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Rational Cubic B-Spline Interpolation and Its Applications in Computer Aided Geometric Design

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Abstract

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in Computer Aided Geometric Design

A Non-Uniform Rational B-spline (NURB) is a vector-valued function of the form

$$\mathbf{s}(u) = \frac{\sum_{i=0}^{L+n-1} \tilde{w}_i \mathbf{p}_i N_i^n(u)}{\sum_{i=0}^{L+n-1} \tilde{w}_i N_i^n(u)}, \qquad 0 \le u \le 1.$$

where the $\tilde{w}_i's$ are scalars called the *weights* of the NURB, the $\mathbf{p}_i's$ are vectors in $\mathbf{E}^k$ (usually $k = 2$ or 3) called the *control points* of the NURB, and the N_i^n's are the usual scalar-valued B-spline base functions associated with some given knot sequence.

Because of the flexibility that the weights and the control points provide, NURBS have recently become very popular tools for the design of curves and surfaces. If the weights are positive then the NURB will lie in the convex hull of its control points and will not possess singularities. Thus it is desireable to have positive weights.

In utilizing a NURB a designer may desire that it pass through a set of data points $\{\mathbf{x}_i\}$. This interpolation problem is solved by the assigning of weights to each data point. Up to now little has been known regarding the relationship between these assigned weights and the weights of the corresponding interpolating NURB. In this thesis this relationship is explored. Sufficient conditions are developed to produce interpolating NURBS which have positive weights. Applications to the problems of degree reduction and curve fairing are presented. Both theoretical and computational results are presented.

To my wife, Dong Liu, my lovely son, Qian Wu

and my parents

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Chapter 1

Preliminaries

1.1 Introduction

A $2D$(two dimensional) nth degree *Bézier curve* is given by (see [9, 10, 11, 27, 35])

$$\mathbf{b}^n(u) = \sum_{i=0}^{n} \mathbf{b}_i B_i^n(u), \qquad 0 \le u \le 1, \tag{1.1.1}$$

where the $\mathbf{b}_i \in \mathbf{E}^2$ are called *Bézier points* that form the *control polygon.*

The $B_i^n(u)'s$ are the Bernstein polynomials of degree n, defined explicitly by:

$$B_i^n(u) = \begin{pmatrix} n \\ i \end{pmatrix} u^i (1-u)^{n-i}, \qquad 0 \le u \le 1, \tag{1.1.2}$$

where the binomial coefficients are given by

$$\begin{pmatrix} n \\ i \end{pmatrix} = \begin{cases} \frac{n!}{i!(n-i)!} & \text{if } 0 \le i \le n \\ 0 & \text{otherwise.} \end{cases} \tag{1.1.3}$$

A rational Bézier curve of degree n in $\mathbf{E}^2$ is the projection of an nth degree Bézier curve in $\mathbf{E}^3$ into the hyperplane $\tilde{w} = 1$. We view this $3D$ hyperplane as a copy of $\mathbf{E}^2$, we assume that a point in $\mathbf{E}^3$ is given by its coordinates $[\mathbf{y}_i \ \ \tilde{w}_i]$, where $\mathbf{y}_i \in \mathbf{E}^2$. It was proved that a $2D$ nth degree rational Bézier curve is given by (see [9, 10, 11, 27, 35])

$$\mathbf{x}(u) = \frac{\sum_{i=0}^{n} \tilde{w}_i \mathbf{b}_i B_i^n(u)}{\sum_{i=0}^{n} \tilde{w}_i B_i^n(u)}, \qquad 0 \le u \le 1, \tag{1.1.4}$$

where the $\tilde{w}_i$ are called weights associated with $\mathbf{b}_i \in \mathbf{E}^2$ that form the control polygon for $\mathbf{x}(u)$. This control polygon is the projection through the origin of the 3D control polygon formed by $[\tilde{w}_i\mathbf{b}_i \ \ \tilde{w}_i]$ into the hyperplane $\tilde{w} = 1$.

If all of the weights are equal to one, we obtain the standard integral Bézier curve. The $\tilde{w}_i$ are typically used as *shape parameters.* If we increase one weight $\tilde{w}_i$, the curve is pulled toward the corresponding control point $\mathbf{b}_i$ (see [9, 10, 11, 27, 35]).

If any $\tilde{w}_i$ is negative, the rational Bézier curves may have a singularity. If all the $\tilde{w}_i'$s are positive, the rational Bézier curve has the convex hull property, i.e. the curve lies in the convex hull of its control points. Thus we desire to have $\tilde{w}_i > 0$ for all i. The rational Bézier curves also have several other properties that their integral counterparts possess; for example, they are invariant under affine parameter transformations, and interpolate to the endpoints of their control polygon.

In addition they possess invariance under perspective parameter transformations, which are defined as follows:(see [22])

Let $\mathbf{c} \in \mathbf{E}^3$ be the center of a projection and the perspective plane be given by the point $\mathbf{a} \in \mathbf{E}^3$ and by the normal vector $\mathbf{N} \in \mathbf{R}^3$. Then the projection of a point $\mathbf{z}$ is

$$\pi(\mathbf{z}) = (1-\alpha)\mathbf{z} + \alpha\mathbf{c}, \quad \text{where} \quad \alpha = \frac{(\mathbf{z}-\mathbf{a})\cdot\mathbf{N}}{(\mathbf{z}-\mathbf{c})\cdot\mathbf{N}}.$$

Also unlike their integral counterparts rational Bézier curves can represent conic curves exactly (see [10, 11, 27, 35]).

A $2D$ nth degree *B-spline curve* is given by (see [9, 10, 11, 27, 35])

$$\mathbf{bs}(u) = \sum_{i=0}^{L+n-1} \mathbf{p}_i N_i^n(u), \qquad 0 \le u \le 1, \tag{1.1.5}$$

where the $\mathbf{p}_i \in \mathbf{E}^2$ are called *the control points* of the curve which form the *control polygon* and $\{N_i^n(u)\}_0^{L+n-1}$ are the normalized B-spline basis functions of degree n, defined by the following:

$$N_i^0(u) = \begin{cases} 1 & \text{if } \tilde{u}_{i-1} \le u \le \tilde{u}_i \\ 0 & \text{otherwise,} \end{cases} \qquad \text{where} \quad i = 0, \ldots, L+2n-2, \tag{1.1.6}$$

$$N_j^n(u) = \frac{u - \tilde{u}_{j-1}}{\tilde{u}_{j+n-1} - \tilde{u}_{j-1}} N_j^{n-1}(u) + \frac{\tilde{u}_{j+n} - u}{\tilde{u}_{j+n} - \tilde{u}_j} N_{j+1}^{n-1}(u), \tag{1.1.7}$$

where $\{\tilde{u}_0 \le \tilde{u}_1, \ldots, \tilde{u}_{L+2n-3} \le \tilde{u}_{L+2n-2}\}$ is a given nondecreasing sequence. The $\tilde{u}_i' s$ are called *knots* which in this context means that each interval $[\tilde{u}_i, \tilde{u}_{i+1}]$ is mapped onto a polynomial curve segment (here a polynomial curve of degree $\leq n$) by the B-spline function $N_j^n(u)$. The B-spline curve is (at least) C^{n-r} at its knots with multiplicity r.

A $2D$ nth degree rational B-spline curve is the projection through the origin of a $3D$ integral B-spline curve into the hyperplane $\tilde{w} = 1$. It is given by

$$\mathrm{s}(u) = \frac{\sum_{i=0}^{L+n-1} \tilde{w}_i \mathbf{p}_i N_i^n(u)}{\sum_{i=0}^{L+n-1} \tilde{w}_i N_i^n(u)}, \qquad 0 \leq u \leq 1, \tag{1.1.8}$$

where the $\tilde{w}_i$ are called *weights* associated with each $\mathbf{p}_i \in \mathbf{E}^2$ that are called *the control points* of the curve which form the *control polygon.* The function $\mathrm{s}(u)$ has been called a NURB for Non-Uniform Rational B-spline.

If all of the weights are equal to one, we obtain the standard integral B-spline curve. As with the rational Bézier curves, the weights of the rational B-spline curves can be used as *shape parameters.* Also it is desirable to have these weights positive. Rational B-spline curves have similar properties to those of rational Bézier curves.

Both rational Bézier curves and rational B-spline curves have achieved widespread acceptance and popularity in the CAD/CAM and graphics community (see [9, 10, 11, 27, 35]).

In this dissertation we are interested in C^2 rational cubic B-spline interpolation (see [10, 11, 27, 35]).

Given $2D$ points $\mathbf{x}_i$ and their weights w_i, we transform them to $3D$ points $[w_i\mathbf{x}_i \;\; w_i]$ and perform $3D$ nonrational algorithms (for example interpolation and degree reduction). The result of these procedures will be a set of $3D$ points $[\mathbf{y}_i \;\; v_i]$. By the projection of this result into the hyperplane $\tilde{w} = 1$, we obtain $2D$ points $\mathbf{y}_i/v_i$. The weights of these $2D$ points are the scalars v_i.

The rational cubic interpolation problem in the context of rational B-splines is the following (see [10, 11, 35]):

Given: $2D$ data points $\mathbf{x}_1, \ldots, \mathbf{x}_{L+1}$, parameter values $u_1, \ldots, u_{L+1}$.

Find: a C^2 rational cubic B-spline curve:

$$\mathbf{L}(u) = \frac{\sum_{i=0}^{L+2} v_i \mathbf{d}_i N_i^3(u)}{\sum_{i=0}^{L+2} v_i N_i^3(u)} \qquad 0 \leq u \leq 1 \tag{1.1.9}$$

such that

$$\mathbf{L}(u_i) = \mathbf{x}_i, \qquad i = 1, \ldots, L+1. \tag{1.1.10}$$

The knot sequence of $\mathbf{L}(u)$ is $0 = u_1 < u_2 < \ldots < u_L < u_{L+1} = 1$, where u_1 and u_{L+1} are the knots of multiplicity three, which ensures that $\mathbf{L}(u)$ interpolates the end points and is tangential at the end points to the first and last legs of its control polygon (see [2]).

The $\{\mathbf{d}_i\}_0^{L+2}$ are the control points of the interpolating rational cubic curve and $\{v_i\}_0^{L+2}$ are the weights of the resulting C^2 rational cubic B-spline curve, (which shall be called *rational weights* for simplicity from now on), associated with $\mathbf{d}_i$. Also we shall call $\mathbf{L}(u)$ the *resulting rational function* or the *resulting rational curve* and we shall call $\sum_{i=0}^{L+2} v_i N_i^3(u)$ the *resulting weight function* from now on. We shall use $\mathbf{L}(u)$ to denote Rational cubic B-spline interpolant in the titles of the figures in this thesis.

Let b_1 and b_{3L-1} be constants determined by the given end conditions of the interpolation and let all components of $\mathbf{b}_1 \in \mathbf{E}^2$ and $\mathbf{b}_{3L-1} \in \mathbf{E}^2$ be the constant vectors determined by the given end conditions of the interpolation respectively.

In order to interpolate with rational curves, we need to assign weights $w_1, \ldots, w_{L+1}$ to the corresponding data points. Then we transform $\mathbf{x}_1, \ldots, \mathbf{x}_{L+1}$ and $w_1, \ldots, w_{L+1}$ to $3D$ points $[w_i\mathbf{x}_i \;\; w_i]^T$ and perform $3D$ cubic B-spline interpolation as follows

$$\begin{cases} v_0 = w_1, & v_{L+2} = w_{L+1}, \\ v_1 = b_1, & v_{L+1} = b_{3L-1}, \\ \sum_{i=0}^{L+2} v_i N_i^3(u_j) = w_j & (j = 2, \ldots, L), \end{cases} \tag{1.1.11}$$

$$\begin{cases} v_0\mathbf{d}_0 = w_1\mathbf{x}_1, & v_{L+2}\mathbf{d}_{L+2} = w_{L+1}\mathbf{x}_{L+1}, \\ v_1\mathbf{d}_1 = b_1\mathbf{b}_1, & v_{L+1}\mathbf{d}_{L+1} = b_{3L-1}\mathbf{b}_{3L-1}, \\ \sum_{i=0}^{L+2} v_i\mathbf{d}_i N_i^3(u_j) = w_j\mathbf{x}_j & (j = 2, \ldots, L). \end{cases} \tag{1.1.12}$$

Thus the preimage of $\mathbf{L}(u)$, which is formed by the control points $\{[v_i\mathbf{d}_i \;\; v_i]\}_0^{L+2}$, interpolates to the data $[w_j\mathbf{x}_j \;\; w_j]^T$. By projecting this into the hyperplane $\tilde{w} = 1$, we obtain $2D$ points $v_i\mathbf{d}_i/v_i = \mathbf{d}_i$. The rational weights of these $2D$ points are the numbers v_i. Thus $\mathbf{L}(u_j) = \mathbf{x}_j$ and $\sum_{i=0}^{L+2} v_i N_i^3(u_j) = w_j$, $j = 1, \ldots, L+1$.

If we change a B-spline which has the control points $\{[v_i\mathbf{d}_i \;\; v_i]\}_0^{L+2}$ into piecewise Bézier form, $[b_1\mathbf{b}_1 \;\; b_1]$ will be the second control point in its first segment, and $[b_{3L-1}\mathbf{b}_{3L-1} \;\; b_{3L-1}]$ will be the next to last control point in its last segment. Thus we can obtain the first derivative of the each component of the preimage of $\mathbf{L}(u)$ at the end points in terms of b_1 and $b_1\mathbf{b}_1$, b_{3L-1} and $b_{3L-1}\mathbf{b}_{3L-1}$ respectively and then we can obtain b_1 and $\mathbf{b}_1$, b_{3L-1} and $\mathbf{b}_{3L-1}$. We first consider b_1. Let $\Delta_1 = u_2 - u_1$, we have (see [10]):

$$\frac{d}{du}(\sum_{i=0}^{L+2} v_i N_i^3(u_1)) = \frac{1}{\Delta_1} 3(b_1 - v_0),$$

i.e.

$$\frac{d}{du}(\sum_{i=0}^{L+2} v_i N_i^3(u_1)) = \frac{1}{\Delta_1} 3(b_1 - w_1), \tag{1.1.13}$$

since $v_0 = w_1$.

Given $w_i = w(u_i)$, $i = 1, \ldots, L+1$, we have $\frac{d}{du} w(u_1)$ and thus we can obtain the following equation involving b_1 by the clamped end conditions, (which equate the derivatives of the interpolant and the curve to be interpolated at each end point):

$$\frac{d}{du}(\sum_{i=0}^{L+2} v_i N_i^3(u_1)) = \frac{d}{du} w(u_1),$$

i.e.

$$\frac{1}{\Delta_1} 3(b_1 - w_1) = \frac{d}{du} w(u_1). \tag{1.1.14}$$

Thus we can obtain b_1 from equation (1.1.14). Similarly we can find b_{3L-1}, and then $\mathbf{b}_1$ and $\mathbf{b}_{3L-1}$.

If we choose the Bessel end conditions for interpolation, (which give the approximate first derivatives at each end point), we have (see [10])

$$b_1 = \frac{1}{3}[\frac{\Delta_2}{\Delta_1 + \Delta_2} w_1 + w_1 + \frac{\Delta_1 + \Delta_2}{\Delta_2} w_2 - \frac{\Delta_1^2}{\Delta_2(\Delta_1 + \Delta_2)} w_3], \tag{1.1.15}$$

$$b_{3L-1} = \frac{1}{3}[\frac{\Delta_{L-1}}{\Delta_{L-1}+\Delta_L}w_{L+1} + w_{L+1} + \frac{\Delta_{L-1}+\Delta_L}{\Delta_{L-1}}w_L - \frac{\Delta_L^2}{\Delta_{L-1}(\Delta_{L-1}+\Delta_L)}w_{L-1}], \quad (1.1.16)$$

$$b_1\mathbf{b}_1 = \frac{1}{3}[\frac{\Delta_2}{\Delta_1+\Delta_2}w_1\mathbf{x}_1 + w_1\mathbf{x}_1 + \frac{\Delta_1+\Delta_2}{\Delta_2}w_2\mathbf{x}_2 - \frac{\Delta_1^2}{\Delta_2(\Delta_1+\Delta_2)}w_3\mathbf{x}_3], \quad (1.1.17)$$

$$b_{3L-1}\mathbf{b}_{3L-1} = \frac{1}{3}[\frac{\Delta_{L-1}}{\Delta_{L-1}+\Delta_L}w_{L+1}\mathbf{x}_{L+1} + w_{L+1}\mathbf{x}_{L+1} + \frac{\Delta_{L-1}+\Delta_L}{\Delta_{L-1}}w_L\mathbf{x}_L - \frac{\Delta_L^2}{\Delta_{L-1}(\Delta_{L-1}+\Delta_L)}w_{L-1}\mathbf{x}_{L-1}]. \quad (1.1.18)$$

Note the knots are $\{\tilde{u}_0, \ldots, \tilde{u}_{L+2}\}$ under the definition in (1.1.7) and we have a relationship between $\{0 = u_1 < u_2 < u_3 < \ldots < u_{L-1} < u_L < u_{L+1} = 1\}$, where u_1 and u_{L+1} are the knots of $\mathbf{L}(u)$ with multiplicity three, and $\{\tilde{u}_0, \ldots, \tilde{u}_{l+2}\}$:

$u_1 = \tilde{u}_0$, $u_1 = \tilde{u}_1$, $u_1 = \tilde{u}_2$ and

$$u_i = \tilde{u}_{i+1}, \qquad i = 2, \ldots L+1. \quad (1.1.19)$$

If some v_i are negative, the resulting curve $\mathbf{L}(u)$ may have singularities. We rewrite (1.1.9) as

$$\mathbf{L}(u) = \sum_{i=0}^{L+2} \frac{v_i N_i^3(u)}{\sum_{j=0}^{L+2} v_j N_j^3(u)} \mathbf{d}_i \quad (1.1.20)$$

and we consider

$$\frac{v_i N_i^3(u)}{\sum_{j=0}^{L+2} v_j N_j^3(u)} \; , \quad i = 0, \ldots, L+2 \; ,$$

as basis functions. If all the v_i's are positive, the resulting curve $\mathbf{L}(u)$ will have the convex hull property. If $v_j = 0$ for some j, then from (1.1.12) the resulting control point $\mathbf{d}_j$ will be called "a point at infinity" in the projective geometry sense. In this case the resulting curve would lose the convex hull property. Thus we are only interested in positive v_i's.

1.2 A Brief Survey

Farin stated in [9]:"Ask anyone in the CAD/CAM industry or graphics about the most promising curve or surface form. The answer is invariably 'NURBS' - nonuniform rational B-splines." Some reasons given by Piegl in [27] for that are as follows:

> They offer a common mathematical form for representing and designing both standard analytic shapes (conics, quadrics, surface of revolution, etc.) and free-form curves and surfaces. Therefore, both analytic and free-form shapes are represented precisely, and a unified database can store both.
>
> By manipulating the control points as well as the weights, NURBS provide the flexibility to design a large variety of shapes.
>
> NURBS have clear geometric interpretations, making them particularly useful for designers, who have a very good knowledge of geometry - especially descriptive geometry.
>
> NURBS have a powerful geometric tool kit (knot insertion/refinement /removal, degree elevation, splitting, etc.), which can be used throughout to design, analyze, process, and interrogate objects.
>
> NURBS are invariant under any scaling, rotation, translation, or shear as well as parallel and perspective projections.

> NURBS are genuine generalizations of nonrational B-spline forms as well as rational and nonrational Bézier curves and surfaces.

The definition and process of C^2 rational cubic B-Spline interpolation in CAGD (Computer Aided Geometric Design) were given in [10, 11, 27].

Following the procedures given in [10, 11, 27], when we perform rational cubic B-Spline interpolation, we first transform the $2D$ data and the assigned weights to $3D$ points. Then we execute integral cubic B-Spline interpolation, and finally perform a homogeneous projection back into $2D$. Thus, rational cubic B-Spline interpolation may inherit some properties of integral cubic B-Spline interpolation. We prove in Chapter 2 that it inherits a minimum norm property, and a best approximation property and we derive an error estimate.

Farin conjectured in [10, 11] that " it seems reasonable to assign high weights in regions where the interpolant is expected to curve sharply". We support this conjecture in Chapter 3.

Farin in [10, 11] shows, by an example, that rational cubic B-spline interpolation may produce a rational curve which has negative weights. In Chapter 4 we present a sufficient condition on the assigned weights which ensure the positivity of the rational weights of the resulting rational interpolant. In Chapter 5 we provide three methods to modify assigned positive weights that led to NURBS with some negative rational weights in such a way that the new NURBS will have positive rational weights. Several examples are presented to illustrate and to compare these three methods.

In [27] Piegl wrote:“ We distinguish between two kinds of interpolation. In the first, we have pure data points unrelated to any other entities in the system; in the second, we have data points from another process that are related to other entities such as section curves. In the first case, I recommend using nonrational curves (except when specific local interpolants seem more suitable than a general method). In the second case, 'true' rational curves have to be computed if the existing entities are rational curves as well.” We consider these two kinds of applications. We present an application of rational B-Spline in Chapter 3: we present two examples to show that sometimes we can consider the assigned weights as *tension parameters* in the interpolation process. In Chapter 6 we present an application of rational cubic B-spline interpolation to the degree reduction of rational Bézier curves and rational B-spline curves of degree greater than 3. In Chapter 6 we also prove that we always can find suitable parameters for interpolation, which will guarantee positive rational weights of the resulting rational curve. Algorithms and examples are given in Chapter 6 which demonstrate these procedures.

Chapter 2

The Properties of a Rational Cubic B-Spline Interpolant

2.1 Introduction

Cubic B-spline interpolants have been well investigated (see [30, 32]). Some of their most important properties are a minimum norm property, a best approximation property and an error estimate. These properties make cubic B-spline interpolation useful in many fields. In this section we discuss a C^2 cubic rational B-spline interpolant defined by (1.1.9) (see [10, 11, 27]). We prove that it has similar properties to those of the interpolatory cubic B-spline function.

2.2 The Properties of a Cubic Rational B-Spline Interpolant

Let $\mathbf{g}(u) = ((\mathbf{g}(u))_1, (\mathbf{g}(u))_2)$, where $(\mathbf{g}(u))_j \in C^2[0,1]$, for $j = 1, 2$, and let $k(u) \in C^2[0,1]$. For any given parameter values $U = (u_1, \ldots u_{L+1})$, where $u_1 = 0$

and $u_{L+1} = 1$, we let $w_i = k(u_i)$ and $\mathbf{x}_i = \mathbf{g}(u_i)/k(u_i)$, i.e. $w_i\mathbf{x}_i = \mathbf{g}(u_i)$ in interpolation (1.1.11) and (1.1.12). Let b_1 and b_{3L-1} be determined by $k(u)$ under the clamped end conditions and $\hat{\mathbf{b}}_1$ and $\hat{\mathbf{b}}_{3L-1}$ be determined by $\mathbf{g}(u)$ under the clamped end conditions respectively.

Now we have three theorems for rational cubic B-spline interpolation:

Theorem 2.1 *(minimum norm property) Let $\hat{v}(u)$ be the cubic B-spline function interpolating to $k(u)$ under the clamped end conditions by (1.1.11) and $\hat{v}(u) > 0$ and $k(u) > 0$ for all $u \in [0,1]$. Let $\mathbf{y}(u) = ((\mathbf{y}(u))_1, (\mathbf{y}(u))_2)$ be the cubic B-spline function interpolating to $\mathbf{g}(u) = ((\mathbf{g}(u))_1, (\mathbf{g}(u))_2)$ under the clamped end conditions by (1.1.12). Then we have*

$$\int_0^1 [\frac{d^2}{du^2}(\mathbf{L}(u))_j]^2 du \le \int_0^1 [\frac{d^2}{du^2}(\frac{\mathbf{g}(u)}{k(u)})_j]^2 du, \quad j = 1, 2, \tag{2.2.1}$$

where $\hat{v}(u) = \sum_{i=0}^{L+2} v_i N_i^3(u)$ from (1.1.11), $\mathbf{y}(u) = \sum_{i=0}^{L+2} v_i \mathbf{d}_i N_i^3(u)$ from (1.1.12), and $\mathbf{L}(u) = \frac{\mathbf{y}(u)}{\hat{v}(u)}$, which interpolates to $\frac{\mathbf{g}(u)}{k(u)}$ under the clamped end conditions by (1.1.10)

Proof. Consider the integral

$$\int_0^1 [\frac{d^2}{du^2}(\mathbf{L}(u))_j - \frac{d^2}{du^2}(\frac{\mathbf{g}(u)}{k(u)})_j]^2 du = \int_0^1 [\frac{d^2}{du^2}(\frac{\mathbf{g}(u)}{k(u)})_j]^2 du - \int_0^1 [\frac{d^2}{du^2}(\mathbf{L}(u))_j]^2 du$$
$$+ 2\int_0^1 [\frac{d^2}{du^2}(\mathbf{L}(u))_j - \frac{d^2}{du^2}(\frac{\mathbf{g}(u)}{k(u)})_j] \frac{d^2}{du^2}(\mathbf{L}(u))_j du. \tag{2.2.2}$$

Note that

$$[\frac{d}{du}(\mathbf{L}(u))_j - \frac{d}{du}(\frac{\mathbf{g}(u)}{k(u)})_j] =$$

$$[\frac{\frac{d}{du}((\mathbf{y}(u))_j)\hat{v}(u) - (\mathbf{y}(u))_j\frac{d}{du}\hat{v}(u)}{(\hat{v}(u))^2} - \frac{\frac{d}{du}((\mathbf{g}(u))_j)k(u) - (\mathbf{g}(u))_j\frac{d}{du}k(u)}{(k(u))^2}].$$

From the clamped end conditions and the interpolatory conditions (1.1.11) and (1.1.12) it follows that

$$[\frac{d}{du}(\mathbf{L}(u))_j - \frac{d}{du}(\frac{\mathbf{g}(u)}{k(u)})_j] = 0, \qquad \text{when } u = 0 \text{ and } u = 1. \tag{2.2.3}$$

Integrating by parts twice and using the interpolatory conditions and (2.2.3), the last term in (2.2.2) becomes

$$\begin{aligned}
&\int_0^1 [\frac{d^2}{du^2}(\mathbf{L}(u))_j - \frac{d^2}{du^2}(\frac{\mathbf{g}(u)}{k(u)})_j]\frac{d^2}{du^2}(\mathbf{L}(u))_j du \\
&= \sum_{i=1}^{n}\int_{u_{i-1}}^{u_i} [\frac{d^2}{du^2}(\mathbf{L}(u))_j - \frac{d^2}{du^2}(\frac{\mathbf{g}(u)}{k(u)})_j]\frac{d^2}{du^2}(\mathbf{L}(u))_j du \\
&= \sum_{i=1}^{n}\{[\frac{d}{du}(\mathbf{L}(u))_j - \frac{d}{du}(\frac{\mathbf{g}(u)}{k(u)})_j]\frac{d^2}{du^2}(\mathbf{L}(u))_j|_{u_{i-1}}^{u_i} \\
&\quad - [\mathbf{L}(u))_j - (\frac{\mathbf{g}(u)}{k(u)})_j]\frac{d^3}{du^3}(\mathbf{L}(u))_j|_{u_{i-1}}^{u_i} \\
&\quad + \int_{u_{i-1}}^{u_i} [(\mathbf{L}(u))_j - \frac{\mathbf{g}(u)}{k(u)})_j]\frac{d^4}{du^4}(\mathbf{L}(u))_j du\} \\
&= [\frac{d}{du}(\mathbf{L}(u))_j - \frac{d}{du}(\frac{\mathbf{g}(u)}{k(u)})_j]\frac{d^2}{du^2}(\mathbf{L}(u))_j|_0^1 = 0.
\end{aligned}$$

Now, (2.2.1) follows from (2.2.2).

Theorem 2.2 *(best approximation property) Let $\hat{v}(u)$ be the cubic B-spline function interpolating to $k(u)$ under the clamped end conditions by (1.1.11) and $\hat{v}(u) > 0$ and $k(u) > 0$ for all $u \in [0,1]$. Let $\mathbf{y}(u) = ((\mathbf{y}(u))_1, (\mathbf{y}(u))_2)$ be the cubic B-spline function interpolating to $\mathbf{g}(u) = ((\mathbf{g}(u))_1, (\mathbf{g}(u))_2)$ under the clamped end conditions by (1.1.12). Let $\mathbf{L}(u) = \frac{\mathbf{y}(u)}{\hat{v}(u)}$, which interpolates to $\frac{\mathbf{g}(u)}{k(u)}$ under the clamped* *end conditions by (1.1.10). Then for any rational B-spline $L^*(u)$ with respect to the same partition as $\mathbf{L}(u)$ we have*

$$\int_0^1 [\frac{d^2}{du^2}(\mathbf{L}(u))_j - \frac{d^2}{du^2}(\frac{\mathbf{g}(u)}{k(u)})_j]^2 du$$
$$\leq \int_0^1 [\frac{d^2}{du^2}(L^*(u))_j - \frac{d^2}{du^2}(\frac{\mathbf{g}(u)}{k(u)})_j]^2 du, \quad j = 1, 2. \tag{2.2.4}$$

Proof. Consider the integral

$$\int_0^1 [\frac{d^2}{du^2}(\mathbf{L}(u))_j - \frac{d^2}{du^2}(\frac{\mathbf{g}(u)}{k(u)})_j]^2 du$$
$$= \int_0^1 [\frac{d^2}{du^2}(\mathbf{L}(u))_j - \frac{d^2}{du^2}(L^*(u))_j + \frac{d^2}{du^2}(L^*(u))_j - \frac{d^2}{du^2}(\frac{\mathbf{g}(u)}{k(u)})_j]^2 du$$
$$= \int_0^1 [\frac{d^2}{du^2}(\mathbf{L}(u))_j - \frac{d^2}{du^2}(L^*(u))_j]^2 du + \int_0^1 [\frac{d^2}{du^2}(L^*(u))_j - \frac{d^2}{du^2}(\frac{\mathbf{g}(u)}{k(u)})_j]^2 du$$
$$+ 2\int_0^1 [\frac{d^2}{du^2}(\mathbf{L}(u))_j - \frac{d^2}{du^2}(L^*(u))_j][\frac{d^2}{du^2}(L^*(u))_j - \frac{d^2}{du^2}(\frac{\mathbf{g}(u)}{k(u)})_j] du.$$

As in the proof Theorem 2.1, we can show that the last integral is zero. Then (2.2.4) follows.

Before we present a convergence theorem for the B-spline rational cubic interpolant defined by (1.1.9), we introduce some definitions and Lemmas.

Definition 2.1 *For $f(u) \in C[0,1]$, let $\|f(u)\|_\infty$ denote the uniform norm of f defined by*

$$\|f(u)\|_\infty = \max_{0 \leq u \leq 1} | f(u) | .$$

For $\mathbf{g}(u) = ((\mathbf{g}(u))_1, (\mathbf{g}(u))_2)$, where $(\mathbf{g}(u))_j \in C[0,1]$, $j = 1, 2$, we define the following norm:

$$||| \mathbf{g}(u) |||_\infty = \max_{1 \leq j \leq 2} \| (\mathbf{g}(u))_j \|_\infty .$$

Now we introduce a definition from [32]

Definition 2.2 *Let* $D^t\phi(x) \equiv \frac{d^t\phi}{dx^t}(x)$. *For each nonnegative integer* t *and for each* p, $1 \le p \le \infty$, *we will let* $PC^{t,p}[0,1]$ *be the set of all real-valued functions* $\phi(x)$ *such that:*

(1) $\phi(x)$ *is* $t-1$ *times continuously differentiable,*

(2) there exists a partition

$$0 = \xi_0 < \xi_1 < \ldots < \xi_s < \xi_{s+1} = 1$$

such that on each open subinterval (ξ_i, ξ_{i+1}), $0 \le i \le s$, $D^{t-1}\phi(x)$ *is continuously differentiable, and*

(3) the L_p-*norm of* $D^t\phi(x)$ *is finite, i.e.,*

$$\|D^t\phi(x)\|_p \equiv (\sum_{i=0}^{s}\int_{\xi_i}^{\xi_{i+1}} \mid D^t\phi(x) \mid^p dx)^{1/p} < \infty.$$

For the special case of $p = \infty$, *we require that*

$$\|D^t\phi(x)\|_{\infty,s} \equiv \max_{0\le i\le s} \sup_{x\in(\xi_i,\xi_{i+1})} \mid D^t\phi(x) \mid < \infty.$$

Definition 2.3 *For* $\mathbf{g}(u) = ((\mathbf{g}(u))_1, (\mathbf{g}(u))_2)$, *where* $(\mathbf{g}(u))_j \in PC^{t,\infty}[0,1]$, *for* $j = 1, 2$, *we define*

$$\||\, D^t\mathbf{g}(u)\, \||_{\infty,s} = \max_{1\le j\le 2} \|\, (D^t\mathbf{g}(u))_j\|_{\infty,s}.$$

From Theorem 4.7 and Theorem 4.8 in [32], we have the following:

Lemma 2.1 *Let $\hat{v}(u)$ be the cubic B-spline function interpolating to $k(u)$ under the clamped end conditions by (1.1.11) and $\hat{v}(u) > 0$ and $k(u) > 0$ for all $u \in [0,1]$. Let $\mathbf{y}(u) = ((\mathbf{y}(u))_1, (\mathbf{y}(u))_2)$ be the cubic B-spline function interpolating to $\mathbf{g}(u) = ((\mathbf{g}(u))_1, (\mathbf{g}(u))_2)$ under the clamped end conditions by (1.1.12). If $k(u) \in PC^{2,\infty}[0,1]$ and $(\mathbf{g}(u))_j \in PC^{2,\infty}[0,1]$, for $j = 1,2$. Then we have*

$$\|\hat{v}(u) - k(u)\|_\infty \leq \frac{2}{3}\|D^2 k(u)\|_{\infty,s} h^2 \tag{2.2.5}$$

and

$$||| \mathbf{y}(u) - \mathbf{g}(u) |||_\infty \leq \frac{2}{3} ||| D^2 \mathbf{g}(u) |||_{\infty,s} h^2. \tag{2.2.6}$$

If $k(u) \in PC^{(4,\infty)}[0,1]$ and $(\mathbf{g}(u))_j \in PC^{4,\infty}[0,1]$, for $j = 1,2$, we have

$$\|\hat{v}(u) - k(u)\|_\infty \leq \frac{5}{384}\|D^4 k(u)\|_{\infty,s} h^4 \tag{2.2.7}$$

and

$$||| \mathbf{y}(u) - \mathbf{g}(u) |||_\infty \leq \frac{5}{384} ||| D^4 \mathbf{g}(u) |||_{\infty,s} h^4. \tag{2.2.8}$$

Lemma 2.2 *Let $\hat{v}(u)$ be the cubic B-spline function interpolating to $k(u)$ under the clamped end conditions by (1.1.11) and $\inf_{0\leq u\leq 1} k(u) \geq z > 0$, where z is a constant, $k(u) \in PC^{2,\infty}[0,1]$ and $h > 0$ is sufficiently small such that*

$$\frac{1}{2}z \geq \frac{2}{3}\|D^2 k(u)\|_{\infty,s} h^2. \tag{2.2.9}$$

Then we have

$$\inf_{0\leq u\leq 1} \hat{v}(u) \geq \frac{1}{2}z; \tag{2.2.10}$$

Furthermore if $k(u) \in PC^{4,\infty}[0,1]$ *and* h *is sufficiently small such that*

$$\frac{1}{2}z \geq \frac{5}{384}\|D^4k(u)\|_{\infty,s}h^4, \tag{2.2.11}$$

then we also have (2.2.10) .

Proof. From Lemma 2.1 and the hypothesis of this Lemma it follows that

$$\hat{v}(u) = k(u) - (k(u) - \hat{v}(u)) \geq z - \frac{2}{3}\|D^2k(u)\|_{\infty,s}h^2 \geq \frac{1}{2}z,$$

or

$$\hat{v}(u) = k(u) - (k(u) - \hat{v}(u)) \geq z - \frac{5}{384}\|D^4k(u)\|_{\infty,s}h^4 \geq \frac{1}{2}z.$$

Now we present the following theorem:

Theorem 2.3 *(error estimate) Let* $\hat{v}(u)$ *be the cubic B-spline function interpolating to* $k(u)$ *under the clamped end conditions by (1.1.11), where* $k(u) \in PC^{2,\infty}[0,1]$ *and* $\inf_{0\leq u\leq 1} k(u) \geq z > 0$*, where* z *is a constant and* $h > 0$ *is sufficiently small such that (2.2.9) holds. Let* $\mathbf{y}(u) = ((\mathbf{y}(u))_1, (\mathbf{y}(u))_2)$ *be the cubic B-spline function interpolating to* $\mathbf{g}(u) = ((\mathbf{g}(u))_1, (\mathbf{g}(u))_2)$*, where* $(\mathbf{g}(u))_j \in PC^{2,\infty}[0,1]$*, for* $j = 1, 2$*, under the clamped end conditions by (1.1.12). Let* $\mathbf{L}(u) = \frac{\mathbf{y}(u)}{\hat{v}(u)}$*, which interpolates to* $\frac{\mathbf{g}(u)}{k(u)}$ *under the clamped end conditions by (1.1.10), we have*

$$\||| \mathbf{L}(u) - \frac{\mathbf{g}(u)}{k(u)} |||_\infty \leq$$

$$\frac{4}{3} \frac{(||| D^2\mathbf{g}(u) |||_{\infty,s} \|k(u)\|_\infty + \|D^2k(u)\|_{\infty,s} ||| \mathbf{g} |||_\infty)}{z^2} h^2. \quad (2.2.12)$$

Furthermore, if $k(u) \in PC^{4,\infty}[0,1]$, $(\mathbf{g}(u))_j \in PC^{4,\infty}[0,1]$, *for* $j = 1,2$, *and* $h > 0$ *is sufficiently small such that (2.2.11) holds, then we have*

$$||| \mathbf{L}(u) - \frac{\mathbf{g}(u)}{k(u)} |||_\infty \leq$$

$$\frac{5(||| D^4\mathbf{g}(u) |||_{\infty,s} \|k(u)\|_\infty + \|D^4k(u)\|_{\infty,s} ||| \mathbf{g} |||_\infty)}{192z^2} h^4. \quad (2.2.13)$$

Proof. We have

$$||| \mathbf{L}(u) - \frac{\mathbf{g}(u)}{k(u)} |||_\infty = ||| \frac{\mathbf{y}(u)}{\hat{v}(u)} - \frac{\mathbf{g}(u)}{k(u)} |||_\infty$$

$$= ||| \frac{\mathbf{y}(u)k(u) - \mathbf{g}(u)\hat{v}(u)}{\hat{v}(u)k(u)} |||_\infty$$

$$= ||| \frac{\mathbf{y}(u)k(u) - \mathbf{g}(u)k(u) + \mathbf{g}(u)k(u) - \mathbf{g}(u)\hat{v}(u)}{\hat{v}(u)k(u)} |||_\infty .$$

From the hypothesis of this theorem and Lemma 2.2 it follows that

$$\|\frac{1}{\hat{v}(u)k(u)}\|_\infty \leq \frac{2}{z^2}. \quad (2.2.14)$$

From Lemma 2.1 it follows that

$$||| \mathbf{y}(u)k(u) - \mathbf{g}(u)k(u) + \mathbf{g}(u)k(u) - \mathbf{g}(u)\hat{v}(u) |||_\infty$$

$$\leq ||| \mathbf{y}(u)k(u) - \mathbf{g}(u)k(u) |||_\infty + ||| \mathbf{g}(u)k(u) - \mathbf{g}(u)\hat{v}(u) |||_\infty$$

$$\leq ||| \mathbf{y}(u) - \mathbf{g}(u) |||_\infty \|k(u)\|_\infty + \leq ||| \mathbf{g}(u) |||_\infty \|\hat{v}(u) - k(u)\|_\infty$$

$$\leq \frac{2}{3}(||| D^2\mathbf{g}(u) |||_{\infty,s} \|k(u)\|_\infty + \|D^2k(u)\|_{\infty,s} ||| \mathbf{g}(u) |||_\infty)h^2. \quad (2.2.15)$$

From the combination of (2.2.14) and (2.2.15) this theorem follows.

The proof of the second part of this theorem is similar to the proof of the first part and is thus omitted.

Remark: For any function $\mathbf{f}(u) = ((\mathbf{f}(u))_1, (\mathbf{f}(u))_2)$, where $(\mathbf{f}(u))_j \in PC^{2,\infty}[0,1]$, for $j = 1,2$, or $(\mathbf{f}(u))_j \in PC^{4,\infty}[0,1]$, for $j = 1,2$, we can let $\mathbf{g}(u) = k(u)\mathbf{f}(u)$, where $k(u)$ satisfied the hypothesis of Theorem 2.3. From Theorem 2.3, we always can find $\mathbf{L}(u)$ to approximate $\frac{\mathbf{g}(u)}{k(u)}$, i.e $\mathbf{f}(u)$, within a given tolerance $\epsilon > 0$, if $h > 0$ is small enough such that ((2.2.9)) holds and the right hand side of (2.2.12) is less than ϵ, or if $h > 0$ is small enough such that (2.2.11) holds holds and the right hand side of (2.2.13) is less than ϵ. Specifically, we let $\mathbf{f}(u)$ be an nth degree Bézier curve or an nth degree B-spline curve, where $n > 2$, and let $k(u)$ not be constant. We always can find a rational cubic B-spline $\mathbf{L}(u)$ to approximate $\mathbf{f}(u)$ to within $\epsilon > 0$.

Chapter 3

The Assigned Weights of Rational Cubic B-Spline Interpolation

3.1 Introduction

In this Chapter we discuss how the weights assigned to the data to be interpolated affect the resulting rational curve and present a new application of a rational cubic B-Spline interpolation in CAGD by considering the assigned weights as *tension parameters.* We shall consider the Bessel end conditions for interpolation.

Farin indicated that " It seems reasonable to assign high weights in regions where the interpolant is expected to curve sharply" (See [10] and [11]). This claim can be supported by investigating how the changes of the assigned $w_i's$ affect the shape of the resulting rational curve in interpolation (1.1.9). We shall investigate how changes of some assigned weights w_i will affect $\mathbf{L}(u)$ defined by (1.1.9). In this case the definition of (1.1.9) implies that $\mathbf{L}(u_i)$ will not be changed for all i and thus $\mathbf{L}(u)$ will only be changed between the knots.

Consider the parametric cubic curve:

$$x = 5t - 11.5t^2 + 7.5t^3, \qquad (0 \leq t \leq 1),$$

$$y = 2t - t^2 - 0.67t^3,$$

presented in [34]. This curve does not have an inflection point, but has a loop, while its x-coordinate has an inflection point. We cannot in general deduce the geometric characteristics of the parametric curve only from the properties of its individual components. We will investigate how the curvature of the resulting rational curve will be changed when some assigned weights are changed. Focusing on $[u_{j-1}, \ u_{j+1}]$, we will prove that when only one assigned weight w_j increases, the magnitude of the curvature of the resulting rational curve $\mathbf{L}(u)$ will increase at u_j, thus the shape of the resulting rational curve $\mathbf{L}(u)$ will be changed at u_j in such a way that the resulting rational curve will bend more sharply (i.e. the direction of the tangent line changes rapidly) in a neighborhood of u_j; and when only w_j decreases, the magnitude of the curvature of the resulting rational curve $\mathbf{L}(u)$ will decrease at u_j. Thus the shape of the resulting rational curve $\mathbf{L}(u)$ will be changed at u_j in such a way that the resulting rational curve will be more flat in a neighborhood of u_j. We can extend this analysis to consider the effect of changing more than one assigned weight. We will present an application of the rational cubic B-Spline curve interpolation: sometimes we can consider the assigned weights as *tension parameters* in the interpolation to control the tension distribution. We present two typical examples to illustrate how this works.

3.2 The Assigned Weights of Rational Cubic B-Spline Interpolation

First we introduce some notation. Consider (see [10])

$$\frac{d}{du}N_i^n(u) = \frac{n}{\tilde{u}_{n+i-1} - \tilde{u}_{i-1}}N_i^{n-1}(u) - \frac{n}{\tilde{u}_{n+i} - \tilde{u}_{i+1}}N_{i+1}^{n-1}(u), \tag{3.2.1}$$

$$\frac{d}{du}(\sum_{i=0}^{L+n-1} d_i N_i^n(u)) = n\sum_{i=1}^{L+n-1}\frac{\Delta d_{i-1}}{\tilde{u}_{n+i-1} - \tilde{u}_{i-1}}N_i^{n-1}(u). \tag{3.2.2}$$

Let $\Delta\hat{d}_{i-1} = \frac{\Delta d_{i-1}}{\tilde{u}_{n+i-1}-\tilde{u}_{i-1}}$ and $\Delta^2\hat{d}_{i-1} = \Delta\hat{d}_i - \Delta\hat{d}_{i-1}$, we have

$$\frac{d^2}{du^2}(\sum_{i=0}^{L+n-1} d_i N_i^n(u)) = n(n-1)\sum_{i=2}^{L+n-1}\frac{\Delta^2\hat{d}_{i-2}}{\tilde{u}_{n-1+i-1} - \tilde{u}_{i-1}}N_i^{n-2}(u). \tag{3.2.3}$$

Note the knots are $\tilde{u}_0, \ldots, \tilde{u}_{L+2}$ under the above definition and we have a relationship between $0 = u_1 < u_2 < u_3 < \ldots < u_{L-1} < u_L < u_{L+1} = 1$, (where u_1 and u_{L+1} are the knots of $\mathbf{L}(u)$ with multiplicity three), and $\tilde{u}_0, \ldots, \tilde{u}_{L+2}$: $u_1 = \tilde{u}_0$, $u_1 = \tilde{u}_1$, $u_1 = \tilde{u}_2$ and

$$u_i = \tilde{u}_{i+1}, \quad i = 2, \ldots L+1. \tag{3.2.4}$$

For the cubic B-spline and knot u_j we have

$$\frac{d}{du}(\sum_{i=0}^{L+2} d_i N_i^3(u_j)) = 3[\frac{\Delta d_{j-1}}{\tilde{u}_{j+2} - \tilde{u}_{j-1}}N_j^2(u_j) + \frac{\Delta d_j}{\tilde{u}_{j+3} - \tilde{u}_j}N_{j+1}^2(u_j)]. \tag{3.2.5}$$

Since $N_j^1(u_j) = 1$ and $N_i^1(u_j) = 0$ when $i \neq j$, we have

$$\frac{d^2}{du^2}(\sum_{i=0}^{L+2} d_i N_i^3(u_j)) = \frac{6}{\tilde{u}_{j+1} - \tilde{u}_{j-1}}[\frac{\Delta d_j}{\tilde{u}_{j+3} - \tilde{u}_j} - \frac{\Delta d_{j-1}}{\tilde{u}_{j+2} - \tilde{u}_{j-1}}]. \tag{3.2.6}$$

For equidistant knots, let $\hat{N} = N_i^2(u_i) = N_{i+1}^2(u_i)$ for all i, and let $\delta\tilde{u} = \tilde{u}_{i+1} - \tilde{u}_i$ which are equal for all i. We have

$$\frac{d}{du}(\sum_{i=0}^{L+2} d_i N_i^3(u_j)) = \frac{\hat{N}}{\delta\tilde{u}}(d_{j+1} - d_{j-1}), \tag{3.2.7}$$

$$\frac{d^2}{du^2}(\sum_{i=0}^{L+2} d_i N_i^3(u_j)) = \frac{1}{(\delta\tilde{u})^2}\Delta^2 d_{j-1}. \tag{3.2.8}$$

Let $\mathbf{L}(u) = ((\mathbf{L}(u))_1, (\mathbf{L}(u))_2)$. The signed curvature of $\mathbf{L}(u)$ is defined by (see [10])

$$\kappa(u) = \frac{\frac{d^2}{du^2}(\mathbf{L}(u))_1 \frac{d}{du}(\mathbf{L}(u))_2 - \frac{d^2}{du^2}(\mathbf{L}(u))_2 \frac{d}{du}(\mathbf{L}(u))_1}{[(\frac{d}{du}(\mathbf{L}(u))_1)^2 + (\frac{d}{du}(\mathbf{L}(u))_2)^2]^{3/2}}. \tag{3.2.9}$$

We first investigate one simple case: we only change one assigned weight w_j by $\delta w_j > 0$ to see how this change affects the resulting curve. Although the changes of $(\sum_{i=0}^{L+2} v_i N_i^3(u))$ defined by (1.1.11), $(\sum_{i=0}^{L+2} v_i \mathbf{d}_i N_i^3(u))$ defined by (1.1.12) and $\mathbf{L}(u)$ defined by (1.1.9) are global, when only w_j changes, the effect of this change decays exponentially (See [30]). Thus we focus on the changes of $\mathbf{L}(u)$ in $(u_{j-1},\ u_{j+1})$. Now for equidistant knots, we have the following Lemma:

Lemma 3.1 *Assume that the resulting weight function defined by (1.1.11) remains positive, when w_j changes. For an equidistant parametrization, if w_j is increased by $\delta w_j > 0$, where $j \neq 1, or\ L+1$, then the magnitude of the curvature of $\mathbf{L}(u)$ will increase at u_j. If w_i is decreased by $\delta w_j > 0$, then magnitude of the curvature of $\mathbf{L}(u)$ will decrease at u_j. The sign of the curvature of $\mathbf{L}(u)$ at u_j remains the same, when w_j changes*

Proof. First we consider the following problem:

Problem 1: for given data points $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_{L+1})$, and weights $W = (w_1, \ldots w_{L+1})$, let $\hat{v}(u) = \sum_{i=0}^{L+2} v_i N_i^3(u)$ be defined by (1.1.11) and let $\mathbf{y}(u) = \sum_{i=0}^{L+2} v_i \mathbf{d}_i N_i^3(u)$ be defined by (1.1.12), and let $\mathbf{L}(u) = \frac{\mathbf{y}(u)}{\hat{v}(u)}$ defined by (1.1.9). We have (see [10])

$$\frac{d}{du}\mathbf{L}(u) = \frac{1}{\hat{v}(u)}[\frac{d}{du}\mathbf{y}(u) - \frac{d}{du}\hat{v}(u)\mathbf{L}(u)], \tag{3.2.10}$$

$$\frac{d^2}{du^2}\mathbf{L}(u) = \frac{1}{\hat{v}(u)}[\frac{d^2}{du^2}\mathbf{y}(u) - 2\frac{d}{du}\hat{v}(u)\frac{d}{du}\mathbf{L}(u) - \frac{d^2}{du^2}(\hat{v}(u))\mathbf{L}(u)]. \tag{3.2.11}$$

Note that

$$(-2\frac{d}{du}\hat{v}(u)\frac{d}{du}\mathbf{L}(u))_1(\frac{d}{du}\mathbf{L}(u))_2 - (-2\frac{d}{du}\hat{v}(u)\frac{d}{du}\mathbf{L}(u))_2(\frac{d}{du}\mathbf{L}(u))_1 = 0$$

and $\hat{v}(u_i) = w_i$. Let

$$nu(u_j) = \frac{1}{w_j^2}[(\frac{d^2}{du^2}\mathbf{y}(u_j) - \frac{d^2}{du^2}\hat{v}(u_j)\mathbf{L}(u_j))_1(\frac{d}{du}\mathbf{y}(u_j) - \frac{d}{du}\hat{v}(u_j)\mathbf{L}(u_j))_2$$
$$- (\frac{d^2}{du^2}\mathbf{y}(u_j) - \frac{d^2}{du^2}\hat{v}(u_j)\mathbf{L}(u_j))_2(\frac{d}{du}\mathbf{y}(u_j) - \frac{d}{du}\hat{v}(u_j)\mathbf{L}(u_j))_1]$$

and

$$de(u_j) = \frac{1}{w_j^3}[(\frac{d}{du}\mathbf{y}(u_j) - \frac{d}{du}\hat{v}(u_j)\mathbf{L}(u_j))_1^2 + (\frac{d}{du}\mathbf{y}(u_j) - \frac{d}{du}\hat{v}(u_j)\mathbf{L}(u_j))_2^2]^{3/2}$$

Then, by (3.2.10) and (3.2.11), we have

$$\kappa(u_j) = \frac{nu(u_j)}{de(u_j)}. \tag{3.2.12}$$

When we increase w_j by $\delta w_j > 0$, we can consider the resulting interpolation as the combination of the following two problems: one is problem 1, and the other, which we shall call problem 2, is the rational cubic B-spline interpolation with all $w_i = 0$ except $w_j = \delta w_j$. We denote this rational cubic B-spline interpolation as $\delta\hat{v}(u) = \sum_{i=0}^{L+2} \delta v_i N_i^3(u)$ defined by (1.1.11) and $\delta\mathbf{y}(u) = \sum_{i=0}^{L+2} \delta v_i \delta\mathbf{d}_i N_i^3(u)$ defined by (1.1.12) and the result of the whole interpolation problem as $\hat{\mathbf{L}}(u) = \frac{\mathbf{y}(u)+\delta\mathbf{y}(u)}{\hat{v}(u)+\delta\hat{v}(u)}$ defined by (1.1.9).

From the hypothesis of this theorem it follows that $(\hat{v}(u) + \delta\hat{v}(u)) > 0$, i.e., $\hat{\mathbf{L}}(u)$ has a second derivative.

First from (3.2.7) we have

$$\frac{d}{du}\delta\hat{v}(u_j) = \frac{\hat{N}}{\delta\tilde{u}}(\delta v_{j+1} - \delta v_{j-1}).$$

From (3.2.8) we have

$$\frac{d^2}{du^2}\delta\hat{v}(u_j) = \frac{1}{\delta(\tilde{u})^2}(\delta v_{j+1} - 2\delta v_j + \delta v_{j-1}).$$

Also we can consider the interpolation (1.1.12) in this case as the following problem: let all $w_i\mathbf{x}_i = \mathbf{0}$ except $w_j\mathbf{x}_j = \delta w_j\mathbf{x}_j$. If $j \neq 2$ or 3, we have $b_1 = 0$ from (1.1.15) and $b_1\mathbf{b}_1 = \mathbf{0}$ from (1.1.17), i.e., we have $b_1\mathbf{b}_1 = b_1\mathbf{x}_j$. If $j = 2$ We have $b_1 = \frac{\Delta_1+\Delta_2}{\Delta_2}\delta w_j$ from (1.1.15) and $b_1\mathbf{b}_1 = \frac{\Delta_1+\Delta_2}{\Delta_2}\delta w_j\mathbf{x}_j$ from (1.1.17), i.e., we have $b_1\mathbf{b}_1 = b_1\mathbf{x}_j$. Similarly we have $b_1\mathbf{b}_1 = b_1\mathbf{x}_j$, when $j = 3$, and we have $b_{3L-1}\mathbf{b}_{3L-1} = b_{3L-1}\mathbf{x}_j$.

Now suppose that for $i \neq j$ $w_i = 0$ and $w_j = \delta w_j$, and for $i \neq j$ $w_i\mathbf{x}_i = \mathbf{0}$ and $w_j\mathbf{x}_j = \delta w_j\mathbf{x}_j$, $b_1\mathbf{b}_1 = b_1\mathbf{x}_j$ and $b_{3L-1}\mathbf{b}_{3L-1} = b_{3L-1}\mathbf{x}_j$. We can obtain the result of (1.1.12) by multiplying both sides of (1.1.11) by $\mathbf{x}_j$. Thus we have $\delta\mathbf{d}_i = \mathbf{x}_j$ for all i. Since $\hat{\mathbf{L}}(u_j) = \mathbf{x}_j$, we now have

$$\frac{d}{du}\delta\mathbf{y}(u_j) = \frac{\hat{N}}{\delta\tilde{u}}(\delta\mathbf{d}_{j+1}\delta v_{j+1} - \delta\mathbf{d}_{j-1}\delta v_{j-1})$$

$$= \frac{\hat{N}}{\delta\tilde{u}}(\delta v_{j+1} - \delta v_{j-1})\mathbf{x}_j = \frac{d}{du}\delta\hat{v}(u_j)\hat{\mathbf{L}}(u_j),$$

and

$$\frac{d}{du}(\mathbf{y}(u_j) + \delta\mathbf{y}(u_j)) = \frac{d}{du}\mathbf{y}(u_j) + \frac{d}{du}\delta\hat{v}(u_j)\hat{\mathbf{L}}(u_j). \tag{3.2.13}$$

Also we have

$$\frac{d^2}{du^2}\delta\mathbf{y}(u_j) = \frac{1}{\delta(\tilde{u})^2}(\delta\mathbf{d}_{j+1}\delta v_{j+1} - 2\delta\mathbf{d}_j\delta v_j + \delta\mathbf{d}_{j-1}\delta v_{j-1})$$

$$= \frac{1}{\delta(\tilde{u})^2}(\delta v_{j+1} - 2\delta v_j + \delta v_{j-1})\mathbf{x}_j = \frac{d^2}{du^2}\delta\hat{v}(u_j)\hat{\mathbf{L}}(u_j),$$

$$\frac{d^2}{du^2}(\mathbf{y}(u_i) + \delta\mathbf{y}(u_j)) = \frac{d^2}{du^2}\mathbf{y}(u_i) + \frac{d^2}{du^2}\delta\hat{v}(u_j)\hat{\mathbf{L}}(u_j). \tag{3.2.14}$$

Now $\hat{v}(u_j) = w_j$, $\delta\hat{v}(u_j) = \delta w_j$ and $\hat{\mathbf{L}}(u_j) = \mathbf{L}(u_j)$. It follows that

$$\frac{d}{du}\hat{\mathbf{L}}(u_j) = \frac{1}{\hat{v}(u_j) + \delta\hat{v}(u_j)}[\frac{d}{du}(\mathbf{y}(u_j) + \delta\mathbf{y}(u_j)) - \frac{d}{du}(\hat{v}(u_j) + \delta\hat{v}(u_j))\hat{\mathbf{L}}(u_j)]$$

$$= \frac{1}{(w_j + \delta w_j)}[\frac{d}{du}\mathbf{y}(u_j) - \frac{d}{du}\hat{v}(u_j)\mathbf{L}(u_j)]. \tag{3.2.15}$$

Thus,

$$[(\frac{d}{du}(\hat{\mathbf{L}}(u))_1)^2 + (\frac{d}{du}(\hat{\mathbf{L}}(u))_2)^2]^{3/2}$$

$$= \frac{1}{(w_j + \delta w_j)^3}[(\frac{d}{du}\mathbf{y}(u_j) - \frac{d}{du}\hat{v}(u_j)\mathbf{L}(u_j))_1^2 + (\frac{d}{du}\mathbf{y}(u_j) - \frac{d}{du}\hat{v}(u_j)\mathbf{L}(u_j))_2^2]^{3/2}$$

$$= \frac{w_j^3}{(w_j + \delta w_j)^3}de(u_j).$$

Also we have

$$\frac{d^2}{du^2}\hat{\mathbf{L}}(u_j) = \frac{1}{\hat{v}(u_j)+\delta\hat{v}(u_j)}[\frac{d^2}{du^2}(\mathbf{y}(u_j)+\delta\mathbf{y}(u_j)) - 2\frac{d}{du}(\hat{v}(u_j)+\delta\hat{v}(u_j))\frac{d}{du}\hat{\mathbf{L}}(u_j)$$

$$-\frac{d^2}{du^2}(\hat{v}(u_j)+\delta\hat{v}(u_j))\hat{\mathbf{L}}(u_j)]$$

$$= \frac{1}{w_j+\delta w_j}[\frac{d^2}{du^2}(\mathbf{y}(u_j) - 2\frac{d}{du}(\hat{v}(u_j)+\delta\hat{v}(u_j))\frac{d}{du}\hat{\mathbf{L}}(u_j) - \frac{d^2}{du^2}\hat{v}(u_j)\hat{\mathbf{L}}(u_j)].$$

Since

$$(-2\frac{d}{du}(\hat{v}(u)+\delta\hat{v}(u))\frac{d}{du}\hat{\mathbf{L}}(u))_1(\frac{d}{du}\hat{\mathbf{L}}(u))_2-(-2\frac{d}{du}(\hat{v}(u)+\delta\hat{v}(u))\frac{d}{du}\hat{\mathbf{L}}(u))_2(\frac{d}{du}\hat{\mathbf{L}}(u))_1$$

$$=0$$

and $\hat{\mathbf{L}}(u_j) = \mathbf{L}(u_j)$, we have

$$\frac{d^2}{du^2}(\hat{\mathbf{L}}(u_j))_1\frac{d}{du}(\hat{\mathbf{L}}(u_j))_2 - \frac{d^2}{du^2}(\hat{\mathbf{L}}(u_j))_2\frac{d}{du}(\hat{\mathbf{L}}(u_j))_1 = \frac{w_j^2}{(w_j+\delta w_j)^2}nu(u_j).$$

Now we have the curvature, denoted by $\hat{\kappa}(u_j)$, of the resulting curve as follows:

$$\hat{\kappa}(u_j) = (\frac{w_j+\delta w_j}{w_j})\frac{nu(u_j)}{de(u_j)} = (1+\frac{\delta w_j}{w_j})\kappa(u_j). \tag{3.2.16}$$

From (3.2.16) it follows that the magnitude of the curvature will increase by $\frac{\delta w_j}{w_j}\kappa(u_j)$ and the curvature sign is unchanged at u_j, when the weight w_j increases by δw_j. The proof of the remaining part of this Lemma is similar to the above argument and thus is omitted.

From the above proof we know that, when we only let w_j be increased by δw_j, from [30] the resulting interpolant of problem 2 is an oscillatory function such that the sign of δv_j is the same as the sign of δw_j at $u = u_j$ and the interpolant decays exponentially, and changes its sign alternatively per data point as u moves away from u_j in (1.1.11).

Now for a general parametrization, we have the following:

Theorem 3.1 *Assume that the resulting weight function defined by (1.1.11) remains positive, when w_j changes. For a general parametrization, if w_j is increased by $\delta w_j > 0$, where $j \neq 1$, or $L+1$, then the magnitude of the curvature of $\mathbf{L}(u)$ will increase at u_j. If w_j is decreased by $\delta w_j > 0$, then the magnitude of the curvature of $\mathbf{L}(u)$ will decrease at u_j. The sign of curvature of $\mathbf{L}(u)$ at u_j remains the same, when w_j changes.*

Proof. We only discuss problem 2 in the proof of Lemma 3.1 in this case and we use the same notation as there. First we have by (3.2.5)

$$\frac{d}{du}\delta\hat{v}(u_j) = 3[\frac{\Delta\delta v_{j-1}}{\tilde{u}_{j+2} - \tilde{u}_{j-1}} N_j^2(u_j) + \frac{\Delta\delta v_j}{\tilde{u}_{j+3} - \tilde{u}_j} N_{j+1}^2(u_j)]. \tag{3.2.17}$$

As in the proof of Lemma 3.1 we consider the interpolation (1.1.12) in this case as the following problem: let $w_i\mathbf{x}_i = \mathbf{0}$ for all $i \neq j$ and $w_j\mathbf{x}_j = \delta w_j\mathbf{x}_j$. We have $b_1\mathbf{b}_1 = b_1\mathbf{x}_j$ and $b_{3L-1}\mathbf{b}_{3L-1} = b_{3L-1}\mathbf{x}_j$. Now assume that $w_i = 0$ for all $i \neq j$ and $w_j = \delta w_j$, and assume that $w_i\mathbf{x}_i = \mathbf{0}$ for all $i \neq j$ and $w_j\mathbf{x}_j = \delta w_j\mathbf{x}_j$, $b_1\mathbf{b}_1 = b_1\mathbf{x}_j$ and $b_{3L-1}\mathbf{b}_{3L-1} = b_{3L-1}\mathbf{x}_j$. We can obtain the result of (1.1.12) by multiplying both sides of (1.1.11) by $\mathbf{x}_j$. Thus $\delta\mathbf{d}_i = \mathbf{x}_j$ for all i, and

$$\frac{d}{du}\delta\mathbf{y}(u_j) = 3[\frac{\Delta(\delta v_{j-1}\delta\mathbf{d}_{j-1})}{\tilde{u}_{j+2} - \tilde{u}_{j-1}} N_j^2(u_j) + \frac{(\Delta\delta v_j\delta\mathbf{d}_j)}{\tilde{u}_{j+3} - \tilde{u}_j} N_{j+1}^2(u_j)]$$

$$= 3[\frac{\Delta\delta v_{j-1}\mathbf{x}_j}{\tilde{u}_{j+2}-\tilde{u}_{j-1}}N_j^2(u_j) + \frac{\Delta\delta v_j\mathbf{x}_j}{\tilde{u}_{j+3}-\tilde{u}_j}N_{j+1}^2(u_j)]$$
$$= 3[\frac{\Delta\delta v_{j-1}}{\tilde{u}_{j+2}-\tilde{u}_{j-1}}N_j^2(u_j) + \frac{\Delta\delta v_j}{\tilde{u}_{j+3}-\tilde{u}_j}N_{j+1}^2(u_j)]\mathbf{x}_j.$$

Now (3.2.17) and $\mathbf{x}_j = \hat{\mathbf{L}}(u_j)$ follows that

$$\frac{d}{du}\delta\mathbf{y}(u_j) = \frac{d}{du}\delta\hat{v}(u_j)\hat{\mathbf{L}}(u_j).$$

Note $\hat{v}(u_j) = w_j$, $\delta\hat{v}(u_j) = \delta w_j$ and $\hat{\mathbf{L}}(u_j) = \mathbf{L}(u_j)$, thus

$$\frac{d}{du}\hat{\mathbf{L}}(u_j) = \frac{1}{\hat{v}(u_j)+\delta\hat{v}(u_j)}[\frac{d}{du}(\mathbf{y}(u_j)+\delta\mathbf{y}(u_j)) - \frac{d}{du}(\hat{v}(u_j)+\delta\hat{v}(u_j))\hat{\mathbf{L}}(u_j)]$$
$$= \frac{1}{(w_j+\delta w_j)}[\frac{d}{du}\mathbf{y}(u_j) - \frac{d}{du}\hat{v}(u_j)\hat{\mathbf{L}}(u_j)]. \tag{3.2.18}$$

We have following by (3.2.6)

$$\frac{d^2}{du^2}\delta\hat{v}(u_j) = \frac{6}{\tilde{u}_{j+1}-\tilde{u}_{j-1}}[\frac{\Delta\delta v_j}{\tilde{u}_{j+3}-\tilde{u}_j} - \frac{\Delta\delta v_{j-1}}{\tilde{u}_{j+2}-\tilde{u}_{j-1}}]. \tag{3.2.19}$$

Now from (3.2.6) and $\mathbf{x}_j = \hat{\mathbf{L}}(u_j)$ it follows that

$$\frac{d^2}{du^2}\delta\mathbf{y}(u_j) = \frac{6}{\tilde{u}_{j+1}-\tilde{u}_{j-1}}[\frac{\Delta(\delta v_j\delta\mathbf{d}_j)}{\tilde{u}_{j+3}-\tilde{u}_j} - \frac{\Delta(\delta v_j\delta\mathbf{d}_{j-1})}{\tilde{u}_{j+2}-\tilde{u}_{j-1}}]$$
$$= \frac{6}{\tilde{u}_{j+1}-\tilde{u}_{j-1}}[\frac{\Delta\delta v_j}{\tilde{u}_{j+3}-\tilde{u}_j} - \frac{\Delta\delta v_{j-1}}{\tilde{u}_{j+2}-\tilde{u}_{j-1}}]\mathbf{x}_j$$
$$= \frac{6}{\tilde{u}_{j+1}-\tilde{u}_{j-1}}[\frac{\Delta v_j}{\tilde{u}_{j+3}-\tilde{u}_j} - \frac{\Delta v_{j-1}}{\tilde{u}_{j+2}-\tilde{u}_{j-1}}]\hat{\mathbf{L}}(u_j)$$
$$= \frac{d^2}{du^2}\delta\hat{v}(u_j)\hat{\mathbf{L}}(u_j). \tag{3.2.20}$$

Thus,

$$\frac{d^2}{du^2}\hat{\mathbf{L}}(u_j) = \frac{1}{\hat{v}(u_j)+\delta\hat{v}(u_j)}[\frac{d^2}{du^2}(\mathbf{y}(u_j)+\delta\mathbf{y}(u_j)) - 2\frac{d}{du}(\hat{v}(u_j)+\delta\hat{v}(u_j))\frac{d}{du}\hat{\mathbf{L}}(u_j)$$

$$-\frac{d^2}{du^2}(\hat{v}(u_j)+\delta\hat{v}(u_j))\hat{\mathbf{L}}(u_j)]$$
$$=\frac{1}{w_j+\delta w_j}[\frac{d^2}{du^2}(\mathbf{y}(u_j)-2\frac{d}{du}(\hat{v}(u_j)+\delta\hat{v}(u_j))\frac{d}{du}\hat{\mathbf{L}}(u_j)-\frac{d^2}{du^2}\hat{v}(u_j)\hat{\mathbf{L}}(u_j)].$$

The proof of the remaining part of this theorem is exactly as in the proof of Lemma 3.1 and thus is omitted.

Since $\hat{\kappa}(u)$ is a continuous function at u_j, $\hat{\kappa}(u)$ will increase in a neighborhood of u_j, when w_j increases. Thus the shape of $\mathbf{L}(u)$ will be changed at u_j in such a way that the resulting curve will bend more sharply (i.e. the change of the direction of the tangent line will be more rapid) in a neighborhood of u_j. And $\hat{\kappa}(u)$ will decrease in a neighborhood of u_j, when w_j decreases. Thus the shape of $\mathbf{L}(u)$ will be changed at u_j in such a way that the resulting curve will be more flat in a neighborhood of u_j.

If more than one weight changes, we can apply the above result one weight at a time to see the affects of their changes. We now have revealed the relationship between the shape of the resulting curve and the weights to be interpolated by the above theorems and we have confirmed Farin's conjecture: " it seems reasonable to assign high weights in regions where the interpolant is expected to curve sharply."

Farin further observed that "there is a limit to the assignment of weights: if all of them are very high, this will not have a significant effect, since a common factor in all weights will simply cancel out." (See [10] and [11]). This can be observed in the following way: if all of the weights are high, for example if $a > 0$ is added to each w_i, the v_i will be $(v_i + a)$ for all i from the linear precision of B-spline interpolation. The basis functions in (1.1.20) became $\frac{(v_i+a)N_i^3(u)}{\sum_{i=0}^{L+2}(v_i+a)N_i^3(u)} \rightarrow N_i^3(u)$, when $a \rightarrow \infty$, i.e (1.1.20) becomes an integral cubic B-spline.

In Figure 3.1 and Figure 3.2, all the assigned weights of the middle curve are 1, the sharp curve has the assigned weight increased at the middle point by 0.4, the flat curve has the assigned weight decreased at the middle point by 0.4. Figure 3.3, Figure 3.4 and Figure 3.5 are their curvature plots respectively.

Figure 3.1 illustrates how the shapes change when the assigned weights change. The middle curve has the assigned weights $w_i = 1$ for all i. When the assigned weight increases at the middle point by 0.4, the resulting rational curve bends more sharply near the middle point. When the assigned weight decreases at the middle point by 0.4, the resulting rational curve becomes more linear near the middle point. Figure 3.2 shows the 3D preimage of the above curve and we can see that the 3D preimage of the resulting rational curves interpolates the data and the corresponding assigned weights. Figure 3.3 shows the curvature plot of the resulting rational curve with assigned weights $w_i = 1$ for all i. Figure 3.4 illustrates that the magnitude of the curvature of the resulting rational curve increases in a neighborhood of the middle point and the curvature sign does not changed, when the assigned weight at the middle point increases by 0.4. Figure 3.5 illustrates that the magnitude of the curvature of the resulting rational curve decreases in a neighborhood of the middle point and keeps the curvature sign, when the assigned weight at the middle point decreases by 0.4.

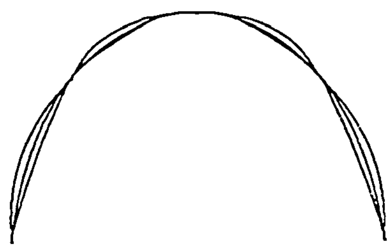

Figure 3.1: Affect of changing the assigned weights on the shape of the curve.

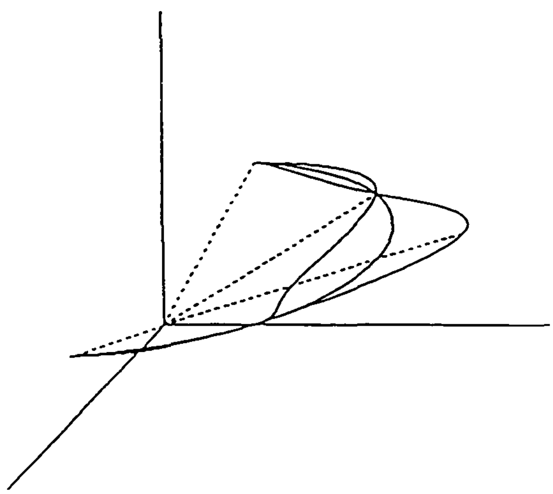

Figure 3.2: The $3D$ preimage of Figure 3.1.

Figure 3.3: Curvature plot: middle assigned weight is 1.

Figure 3.4: Curvature plot: middle assigned weight is 1.4.

Figure 3.5: Curvature plot: middle assigned weight is 0.6.

3.3 Using The Assigned Weights of Rational Cubic B-Spline Interpolation as Tension Parameters

If we change assigned weights and do not change the data, $\mathbf{L}(u_i)$ will not be changed and $\mathbf{L}(u)$ will only be changed between the knots after rational cubic B-Spline interpolation. Furthermore, since the curvatures of $\mathbf{L}(u_i)$ will be changed in the way indicated in Lemma 3.1 and Theorem 3.1, we can use the assigned weights as *tension parameters* to control the tension distribution.

Even convex data points (i.e., points such that the polygon formed by joining adjacent data points with straight lines is convex) may produce unwanted inflection points on the corresponding interpolatory cubic spline, as illustrated in Figure 3.6. The spline function under tension, which has the convexity preserving property, i.e. the resulting curve will preserve the convexity of the given data, was developed by many authors (see [5], [14],[15], [24], [31] and [33]) to deal with this problem. In general, the spline function under tension is not in C^2.

From Figure 3.6 and Figure 3.7 (the curvature plot of Figure 3.6), we know that we need to increase the magnitude of the curvature at points on either side of the middle point and decrease the magnitude of the curvature at the middle point of the curve. We let the assigned weights be 2 at the points on either side of the middle point and let the other assigned weights be 1. Figure 3.8 is the graph of the resulting curve and Figure 3.9 is its curvature plot. This change of assigned weights removes two inflection points of Figure 3.7, preserves the data convexity, and produces a curvature of 0 in a neighborhood of the middle point in Figure 3.9. Comparing their graphs in Figure 3.8 and in Figure 3.6, we can observe that comparing the resulting curve to a cubic spline interpolant, the former is more linear in a neighborhood of the middle point, and the direction of its tangent line changes more rapidly in a neighborhood of the points on either side of the middle point. The resulting curve is C^2.

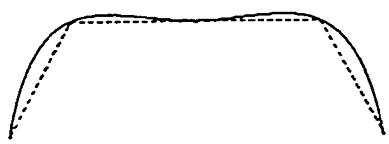

Figure 3.6: Original cubic B-spline interpolant.

Figure 3.7: Curvature plot of Figure 3.6.

Figure 3.8: $\mathbf{L}(u)$. Assigned weights are $1, 2, 1, 2$ and 1.

Figure 3.9: Curvature plot of Figure 3.8.

We next consider a second example from [10]. A chord length parameterization, whose parameterization is defined by

$$\frac{\Delta_i}{\Delta_{i+1}} = \frac{\|\Delta \mathbf{x}_i\|}{\|\Delta \mathbf{x}_{i+1}\|},$$

frequently produces better results than uniform knot spacing. As opposed to the uniform parameterization, it has been proven [7] that chord length parameterization (in connection with natural end conditions) cannot produce curves with corners. Farin shows that the chord length parameterization (in connection with Bessel end conditions) yields the "roundest" curve, exhibiting the inflection points in Figure 3.10 and its curvature plot Figure 3.11.

From Figure 3.10 and Figure 3.11 (the curvature plot of Figure 3.10) we know that we need to decrease the magnitudes of the curvature at both middle points (the sixth and the seventh point from the left). We let the weight be 0.2 at the sixth point and let the assigned weight be 0.19 at the seventh point, and let the other assigned weights be 1. Figure 3.12 is the graph of the resulting curve and Figure 3.13 is its curvature plot. Comparing their graphs (Figure 3.12 and Figure 3.10), we can observe that the resulting curve in Figure 3.12 is a better fit to the data points than the cubic spline in Figure 3.10 in the sense that the resulting curve in Figure 3.12 bends more sharply between the two middle points and is more linear elsewhere. The curvature plot of Figure 3.12 is better than the curvature plot of Figure 3.10, since the former exhibits fewer inflection points and is smoother than the latter. Thus the curve in Figure 3.12 is fairer than the curve in Figure 3.10. In Figure 3.12 we have produced a curve that has a corner unlike the graph of Figure 3.10. The resulting curve in Figure 3.12 is C^2.

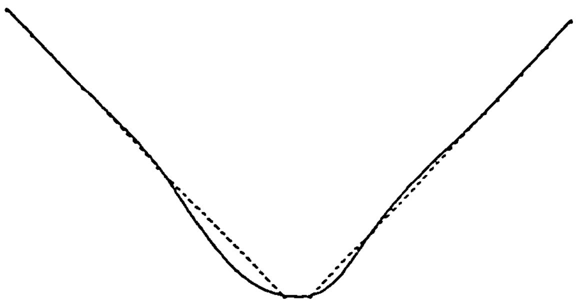

Figure 3.10: Original cubic B-spline interpolant.

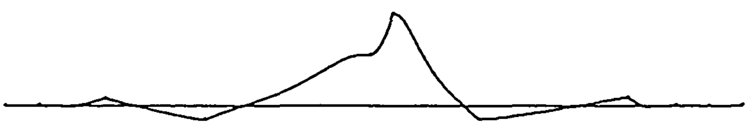

Figure 3.11: Curvature plot of Figure 3.10.

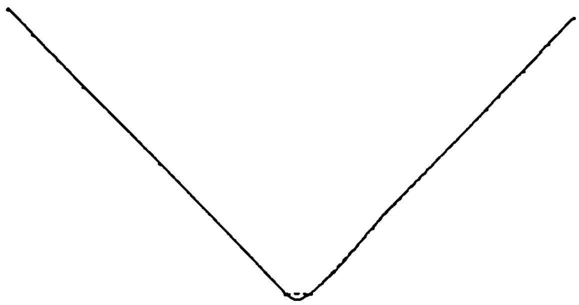

Figure 3.12: $\mathbf{L}(u)$. Assigned weights are 1, 1, 1, 1, 1, 0.2, 0.19, 1, 1, 1, 1, 1 and 1.

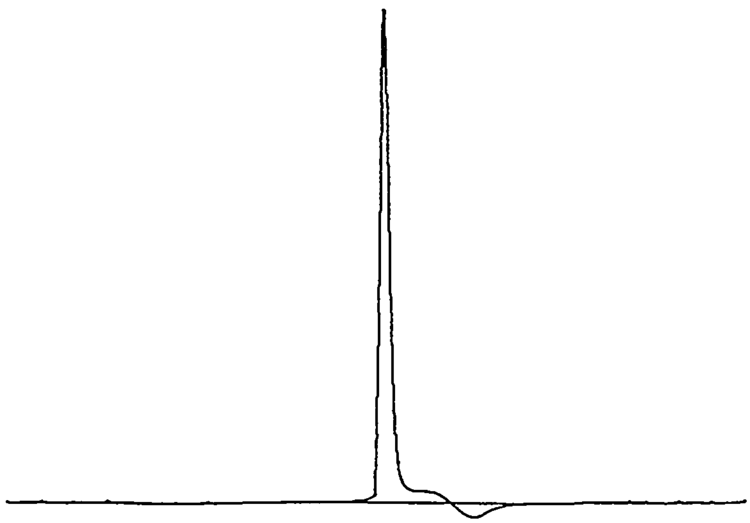

Figure 3.13: Curvature plot of Figure 3.12.

Chapter 4

Sufficient Conditions to Guarantee the Positivity of the Weights of a Rational Interpolant

4.1 Introduction

In order to use C^2 rational cubic B-spline interpolation as in Chapter 1 to produce a curve with a desired shape, a designer may assign relatively high weights at some points. But from [20] we know that if we increase the assigned weight w_j by $\delta w_j > 0$ at the point u_j, the rational weights of the interpolant at the points on either side of u_j will decrease by δv_{j-1} and δv_{j+1} respectively. If δw_j is too large, one or both of these rational weights may become negative.

In the first example of Section 3.3 we try to obtain a better result by increasing the assigned weights to 7 at the points on either side of the middle point and letting the other weights be 1. The rational weight at the middle point of the resulting curve now becomes -4. Figure 4.1 shows that its graph produced extra loops at the points on either side of the middle point. Figure 4.2 (the curvature plot of the curve in Figure 4.1) shows that singularities are created and new inflection points occur. All of these problem are caused by the negative weights of the resulting rational interpolant. Figure 4.3 shows the preimage of the curves in Figures 3.6 and 4.1. We can see that the perspective projection of the preimage of the curve in Figure 4.1 will have loops and cusps. A discussion about loops and the cusps of a cubic spline can be found in [34]. In this Chapter we present sufficient conditions on the assigned weights which ensure the positivity of the weights of the resulting rational interpolant.

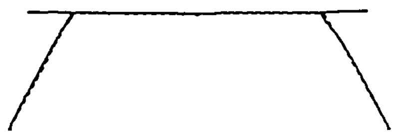

Figure 4.1: $\mathbf{L}(u)$. Assigned weights are $1, 7, 1, 7$ and 1.

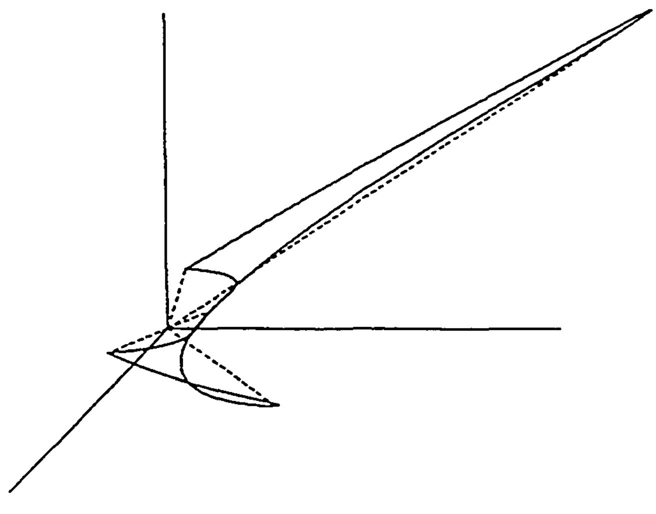

Figure 4.2: Curvature plot of Figure 4.1.

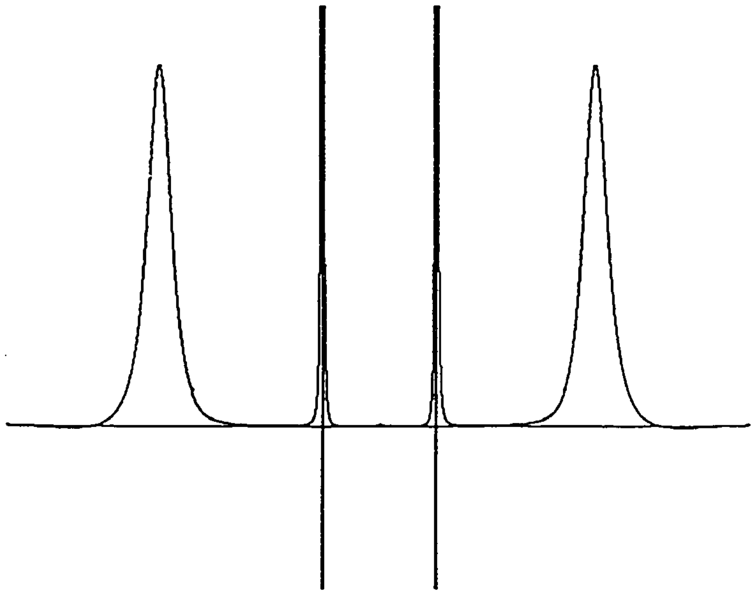

Figure 4.3: The $3D$ preimage of Figure 4.1.

4.2 Sufficient Conditions

We want to explore the relation between the assigned weights, $\{w_i\}$, and the weights of the corresponding rational interpolant, $\{v_i\}$. We start with (1.1.11). Given assigned weights, $w_1, \ldots, w_{L+1}$, associated with given data $\mathbf{x}_1, \ldots, \mathbf{x}_{L+1}$, the rational weights $v_0, \ldots, v_{L+2}$ are the solution of the following system of linear equations:

$$\begin{cases} v_0 = w_1, & v_{L+2} = w_{L+1}, \\ v_1 = b_1, & v_{L+1} = b_{3L-1}, \\ \sum_{i=0}^{L+2} v_i N_i^3(u) = w_i & j = 2, \ldots, L. \end{cases} \tag{4.2.1}$$

However, (4.2.1) can be reformulated as (cf. (9.7) in [10])

$$\begin{pmatrix} 1 & 0 & 0 & & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \alpha_L & \beta_L & \gamma_L \\ 0 & & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_L \\ v_{L+1} \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_L \\ r_{L+1} \end{pmatrix}, \tag{4.2.2}$$

where the coefficient matrix is tridiagonal and

$r_1 = b_1$, $r_i = (\Delta_{i-1} + \Delta_i)w_i$, $r_{L+1} = b_{3L-1}$,

$\Delta_i = \Delta u_i = u_{i+1} - u_i$, $u_1 = 0$, $u_{L+1} = 1$, $\Delta_0 = \Delta_{L+1} = 0$,

$v_0 = w_1$, $v_{L+2} = w_{L+1}$,

$$\begin{aligned} \alpha_i &= \tfrac{\Delta_i^2}{\Delta_{i-2}+\Delta_{i-1}+\Delta_i}, \\ \beta_i &= \tfrac{\Delta_i(\Delta_{i-2}+\Delta_{i-1})}{\Delta_{i-2}+\Delta_{i-1}+\Delta_i} + \tfrac{\Delta_{i-1}(\Delta_i+\Delta_{i+1})}{\Delta_{i-1}+\Delta_i+\Delta_{i+1}}, \\ \gamma_i &= \tfrac{\Delta_{i-1}^2}{\Delta_{i-1}+\Delta_i+\Delta_{i+1}}, \\ &\text{for } i = 2, \ldots, L. \end{aligned} \tag{4.2.3}$$

Lemma 4.1 *If the parameter values $u_1 \ldots u_{L+1}$ satisfy the conditions*

$$\Delta_{i-1} < (\Delta_i + \Delta_{i+1}) \textit{ and } \Delta_i < (\Delta_{i-2} + \Delta_{i-1}) \quad \textit{for } i = 2, \ldots, L, \tag{4.2.4}$$

then

$$\beta_i > \alpha_i + \gamma_i \quad \textit{for } i = 2, \ldots, L.$$

Proof. For $i = 2, \ldots, L$,

$$\begin{aligned} \beta_i - \alpha_i + \gamma_i &= \frac{\Delta_i(\Delta_{i-2} + \Delta_{i-1})}{\Delta_{i-2} + \Delta_{i-1} + \Delta_i} + \frac{\Delta_{i-1}(\Delta_i + \Delta_{i+1})}{\Delta_{i-1} + \Delta_i + \Delta_{i+1}} \\ &\quad - (\frac{\Delta_i^2}{\Delta_{i-2} + \Delta_{i-1} + \Delta_i} + \frac{\Delta_{i-1}^2}{\Delta_{i-1} + \Delta_i + \Delta_{i+1}}) \\ &= \frac{\Delta_i(\Delta_{i-2} + \Delta_{i-1} - \Delta_i)}{\Delta_{i-2} + \Delta_{i-1} + \Delta_i} + \frac{\Delta_{i-1}(\Delta_i + \Delta_{i+1} - \Delta_{i-1})}{\Delta_{i-1} + \Delta_i + \Delta_{i+1}}. \end{aligned}$$

Then $\Delta_{i-2}+\Delta_{i-1}+\Delta_i > 0$, $\Delta_{i-1}+\Delta_i+\Delta_{i+1} > 0$, and $\Delta_i > 0$, for $i = 2, \ldots, L$. It follows from (4.2.4) that $\beta_i - \alpha_i + \gamma_i > 0$, i.e., $\beta_i > \alpha_i + \gamma_i$, for $i = 2, \ldots, L$.

Definition 4.1 *A Matrix $A = (a_{ij})_{n \times n}$ is said to be a positive matrix, (denoted by $A > 0$), if all $a_{ij} > 0$ for $1 \leq i, j, \leq n$.*

Definition 4.2 *For $A = (a_{ij})$, let $\|A\|_\infty$ denote the row norm of A defined by*

$$\|A\|_\infty = \max_i \sum_{k=1}^{n} | a_{ik} | .$$

The following lemma is well-known (see [34] p. 37).

Lemma 4.2 *If $\|A\|_\infty < 1$ and $A > 0$, then $(I - A^2)^{-1}$ exists and $(I - A^2)^{-1} > 0$.*

Theorem 4.1 *If the parameter values $u_1 \dots u_{L+1}$ satisfy (4.2.4), $v_0 = w_1 > 0$, $v_{L+2} = w_{L+1} > 0$, $b_1 > 0$, $b_{3L-1} > 0$,*

$$(\Delta_1 + \Delta_2)w_2 - [b_1\alpha_2 + (\Delta_1 + \Delta_2)w_3\tfrac{\gamma_2}{\beta_2}] > 0,$$

$$(\Delta_{i-1} + \Delta_i)w_i - [(\Delta_{i-2} + \Delta_{i-1})w_{i-1}\tfrac{\alpha_i}{\beta_i} + (\Delta_i + \Delta_{i+1})w_{i+1}\tfrac{\gamma_i}{\beta_i}] > 0$$

$$i = 2, \dots, L-1, \tag{4.2.5}$$

and

$$(\Delta_{L-1} + \Delta_L)w_L - [(\Delta_{L-2} + \Delta_{L-1})w_{L-1}\tfrac{\alpha_L}{\beta_L} + b_{3L-1}\gamma_L] > 0,$$

then $v_i > 0$ for $i = 0, \dots, L+2$.

Proof. By Lemma 4.1 for $i = 2, \dots, L$, $\beta_i > 0$. From (4.2.2) it follows that

$$\begin{pmatrix} 1 & 0 & 0 & & 0 \\ \frac{\alpha_2}{\beta_2} & 1 & \frac{\gamma_2}{\beta_2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{\alpha_L}{\beta_L} & 1 & \frac{\gamma_L}{\beta_L} \\ 0 & & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ \frac{v_2}{\beta_2} \\ \vdots \\ \frac{v_L}{\beta_L} \\ v_{L+1} \end{pmatrix} = \begin{pmatrix} r_1 \\ \frac{r_2}{\beta_2} \\ \vdots \\ \frac{r_L}{\beta_L} \\ r_{L+1} \end{pmatrix}. \tag{4.2.6}$$

Let $$A = \begin{pmatrix} 0 & 0 & 0 & & 0 \\ \frac{\alpha_2}{\beta_2} & 0 & \frac{\gamma_2}{\beta_2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{\alpha_L}{\beta_L} & 0 & \frac{\gamma_L}{\beta_L} \\ 0 & & 0 & 0 & 0 \end{pmatrix},$$

$$V = \begin{pmatrix} v_1 \\ \frac{v_2}{\beta_2} \\ \vdots \\ \frac{v_L}{\beta_L} \\ v_{L+1} \end{pmatrix}, \quad F = \begin{pmatrix} r_1 \\ \frac{r_2}{\beta_2} \\ \vdots \\ \frac{r_L}{\beta_L} \\ r_{L+1} \end{pmatrix}.$$

Now (4.2.6) becomes

$$(I + A)V = F. \tag{4.2.7}$$

Thus,

$$(I - A^2)V = (I - A)F. \tag{4.2.8}$$

From (4.2.4) and Lemma 4.1 it follows that

$$\left|\frac{\alpha_i}{\beta_i}\right| + \left|\frac{\gamma_i}{\beta_i}\right| = \frac{\alpha_i + \gamma_i}{\beta_i} < 1, \quad (i = 2, \ldots, L).$$

Also

$$\|A\|_\infty = \max_i \sum_{k=1}^{L+1} | a_{ik} | = \max_{2 \le i \le L} \frac{\alpha_i + \gamma_i}{\beta_i} < 1.$$

Now from Lemma 4.2 it follows that $(I - A^2)^{-1}$ exists and

$$(I - A^2)^{-1} > 0. \tag{4.2.9}$$

So (4.2.8) implies that

$$V = (I - A^2)^{-1}(I - A)F. \tag{4.2.10}$$

If $(I - A)F > 0$, it follows from (4.2.9) and (4.2.10) that $V > 0$, which implies that $v_{i+1} > 0$ for $i = 0, \ldots, L$. Since $v_0 = w_0 > 0, \quad v_{L+2} = w_L > 0$, it suffices to prove $v_i > 0$ for $i = 0, \ldots, L+2$.

Note that $(I - A)F > 0$, if $r_1 > 0, \quad r_{L+1} > 0$ and

$$\frac{r_2}{\beta_2} - [r_1 \frac{\alpha_2}{\beta_2} + r_3 \frac{\gamma_2}{\beta_2^2}] > 0,$$

$$\frac{r_i}{\beta_i} - [r_{i-1} \frac{\alpha_i}{\beta_i^2} + r_{i+1} \frac{\gamma_i}{\beta_i^2}] > 0, \quad i = 3, \ldots, L-1,$$

$$\frac{r_L}{\beta_L} - [r_{L-1} \frac{\alpha_L}{\beta_L^2} + r_{L+1} \frac{\gamma_L}{\beta_L}] > 0.$$

which is equivalent to our assumption that $b_1 > 0, \quad b_{3L-1} > 0$ and (4.2.5). The Theorem is proved

Remark: Condition (4.2.4) of Theorem 4.1 is not as restrictive as it appears to be. Parameter values $U = \{u_i\}$ whose distribution is uniform obey (4.2.4). In nonuniform distributions (4.2.4) means that no subinterval has a width Δ_i that is greater than or equal to the sum of the widths of the its two nearest previous subintervals or its two nearest following subintervals. From now on, unless stated otherwise we shall assume that $U = \{u_i\}$ satisfies (4.2.4).

We mentioned that the values of b_1 and b_{3L-1} are determined by the interpolation end condition. We will discuss Theorem 4.1 under the Bessel end conditions and under the Clamped end conditions in Chapter 5 and Chapter 6 respectively.

Now we prove the following Lemma about the $\{w_i\}$ which will be useful later.

Lemma 4.3 *The hypothesis of Theorem 4.1 implies that* $w_i > 0$ *for* $i = 1, \ldots, L+1$

Proof. From the hypothesis of Theorem 4.1 it follows that $v_i > 0$ for $i = 0, \ldots, L+2$, i.e. $V > 0$ in (4.2.7). It follows that $F > 0$, i.e. $w_i > 0$, since $(I + A) > 0$ in (4.2.7).

Chapter 5

Methods and Computational Examples

5.1 Introduction

In this Chapter we provide three methods to modify assigned positive weights that have caused some of the weights of the resulting rational function to be negative, in such a way that the sufficient conditions in Theorem 4.1 are satisfied. As we mentioned before, the resulting rational curves will still interpolate the given data points. Only the shapes of the resulting curve between the data points will be changed. Several examples are provided to illustrate and to compare these three methods.

5.2 Adjusting the Assigned Weights

Method 1, which modifies the assigned weights to produce positive rational weights, is to add a large positive number a to each of the assigned weights. The linear precision of B-spline interpolation implies that the resulting rational weights will be $v_i + a$ and will all be positive if $a > \max\{-v_i : v_i < 0\}$. However, while this method is simple, it may produce a curve that differs greatly from that desired by the designer. (See [10] and [11] and Chapter 3). Thus we consider two alternatives which have the feature that the assigned weights are modified as little as is possible.

We now consider the Bessel end conditions for interpolation. The methods that we obtain for the Bessel end condition can be easily extended to other end conditions with little modification.

Theorem 4.1 becomes the following Lemma under the Bessel end conditions on the next page.

Lemma 5.1 *If the parameter values $u_1 \dots u_{L+1}$ satisfy (4.2.4) and $v_0 = w_1 > 0$, $v_{L+2} = w_{L+1} > 0$,*

$$b_1 = \frac{1}{3}[\frac{\Delta_2}{\Delta_1+\Delta_2}w_1 + w_1 + \frac{\Delta_1+\Delta_2}{\Delta_2}w_2 - \frac{\Delta_1^2}{\Delta_2(\Delta_1+\Delta_2)}w_3] > 0,$$

$$b_{3L-1} = \frac{1}{3}[\frac{\Delta_{L-1}}{\Delta_{L-1}+\Delta_L}w_{L+1} + w_{L+1} + \frac{\Delta_{L-1}+\Delta_L}{\Delta_{L-1}}w_L - \frac{\Delta_L^2}{\Delta_{L-1}(\Delta_{L-1}+\Delta_L)}w_{L-1}] > 0,$$

$$(\Delta_1 + \tfrac{2}{3}\Delta_2)w_2 - \frac{\Delta_1+2\Delta_2}{3(\Delta_1+\Delta_2)}\alpha_2 w_1 + [\frac{\Delta_1^2\alpha_2}{3\Delta_2(\Delta_1+\Delta_2)} - (\Delta_2+\Delta_3)\frac{\gamma_2}{\beta_2}]w_3 > 0,$$

$$(\Delta_{i-1} + \Delta_i)w_i - [(\Delta_{i-2} + \Delta_{i-1})w_{i-1}\frac{\alpha_i}{\beta_i} + (\Delta_i + \Delta_{i+1})w_{i+1}\frac{\gamma_i}{\beta_i}] > 0 \qquad (5.2.1)$$

$$i = 2, \dots, L-1,$$

$$(\tfrac{2}{3}\Delta_{L-1} + \Delta_L)w_L + [\frac{\Delta_L^2}{3\Delta_{L-1}(\Delta_{L-1}+\Delta_L)}\gamma_L - (\Delta_{L-2}+\Delta_{L-1})\frac{\alpha_L}{\beta_L}]w_{L-1} - \frac{2\Delta_{L-1}+\Delta_L}{3(\Delta_{L-1}+\Delta_L)}\gamma_L w_{L+1} > 0,$$

then $v_i > 0$ for $i = 0, \dots, L+2$.

Lemma 5.2 *Suppose that the parameter values $u_1 \dots u_{L+1}$ satisfy the conditions*

$$\Delta_2 \geq \Delta_1 \quad and \quad \Delta_L \geq \Delta_{L+1}. \qquad (5.2.2)$$

Then for $w_1 > 0$ and $w_2 > 0$,

$$(\Delta_1 + \frac{2}{3}\Delta_2)w_2 - \frac{\Delta_1 + 2\Delta_2}{3(\Delta_1+\Delta_2)}\alpha_2 w_1 + [\frac{\Delta_1^2\alpha_2}{3\Delta_2(\Delta_1+\Delta_2)} - (\Delta_2+\Delta_3)\frac{\gamma_2}{\beta_2}]w_3 > 0 \qquad (5.2.3)$$

implies that $b_1 > 0$. If $w_L > 0$ and $w_{L+1} > 0$, then

$$(\frac{2}{3}\Delta_{L-1} + \Delta_L)w_L + [\frac{\Delta_L^2}{3\Delta_{L-1}(\Delta_{L-1}+\Delta_L)}\gamma_L - (\Delta_{L-2}+\Delta_{L-1})\frac{\alpha_L}{\beta_L}]w_{L-1} - \frac{2\Delta_{L-1}+\Delta_L}{3(\Delta_{L-1}+\Delta_L)}\gamma_L w_{L+1} > 0 \qquad (5.2.4)$$

implies that $b_{3L-1} > 0$.

Proof. If $w_3 \le 0$, considering $w_1 > 0$, $w_2 > 0$, and (1.1.15), we have $b_1 > 0$.

Now if $w_3 > 0$ and $b_1 \le 0$, we have the following from (1.1.15)

$$w_2 \le -w_1 - \frac{\Delta_2}{\Delta_1 + \Delta_2} w_1 + \frac{\Delta_1^2}{(\Delta_1 + \Delta_2)^2} w_3. \tag{5.2.5}$$

Substituting (5.2.5) into (5.2.3), we should have

$$(\Delta_1 + \frac{2}{3}\Delta_2)(-w_1 - \frac{\Delta_2}{\Delta_1 + \Delta_2} w_1) + (\Delta_1 + \frac{2}{3}\Delta_2)\frac{\Delta_1^2}{(\Delta_1 + \Delta_2)^2} w_3$$
$$- \frac{\Delta_1 + 2\Delta_2}{3(\Delta_1 + \Delta_2)}\alpha_2 w_1 + [\frac{\Delta_1^2 \alpha_2}{3\Delta_2(\Delta_1 + \Delta_2)} - (\Delta_2 + \Delta_3)\frac{\gamma_2}{\beta_2}] w_3 > 0,$$

i.e.,

$$- [\Delta_1 + \frac{2}{3}\Delta_2 + (\Delta_1 + \frac{2}{3}\Delta_2)\frac{\Delta_2}{\Delta_1 + \Delta_2} + \frac{\Delta_1 + 2\Delta_2}{3(\Delta_1 + \Delta_2)}\alpha_2] w_1$$
$$+ [\frac{\Delta_1^2}{(\Delta_1 + \Delta_2)} - (\Delta_2 + \Delta_3)\frac{\gamma_2}{\beta_2}] w_3 > 0.$$

Equivalently we have

$$(\Delta_2 + \Delta_3)\frac{\gamma_2}{\beta_2} > \frac{1}{2}\Delta_1.$$

Also from (5.2.2) we have

$$\frac{\Delta_1^2}{(\Delta_1 + \Delta_2)} \le \frac{1}{2}\Delta_1.$$

These two inequalities imply that

$$[\frac{\Delta_1^2}{(\Delta_1 + \Delta_2)} - (\Delta_2 + \Delta_3)\frac{\gamma_2}{\beta_2}] w_3 < 0.$$

Since $w_1 > 0$, we have

$$- [\Delta_1 + \frac{2}{3}\Delta_2 + (\Delta_1 + \frac{2}{3}\Delta_2)\frac{\Delta_2}{\Delta_1 + \Delta_2} + \frac{\Delta_1 + 2\Delta_2}{3(\Delta_1 + \Delta_2)}\alpha_2] w_1$$

$$+\left[\frac{\Delta_1^2}{(\Delta_1+\Delta_2)} - (\Delta_2+\Delta_3)\frac{\gamma_2}{\beta_2}\right]w_3 < 0.$$

This is a contradiction. Thus it follows that $b_1 > 0$.

The proof of that $b_{3l-1} > 0$ is similar to the above and is thus omitted.

The following Lemma corresponds to Lemma 4.3.

Lemma 5.3 *The hypothesis of Lemma 5.1 implies that* $w_i > 0$ *for* $i = 1, \dots, L+1$.

Whenever a given set of assigned weights $\{w_i\}_{i=1}^{L+1}$ causes some $v_i < 0$, we propose three procedures to modify $\{w_i\}_{i=1}^{L+1}$ in such a way that all $v_i > 0$. We first prove that when all $w_i = 1$, (5.2.1) holds:

Lemma 5.4 *We have*

$$\begin{aligned}
&\tfrac{1}{3}\left[\tfrac{\Delta_2}{\Delta_1+\Delta_2} + 1 + \tfrac{\Delta_1+\Delta_2}{\Delta_2} - \tfrac{\Delta_1^2}{\Delta_2(\Delta_1+\Delta_2)}\right] > 0,\\
&\tfrac{1}{3}\left[\tfrac{\Delta_{L-1}}{\Delta_{L-1}+\Delta_L} + 1 + \tfrac{\Delta_{L-1}+\Delta_L}{\Delta_{L-1}} - \tfrac{\Delta_L^2}{\Delta_{L-1}(\Delta_{L-1}+\Delta_L)}\right] > 0,\\
&(\Delta_1 + \tfrac{2}{3}\Delta_2) - \tfrac{\Delta_1+2\Delta_2}{3(\Delta_1+\Delta_2)}\alpha_2 + \left[\tfrac{\Delta_1^2\alpha_2}{3\Delta_2(\Delta_1+\Delta_2)} - (\Delta_2+\Delta_3)\tfrac{\gamma_2}{\beta_2}\right] > 0,\\
&(\Delta_{i-1}+\Delta_i) - \left[(\Delta_{i-2}+\Delta_{i-1})\tfrac{\alpha_i}{\beta_i} + (\Delta_i+\Delta_{i+1})\tfrac{\gamma_i}{\beta_i}\right] > 0 \qquad (5.2.6)\\
&\qquad i = 2, \dots, L-2,\\
&(\tfrac{2}{3}\Delta_{L-1} + \Delta_L) + \left[\tfrac{\Delta_L^2}{3\Delta_{L-1}(\Delta_{L-1}+\Delta_L)}\gamma_L - (\Delta_{L-2}+\Delta_{L-1})\tfrac{\alpha_L}{\beta_L}\right]\\
&\qquad\qquad - \tfrac{2\Delta_{L-1}+\Delta_L}{3(\Delta_{L-1}+\Delta_L)}\gamma_L > 0.
\end{aligned}$$

Proof. The first two inequalities of (5.2.6) hold from the following respectively:

$$\frac{\Delta_1+\Delta_2}{\Delta_2} - \frac{\Delta_1^2}{\Delta_2(\Delta_1+\Delta_2)} = \frac{(\Delta_1+\Delta_2)^2}{\Delta_2(\Delta_1+\Delta_2)} - \frac{\Delta_1^2}{\Delta_2(\Delta_1+\Delta_2)} > 0,$$

$$\frac{\Delta_{L-1}+\Delta_L}{\Delta_{L-1}} - \frac{\Delta_L^2}{\Delta_{L-1}(\Delta_{L-1}+\Delta_L)} = \frac{(\Delta_{L-1}+\Delta_L)^2}{\Delta_{L-1}} - \frac{\Delta_L^2}{\Delta_{L-1}(\Delta_{L-1}+\Delta_L)} > 0.$$

Since $\Delta_0 = 0$, $\Delta_1 > 0$, $\frac{\Delta_2(\Delta_0+\Delta_1)}{\Delta_0+\Delta_1+\Delta_2} > 0$, we have

$$\Delta_1 + \frac{2}{3}\Delta_2 = \frac{2}{3}\frac{\Delta_2(\Delta_1+\Delta_2)^2}{(\Delta_1+\Delta_2)^2} + (\Delta_2+\Delta_3)\frac{\frac{\Delta_1^2}{\Delta_1+\Delta_2+\Delta_3}}{\frac{\Delta_1(\Delta_2+\Delta_3)}{\Delta_1+\Delta_2+\Delta_3}}$$

$$> \frac{\Delta_1+2\Delta_2}{3(\Delta_1+\Delta_2)}\frac{\Delta_2^2}{(\Delta_0+\Delta_1+\Delta_2)} + (\Delta_2+\Delta_3)\frac{\frac{\Delta_1^2}{\Delta_1+\Delta_2+\Delta_3}}{\frac{\Delta_2(\Delta_0+\Delta_1)}{\Delta_0+\Delta_1+\Delta_2}+\frac{\Delta_1(\Delta_2+\Delta_3)}{\Delta_1+\Delta_2+\Delta_3}}$$

$$= \frac{\Delta_1+2\Delta_2}{3(\Delta_1+\Delta_2)}\alpha_2 + (\Delta_2+\Delta_3)\frac{\gamma_2}{\beta_2}.$$

It follows that the third inequality of (5.2.6) holds.

Since $\frac{\Delta_i(\Delta_{i-2}+\Delta_{i-1})}{\Delta_{i-2}+\Delta_{i-1}+\Delta_i} > 0$ and $\frac{\Delta_{i-1}(\Delta_i+\Delta_{i+1})}{\Delta_{i-1}+\Delta_i+\Delta_{i+1}} > 0$ we have

$$\Delta_{i-1}+\Delta_i = (\Delta_{i-2}+\Delta_{i-1})\frac{\frac{\Delta_i^2}{\Delta_{i-2}+\Delta_{i-1}+\Delta_i}}{\frac{\Delta_i(\Delta_{i-2}+\Delta_{i-1})}{\Delta_{i-2}+\Delta_{i-1}+\Delta_i}} + (\Delta_i+\Delta_{i+1})\frac{\frac{\Delta_{i-1}^2}{\Delta_{i-1}+\Delta_i+\Delta_{i+1}}}{\frac{\Delta_{i-1}(\Delta_i+\Delta_{i+1})}{\Delta_{i-1}+\Delta_i+\Delta_{i+1}}}$$

$$> (\Delta_{i-2}+\Delta_{i-1})\frac{\frac{\Delta_i^2}{\Delta_{i-2}+\Delta_{i-1}+\Delta_i}}{\frac{\Delta_i(\Delta_{i-2}+\Delta_{i-1})}{\Delta_{i-2}+\Delta_{i-1}+\Delta_i}+\frac{\Delta_{i-1}(\Delta_i+\Delta_{i+1})}{\Delta_{i-1}+\Delta_i+\Delta_{i+1}}}$$

$$+ (\Delta_i+\Delta_{i+1})\frac{\frac{\Delta_{i-1}^2}{\Delta_{i-1}+\Delta_i+\Delta_{i+1}}}{\frac{\Delta_i(\Delta_{i-2}+\Delta_{i-1})}{\Delta_{i-2}+\Delta_{i-1}+\Delta_i}+\frac{\Delta_{i-1}(\Delta_i+\Delta_{i+1})}{\Delta_{i-1}+\Delta_i+\Delta_{i+1}}}$$

$$= (\Delta_{i-2}+\Delta_{i-1})\frac{\alpha_i}{\beta_i} + (\Delta_i+\Delta_{i+1})\frac{\gamma_i}{\beta_i} \quad (i = 3, \ldots, L-1).$$

Since $\Delta_{L+1} = 0$, $\Delta_L > 0$, $\frac{\Delta_{L-1}(\Delta_L+\Delta_{L+1})}{\Delta_{L-1}+\Delta_L+\Delta_{L+1}} > 0$, we have

$$\frac{2}{3}\Delta_{L-1} + \Delta_L = (\Delta_{L-2}+\Delta_{L-1})\frac{\frac{\Delta_L^2}{\Delta_{L-2}+\Delta_{L-1}+\Delta_L}}{\frac{\Delta_L(\Delta_{L-2}+\Delta_{L-1})}{\Delta_{L-2}+\Delta_{L-1}+\Delta_L}} + \frac{2}{3}\frac{\Delta_{L-1}(\Delta_{L-1}+\Delta_L)^2}{(\Delta_{L-1}+\Delta_L)^2}$$

$$> (\Delta_{L-2}+\Delta_{L-1})\frac{\frac{\Delta_L^2}{\Delta_{L-2}+\Delta_{L-1}+\Delta_L}}{\frac{\Delta_L(\Delta_{L-2}+\Delta_{L-1})}{\Delta_{L-2}+\Delta_{L-1}+\Delta_L}+\frac{\Delta_{L-1}(\Delta_L+\Delta_{L+1})}{\Delta_{L-1}+\Delta_L+\Delta_{L+1}}} + \frac{2\Delta_{L-1}+\Delta_L}{3(\Delta_{L-1}+\Delta_L)}\gamma_L$$

$$= (\Delta_{L-2}+\Delta_{L-1})\frac{\alpha_L}{\beta_L} + \frac{2\Delta_{L-1}+\Delta_L}{3(\Delta_{L-1}+\Delta_L)}\gamma_L.$$

So the last inequality of (5.2.6) holds. This proves Lemma 5.4.

Now we present method 2. Let

$$
\begin{aligned}
&b_1 = \tfrac{1}{3}\left(\tfrac{\Delta_2}{\Delta_1+\Delta_2} w_1 + w_1 + \tfrac{\Delta_1+\Delta_2}{\Delta_2} w_2 - \tfrac{\Delta_1^2}{\Delta_2(\Delta_1+\Delta_2)} w_3\right) = h_1,\\
&(\Delta_1 + \tfrac{2}{3}\Delta_2) w_2 - \tfrac{\Delta_1+2\Delta_2}{3(\Delta_1+\Delta_2)} \alpha_2 w_1 + \left(\tfrac{\Delta_1^2\alpha_2}{3\Delta_2(\Delta_1+\Delta_2)} - (\Delta_2+\Delta_3)\tfrac{\gamma_2}{\beta_2}\right) w_3 = h_2,\\
&(\Delta_{i-1}+\Delta_i) w_i - \left[(\Delta_{i-2}+\Delta_{i-1}) w_{i-1} \tfrac{\alpha_i}{\beta_i} + (\Delta_i + \Delta_{i+1}) w_{i+1} \tfrac{\gamma_i}{\beta_i}\right] = h_i \\
&\qquad\qquad i = 3, \ldots, L-1, \\
&(\tfrac{2}{3}\Delta_{L-1} + \Delta_L) w_L + \left(\tfrac{\Delta_L^2}{3\Delta_{L-1}(\Delta_{L-1}+\Delta_L)} \gamma_L - (\Delta_{L-2}+\Delta_{L-1}) \tfrac{\alpha_L}{\beta_L}\right) w_{L-1} \\
&\qquad\qquad\qquad\qquad - \tfrac{2\Delta_{L-1}+\Delta_{L+1}}{3(\Delta_{L-1}+\Delta_L)} \gamma_L w_{L+1} = h_L.
\end{aligned}
\tag{5.2.7}
$$

Method 2

Input:

$w_1, \ldots, w_{L+1}$, where $w_i > 0$ for all i, (the assigned weights to $\{\mathbf{x}_i\}_{i=1}^{L+1}$),

$u_1, \ldots, u_{L+1}$, (given parameter values),

$h > 0$ (a given small constant),

Note: If $h_i \leq 0$ in (5.2.7), i.e the ith inequality in (5.2.1) is not true,

this method will modify $\{w_i\}$ to let $h_i > h$ in (5.2.7) and then

the ith inequality in (5.2.1) will be true.

Output:

$\hat{w}_1, \ldots, \hat{w}_{L+1}$, where $\hat{w}_i > 0$ for all i,

(the modified assigned weights to $\{\mathbf{x}_i\}_{i=1}^{L+1}$which satisfy (5.2.1))

b_1 and b_{3L-1} (which will be used in the interpolation formula (4.2.2)).

Step 1:

If $h_i \leq 0$ for $i = 1, \ldots, L$, increase w_i in such a way that the new $h_{i-1} \geq h$, and then decrease w_{i+1} if necessary, such that $h_i = h$.

Step 2:

Obtain b_{3L-1} by (1.1.16). If $b_{3L-1} > 0$ then set $\hat{w}_i = w_i$, for $i = 1, \ldots, L+1$. Otherwise, If $b_{3L-1} \leq 0$, set $z = \mid b_{3L-1} \mid$ and then set $\hat{w}_i = w_i + h + z$ for $i = 1, \ldots, L+1$; $b_{3L-1} = b_{3L-1} + h + z$, and obtain b_1 by (1.1.15)

Discussion of Step 1: Step 1 is a recursive process to correct h_i, if $h_i \leq 0$. We check each $h_i \quad i = 1, \ldots, L$ to see if condition (5.2.1) holds or not. If $h_i > 0$, i.e. the ith inequality in (5.2.1) is true, do not change w_i. If $h_i \leq 0$, i.e. the ith inequality in (5.2.1) is not true, we first increase w_i in such a way that the previous inequalities still hold for w_i after updating w_i, i.e. the new value of h_{i-1} is still greater than or equal to h (which ensures that the previous inequalities still hold after correcting the ith inequality). Then decrease w_{i+1} if necessary, such that the new $h_i = h$ and the ith inequality in (5.2.1) is true. In this way we can get new weights $\{\hat{w}_i\}_1^L$ and b_1 that ensure that all $h_i > 0$ in (5.2.1) through Step 1.

From our sufficient condition on $\{w_i\}_{i=1}^{L+1}$, we can see that if the differences between one weight and its neighboring weights are too large, negative resulting weights will occur. But it is this kind of large difference that gives rise to a sharp bending interpolation curve which may be desired by the designer. When we change the weights $\{w_i\}$ to $\{\hat{w}_i\}$, we try to change them in such a way that $\sum_{i=0}^{L+1} |\hat{w}_i - w_i|$ will be small and such that $\{\hat{w}_i\}$ will satisfy (5.2.1). Assume that the ith inequality is not true. Lemma 5.4 implies that the coefficient of w_i is greater than that of w_{i+1}. So we increase w_i as much as possible first, and then decrease w_{i+1} if necessary. If the coefficient of w_{i-1} is less than the one of w_{i+1}, the change of h_i by our Algorithm is optimal in the sense that $\sum_{i=0}^{L+1} |\hat{w}_i - w_i|$ is least. (Note that the $\hat{w}_i$ here is not the final value of $\hat{w}_i$ that we shall output. We use $\hat{w}_i$ for convenience in our discussion). If the coefficient of w_{i-1} is greater than that of w_{i+1}, the change of h_i by our Algorithm is not optimal in the same sense. Now we estimate

$$[(\Delta_{i-2} + \Delta_{i-1})\frac{\alpha_i}{\beta_i} - (\Delta_i + \Delta_{i+1})\frac{\gamma_i}{\beta_i}].$$

First we have

$$\begin{aligned}(\Delta_{i-2} + \Delta_{i-1})\frac{\alpha_i}{\beta_i} &= (\Delta_{i-2} + \Delta_{i-1})\frac{\frac{\Delta_i^2}{\Delta_{i-2}+\Delta_{i-1}+\Delta_i}}{\frac{\Delta_i(\Delta_{i-2}+\Delta_{i-1})}{\Delta_{i-2}+\Delta_{i-1}+\Delta_i} + \frac{\Delta_{i-1}(\Delta_i+\Delta_{i+1})}{\Delta_{i-1}+\Delta_i+\Delta_{i+1}}} \\ &< (\Delta_{i-2} + \Delta_{i-1})\frac{\frac{\Delta_i^2}{\Delta_{i-2}+\Delta_{i-1}+\Delta_i}}{\frac{\Delta_i(\Delta_{i-2}+\Delta_{i-1})}{\Delta_{i-2}+\Delta_{i-1}+\Delta_i}} = \Delta_i.\end{aligned}$$

From (4.2.4) we can obtain

$$(\Delta_i + \Delta_{i+1})\gamma_i = (\Delta_i + \Delta_{i+1})\frac{\Delta_{i-1}^2}{\Delta_{i-1} + \Delta_i + \Delta_{i+1}} > \frac{\Delta_{i-1}^2}{2},$$

$$\begin{aligned}0 < \beta_i &= \frac{\Delta_i(\Delta_{i-2} + \Delta_{i-1})}{\Delta_{i-2} + \Delta_{i-1} + \Delta_i} + \frac{\Delta_{i-1}(\Delta_i + \Delta_{i+1})}{\Delta_{i-1} + \Delta_i + \Delta_{i+1}} \\ &< \frac{\Delta_{i-2} + \Delta_{i-1} + \Delta_i + \Delta_{i+1}}{2},\end{aligned}$$

and then we have

$$(\Delta_i + \Delta_{i+1})\frac{\gamma_i}{\beta_i} > \frac{\Delta_{i-1}^2}{\Delta_{i-2} + \Delta_{i-1} + \Delta_i + \Delta_{i+1}} > 0.$$

It follows that

$$0 < [(\Delta_{i-2} + \Delta_{i-1})\frac{\alpha_i}{\beta_i} - (\Delta_i + \Delta_{i+1})\frac{\gamma_i}{\beta_i}] < \Delta_i - \frac{\Delta_{i-1}^2}{\Delta_{i-2} + \Delta_{i-1} + \Delta_i + \Delta_{i+1}}.$$

Considering $u_1 = 0$ and $u_{L+1} = 1$, we have $0 < (\Delta_i - \frac{\Delta_{i-1}^2}{\Delta_{i-2}+\Delta_{i-1}+\Delta_i+\Delta_{i+1}}) < \Delta_i$. Thus if Δ_i is sufficiently small then $(\Delta_i - \frac{\Delta_{i-1}^2}{\Delta_{i-2}+\Delta_{i-1}+\Delta_i+\Delta_{i+1}})$ is small and so is $\sum_{i=0}^{L+1} |\hat{w}_i - w_i|$.

Discussion of Step 2: Step 2 will produce an output. Given the results of Step 1, if $b_{3L-1} > 0$, from Lemma 5.3 it follows that $w_i > 0$ for $i = 1, \ldots, L+1$, and we let $\hat{w}_i = w_i$, $i = 0, \ldots, L$ that satisfy the conditions of Lemma 5.1 and that are the weights that we desire.

In other cases we have

$$\frac{1}{3}[\frac{\Delta_2}{\Delta_1+\Delta_2}w_1 + w_1 + \frac{\Delta_1+\Delta_2}{\Delta_2}w_2 - \frac{\Delta_1^2}{\Delta_2(\Delta_1+\Delta_2)}w_3] > 0,$$

$$(\Delta_1 + \tfrac{2}{3}\Delta_2)w_2 - \frac{\Delta_1+2\Delta_2}{3(\Delta_1+\Delta_2)}\alpha_2 w_1 + (\frac{\Delta_1^2\alpha_2}{3\Delta_2(\Delta_1+\Delta_2)} - (\Delta_2+\Delta_3)\frac{\gamma_2}{\beta_2})w_3 > 0,$$

$$(\Delta_{i-1} + \Delta_i)w_i - [(\Delta_{i-2} + \Delta_{i-1})w_{i-1}\frac{\alpha_i}{\beta_i} + (\Delta_i + \Delta_{i+1})w_{i+1}\frac{\gamma_i}{\beta_i}] > 0$$

$$i = 3, \ldots, L-1,$$

$$(\tfrac{2}{3}\Delta_{L-1} + \Delta_L)w_L + (\frac{\Delta_L^2}{3\Delta_{L-1}(\Delta_{L-1}+\Delta_L)}\gamma_L - (\Delta_{L-2} + \Delta_{L-1})\frac{\alpha_L}{\beta_L})w_{L-1}$$
$$- \frac{2\Delta_{L-1}+\Delta_L}{3(\Delta_{L-1}+\Delta_L)}\gamma_L w_{L+1} > 0.$$

Since $h + z > 0$ then by Lemma 5.4 it follows that

$$\frac{1}{3}[\frac{\Delta_2}{\Delta_1+\Delta_2}(h+z)+(h+z)+\frac{\Delta_1+\Delta_2}{\Delta_2}(h+z)-\frac{\Delta_1^2}{\Delta_2(\Delta_1+\Delta_2)}(h+z)]>0,$$

$$(\Delta_1+\tfrac{2}{3}\Delta_2)(h+z)-\frac{\Delta_1+2\Delta_2}{3(\Delta_1+\Delta_2)}\alpha_2(h+z)$$
$$+[\frac{\Delta_1^2\alpha_2}{3\Delta_2(\Delta_1+\Delta_2)}-(\Delta_2+\Delta_3)\frac{\gamma_2}{\beta_2}](h+z)>0,$$

$$(\Delta_{i-1}+\Delta_i)(h+z)-[(\Delta_{i-2}+\Delta_{i-1})(h+z)\frac{\alpha_i}{\beta_i}+(\Delta_i+\Delta_{i+1})(h+z)\frac{\gamma_i}{\beta_i}]>0$$
$$i=3,\dots,L-1,$$

$$(\tfrac{2}{3}\Delta_{L-1}+\Delta_L)(h+z)+(\frac{\Delta_L^2}{3\Delta_{L-1}(\Delta_{L-1}+\Delta_L)}\gamma_L-(\Delta_{L-2}+\Delta_{L-1})\frac{\alpha_L}{\beta_L})(h+z)$$
$$-\frac{2\Delta_{L-1}+\Delta_{L+1}}{3(\Delta_{L-1}+\Delta_L)}\gamma_L(h+z)>0.$$

Combining the above two systems, it follows that $\hat{w}_i = w_i + h + z$ $i = 0, \dots, L$ satisfy (5.2.1). Now considering $b_{3L-1} = b_{3L-1} + h + z > 0$, from Lemma 5.3 it follows that $\hat{w}_i > 0$ for $i = 1, \dots, L+1$.

From the above discussion we have the following:

Theorem 5.1 *Method 2 produces positive assigned weights $\{\hat{w}_i\}_{i=1}^{L+1}$ that satisfy (5.2.1). Hence the corresponding rational interpolant has positive weights V.*

We assume that the parameter values $u_1 \dots u_{L+1}$ satisfy the conditions $\Delta_2 \geq \Delta_1$ and , $\Delta_L \geq \Delta_{L+1}$. Now, we introduce a third method to modify the assigned weights in the least squares sense. We have the following:

Problem L2:

Given assigned weights $w_i > 0$; $i = 1, \dots, L+1$ that do not produce positive weights v_i.

Find weights $\{\hat{w}_i\}_{i=1}^{L+1} \in R^{L+1}$ such that the corresponding rational function formed by $\sum_{i=0}^{L+2} v_i N_i^3(u)$ has positive coefficients and of all such weights $\sum_{i=1}^{L+1} (\hat{w}_i - w_i)^2$ is a minimum.

As an alternative to this problem we consider the following:

Problem L2b:

Given assigned weights $w_i > 0$; $i = 1, \ldots, L+1$ that do not produce positive rational weights v_i.

Find weights $\{\hat{w}_i\}_{i=1}^{L+1} \in R^{L+1}$ such that conditions (5.2.1) are satisfied and all $\hat{w}_i > 0$ and such that $\sum_{i=1}^{L+1}(\hat{w}_i - w_i)^2$ is a minimum.

We will not change w_1 and w_{L+1} under the Bessel interpolation end conditions. We have the following Lemma under these assumptions:

Lemma 5.5 *For given $w_i > 0$; $i = 1, \ldots, L+1$ that do not produce positive weights v_i; $i = 0, \ldots, L+2$, if we find any new weights x_i; $i = 1, \ldots, L+1$, with $x_1 = w_1$ and $x_{L+1} = w_{L+1}$, that satisfy condition (5.2.1), then $b_1 > 0$, $b_{3L-1} > 0$, and then $x_i > 0$ for all i.*

Proof. First we wish to show that $x_2 > 0$. Note that all the $w_i's$ are $x_i's$ now and later on in the proof of this Lemma in (5.2.1)). Since $(\Delta_1 + \frac{2}{3}\Delta_2) > (\Delta_2 + \Delta_3)\frac{\gamma_2}{\beta_2}$ from the proof of of Lemma 5.4 and $\frac{\Delta_1^2 \alpha_2}{3\Delta_2(\Delta_1+\Delta_2)} < (\Delta_2 + \Delta_3)\frac{\gamma_2}{\beta_2}$. If $x_2 < 0$, then since x_2 and x_3 satisfy the third inequality of (5.2.1) and $x_1 > 0$, it follows that $x_3 < x_2 < 0$. Since $\Delta_{i-1} > (\Delta_i + \Delta_{i+1})\frac{\gamma_i}{\beta_i}$ and $\Delta_i > (\Delta_{i-2} + \Delta_{i-1})\frac{\alpha_i}{\beta_i}$ from the proof of Lemma 5.4 in

$$(\Delta_{i-1}+\Delta_i)x_i - [(\Delta_{i-2}+\Delta_{i-1})x_{i-1}\frac{\alpha_i}{\beta_i} + (\Delta_i+\Delta_{i+1})x_{i+1}\frac{\gamma_i}{\beta_i}] > 0$$

of (5.2.1), $i = 3, \ldots, L-1$, when $i = 3$ we have $x_4 < x_3 < 0$, since $x_3 < x_2 < 0$. The proof continues inductively to obtain $x_L < x_{L-1} < \ldots < x_3 < x_2 < 0$. But the last inequality of (5.2.1) can not hold when $x_L < x_{L-1} < 0$ from the proof of Lemma 5.4 and $x_{L+1} > 0$ and $\frac{\Delta_L^2}{3\Delta_{L-1}(\Delta_{L-1}+\Delta_L)}\gamma_L < (\Delta_{L-2}+\Delta_{L-1})\frac{\alpha_L}{\beta_L}$. This is a contradiction. Thus it follows that $x_2 > 0$. Since the x_i; $i = 1, \ldots, L+1$, where $x_1 = w_1$ and $x_{L+1} = w_{L+1}$, satisfy condition (5.2.1) and $x_2 > 0$ we have $b_1 > 0$ from Lemma 5.2. Similarly we can prove that $x_L > 0$ and then $b_{3L-1} > 0$. Now since $b_1 > 0$ and $b_{3L-1} > 0$ and the fact that x_i; $i = 1, \ldots, L+1$, satisfy condition (5.2.1) it follows that all $x_i > 0$ from Lemma 5.3.

From Lemma 5.5 Problem L2b can be reduced to the following minimization problem:

$$\min_{x\in\mathcal{D}} F(x) = \min_{x\in\mathcal{D}} \sum_{i=2}^{L}(x_i - w_i)^2, \tag{5.2.8}$$

where

$\mathcal{D} := \{x | a_i(x) > d_i; \;\; i = 2, \ldots, L\} \subset R^{L-1}$,

$x = [x_2, \ldots, x_L]^T$ and

$a_2(x) = (\Delta_1 + \frac{2}{3}\Delta_2)x_2 + [\frac{\Delta_1^2 \alpha_2}{3\Delta_2(\Delta_1+\Delta_2)} - (\Delta_2 + \Delta_3)\frac{\gamma_2}{\beta_2}]x_3$,

$a_i(x) = (\Delta_{i-1} + \Delta_i)x_i - [(\Delta_{i-2} + \Delta_{i-1})x_{i-1}\frac{\alpha_i}{\beta_i} + (\Delta_i + \Delta_{i+1})x_{i+1}\frac{\gamma_i}{\beta_i}]$

$(i = 3, \ldots, L\text{-}1)$,

$a_L(x) = (\frac{2}{3}\Delta_{L-1} + \Delta_L)x_L + [\frac{\Delta_L^2}{3\Delta_{L-1}(\Delta_{L-1}+\Delta_L)}\gamma_L - (\Delta_{L-2} + \Delta_{L-1})\frac{\alpha_L}{\beta_L}]x_{L-1}$,

$d_2 = \frac{\Delta_1+2\Delta_2}{3(\Delta_1+\Delta_2)}\alpha_2 w_1$, $d_i = 0$; $(i = 2, \ldots, \mathrm{L}-1)$ and $d_L = \frac{2\Delta_{L-1}+\Delta_L}{3(\Delta_{L-1}+\Delta_L)}\gamma_L w_{L+1}$.

Definition 5.1 *Let*

$$\begin{pmatrix} a_2(x) \\ \vdots \\ a_i(x) \\ \vdots \\ a_L(x) \end{pmatrix} = \hat{A}x.$$

Where the coefficient matrix $\hat{A} = (\hat{a}_{i,j})$ *is the following tridiagonal matrix:*

$$\begin{pmatrix} (\Delta_1+\frac{2}{3}\Delta_2) & \hat{a}_{2,3} & 0 & & & 0 \\ -(\Delta_1+\Delta_2)\frac{\alpha_3}{\beta_3} & (\Delta_2+\Delta_3) & -(\Delta_3+\Delta_4)\frac{\gamma_3}{\beta_3} & & & \\ \ddots & \ddots & \ddots & & & \\ & -(\Delta_{i-2}+\Delta_{i-1})\frac{\alpha_i}{\beta_i} & (\Delta_{i-1}+\Delta_i) & -(\Delta_i+\Delta_{i+1})\frac{\gamma_i}{\beta_i} & & \\ & \ddots & \ddots & \ddots & & \\ & & -(\Delta_{L-3}+\Delta_{L-2})\frac{\alpha_{L-1}}{\beta_{L-1}} & (\Delta_{L-2}+\Delta_{L-1}) & -(\Delta_{L-1}+\Delta_L)\frac{\gamma_{L-1}}{\beta_{L-1}} & \\ 0 & & 0 & \hat{a}_{L,L-1} & (\frac{2}{3}\Delta_{L-1}+\Delta_L) & \end{pmatrix},$$

where

$$\hat{a}_{2,3} = \frac{\Delta_1^2\alpha_2}{3\Delta_2(\Delta_1+\Delta_2)} - (\Delta_2+\Delta_3)\frac{\gamma_2}{\beta_2},$$

$$\hat{a}_{L,L-1} = \frac{\Delta_L^2}{3\Delta_{L-1}(\Delta_{L-1}+\Delta_L)}\gamma_L - (\Delta_{L-2}+\Delta_{L-1})\frac{\alpha_L}{\beta_L}.$$

Problem (5.2.8) is a well discussed least-distance problem whose object function is convex and its domain $\mathcal{D}$ is a convex set (see [1, 6, 18, 23, 28, 39, 37, 38]).

For problem (5.2.8) we have

Lemma 5.6 *$\mathcal{D}$ is nonempty*

Proof. Let $x_2 = \ldots = x_L = \max(w_1, w_{L+1})$. From Lemma 5.4 we have $a_i(x) > d_i$, $i = 2, \ldots, L$, where $x = (x_2, \ldots, x_L)^T$.

Lemma 5.7 *If $\phi(x)$ is a convex function defined on a convex set $S \subset R^n$, then any local minimum is a global minimum.*

Proof. There is a proof of this conclusion under the condition that $\phi(x)$ is assumed to be explicitly quasiconvex, (a weaker hypotheses than that of this lemma), in [21].

Lemma 5.8 *If problem (5.2.8) has a solution, then the solution is unique and is given by $x_i = w_i$, for $i = 2, \ldots, L$.*

Proof. The set D is open and nonempty. Hence, if (5.2.8) is solvable, then any solution is a local minimum of $F(x)$. Since $F(x)$ is convex and D is convex and from Lemma 5.7 it follows that any local minimum of $F(x)$ is its global minimum and the global minimum is unique.

Sometimes the solution of an optimization problem with inequality constraints is on the boundary of a feasible domain and thus (5.2.8) may not have a solution.

In order to solve problem (5.2.8) we shall consider the following optimization problem:

$$\min_{x \in \tilde{D}} F(x) = \min_{x \in \tilde{D}} \sum_{i=2}^{L} (x_i - w_i)^2, \tag{5.2.9}$$

where

$$\tilde{D} := \{x | a_i(x) \geq d_i; \;\; i = 2, \ldots, L\} \subset R^{L-1} \text{ and is closed.}$$

Also we have

Lemma 5.9 *$\tilde{D}$ is nonempty.*

Proof. See the proofs of Lemma 5.6.

As in problem (5.2.8) we have the following:

Lemma 5.10 *If problem (5.2.9) has a solution, then the solution is unique.*

If $\hat{x} \in \mathcal{D}$, where $\hat{x} = [\hat{x}_2, \ldots, \hat{x}_L]^T$ is the solution of (5.2.9), then $\hat{x}$ is used as the solution of (5.2.8).

If $\hat{x}$ is not in $\mathcal{D}$, then $a_i(\hat{x}) = d_i$ for some i, i.e., the sufficient condition (5.2.1) in Lemma 5.1 is not satisfied, and thus we can not guarantee that all $v_i > 0$. Also there is the possibility that some of the components of $\hat{x}$ are 0. Thus $\sum_{i=0}^{L+2} v_i N_i^3(u_j) = \hat{x}_j = 0$ in interpolation (1.1.11) and $\sum_{i=0}^{L+2} v_i \mathbf{d}_i N_i^3(u_j) = \hat{x}_j \mathbf{x}_j = 0$ in interpolation (1.1.12) for some j and thus it will result that the resulting curve $\mathbf{L}(\mathbf{u}) = \frac{\sum_{i=0}^{L+2} v_i \mathbf{d}_i N_i^3(u)}{\sum_{i=0}^{L+2} v_i N_i^3(u)}$ obtained by interpolation (1.1.9) defined by (1.1.11) and (1.1.12) will not be defined at u_j for some j, i.e. it will result in the instability of the interpolation (1.1.9) defined by (1.1.11) and (1.1.12).

Let $\tilde{x} = \hat{x} + \epsilon$, where ϵ is any given positive number. $\tilde{x}$ satisfies $a_i(\tilde{x}) > d_i$ for $i = 2, \ldots, L$ from Lemma 5.4 and thus $\tilde{x} > 0$ from Lemma 5.5 and $\tilde{x} \in \mathcal{D}$. Note from the definition of (5.2.8) we have

$$\min_{x \in \mathcal{D}} \sum_{i=2}^{L} (x_i - w_i)^2 \leq \sum_{i=2}^{L} (\tilde{x}_i - w_i)^2$$

$$= \sum_{i=2}^{L} (\hat{x}_i - w_i)^2 + 2[\sum_{i=2}^{L} (\hat{x}_i - w_i)]\epsilon + (L+1)\epsilon^2$$

$$= \min_{x \in \bar{\mathcal{D}}} \sum_{i=2}^{L} (x_i - w_i)^2 + 2[\sum_{i=2}^{L} (\hat{x}_i - w_i)]\epsilon + (L+1)\epsilon^2.$$

Thus we can use $\tilde{x}$ as a good approximate solution to problem (5.2.8), when we choose ϵ sufficient small.

In order to solve (5.2.9), we need the following Lemmas:

Lemma 5.11 *The matrix $\hat{A}$ is strictly diagonally dominant.*

Proof. From Lemma 5.4 it follows that

$$| (\Delta_0 + \tfrac{2}{3}\Delta_1) | > | \hat{a}_{1,2} | = | \frac{\Delta_2^2 \alpha_1}{3\Delta_1(\Delta_0+\Delta_1)} - (\Delta_1 + \Delta_2)\frac{\gamma_1}{\beta_1} |,$$

$$| \Delta_{i-1} + \Delta_i | > | -(\Delta_{i-2} + \Delta_{i-1})\frac{\alpha_i}{\beta_i} | + | -(\Delta_i + \Delta_{i+1})\frac{\gamma_i}{\beta_i} | \quad , 3 \leq i \leq L-2,$$

$$| (\tfrac{2}{3}\Delta_{L-1} + \Delta_L) | > | \hat{a}_{L-1,L-2} | = | \frac{\Delta_L^2}{3\Delta_{L-1}(\Delta_{L-1}+\Delta_L)}\gamma_L - (\Delta_{L-2} + \Delta_{L-1})\frac{\alpha_L}{\beta_L} | .$$

Lemma 5.11 is proved

Lemma 5.12 *The row vectors of the coefficient matrix $\hat{A}$ are linear independent*

Proof. Let $c = (c_2, \ldots, c_L)^T$, where c_i $(i = 2, \ldots, L)$ are any real number. From Lemma 5.11 we have

$$\hat{A}c = 0 \Longrightarrow c = 0,$$

i.e., the $(L-1)$ column vectors of $\hat{A}$ are linear independent and so are the row vectors of $\hat{A}$, since $\hat{A}$ is $(L-1) \times (L-1)$. Lemma 5.12 is proved.

Problem (5.2.9) is a least distance problem with linear constraints. The row vectors of the coefficient matrix are linear independent. There are several efficient methods to solve this problem (see [6, 18, 23, 28, 39, 37, 38]).

Here we use QPROG/DQPROG in IMSL MATH/LIBRARY ([39]). QPROG/DQPROG is based on M.J.D. Powell's implementation of the Goldfarb and Idnani (1983) dual quadratic programming (QP) algorithm for convex QP problems subject to general linear equality/inequality constraints, i.e., a problem of the form

$$\min_{x \in R^n} g^T x + \frac{1}{2} x^T Q x \tag{5.2.10}$$

such that

$$A_2 x = \hat{b}_2$$

and

$$A_3 x \geq \hat{b}_3,$$

given the vectors $\hat{b}_2$, $\hat{b}_3$ and g and the matrix Q, A_2 and A_3. Q is required to be positive definite. For more details, see Powell (1983a. 1983b).

For using QPROG/DQPROG, we set $y_i = x_i - w_i$, $i = 2, \ldots, L$ and $y = \ (y_2, \ldots, y_L)^T$ and we change (5.2.9) into

$$\min_{y \in \hat{D}} \hat{F}(y) = \min_{y \in \hat{D}} \sum_{i=2}^{L} y_i^2, \tag{5.2.11}$$

where

$$\hat{D} : \{y | a_i(y) \geq \hat{d}_i \ i = 2, \ldots, L\} \subset R^{L-1},$$

$$\hat{d}_2 = d_2 - (\Delta_1 + \frac{2}{3}\Delta_2) w_2 - [\frac{\Delta_1^2 \alpha_2}{3\Delta_2(\Delta_1 + \Delta_2)} - (\Delta_2 + \Delta_3)\frac{\gamma_2}{\beta_2}] w_3,$$

$$\hat{d}_i = d_i - (\Delta_{i-1} + \Delta_i)w_i + [(\Delta_{i-2} + \Delta_{i-1})w_{i-1}\frac{\alpha_i}{\beta_i} + (\Delta_i + \Delta_{i+1})w_{i+1}\frac{\gamma_i}{\beta_i}]$$

$$i = 3, \ldots, L-1,$$

$$\hat{d}_L = d_L - (\frac{2}{3}\Delta_{L-1} + \Delta_L)w_L - [\frac{\Delta_L^2}{3\Delta_{L-1}(\Delta_{L-1} + \Delta_L)}\gamma_L - (\Delta_{L-2} + \Delta_{L-1})\frac{\alpha_L}{\beta_L}]w_{L-1}.$$

Now $\hat{b}_2$ is a zero vector, $\hat{b}_3 = (\hat{d}_2, \ldots, \hat{d}_L)^T$, the g is a zero vector, $Q = 2I$, A_2 is a zero matrix and $A_3 = \hat{A}$ which is as in Lemma 5.11.

After we obtain new weights x, we will check if $x_i > 0$ for $i = 2, \ldots, L$: if not, let $x + \epsilon$ be the solution to problem (5.2.8), where $\epsilon > 0$ is given; otherwise we will check if $v_i > 0$ for $i = 0, \ldots, L+2$: if it is, then x is the solution to problem (5.2.8), otherwise let $x + \epsilon$ be the solution to problem (5.2.8). Then we obtain b_1 and b_{3L-1} which will be used in the interpolation formula (4.2.2).

5.3 Computational Examples and Discussion

In this section we present several computational examples to demonstrate inherent problems in choosing the weights w_i for the data points x_i in the interpolation scheme (1.1.9) and to show that the effectiveness of three methods.

First we consider the example, in Figure 4.1, at the beginning of Chapter 4. Figure 5.3 shows the curve with weights obtained by applying method 2 to the weights of the curve in Figure 4.1 and Figure 5.4 is its curvature plot. The value of h in method 2 is 0.1. There are no extra loops any more at the points on either side of the middle point and there are no singularities or points of inflection in its curvature plot (Figure 5.4). Data convexity has been preserved. The curve is a better fit to the data than the curve in Figure 3.8 in that the resulting curve is more flat in a neighborhood of the middle point, and the direction of its tangent line changes rapidly in a neighborhood of the points on either side of the middle point.

Figure 5.5 shows the curve with weights obtained by applying method 3 to the weights of the curve in Figure 4.1. Figure 5.6 is its curvature plot. There are no extra loops any more at the points on either side of the middle point and there are no singularities or points of inflection in its curvature plot, and data convexity has been preserved. The curve is a better fit to the data than the curve in Figure 3.8 in that the resulting curve is more flat in a neighborhood of the middle point, and the direction of its tangent line changes rapidly in a neighborhood of the points on either side of the middle point.

Figure 5.1 shows the curve with weights obtained by applying method 1 to the weights of the curve in Figure 4.1, i.e., by adding 4.2 to each of the original weights, since the resulting negative weight of the curve (Figure 4.1) is -4. Figure 5.2 is its curvature plot. There are no extra loops any more at the points on either side of the middle point and there are no singularities or points of inflection in its curvature plot Figure 5.2. and it preserves data convexity. There is no significant improvement in the graph of the curve in Figure 5.1 relative to the curve in Figure 3.8 since all of the weights have been increased. (See [10] and [11] and Chapter 3).

Farin used curvature plots for the following definition of a fair curve in [11]: " A curve is called fair if its curvature plot is continuous and consists of only a few monotone pieces." The number of curvature extrema of a fair curve should thus be small. Comparing the curvature plot in Figure 4.2 and the curvature plots in Figure 5.2, Figure 5.4, and Figure 5.6, we can see that all three methods not only produced new weights which ensure positive resulting weights, but also faired the original curve.

All of the resulting weights in Figure 5.1, Figure 5.3 and Figure 5.5 are positive and the curves in these figures have the convex hull property. The curve in Figure 4.1 did not have the convex hull property.

All the data and weights of these graphs are presented in Table 5.1 on page 79.

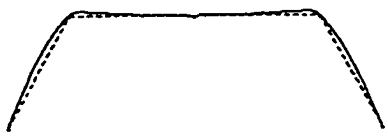

Figure 5.1: $\mathbf{L}(u)$. Assigned weights are modified by method 1.

Figure 5.2: Crvature plot of Figure 5.1.

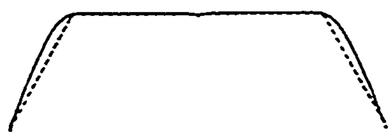

Figure 5.3: $\mathbf{L}(u)$. Assigned weights are modified by method 2.

Figure 5.4: Curvature plot of Figure 5.3.

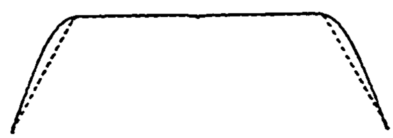

Figure 5.5: $\mathbf{L}(u)$. Assigned weights are modified by method 3.

Figure 5.6: Curvature plot of Figure 5.5.

Table 5.1: Data and weights for Figures 3.6, 3.8, 4.1, 5.1, 5.3 and 5.5

In table Weight below, each row gives the weights to be interpolated for the curves corresponding to Figure 3.6, Figure 3.8, Figure 4.1, Figure 5.1, Figure 5.3, and Figure 5.5 respectively.

In table Data, the first row gives the data values to be interpolated in the x direction and the second row gives the data values to be interpolated in the y direction for Figure 3.6, Figure 3.8, Figure 4.1, Figure 5.1, Figure 5.3, Figure 5.5.

Each table below gives the weights and the control points of the resulting rational curves that appear on the following pages. The numbers in the first row, denoted by v, are the weights of the resulting rational curves. The numbers in the second row, denoted by x, are the control points of the resulting rational curves in the x direction. The numbers in the third row, denoted by y, are the control points of the resulting rational curves in the y direction.

Weight:

1	1	1	1	1
1	2	1	2	1
1	7	1	7	1
5.2	11.2	5.2	11.2	5.2
1	7	3.7	7	1
1	6.46	3.68	6.46	1

Data:

0.5	1.5	3.5	5.5	6.5
1	2.732	2.732	2.732	1

Figure 3.6:

v	1	1	1	1	1	1	1
x	0.5	0.667	1.286	3.5	5.714	6.333	6.5
y	1	1.866	3.165	2.516	3.165	1.866	1

Figure 3.8:

v	1	1.633	2.583	0.508	2.583	1.633	1
x	0.5	0.949	1.537	3.5	5.463	6.051	6.5
y	1	2.202	2.900	2.306	2.900	2.202	1

Figure 4.1:

v	1	5	11.000	-4.000	11.000	5.000	1
x	0.5	1.333	1.740	3.5	5.260	5.667	6.5
y	1	2.559	2.771	2.786	2.771	2.559	1

Figure 5.1:

v	5.2	9.2	15.2	0.200	15.200	9.2	5.2
x	0.5	1.029	1.615	3.500	5.385	5.971	6.5
y	1	2.243	2.880	-2.897	2.880	2.243	1

Figure 5.3:

v	1	4.55	9.875	0.612	9.875	4.55	1
x	0.5	1.119	1.540	3.5	5.46	5.88	6.5
y	1	2.542	2.776	2.379	2.776	2.542	1

Figure 5.5:

v	1	4.193	8.983	1.028	8.983	4.193	1
x	0.5	1.088	1.517	3.500	5.483	5.912	6.5
y	1	2.525	2.780	2.521	2.780	2.525	1

Farin in [11] presented an example which resulted in negative weights after interpolation. We will apply our three methods to this example.

Figure 5.7 - Figure 5.11 are the resulting weight functions. Their assigned weights and resulting rational weights are in Table 5.2. The resulting rational weight function in Figure 5.7 has two negative weights but is positive everywhere. However the resulting rational weight function in Figure 5.8 has two negative weights and is negative on some subset of the interval. Figure 5.9, Figure 5.10 and Figure 5.11 are the resulting weight functions whose modified assigned weights are obtained by applying method 2 ($h = 0.01$), method 3 and method 1 ($a = 2$) to the assigned weights in Figure 5.7 respectively. All the curves in these three Figures show that the resulting rational weights are now positive.

Table 5.2: Assigned weights for rational weight functions

In each of the following tables, the first row gives weights to be interpolated, the second row gives the resulting rational weights after interpolation.

Fig.5.7		1	1	1	1	7	1	1	1	1	
	1	1	0.786	1.75	-1.786	11.393	-1.786	1.75	0.786	1	1
Fig.5.8		1	1	1	1	9	1	1	1	1	
	1	1	0.714	2	-2.714	14.857	-2.714	2	0.714	1	1
Fig.5.9		1	1	1	2.04	7	2.04	1	1	1	
	1	1	0.934	1.23	0.146	10.427	0.146	1.23	0.934	1	1
Fig.5.10		1	1	0.789	1.842	6.579	1.842	0.789	1	1	
	1	1.035	1.01	0.912	0.078	9.830	0.078	0.912	1.01	1.035	1
Fig.5.11		3	3	3	3	11	3	3	3	3	
	3	3	2.786	3.75	0.214	13.393	0.214	3.75	2.786	3	3

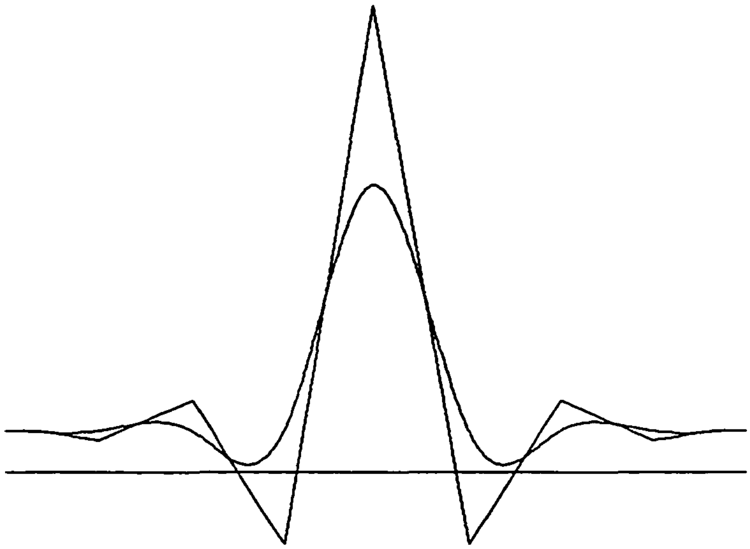

Figure 5.7: Weight function with two negative rational weights

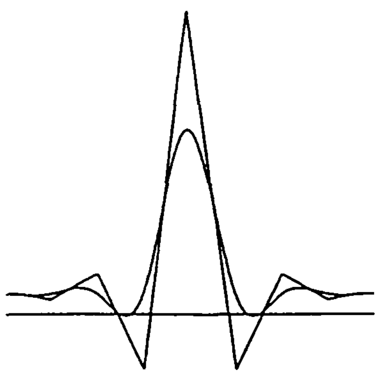

Figure 5.8: Weight function is negative on some subintervals

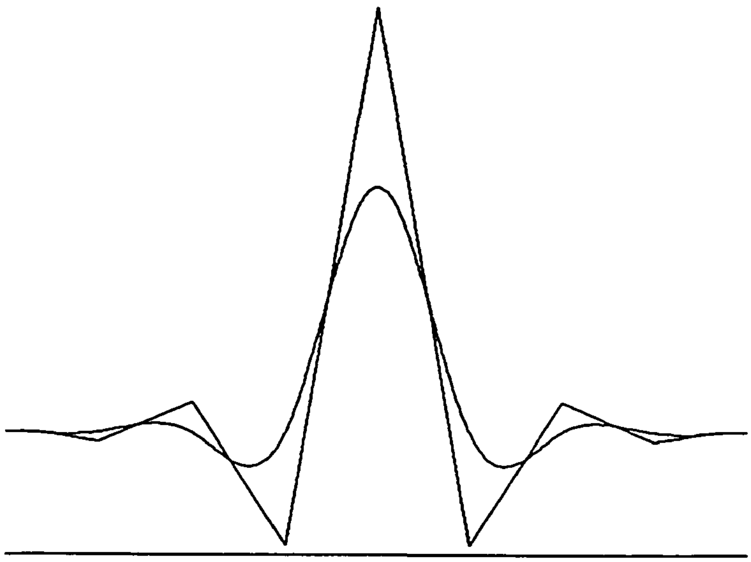

Figure 5.9: Weight function. Assigned weights are modified by method 1

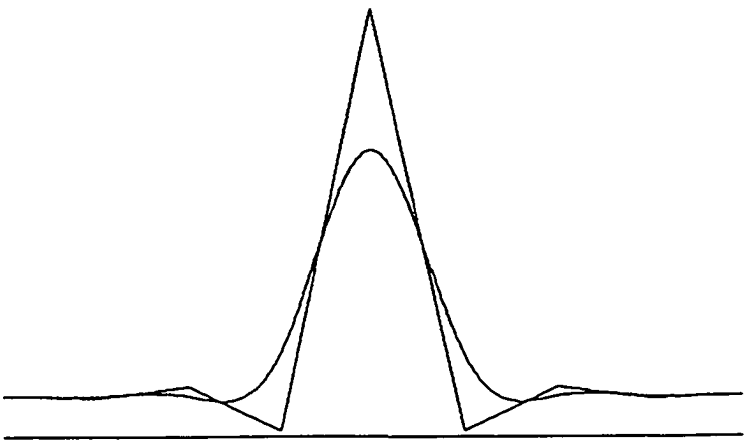

Figure 5.10: Weight function. Assigned weights are modified by method 2

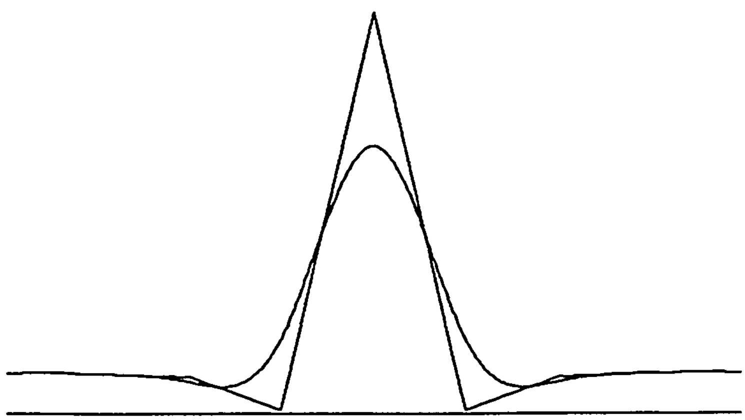

Figure 5.11: Weight function. Assigned weights are modified by method 3

Table 5.3: Affect of choosing of h in method 2 on the rational weights

case 1		1	1	1	2.4	7	2.4	1	1	1	
h=.1	1	1	0.986	1.05	0.814	10.093	0.814	1.05	0.986	1	1
case 2		1	1	1	2.04	7	2.04	1	1	1	
h=.01	1	1	0.934	1.23	0.146	10.427	0.146	1.23	0.934	1	1
case 3		1	1	1	2.004	7	2.004	1	1	1	
h=.001	1	1	0.929	1.248	0.079	10.461	0.079	1.248	0.929	1	1
case 4		1	1	1	2	7	2	1	1	1	
h=0	1	1	0.929	1.25	0.0714	10.464	0.071	1.25	0.929	1	1
case 5		1	1	1	1.996	7	1.996	1	1	1	
h=-.001	1	1	0.928	1.252	0.064	10.468	0.064	1.252	0.928	1	1
case 6		1	1	1	1.96	7	1.96	1	1	1	
h=-.01	1	1	0.923	1.27	-0.003	10.501	-0.003	1.27	0.923	1	1

Now we choose different values of h, and apply method 2 to the weights of Figure 5.7. The results are shown in Table 5.3. There are no large differences in the weights after interpolation (second row) from case 1 through case 3, when h assumed the value 0.1, 0.01 or 0.001 respectively. Thus method 2 was not sensitive to changes in h. When $h = -0.001$ in case 5 of Table 3, the weights after interpolation are still positive. Thus the sufficient condition in Chapter 4 is not a necessary condition. In case 6 when $h = -0.01$, v_5 and v_7 are negative.

Figure 5.12 shows the resulting curve with the given weights of Figure 5.7 and the given data in Table 5.4 (see [11]). Figure 5.13 is its curvature plot. Figure 5.14, Figure 5.16 and Figure 5.18 show the resulting curve with weights obtained by applying our three methods to the given weights of Figure 5.12 respectively and the same given data of Figure 5.12. Figure 5.15, Figure 5.17 and Figure 5.19 are their corresponding curvature plots. We also considered an integral cubic B-spline interpolant with the same given data of Figure 5.12 (Figure 5.20 for the curve and Figure 5.21 for the curvature plot). All of the resulting data are in Table 5.4.

Comparing Figure 5.12 with Figure 5.20 and Figure 5.21, shows that the high weight at a point with a * caused the curve to have a sharp bend in its neighborhood (the direction of the tangent line changes rapidly there) and its curvature plot Figure 5.13 shows that the magnitude of the curvature increases in neighborhoods on both sides of this point. The fifth and seventh weights obtained by interpolation are negative. Even though there are no singularities in this example, the resulting negative rational weights cause the loss of the convex hull property. The curves obtained by interpolation with modified weights by method 2 and method 3 (see Figure 5.16, Figure 5.18) are similar in shape in the neighborhoods on either side of a $*$ to the curves obtained by interpolation with the original assigned weights. However, the curve obtained by method 1 does not bend as sharply (see Figure 5.14). Comparing the curvature plot in Figure 5.13 and the curvature plots in Figure 5.15, Figure 5.17, and Figure 5.19, we can see that the three methods not only produce new weights which ensure positive rational weights, but also fair the original curve.

Table 5.4: Data and weights for Figures 5.12, 5.14, 5.16, 5.18 and 5.20.

In each of the following two-dimensional interpolation examples using (1.1.9), the first row gives the data values to be interpolated in the x direction, and the second row gives the data values to be interpolated in the y direction. For each Figure, the first row gives the control points of the resulting rational curve in the x direction after interpolation, and the second row gives the control points of the resulting rational curve in the y direction after interpolation. The assigned weights of Figure 5.12, Figure 5.14, Figure 5.16 and Figure 5.18 are from the given weights of Figure 5.7, Figure 5.9, Figure 5.10, and Figure 5.11 in Table 5.2 respectively. In Figure 5.20 the weights are all 1.

		1.9	1	1	1	2	3	3	3	2.1	
		3.19	2.7	1.42	0.85	0.35	0.85	1.42	2.7	3.19	
Fig.5.12	1.9	1.45	0.435	1.502	2.718	2	1.282	2.498	3.565	2.55	2.1
	3.19	3.158	3.655	0.806	-0.002	0.322	-0.002	0.806	3.655	3.158	3.19
Fig.5.14	1.9	1.45	0.701	1.266	-2.223	2	6.223	2.734	3.299	2.55	2.1
	3.19	3.158	3.094	1.022	2.422	0.263	2.422	1.022	3.094	3.158	3.19
Fig.5.16	1.9	1.45	0.546	1.657	-18.276	2	22.276	2.343	3.454	2.55	2.1
	3.19	3.158	3.209	0.788	11.294	0.274	11.294	0.788	3.209	3.158	3.19
Fig.5.18	1.9	1.435	0.593	1.839	-33.085	2	37.085	2.161	3.407	2.565	2.1
	3.19	3.099	3.075	0.567	19.941	0.273	19.941	0.567	3.075	3.099	3.19
Fig.5.20	1.9	1.45	0.771	1.128	0.718	2	3.282	2.872	3.229	2.55	2.1
	3.19	3.158	2.947	1.149	0.979	0.036	0.979	1.149	2.947	3.158	3.19

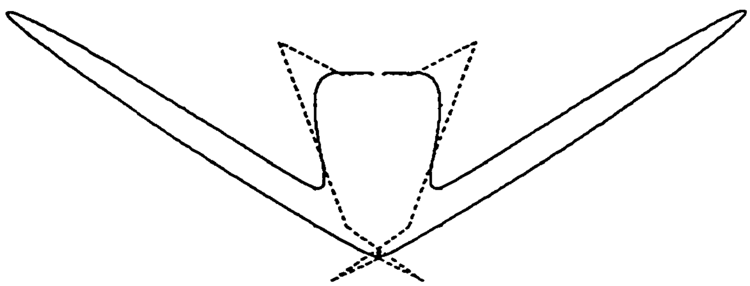

Figure 5.12: $\mathbf{L}(u)$. Assigned weights are 1, 1, 1, 1, 7, 1, 1, 1 and 1.

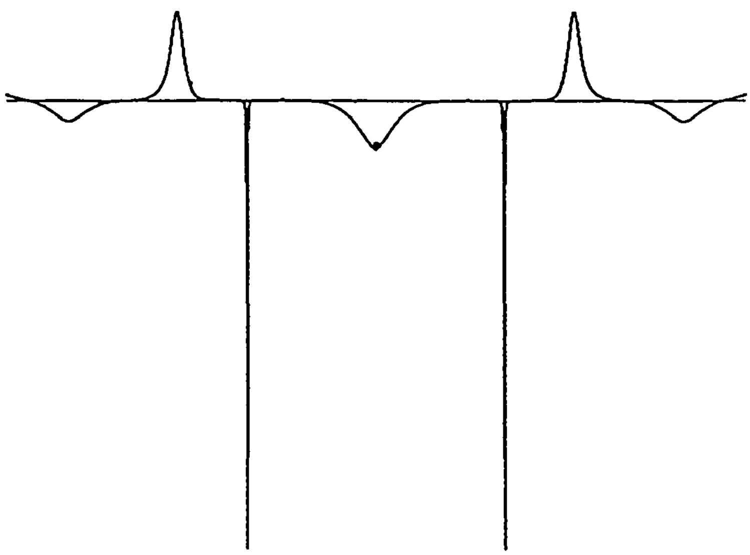

Figure 5.13: Curvature plot of Figure 5.12.

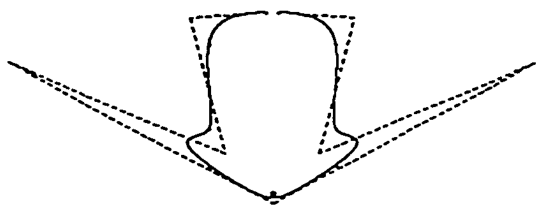

Figure 5.14: $\mathbf{L}(u)$. Assigned weights modified by method 1.

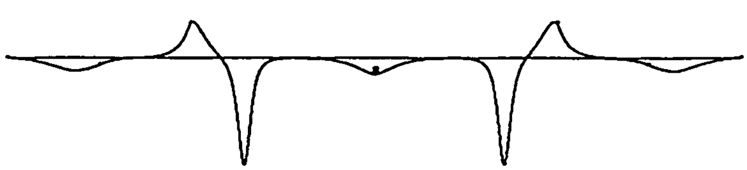

Figure 5.15: Curvature plot of Figure 5.14.

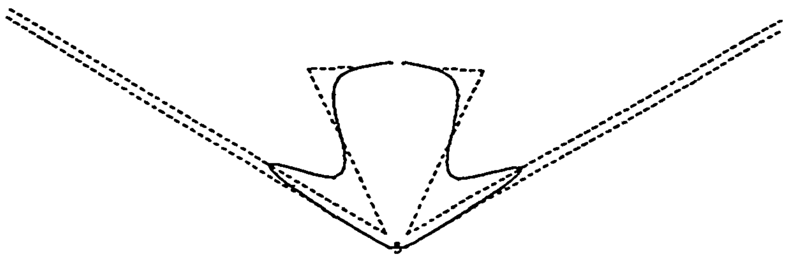

Figure 5.16: $\mathbf{L}(u)$. Assigned weights modified by method 2.

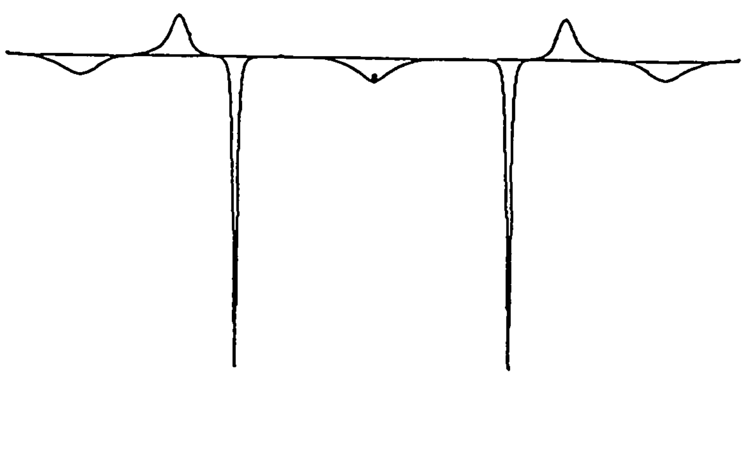

Figure 5.17: Curvature plot of Figure 5.16.

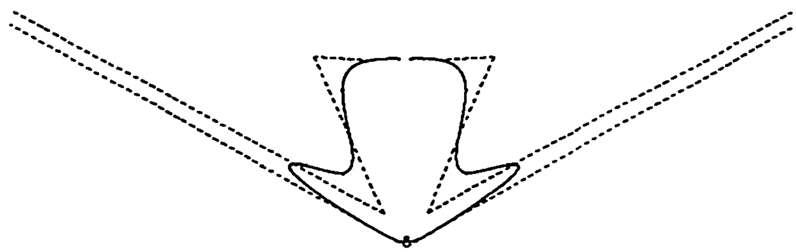

Figure 5.18: $\mathbf{L}(u)$. Assigned weights modified by method 3.

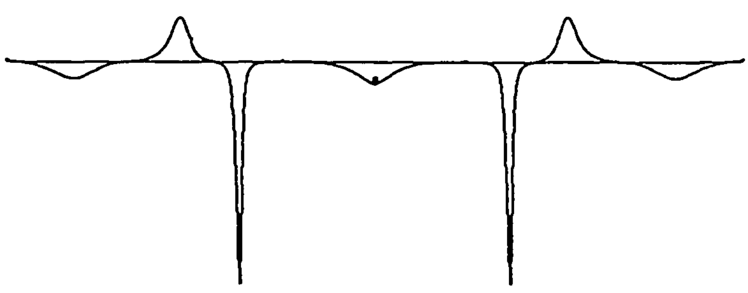

Figure 5.19: Curvature plot of Figure 5.18.

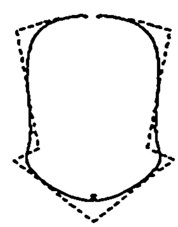

Figure 5.20: $\mathbf{L}(u)$. All assigned weights are 1.

Figure 5.21: Curvature plot of Figure 5.20.

Chapter 6

Degree Reduction of Rational Bézier and Rational B-Spline Curves

6.1 Introduction

Because of the limitation on the maximum polynomial degree that certain computer systems can store and work with, often a curve of high degree must be approximated by a number of curves of lower degree. This process of approximation is called degree reduction. (see [3, 4, 10, 36]). One of the main uses of a degree reduction algorithm is in rendering, curve-curve, curve-surface intersection calculations.

To achieve degree reduction of a rational curve we follow the procedure mentioned in Chapter 1: Given $2D$ control points $\{\mathbf{p}_i\}$ and their weights $\{w_i\}$, we transform them to $3D$ control points $\{[w_i\mathbf{p}_i \;\; w_i]\}$ and use $3D$ algorithms for degree reduction. The result of this procedure will be a set of $3D$ control points $\{[\mathbf{y}_i \;\; v_i]\}$. From these, we obtain $2D$ control points $\{\mathbf{y}_i/v_i\}$. The weights of these $2D$ control points are the numbers $\{v_i\}$.

The authors in [36, 3] use interpolation by choosing Chebyshev interpolation points or approximation in the least squares sense to reduce the degree of a Bézier curves. All of the resulting curves in [3, 4, 36] are in $C[0,1]$.

We consider degree reduction of a rational Bézier curve or a rational B-spline curve of degree greater than 3 by using rational cubic B-spline interpolation. The resulting curves are in $C^2[0,1]$. A rational cubic B-spline is acceptable in most computer systems.

We shall prove that we always can find parameter values to interpolate at so that the resulting rational cubic B-spline will have positive weights.

6.2 Degree Reduction

Degree reduction of $\mathbf{x}(u) = \frac{\sum_{i=0}^{n} \bar{w}_i \mathbf{b}_i B_i^n(u)}{\sum_{i=0}^{n} \bar{w}_i B_i^n(u)}$ and $\mathbf{s}(u) = \frac{\sum_{i=0}^{L+n-1} \bar{w}_i \mathbf{p}_i N_i^n(u)}{\sum_{i=0}^{L+n-1} \bar{w}_i N_i^n(u)}$ with degree greater than 3 is achieved by (1.1.9) defined by (1.1.11) and (1.1.12) to approximate the curve.

First we choose $u_1, \ldots, u_{L+1}$ such that $0 = u_1 < u_2 < \ldots < u_L < u_{L+1} = 1$. For degree reduction of $\mathbf{x}(u)$, we let

$$w_j = \sum_{i=0}^{n} \tilde{w}_i B_i^n(u_j) \qquad j = 1, \ldots, L+1 \tag{6.2.1}$$

and

$$w_j \mathbf{x}_j = w_j \mathbf{x}(u_j) = \sum_{i=0}^{n} \tilde{w}_i \mathbf{b}_i B_i^n(u_j) \qquad j = 1, \ldots, L+1. \tag{6.2.2}$$

We obtain a C^2 rational cubic B-spline approximation to $\mathbf{x}(u)$ by (1.1.9).

Similarly for degree reduction of $\mathbf{s}(u)$, we let

$$w_j = \sum_{i=0}^{L+n-1} \tilde{w}_i N_i^n(u_j) \qquad j = 1, \ldots, L+1 \tag{6.2.3}$$

and

$$w_j \mathbf{x}_j = w_j \mathbf{s}(u_j) = \sum_{i=0}^{L+n-1} \tilde{w}_i \mathbf{p}_i N_i^n(u_j) \qquad j = 1, \ldots, L+1. \tag{6.2.4}$$

We obtain a C^2 rational cubic B-spline approximation to $\mathbf{s}(u)$ by (1.1.9).

We will apply Theorem 2.3 to this degree reduction to obtain an error estimate.

Definition 6.1 *For the real numbers y_i, $i = 0, \ldots, m$, define*

$$y_{\inf} = \inf_{0 \le i \le m} y_i.$$

Definition 6.2 *For $w = (w_0, \ldots, w_m)$, define the norm:*

$$\|w\| = \max_{0 \le i \le m} | w_i |$$

Let $\mathbf{x} = (\mathbf{x}_0, \ldots, \mathbf{x}_m)$, *where* $\mathbf{x}_i = ((\mathbf{x}_i)_1, (\mathbf{x}_i)_2) \in E^2$, *for* $i = 0, \ldots, m$. *Define the norm:*

$$||| \mathbf{x} |||= \max_{1\le j\le 2} \max_{0\le i\le m} | (\mathbf{x}_i)_j | .$$

The convex hull property of Bézier curves and B-splines yield the following:

Lemma 6.1 *If all* $\tilde{w}_i > 0$, *then*

$$0 < \tilde{w}_{\text{inf}} \le \inf_{0\le u\le 1} \sum_{i=0}^{n} \tilde{w}_i B_i^n(u), \tag{6.2.5}$$

$$0 < \tilde{w}_{\text{inf}} \le \inf_{0\le u\le 1} \sum_{i=0}^{L+n-1} \tilde{w}_i N_i^n(u). \tag{6.2.6}$$

Lemma 6.2

$$\|\tilde{w}\| \ge \| \sum_{i=0}^{n} \tilde{w}_i B_i^n(u)\|_\infty, \tag{6.2.7}$$

$$\|\tilde{w}\| \ge \| \sum_{i=0}^{L+n-1} \tilde{w}_i N_i^n(u)\|_\infty. \tag{6.2.8}$$

Lemma 6.3

$$||| \tilde{w}\mathbf{b} |||\ge||| \sum_{i=0}^{n} \tilde{w}_i \mathbf{b}_i B_i^n(u) |||_\infty, \tag{6.2.9}$$

$$||| \tilde{w}\mathbf{p} |||\ge||| \sum_{i=0}^{L+n-1} \tilde{w}_i \mathbf{p}_i N_i^n(u) |||_\infty, \tag{6.2.10}$$

where $||| \,.\, |||_\infty$ *is as defined on page 15.*

Definition 6.3 *Let* $\Delta_i = u_{i+1} - u_i$,

$$h = \max_{0 \le i \le L-1} \Delta_i,$$

and let

$$\Delta^r q_j = \Delta^{r-1} q_{j+1} - \Delta^{r-1} q_j = \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} q_{i+j}, \tag{6.2.11}$$

where $q_i \in E^k$ $k = 1, 2$.

We have

$$\frac{d^r}{du}(\sum_{i=0}^{n} \tilde{w}_i B_i^n(u)) = \frac{n!}{(n-r)!} \sum_{i=0}^{n-r} \Delta^r \tilde{w}_i B_i^n(u), \tag{6.2.12}$$

and

$$\frac{d^r}{du}(\sum_{i=0}^{n} \tilde{w}_i \mathbf{b}_i B_i^n(u)) = \frac{n!}{(n-r)!} \sum_{i=0}^{n-r} \Delta^r (\tilde{w}_i \mathbf{b}_i) B_i^n(u). \tag{6.2.13}$$

The convex hull property of Bézier curve yields the following:

Lemma 6.4 *We have*

$$\|\frac{d^r}{du}(\sum_{i=0}^{n} \tilde{w}_i B_i^n(u))\|_\infty \le \frac{n!}{(n-r)!} \|\Delta^r \tilde{w}\|, \tag{6.2.14}$$

and

$$|||\ \frac{d^r}{du}(\sum_{i=0}^{n} \tilde{w}_i \mathbf{b}_i B_i^n(u))\ |||_\infty \le \frac{n!}{(n-r)!}\ |||\ \Delta^r(\tilde{w}\mathbf{b})\ |||, \tag{6.2.15}$$

where

$$\|\Delta^r \tilde{w}\| = \max_{0 \le i \le n-r} |\ \Delta^r \tilde{w}_i\ |,$$

and

$$||| \Delta^r(\tilde{w}\mathbf{b}) |||= \max_{1\le j\le 3} \|(\Delta^r(\tilde{w}\mathbf{b}))_j\|,$$

where

$$\|(\Delta^r(\tilde{w}\mathbf{b}))_j\| = \max_{0\le i\le n-r} | (\Delta^r(\tilde{w}_i\mathbf{b}_i))_j | .$$

Let

$$R(u) = \sum_{i=0}^{L+n-1} \tilde{w}_i N_i^n(u)$$

and

$$\mathbf{M}(u) = \sum_{i=0}^{L+n-1} \tilde{w}_i \mathbf{p}_i N_i^n(u).$$

Consider

$$\frac{d}{du}R(u) = \frac{d}{du}\Big(\sum_{i=0}^{L+n-1} \tilde{w}_i N_i^n(u)\Big) = n\sum_{i=1}^{L+n-1} \frac{\Delta\tilde{w}_i}{\tilde{u}_{n+i-1}-\tilde{u}_{i-1}} N_i^{n-1}(u) \qquad (6.2.16)$$

and

$$\frac{d}{du}\mathbf{M}(u) = \frac{d}{du}\Big(\sum_{i=0}^{L+n-1} \tilde{w}_i \mathbf{p}_i N_i^n(u)\Big) = n\sum_{i=1}^{L+n-1} \frac{\Delta(\tilde{w}_i\mathbf{p}_i)}{\tilde{u}_{n+i-1}-\tilde{u}_{i-1}} N_i^{n-1}(u). \qquad (6.2.17)$$

Note that

$$\sum_{i=1}^{L+n-1} N_i^{n-1}(u) = 1$$

and let

$$\tilde{h} = \inf_{1\le i\le L+n-1} (\tilde{u}_{n+i-1} - \tilde{u}_{i-1}).$$

We have the following Lemma

Lemma 6.5 *If $R(u) \in PC^{1,\infty}[0,1]$ and $(\mathbf{M}(u))_j \in PC^{1,\infty}[0,1]$, for $j = 1, 2$,*

$$\|DR(u)\|_{\infty,s} = \|\frac{d}{du}(\sum_{i=0}^{L+n-1} \tilde{w}_i N_i^n(u))\|_{\infty,s} \le \frac{n}{\bar{h}}\|\Delta\tilde{w}\| \tag{6.2.18}$$

and

$$||| D\mathbf{M}(u) |||_{\infty,s} = ||| \frac{d}{du}(\sum_{i=0}^{n} \tilde{w}_i \mathbf{p}_i N_i^n(u)) |||_{\infty,s} \le \frac{n}{\bar{h}} ||| \Delta(\tilde{w}\mathbf{p}) |||, \tag{6.2.19}$$

where $\|.\|_{\infty,s}$ and $||| \, . \, |||_{\infty,s}$ are as defined on page 16.

Similarly we can obtain the uniform norm (defined as above) of the higher order derivative of $R(u)$ and $\mathbf{M}(u)$.

Corresponding to Lemma 2.2 we have the following:

Lemma 6.6 *If all $\tilde{w}_i > 0$ and h is sufficiently small such that*

$$\tilde{w}_{\text{inf}} \ge \frac{5}{192} \frac{n!}{(n-4)!} \|\Delta^4 \tilde{w}\| h^4, \tag{6.2.20}$$

then we have

$$\inf_{0 \le u \le 1} (\sum_{i=0}^{L+2} v_i N_i^3(u)) \ge \frac{1}{2} \tilde{w}_{\text{inf}}, \tag{6.2.21}$$

Proof. This follows from Lemma 6.1 and letting $z = \tilde{w}_{\text{inf}}$ in Lemma 2.2

Lemma 6.7 *If all $\tilde{w}_i > 0$ and h is sufficiently small such that*

$$\frac{1}{2}\tilde{w}_{\text{inf}} \ge \frac{2}{3} \|D^2 R(u)\|_{\infty,s} h^2, \tag{6.2.22}$$

where $R(u) \in PC^{2,\infty}[0,1]$, *then we have*

$$\inf_{0\le u\le 1} \sum_{i=0}^{L+2} v_i N_i^3(u) \ge \frac{1}{2}\tilde{w}_{\inf}. \tag{6.2.23}$$

Also if all $\tilde{w}_i > 0$ *and* $R(u) \in PC^{4,\infty}[0,1]$ *and* h *is sufficiently small such that*

$$\tilde{w}_{\inf} \ge \frac{5}{192}\|D^4 R(u)\|_{\infty,s} h^4, \tag{6.2.24}$$

then we have (6.2.23) .

Proof. From Lemma 6.1 and letting $z = \tilde{w}_{\inf}$ in Lemma 2.2 the result of this Lemma follows.

From Lemma 6.6 and Theorem 2.3 we immediately obtain the following convergence theorem:

Theorem 6.1 *If all* $\tilde{w}_i > 0$ *and* h *is sufficiently small such that (6.2.20) holds, then we have*

$$||| \, L(u) - \mathbf{x}(u) \, |||_\infty \le \frac{5n!(\|\tilde{w}\| \; ||| \, \Delta^4(\tilde{w}\mathbf{b}) \, ||| + \|\Delta^4 \tilde{w}\| \; ||| \, (\tilde{w}\mathbf{b}) \, |||)}{192(n-4)!\tilde{w}_{\inf}^2} h^4. \tag{6.2.25}$$

From Lemma 6.7 and Theorem 2.3 we immediately obtain the following convergence theorem:

Theorem 6.2 *If all* $\tilde{w}_i > 0$ *and* $R(u) \in PC^{2,\infty}[0,1]$, $(\mathbf{M}(u))_j \in PC^{2,\infty}[0,1]$, *for* $j = 1, 2$, *and* h *is sufficient small such that (6.2.22) holds, then we have*

$$||| \, L(u) - \mathbf{s}(u) \, |||_\infty \le \frac{(\|\tilde{w}\| \; ||| \, D^2\mathbf{M}(u) \, |||_{\infty,s} + \|D^2 R(u)\|_{\infty,s} \; ||| \, (\tilde{w}\mathbf{p}) \, |||)}{3\tilde{w}_{\inf}^2} h^2. \tag{6.2.26}$$

Furthermore, if $R(u) \in PC^{4,\infty}[0,1]$, $(\mathbf{M}(u))_j \in PC^{4,\infty}[0,1]$, *for* $j = 1, 2$, *and* h *is sufficiently small such that (6.2.24) holds, then we have*

$$||| L(u) - \mathbf{s}(u) |||_\infty \leq \frac{5(\|\tilde{w}\| \; ||| D^4\mathbf{M}(u) |||_{\infty,s} + \|D^4R(u)\|_{\infty,s} \; ||| (\tilde{w}\mathbf{p}) |||)}{192\tilde{w}_{\inf}^2} h^4. \tag{6.2.27}$$

6.3 Weights

We shall prove that we always can find interpolation parameter values which will guarantee positive weights of the resulting curves.

As indicated in section 1 of Chapter 1 we use cubic B-spline interpolation in 3 dimensions to achieve 2 dimensional rational cubic B-spline interpolation. In this section we will discuss the third component of a 3D cubic B-spline interpolation, since we are primarily interested in the weights.

clamped end conditions are used in the interpolation in this chapter. We will find b_1 and b_{3L-1} in the interpolation (1.1.11) for (6.2.1). If we change a B-spline which has the control points $\{[v_i\mathbf{d}_i \;\; v_i]\}_0^{L+2}$ into piecewise Bézier form, b_1 will be the fourth component of the second control point in its first segment, and b_{3L-1} will be the fourth component of the next to last control point in its last segment. Thus we have (see [10]):

$$\frac{d}{du}\left(\sum_{i=0}^{L+2} v_i N_i^3(0)\right) = \frac{1}{\Delta_1} 3(b_1 - v_0),$$

i.e.,

$$\frac{d}{du}\left(\sum_{i=0}^{L+2} v_i N_i^3(0)\right) = \frac{1}{\Delta_0} 3(b_1 - \tilde{w}_0), \qquad \text{since} \qquad v_0 = w_1 = \tilde{w}_0.$$

For the Bézier curve $\sum_{i=0}^{n} \tilde{w}_i B_i^n$ we have

$$\frac{d}{du}\left(\sum_{i=0}^{n} \tilde{w}_i B_i^n(0)\right) = n(\tilde{w}_1 - \tilde{w}_0).$$

Under the clamped end conditions we have

$$\frac{d}{du}\left(\sum_{i=0}^{n} \tilde{w}_i B_i^n(0)\right) = \frac{d}{du}\left(\sum_{i=0}^{L+2} v_i N_i^3(0)\right).$$

It follows that

$$b_1 = \frac{n}{3}(\tilde{w}_1 - \tilde{w}_0)\Delta_1 + \tilde{w}_0. \qquad (6.3.28)$$

Similarly, we obtain

$$b_{3L-1} = \frac{n}{3}(\tilde{w}_{n-1} - \tilde{w}_n)\Delta_L + \tilde{w}_n. \qquad (6.3.29)$$

Now we have the following Theorem:

Theorem 6.3 *If the parameter values $u_0 \ldots u_L$ satisfy the condition*

$$\Delta_{i-1} < (\Delta_i + \Delta_{i+1}), \ \ \Delta_i < (\Delta_{i-2} + \Delta_{i-1}) \qquad \textit{for all } i = 2, \ldots, L, \qquad (6.3.30)$$

$$\Delta_1 = \Delta_2, \qquad \Delta_{L-1} = \Delta_L, \qquad (6.3.31)$$

$$\Delta_i \leq m\Delta_{i-1}, \qquad \textit{for } i = 2, \ldots, L, \qquad (6.3.32)$$

and

$$h < \min\left(\frac{\tilde{w}_{\inf}}{(m+1)n\|\Delta\tilde{w}\|}, \frac{3\tilde{w}_0}{4n \mid \tilde{w}_1 - \tilde{w}_0 \mid}, \frac{3\tilde{w}_n}{4n \mid \tilde{w}_{n-1} - \tilde{w}_n \mid}\right), \qquad (6.3.33)$$

where m is an arbitrarily fixed number greater than or equal to one, then $v_i > 0$ for all i in (1.1.11) for (6.2.1).

Proof. From Lemma 6.1 it follows that all w_i defined in (6.2.1) are positive. So $v_0 = w_1 => 0$ and $v_{L+2} = w_{L+1} => 0$. By (6.3.33) it follows that

$$b_1 = \frac{n}{3}(\tilde{w}_1 - \tilde{w}_0)\Delta_1 + \tilde{w}_0 > \frac{3}{4}\tilde{w}_0 > 0$$

and

$$b_{3L-1} = \frac{n}{3}(\tilde{w}_{n-1} - \tilde{w}_n)\Delta_L + \tilde{w}_n > \frac{3}{4}\tilde{w}_n > 0.$$

Now we prove that (4.2.5) in Theorem 4.1 are satisfied. From

$$\Delta_i \le m\Delta_{i-1}, \quad \text{for } i = 2, \ldots, L,$$

it follows that

$$\frac{\Delta_{i-1}}{\Delta_{i-1} + \Delta_i} \ge \frac{\Delta_{i-1}}{(m+1)\Delta_{i-1}} = \frac{1}{m+1}, \quad \text{for } i = 2, \ldots, L. \tag{6.3.34}$$

Let

$$\eta = \frac{w_{\text{inf}}}{m+1}. \tag{6.3.35}$$

Since $\sum_{i=0}^{n} \tilde{w}_i B_i^n(u)$ is a continuous function on $[0,1]$, we have

$$\begin{aligned} \left| \sum_{i=0}^{n} \tilde{w}_i B_i^n(u_i) - \sum_{i=0}^{n} \tilde{w}_i B_i^n(u_j) \right| &\le \left\| \frac{d}{du}\left(\sum_{i=0}^{n} \tilde{w}_i B_i^n(u)\right) \right\|_\infty \mid u_i - u_j \mid \\ &\le n\|\Delta\tilde{w}\| \mid u_i - u_j \mid, \quad \forall u_i, u_j \in [0,1]. \end{aligned}$$

It follows that, if h satisfies (6.3.33), we have

$$\mid w_i - w_{i+1} \mid < \eta, \quad \text{for } i = 1, \ldots, L. \tag{6.3.36}$$

where w_i is defined by (6.2.1). We assume h satisfies (6.3.33) from now on.

First we examine the first inequality of (4.2.5). Considering that (6.3.33) holds and $\tilde{w}_0 = w_1$, we have

$$b_1 = \frac{n}{3}(\tilde{w}_1 - \tilde{w}_0)\Delta_1 + \tilde{w}_0 < \frac{5}{4}w_1.$$

We rewrite the first inequality of (4.2.5) as the following:

$$(\Delta_1 + \Delta_2)w_2 - [\frac{5}{4}w_1\frac{\Delta_2^2}{\Delta_1 + \Delta_2} + (\Delta_2 + \Delta_3)w_3\frac{\gamma_2}{\beta_2}] > 0. \tag{6.3.37}$$

From (6.3.36) it follows that $| w_1 - w_2 | < \eta$ and $| w_2 - w_3 | < \eta$. We only discuss the case where $w_1 < w_2 + \eta$ and $w_3 < w_2 + \eta$. The proof of the other cases are similar and are omitted. In this case,

$$\begin{aligned}&(\Delta_1 + \Delta_2)w_2 - [\frac{5}{4}w_1\frac{\Delta_2^2}{\Delta_1 + \Delta_2} + (\Delta_2 + \Delta_3)w_3\frac{\gamma_2}{\beta_2}] \\ &\quad > (\Delta_1 + \Delta_2)w_2 - [\frac{5}{4}(w_2 + \eta)\frac{\Delta_2^2}{\Delta_1 + \Delta_2} + (\Delta_2 + \Delta_3)(w_2 + \eta)\frac{\gamma_2}{\beta_2}].\end{aligned} \tag{6.3.38}$$

Now $\Delta_1 = \Delta_2$ and from the definition of η it follows that

$$\begin{aligned}\Delta_2 w_2 - \frac{5}{4}(w_2 + \eta)\frac{\Delta_2^2}{\Delta_1 + \Delta_2} &= \Delta_2 w_2 - \frac{5}{4}\frac{\Delta_2^2}{\Delta_1 + \Delta_2}w_2 - \frac{5}{4}\frac{\Delta_2^2}{\Delta_1 + \Delta_2}\eta \\ &= \Delta_2 w_2 - \frac{5}{8}\Delta_2 w_2 - \frac{5}{8}\Delta_2\eta \\ &= \frac{3}{8}\Delta_2 w_2 - \frac{5}{8}\Delta_2\eta > 0\end{aligned}$$

and

$$\Delta_1 w_2 - (\Delta_2 + \Delta_3)(w_2 + \eta)\frac{\gamma_2}{\beta_2}$$

$$= (\Delta_2 + \Delta_3)\frac{\gamma_2}{\frac{\Delta_1(\Delta_2+\Delta_3)}{\Delta_1+\Delta_2+\Delta_3}} w_2 - (\Delta_2 + \Delta_3)\frac{\gamma_2}{\beta_2}(w_2 + \eta)$$

$$= (\Delta_2 + \Delta_3)\frac{\Delta_2}{\Delta_2 + \Delta_3}\frac{\frac{\Delta_1^2}{\Delta_1+\Delta_2}}{\frac{\Delta_1(\Delta_2+\Delta_3)}{\Delta_1+\Delta_2+\Delta_3}} w_2 - (\Delta_2 + \Delta_3)\frac{\gamma_2}{\beta_2}\eta$$

$$> (\Delta_2 + \Delta_3)\frac{\gamma_2}{\frac{\Delta_1(\Delta_2+\Delta_3)}{\Delta_1+\Delta_2+\Delta_3}}\frac{\Delta_2}{\Delta_2 + \Delta_3} w_2 - (\Delta_2 + \Delta_3)\frac{\gamma_2}{\beta_2}\eta \geq 0.$$

So (6.3.38) > 0, i.e. the first inequality of (4.2.5) holds.

Now we examine the second inequality of (4.2.5). From (6.3.36) it follows that $| w_i - w_{i-1} | < \eta$ and $| w_i - w_{i+1} | < \eta$. We only discuss the case where $w_{i-1} < w_i + \eta$ and $w_{i+1} < w_i + \eta$. The proof of the other cases are similar and are omitted. In this case,we have

$$(\Delta_{i-1} + \Delta_i)w_i - [(\Delta_{i-2} + \Delta_{i-1})w_{i-1}\frac{\alpha_i}{\beta_i} + (\Delta_i + \Delta_{i+1})w_{i+1}\frac{\gamma_i}{\beta_i}]$$

$$> (\Delta_{i-1} + \Delta_i)w_i - [(\Delta_{i-2} + \Delta_{i-1})(w_i + \eta)\frac{\alpha_i}{\beta_i} + (\Delta_i + \Delta_{i+1})(w_i + \eta)\frac{\gamma_i}{\beta_i}]$$

$$= w_i[(\Delta_{i-1} + \Delta_i) - (\Delta_{i-2} + \Delta_{i-1})\frac{\alpha_i}{\beta_i} - (\Delta_i + \Delta_{i+1})\frac{\gamma_i}{\beta_i}]$$

$$-\eta(\Delta_{i-2} + \Delta_{i-1})\frac{\alpha_i}{\beta_i} - \eta(\Delta_i + \Delta_{i+1})\frac{\gamma_i}{\beta_i}.$$

First we have

$$\Delta_{i-1} - (\Delta_i + \Delta_{i+1})\frac{\gamma_i}{\beta_i} = \frac{\Delta_{i-1}}{\Delta_i}(\Delta_{i-2} + \Delta_{i-1})\frac{\alpha_i}{\beta_i}$$

$$> \frac{\Delta_{i-1}}{\Delta_{i-1} + \Delta_i}(\Delta_{i-2} + \Delta_{i-1})\frac{\alpha_i}{\beta_i},$$

i.e.,

$$\Delta_{i-1} - (\Delta_i + \Delta_{i+1})\frac{\gamma_i}{\beta_i} > \frac{\Delta_{i-1}}{\Delta_{i-1} + \Delta_i}(\Delta_{i-2} + \Delta_{i-1})\frac{\alpha_i}{\beta_i}. \qquad (6.3.39)$$

Similarly, we have

$$\Delta_i - (\Delta_{i-2} + \Delta_{i-1})\frac{\alpha_i}{\beta_i} > \frac{\Delta_i}{\Delta_i + \Delta_{i+1}}(\Delta_i + \Delta_{i+1})\frac{\gamma_i}{\beta_i}. \tag{6.3.40}$$

From (6.3.39) and (6.3.40) it follows that

$$\begin{aligned} w_i[(\Delta_{i-1} + \Delta_i) - (\Delta_{i-2} + \Delta_{i-1})\frac{\alpha_i}{\beta_i} - (\Delta_i + \Delta_{i+1})\frac{\gamma_i}{\beta_i}] \\ - \eta(\Delta_{i-2} + \Delta_{i-1})\frac{\alpha_i}{\beta_i} - \eta(\Delta_i + \Delta_{i+1})\frac{\gamma_i}{\beta_i} \\ > w_i\frac{\Delta_{i-1}}{\Delta_{i-1} + \Delta_i}(\Delta_{i-2} + \Delta_{i-1})\frac{\alpha_i}{\beta_i} + w_i\frac{\Delta_i}{\Delta_i + \Delta_{i+1}}(\Delta_i + \Delta_{i+1})\frac{\gamma_i}{\beta_i} \\ -\eta(\Delta_{i-2} + \Delta_{i-1})\frac{\alpha_i}{\beta_i} - \eta(\Delta_i + \Delta_{i+1})\frac{\gamma_i}{\beta_i} \geq 0 \text{ for } i = 2, \ldots, L-2. \end{aligned}$$

The last inequality is from the definition of η, Lemma 6.1 and (6.3.34). Thus the second inequality of (4.2.5) holds.

The proof of the third inequality of (4.2.5) is similar to the proof of the first inequality of (4.2.5) and is thus omitted.

Now Theorem 4.1 implies Theorem 6.3.

Remark: If $\tilde{w}_1 = \tilde{w}_0$ we choose

$$h < \min(\frac{\tilde{w}_{\inf}}{(m+1)n\|\Delta\tilde{w}\|}, \frac{3\tilde{w}_n}{4n \mid \tilde{w}_{n-1} - \tilde{w}_n \mid})$$

in (6.3.33). The other cases are similar to the above.

We can find b_1 and b_{3L-1} in the interpolation (1.1.11) for (6.2.3) under the clamped end conditions in a similar way as follows:

$$b_1 = \frac{n}{3}(\tilde{w}_1 - \tilde{w}_0)\frac{\Delta_1}{\tilde{h}_1} + \tilde{w}_0$$

and

$$b_{3L-1} = \frac{n}{3}(\tilde{w}_{l+n-2} - \tilde{w}_{L+n-1})\frac{\Delta_L}{\tilde{h}_l} + \tilde{w}_{L+n-1},$$

where $\tilde{h}_1$ is the width of the first segment of the piecewise Bézier form of a B-spline which has the control points $\{[\tilde{w}_i\mathbf{p}_i \;\; \tilde{w}_i]\}_0^{L+n-1}$, and $\tilde{h}_l$ is the width of the last segment of the piecewise Bézier form of a B-spline which has the control points $\{[\tilde{w}_i\mathbf{p}_i \;\; \tilde{w}_i]\}_0^{L+n-1}$.

Similarly, we can obtain

Theorem 6.4 *If the parameter values $u_1 \ldots u_{L+1}$ satisfy the condition*

$$\Delta_{i-1} < (\Delta_i + \Delta_{i+1}), \quad \Delta_i < (\Delta_{i-2} + \Delta_{i-1}), \quad \textit{for all } i = 1, \ldots, L, \tag{6.3.41}$$

$$\Delta_1 = \Delta_2, \qquad \Delta_{L-1} = \Delta_L,$$

$$\Delta_i \leq m\Delta_{i-1} \quad i = 2, \ldots, L, \tag{6.3.42}$$

and

$$h < \min(\frac{\tilde{h}\tilde{w}_{\inf}}{(m+1)n\|\Delta\tilde{w}\|}, \frac{3\tilde{h}_1\tilde{w}_0}{4n \mid \tilde{w}_1 - \tilde{w}_0 \mid}, \frac{3\tilde{h}_l\tilde{w}_{L+n-1}}{4n \mid \tilde{w}_{l+n-2} - \tilde{w}_{L+n-1} \mid}), \tag{6.3.43}$$

where m is an arbitrary fixed number greater than or equal to 1, then we have $v_i > 0$ in (1.1.11) for (6.2.3)

6.4 Algorithms and Examples

Given a rational Bézier curve of degree greater than three we have the following algorithm to find a cubic rational B-spline with positive weights which approximates the original rational Bézier curve within a given tolerance $\epsilon > 0$.

Algorithm I:

1. $j = 1$

Let h_1 be a value of h that satisfies (6.2.20) in Lemma 6.6,

let h_2 satisfy

$$\frac{5n!(\|\tilde{w}\| \; ||| \; \Delta^4(\tilde{w}\mathbf{b}) \; ||| + \|\Delta^4\tilde{w}\| \; ||| \; (\tilde{w}\mathbf{b}) \; |||)}{192(n-4)!\tilde{w}_{\inf}^2} h_2^4 \leq \epsilon \tag{6.4.44}$$

and let

$$\hat{h}_1 = \min(h_1, h_2).$$

2. Choose parameters $\{u_i\}_{i=1}^{L+1}$ such that $\Delta_1 = \Delta_2$, $\Delta_{L-1} = \Delta_L$,

Δ_i, $i = 2, \ldots, L$ satisfy (6.3.30) and all $\Delta_i \leq \hat{h}_j$.

3. Obtain the $\tilde{w}_i's$ by (6.2.1) and then examine if the $\tilde{w}_i's$ satisfy (4.2.5). If so, then go to step **5**.

4. $j = j + 1$, $\hat{h}_j = \frac{1}{2}\hat{h}_{j-1}$ go to step **2**.

5. Interpolate by using (1.1.9) defined by (1.1.11) and (1.1.12) for (6.2.1) and (6.2.2) respectively.

The definition of h_1, h_2 and $\hat{h}_1$ ensure that if all $\Delta_i \le \hat{h}_1$, then the interpolation error will be less than ϵ by Theorem 6.1.

Theorem 6.3 tells us that if the parameter values of the interpolant satisfy the parameter condition of Theorem 6.3, and h satisfies (6.3.33), we will obtain all $v_i > 0$ by (1.1.11) for (6.2.1). Condition (6.3.33) is much stricter than (6.2.20) and (6.4.44) in general cases, since the power of h in (6.2.20) and (6.4.44) is four. (See the following examples). Also (6.3.33) is a sufficient condition and thus even if (6.3.33) is not satisfied, all v_i may be positive (see the following examples). An alternative is not to find h which satisfies (6.3.33), but instead to consider an iterative method to find h which is as large as possible while still yielding positive weights. Here $\hat{h}_1$ is a good initial choice, since the definition of h_1 and Lemma 6.6 imply that the resulting weight function

$$\sum_{i=0}^{L+2} v_i N_i^3(u) > 0, \qquad \text{for all } u \in [0,1].$$

In the first loop, Step **2** ensures that the $\Delta_i's$ satisfy (6.3.30) of Theorem 6.3 and all $\Delta_i \le \hat{h}_1$. We always can choose m large enough such that (6.3.32) holds. Thus, in practice we ignore (6.3.32). The interpolation error will be less than ϵ by Theorem 6.1, since all $\Delta_i \le \hat{h}_1$. From the second loop, we can choose the middle points of each segment as extra parameter values and combine them with the original ones to form new parameter values. The new parameter values still satisfy the parameter condition of Theorem 6.3 and all $\Delta_i \le \hat{h}_j \le \hat{h}_1$.

Step **3** check if $\tilde{w}_i$ satisfy (4.2.5) instead of interpolating to save time to check if all v_i are positive now.

Step **4** ensure that $\hat{h}_j \to 0$ as $j \to \infty$. $\hat{h}_j \to 0$ and Theorem 6.3 implies that algorithm 1 will produce all $v_i > 0$.

Step **5** produce a C^2 cubic rational B-spline. The definitions of h_1, h_2, $\hat{h}_j$ and step **2** ensure that the interpolation error is less than ϵ by Theorem 6.1. Thus from the above discussion we have the following:

Theorem 6.5 *Algorithm I produces a C^2 rational cubic B-spline with positive weights which approximates a given rational Bézier curve of degree greater than three within a given tolerance $\epsilon > 0$.*

We have a similar algorithm for the degree reduction of a rational B-spline curve within a given tolerance $\epsilon > 0$:

Algorithm II:

1. $j = 1$

Let h_1 be the value of h that satisfies (6.2.22) in Lemma 6.7, let h_2 satisfy

$$\frac{(\|\tilde{w}\| \,|||\, D^2\mathbf{M}(u) \,|||_{\infty,s} + \|D^2 R(u)\|_{\infty,s} \,|||\, (\tilde{w}\mathbf{p}) \,|||)}{3\tilde{w}^2_{\text{inf}}} h_2^2 \le \epsilon, \tag{6.4.45}$$

where $(\mathbf{M}(u))_j \in PC^{2,\infty}[0,1]$, for $j = 1,2$ and 3, and $R(u) \in PC^{2,\infty}[0,1]$.

If $(\mathbf{M}(u))_j \in PC^{4,\infty}[0,1]$, for $j = 1,2$ and 3, and $R(u) \in PC^{4,\infty}[0,1]$. Let h_1 be the value of h that satisfies (6.2.24) in Lemma 6.7

and let h_2 satisfy

$$\frac{5(\|\tilde{w}\| \,|||\, D^4\mathbf{M}(u) \,|||_{\infty,s} + \|D^4 R(u)\|_{\infty,s} \,|||\, (\tilde{w}\mathbf{p}) \,|||)}{192\tilde{w}^2_{\text{inf}}} h_2^4 \le \epsilon \tag{6.4.46}$$

and let

$$\hat{h}_1 = \min(h_1, h_2).$$

2. Choose parameters $\{u_i\}_{i=1}^{L+1}$ such that $\Delta_1 = \Delta_2$, $\Delta_{L-1} = \Delta_L$, Δ_i, $i = 2, \ldots, L$ satisfy (6.3.41) and all $\Delta_i \leq \hat{h}_j$.

3. Examine if $\tilde{w}_i, s$ obtained by (6.2.3) satisfy (4.2.5). If so, then go to step **5.**

4. $j = j + 1$, $\hat{h}_j = \frac{1}{2}\hat{h}_{j-1}$ go to step **2.**

5. Interpolate by using (1.1.9) defined by (1.1.11) and (1.1.12) for (6.2.3) and (6.2.4) respectively.

We have the following:

Theorem 6.6 *Algorithm II produces a C^2 rational cubic B-spline with positive weights which approximates a given rational B-spline curve of degree greater than three within a given tolerance $\epsilon > 0$.*

Example 6.1 is for degree reduction of a rational Bézier curve of degree 8. Given $\epsilon = 0.1$, we obtain $h_1 \leq 0.088$ by (6.2.20) and $h_2 \leq 0.048$ by (6.4.44), so $\hat{h}_1 = 0.048$. We choose equal $\Delta_i's$ so m may be chosen as 1 in (6.3.32). In order to satisfy (6.3.34) in Theorem 6.3 , h should be less than 0.0048. But we choose $\hat{h}_1 = 0.048$. Thus we must subdivide the interpolation interval into 22 subintervals(i.e. L in (1.1.9) is 22). In this example all $v_i > 0$. Actually, even if we choose 8 subintervals, we still have all $v_i > 0$.

Figure 6.1 is the graph of the given 8th degree rational Bézier curve. The bottom of the heart in Figure 6.1 has a corner. The weight of the control point there is 14 and the weights of the other control points range from 1 to 1.2. Figure 6.2 is the graph of the resulting rational cubic B-spline, with 22 subintervals. There are no apparent significant differences between Figure 6.1 and Figure 6.2. Figure 6.3 is the graph of the resulting rational cubic B-spline, (with 8 subintervals). This function has positive rational weights.

Example 6.2 is for degree reduction of a rational B-spline in two dimensions. We consider a rational B-spline of degree 4 with 19 simple knots. Given $\epsilon = 0.1$, we obtained $h_1 \leq 0.246$ by (6.2.20) and $h_2 \leq 0.014498$ by (6.4.46), so $\hat{h}_1 = 0.014498$. We choose equal Δ_i's so m may be chosen as 1 in (6.3.32). In order to satisfy (6.3.43) in Theorem 6.4, h should be less than 0.001389!. But we choose $\hat{h}_1 =$ 0.014498, requiring 69 subintervals.

Figure 6.4 is the graph of the given 4th degree rational B-spline. The mouth of the face in Figure 6.4 has a corner. The weight of the control point there is 12 and the weights of the control points near it are as low as 1. Figure 6.5 is the graph of the resulting rational cubic B-spline with 69 subintervals. There are no significant differences between Figure 6.4 and Figure 6.5. In this example all $v_i > 0$. We know that (6.4.46) is a sufficient condition. Figure 6.6 is the graph of the resulting rational cubic B-spline curve with 24 subintervals and its weights are all positive. However there are small differences between Figure 6.4 and Figure 6.6. Figure 6.7

is the graph of the resulting rational cubic B-spline curve with 12 subintervals and one of its rational weights is negative.

Example 6.3 is for degree reduction of a rational Bézier curve in three dimensions. Farin found that an octant of the sphere can be exactly represented by a certain rational quartic Bézier triangular patch, and then eight copies of this octant patch can be assembled to represent the whole sphere. There are no singularities at the north and south poles in this representation of the whole sphere. We know that the boundary curves of a triangular patch are determined by the boundary control vertices (having at least one zero as a subscript), i.e. the three boundary curves of the rational quartic Bézier triangular patch representation of the octant of the sphere are the rational quartic curves. We use these three rational quartic curves as an example. Given $\epsilon = 0.1$ we obtain $h_1 \leq 1.23$ by (6.2.20) and $h_2 \leq 0.8$ by (6.4.44), so $\hat{h}_1 = 0.8$. We choose equal Δ_i's so m may be chosen as 1 in (6.3.32). In order to satisfy (6.3.34)

in Theorem 6.3 , h should be less than 0.68. But we choose $\hat{h}_1 = 0.8$, so we must subdivide the interpolation interval into 2 subintervals(i.e. L in (1.1.9) is 2). In this example all $v_i > 0$. Actually, even if we choose the original interval, we still have all $v_i > 0$.

Figure 6.8 is the graph of the three rational quartic Bézier curves. Figure 6.9 is the graph of the resulting rational cubic B-spline curve with 2 subintervals. There are no apparent significant differences between Figure 6.8 and Figure 6.9.

Figure 6.10 is the graph of a resulting rational cubic B-spline curve with the original interval and its weights are still positive. There are few differences between Figures 6.8 and 6.10.

Example 6.4 is for degree reduction of a rational B-spline curve in three dimensions. We use a rational B-spline curve of degree 4 with 26 distinct knots as a example. Given $\epsilon = 0.1$, we obtain $h_1 \leq 0.447$ by (6.2.20) and $h_2 \leq 0.0243$ by (6.4.46), so $\hat{h}_1 = 0.0243$. we choose equal $\Delta_i's$ so m may be chosen as 1 in (6.3.32). In order to satisfy (6.3.43) in Theorem 6.4, h should be less than 0.0031. But we choose $\hat{h}_1 = 0.0243$, so we need 42 subintervals.

Figure 6.11 is the graph of the given 4th degree rational B-spline curve. Figure 6.12 is the graph of the resulting rational cubic B-spline with 42 subintervals. There are no significant differences between Figure 6.11 and Figure 6.12. In this example all $v_i > 0$. Figure 6.13 is the graph of the resulting rational cubic B-spline curve with 28 subintervals and its weights are all positive, but there is little difference between Figure 6.12 and Figure 6.13. Figure 6.14 is the graph of the resulting rational cubic B-spline curve with 16 subintervals and two of its rational weights are negative.

In these four examples both algorithms produce rational cubic B-spline curves that approximate the original curves within a tolerance of $\epsilon > 0$ and whose weights are all positive.

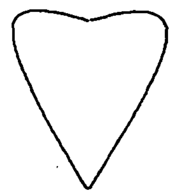

Figure 6.1: $8th$ degree rational Bézier curve.

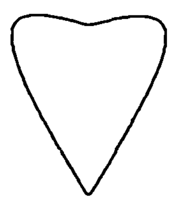

Figure 6.2: $\mathbf{L}(u)$ with 22 subintervals.

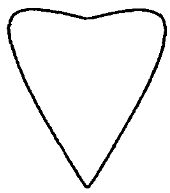

Figure 6.3: $\mathbf{L}(u)$ with 8 subintervals.

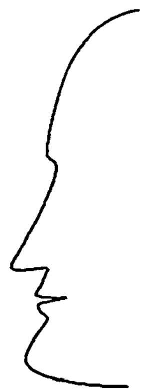

Figure 6.4: $4th$ degree rational B-spline with 19 knots.

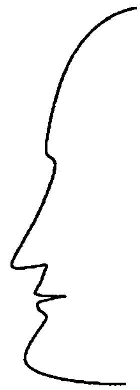

Figure 6.5: $\mathbf{L}(u)$ with 69 subintervals.

Figure 6.6: $\mathbf{L}(u)$with 24 subintervals.

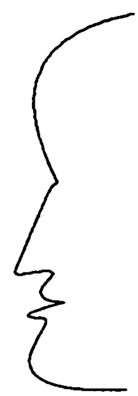

Figure 6.7: $\mathbf{L}(u)$ with 12 subintervals.

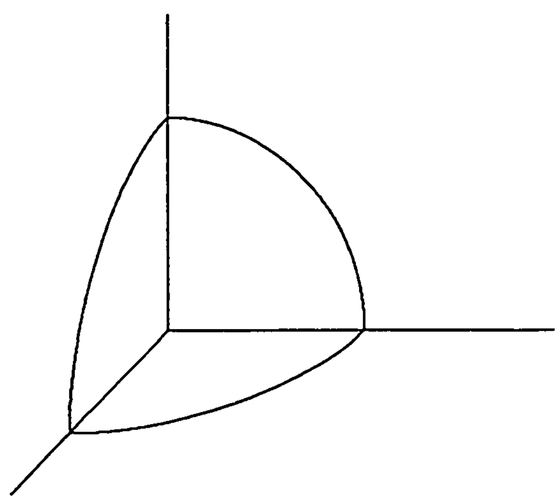

Figure 6.8: $4th$ degree rational Bézier curves (three boundary curves).

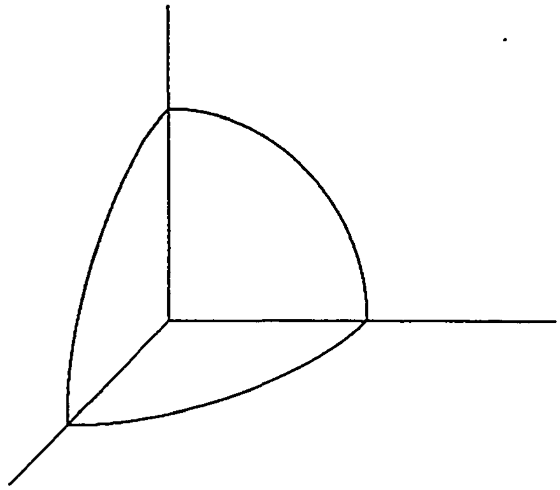

Figure 6.9: $\mathbf{L}(u)'s$ (three boundary curves), each with 2 subintervals.

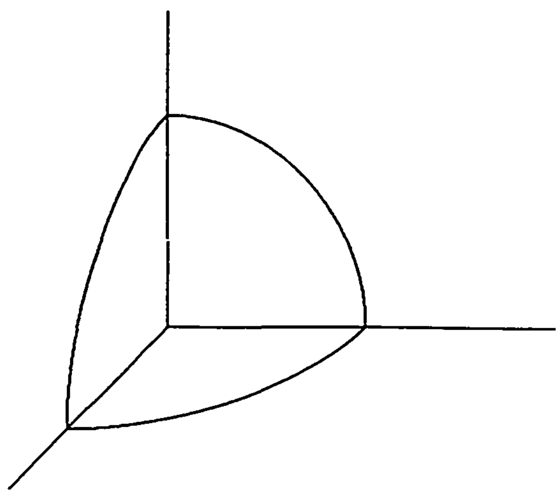

Figure 6.10: $\mathbf{L}(u)'s$ (three boundary curves), each with 1 subinterval.

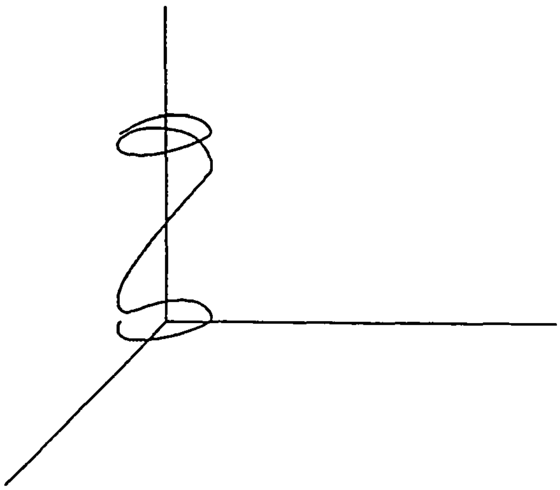

Figure 6.11: $4th$ degree rational B-spline with 26 knots.

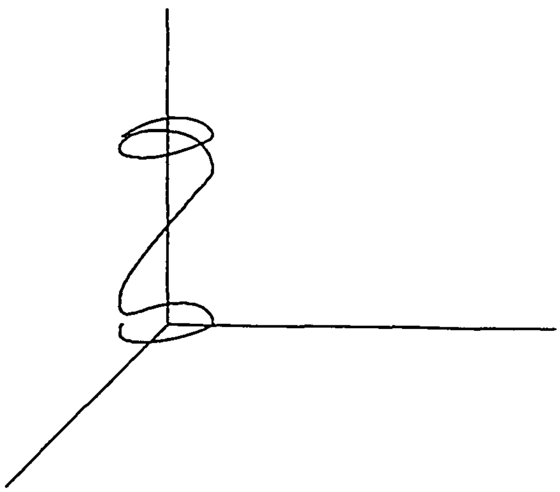

Figure 6.12: $\mathbf{L}(u)$ with 42 subintervals.

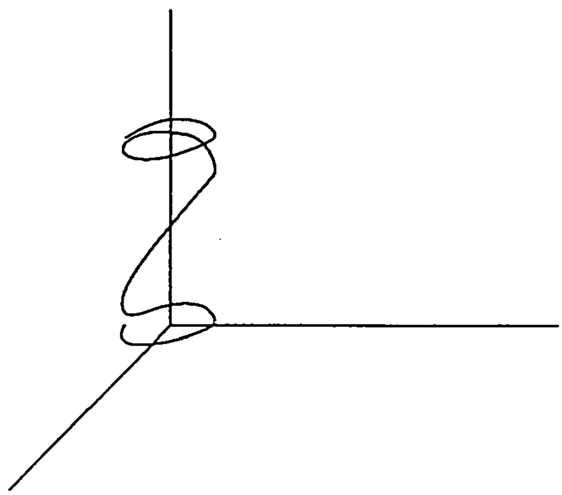

Figure 6.13: $\mathbf{L}(u)$ with 28 subintervals.

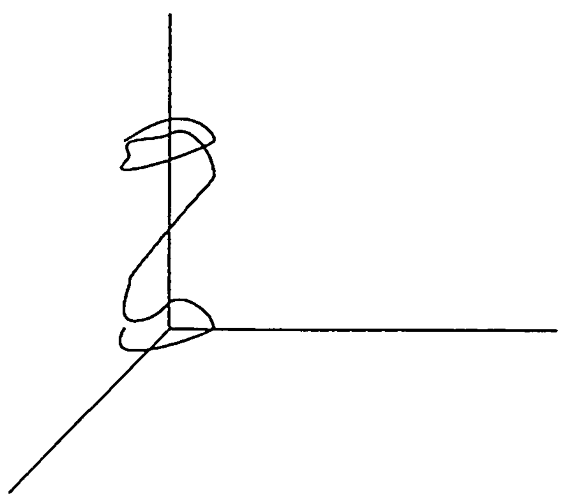

Figure 6.14: $\mathbf{L}(u)$ with 16 subintervals.

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