms of R . Similar values are obtained for the 6×5 7×5 etc.

Table 3.4. Two designs for $v=5$ relative to universal optimality

| ρ_{10} | $A\text{-eff}$ | $E\text{-eff}$ | $D\text{-eff}$ | $R\text{-eff}$ |
|-------------|-----------------|----------------|----------------|----------------|
| 0.1 | (.9998, .99998) | (.990, .994) | (.9995, .9999) | (.971, .991) |
| 0.2 | (.999, .9999) | (.984, .988) | (.998, .9998) | (.951, .983) |
| 0.3 | (.999, .9998) | (.979, .985) | (.997, .999) | (.936, .978) |
| 0.4 | (.998, .9995) | (.974, .983) | (.995, .998) | (.924, .977) |
| 0.5 | (.997, .999) | (.968, .983) | (.991, .996) | (.914, .978) |

first entry is for 4×5 , second for 5×5 .

Regarding the 5×5 square relative to his process (c_2), which is very close to the process considered here, Martin (1986) remarks that it "... has almost perfect second order balance. This was the optimal design found, and it seems unlikely that any better design exists". We now see that the reason for the near second order balance is that this degree of balance is achieved on the torus, and that this design is a member of a family of torus designs for 5 treatments that exhibit high efficiency and balance.

Next we examine a 6×8 design for 8 varieties based on the section 3.4 approach (Table 3.5). The design, constructed using $a = (0, 1, 5, 4)$ and $b = (0, 3, 1)$, is

| | | | | | | | |
|---|---|---|---|---|---|---|----|
| 0 | 3 | 1 | 4 | 2 | 5 | 3 | 6 |
| 1 | 4 | 2 | 5 | 3 | 6 | 4 | 7 |
| 5 | 0 | 6 | 1 | 7 | 2 | 0 | 3 |
| 4 | 7 | 5 | 0 | 6 | 1 | 7 | 2 |
| 5 | 0 | 6 | 1 | 7 | 2 | 0 | 3 |
| 1 | 4 | 2 | 5 | 3 | 6 | 4 | 7. |

As compared to the designs for 5 varieties the behavior here is less satisfactory, reflecting the poorer approximation to neighbor balance. The A -efficiencies

still exceed 99%, however.

Table 3.5. 6×8 for $v=8$ relative to universal optimality

| ρ_{10} | <i>A</i> -eff | <i>E</i> -eff | <i>D</i> -eff | <i>R</i> -eff |
|-------------|---------------|---------------|---------------|---------------|
| 0.1 | .999 | .952 | .996 | .914 |
| 0.2 | .997 | .917 | .988 | .857 |
| 0.3 | .995 | .891 | .979 | .818 |
| 0.4 | .993 | .873 | .970 | .791 |
| 0.5 | .990 | .859 | .958 | .772 |

To better see the effect of controlling neighbors consider this design due to Preece (1976):

| | | | | | |
|---|---|---|---|---|---|
| 4 | 8 | 2 | 1 | 6 | 9 |
| 3 | 7 | 9 | 5 | 2 | 4 |
| 6 | 5 | 1 | 8 | 3 | 7 |
| 2 | 4 | 3 | 7 | 8 | 6 |
| 1 | 9 | 8 | 4 | 5 | 3 |
| 5 | 6 | 7 | 2 | 9 | 1 |

This is a generalized Youden design with the additional property that each 3×3 corner is a complete replicate, so that like varieties are very well separated. There is, however, a single like diagonal neighbor pair. The design of example 3.2 is compared to this design (first value) and to the hypothetical universally optimum design (second value) in Table 3.6. Gains in A-efficiency for our (torus) neighbor-balanced design are small, but are more substantial with respect to the other criteria.

Table 3.6. Comparison of 6×6 designs for $v=9$

| ρ_{10} | <i>A-eff</i> | <i>E-eff</i> | <i>D-eff</i> | <i>R-eff</i> |
|-------------|---------------|---------------|---------------|---------------|
| 0.1 | (1.004, .999) | (1.095, .963) | (1.021, .996) | (1.144, .931) |
| 0.2 | (1.010, .997) | (1.166, .933) | (1.047, .988) | (1.251, .880) |
| 0.3 | (1.016, .995) | (1.212, .908) | (1.068, .979) | (1.321, .838) |
| 0.4 | (1.020, .993) | (1.236, .888) | (1.081, .970) | (1.363, .805) |
| 0.5 | (1.022, .991) | (1.248, .872) | (1.090, .961) | (1.385, .782) |

Next, a torus-constructed design for $v = 10$ is compared to a design with combined row/column balance in the plane (Table 3.7, which has the same format as Table 3.6). For the torus design take $a = (0, 7, 1)$ and $b = (0, 1, 3, 2, 7, 5)$, and for the competitor the Latin square $B(c, d)$ where $c = (0, 1, 3, 2, 4, 9, 8, 6, 7, 5)$ and $d = (0, 3, 9, 6, 2, 7, 4, 8, 1, 5)$; neither design is especially well-balanced in the diagonals. Differences are seen mainly in the *E* and *R* criteria, where $B(c, d)$ is superior.

Table 3.7. Comparison of 10×10 designs for $v = 10$

| ρ_{10} | <i>A-eff</i> | <i>E-eff</i> | <i>D-eff</i> | <i>R-eff</i> |
|-------------|----------------|--------------|---------------|--------------|
| 0.1 | (1.001, .9996) | (.983, .966) | (1.006, .998) | (.970, .938) |
| 0.2 | (1.000, .999) | (.969, .942) | (1.007, .994) | (.947, .897) |
| 0.3 | (.999, .998) | (.955, .929) | (1.003, .991) | (.923, .874) |
| 0.4 | (.998, .997) | (.943, .922) | (.996, .987) | (.899, .860) |
| 0.5 | (.997, .996) | (.935, .917) | (.989, .975) | (.881, .851) |

Tables 3.4 and 3.7 suggest that, when possible, planar balance of neighbors is desirable. When planar balance is not possible, torus designs offer a very close approximation.

The planar designs so far examined have been obtained by separating torus designs between two rows and two columns, and it should be noted that where this is done can affect the planar behavior. We have also discussed how the balanced neighbor counts of the torus can be maintained in the plane by bordering; in some cases it may be desirable to increase the size of an unbordered planar design by addition of one of these potential borders as an actual row or column of the design. This is especially relevant to corollary 3.2 designs, which suffer the greatest departure from neighbor balance in their planar versions because of the repeated set of neighbors lost by the separation of two rows. Adding, say, the row that would serve as the north border of a $(v-1)/4 \times v(v-1)/4$ unbordered design with the property that each pair of neighbors occurs $(v+3)/4$ or $(v-1)/4$ times in rows and columns combined, and $(v-1)/4$ or $(v-5)/4$ times in diagonals. This is the closest approximation to exact neighbor balance achievable in an unbordered planar design of this size.

To illustrate this we evaluate 3×39 and 4×39 designs for $v = 13$ (see Table 3.8). The designs are constructed using $a = (0, 1, 4, 8)$ and $b = (0, 6, 11, 9)$, and the column separation is between the first two columns of $B_1 = B(a, b)$. Both designs perform well, the 4×39 design holding a slight advantage in A -efficiency and, as expected, a somewhat stronger edge in the E and R criteria.

Table 3.8. Two designs for $v = 13$ relative to hypothetical optimum

| ρ_{10} | <i>A</i> -eff | <i>E</i> -eff | <i>D</i> -eff | <i>R</i> -eff |
|-------------|---------------|---------------|---------------|---------------|
| 0.1 | (.998, .9998) | (.939, .981) | (.991, .999) | (.894, .963) |
| 0.2 | (.995, .999) | (.895, .967) | (.973, .996) | (.823, .935) |
| 0.3 | (.993, .999) | (.866, .956) | (.956, .993) | (.776, .916) |
| 0.4 | (.990, .998) | (.847, .948) | (.941, .988) | (.745, .903) |
| 0.5 | (.988, .998) | (.836, .943) | (.928, .983) | (.726, .894) |

first entry is for 3×39 , second for 4×39 .

In conclusion, the calculations given here relative to an unattainable bound indicate that optimum torus designs can be excellent planar designs.

Chapter 4

Balanced Incomplete Block Designs

With Nested Rows and Columns

4.1. INTRODUCTION

Optimal two dimensional designs have been considered in Chapter 2, with a model without row and column effects, under generalized least squares analysis assuming a known error covariance structure. The torus approach has been taken in Chapter 3 for optimality arguments of comparatively smaller size designs and for the high efficiency of planar versions of torus designs. From the optimality results of these chapters, and the findings of several other papers (e.g. Kiefer and Wynn (1981), Gill and Shukla (1985b), and Martin (1986)) it is clear that optimal properties of a design depends on the assumed error covariance matrix. It turns out that a design in some sense optimal under one covariance structure

may be poor under some other covariance structures. Kiefer & Wynn (1981) suggest taking the minimax approach if the form of correlation is not known to the experimenter.

However, as discussed in Chapter 1, standard row-column designs of small sizes (ignoring correlations) including row and column effects in the model may be used as well. Jarrett & Hall (1982) advocated the use of several blocks each consisting of p rows and q columns for reasonably small p and q with elimination of heterogeneity due to rows and columns within each block. Also, the detailed studies of Pearce (1978, 1980) and Kempton and Howes (1981) demonstrate that nested row-column designs under the standard least squares analysis could be an useful alternative, in particular, to designs of Chapters 2 and 3. This motivates us to consider standard nested row-column designs in relation to two dimensional designs for correlated observations. In addition, nested row-column designs have applications in the situations where there are more sources of variation than can be controlled by ordinary blocking. Such sources of variation may make a significant contribution to the variability in the experimental material, and should be controlled, whenever possible. These designs can be viewed as a generalization of lattice square designs. The remainder of this dissertation will be devoted to block designs with nested rows and columns.

Block designs with nested rows and columns are designs for v treatments in

b blocks of size $k = pq$, where each block is composed of p rows and q columns. Two important subclasses of these designs are balanced and partially balanced incomplete block designs with nested rows and columns. In this chapter we consider only the balanced designs, for which

- (a) a treatment occurs at most once in each block,
- (b) each treatment occurs in r blocks, and
- (c) $pqrI_v - pN_1N_1' - qN_2N_2' + NN' = aI_v - \lambda J_v$.

(4.1)

Here I_v is the $v \times v$ identity matrix, J_v is the $v \times v$ matrix of 1's, N_1 , N_2 , and N are respectively the treatment-row, treatment-column and treatment-block incidence matrices, and a and λ are integers. Balanced incomplete block design with nested rows and columns will be denoted by $BIBRC(v, b, r, p, q, \lambda)$, or $BIBRC$ for short. These designs were introduced by Singh and Dey (1979) for the elimination of heterogeneity in two directions within each block.

The analysis is based on the following additive linear model :

$$y_{i_j, i_l} = \mu + \beta_i + \rho_{i_j} + \gamma_{i_l} + \tau_{[i_j, i_l]} + \epsilon_{i_j, i_l},$$

where y_{i_j, i_l} is the response obtained from the (i_j, i_l) -th cell of the i -th block, μ is the general mean, β_i is the effect of the i -th block, ρ_{i_j} is the effect of the j -th row in the i -th block, γ_{i_l} is the effect of the l -th column in the i -th block, τ is the effect for the treatment applied to plot (i_j, i_l) and the ϵ 's are uncorrelated errors with zero expectation and variance σ^2 .

The information matrix for estimating treatment effects with an equireplicate design, after eliminating the effects of rows, columns and blocks, is

$$C = rI_v - N_1N_1'/q - N_2N_2'/p + NN'/(pq) \quad (4.2)$$

Condition (c) of (4.1) says that the C-matrix in (4.2) for estimation of treatment contrasts has the same form as that of an ordinary balanced incomplete block design (in the absence of nested rows and columns); hence the balance. Further details of analysis and related discussions can be found in the original paper by Singh and Dey (1979) (also see John (1987), Dey (1987)). Constructions have been given by Singh and Dey (1979), Street (1981), Agrawal and Prasad (1982, 1983) and Cheng (1986). In section 4.2 we present a new technique for the construction of *BIBRC* designs, based on the method of differences, that takes advantage of the fact that if $p = q$ a sufficient condition for (c) to hold is that both (N_1, N_2) and N are incidence matrices for balanced incomplete block designs.

For vectors a and b of lengths n_1 and n_2 respectively, we shall continue to use $B(a, b)$ to denote a $n_1 \times n_2$ array whose (i, j) -th element is equal to the sum of the i -th element of a and the j -th element of b . $R(a, b)$ will be used to denote the vector list (by vector list is meant a vector considered as a list of elements, so that two vector lists are considered equal if one is a permutation of the other) whose elements are obtained by adding the elements of a to the elements of b .

Also $1 + yA$ will be used to denote $\{1 + yz | z \in A\}$.

4.2. CONSTRUCTION OF BIBRC DESIGNS.

The construction method may be summarized as follows.

CONSTRUCTION. Let G be an abelian group of order v . Suppose that we can find two sets of vectors b_1, \dots, b_m and b'_1, \dots, b'_m on G which are m -supplementary difference sets of BIB designs, i.e.

- (i) each b_j has p distinct elements of G and each b'_j has q distinct elements of G , and
- (ii) each nonzero element of G occurs $mp(p-1)/(v-1)$ times among the symmetric differences arising from the b_j 's and $mq(q-1)/(v-1)$ times among the symmetric differences arising from the b'_j 's.

Suppose further that

- (iii) the m vectors $R_i = R(d_i, d'_i)$ are together composed of

$\lambda = mpq(p-1)(q-1)/(v-1)$ occurrences of each nonzero element of G , where d_i and d'_i are the vectors of symmetric differences corresponding to b_i and b'_i respectively.

Then there exists a $BIBRC(v, b, r, p, q, \lambda)$ design with $b = mv$, $r = mpq$, p , q , and λ .

PROOF: Define m initial $p \times q$ blocks $B_i = B(b_i, b'_i)$. These m blocks, when

developed, give our design. By (i) and (ii), N_1 and N_2 are incidence matrices of *BIB* designs. The symmetric differences arising from B_i are q copies of d_i (from columns), p copies of d'_i (from rows), and the elements of R_i (all diagonal differences), so by (ii) and (iii), N is the incidence matrix of a *BIB* design. This completes the proof. \square

Example 4.1. For $v = 19$ and $G = Z_{19}$ write $b_1 = (0, 2, 3, 14)$, $b_2 = (0, 4, 6, 9)$, $b_3 = (0, 1, 7, 11)$, $b'_1 = (0, 6, 10)$, $b'_2 = (0, 1, 12)$, and $b'_3 = (0, 2, 5)$. The initial blocks are

$$B(b_1, b'_1) = \begin{bmatrix} 0 & 6 & 10 \\ 2 & 8 & 12 \\ 3 & 9 & 13 \\ 14 & 1 & 5 \end{bmatrix}, \quad B(b_2, b'_2) = \begin{bmatrix} 0 & 1 & 12 \\ 4 & 5 & 16 \\ 6 & 7 & 18 \\ 9 & 10 & 2 \end{bmatrix}, \quad B(b_3, b'_3) = \begin{bmatrix} 0 & 2 & 5 \\ 1 & 3 & 6 \\ 7 & 9 & 12 \\ 11 & 13 & 16 \end{bmatrix}.$$

A BIBRC(19, 57, 36, 4, 3, 12) is found by successively adding $0, \dots, 18 \pmod{19}$ to the initial blocks.

If $p = q$, it is sufficient for (ii) that the $2mp(p-1)$ combined symmetric differences from the b_i 's and b'_i 's are balanced, a fact which is taken advantage of in section 3 below.

Applying this technique we obtain several infinite series of designs as presented in the following Theorems and corollaries. In all cases G will be taken as the finite field GF_q with primitive element x .

THEOREM 4.1. Let $v = 2tm + 1$ be a prime power and write $x^{u_i} = 1 - x^{mi}$.

- (a) If there exists a positive integer $u \not\equiv (u_i - u_j) \pmod{m}$ for $i, j = 1, \dots, t$, then there exists a BIBRC with $b = mv$, $r = 4mt^2$, $p = q = 2t$, and $\lambda = 2t(2t - 1)^2$.
- (b) If there exists a positive integer $u \not\equiv \pm u_i, (u_i - u_j) \pmod{m}$ for $i, j = 1, \dots, t$, then there exists a BIBRC with $b = mv$, $r = m(2t + 1)^2$, $p = q = 2t + 1$, and $\lambda = 2t(2t + 1)^2$.
- (c) If there exists a positive integer $u \not\equiv u_i, (u_i - u_j) \pmod{m}$ for $i, j = 1, \dots, t$, then there exists a BIBRC with $b = mv$, $r = 2mt(2t + 1)$, $p = 2t$, $q = 2t + 1$, and $\lambda = 2t(2t + 1)(2t - 1)$.

PROOF: (a) Let $b_1 = (x^0, x^m, \dots, x^{(2t-1)m})$, $b_i = x^{i-1}b_1$, $b'_1 = (x^u, x^{m+u}, \dots, x^{(2t-1)m+u})$, and $b'_i = x^{i-1}b'_1$. Sprott (1954) has shown that b_1, \dots, b_m (and hence b'_1, \dots, b'_m) are m -supplementary difference sets for a BIBD for which $d_1 = b_1 \otimes (1 + x^0, 1 \pm x^m, \dots, 1 \pm x^{(t-1)m})$. Using $x^{tm} = -1$ we obtain $1 + x^{im} = x^{im}(1 - x^{(t-i)m}) = x^{im+u_{t-i}}$ so that $d_1 = b_1 \otimes \omega$ where $\omega = (x^{u_1}, x^{u_1}, \dots, x^{u_{t-1}}, x^{u_{t-1}}, x^{u_t})$.

Now $R_i = R(d_i, d'_i) = x^{i-1}R(d_1, d'_1) = x^{i-1}R(d_1, x^u d_1)$, so $\{R_1, \dots, R_m\} = (x^0, x^1, \dots, x^{m-1}) \otimes R(d_1, x^u d_1)$. Using $R(x^{n_1} b_1, x^{n_2} b_1) = x^{n_1} b_1 \otimes (1 + x^{n_2 - n_1} b_1)$ (which can be verified easily), we obtain $R(d_1, x^u d_1) = \{\omega_j b_1 \otimes (1 + \omega_\ell \omega_j^{-1} x^u b_1) \mid j, \ell = 1, \dots, 2t - 1\}$, ω_j being the

j -th component of ω . Since $(x^0, x^1, \dots, x^{m-1}) \otimes b_1 = G - \{0\}$, condition (iii) is satisfied and the proof is completed.

(b) Adjoin 0 to the b_i 's and b'_i 's in the proof of (a).

(c) Adjoin 0 to the b'_i 's in the proof of (a). \square

The conditions on u are needed to give $d_i \cap d'_i = \emptyset$, insuring the binary property (i.e. a treatment occurs at most once in each block). Sufficient (but often not necessary) conditions for the existence of u , and hence the designs of Theorem 1, are $m \geq t(t-1) + 2$, $m \geq t(t+1) + 2$, and $m \geq t^2 + 2$ for (a), (b), and (c) respectively.

The following two corollaries illustrate the application of this theorem. Note that the sufficient conditions are not necessary in corollaries 4.1(b), 4.2(a), and 4.2(c).

COROLLARY 4.1. *Let $v = 4m + 1$ be a prime power.*

(a) *If $m \geq 4$, then there exists a BIBRC with $b = v(v-1)/4$, $r = 4(v-1)$, $p = q = 4$, and $\lambda = 36$.*

(b) *If $m \geq 7$, then there exists a BIBRC with $b = v(v-1)/4$, $r = 25(v-1)/4$, $p = q = 5$, and $\lambda = 100$.*

(c) *If $m \geq 6$, then there exists a BIBRC with $b = v(v-1)/4$, $r = 5(v-1)$, $p = 4$, $q = 5$, and $\lambda = 60$.*

Agrawal & Prasad (1982, 1983) have constructed 4×4 designs with the same r and prime power $v = 16m + 1$, and with $r = 8(v - 1)$ for prime power $v = 2m + 1$. Similar comparisons for the 4×5 and 5×5 designs may be made to the results of Agrawal & Prasad (1982, 1983) and Street (1981) : in each case our series are for smaller r or a less sparse series of v .

COROLLARY 4.2. *Let $v = s^n$ be an odd prime power.*

- (a) *If $n \geq 2$, then there exists a BIBRC with $b = s^n(s^n - 1)/(s - 1)$, $r = (s - 1)(s^n - 1)$, $p = q = s - 1$, and $\lambda = (s - 1)(s - 2)^2$.*
- (b) *If $n \geq 3$, then there exists a BIBRC with $b = s^n(s^n - 1)/(s - 1)$, $r = s^2(s^n - 1)/(s - 1)$, $p = q = s$, and $\lambda = (s - 1)s^2$.*
- (c) *If $n \geq 2$, then there exists a BIBRC with $b = s^n(s^n - 1)/(s - 1)$, $r = s(s^n - 1)$, $p = s - 1$, $q = s$, and $\lambda = s(s - 1)(s - 2)$.*

Corollary 4.2 follows upon taking $m = (s^n - 1)/(s - 1)$, in which case $u_i \equiv 0 \pmod{m}$ for every i . The condition $n \geq 3$ in (b) is required to give incomplete blocks.

THEOREM 4.2. *Let $v = 2tm + 1$ be a prime power where $t > 1$ is odd, and write $x^{u_i} = 1 - x^{2mi}$.*

- (a) *If there exists a positive integer $u \not\equiv (u_i - u_j) \pmod{m}$ for $i, j = 1, \dots, (t - 1)/2$, then there exists a BIBRC with $b = mv$, $r = mt^2$, $p = q = t$, and*

$$\lambda = \frac{1}{2}t(t-1)^2.$$

- (b) If there exists a positive integer $u \not\equiv \pm u_i, (u_i - u_j) \pmod{m}$ for $i, j = 1, \dots, (t-1)/2$, then there exists a BIBRC with $b = mv, r = m(t+1)^2, p = q = t+1$, and $\lambda = \frac{1}{2}t(t+1)^2$.
- (c) If there exists a positive integer $u \not\equiv u_i, (u_i - u_j) \pmod{m}$ for $i, j = 1, \dots, (t-1)/2$, then there exists a BIBRC with $b = mv, r = mt(t+1), p = t, q = t+1$, and $\lambda = \frac{1}{2}t(t^2 - 1)$.

PROOF: Here we prove (b) first. Let $b_1 = (0, x^0, x^{2m}, \dots, x^{(2t-2)m}) = (0, b_1^*)$.

The symmetric differences arising from b_1 are the nonzero elements of $B(b_1, -b_1)$.

Hence $d_1 = b_1^* \otimes (\pm x^0, 1 + x^m, 1 + x^{3m}, \dots, 1 + x^{(t-2)m},$

$1 + x^{(t+2)m}, 1 + x^{(t+4)m}, \dots, 1 + x^{(2t-1)m})$ (also see Sprott, 1954). Using $x^{tm} =$

-1 , and for odd i , $1 + x^{im} = x^{im+u(t-i)/2}$, gives

$d_1 = (x^0, x^m, \dots, x^{(2t-1)m}) \otimes (x^0, x^{u_1}, \dots, x^{u_{(t-1)/2}}) = (b_1^*, x^m b_1^*) \otimes \omega$, say. Tak-

ing $b_i = x^{i-1} b_1$ and $b'_i = x^{u+i-1} b_1, i = 1, \dots, m$ it is easily seen that each of rows

and columns will be BIBDs. The diagonal differences are $R_i = x^{i-1} R(d_1, x^u d_1)$

for $i = 1, \dots, m$. Upon noting that $(b_1^*, x^m b_1^*)$ is the b_1 of Theorem 1, we obtain

$R(d_1, x^u d_1) = [\omega_j (b_1^*, x^m b_1^*) \otimes \{1 + \omega_\ell \omega_j^{-1} x^u (b_1^*, x^m b_1^*)\} | j, \ell = 1, \dots, (t+1)/2]$, so

$\{R_1, \dots, R_m\} = (x^0, x^1, \dots, x^{m-1}) \otimes R(d_1, x^u d_1)$ is a balanced list.

(a) Delete 0 from the b_i 's and b'_i 's in the proof of (b).

(c) Delete 0 from the b_i 's in the proof of (b). \square

Sufficient conditions for the existence of u in (a), (b), and (c) of Theorem 4.2 are $m \geq \frac{1}{4}(t-1)(t-3) + 2$, $m \geq \frac{1}{4}(t^2-1) + 2$, and $m \geq \frac{1}{4}(t-1)^2 + 2$ respectively.

Three corollaries are presented as applications of Theorem 4.2. In section 4.3 it will be shown how, under certain conditions, the number of blocks in the first two theorems may be reduced.

COROLLARY 4.3. *Let $v = 6m + 1$ be a prime power.*

- (a) *If $m \geq 2$, then there exists BIBRC with $b = v(v-1)/6$, $r = 3(v-1)/2$, $p = q = 3$, and $\lambda = 6$.*
- (b) *If $m \geq 4$, then there exists a BIBRC with $b = v(v-1)/6$, $r = 8(v-1)/3$, $p = q = 4$, and $\lambda = 24$.*
- (c) *If $m \geq 3$, then there exists a BIBRC with $b = v(v-1)/6$, $r = 2(v-1)$, $p = 3$, $q = 4$, and $\lambda = 12$.*

The 4×4 's constructed here may be compared to those of corollary 4.1 and the previously mentioned designs of Agrawal & Prasad (1982, 1983). Designs with the same parameters as the 3×3 and 3×4 series are constructed by Street (1981), but for $m \geq 3$ and $m \geq 4$, respectively.

COROLLARY 4.4. Let $v = 10m + 1$ be a prime power.

(a) If $m \geq 4$, then there exists a BIBRC with $b = v(v - 1)/10$,

$$r = 5(v - 1)/2, p = q = 5, \text{ and } \lambda = 40.$$

(b) If $m \geq 7$, then there exists a BIBRC with $b = v(v - 1)/10$,

$$r = 18(v - 1)/5, p = q = 6, \text{ and } \lambda = 90.$$

(c) If $m \geq 6$, then there exists a BIBRC with $b = v(v - 1)/10$,

$$r = 3(v - 1), p = 5, q = 6, \text{ and } \lambda = 60.$$

COROLLARY 4.5. Let $v = s^n$ ($n \geq 2$) be an odd prime power, and let

$$s = 4\alpha + 3.$$

(a) If $\alpha \geq 1$, then there exists a BIBRC with $b = s^n(s^n - 1)/(s - 1)$,

$$r = (s^n - 1)(s - 1)/4, p = q = (s - 1)/2, \text{ and } \lambda = (s - 3)^2(s - 1)/16.$$

(b) If $\alpha \geq 0$, then there exists a BIBRC with $b = s^n(s^n - 1)/(s - 1)$,

$$r = (s + 1)^2(s^n - 1)/\{4(s - 1)\}, p = q = (s + 1)/2, \text{ and}$$

$$\lambda = (s + 1)^2(s - 1)/16.$$

(c) If $\alpha \geq 1$, then there exists a BIBRC with $b = s^n(s^n - 1)/(s - 1)$,

$$r = (s + 1)(s^n - 1)/4, p = (s - 1)/2, q = (s + 1)/2, \text{ and}$$

$$\lambda = (s^2 - 1)(s - 3)/16.$$

Combining the constructions of Theorems 4.1 and 4.2 gives

THEOREM 4.3. Let $v = 2tm + 1$ be a prime power where $t > 1$ is odd and write $x^{u_i} = 1 - x^{m_i}$.

- (a) If there exists a positive integer $u \not\equiv (u_{2i} - u_j) \pmod{m}$ for $i = 1, \dots, (t-1)/2$ and $j = 1, \dots, t$, then there exists a BIBRC with $b = mv$, $r = 2mt^2$, $p = t$, $q = 2t$, and $\lambda = t(t-1)(2t-1)$.
- (b) If there exists a positive integer $u \not\equiv u_{2i}, (u_{2i} - u_j) \pmod{m}$ for $i = 1, \dots, (t-1)/2$ and $j = 1, \dots, t$, then there exists a BIBRC with $b = mv$, $r = mt(2t+1)$, $p = t$, $q = 2t+1$, and $\lambda = t(t-1)(2t+1)$.
- (c) If there exists a positive integer $u \not\equiv -u_j, (u_{2i} - u_j) \pmod{m}$ for $i = 1, \dots, (t-1)/2$ and $j = 1, \dots, t$, then there exists a BIBRC with $b = mv$, $r = 2mt(t+1)$, $p = t+1$, $q = 2t$, and $\lambda = t(t+1)(2t-1)$.
- (d) If there exists a positive integer $u \not\equiv u_{2i}, -u_j, (u_{2i} - u_j) \pmod{m}$ for $i = 1, \dots, (t-1)/2$ and $j = 1, \dots, t$, then there exists a BIBRC with $b = mv$, $r = m(t+1)(2t+1)$, $p = t+1$, $q = 2t+1$, and $\lambda = t(t+1)(2t+1)$.

PROOF: In each case take the b_i 's from Theorem 4.2 and the b'_i 's from

Theorem 4.1 . \square

Sufficient conditions for the existence of u in (a)-(d) of Theorem 4.3 are $m \geq \frac{1}{2}(t-1)^2 + 2$, $m \geq \frac{1}{2}t(t-1) + 2$, $m \geq \frac{1}{2}(t^2 + 5)$, and $m \geq \frac{1}{2}t(t+1) + 2$, respectively.

All of the designs given in this section also serve as nested balanced incomplete block designs as defined by Preece (1967).

4.3. DESIGNS WITH FEWER REPLICATES

In this section we turn our attention to reducing the number of blocks for sub-series of the Theorems 4.1 and 4.2 designs.

THEOREM 4.4. *Let $v = 4tm + 1$ be a prime power and write $x^{u_i} = 1 - x^{2mi}$.*

(a) If $u_i - u_j \not\equiv m \pmod{2m}$ for $i, j = 1, \dots, t$, then there exists a BIBRC with

$$b = mv, r = 4mt^2, p = q = 2t, \text{ and } \lambda = t(2t - 1)^2.$$

(b) If $u_i, u_i - u_j \not\equiv m \pmod{2m}$ for $i, j = 1, \dots, t$, then there exists a BIBRC

$$\text{with } b = mv, r = m(2t + 1)^2, p = q = 2t + 1, \text{ and } \lambda = t(2t + 1)^2.$$

PROOF: (a) In Theorem 4.1(a) write $v = 2tm_0 + 1$ where $m_0 = 2m$; it will be shown that with the conditions of this theorem and proper choice of u , it is sufficient to use just the first m initial blocks given there. With $b_1 = (x^0, x^{2m}, \dots, x^{(4t-2)m})$ take $b_i = x^{i-1}b_1$, $b'_1 = (x^m, x^{3m}, \dots, x^{(4t-1)m})$, and $b'_i = x^{i-1}b'_1$ (i.e. take $u = m = m_0/2$ in Theorem 4.1(a)), $i = 1, \dots, m$. Then $d_1 = b_1 \otimes (x^\gamma, x^\gamma, x^{u_t})$ where $x^\gamma = (x^{u_1}, \dots, x^{u_{t-1}})$, so the b_i 's and b'_i 's are together a $2m$ -supplementary difference set, which by the conditions on the u_j 's satisfy $d_i \cap d'_i = \emptyset$.

The diagonal differences are $\{R_1, \dots, R_m\} = (x^0, x^1, \dots, x^{m-1}) \otimes$

$R(d_1, x^m d_1)$. Now $R(d_1, x^m d_1)$ can be written as four copies of R_{11} , two copies of each of R_{12} and R_{21} , and one copy of R_{22} , where

$$R_{11} = R(x^\gamma \otimes b_1, x^\gamma \otimes x^m b_1), R_{12} = R(x^{u_t} b_1, x^\gamma \otimes x^m b_1),$$

$R_{21} = R(x^\gamma \otimes b_1, x^{u_t+m} b_1)$ and $R_{22} = R(x^{u_t} b_1, x^{u_t+m} b_1)$. Factor these as follows, using $R(x^{n_1} b_1, x^{n_2} b_1) = x^{n_1} b_1 \otimes (1 + x^{n_2-n_1} b_1)$ and $x^m b_1 = x^{-m} b_1 = b'_1$:

$$\begin{aligned} R_{11} &= [x^{u_j} \{R(b_1, x^{u_t-u_j+m} b_1), x^m R(x^{u_t-u_j-m} b_1, b_1)\} | j < \ell = 1, \dots, t-1] \\ &= \{x^{u_j} (b_1, b'_1) \otimes (1 + x^{u_t-u_j} b'_1) | j < \ell = 1, \dots, t-1\}, \end{aligned}$$

$$R_{12} = \{x^{u_t} b_1 \otimes (1 + x^{u_j-u_t} b'_1) | j = 1, \dots, t-1\},$$

$$R_{21} = \{x^{u_t} b'_1 \otimes (1 + x^{u_j-u_t} b'_1) | j = 1, \dots, t-1\}, \text{ and } R_{22} = x^{u_t} b_1 \otimes (1 + b'_1).$$

Since $(x^0, x^1, \dots, x^{m-1}) \otimes (b_1, b'_1) = GF_v - \{0\}$, $(x^0, x^1, \dots, x^{m-1}) \otimes R_{11}$ is a balanced list, as is $(x^0, x^1, \dots, x^{m-1}) \otimes (R_{12}, R_{21})$. The proof is complete if $(x^0, x^1, \dots, x^{m-1}) \otimes b_1 \otimes (1 + b'_1)$ is a balanced list, which will be the case provided the elements of $1 + b'_1 = 1 + x^m b_1$ can be partitioned into t pairs of the form $x^{q_i}, x^{q_i+k_i m}$ $i = 1, \dots, t$ where k_i is odd. One such partition is

$$1 + x^m, 1 + x^{(4t-1)m} = x^{(4t-1)m} (1 + x^m)$$

$$1 + x^{3m}, 1 + x^{(4t-3)m} = x^{(4t-3)m} (1 + x^{3m})$$

\vdots

$$1 + x^{(2t-1)m}, 1 + x^{(2t+1)m} = x^{(2t+1)m} (1 + x^{(2t-1)m}).$$

(b) Adjoin 0 to the b_i 's and b'_i 's in (a). \square

As an example of Theorem 4.4, $t = 2$ gives 4×4 blocks with $r = 2(v - 1)$, and 5×5 blocks with $r = 25(v - 1)/8$, for $v = 8m + 1$ a prime power (compare corollary 4.1). For $v < 500$ the conditions fail to hold for $v = 81$, and for $v = 289$ in the 5×5 case. Setting $t = 3$, the conditions for $v = 12m + 1$ in 6×6 and 7×7 blocks also fail for two values of $v < 500$: $v = 37$ in 6×6 blocks, and $v = 169$. Corresponding to corollary 4.2 we have

COROLLARY 4.6. *Let $v = s^n$ be an odd prime power where n is even.*

- (a) *If $n \geq 2$, then there exists a BIBRC with $b = s^n(s^n - 1)/\{2(s - 1)\}$, $r = (s - 1)(s^n - 1)/2$, $p = q = s - 1$, and $\lambda = (s - 2)^2(s - 1)/2$.*
- (b) *If $n \geq 4$, then there exists a BIBRC with $b = s^n(s^n - 1)/\{2(s - 1)\}$, $r = s^2(s^n - 1)/\{2(s - 1)\}$, $p = q = s$, and $\lambda = s^2(s - 1)/2$.*

THEOREM 4.5. *Let $v = 4tm + 1$ be a prime power where $t > 1$ is odd and write $x^{u_i} = 1 - x^{4mi}$.*

- (a) *If $u_i - u_j \not\equiv m \pmod{2m}$ for $i, j = 1, \dots, (t - 1)/2$, then there exists a BIBRC with $b = mv$, $r = mt^2$, $p = q = t$, and $\lambda = t(t - 1)^2/4$.*
- (b) *If $u_i, u_i - u_j \not\equiv m \pmod{2m}$ for $i, j = 1, \dots, (t - 1)/2$, then there exists a BIBRC with $b = mv$, $r = m(t + 1)^2$, $p = q = t + 1$, and $\lambda = t(t + 1)^2/4$.*

PROOF: Theorem 4.5 stands in relation to Theorem 4.2 as Theorem 4.4 to Theorem 1, and the proof is similar. The initial blocks are $B(b_i, b'_i)$ where for

(a), $b_1 = (x^0, x^{4m}, \dots, x^{4(t-1)m})$, $b_i = x^{i-1}b_1$, and $b'_i = x^{m+i-1}b_1$, and for (b), 0 is adjoined to the b_i 's and b'_i 's of (a). \square

COROLLARY 4.7. *Let $v = 12m + 1$ be a prime power.*

(a) *There exists a BIBRC with $b = v(v-1)/12$, $r = 3(v-1)/4$, $p = q = 3$, and $\lambda = 3$.*

(b) *If $m \geq 2$, then there exists a BIBRC with $b = v(v-1)/12$, $r = 4(v-1)/3$, $p = q = 4$, and $\lambda = 12$.*

PROOF: The proof of (a) is immediate from Theorem 4.5 since there is only a single u_i . For (b) the condition is $u_1 \not\equiv m \pmod{2m}$ where $x^{u_1} = 1 - x^{4m}$. If this fails take the b_i 's as given in the Theorem, but $b'_i = xb_i$. Then $d_1 = (x^0, x^{2m}, \dots, x^{10m}) \otimes (x^0, x^m)$, showing that the rows and columns will each be BIBDs. Now $R_i = x^{i-1}R(d_1, xd_1)$ simplifies to $x^{i-1}(x^0, x^m, \dots, x^{11m}) \otimes (1+x, 1+x^{m+1}, \dots, 1+x^{11m+1})$ and the diagonal differences are balanced as well. \square

COROLLARY 4.8. *If $v = 20m + 1$ ($m \geq 2$) is a prime power, then there exists a BIBRC with $b = v(v-1)/20$, $r = 5(v-1)/4$, $p = q = 5$, and $\lambda = 20$.*

PROOF: The condition is $\delta = u_2 - u_1 \not\equiv m \pmod{2m}$ where $x^\delta = 1 + x^{4m}$. If this fails take the b_i 's as given in Theorem 4.5 but $b'_i = xb_i$. The proof follows easily upon noting that $x^{-u_1}d_1 = (x^0, x^m, x^{2m}, x^{3m}) \otimes b_1$. \square

The conditions for $v = 20m + 1$ in 6×6 blocks fail twice for $v < 500$ ($v = 41, 61$). Corollaries 4.7 and 4.8 should be compared to corollaries 4.3 and 4.4, respectively. We also have (compare corollary 4.5),

COROLLARY 4.9. *Let $v = s^n$ be an odd prime power where n is even, and $s = 4\alpha + 3$ ($\alpha \geq 1$). There exist BIBRCs with*

$$(a) \ b = s^n(s^n - 1)/\{2(s - 1)\}, \ r = (s - 1)(s^n - 1)/8, \ p = q = (s - 1)/2, \text{ and} \\ \lambda = (s - 3)^2(s - 1)/32, \text{ and}$$

$$(b) \ b = s^n(s^n - 1)/\{2(s - 1)\}, \ r = (s + 1)^2(s^n - 1)/\{8(s - 1)\}, \\ p = q = (s + 1)/2, \text{ and } \lambda = (s + 1)^2(s - 1)/32.$$

Finally, we note that stricter conditions on the u_i 's can for a given v , p , and q , give designs with yet smaller r than obtained in Theorems 4.1-4.5, or designs that cannot be constructed by those theorems. Unfortunately such conditions are often not satisfied. One example worthy of note is for $v = 169$ and $p = q = 4$ (compare corollary 4.7) where $r = 112$ is attained by using the fact that $u_1 \equiv m \pmod{2m}$. The required vectors are $b_1 = (0, x^0, x^{56}, x^{112})$, $b_i = x^{2i-2}b_1$, and $b'_i = xb_i$, $i = 1, \dots, 7$. For $v = 169$ in 6×6 blocks (Theorem 4.4 fails) take $b_1 = (x^0, x^{28}, \dots, x^{140})$, $b_i = x^{2i-2}b_1$, and $b'_i = x^3b_i$, $i = 1, \dots, 14$; for 7×7 blocks adjoin 0 to each of these vectors. For $v = 289$ in 5×5 blocks (again Theorem 4.4 fails) take $b_1 = (0, x^0, x^{72}, x^{144}, x^{216})$, $b_i = x^{2i-2}b_1$, and $b'_i = x^7b_i$,

$i = 1, \dots, 36$. We have not been able to get designs for $v = 41$ or 61 in 6×6 blocks by these techniques (see following corollary 4.8).

Table 4.1 lists the designs constructed in this chapter for $v \leq 101$ and $3 \leq p \leq q$. The series of 2×2 designs of Theorems 4.1 and 4.4 have the same r as those constructed by Agrawal & Prasad (1982, 1983), and Theorem 4.1 gives 2×3 's with the same r as Agrawal & Prasad (1983, Theorem 5), so those designs are not included. Only the construction with the smallest r is given, and r_m is the smallest replication for given v , p , and q the author has found in the literature (including this work). Sources are listed for designs with replications less than or equal to that constructed here; of the 149 designs in this range, 80 appear to be new. The sources are P for Preece (1967), $SD2$ for Theorem 2 of Singh & Dey (1979), $S6$ for Theorem 6 of Street (1981), $AP1$, $AP2$, and $AP4$ for Theorems 1, 2, and 4 of Agrawal & Prasad (1982), $AP5$ for Theorem 5 of Agrawal & Prasad (1983), IJ for Ipinyomi & John (1985), and $C1$ for Theorem 2.1 of Cheng (1986). Cheng's result combines a $BIBD$ with a $BIBRC$ to give a new $BIBRC$, both row-column designs having the same v ; when the referenced design does not explicitly appear in his paper, the dimensions $p \times q$ of the required initial $BIBRC$ are listed in parentheses, as well as a source if that design does not appear in this table. An extensive list of $BIBDs$ is available in Mathon & Rosa (1985).

Table 4.1. Constructed *BIBRCs* for $v \leq 101$

| v | p | q | r | r_m | Method | Source |
|-----|-----|-----|-----|-------|------------|-------------------------|
| 11 | 3 | 3 | 45 | 45 | <i>Th1</i> | <i>AP5</i> |
| 13 | 3 | 3 | 9 | 9 | <i>Th5</i> | <i>IJ</i> |
| 17 | 3 | 3 | 36 | 36 | <i>Th4</i> | — |
| 17 | 4 | 4 | 32 | 32 | <i>Th4</i> | — |
| 19 | 3 | 3 | 27 | 9 | <i>Th2</i> | <i>P</i> |
| 19 | 3 | 4 | 36 | 36 | <i>Th2</i> | — |
| 23 | 3 | 3 | 99 | 99 | <i>Th1</i> | <i>AP5</i> |
| 25 | 3 | 3 | 18 | 18 | <i>Th5</i> | — |
| 25 | 3 | 4 | 48 | 24 | <i>Th2</i> | <i>AP1</i> |
| 25 | 4 | 4 | 32 | 32 | <i>Th5</i> | — |
| 25 | 3 | 6 | 72 | 72 | <i>Th3</i> | — |
| 25 | 4 | 5 | 120 | 12 | <i>Th1</i> | <i>C1</i> |
| 25 | 3 | 7 | 84 | 84 | <i>Th3</i> | — |
| 27 | 3 | 3 | 117 | 52 | <i>Th1</i> | <i>SD2</i> |
| 29 | 3 | 3 | 63 | 63 | <i>Th4</i> | — |
| 29 | 4 | 4 | 112 | 112 | <i>Th1</i> | <i>C1(AP1, 4 × 7)</i> |
| 29 | 4 | 5 | 140 | 140 | <i>Th1</i> | — |
| 29 | 5 | 5 | 175 | 175 | <i>Th1</i> | — |
| 31 | 3 | 3 | 45 | 45 | <i>Th2</i> | <i>S6</i> |
| 31 | 3 | 4 | 60 | 60 | <i>Th2</i> | <i>S6, C1(P, 3 × 5)</i> |
| 31 | 4 | 4 | 80 | 80 | <i>Th2</i> | — |
| 31 | 3 | 6 | 90 | 90 | <i>Th3</i> | — |
| 31 | 3 | 7 | 105 | 105 | <i>Th3</i> | — |
| 37 | 3 | 3 | 27 | 27 | <i>Th5</i> | — |
| 37 | 3 | 4 | 72 | 36 | <i>Th2</i> | <i>AP1</i> |
| 37 | 4 | 4 | 48 | 48 | <i>Th5</i> | — |

Table 4.1. (continued)

| v | p | q | r | r_m | Method | Source |
|-----|-----|-----|-----|-------|------------|-------------------------|
| 37 | 3 | 6 | 108 | 108 | <i>Th3</i> | <i>AP4, S6</i> |
| 37 | 4 | 5 | 180 | 180 | <i>Th1</i> | — |
| 37 | 3 | 7 | 126 | 126 | <i>Th3</i> | — |
| 37 | 4 | 6 | 144 | 144 | <i>Th3</i> | — |
| 37 | 5 | 5 | 225 | 225 | <i>Th1</i> | — |
| 37 | 4 | 7 | 168 | 168 | <i>Th3</i> | — |
| 41 | 3 | 3 | 90 | 90 | <i>Th4</i> | — |
| 41 | 4 | 4 | 80 | 80 | <i>Th4</i> | <i>SD2</i> |
| 41 | 4 | 5 | 200 | 40 | <i>Th1</i> | <i>AP1</i> |
| 41 | 5 | 5 | 50 | 50 | <i>Th5</i> | — |
| 43 | 3 | 3 | 63 | 63 | <i>Th2</i> | <i>S6, C1(AP2, 3×7)</i> |
| 43 | 3 | 4 | 84 | 84 | <i>Th2</i> | <i>S6, C1(AP2, 3×7)</i> |
| 43 | 4 | 4 | 112 | 112 | <i>Th2</i> | — |
| 43 | 3 | 6 | 126 | 126 | <i>Th3</i> | <i>S6, C1(3×7)</i> |
| 43 | 3 | 7 | 147 | 21 | <i>Th3</i> | <i>AP2</i> |
| 43 | 4 | 6 | 168 | 168 | <i>Th3</i> | <i>C1(AP1, 6×7)</i> |
| 43 | 4 | 7 | 196 | 196 | <i>Th3</i> | — |
| 47 | 3 | 3 | 207 | 207 | <i>Th1</i> | <i>AP5</i> |
| 49 | 3 | 3 | 36 | 36 | <i>Th5</i> | — |
| 49 | 3 | 4 | 96 | 48 | <i>Th2</i> | <i>AP1</i> |
| 49 | 4 | 4 | 64 | 64 | <i>Th5</i> | — |
| 49 | 3 | 6 | 144 | 144 | <i>Th3</i> | <i>S6</i> |
| 49 | 4 | 5 | 240 | 240 | <i>Th1</i> | — |
| 49 | 3 | 7 | 168 | 48 | <i>Th3</i> | <i>C1(4×7)</i> |
| 49 | 4 | 6 | 192 | 96 | <i>Th3</i> | <i>C1(4×7)</i> |
| 49 | 5 | 5 | 150 | 150 | <i>Th4</i> | — |
| 49 | 4 | 7 | 224 | 16 | <i>Th3</i> | <i>C1</i> |
| 49 | 6 | 6 | 144 | 144 | <i>Th4</i> | <i>C1(6×7)</i> |
| 49 | 6 | 7 | 336 | 24 | <i>Th1</i> | <i>C1</i> |
| 53 | 3 | 3 | 117 | 117 | <i>Th4</i> | — |
| 53 | 4 | 4 | 208 | 208 | <i>Th1</i> | <i>C1(AP1, 4×13)</i> |

Table 4.1. (continued)

| v | p | q | r | r_m | Method | Source |
|-----|-----|-----|-----|-------|--------|----------------------------|
| 53 | 4 | 5 | 260 | 260 | $Th1$ | — |
| 53 | 5 | 5 | 325 | 325 | $Th1$ | — |
| 59 | 3 | 3 | 261 | 261 | $Th1$ | $AP5$ |
| 61 | 3 | 3 | 45 | 45 | $Th5$ | — |
| 61 | 3 | 4 | 120 | 60 | $Th2$ | $AP1$ |
| 61 | 4 | 4 | 80 | 80 | $Th5$ | $SD2$ |
| 61 | 3 | 6 | 180 | 180 | $Th3$ | $S6$ |
| 61 | 4 | 5 | 300 | 60 | $Th1$ | $AP1$ |
| 61 | 3 | 7 | 210 | 210 | $Th3$ | $S6$ |
| 61 | 4 | 6 | 240 | 240 | $Th3$ | $C1(5 \times 6)$ |
| 61 | 5 | 5 | 75 | 75 | $Th5$ | — |
| 61 | 4 | 7 | 280 | 280 | $Th3$ | — |
| 61 | 5 | 6 | 180 | 60 | $Th2$ | $AP1$ |
| 61 | 6 | 6 | 180 | 180 | $Th4$ | — |
| 61 | 6 | 7 | 420 | 420 | $Th1$ | — |
| 61 | 7 | 7 | 245 | 245 | $Th4$ | — |
| 67 | 3 | 3 | 99 | 99 | $Th2$ | $S6$ |
| 67 | 3 | 4 | 132 | 132 | $Th2$ | $S6$ |
| 67 | 4 | 4 | 176 | 176 | $Th2$ | — |
| 67 | 3 | 6 | 198 | 198 | $Th3$ | $S6, C1(AP2, 3 \times 11)$ |
| 67 | 3 | 7 | 231 | 231 | $Th3$ | $S6$ |
| 67 | 4 | 6 | 264 | 264 | $Th3$ | — |
| 67 | 4 | 7 | 308 | 308 | $Th3$ | — |
| 67 | 6 | 6 | 396 | 396 | $Th1$ | $C1(AP1, 6 \times 11)$ |
| 67 | 6 | 7 | 462 | 462 | $Th1$ | — |
| 67 | 7 | 7 | 539 | 539 | $Th1$ | — |
| 71 | 3 | 3 | 315 | 315 | $Th1$ | $AP5, C1(S6, 3 \times 7)$ |
| 71 | 5 | 5 | 175 | 175 | $Th2$ | $S6$ |
| 71 | 5 | 6 | 210 | 210 | $Th2$ | $S6, C1(AP2, 5 \times 7)$ |
| 71 | 6 | 6 | 252 | 252 | $Th2$ | — |

Table 4.1. (continued)

| v | p | q | r | r_m | Method | Source |
|-----|-----|-----|-----|-------|------------|--------------------------|
| 73 | 3 | 3 | 54 | 36 | <i>Th5</i> | <i>SD2</i> |
| 73 | 3 | 4 | 144 | 72 | <i>Th2</i> | <i>AP1</i> |
| 73 | 4 | 4 | 96 | 96 | <i>Th5</i> | — |
| 73 | 3 | 6 | 216 | 216 | <i>Th3</i> | <i>AP4, S6</i> |
| 73 | 4 | 5 | 360 | 360 | <i>Th1</i> | — |
| 73 | 3 | 7 | 252 | 252 | <i>Th3</i> | <i>S6</i> |
| 73 | 4 | 6 | 288 | 288 | <i>Th3</i> | <i>AP4</i> |
| 73 | 5 | 5 | 225 | 225 | <i>Th4</i> | — |
| 73 | 4 | 7 | 336 | 336 | <i>Th3</i> | — |
| 73 | 6 | 6 | 216 | 216 | <i>Th4</i> | — |
| 73 | 6 | 7 | 504 | 504 | <i>Th1</i> | — |
| 73 | 7 | 7 | 294 | 294 | <i>Th4</i> | — |
| 79 | 3 | 3 | 117 | 117 | <i>Th2</i> | <i>S6</i> |
| 79 | 3 | 4 | 156 | 156 | <i>Th2</i> | <i>S6, C1(AP2, 3×13)</i> |
| 79 | 4 | 4 | 208 | 208 | <i>Th2</i> | — |
| 79 | 3 | 6 | 234 | 234 | <i>Th3</i> | <i>S6</i> |
| 79 | 3 | 7 | 273 | 273 | <i>Th3</i> | <i>S6</i> |
| 79 | 4 | 6 | 312 | 312 | <i>Th3</i> | — |
| 79 | 4 | 7 | 364 | 364 | <i>Th3</i> | — |
| 79 | 6 | 6 | 468 | 468 | <i>Th1</i> | — |
| 79 | 6 | 7 | 546 | 546 | <i>Th1</i> | — |
| 79 | 7 | 7 | 637 | 637 | <i>Th1</i> | — |
| 81 | 3 | 3 | 180 | 40 | <i>Th4</i> | <i>SD2</i> |
| 81 | 4 | 4 | 320 | 80 | <i>Th1</i> | <i>SD2</i> |
| 81 | 4 | 5 | 400 | 80 | <i>Th1</i> | <i>AP1</i> |
| 81 | 5 | 5 | 100 | 100 | <i>Th5</i> | — |
| 81 | 5 | 6 | 240 | 240 | <i>Th2</i> | <i>S6</i> |
| 81 | 6 | 6 | 144 | 144 | <i>Th5</i> | — |
| 81 | 5 | 10 | 400 | 400 | <i>Th3</i> | — |
| 81 | 5 | 11 | 440 | 440 | <i>Th3</i> | — |
| 81 | 6 | 10 | 480 | 480 | <i>Th3</i> | — |

Table 4.1. (continued)

| v | p | q | r | r_m | Method | Source |
|-----|-----|-----|-----|-------|------------|------------------------|
| 81 | 8 | 8 | 320 | 320 | <i>Th4</i> | $C1(8 \times 9)$ |
| 81 | 6 | 11 | 528 | 528 | <i>Th3</i> | — |
| 81 | 8 | 9 | 720 | 40 | <i>Th1</i> | $C1$ |
| 83 | 3 | 3 | 369 | 369 | <i>Th1</i> | $AP5$ |
| 89 | 3 | 3 | 198 | 198 | <i>Th4</i> | — |
| 89 | 4 | 4 | 176 | 176 | <i>Th4</i> | — |
| 89 | 4 | 5 | 440 | 440 | <i>Th1</i> | $C1(AP1, 4 \times 11)$ |
| 89 | 5 | 5 | 275 | 275 | <i>Th4</i> | — |
| 97 | 3 | 3 | 72 | 72 | <i>Th5</i> | — |
| 97 | 3 | 4 | 192 | 96 | <i>Th2</i> | $AP1$ |
| 97 | 4 | 4 | 128 | 128 | <i>Th5</i> | — |
| 97 | 3 | 6 | 288 | 288 | <i>Th3</i> | $S6$ |
| 97 | 4 | 5 | 480 | 480 | <i>Th1</i> | — |
| 97 | 3 | 7 | 336 | 336 | <i>Th3</i> | $S6$ |
| 97 | 4 | 6 | 384 | 384 | <i>Th3</i> | $AP4$ |
| 97 | 5 | 5 | 300 | 300 | <i>Th4</i> | — |
| 97 | 4 | 7 | 448 | 448 | <i>Th3</i> | — |
| 97 | 6 | 6 | 288 | 288 | <i>Th4</i> | — |
| 97 | 6 | 7 | 672 | 672 | <i>Th1</i> | — |
| 97 | 7 | 7 | 392 | 392 | <i>Th4</i> | — |
| 97 | 8 | 8 | 384 | 384 | <i>Th4</i> | — |
| 97 | 8 | 9 | 864 | 864 | <i>Th1</i> | — |
| 97 | 9 | 9 | 486 | 486 | <i>Th4</i> | — |
| 101 | 3 | 3 | 225 | 225 | <i>Th4</i> | — |
| 101 | 4 | 4 | 400 | 400 | <i>Th1</i> | $C1(4 \times 5)$ |
| 101 | 4 | 5 | 500 | 100 | <i>Th1</i> | $AP1$ |
| 101 | 5 | 5 | 125 | 125 | <i>Th5</i> | — |
| 101 | 5 | 6 | 300 | 300 | <i>Th2</i> | $S6$ |
| 101 | 6 | 6 | 180 | 180 | <i>Th5</i> | — |
| 101 | 5 | 10 | 500 | 500 | <i>Th3</i> | $AP4, S6$ |
| 101 | 5 | 11 | 550 | 550 | <i>Th3</i> | — |

Chapter 5

Partially balanced nested row-column designs

5.1. INTRODUCTION

Chapter 4 dealt with the construction of *BIBRC*'s. A general method for the construction of *BIBRC* designs was developed there. Unfortunately the series of designs obtained by the construction were limited to prime power numbers for the treatments, as were those of Singh & Dey (1979), Street(1981), and Agrawal and Prasad (1982a, 1983) and ~~Cheng (1986)~~. It would be desirable to have nested row-column designs for as many v as possible, and where possible, to reduce the number of replications, though a price must be paid in terms of balance to achieve this goal.

The C -matrix for a balanced incomplete block design with nested rows and columns as given in Chapter 4, has the same form as that of an ordinary balanced

incomplete block design (in the absence of nested rows and columns); hence the balance. An obvious alternative to requiring balance is to obtain nested row-column designs with partial balance in a manner directly analogous to the generalization from balanced to partially balanced incomplete block designs. Define an n -class partially balanced incomplete block design with nested rows and columns as a nested row-column design for which the C -matrix (4.2) has the same form as that of an n -class partially balanced incomplete block design. Hence an n -class association scheme is defined on the treatments, and $(C)_{ij}$ is constant over all pairs of treatments (i, j) which are ℓ -th associates. The analysis and constructions of partially balanced incomplete block design with nested rows and columns, *PBIBRC*, have been considered by Street(1981), Agrawal and Prasad (1982b) and Ipinyomi and John (1985). Some discussion can be found in John (1987). In this chapter constructions of *PBIBRC* designs based on rectangular, Latin square and pseudocyclic association schemes are considered. Several constructions for rectangular and Latin square type nested row-column designs are given in section 5.2. In section 5.3 the method of differences has been used to obtain some families of *PBIBRC*s based on the pseudocyclic association scheme. For background on partially balanced incomplete block designs and association schemes see, for example, Raghavarao (1971).

5.2. RECTANGULAR AND LATIN SQUARE TYPE DESIGNS

WITH NESTED ROWS AND COLUMNS

Definition 5.1 : Rectangular association scheme. Let $v = v_1 v_2$ treatments be denoted by ordered pairs (α, β) , $\alpha = 1, 2, \dots, v_1$, and $\beta = 1, 2, \dots, v_2$. Distinct treatments (α, β) and (α', β') are first associates if $\alpha = \alpha'$, second associates if $\beta = \beta'$, and third associates otherwise.

When $v_1 = v_2$ the rectangular association scheme reduces to the 2-associate class Latin square scheme by combining the first and second associate classes. Nested row-column designs based on these association schemes will be denoted by $RTRC(v, b, r, p, q)$ and $LSRC(v, b, r, p, q)$, or by $RTRC$ and $LSRC$ for short.

In constructing $RTRC$'s Agrawal & Prasad (1982b) introduce the concept of a "series A " design : a $BIBRC$ is said to belong to series A if each of N_1, N_2 , and N is the incidence matrix of a BIB design. For the purposes of the following theorems, say that a $BIBRC$ with square blocks (i.e. $p=q$) belongs to "series B " if both (N_1, N_2) and N are incidence matrices of $BIBD$'s, and to "series AB " if it belongs to both series A and series B . Theorems 4.1-4.3 of Chapter 4 give infinite series of designs of series A and series AB , as does Theorem 6 of Street (1981). Infinite series of designs belonging to series B are given by Theorems 4.4 and 4.5 of Chapter 4. These designs can be combined to construct $RTRC$'s and

LSRC's as shown in Theorems 5.1-5.3.

THEOREM 5.1. *If there exists a $BIBRC(v_1, b_1, r_1, p_1, q_1)$ and a $BIBRC(v_2, b_2, r_2, p_2, q_2)$ both belonging to series A, then there exists an $RTRC(v_1 v_2, b_1 b_2, r_1 r_2, p_1 p_2, q_1 q_2)$.*

PROOF: Denoting the two *BIBRC*'s by D_1 and D_2 , let A_1, A_2, \dots, A_{b_1} be the blocks of D_1 and C_1, C_2, \dots, C_{b_2} be the blocks of D_2 . From each pair A_i, C_j form a new block by replacing $(A_i)_{uw} = a_{iuw}$ by

$$\begin{bmatrix} (a_{iuw}, c_{j11}) & (a_{iuw}, c_{j12}) & \dots & (a_{iuw}, c_{j1q_2}) \\ (a_{iuw}, c_{j21}) & (a_{iuw}, c_{j22}) & \dots & (a_{iuw}, c_{j2q_2}) \\ \vdots & \vdots & \ddots & \vdots \\ (a_{iuw}, c_{jp_21}) & (a_{iuw}, c_{jp_22}) & \dots & (a_{iuw}, c_{jp_2q_2}) \end{bmatrix}$$

where $(C_j)_{lm} = c_{jlm}$. We thus get $b_1 b_2$ blocks of size $p_1 p_2 \times q_1 q_2$, which are the blocks of the *RTRC* design.

To see this, let λ_1^r, λ_1^c , and λ_1^b be the number of times that a pair of treatments occur respectively in rows, columns, and blocks of D_1 , and let λ_2^r, λ_2^c , and λ_2^b be the corresponding numbers in D_2 . Among the $b_1 b_2 p_1 p_2$ rows every pair of treatments (α, β) and (α, β') , $\beta \neq \beta'$ occurs together $r_1 \lambda_2^r$ times, every pair of treatments (α, β) and (α', β) , $\alpha \neq \alpha'$ occurs together $r_2 \lambda_1^r$ times, and every pair of treatments (α, β) and (α', β') , $\alpha \neq \alpha', \beta \neq \beta'$ occurs together $\lambda_1^r \lambda_2^r$ times, so N_1 is the incidence matrix of a rectangular design with treatment concurrence numbers $r_1 \lambda_2^r, r_2 \lambda_1^r$, and $\lambda_1^r \lambda_2^r$. Likewise N_2 and N are incidence matrices of

rectangular designs with concurrence numbers $r_1\lambda_2^c$, $r_2\lambda_1^c$, and $\lambda_1^c\lambda_2^c$, and $r_1\lambda_2^b$, $r_2\lambda_1^b$, and $\lambda_1^b\lambda_2^b$ respectively. Hence our C -matrix has the desired form and the proof is complete. \square

THEOREM 5.2. *If there exists a $BIBRC(v_1, b_1, r_1, p, p)$ belonging to series B and a $BIBRC(v_2, b_2, r_2, q, q)$ belonging to series AB , then there exists an $RTRC(v_1v_2, b_1b_2, r_1r_2, pq, pq)$.*

PROOF: Proceed as in Theorem 5.1 with $p_1 = q_1 = p$, $p_2 = q_2 = q$, the A_i 's as the blocks of the series B design, and the C_j 's as the blocks of the series AB design. \square

As an example of Theorem 5.2, let the series B design be the $BIBRC(5, 5, 4, 2, 2)$ of Agrawal & Prasad (1982a), and the series AB design the $BIBRC(s^2, s+1, s+1, s, s)$ described by Singh & Dey (1979), where s is any prime power. Then we have a $RTRC(5s^2, 5(s+1), 4(s+1), 2s, 2s)$. Setting $s = 2$ gives a design for 20 treatments in 4×4 blocks with 12 replicates; a $BIBRC(20, b, r, 4, 4)$ must have r a multiple of 76. We also have

COROLLARY 5.1. *If there exists a $BIBRC(v, b, r, p, p)$ of series AB , then there exists a $LSRC(v^2, b^2, r^2, p^2, p^2)$.*

PROOF: In Theorem 5.1 take the series B design as the series AB design. \square

The next result combines a series A design with an ordinary BIB .

THEOREM 5.3. *If there exists a $BIBRC(v, b_1, r_1, p, q)$ of series A and a $BIB(v, b_2, r_2, k_2, \lambda_2)$, then there exists a $LSRC(v^2, 2b_1b_2, 2r_1r_2, p, qk_2)$.*

PROOF: Let A_1, A_2, \dots, A_{b_1} be the blocks of the $BIBRC$ and C_1, C_2, \dots, C_{b_2} the blocks of the BIB . Let a_{iuv} be the uv -th element of the i -th block A_i , and c_{jh} be the h -th element of the j -th block C_j . Using C_j obtain b_1 blocks by replacing a_{iuv} by $((a_{iuv}, c_{j1}), (a_{iuv}, c_{j2}), \dots, (a_{iuv}, c_{jk}))$ for all i, u , and v . Varying $j = 1, 2, \dots, b_2$ gives b_1b_2 blocks of size $p \times qk_2$. To these adjoin another b_1b_2 blocks obtained by replacing (a_{iuv}, c_{jh}) by (c_{jh}, a_{iuv}) in the former b_1b_2 blocks. These $2b_1b_2$ blocks give the required design. The derivation of counts is straightforward. \square

The following corollary is immediate from the above theorem.

COROLLARY 5.2. *If there exists a $BIBRC(v, b, r, p, q)$ of series A , then there exists a $LSRC(v^2, 2b, 2r, p, qv)$.*

Starting with a series A design as given in corollary 5.2, Theorem 4.2 of Agrawal & Prasad (1982b) gives $RTRC(v^2, (v-1)b, (v-1)r, p, qv)$ with $(v-1)/2$ times the number of replications obtained here. Their construction is more flexible, however, in allowing for other than a square number of treatments.

The last result of this section combines two *BIBRC*'s while keeping the block size fixed.

THEOREM 5.4. *If there exists a $BIBRC(v_i, b_i, r_i, p, q)$ for $i = 1, 2$, then there exists a $RTRC(v_1v_2, b_1v_2 + b_2v_1, r_1 + r_2, p, q)$.*

PROOF: Denoting the two *BIBRC*'s by D_1 and D_2 , let A_1, A_2, \dots, A_{b_1} be the blocks of D_1 , and C_1, C_2, \dots, C_{b_2} be the blocks of D_2 , $A_i = (a_{iuw})$, $C_j = (c_{jws})$. Using A_i form v_2 blocks $A_{ih} = (h, a_{iuh})$, $h = 1, 2, \dots, v_2$, (h, a_{iuh}) being the (u, w) -th element of A_i . Similarly using C_j form v_1 blocks $C_{js} = (c_{jws}, s)$, $s = 1, 2, \dots, v_1$. The resulting $b_1v_2 + b_2v_1$ blocks give the desired design.

Let β and β' be any two distinct symbols of D_1 . Then, by construction, the number of times that (h, β) and (h, β') occur together in rows, columns, and blocks arising from D_1 is equal to the number of times that β and β' occur together respectively in rows, columns, and blocks of D_1 . That is, counts for first associate pairs of treatments in the blocks arising from D_1 are the same as the counts for the *BIBRC* D_1 . Similar results hold for second associate pairs (α, s) and (α', s) in the blocks arising from D_2 , where α and α' are two distinct symbols of D_2 . The proof is completed upon noting that no first associate pair of treatments occurs in the blocks arising from D_2 , no second associate pair occurs in the blocks arising from D_1 , and the third associate pairs do not occur in any

block. \square

COROLLARY 5.3. *If there exists a $BIBRC(v, b, r, p, q)$, then there exists a $LSRC(v^2, 2bv, 2r, p, q)$.*

PROOF: Take $D_1 = D_2$ in Theorem 5.4. \square

For example, the series $BIBRC(v, v(v-1)/4, v-1, 2, 2)$ of Agrawal & Prasad (1982a) for $v = 4t + 1$ a prime power yields the series $LSRC(v^2, v^2(v-1)/2, 2(v-1), 2, 2)$ with $2/(v+1)$ of the replicates needed for a balanced design.

5.3. PSEUDOCYCLIC DESIGNS WITH NESTED ROWS AND COLUMNS

Definition 5.2 : Pseudocyclic association scheme. Let the v symbols be denoted by the elements of the Galois field GF_v , where $v = 4e + 1$ is a prime power. The first associates of y are $y + q_1, y + q_2, \dots, y + q_{(v-1)/2}$, where $q_1, q_2, \dots, q_{(v-1)/2}$ are the distinct quadratic residues of GF_v , and the second associates of y are the remaining field elements except itself.

By this definition, two symbols are first associates if their difference is a quadratic residue, and second associates if their difference is a quadratic non-residue. Nested row-column designs based on the pseudocyclic scheme will be denoted by $PCRC(v, b, r, p, q)$, or $PCRC$ for short.

For the first two constructions in this section, the notations of Chapter 4 and a modified version of the difference method described there will be used. Let b_1, b_2, \dots, b_m and b'_1, b'_2, \dots, b'_m be a set of $2m$ initial blocks of size p for a *PBIB* design based on the pseudocyclic association scheme. Let d_i and d'_i be the vectors of symmetric differences for b_i and b'_i respectively. Then $B_i = B(b_i, b'_i)$, $i = 1, 2, \dots, m$, are a set of initial blocks for a *PCRC* if and only if

- (i) $d_i \cap d'_i = \emptyset$ for $i = 1, 2, \dots, m$, and,
- (ii) $R_i = R(d_i, d'_i)$ for $i = 1, 2, \dots, m$ collectively contain each quadratic residue equally often and each quadratic non-residue equally often.

The detailed proofs of Theorems 5.5 and 5.6 are similar to those of Theorems 4.4 and 4.5 of chapter 4.

THEOREM 5.5. *Let $v = 8tm + 1$ be a prime power and write $x^{u_i} = 1 - x^{4mi}$.*

- (a) *If $u_i - u_j \not\equiv 2m \pmod{4m}$ for $i, j = 1, 2, \dots, t$, then there exists a*

$$PCRC(v, mv, 4mt^2, 2t, 2t).$$

- (b) *If $u_i, u_i - u_j \not\equiv 2m \pmod{4m}$ for $i, j = 1, 2, \dots, t$, then there exists a*

$$PCRC(v, mv, m(2t+1)^2, 2t+1, 2t+1).$$

PROOF: (a) With $b_1 = (x^0, x^{4m}, \dots, x^{4(2t-1)m})$ take $b_i = x^{2i-2}b_1$ and $b'_i = x^{2i-2+2m}b_1$.

(b) Adjoin 0 to the b_i 's and b_i' 's in (a). \square

Theorem 5.5 with $t = 1$ gives 2×2 blocks and 3×3 blocks for $v = 8m + 1$ a prime power. In the 2×2 case the conditions always hold, and for $v < 500$ the conditions fail to hold only for $v = 25$ in the 3×3 case. Taking $t = 2$, we obtain 4×4 blocks and 5×5 blocks for $v = 16m + 1$ a prime power. For $v < 500$ the conditions fail to hold for $v = 81$, and for $v = 17$ and 289 in the 5×5 case. Taking $t = 3$, the conditions for $v = 24m + 1 < 500$ in both 6×6 and 7×7 blocks fail for $v = 25$ and 169.

To illustrate Theorem 5.5, put $t = 2$ and $m = 1$; using $x = 3$, $b_1 = (1, 13, 16, 4)$ and $b_1' = (9, 15, 8, 2)$. A $PCRC(17, 17, 16, 4, 4)$ is obtained by developing (mod 17) the initial block

$$B = \begin{bmatrix} 10 & 16 & 9 & 3 \\ 5 & 11 & 4 & 15 \\ 8 & 14 & 7 & 1 \\ 13 & 2 & 12 & 6 \end{bmatrix}.$$

THEOREM 5.6. *Let $v = 8tm + 1$ be a prime power where $t > 1$ is odd and write $x^{u_i} = 1 - x^{4mi}$.*

(a) *If $u_{2i} - u_{2j} \not\equiv 2m \pmod{4m}$ for $i, j = 1, 2, \dots, (t-1)/2$, then there exists a*

$$PCRC(v, mv, mt^2, t, t).$$

(b) *If $u_{2i}, u_{2i} - u_{2j} \not\equiv 2m \pmod{4m}$ for $i, j = 1, 2, \dots, (t-1)/2$, then there exists a $PCRC(v, mv, m(t+1)^2, t+1, t+1)$.*

PROOF: (a) With $b_1 = (x^0, x^{8m}, \dots, x^{8(t-1)m})$ take $b_i = x^{2i-2}b_1$ and $b'_i = x^{2i-2+2m}b_1$.

(b) Adjoin 0 to the b_i 's and b'_i 's in (a). \square

COROLLARY 5.4. *Let $v = 24m + 1$ be a prime power. There exist*

(a) a $PCRC(v, mv, 9m, 3, 3)$, and (b) a $PCRC(v, mv, 16m, 4, 4)$.

PROOF: (a) is immediate, and for (b) the condition is $u_2 \not\equiv 2m \pmod{4m}$ where $x^{u_2} = 1 - x^{8m}$. If this fails take the b_i 's as given in Theorem 5.6, but $b'_i = xb_i$.

The result is then easily established. \square

Example 5.2. Taking $t = 3$, $m = 1$ in the above corollary, we obtain a $PCRC$ with $v = b = 25$, $p = q = 3$ by developing

$$B = \begin{bmatrix} 9 & 21 & 6 \\ 14 & 17 & 5 \\ 13 & 22 & 1 \end{bmatrix}$$

over GF_{5^2} where entries are powers of the primitive root x with primitive polynomial $f(x) = x^2 + 2x + 3$.

COROLLARY 5.5. *If $v = 40m + 1$ is a prime power, then there exists a $PCRC(v, mv, 25m, 5, 5)$.*

PROOF: The condition is $\delta = u_4 - u_2 \not\equiv 2m \pmod{4m}$ where $x^\delta = 1 + x^{8m}$. If this fails take the b_i 's as given in Theorem 5.6 but $b'_i = xb_i$. Again the result follows easily. \square

The conditions for $v = 40m + 1$ in 6×6 blocks do not always hold, but for $v < 500$ they fail only for $v = 41$. A final example of the above theorems is

COROLLARY 5.6. *Let $v = s^h$ be an odd prime power such that $(s^h - 1)/4(s - 1)$ is an integer. Then there exist PCRCs with $b = s^h(s^h - 1)/\{4(s - 1)\}$ and*

- (a) $r = s^2(s^h - 1)/\{4(s - 1)\}$ and $p = q = s$ if $h > 2$,
- (b) $r = (s - 1)(s^h - 1)/4$ and $p = q = s - 1$,
- (c) $r = (s + 1)^2(s^h - 1)/\{16(s - 1)\}$ and $p = q = (s + 1)/2$ if $s \equiv 3 \pmod{4}$,
- (d) $r = (s - 1)(s^h - 1)/16$ and $p = q = (s - 1)/2$ if $s \equiv 3 \pmod{4} > 3$.

PROOF: The results (a) and (b) follow from Theorem 5.5, and (c) and (d) from Theorem 5.6, by taking $m = (s^h - 1)/\{4(s - 1)\}$ and $t = (s - 1)/2$. \square

THEOREM 5.7. *Let $v = 4m + 1$ be a prime power. Then there exist PCRCs(v, mv, mpq, p, q) for all p and q such that $pq < v$.*

PROOF: Take any pq distinct elements of GF_v and arbitrarily arrange them into an $p \times q$ array B_1 . Let y_1 and y_2 be any two elements of B_1 . Then $\pm(y_1 - y_2) = (x^0, x^{2m}) \otimes (y_1 - y_2)$, where x is a primitive root of the field. Hence the design is obtained by developing $B_i = x^{2i-2}B_1$, $i = 1, 2, \dots, m$ over the field. \square

The next construction juxtaposes initial blocks for pseudocyclic designs in such a way that a PCRC is obtained.

LEMMA 5.1. (Sprott(1955)) Let $v = 4s(4\gamma + 1) + 1$ be a prime power. Development of the s initial blocks

$$b_i = (x^{2i}, x^{2i+4s}, x^{2i+8s}, \dots, x^{2i+16\gamma s}), i = 0, 1, \dots, (s-1)$$

over GF_v , gives a partially balanced incomplete block design based on the pseudocyclic association scheme.

THEOREM 5.8. Let $v = 4m(4\alpha + 1)(4\beta + 1) + 1$ be a prime power.

If $(4\alpha + 1)$ and $(4\beta + 1)$ are relatively prime then there exists a

$PCRC(v, mv, mpq, 4\alpha + 1, 4\beta + 1)$.

PROOF: Write $v = 4m(4\gamma_1 + 1) + 1$ where $\gamma_1 = 4\alpha\beta + \alpha + \beta$. Taking $\gamma = \gamma_1$ and $s = m$ in lemma 1, b_i can be arranged into the $(4\alpha + 1) \times (4\beta + 1)$ array

$$B_i = \begin{bmatrix} x^{2i} & x^{2i+4m(4\alpha+1)} & \dots & x^{\iota_1} \\ x^{2i+4m(4\beta+1)} & x^{2i+4m(4\beta+1+4\alpha+1)} & \dots & x^{\iota_2} \\ \vdots & \vdots & \ddots & \vdots \\ x^{2i+16m\alpha(4\beta+1)} & x^{2i+4m(4\alpha(4\beta+1)+4\alpha+1)} & \dots & x^{\iota_{(4\alpha+1)}} \end{bmatrix}$$

where $\iota_\ell = 2i + 16m\beta(4\alpha + 1) + 4m(\ell - 1)(4\beta + 1)$, $\ell = 1, 2, \dots, (4\alpha + 1)$ and $i = 1, 2, \dots, m$. Our mv blocks are the developed B_i 's. By the construction, N is the incidence matrix for the lemma 1 design with $\gamma = \gamma_1$, and $s = m$. Now writing $v = 4m_1(4\beta + 1) + 1$ where $m_1 = m(4\alpha + 1)$, it can be seen that the rows of the B_i 's are the initial blocks of the lemma 1 design with $\gamma = \beta$ and $s = m_1$, so that N_1 is the incidence matrix for that design. Likewise N_2 is the incidence

matrix for the lemma 1 design with $\gamma = \alpha$ and $s = m_2$ where $m_2 = m(4\beta + 1)$.

□

Our final construction generalizes Theorem 5.8 to n -associate class
PBIB – RC designs.

LEMMA 5.2. (*Sprott(1955)*) Let $v = 2nm(2n\gamma + 1) + 1$ be an odd prime power.

If

$$\{\theta_s | x^{\theta_s} = x^{2nms} - 1, s = 1, 2, \dots, a\gamma\}$$

is composed of γ_i occurrences of the residue class of $(i-1) \pmod{n}$, then development of the m initial blocks

$$b_u = (x^{un}, x^{(u+2m)n}, x^{(u+4m)n}, \dots, x^{(u+4mn\gamma)n}), u = 0, 1, \dots, (m-1).$$

over GF_v gives a *PBIB* design with n associate classes and with parameters

$$v = 2nm(2n\gamma + 1), b = mv, k = 2n\gamma + 1, n_i = 2m(2n\gamma + 1), \lambda_i = \gamma_i \text{ and}$$

$P_i = (p_{jj}^i)$, where p_{jj}^i are the number of expressions

$$x^{\tau n + j - j'} - 1 = x^z,$$

where $z \equiv i - j \pmod{n}$, and τ ranges from 0 to $(v - n - 1)/n$.

THEOREM 5.9. *If $v = 2nm(2n\gamma_1 + 1)(2n\gamma_2 + 1) + 1$ is prime power such that $(2n\gamma_1 + 1)$ and $(2n\gamma_2 + 1) + 1$ are relative primes, then there exists a PBIBRC design with parameters $v = 2nm(2n\gamma + 1), b = mv, p = 2n\gamma_1 + 1, q = 2n\gamma_2 + 1$ based on the association scheme of lemma 5.2.*

PROOF: The proof proceeds along the lines of Theorem 5.8 but uses lemma 5.2 instead of lemma 5.1. The initial blocks are

$$B_i = \begin{bmatrix} x^{in} & x^{i+2m_2} & \dots & x^{(i+4m_2n\gamma_2)n} \\ x^{(i+(i+2m_2))n} & x^{(i+2m_2+(i+2m_1))n} & \dots & x^{(i+4m_2n\gamma_2+(i+2m_1))n} \\ \vdots & \vdots & \ddots & \vdots \\ x^{(i+4m_1n\gamma_1)n} & x^{(i+2m_2+4m_1n\gamma_1)n} & \dots & x^{(i+(4m_2\gamma_2+4m_1\gamma_1)n)n} \end{bmatrix}.$$

Chapter 6

Concluding Remarks

Optimal designs in two-dimensional layouts when the errors follow the second order autonormal process have been investigated in the first half of this dissertation. For the non-stationary planar process conditions for optimal designs, and an efficiency bound for a specific class of designs, have been established theoretically. Though these design problems have been previously investigated by Martin (1982,1986), and Gill & Shukla (1985b), they do not give constructions except for a few isolated examples. We have successfully developed several techniques that produce infinite series of universally optimum designs on the torus and highly efficient designs in the plane for the proposed models.

It would be unwise to claim that the assumed correlation covers an exact behaviour in real experiments. However, the designs constructed here are expected to be highly efficient for some other correlation structures, and for both

generalized and ordinary least squares. For the error covariance structure c_2 of Martin (1986), which is different than the second order conditional autonormal process considered in Chapter 2 of this dissertation, his enumeration suggests that under generalized least squares a design with no like neighbors in rows, columns and diagonals has high A -value, and that these designs with nearest neighbor balance in rows and columns and good neighbor balance in diagonals have high D -values. Each Latin square design constructed in Chapter 2 is therefore expected to perform well with respect to the A and D criteria; the degree of performance for the D criterion of course depends on the dispersion of the diagonal neighbor counts. For general guidelines to other properties of designs like those constructed in Chapters 2 and 3 for a variety of covariance structures under both generalized and ordinary least squares, see Martin (1986), especially sections 12 and 13. Theoretical investigation of optimal designs for these and other error covariance structures, and the construction of optimal designs, remain open and challenging problems.

In practice, it is very unlikely that the errors will follow a well defined covariance structure, and a design optimal with respect to one covariance structure may perform very poorly for some other covariance structures (see Martin (1986), sections 12 and 13). Furthermore, the design properties depend also on whether generalized or ordinary least squares is used for the assumed model. Hence the

use of these designs depends on the confidence of the experimenter as to how good the assumed model represents the data. For this reason some practicing statisticians have advocated use of the less efficient ordinary least squares, ignoring correlations, but eliminating heterogeneity in two directions within each rectangular block. These are the nested row-column designs considered in Chapters 4 and 5. The ordinary least squares approach with elimination of heterogeneity due to rows and columns could be preferred to generalized least squares on the grounds of robustness of the designs to departures from conveniently assumed (for theoretical reasons) error covariance structures. Apart from the inconclusive studies of Pearce (1978), and Kempton & Howes (1981) on the comparison of these two approaches, nothing else (to the best of the author's knowledge) is known so far. Further research needs to be done to judge the performance of nested row-column designs as alternatives to two-dimensional designs for correlated errors.

Besides being alternatives to neighbor designs, nested row-column designs are required more frequently nowadays in agricultural experiments, especially when the number of treatments is large. There has been limited progress in the construction of nested row-column designs that satisfy desirable statistical properties. Several methods for the construction of balanced and partially balanced incomplete block designs with nested rows and columns have been developed in

Chapters 4 and 5, contributing many new designs in this area. The balanced nested row-column designs, like those given in Chapter 4, have been limited to prime power numbers of treatments, except for some 2×2 's of Agrawal & Prasad (1983) and a few designs listed by various authors for some composite numbers of treatments. Construction problems of these designs for composite numbers of treatments are very much open to design constructionists.

The designs constructed in Chapters 2-5 can be used for other purposes also. Sets of v Latin squares for v treatments with row, column and diagonal neighbor balance (like neighbors occur on the diagonals, though not always desirable) for use as directional polycross designs have been previously given by Olesen (1976), and Morgan (1988a, 1988b). Polycross designs balanced for nearest row, column, and for diagonal neighbors with no like neighbors in any direction have not been constructed elsewhere. The designs of Chapter 2 satisfy these properties in the non-directional case; directional designs can be obtained by appropriate rotations of the designs given.

The torus designs of Chapter 3 can be layed out as fully bordered designs. Bordering the first and last rows (columns) by the last and first rows (columns) respectively, the torus neighbor balance properties can be preserved in the plane. Bordered designs have recently been proposed by a number of authors to facilitate (on the grounds of stationarity) the analysis of designs for correlated ob-

servations, and to achieve neighbor balance leading to improved precision of the estimates of treatment effects. Also, for the purpose of polycross experimentation, fully bordered designs have been previously considered by Freeman (1979), Afsarinejad & Seegar (1988), and Morgan (1988c). If one wished to consider side bordered designs (see Freeman (1979), Morgan (1988a) for example), this can be done too.

Recall that a nested balanced incomplete block design (Preece, 1967) is an arrangement of v treatments in b blocks of size pq , where the pq plots are further grouped into p rows, called “subblocks”, of q plots each such that the blocks (ignoring subblocks) give a balanced incomplete block design, and the subblocks (ignoring blocks) give a balanced incomplete block design. Taking either the rows or the columns as subblocks, the *BIBRCs* of Theorems 4.1-4.3 can also be used as nested balanced incomplete block designs.

While constructing designs in Chapters 2-5, every attempt has been made to keep the number of replicates and blocks as small as possible, but not at the cost of the desired optimality properties. In some cases the attempt to construct designs with minimum possible size has not been successful and consequently some designs use large numbers of replicates. Even if minimal size designs are obtained, they will have fairly large numbers of replicates in some cases, especially when v and pq are large. The optimality properties of designs requiring

large numbers of replicates and blocks are often of little consequence to practicing statisticians. Instead they prefer smaller sized, less efficient designs, trading optimality properties for replicates. But there are no established rules that balance the gain in replicates against the loss in efficiency and variance balance. Furthermore, in the absense of optimal designs, one would not know how much efficiency is lost with a given design. Hence it is important to have such designs available for use in practice, affording workers an opportunity to compare (to measure the degree of inefficiency of) their smaller sized designs against optimal or high efficiency designs, as those given in this dissertation.

Appendix I-Maximization of $\text{tr}(C)$ for equireplicate squares

The problem is to make the y_i^k 's as equal as possible,

$y_i^k = (c_i^k + e_i^k + m_i^k) - (2c_i^k + 3e_i^k + 4m_i^k)\beta_1 - (c_i^k + 2e_i^k + 4m_i^k)\beta_2$. With $b = v = p = q$, $\bar{y} = v - 4(v - 1)\beta_1 - 4(v - 2)\beta_2 - \frac{4}{v}\beta_2$. The proposed values for (c_i^k, e_i^k, m_i^k) are $(c_i^k, e_i^k, m_i^k) = (0, 4, v - 4) \Rightarrow y_i^k = v - 4(v - 1)\beta_1 - 4(v - 2)\beta_2 = \bar{y}_2 > \bar{y}$ and $(c_i^k, e_i^k, m_i^k) = (1, 2, v - 3) \Rightarrow y_i^k = v - 4(v - 1)\beta_1 - 4(v - 2)\beta_2 - \beta_2 = \bar{y}_1 < \bar{y}$. It will be shown that for any other choice of these three counts and equireplicate squares $y_i^k \notin (\bar{y}_1, \bar{y}_2)$. Equal replication gives

$r_i^k = v = c_i^k + e_i^k + m_i^k \Rightarrow y_i^k = v - (2v + e_i^k + 2m_i^k)\beta_1 - (v + e_i^k + 3m_i^k)\beta_2$, which can be examined for each possible c_i^k . For instance, if $c_i^k = 0$ then $m_i^k = v - e_i^k$, so $y_i^k = v - 4(v - 1)\beta_1 - 4(v - 2)\beta_2 - (4 - e_i^k)\beta_1 - (8 - 2e_i^k)\beta_2$ from which $y_i^k < \bar{y}_1$ for $e_i^k \leq 3$ and $y_i^k \geq \bar{y}_2$ for $e_i^k \geq 4$. Likewise if $c_i^k = 1$ then $m_i^k = v - e_i^k - 1$ so $y_i^k = v - 4(v - 1)\beta_1 - 4(v - 2)\beta_2 - (2 - e_i^k)\beta_1 - (5 - 2e_i^k)\beta_2$ and $y_i^k \leq \bar{y}_1$ for $e_i^k \leq 2$, $y_i^k > \bar{y}_2$ for $e_i^k \geq 3$. Similar steps show that if $c_i^k = 2$ then $y_i^k < \bar{y}_1$ for $e_i^k = 0$ and $y_i^k > \bar{y}_2$ for $e_i^k \geq 1$, and that for $c_i^k = 3$ or 4 , $y_i^k > \bar{y}_2$ for $e_i^k \geq 0$.

**Appendix II- A Theorem 2.1 design does not maximize $\text{tr}(C)$
over the non-equireplicate squares**

Given a design satisfying Theorem 2.1, it will be shown that it is possible to increase $\text{tr}(C)$ for some values of β_1, β_2 . Let t_1 be a treatment occurring in a corner of square 1, and t_2 a treatment not occurring in a corner of that square. Choose any interior plot containing t_2 which is not bordered by t_1 : change that plots treatment to t_1 . Square 1 and indeed the entire design is no longer equireplicate.

$\text{Tr}(C)$ is increased if and only if $(y_{t_1}^1 - \bar{y})^2 + (y_{t_2}^1 - \bar{y})^2$ is decreased. In the original design $(y_{t_1}^1 - \bar{y})^2 = -(1 - \frac{4}{v})\beta_2$ and $(y_{t_2}^1 - \bar{y}) = \frac{4}{v}\beta_2$ while in the new design $(y_{t_1}^1 - \bar{y}) = 1 - 4\beta_1 - 5\beta_2 + \frac{4}{v}\beta_2$ and $(y_{t_2}^1 - \bar{y}) = -1 + 4\beta_1 + 4\beta_2 + \frac{4}{v}\beta_2$. Both of these deviations from \bar{y} are smaller than their predecessors and if and only if

$$0 < 1 - 4\beta_1 - 4\beta_2 < \min\{(2 - \frac{8}{v})\beta_2, \frac{8}{v}\beta_2\}.$$

For instance, $\beta_1 = \beta_2 = 0.12$ satisfies this for $4 < v < 24$.

Appendix III- Integers $m \equiv 2 \pmod{4}$ satisfying $\chi(x^m - 2) = 1$.

For $v = 8t + 5$, there are $2t + 1$ odd values m' such that $\chi(x^{m'} - 1) = -1$ (Storer, 1967 pg. 30). Since 2 is a quadratic non-residue, $\chi(x^{m'-1}) = -\chi(2x^{m'} - 2) = -\chi(x^m - 2)$ for even m , so there are $2t + 1$ pairs $x^m - 2, x^m$ such that $\chi(x^m - 2) = \chi(x^m) = 1$. One of these pairs is $m = 0$, which upon multiplication by $-1 = x^{4t+2}$ reproduces itself. For any other pair, multiplication by -1 gives $-x^m, -x^m + 2 = x^{m'} - 2, x^{m'}$ for some even m' , showing that (since $4t + 2 \equiv 2 \pmod{4}$) the remaining $2t$ pairs may be divided into t pairs for which $m \equiv 2 \pmod{4}$, and t for which $m \equiv 0 \pmod{4}$.

Appendix IV - Proof of lemma 3.2

Write $v = q^n = 4m + 1$, q is prime. Partition GF_v into the q^{n-1} disjoint ordered cycles of length q given by the cosets of $c_o := (0, 1, 2, \dots, q-1)$, where each cycle (y_1, y_2, \dots, y_q) is ordered so that $y_j - y_{j-1} = 1$. Then replace each y by $\chi(y)$ where

$$\chi(y) = \begin{cases} 1, & \text{if } y \text{ is a quadratic residue} \\ -1 & \text{if } y \text{ is a quadratic non-residue} \\ 0 & \text{if } y=0. \end{cases}$$

Now, all q^{n-1} cycles taken together have $4m + 1$ pairs of ordered, adjacent elements. Of these, m are $(-1, 1)$, m are $(1, -1)$, m are $(-1, -1)$, and $m - 1$ are $(1, 1)$ (Storer, 1967).

We will first show that there is at least one ordered triple $(-1, -1, 1)$ or $(1, -1, -1)$. Suppose this is not so. Then if a cycle contains two consecutive -1's it must contain only -1's, and each such cycle will contain q pairs $(-1, -1)$. Hence the number of $(-1, -1)$ pairs is a multiple of $q \Rightarrow q|m$, i.e. $q|(q^n - 1)/4$, which is impossible.

Now let $n > 1$. We wish to show there is a triple $(1, -1, -1)$ or $(-1, -1, 1)$ not arising from c_o . The nonzero elements of c_o are in some order x^{iu} for $i = 1, 2, \dots, q-1$, where $u = (q^n - 1)/(q - 1)$. If n is even these are all

quadratic residues and the result is established. If n is odd these have $(q-1)/2$ quadratic non-residues, so that the number of $(-1, -1)$ pairs in $\chi(c_o)$ is j , say, and the number of such pairs in the other $q^{n-1} - 1$ cycles is $m - j$, where j is between 0 and $(q-3)/2$. Suppose none of the desired triple occurs outside of $\chi(c_o)$. Then arguing as above, if two consecutive -1 's occur in one of these cycles, the cycle contains only -1 's. Hence $q|(m-j)$, i.e. $4q|(q^n - 4j - 1) \Rightarrow j = (q-1)/4 \Rightarrow$ there are $(q^{n-1} - 1)/4$ cycles composed solely of -1 's and $3(q^{n-1} - 1)/4$ cycles containing no $(-1, -1)$ pairs.

We complete the proof by counting ordered triples among the $q^{n-1} - 1$ cycles excluding $\chi(c_o)$. There are $(q^n - q)/4$ triples $(-1, -1, 1)$, q from each of the cycles composed only of -1 's. There are $(q^n - q)/4$ triples $(1, -1, 1)$, one for each occurrence of -1 not in $\chi(c_o)$ or the cycles of -1 's. To count the number of $(1, 1, -1)$ triples, note that since $(1, -1, -1)$ does not occur in these cycles, this is equal to the number of $(1, -, -1)$ triples where the middle element is arbitrary, i.e. this is the number of pairs $(w, w+2)$, $w \notin GF_q$, such that w is a quadratic residue and $w+2$ is not. Multiplying by 2^{-1} , this is the number of $(1, -1)$ pairs if 2 is quadratic, or the number of $(-1, 1)$ if 2 is not quadratic, among these cycles. In either case this is just the number of -1 's in the $3(q^{n-1} - 1)/4$ cycles excluding $\chi(c_o)$ and the cycles of -1 's, since for these cycles, every -1 is preceded and succeeded by a 1. We then easily count that this number is $(q^n - q)/4$. A

similar argument shows that the number of $(-1, 1, 1)$ triples is also $(q^n - q)/4$.

With the q triples of $\chi(c_o)$, all triples are now accounted for. In particular, after excluding $\chi(c_o)$, there are no triples of the form $(1, 1, 1)$, $(-1, 1, -1)$, $(-1, -1, 1)$ or $(1, -1, -1)$. So we see that our cycles are $\chi(c_o)$, $(q^{n-1} - 1)/4$ cycles of -1 's, and $3(q^{n-1} - 1)/4$ cycles composed of consecutive triples $(1, 1, -1)$. This implies that $3|q$. Since q is prime, $q = 3$. But with odd n , $3^n \not\equiv 1 \pmod{4}$, a contradiction.

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