Summer 1989

Best Approximation With Geometric Constraints

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BEST APPROXIMATION WITH GEOMETRIC CONSTRAINTS

by

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A Dissertation Submitted to the Faculty of
Old Dominion University in Partial Fulfillment of the
Requirements for the Degree of

DOCTOR OF PHILOSOPHY
in
Computational and Applied Mathematics

July, 1989

Approved by :

S. E. Weinstein (Director)

ABSTRACT

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This is a study of best approximation with certain geometric constraints. Two major problem areas are considered: best L_p approximation to a function in $L_p[0,1]$ by convex functions, n-convex functions, (m,n)-convex functions and (m,n)-convex splines, for $1 \leq p < \infty$, and best uniform approximation to a continuous function by convex functions, quasi-convex functions and piecewise monotone functions.

Acknowledgments

I wish to express my deeply sincere gratitude to Dr. S. E. Weinstein, my Ph. D. advisor, for his continuous encouragement and significant guidance in various areas of numerical analysis and approximation theory and to Dr. J. J. Swetits, one of my committee members, for his generous suggestions and constructive criticism of my research which made great improvements of this dissertation. I would also like to thank Dr. H. Kaneko and Dr. M. Bartelt for serving on my committee. A special note of appreciation is due to Dr. J. Tweed, Chairman of Department of Mathematics and Statistics, for providing a very nice academic environment and financial support during the last three years.

Particular thanks are in order for Professor Yuesheng Li and Professor Youqain Huang of Zhongshan University, P. R. China, and Dr. C. A. Micchelli of IBM for their encouragement and recomendation which made it possible that I pursued my further study in the United States.

I wish to thank my wife Yumei Hou, without whose understanding and support this work would never have been completed. This dissertation is dedicated to my wife.

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Chapter 1: Introduction

1.1. Notation

This dissertation is devoted to a study of best approximation with certain geometric constraints. More precisely speaking, the basic problem that will be considered is best approximation of a given function f in a Banach space, for example, $C[a,b]$ or $L_p[a,b]$ for $1 \leq p < \infty$, by a set of functions which satisfy convex, quasi-convex, n-convex, or piecewise monotone constraints.

Let X be a Banach space. In the following chapters, X will be specified to be $C[a,b]$, $B[a,b]$ or $L_p[a,b]$ for $1 \leq p < \infty$, where $C[a,b]$ is the space of continuous functions with the supremum norm, $B[a,b]$ is the space of bounded functions with the supremum norm and $L_p[a,b]$ is the space of pth power Lebesgue integrable functions with the norm $\|f\|_p = \{\int_a^b |f(x)|^p dx\}^{1/p}$. Let K be a nonempty subset of X, which will be specifically defined in the concrete context. A function $g^* \in K$ is said to be *a best approximation* to $f \in X$ from K, if g^* satisfies the following condition:

$$\|f - g^*\|_X = \inf\{\|f - g\|_X : g \in K\},$$

where $\|.\|_X$ is the appropriate norm on X.

1.2. A Brief Survey

There has been much interest in best approximation by monotone, convex, quasi-convex, n-convex, or piecewise monotone functions. In the 1960's and 1970's, much attention was paid to uniform approximation by finite dimensional subsets subject to certain constraints (see survey papers [7] and [28]). Recently, an increasing number of papers were devoted to best approximation by the whole set of monotone, convex or generalized-convex functions.

For monotone approximation, constructive solutions to best L_p approximation were presented in [21] for $p = 1$ and in [50] for $1 < p < \infty$, a characterization of best monotone L_p approximation for $1 \leq p < \infty$, was proved in [52], and the uniqueness of best monotone L_1 approximation was shown in [48]. A representation of the error of monotone least square approximation was established in [51]. Some further properties of best monotone L_p approximation were investigated in [25]. The L_∞ case was considered in [53], [58], and [59].

For convex approximation, the existence and uniqueness of a best convex L_1 approximation to a continuous function were presented in [24]. The existence of a best uniform convex approximation to a bounded function was demonstrated in

[56]. The characterization of best uniform convex approximation to a continuous function was announced in [6] and proved in [69].

Burchard [6] and Brown [3] characterized a best n-convex uniform approximation. Certain aspects of best n-convex uniform approximation were discussed in [71]. With certain additional restrictions, Zwick [70] presented a partial characterization of a best n-convex L_1 approximation and proved the uniqueness of the best n-convex L_1 approximation. The existence of best n-convex L_1 approximation was proved in [22] and [62], by different approaches.

Ubhaya first considered the problem of best approximation by quasi-convex functions. In [63], it was proved that a function is quasi-convex on $[a, b]$ if and only if it is nonincreasing on $[a, p)$ (or $[a, p]$) and nondecreasing on $[p, b]$ (or $(p, b]$), with some $p \in [a, b]$. With this result, Ubhaya employed his results on best monotone uniform approximation [58, 59] to study best quasi-convex uniform approximation. Best L_p approximation for $1 \leq p < \infty$ by quasi-convex functions was also considered by Ubhaya in [61]. A more general result concerning the existence of best L_p approximation from a nonconvex subset was presented in [62].

Best piecewise monotone uniform approximation is a natural generalization of best quasi-convex uniform approximation. However, this topic has not yet drawn a lot of attention. Two papers that the author found to be connected with this topic are [44] and [45]. The discrete case was considered by Cullinan and Powell [9].

1.3. A General Description of This Thesis

This thesis will deal with two major problem areas: chapters 2 through 5 are devoted to the problems of best L_p approximation by convex, n-convex, and (m,n)-convex functions; chapters 6 through 9 are concerned with the problems of best uniform approximation to a continuous function by convex, quasi-convex and piecewise monotone functions.

In Chapter 2, a characterization of a best L_p approximation, for $1 \leq p < \infty$, by convex functions is presented and some structural properties of the best convex approximations are established. In Chapter 3, best n-convex L_p approximation, for $1 \leq p < \infty$, is considered. A characterization of best n-convex L_p approximation for $1 \leq p < \infty$ is proved and some properties of best approximation are discussed. In Chapter 4, the existence of a best L_1 approximation with multiple constraints is proved and a characterization of this best approximation is established. This characterization is used to discover some relationship between best convex L_p approximation and best monotone convex L_p approximation. In Chapter 5, a characterization of best L_p approximation by multiply constrained splines for $1 \leq p < \infty$ and a sufficient condition for the uniqueness of best L_1 approximation are established. In the last 4 chapters, best uniform approximations by convex functions, quasi-convex functions, and piecewise monotone functions are studied. In Chapter 6, a duality theorem is established that expresses the error of the best convex uniform approximation in terms of the supremum of a linear functional and it is used to investigate some properties of best approximation. In Chapter 7, a similar duality theorem is established to characterize both best quasi-convex uniform approximation and the set of optimal knots. In Chapter 8, a duality theorem is established that gives a representation of the error of best piecewise monotone uniform approximation. This duality leads to characterizations of both a best piecewise monotone uniform approximation and the set of knot vectors of best piecewise monotone uniform approximation. Chapter 9 is a continuation of Chapter 8, where an alternative characterization of the set of knot vectors of best piecewise monotone uniform approximation is proved and an algorithm for the computation of a best piecewise monotone approximation is developed by this characterization.

The main approaches used in L_p approximations are general forms of integration by parts which will be established in the corresponding chapters, and the following duality theorem of best L_p approximation, for $1 \leq p < \infty$, by a convex set:

THEOREM A [12]. *Let $f \in L_p = L_p[0,1]$, for $1 \leq p < \infty$. Let K_p be a convex set in L_p. Then,*

(i) for $1 < p < \infty$, $g_p^* \in K_p$ is a L_p best approximation to f from K_p if and only if

$$\int_0^1 (g_p^* - g)(f - g_p^*)|f - g_p^*|^{p-2} \geq 0, \quad \text{for all } g \in K_p;$$

(ii) for $p = 1$, $g_1^* \in K_1$ is a L_1 best approximation to f from K_1 if and only if there exists a $\phi \in L_\infty$ with $\|\phi\|_\infty = 1$ and $\int_0^1 \phi(f - g_1^*) = \|f - g_1^*\|_1$ satisfying

$$\int_0^1 (g_1^* - g)\phi \geq 0, \quad \textit{for all } \; g \in K_1.$$

The approach used to investigate best uniform approximation is the representation of the error of a best approximation that will be established for each specific problem.

Chapter 2: Best Convex Approximation in L_p for $1 \leq p < \infty$

2.1. Introduction

In this chapter, best L_p approximation to $f \in L_p[0,1]$ from convex functions in $L_p = L_p[0,1]$, for $1 \leq p < \infty$ is considered.

Let $K_p \subset L_p$ denote the set of convex functions in L_p. Then K_p is a closed convex set. $g^* \in K_p$ is called *a best convex L_p approximation* to $f \in L_p$ if

$$\|f - g^*\|_p = \inf\{\|f - g\|_p : \text{ for all } \quad g \in K_p\}. \tag{2.1.1}$$

For $1 < p < \infty$, both the existence and uniqueness of a best convex L_p approximation follow from the facts that K_p is closed and convex in the reflexive space L_p, and that the L_p norms are strictly convex. The existence and uniqueness of a best convex L_1 approximation g^* to $f \in C[a,b]$ is presented in [24]. In addition, it is proved in [24] that g^* must be piecewise linear where it does not agree with f. The existence and uniqueness for best convex L_1 approximation with the continuity condition relaxed is presented in [23].

In this chapter, we characterize best L_p approximation to $f \in L_p[0,1]$ for $1 \leq p < \infty$ by convex functions in $L_p[0,1]$. We show that a best approximation g^* must be

(i) linear on any interval on which $g^* < f$ a.e.,

(ii) a linear spline with at most one knot on any interval on which $g^* > f$ a.e., and

(iii) piecewise linear where $g^* \neq f$ a.e..

Furthermore, it is proved that g^* is also the best convex L_p approximation to f on any maximal subinterval on which $g^* \neq f$ a.e..

Because of our choice of norms, we identify as one function any two functions that differ on a subset of $[0,1]$ of measure zero. Thus, approximation may equivalently be considered on $[0,1]$, or on any subset of $[0,1]$ of full measure (such as $(0,1)$).

2.2. PRELIMINARIES

The duality Theorem A in Chapter 1 provides the following two characterizations for best convex L_p approximation:

(i) For $1 < p < \infty$, g^* is the best L_p approximation from K_p to $f \in L_p$ if and only if for all $g \in K_p$

$$\int_0^1 (g^* - g)(f - g^*)|f - g^*|^{p-2} \geq 0. \tag{2.2.1}$$

(ii) For $p = 1$, g^* is a best L_1 approximation from K_1 to $f \in L_1$ if and only if there exists a $\phi_1 \in L_\infty$, with $||\phi_1||_\infty = 1$, such that

$$\int_0^1 (f - g^*)\phi_1 = ||f - g^*||_1, \tag{2.2.2}$$

and for all $g \in K_1$

$$\int_0^1 (g^* - g)\phi_1 \geq 0. \tag{2.2.3}$$

In addition, if $f(x) \neq g^*(x)$, then $\phi_1(x) = \text{sign}(f(x) - g^*(x))$.

The above characterizations of best convex L_p approximation result from the convexity of the sets K_p but do not utilize the convexity of the functions in these sets. However, the characterization theorem (Theorem 2.1) presented in section 2.3 of this chapter does depend on the convexity of the functions in K_p, and is simpler than the above in the sence that unlike (2.2.1) or (2.2.3) which depends on f, g^* and on all $g \in K_p$, Theorem 2.1 depends solely on f and g^*. With this characterization as our goal we complete this section with three lemmas.

LEMMA 2.1. *For $1 \leq p < \infty$, let g_p^* be a best L_p approximation from K_p to $f \in L_p$. For $1 < p < \infty$, let $\phi_p = (f - g_p^*)|f - g_p^*|^{p-2}$, and for $p = 1$, there exists a $\phi_1 \in L_\infty$, satisfying $||\phi_1||_\infty = 1$, and (2.2.2). For $1 \leq p < \infty$, define*

$$h_p(x) = \int_0^x \phi_p(u)du, \tag{2.2.4}$$

and

$$H_p(x) = \int_0^x h_p(u)du. \tag{2.2.5}$$

Then,

(i) $\int_0^1 g_p^* \phi_p = 0$;

(ii) $\int_0^1 g\phi_p \leq 0$, *for all* $g \in K_p$;

(iii) $H_p(x) \leq 0$, *for all* $x \in [0,1]$;

(iv) $h_p(1) = H_p(1) = 0$;

(v) If $H_p(x) = 0$, *then* $h_p(x) = 0$, *and there exists an* $\epsilon_0 > 0$ *such that* $h_p(t) \geq 0$ *for all* $t \in (x - \epsilon_0, x)$, *and* $h_p(t) \leq 0$ *for all* $t \in (x, x + \epsilon_0)$.

PROOF: (i) It follows from (2.2.1) and (2.2.3) that for $1 \leq p < \infty$,

$$\int_0^1 g_p^* \phi_p \geq \int_0^1 g\phi_p, \quad \text{for all} \quad g \in K_p. \tag{2.2.6}$$

Letting $g = 2g_p^*$ in (2.2.6) we have $\int_0^1 g_p^* \phi_p \leq 0$. Similarly, letting $g = (1/2)g_p^*$ in (2.2.6) we have $\int_0^1 g_p^* \phi_p \geq 0$. Hence, we have (i).

(ii) Combining (i) and (2.2.6) gives (ii).

(iii) For $0 \leq t \leq 1$, define

$$g_t(x) = \begin{cases} t - x & 0 \leq x < t \\ 0 & t \leq x < 1. \end{cases}$$

Then, $g_t \in K_p$ for $1 \leq p < \infty$, and by (ii),

$$\int_0^1 g_t(x)\, \phi_p(x) dx = \int_0^t (t - x)\, \phi_p(x) dx \leq 0.$$

This implies (iii).

(iv) By alternately choosing $g = 1$ and $g = -1$ in (ii), we have $h_p(1) = 0$. From (iii), $H_p(1) \leq 0$. Furthermore, if we let $g = x$ in (ii), by using $h_p(1) = 0$ we find that $H_p(1) \geq 0$. Hence, $H_p(1) = 0$.

(v) If $x = 0$ or 1 then $h_p(x) = H_p(x) = 0$. Thus assume that $0 < x < 1$ and that $H_p(x) = 0$. Then, for sufficiently small $\epsilon > 0$

$$0 \geq H_p(x+\epsilon) = \int_0^{x+\epsilon} h_p(u)du = \int_x^{x+\epsilon} h_p(u)du.$$

Thus, by the continuity of h_p, there exists an $\epsilon_1 > 0$ such that $h_p(t) \leq 0$, for all $t \in (x, x+\epsilon_1)$. Also,

$$0 \geq H_p(x-\epsilon) = \int_0^{x-\epsilon} h_p(u)du = -\int_{x-\epsilon}^{x} h_p(u)du,$$

which implies, by the continuity of h_p, that there exists an ϵ_0 where $0 < \epsilon_0 < \epsilon_1$ such that $h_p(t) \geq 0$ for all $t \in (x-\epsilon_0, x)$. The above two arguments together imply that $h_p(x) = 0$.

We now state some general properties of convex functions, which are needed in section 2.3. A function g is said to be *piecewise linear* on $(0,1)$ if there is a countable union of open intervals, $\{I_n : n = 1, 2, ...\}$, such that g is linear on each I_n and $\overline{\cup_{n=1}^{\infty} I_n} = [0,1]$.

LEMMA 2.2. *Let g be convex on $[0,1]$. Then*

(i) g is absolutely continuous on (a,b) for any $0 < a < b < 1$;

(ii) the right and left derivatives g'_+ and g'_- exist at each point in $(0,1)$, g' exists a.e. in $(0,1)$, and $g' = g'_- = g'_+$ a.e. in $(0,1)$;

(iii) g', g'_-, and $g'_+ \in L_1[a,b]$, whenever $0 < a < b < 1$;

(iv) g'_+ and g'_- are monotone increasing in $(0,1)$, g'_+ is right-continuous and g'_- is left-continuous in $(0,1)$;

(v) if g is not strictly convex on any subinterval of $(0,1)$, then g is piecewise linear on (0,1).

PROOF: Proofs of (i) - (iv) can be found in [41] and [42].

(v) By the hypothesis, there is an open interval on which g is linear. Let $\{I_\alpha\}$ be the collection of all open intervals such that g is linear on each I_α. By Proposition 9 in Royden [42] page 32 (Lindelof), there is a countable subcollection, $\{I_{a_n} : n = 1, 2, \ldots\}$ such that $\cup_\alpha I_\alpha = \cup_{n=1}^{\infty} I_{\alpha_n}$. If $\overline{\cup_{n=1}^{\infty} I_{\alpha_n}} \neq [0,1]$, then the complement of this set in $[0,1]$ contains an interval I. But, then g is linear on a subinterval of I, which is a contradiction. Thus, g is piecewise linear on $(0,1)$.

LEMMA 2.3. *Let $g\phi \in L_1[0,1]$. Assume that g is also of bounded variation on $[a,b]$ whenever $0 < a < b < 1$, and that g is also monotone in both some right neighborhood of 0 and some left neighborhood of 1. Let $h(x) = \int_0^x \phi(t)dt$ satisfy $h(1) = 0$. Then,*

$$\int_0^1 g(x)\phi(x)dx = -\int_0^1 h(x)dg(x). \tag{2.2.7}$$

PROOF: If $g(0^+)$ and $g(1^-)$ are finite, then by integration by parts and the hypothesis that $h(1) = 0$, the conclusion holds.

If $|g(0^+)| = +\infty$ and $|g(1^-)| = +\infty$, then there exist an $\epsilon_0 > 0$ such that $|g(x)|$ is both nondecreasing on $(1 - \epsilon_0, 1)$ and nonincreasing on $(0, \epsilon_0)$. For any ϵ such that $0 < \epsilon < \epsilon_0$,

$$\int_{\epsilon}^{1-\epsilon} g(x)\phi(x)dx = g(1-\epsilon)h(1-\epsilon) - g(\epsilon)h(\epsilon) - \int_{\epsilon}^{1-\epsilon} h(x)dg(x).$$

Since $|g\phi| \in L_1[0,1]$, and since

$$|g(1-\epsilon)h(1-\epsilon)| \leq |g(1-\epsilon)| \int_{1-\epsilon}^{1} |\phi(x)|dx \leq \int_{1-\epsilon}^{1} |g(x)\phi(x)|dx$$

and

$$|g(\epsilon)h(\epsilon)| \leq |g(\epsilon)| \int_{0}^{\epsilon} |\phi(x)|dx \leq \int_{0}^{\epsilon} |g(x)\phi(x)|dx,$$

$|g(1-\epsilon)h(1-\epsilon)| \to 0$, as $\epsilon \to 0$, and $|g(\epsilon)h(\epsilon)| \to 0$, as $\epsilon \to 0$. Hence, equation (2.2.7) holds in this case.

If $|g(0^+)| = +\infty$ and $g(1^-)$ is finite (or if $|g(1^-)| = +\infty$ and $g(0^+)$ is finite) then we can similarly show that $|g(\epsilon)h(\epsilon)| \to 0$, as $\epsilon \to 0$, and $g(1^-)h(1^-) = 0$ (or $|g(1-\epsilon)h(1-\epsilon)| \to 0$ as $\epsilon \to 0$ and $g(0^+)h(0^+) = 0$).

2.3. CHARACTERIZATION

The purpose of this section is to establish the characterization of best convex L_p approximation by using the duality theorem and Lemma 2.3.

THEOREM 2.1 (CHARACTERIZATION). *(a) For $1 < p < \infty$, $g_p^* \in K_p$ is the best convex L_p approximation to $f \in L_p[0,1]$ if and only if*

(i) $h_p(1) = 0$,

(ii) $H_p(1) = 0$,

(iii) $H_p(x) \leq 0$ *for all* $x \in [0,1]$,

(iv) if $H_p(x_0) < 0$ *for some* $x_0 \in (0,1)$, *then* g_p^* *is a straight line in a neighborhood of* x_0.

(b) For $p = 1$, $g_1^* \in K_1$ *is a best convex* L_1 *approximation to* $f \in L_1[0,1]$ *if and only if there exists a* $\phi_1 \in L_\infty$ *satisfying* $\|\phi_1\|_\infty = 1$ *and (2.2.2), such that conditions (i)-(iv) in (a) hold with* $p = 1$.

PROOF: We demostrate the proof for (a) only.

(Necessity) (i), (ii), and (iii) are proved necessary conditions in Lemma 2.1. Hence, it remains to show that the best approximation g_p^* satisfies condition (iv).

To this end we prove the following equation:

$$\int_0^1 g_p^*(x)\phi_p(x)dx = \int_0^1 H_p(x)d(g_p^*)'_+. \tag{2.3.1}$$

By Lemma 2.3, we have

$$\int_0^1 g_p^*(x)\phi_p(x)dx = -\int_0^1 h_p(x)(g_p^*)'_+(x)dx. \tag{2.3.2}$$

Since $H_p(0) = H_p(1) = 0$, by part (v) of Lemma 2.1, there exists an $\epsilon_0 > 0$ such that $h_p(t) \geq 0$ for all $t \in (1-\epsilon_0, 1)$, and $h_p(t) \leq 0$ for all $t \in (0, \epsilon_0)$.

Assume that $(g_p^*)'_+(0^+) = -\infty$ and that $(g_p^*)'_+(1^-) = +\infty$. Let $0 < \epsilon < \epsilon_0$ so that $(g_p^*)'_+(\epsilon) \le 0$ and $(g_p^*)'_+(1-\epsilon) \ge 0$. Then,

$$\begin{aligned}&\int_\epsilon^{1-\epsilon} h_p(x)\,(g_p^*)'_+(x)\,dx\\ &= H_p(1-\epsilon)(g_p^*)'_+(1-\epsilon) - H_p(\epsilon)(g_p^*)'_+(\epsilon) - \int_\epsilon^{1-\epsilon} H_p(x)\,d(g_p^*)'_+.\end{aligned}$$

Also,

$$\begin{aligned}0 \le -H_p(1-\epsilon)(g_p^*)'_+(1-\epsilon) &= (g_p^*)'_+(1-\epsilon)\int_{1-\epsilon}^1 h_p(x)\,dx\\ &\le \int_{1-\epsilon}^1 h_p(x)\,(g_p^*)'_+(x)\,dx,\end{aligned}$$

and

$$0 \le H_p(\epsilon)(g_p^*)'_+(\epsilon) = (g_p^*)'_+(\epsilon)\int_0^\epsilon h_p(x)\,dx \le \int_0^\epsilon h_p(x)\,(g_p^*)'_+(x)\,dx.$$

However,

$$\int_0^1 h_p(x)\,(g_p^*)'_+(x)\,dx = \lim_{\epsilon\to 0}\int_\epsilon^{1-\epsilon} h_p(x)\,(g_p^*)'_+(x)\,dx,$$

and thus,

$$\lim_{\epsilon\to 0}\left\{\int_0^\epsilon h_p(x)\,(g_p^*)'_+(x)\,dx + \int_{1-\epsilon}^1 h_p(x)\,(g_p^*)'_+(x)\,dx\right\} = 0.$$

Noting that $\int_0^\epsilon h_p(x)\,(g_p^*)'_+(x)\,dx \ge 0$ and $\int_{1-\epsilon}^1 h_p(x)\,(g_p^*)'_+(x)\,dx \ge 0$, we deduce that

$$\lim_{\epsilon\to 0}\int_0^\epsilon h_p(x)\,(g_p^*)'_+(x)\,dx = 0,$$

and

$$\lim_{\epsilon\to 0}\int_{1-\epsilon}^1 h_p(x)\,(g_p^*)'_+(x)\,dx = 0.$$

Hence (2.3.1) holds for this case.

We can similarly prove (2.3.1) in the cases

(a) $(g_p^*)'_+(0^+) = -\infty$ and $(g_p^*)'_+(1^-)$ is finite, or

(b) $(g_p^*)'_+(1^-) = +\infty$ and $(g_p^*)'_+(0^+)$ is finite.

Combining (2.3.1) and part (i) of Lemma 2.1, we find

$$\int_0^1 H_p(x)\, d(g_p^*)'_+ = 0. \tag{2.3.3}$$

Now, suppose as in (iv) that $H_p(x_0) < 0$. By the continuity of H_p, there exist x_1 and x_2 with $0 < x_1 < x_0 < x_2 < 1$, such that

$$M_H \equiv \max\{H_p(x) : x_1 \le x \le x_2\} < 0.$$

Hence,

$$0 = \int_0^1 H_p(x)\, d(g_p^*)'_+ \le \int_{x_1}^{x_2} H_p(x) d(g_p^*)'_+ \le M_H[(g_p^*)'_+(x_2) - (g_p^*)'_+(x_1)] \le 0.$$

Consequently, $(g_p^*)'_+(x_2) = (g_p^*)'_+(x_1)$. Finally, by the absolute continuity of g_p^*,

$$\begin{aligned} g_p^*(x) &= \int_{x_1}^{x} (g_p^*)'_+(y)\, dy + g_p^*(x_1) \\ &= (g_p^*)'_+(x_0)(x - x_1) + g_p^*(x_1), \quad \text{for } x \in [x_1, x_2]. \end{aligned}$$

Thus, g_p^* is a linear function in a neighborhood of x_0.

(Sufficiency) For all $g \in K_p$, as in establishing (2.3.1), we can use (i), (ii) and (iii) to show that

$$\int_0^1 g(x)\, \phi_p(x) dx = \int_0^1 H_p(x) dg'_+ \le 0. \tag{2.3.4}$$

Also, condition (iv) and (2.3.1) imply that

$$\int_0^1 g^*(x)\,\phi_p(x)\,dx = 0. \tag{2.3.5}$$

Thus, combining (2.3.4) and (2.3.5) shows that g_p^* is a best L_p approximation to $f \in L_p$ from K_p.

Using integration by parts, we have for $x \in [0,1]$

$$H_p(x) = \int_0^x (x-u)\phi_p(u)du.$$

Hence, we can restate the charactrization:

THEOREM 2.2 (ALTERNATIVE CHARACTERIZATION). *(a) For $1 < p < \infty$, $g_p^* \in K_p$ is the best convex L_p approximation to $f \in L_p[0,1]$ if and only if*

(i) $\int_0^1 \phi_p(u)du = 0$,

(ii) $\int_0^1 u\phi_p(u)du = 0$,

(iii) $\int_0^x (x-u)\phi_p(u)du \leq 0$ *for all* $x \in [0,1]$,

(iv) if $\int_0^{x_0} (x_0 - u)\,\phi_p(u)\,du < 0$ *for some* $x_0 \in (0,1)$*, then* $g_p^*(x)$ *is a straight line in a neighborhood of* x_0.

(b) For $p = 1$, $g_1^* \in K_1$ *is a best convex* L_1 *approximation to* $f \in L_1[0,1]$ *if and only if there exists a* $\phi_1 \in L_\infty$ *satisfying* $\|\phi_1\|_\infty = 1$ *and (2.2.2), such that conditions (i)-(iv) in (a) hold with* $p = 1$.

2.4. Structural Properties of Best Convex L_p Approximation

The following corollaries of Theorem 2.1 give structural properties of a best convex L_p approximation, for $1 \leq p < \infty$. In each case, $f \in L_p$, and g^* denotes a best L_p-approximation to f from K_p, for $1 \leq p < \infty$,

COROLLARY 2.1. *(a) If g^* is strictly convex on $(a,b) \subseteq (0,1)$ then, $g^*(x) = f(x)$ a.e. in (a,b);*

(b) If $g^*(x) \neq f(x)$ a.e. in $(a,b) \subseteq (0,1)$, then g^* is piecewise linear on (a,b).

PROOF: (a) By property (iv) of Theorem 2.1, $H_p(x) \equiv 0$ on (a,b). Therefore, $H_p''(x) = \phi_p(x) = 0$ a.e. in (a,b), which implies that $g^*(x) = f(x)$ a.e. in (a,b).

(b) If $g^*(x) \neq f(x)$ a.e. in (a,b) then by part (a), g^* cannot be strictly convex on any subinterval $I \subseteq (a,b)$. Hence by part (v) of Lemma 2.2, g^* is piecewise linear on (a,b).

LEMMA 2.4. *If $H_p(t) < 0$ for all $t \in (a,b) \subseteq (0,1)$, then g^* is linear on (a,b).*

PROOF: If g^* is not linear on (a,b), then there must exist a point $t \in (a,b)$ such that g^* is not linear on any neighborhood of t. However, by (iv) of Theorem 2.1, g^* is linear on some neighborhood of t, which is a contradiction.

COROLLARY 2.2. *Let (a,b) be any open interval in $(0,1)$.*

(a) If $g^*(t) < f(t)$ a.e. in (a,b), then g^* is linear on (a,b).

(b) If $g^*(t) > f(t)$ a.e. in (a,b), then g^* is a linear spline on (a,b) with at most one knot.

PROOF: (a) Suppose that $g^*(t) < f(t)$ a.e. in $(a,b) \subseteq (0,1)$ and that $H_p(x_0) = 0$ for some $x_0 \in (a,b)$. Then, by part (v) of Lemma 2.1, $h_p(x_0) = 0$. By the hypothesis $\phi_p(t) > 0$ a.e. in $(x_0,b) \subseteq (a,b)$. Thus, h_p is strictly increasing on (x_0,b), which implies that $h_p(t) > h_p(x_0) = 0$ for all $t \in (x_0,b)$. Hence, $H_p(t) > H_p(x_0) = 0$ for all $t \in (x_0,b)$, contradicting property (iii) of Theorem 2.1. Thus $g^*(t) < f(t)$ a.e. on $(a,b) \subseteq (0,1)$ implies that $H_p(t) < 0$ for all $t \in (a,b)$, and by Lemma 2.4, g^* is linear on (a,b).

(b) First suppose that $H_p(t) < 0$ for all $t \in (a,b)$. Then, by Lemma 2.4, g^* is linear on (a,b). Next, suppose that $H_p(x_0) = 0$ for some $x_0 \in (a,b)$. Then $h_p(x_0) = 0$ by part (v) of Lemma 2.1, $g^*(t) > f(t)$ a.e. on (a,b) implies that $\phi_p(x) < 0$ a.e. on (a,b). Thus $H_p'(x) = h_p(x)$ is strictly decreasing on (a,b). Therefore H_p' has a unique zero at x_0 in (a,b). Let $E_0 = (a,x_0) \cup (x_0,b)$. Then $H_p'(t) \neq 0$ for all $t \in E_0$. Thus, by Rolle's Theorem, since $H_p(x_0) = 0$, we have $H_p(t) \neq 0$ for all $t \in E_0$. Hence, by (iii) of Theorem 2.1, $H_p(t) < 0$ for all $t \in E_0$. By Lemma 2.4, g^* must be linear on both (a,x_0) and (x_0,b). Since g^* is continuous, it must be a linear spline on (a,b) with at most one knot at x_0.

REMARK: If $f \in C(0,1)$, define the open sets

$$A_- = \{x \in (0,1) : g^*(x) < f(x)\},$$

and

$$A_+ = \{x \in (0,1) : g^*(x) > f(x)\}.$$

Then A_- and A_+ are each a union of a countable set of disjoint open intervals called components, and thus by Corollary 2.2,

(i) g^* is linear on each component of A_-, and

(ii) g^* is a linear spline, with at most one knot on each component of A_+.

COROLLARY 2.3. *(a) If $[a,b] \subseteq [0,1]$ such that $H_p(a) = H_p(b) = 0$, then, g^* is also the best convex L_p approximation to f on $[a,b]$.*

(b) If $H_p(a) = 0$ for some $a \in (0,1)$ then g^* is also the best convex L_p-approximation to f on both $[0,a]$, and $[a,1]$.

(c) If $g \in K_p$ is a best convex L_p approximation to f on both $[0,a]$ and $[a,1]$, then g is a best convex L_p approximation to f on $[0,1]$.

PROOF: (a) Since $H_p(a) = H_p(b) = 0$, by part (v) of Lemma 2.1, we have $h_p(a) = h_p(b) = 0$. Corresponding to h_p and H_p on $[0,1]$, define $h_{p,a}$ and $H_{p,a}$ by

$$h_{p,a}(x) = \int_a^x \phi_p(u)du, \quad \text{for} \quad x \in [a,b]$$

and

$$H_{p,a}(x) = \int_a^x h_{p,a}(u)du, \quad \text{for} \quad x \in [a,b].$$

Then, for $x \in [a,b]$

$$h_{p,a}(x) = h_p(x) - h_p(a) = h_p(x), \tag{2.4.1}$$

and

$$H_{p,a}(x) = \int_a^x h_p(u)du = H_p(x) - H_p(a) = H_p(x). \qquad (2.4.2)$$

Thus, $h_{p,a}(b) = h_p(b) = 0$, and $H_{p,a}(b) = H_p(b) = 0$. Properties (iii) and (iv) of Theorem 2.1 for f and g^* on $[a,b]$ follow from (2.4.1), (2.4.2) and the corresponding properties for f and g^* on $[0,1]$.

(b) Since $H_p(0) = H_p(1) = 0$, the statement in (b) follows from (a).

(c) This result follows from Theorem 2.1.

By property (iv) of Theorem 2.1, $H_p(x) = 0$ if g^* is not linear in any neighborhood of x. Thus, in Corollary 2.3, the hypothesis that $H_p(a) = 0$, (or $H_p(b) = 0$) can be replaced by the alternative, "g^* is not linear in any neighborhood of a", (or "g^* is not linear in any neighborhood of b").

COROLLARY 2.4. *If g^* is linear on $(a,b) \subseteq (0,1)$, but is not linear on any larger open subinterval of $(0,1)$ containing (a,b), then g^* is the best L_p straight line approximation to f on (a,b).*

PROOF: By the above hypothesis g^* is not a straight line in any neighborhood of either a or b. Thus by property (iv) of Theorem 2.1, $H_p(a) = H_p(b) = 0$. By Corollary 2.3, g^* is the best L_p approximation to f on (a,b), from K_p. Since g^* is a straight line on (a,b), and since K_p contains all the straight lines, g^* must be the best L_p straight line approximation to f on (a,b).

Chapter 3: Best n-Convex Approximation in L_p for $1 \leq p < \infty$

3.1. INTRODUTION

In this chapter, we characterize best L_p approximation to $f \in L_p[0,1]$ from n-convex functions in $L_p[0,1]$, for $1 \leq p < \infty$ and $n = 1, 2, \ldots$. This characterization will be used to derive some addition properties of the best approximations.

A real-valued function g, defined on $[0,1]$, is called *n-convex* if for any $n+1$ distinct points $x_0, x_1, \ldots, x_n$ in $[0,1]$, the nth order divided difference

$$[x_0, x_1, \ldots, x_n]g \geq 0.$$

Thus, 1-convex functions are nondecreasing and 2-convex functions are convex in the usual sense. For $n = 1, 2, \ldots$, let $K_{n,p}$ denote the subset of n-convex functions in L_p.

$g^* \in K_{n,p}$ is called a *best n-convex L_p approximation* to $f \in L_p$ if

$$\| f - g^* \|_p = \inf\{\| f - g \|_p : \text{ for } g \in K_{n,p}\}.$$

The existence of a best n-convex L_1 approximation is proved in [22], and the uniqueness is proved under some additional restrictions in [70]. For $1 < p < \infty$, both the existence and uniqueness of the best n-convex approximation follow from the facts that $K_{n,p}$ is closed and convex in the reflexive space L_p, and that the L_p norms are strictly convex. The characterizations of best 1-convex, and best 2-convex L_p approximations for $1 \leq p < \infty$ are established in [52] and Chapter 2, respectively. With certain additional restrictions, the characterization of a best n-convex L_1 approximation, for $n \geq 1$, is considered in [70].

The existence of a best n-convex uniform approximation was proved independently in [6] and [68]. Burchard [6] and Brown [3] have characterized best uniform n-convex approximation. Some additional properties of best uniform n-convex approximation are considered in [71].

3.2. Some Properties of n-Convex Functions

It is known (eg.[4]) that if g is an n-convex function on $[0,1]$ then $g^{(n-2)}$ exists and is absolutely continuous on any closed subinterval of $(0,1)$, $g_-^{(n-1)}$ exists and is left-continuous and nondecreasing in $(0,1)$, $g_+^{(n-1)}$ exists and is right-continuous and nondecreasing in $(0,1)$, $g^{(n-1)}$ exists a.e. in $(0,1)$ and, $g^{(n-1)} = g_-^{(n-1)} = g_+^{(n-1)}$ a.e. in $(0,1)$. In addition, for each $[a,b] \subset (0,1)$, there is a polynomial p of degree

$n-1$ such that

$$g(x) = p(x) + \{1/(n-1)!\} \int_a^b (x-t)_+^{n-1} \, d\mu(t), \quad x \in [a,b], \tag{3.2.1}$$

where μ is a nonnegative Borel measure defined by $\mu([x,y]) = g_+^{(n-1)}(y) - g_-^{(n-1)}(x)$, for $0 < x \leq y < 1$.

The following lemma is used in section 3.3 to establish the characterization of best n-convex L_p approximation.

LEMMA 3.1. *If g is n-convex on $[0,1]$, then there exist a and b with $0 < a \leq b < 1$ such that g is monotone on both $(0,a)$ and $(b,1)$.*

PROOF: We show that there exists a partition of $[0,1] : 0 = x_0 \leq x_1 \leq ... \leq x_n = 1$, such that if $I_i = (x_i, x_{i+1})$ for $i = 0, ..., n-1$ then for n even, g is nondecreasing on I_i if i is odd, and g is nonincreasing on I_i if i is even; and for n odd, g is nonincreasing on I_i if i is odd, and g is nondecreasing on I_i if i is even.

For $n = 1$, g is nondecreasing on $(0,1)$. For $n = 2$, since g is convex on $[0,1]$, there exists an $x_1 \in [0,1]$ such that g is nonincreasing on $(0, x_1)$ and is nondecreasing on $(x_1, 1)$. The proof is completed by induction on n, observing that for $n \geq 2$ if g is n-convex then g' is $(n-1)$-convex.

DEFINITION 3.1: g is said to be strictly n-convex on the interval I if

$$[x_0, x_1, ..., x_n]g > 0 \text{ for any } x_0 < x_1 < ... < x_n \text{ in } I. \tag{3.2.2}$$

DEFINITION 3.2: A real-valued function g on $[0,1]$ is said to be a spline of degree $n-1$ with countable knots on $[0,1]$, if $g \in C^{n-2}[0,1]$ and there exists a countable set of disjoint open intervals, $\{I_i : i = 1, 2, ...\}$, where $\overline{\cup_{i=1}^{\infty} I_i} = [0,1]$, such that g is a polynomial of degree $\leq n-1$ on each I_i.

LEMMA 3.2. *(i) Let g be n-convex on [0,1]. If for some* $0 < x_0 < x_1 < ... < x_n < 1$, $[x_0, x_1, ..., x_n]g = 0$, *then g is a polynomial of degree* $n-1$ *on* $[x_0, x_n]$.

(ii) Let g be n-convex on $[0,1]$ *but not strictly n-convex on any subinterval of* $(0,1)$. *Then, g is a spline of degree* $n-1$ *with countable knots on* $[0,1]$.

PROOF: (i) There is a polynomial of degree $n-1$ p such that

$$g(x) = p(x) + \{1/(n-1)!\} \int_{x_0}^{x_n} (x-t)_+^{n-1}\, d\mu(t), \quad x \in [x_0, x_1], \tag{3.2.3}$$

where μ is a nonnegative Borel measure. Hence

$$[x_0, x_1, ..., x_n]g = \int_{x_0}^{x_n} B_{0,n}(t)\, d\mu(t), \tag{3.2.4}$$

where $B_{0,n}(t) = [x_0, x_1, ..., x_n](\cdot - t)_+^{n-1}/(n-1)!$ is an nth order B-spline with knots at $x_0, x_1, ..., x_n$. By a property of B-spline, $B_{o,n}(t) > 0$, for all $t \in (x_0, x_n)$. The assumption and (3.2.4) impliy that $\int_{x_0}^{x_n} B_{0,n}(t)\, d\mu(t) = 0$. This equation holds only if $d\mu(t) \equiv 0$. Substituting this into (3.2.3) yields $g(x) = p(x)$ on $[x_0, x_n]$.

(ii) By the hypothesis, there is an open interval on which g is a polynomial of degree $\leq n-1$. Let $\{I_\alpha\}$ be the collection of all open subintervals in $(0,1)$ such that

g is a polynomial of degree $\leq n-1$ on each I_α. By Proposition 9 in Royden [42, p. 32], there is a countable subcollection, $\{I_{\alpha_n} : n = 1, 2, ...\}$ of disjoint open intervals such that $\cup_\alpha I_\alpha = \cup_{n=1}^{\infty} I_{\alpha_n}$. If $\overline{\cup_{n=1}^{\infty} I_{\alpha_n}} \neq [0,1]$, then the complement of this set in [0,1] contains an open interval $I \notin \{I_\alpha\}$. By the hypothesis and Definition 3.1, there exist $x_0 < x_1 < ... < x_n$ in I such that $[x_0, x_1, ..., x_n]g = 0$. Part (i) of this lemma then implies that g is a polynomial of degree $\leq n-1$ on $[x_0, x_n]$, which is a contradiction. Thus, g is a spline of degree $n-1$ with countable knots on $(0,1)$.

3.3. Characterization of n-Convex Approximation

For $1 \leq p < \infty$, $K_{n,p}$ is a closed convex cone in L_p. Thus, by using Theorem A and by reasoning as in the proof of Theorem 2.1 we establish the following characterization theorem.

THEOREM 3.1. *(a) For $1 < p < \infty$, given $g_p^* \in K_{n,p}$ define*

$$\phi_p = (f - g_p^*)|f - g_p^*|^{p-2}, \tag{3.3.1}$$

and

$$H_{p,k}(x) = \{1/(k-1)!\} \int_0^x (t-x)^{k-1} \phi_p(t)dt, \quad k = 1, 2, ..., n. \tag{3.3.2}$$

Then, g_p^* is the best n-convex L_p approximation to $f \in L_p[0,1]$ if and only if

(i) $H_{p,k}(1) = 0$ for $k = 1, 2, ..., n$,

(ii) $H_{p,n}(x) \geq 0$ for all $x \in [0,1]$,

(iii) if $H_{p,n}(x_0) > 0$ *for some* $x_0 \in (0,1)$, *then* g_p^* *is a polynomial of degree* $\leq n-1$ *in a neighborhood of* x_0.

(b) $g_1^* \in K_{n,1}$ *is a best n-convex* L_1 *approximation to* $f \in L_1[0,1]$ *if and only if there exists a* $\phi_1 \in L_\infty$ *with* $\| \phi_1 \|_\infty = 1$, *and* $\int_0^1 (f - g_1^*)\phi_1 = \|f - g_1^*\|_1$, *such that (i), (ii), and (iii) of (a) hold with* $p = 1$.

PROOF: (a) Since $K_{n,p}$ is a convex cone, by the duality theorem (Theorem A), we find that g_p^* is the best approximation from $K_{n,p}$ if and only if

$$\int_0^1 g_p^*(x)\phi_p(x)dx = 0, \tag{3.3.3}$$

and

$$\int_0^1 g(x)\phi_p(x)dx \leq 0, \quad \text{for all } g \in K_{n,p}. \tag{3.3.4}$$

(Necessity) Alternatly, let $g = (1-t)^{k-1}/(k-1)!$ and $g = -(1-t)^{k-1}/(k-1)!$ in (3.3.4), for $k = 1, 2, ..., n$ to prove (i). For each $x \in [0,1]$, define

$$g_x(t) = (-1)^{n-2}(x-t)_+^{n-1}/(n-1)! \quad \text{for } t \in [0,1].$$

Then, $g_x \in K_{n,p}$. Next, using (3.3.4) with $g = g_x$ for each $x \in [0,1]$ yields (ii). To establish (iii), we shall use Lemma 2.3. We have the following recursive relations:

$$H_{p,1}(x) = \int_0^x \phi_p(t)dt, \tag{3.3.5}$$

and

$$H_{p,k}(x) = -\int_0^x H_{p,k-1}(t)dt, \quad \text{for } k = 2, ..., n. \tag{3.3.6}$$

By Lemma 2.3,

$$\int_0^1 g_p^* \phi_p = -\int_0^1 H_{p,1} dg_p^* = -\int_0^1 H_{p,1}(g_p^*)'.$$

To apply Lemma 2.3 again, we first show that $H_{p,1}(g_p^*)' \in L_1[0,1]$. If $(g_p^*)'(1^-)$ and $(g_p^*)'(0^+)$ are both finite then we are done. If $|(g_p^*)'(1^-)| = +\infty$, then by Lemma 3.1, there exists an $a \in (0,1)$ such that $|(g_p^*)'|$ is nondecreasing on $(a,1)$. For $x \in [a,1]$ define

$$p(x) = \begin{cases} \text{sign} H_{p,1}(x) & \text{if } H_{p,1}(x) \neq 0 \\ 1 & \text{if } H_{p,1}(x) = 0 \end{cases} \tag{3.3.6}$$

and let $\sigma = 1$ or -1 so that $|(g_p^*)'(x)| = \sigma(g_p^*)'(x)$ for $x \in [a,1]$. Then,

$$|(g_p^*)'(x)H_{p,1}(x)| = \sigma(g_p^*)'(x)p(x)H_{p,1}(x) \quad \text{for} \quad x \in [a,1].$$

It follows that for any $\epsilon > 0$

$$\begin{aligned} &\int_a^{1-\epsilon} |(g_p^*)'(x)H_{p,1}(x)|dx \\ &= \sigma \int_a^{1-\epsilon} (g_p^*)'(x)p(x)H_{p,1}(x)dx \\ &= \sigma \left\{ \int_a^{1-\epsilon} (g_p^*)'(x)p(x)dx \right\} H_{p,1}(1-\epsilon) - \sigma \int_a^{1-\epsilon} \int_a^t (g_p^*)'(x)p(x)dx\phi_p(t)dt \\ &\leq \sigma[g_p^*(1-\epsilon) - g_p^*(a)]H_{p,1}(1-\epsilon) + \sigma \int_a^{1-\epsilon} [g_p^*(t) - g_p^*(a)]|\phi_p(t)|dt. \end{aligned}$$

If $g_p^*(1^-)$ is finite, then

$$0 \leq \sigma[g_p^*(1-\epsilon) - g_p^*(a)]|H_{p,1}(1-\epsilon)| \to 0, \quad \text{as} \quad \epsilon \to 0. \tag{3.3.7}$$

If $|g_p^*(1^-)| = +\infty$, then by Lemma 3.1, we can choose $\epsilon_0 > 0$ such that $|g_p^*|$ is nondecreasing on $(1-\epsilon_0, 1)$ and $a < 1-\epsilon_0$. Let ϵ satisfy $0 < \epsilon \leq \epsilon_0$. Correspondingly,

$$0 \leq \sigma g_p^*(1-\epsilon)|H_{p,1}(1-\epsilon)| \leq |g_p^*(1-\epsilon)| \int_{1-\epsilon}^{1} |\phi_p| \leq \int_{1-\epsilon}^{1} |g_p^*\phi_p| \to 0 \text{ as } \epsilon \to 0.$$

Hence, (3.3.7) holds in this case.

On the other hand, since $g_p^*\phi_p \in L_1[0,1]$ and a is fixed,

$$\sigma \int_a^{1-\epsilon} [g_p^*(t) - g_p^*(a)]|\phi_p(t)|dt \leq \int_a^{1-\epsilon} |g_p^*\phi_p| + |g_p^*(a)| \int_0^1 |\phi_p| < +\infty.$$

Thus, $\lim_{\epsilon\to 0} \int_a^{1-\epsilon} |(g_p^*)'H_{p,1}| < \infty$. Hence $(g_p^*)'H_{p,1} \in L_1[a,1]$. The possibility that $|(g_p^*)'(0^+)| = +\infty$ can be handled similarly. Thus, we can apply Lemma 2.3 again to obtain

$$\int_0^1 g_p^*\phi_p = -\int_0^1 H_{p,2}(g_p^*)''.$$

This procedure can be repeated to yield the equation

$$\int_0^1 g_p^*\phi_p = -\int_0^1 H_{p,n} d(g_p^*)_+^{(n-1)}. \tag{3.3.8}$$

Combining (3.3.8) and $\int_0^1 g_p^*\phi_p = 0$ we have

$$\int_0^1 H_{p,n} d(g_p^*)_+^{(n-1)} = 0. \tag{3.3.9}$$

From (3.3.9) and the continuity of $H_{p,n}$, it follows that if $H_{p,n}(x_0) > 0$, then $(g_p^*)_+^{(n-1)}$ must be constant in a neighborhood of x_0, thus establishing (iii).

(Sufficiency) For all $g \in K_{n,p}$, as in establishing (3.3.8), we can use (i), and (ii) to show that

$$\int_0^1 g\phi_p = -\int_0^1 H_{p,n} d(g)_+^{(n-1)} \leq 0. \tag{3.3.10}$$

If $g_p^* \in K_{n,p}$ satisfies (i), (ii) and (iii), then (3.3.8) holds and

$$\int_0^1 g_p^*\phi_p = -\int_0^1 H_{p,n} d(g_p^*)_+^{(n-1)} = 0.$$

Thus, g_p^* is the best approximation to f.

(b) The proof of part (b) is similar to that of part (a).

COROLLARY 3.1. *For $1 \leq p < \infty$, let $f \in C[0,1]$ and $g_p^* \in K_{n,p}$ be given, and assume that $f \neq g_p^*$ a.e. in $[0,1]$ and that $f - g_p^*$ has a finite number of sign changes at $\tau_1 < ... < \tau_N$ in $(0,1)$. Let*

$$\phi_p = \begin{cases} (f-g_p^*)|f-g_p^*|^{p-2} & 1 < p < \infty \\ sign(f-g_p^*) & p = 1 \end{cases}$$

and define $H_{p,k}(x)$ as in (3.3.2) for $k = 1, ..., n$. Then, g_p^* is a best L_p approximation to f from $K_{n,p}$ if and only if (i) and (ii) (of Theorem 3.1) hold with $1 \leq p < \infty$, and

(iii)' g_p^* is a spline of degree $n-1$ with simple knots $\xi_1, \xi_2, ..., \xi_r$, the distinct zeros of $H_{p,n}$ in $(0,1)$.

PROOF: Let $g_p^* \in K_{n,p}$ be a best n-convex L_p approximation to f. By the hypothesis, $f - g_p^*$ has N sign changes in $(0,1)$. Thus, by Rolle's Theorem, $H_{p,n}$ has at most $N+n$ zeros in $(0,1)$, counting multiplicities. Let $\xi_1, \xi_2, ..., \xi_r$ be the distinct zeros of $H_{p,n}$ in $(0,1)$, where $r \leq N+n$. Hence, with $\xi_0 = 0$ and $\xi_{r+1} = 1$,

$$\int_0^x (t-x)^{n-1}\phi_p(t)dt > 0, \quad \text{for } \ x \in (\xi_i, \xi_{i+1}),\ i = 0, 1, ..., r.$$

Thus by Theorem 3.1, g_p^* is a polynomial of degree $\leq n-1$ on each subinterval (ξ_i, ξ_{i+1}). Since $g_p^* \in C^{n-2}(0,1)$, g_p^* is a spline of degree $n-1$ with simple knots $\xi_1, \xi_2, ..., \xi_r$.

Conversely, let g_p^* satisfy the assumptions and conditions (i), (ii) and (iii)'. If

$$\int_0^{x_0} (t-x_0)^{n-1}\phi_p(t)dt > 0, \quad \text{for some } \ x_0 \in (0,1),$$

then $x_0 \notin \{\xi_1, \xi_2, ..., \xi_r\}$. Hence, $x_0 \in (\xi_j, \xi_{j+1})$ for some index $j \in \{0, 1, ..., r\}$. By (iii)' g_p^* is a polynomial of degree $\leq n-1$ on (ξ_j, ξ_{j+1}), which is a neighborhood of x_0. Thus by Theorem 3.1, g_p^* is a best n-convex L_p-approximation to f.

Remark: Corollary 3.1, for the case p=1, was also proved by D. Zwick [70], by a different approach.

3.4. Some Additional Properties of Best n-convex Approximations

We now present several structural properties of a best n-convex approximation in L_p which follow from Theorem 3.1. In each of the following results, $1 \leq p < \infty$, $f \in L_p$, and g_p^* denotes a best n-convex L_p-approximation to f from $K_{n,p}$.

LEMMA 3.3. *If for some* $x_0 \in (0,1)$, $H_{p,n-2i}(x_0) = 0$ *for* $i = 0, 1, ..., m$ *with* $n - 2m \geq 3$, *then* $H_{p,n-2i-1}(x_0) = 0$, *and* $H_{p,n-2i-2}(t) \geq 0$ *for all* t *in some neighborhood of* x_0. *In addition, if* $H_{p,n-2m-2}(x_0) > 0$, *then in some neighborhood of* x_0, g_p^* *is a spline of degree* $n-1$ *with a knot at* x_0.

PROOF: For sufficiently small $\epsilon > 0$

$$0 \leq H_{p,n}(x_0 + \epsilon) = -\int_{x_0}^{x_0+\epsilon} H_{p,n-1}(t)dt.$$

By the continuity of $H_{p,n-1}$, there exists an $\epsilon_1 > 0$ such that $H_{p,n-1}(t) \leq 0$ for all $t \in (x_0, x_0 + \epsilon_1)$. Similarly,

$$0 \leq H_{p,n}(x_0 - \epsilon) = \int_{x_0-\epsilon}^{x_0} H_{p,n-1}(t)dt.$$

and there exists an ϵ_2 with $0 < \epsilon_2 < \epsilon_1$ such that $H_{p,n-1}(t) \geq 0$ for all $t \in (x_0 - \epsilon_2, x_0)$. Thus, $H_{p,n-1}(x_0) = 0$. For any $0 < \epsilon < \epsilon_2$,

$$0 \leq -H_{p,n-1}(x_0 + \epsilon) = \int_{x_0}^{x_0+\epsilon} H_{p,n-2}(t)dt,$$

and

$$0 \geq -H_{p,n-1}(x_0 - \epsilon) = -\int_{x_0-\epsilon}^{x_0} H_{p,n-2}(t)dt.$$

Hence,

$$\int_{x_0-\epsilon}^{x_0+\epsilon} H_{p,n-2}(t)dt \geq 0.$$

It follows that there exists an $\epsilon_3 > 0$ such that

$$H_{p,n-2}(t) \geq 0 \;\text{ for }\; t \in (x_0 - \epsilon_3, x_0 + \epsilon_3).$$

Thus, this lemma holds for $i = 0$. Similarly, we can verify it for $i = 1, 2, ..., m$.

In addition, if $H_{p,n-2m-2}(x_0) > 0$, then there exists a $\delta > 0$ such that, $H_{p,n-2m-2}(t) > 0$ on $(x_0 - \delta, x_0 + \delta)$. Hence, $H_{p,n-2m-1}(t)$ is strictly increasing on $(x_0 - \delta, x_0 + \delta)$. Since, $H_{p,n-2m-1}(x_0) = 0$, $H_{p,n-2m-1}(t) < 0$ for $t \in (x_0 - \delta, x_0)$ and $H_{p,n-2m-1}(t) > 0$ for $t \in (x_0, x_0 + \delta)$. Hence, $H_{p,n-2m}(t) > 0$, for $t \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)$. Finally, $H_{p,n}(t) > 0$ for $t \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)$. Thus, g_p^* is a spline of degree $n - 1$ in a neighborhood of x_0 with a knot at x_0.

LEMMA 3.4. *If $H_{p,n}(t) > 0$ for all $t \in (\alpha, \beta) \subseteq (0, 1)$, then g_p^* is a polynomial of degree $\leq n - 1$ on (α, β).*

PROOF: If g_p^* is not a polynomial of degree $\leq n - 1$ on (α, β), then there must exist a point $t \in (\alpha, \beta)$ such that g_p^* is not a polynomial of degree $\leq n - 1$ on any neighborhood of t. However, this contradicts property (iii) of Theorem 3.1.

COROLLARY 3.2. *(i) If g_p^* is strictly n-convex on $(\alpha, \beta) \subseteq (0, 1)$, then $g_p^* = f$ a.e. on (α, β);*

(i) If $g_p^* \neq f$ a.e. on $(\alpha, \beta) \subseteq (0, 1)$, then on (α, β) g_p^* is a spline of degree $n - 1$ with countable knots on (α, β).

PROOF: (i) By property (iii) of Theorem 3.1, $H_{p,n}(x) \equiv 0$ on (α, β). Thus,

$$H_{p,n}^{(n)}(x) = (-1)^{n-1}\phi_p(x) = 0 \text{ a.e. on } (\alpha, \beta),$$

and (i) follows.

(ii) If $g_p^* \neq f$ a.e. on (α, β) then by part (i), g_p^* cannot be strictly n-convex on any subinterval $I \subseteq (\alpha, \beta)$. Hence, by Lemma 3.2$g_p^*$ is a spline of degree $n-1$ with countable knots.

COROLLARY 3.3. *(i) If $H_{p,n-2}(t) < 0$ for all $t \in (\alpha, \beta) \subseteq (0,1)$, then g_p^* is a polynomial of degree $\leq n-1$ on (α, β);*

(ii) If $H_{p,n-2}(t) > 0$ for all $t \in (\alpha, \beta) \subseteq (0,1)$, then g_p^* is a spline of degree $n-1$ with at most one knot on (α, β).

PROOF: (i) Assume for some $x_0 \in (\alpha, \beta)$, $H_{p,n}(x_0) = 0$. By Lemma 3.3, $H_{p,n-2}(x_0) \geq 0$, contradicting the hypothesis. Hence, $H_{p,n}(t) > 0$ for all $t \in (\alpha, \beta)$. Lemma 3.4 implies that g_p^* is a polynomial of degree $\leq n-1$ on (α, β).

(ii) If $H_{p,n}(x_0) = 0$ for some $x_0 \in (\alpha, \beta)$, then by Lemma 3.3, g_p^* is a spline of degree $n-1$ in a neighborhood of x_0. If $H_{p,n}$ has another zero $x_1 \neq x_0$ in (α, β), then by Lemma 3.3, $H_{p,n-1}(x_0) = H_{p,n-1}(x_1) = 0$. Since $H_{p,n-2}(t) > 0$ for all $t \in (\alpha, \beta)$, $H_{p,n-1}$ is strictly increasing on (α, β), which is a contradiction. Thus $H_{p,n}$ has at most one zero in (α, β), which implies that $H_{p,n}(t) > 0$ on $(\alpha, x_0) \cup (x_0, \beta)$. Hence, g_p^* is a spline of degree $n-1$ with at most one knot on (α, β).

COROLLARY 3.4. *(i) If* $[\alpha,\beta] \subseteq [0,1]$ *such that*

$$H_{p,k}(\alpha) = H_{p,k}(\beta) = 0, \quad \text{for} \ \ k = n, n-2, \ldots, 2 \ (\text{or} \ \ 1)$$

then g_p^* *is also the best* n*-convex* L_p *approximation to* f *on* $[\alpha,\beta]$.

(ii) In addition, let $f \in C(\alpha,\beta)$ *and let* $t_1 < \ldots < t_r$ *be distinct zeros of* $H_{p,n}$ *in* (α,β). *Further assume that* g_p^* *satisfies the hypothesis of Corollary 3.1. Then* g_p^* *is a best* L_p *approximation to* f *on* (α,β) *from* $S_{n-1}(t_1,\ldots,t_r)$, *the space of all splines of degree* $n-1$ *with simple knots at* $t_1,\ldots,t_r$.

PROOF: (i) follows directly from Theorem 3.1 and Lemma 3.4.

(ii) By (i) and Corollary 3.1, $g_p^* \in S_{n-1}(t_1,\ldots,t_r)$ on (α,β). For each t_i

$$\begin{aligned}
\int_\alpha^\beta \phi_p(x)(x-t_i)_+^{n-1}/(n-1)!dx &= \int_\alpha^\beta (-1)^{n-1} H_{p,n}^{(n)}(x)(x-t_i)_+^{n-1}/(n-1)!dx \\
&= \int_\alpha^\beta H_{p,n}'(x)(x-t_i)_+^0 dx \\
&= H_{p,n}(\beta) - H_{p,n}(t_i) \\
&= 0.
\end{aligned}$$

Thus, $\int_\alpha^\beta \phi_p(x)s(x)dx = 0$, for all $s \in S_{n-1}(t_1,\ldots,t_r)$. Hence, g_p^* is a best L_p approximation to f on (α,β) from $S_{n-1}(t_1,\ldots,t_r)$.

Chapter 4: Best L_p Approximation for $1 \leq p < \infty$ with Multiple Constraints

4.1. Introduction

In this chapter, we consider the problem of best L_p approximation from a set of functions with multiple constraints. The approximating functions in this chapter are the (m,n)-convex functions. Given $0 \leq m \leq n$. g is said to be (m,n)-*convex* if $(-1)^i g$ is $(m+i)$-convex, for $i = 0,1,...,n-m$. Note that for $n > m$, (m,n)-convex functions are functions with multiple constraints. From the above definition, (n,n)-convex functions are n-convex functions and $(0,n)$-convex functions are n-time monotone functions. For some applications of n-time monotone functions, see [66] and other references therein. In addition, $(0,\infty)$-convex functions are complete monotone functions (see [65]). Morn generally, we define $(m,n)_\sigma$-convexity. Let $\sigma = (\sigma_0, \sigma_1, ..., \sigma_{n-m})$, where each σ_i is 1 or -1. A function g is said to be $(m,n)_\sigma$-*convex* if $\sigma_i(-1)^i g$ is $(m+1)$-convex, for $i = 0,1,...,n-m$.

Let $K^p_{m,n}$ denote the subset of (m,n)-convex functions in $L_p = L_p[0,1]$. Then, $K^p_{m,n}$ is a closed convex cone in L_p. In this chapter, we characterize a best L_p approximation of a function f in $L_p[0,1]$ from $K^p_{m,n}$ and study its applications to monotone convex approximation.

4.2. The Existence of a Best L_1 Approximation

For $1 < p < \infty$, the existence of a unique best L_p approximation from $K^p_{m,n}$ follows from the facts that $K^p_{m,n}$ is closed and convex in the reflexive Banach space L_p and that the L_p norm is strictly convex. The existence of a best (m,n)-convex L_1 approximation will be proved to be a consequence of an existence theorem of a recent paper [62] by Ubhaya. We first state a definition and a theorem that appear in [62]. Let H be the set of all extended real-valued functions on $[0,1]$. We say that $P \subset H$ is sequentially closed if it is closed under pointwise convergence of sequences of functions. We denote by $\overline{P}$, the smallest superset of P which is sequentially closed.

THEOREM U. *Let P be a nonempty set in H. Assume the following two conditions are satisfied:*

(1) $P \cap L_p = \overline{P} \cap L_p$;

(2) There exists a positive integer z which depends upon P only and the following holds: If $k \in P$, there exist an integer $1 \le r \le z$ and points $\{x_i : i = 0, 1, ..., r\}$

with $0 = x_0 < x_1 < \ldots < x_r = 1$ *so that* k *is monotone on each interval* (x_{i-1}, x_i).

Then a best approximation to f *in* L_p *from* $P \cap L_p$ *exists, for* $1 \leq p < \infty$.

THEOREM 4.1. *Let* $f \in L_1[0,1]$. *Then there exists a best* (m,n)*-convex* L_1 *approximation to* f.

PROOF: Let $K_i = \{g \in H : (-1)^i g \text{ is } (m+i)\text{-convex}\}$. Then, $K^1_{m,n} = \cap_{i=0}^{n-m}\{K_i \cap L_1\}$. By Proposition 3.4 of [62], $K_i \cap L_1 = \overline{K_i} \cap L_1$. Hence,

$$\begin{aligned} K^1_{m,n} &= \cap_{i=0}^{n-m}\{\overline{K_i} \cap L_1\} \\ &= \cap_{i=0}^{n-m}\{\overline{K_i \cap L_1}\} \\ &= \overline{\cap_{i=0}^{n-m}\{\overline{K_i \cap L_1}\}} \\ &= \overline{\cap_{i=0}^{n-m}\{K_i \cap L_1\}} \\ &= \overline{K^1_{m,n}}. \end{aligned}$$

Therefore condition (1) in Theorem U is satisfied. In addition, since an (m,n)-convex function is m-convex, by Lemma 3.1, condition (2) is also satisfied. It follows from Theorem U that there exists a best approximation to f from $K^1_{m,n}$ in L_1.

4.3. CHARACTERIZATION OF BEST (m,n)-CONVEX L_p APPROXIMATION

In this section, we establish a characterization of best L_p approximation by (m,n)-convex functions, for $1 \leq p < \infty$. To do this, we first prove the following:

LEMMA 4.1. *Let g be (m,n)-convex on $[0,1]$. Then, $g_{-}^{(n-1)}(1^{-})$ and $g^{(m+i)}(1^{-})$, $i = 0,1,...,n-m-2$, are finite.*

PROOF: Since g is m-convex and $-g$ is $(m+1)$-convex, we find that $g^{(m)}$ is nonincreasing and $g^{(m)}(x) \geq 0$, for all $x \in (0,1)$. Hence, for arbitary small $0 < \epsilon < 1/2$, $0 \leq g^{(m)}(1-\epsilon) \leq g^{(m)}(1/2)$. However, $g^{(m)}(1/2) < +\infty$. It follows that $g^{(m)}(1^{-})$ is finite. The proof can be completed by induction.

For nonnegative integers $m \leq n$, let $N_{m,n} = \{m+1,...,n\}$, and $N_n = N_{0,n}$.

THEOREM 4.2 (CHARACTERIZATION). *For $1 \leq p < \infty$, let $f \in L_p[0,1]$ and let $g_p^* \in K_{m,n}^p$.*

(a) For $1 < p < \infty$, let $\phi_p = (f - g_p^*)|f - g_p^*|^{p-2}$, and

$$H_{p,i}(x) = \{1/(i-1)!\} \int_0^x (x-t)^{i-1}\phi_p(t)dt, \quad x \in [0,1],\ i = 1,2,...,n.$$

Then g_p^* is the best L_p approximation from $K_{m,n}^p$ to f if and only if

(i) $H_{p,i}(1) = 0, \quad i \in N_m$;

(ii) $(-1)^m H_{p,i}(1) \leq 0, \quad i \in N_{m,n}$;

(iii) $(-1)^m H_{p,n}(x) \leq 0, \quad x \in [0,1]$;

(iv) if $(-1)^m H_{p,i}(1) < 0$, *for some* $i \in N_{m,n}$, *then* $g_p^{*(i-1)}(1^{-}) = 0$;

(v) if $(-1)^m H_{p,n}(x) < 0$, *for some* $x \in (0,1)$, *then* g_p^* *is a polynomial of degree* $n-1$ *in a neighborhood of* x.

(b) For $p = 1$, g_1^* is a best L_1 approximation to f from $K_{m,n}^p$ if and only if there exists $\phi_1 \in L_\infty$ with $\|\phi_1\|_\infty = 1$ and $\int_0^1 \phi_1(f - g_1^*) = \|f - g_1^*\|_1$, satisfying the conditions (i)-(v) of part (a) with $p = 1$.

PROOF: (a) Since $K_{m,n}^p$ is a convex cone, by Theorem A, $g_p^* \in K_{m,n}^p$ is a best L_p approximation to f from $K_{m,n}^p$ if and only if

$$\int_0^1 g_p^* \phi_p = 0, \tag{4.3.1}$$

and

$$\int_0^1 g\phi_p \le 0, \quad \text{for all } \; g \in K_{m,n}^p. \tag{4.3.2}$$

(Necessity) First, observe that $(1-x)^{i-1}/(i-1)!$, $-(1-x)^{i-1}/(i-1)! \in K_{m,n}^p$, for $i = 1, 2, ..., m$. By substituting these functions into (4.3.2), we prove (i). Next, since $(-1)^m(1-x)^{i-1}/(i-1)! \in K_{m,n}^p$, $i = m+1, ..., n$, by using (4.3.2), we have (ii). Similarly, $(-1)^m(t-x)_+^{n-1}/(n-1)! \in K_{m,n}^p$, for $t \in [0,1]$, which gives (iii).

To prove (iv) and (v), we establish the following form of integration by parts:

$$\int_0^1 g_p^* \phi_p = \sum_{i=m}^{n-1} (-1)^i H_{p,i+1}(1) g_p^{*(i)}(1^-) + (-1)^n \int_0^1 H_{p,n}\, d(g_{p-}^{*(n-1)}). \tag{4.3.3}$$

A similar reasoning as in the proof of Theorem 3.1 gives

$$\int_0^1 g_p^* \phi_p = (-1)^m \int_0^1 H_{p,m}\, g_p^{*(m)},$$

and $H_{p,m}\, g_p^{*(m)} \in L_1[0,1]$. By Lemma 4.1, $g_p^{*(m)}(1^-)$ is finite, and thus, for arbitary small $\epsilon > 0$, $H_{p,m+1}\, g_p^{*(m+1)} \in L_1[\epsilon, 1]$. Hence,

$$\int_\epsilon^1 H_{p,m}\, g_p^{*(m)}$$
$$= H_{p,m+1}(1) g_p^{*(m)}(1^-) - H_{p,m+1}(\epsilon) g_p^{*(m)}(\epsilon) - \int_\epsilon^1 H_{p,m+1}\, g_p^{*(m+1)}$$

If $g_p^{*(m)}(0^+)$ is finite, then we are done. Othewise we have $|g_p^{*(m)}(0^+)| = +\infty$. Since $g_p^{*(m)}$is nonincreasing, there exists $t \in (0,1)$ such that $|g_p^{*(m)}|$ is nonincreasing on $(0,t)$. Whenever $0 < \epsilon < t$,

$$|H_{p,m+1}(\epsilon) g_p^{*(m)}(\epsilon)| \leq |g_p^{*(m)}(\epsilon)| \int_0^\epsilon |H_{p,m}| \leq \int_0^\epsilon |g_p^{*(m)} H_{p,m}| \to 0, \quad \text{as } \epsilon \to 0.$$

Therefore,

$$\int_0^1 H_{p,m}\, g_p^{*(m)} = H_{p,m+1}(1)\, g_p^{*(m)}(1^-) - \int_0^1 H_{p,m+1}\, g_p^{*(m+1)}$$

and $H_{p,m+1}\, g_p^{*(m+1)} \in L_1[0,1]$. This procedure can be repeated to obtain (4.3.3).

Combining (4.3.1) and (4.3.3) yields

$$\sum_{i=m}^{n-1} (-1)^i H_{p,i+1}(1) g_p^{*(i)}(1^-) + (-1)^n \int_0^1 H_{p,n} d(g_{p-}^{*(n-1)}) = 0. \tag{4.3.4}$$

The definition of (m,n)-convex function together with (ii) and (iii) implies that

$$(-1)^i H_{p,i+1}(1) g_p^{*(i)}(1^-) = 0, \quad i = m, ..., n-1, \tag{4.3.5}$$

and

$$\int_0^1 H_{p,n} d(g_{p-}^{*(n-1)}) = 0. \tag{4.3.6}$$

Equations (4.3.5) and (4.3.6) give (iv) and (v) respectively.

(Sufficiency) Assume $g_p^* \in K_{m,n}^p$ and satisfies conditions (i)-(v). Then, by (4.3.3), (4.3.1) holds. Also, (4.3.3) is true if we replace g_p^* by any $g \in K_{m,n}^p$. Hence (4.3.2) holds by using conditions (i)-(v). Consequently, g_p^* is a best L_p approximation to f from $K_{m,n}^p$.

(b) Since the proof for $p = 1$ is similar, we omit the details.

This theorem can be extended to characterize a best L_p approximation from $(m,n)_\sigma$-convex functions.

4.4. Best Monotone Convex L_p Approximation

As applications of the results in Section 4.3, in this section we consider best L_p approximation by monotone convex functions, and the relationship between best convex L_p approximation and best monotone convex L_p approximation. For $1 \le p < \infty$, let $M_D(a,b) \subset L_p[a,b]$ be the set of nonincreasing convex functions on (a,b) and $M_I(a,b)$ the set of nondecreasing convex functions in $L_p[a,b]$. Thus, $g(x) \in M_D(a,b)$ if and only if $G(x) \equiv g(-x) \in M_I(-b,-a)$. In addition, $g^*(x)$ is a best L_p approximation to f from $M_D(a,b)$ if and only if $G^*(x) \equiv g^*(-x)$ is a best L_p approximation to $F(x) = f(-x)$ from $M_I(-b,-a)$.

Since a nonincreasing convex function is $(1,2)_\sigma$-convex with $\sigma = (-1,-1)$ and a nondecreasing convex function is $(1,2)_\sigma$- convex with $\sigma = (1,-1)$, we have the following two corollaries of Theorem 4.2:

COROLLARY 4.1. *(a) For $1 < p < \infty$, $g^* \in M_D(a,b)$ is the best nonincreasing convex L_p approximation to $f \in L_p[a,b]$ if and only if*

(i) $\int_a^b \phi_p(x)dx = 0$;

(ii) $\int_a^b x\phi_p(x)dx \geq 0$;

(iii) $\int_a^t (t-x)\phi_p(x)dx \leq 0$, *for all* $t \in [a,b]$;

(iv) if $\int_a^b x\phi_p(x)dx > 0$, *then* $g_{p-}^{*'}(b^-) = 0$;

(v) if $\int_a^{t_0} (t_0 - x)\phi_p(x)dx < 0$, *for some* $t_0 \in (a,b)$, *then* g_p^* *is a linear polynomial in a neighborhood of* t_0.

(b) For $p = 1$, $g_1^* \in M_D(a,b)$ is a best nonincreasing convex L_1 approximation to $f \in L_1[a,b]$ if and only if there exists a $\phi_1 \in L_\infty[a,b]$ with $\|\phi_1\|_\infty = 1$, $\int_a^b \phi_1(f - g_1^*) = \|f - g_1^*\|_1$ satisfying conditions (i)-(v) in (a) with $p = 1$.

COROLLARY 4.2. *(a) For $1 < p < \infty$, $g^* \in M_I(a,b)$ is the best nondecreasing convex L_p approximation to $f \in L_p[a,b]$ if and only if*

(i) $\int_a^b \phi_p(x)dx = 0$;

(ii) $\int_a^b x\phi_p(x)dx \leq 0$;

(iii) $\int_t^b (x-t)\phi_p(x)dx \leq 0$, *for all* $t \in [a,b]$;

(iv) if $\int_a^b x\phi_p(x)dx < 0$, *then* $g_{p-}^{*'}(a^+) = 0$;

(v) if $\int_{t_0}^b (x - t_0)\phi_p(x)dx < 0$, *for some* $t_0 \in (a,b)$, *then* g_p^* *is a linear polynomial in a neighborhood of* t_0.

(b) For $p = 1$, $g_1^* \in M_I(a,b)$ is a best nonincreasing convex L_1 approximation to $f \in L_1[a,b]$ if and only if there exists a $\phi_1 \in L_\infty[a,b]$ with $\|\phi_1\|_\infty = 1$, $\int_a^b \phi_1(f - g_1^*) = \|f - g_1^*\|_1$ satisfying conditions (i)-(v) in (a) with $p = 1$.

The next three theorems establish the relationship between best convex L_p approximation and best monotone convex L_p approximation.

THEOREM 4.3. *Let g_p^* be a best convex L_p approximation to $f \in L_p[0,1]$, for $1 \le p < \infty$. Then, there exists a $t \in [0,1]$ such that g_p^* is a best nonincreasing convex L_p approximation to f on $[0,t]$ and a best nondecreasing convex L_p approximation to f on $[t,1]$.*

PROOF: If g_p^* is nonincreasing (nondecreasing) on $(0,1)$, then let $t = 1\,(t = 0)$. Assume that g_p^* is a non-monotone convex function. Let

$$m = \inf\{g_p^*(x) : x \in [0,1]\}.$$

Then the set $A = \{x \in [0,1] : g_p^*(x) = m\}$ is a nonempty and closed interval contained in $(0,1)$. Define $t = \inf A$. Then, g_p^* is nonincreasing on $(0,t)$ and nondecreasing on $(t,1)$. By the definition of t, g_p^* can not be a linear polynomial in any neighborhood of t which contains t as an interior point. The characterization of best convex approximation implies $\int_0^t (t - x)\phi_p(x)dx = 0$. By Corollary 2.3, g_p^* is a best convex approximation to f on both $[0,t]$ and $[t,1]$. Since the set of nonincreasing convex functions in $L_p[0,t]$ is contained in the set of convex functions in $L_p[0,t]$, g_p^* is also a best nondecreasing convex approximation to f on $[0,t]$. Similarly, g_p^* is a best nondecreasing convex approximation to f on $[t,1]$.

THEOREM 4.4. *(a) For $1 < p < \infty$ let $f \in L_p[0,1]$. Let $t \in (0,1)$, $g_D \in M_D(0,t)$ be the best nonincreasing convex L_p approximation to f on $[0,t]$ and $g_I \in M_I(t,1)$ be the best nondecreasing convex L_p approximation to f on $[t,1]$. Define*

$$\phi_{p,D}(x) = [f(x) - g_D(x)]|f(x) - g_D(x)|^{p-2}, \quad \text{for} \quad x \in [0,t],$$

$$\phi_{p,I}(x) = [f(x) - g_I(x)]|f(x) - g_I(x)|^{p-2}, \quad \text{for} \quad x \in [t,1],$$

and

$$g(x) = \begin{cases} g_D(x), & x \in [0,t] \\ g_I(x), & x \in [t,1]. \end{cases}$$

Then, g is the best convex L_p approximation to f on $[0,1]$ if and only if

(i) $g_D(t) = g_I(t)$,

(ii) $\int_0^t (t-x)\phi_{p,D}(x)dx = \int_t^1 (x-t)\phi_{p,I}(x)dx$.

PROOF: Let

$$\phi_p(x) = \begin{cases} \phi_{p,D}(x), & x \in [0,t] \\ \phi_{p,I}(x), & x \in [t,1]. \end{cases}$$

Assume g is the best convex L_p approximation to f on $[0,1]$. Then g is continuous on $(0,1)$ and thus $g_D(t) = g_I(t)$. In addition, by the characterization of best convex L_p approximation, we have $\int_0^1 \phi_p = 0$, and $\int_0^1 x\phi_p(x)dx = 0$. Hence, for $t \in (0,1)$, $\int_0^1 (t-x)\phi_p(x)dx = 0$. It follows from the last equation that (ii) holds.

Condition (i) implies that g is convex on $[0,1]$. By the assumptions, we find

$$\int_0^1 \phi_p(x)dx = \int_0^t \phi_{p,D}(x)dx + \int_t^1 \phi_{p,I}(x)dx = 0,$$

and

$$\int_0^1 x\phi_p(x)dx = \int_0^t (t-x)\phi_{p,D}(x)dx + \int_t^1 (t-x)\phi_{p,I}(x)dx = 0.$$

For $x \in [0,t]$,

$$\int_0^x (x-u)\phi_p(u)du = \int_0^x (x-u)\phi_{p,D}(u)du \le 0,$$

and for $x \in (t,1]$, by condition (ii),

$$\begin{aligned}
&\int_0^x (x-u)\phi_p(u)du \\
&= \int_0^t (x-u)\phi_{p,D}(u)du + \int_t^x (x-u)\phi_{p,I}(u)du \\
&= \int_0^t (t-u)\phi_{p,D}(u)du + \int_t^x (x-t)\phi_{p,I}(u)du + \int_t^x (t-u)\phi_{p,I}(u)du \\
&= \int_t^1 (u-t)\phi_{p,I}(u)du - \int_t^x (u-t)\phi_{p,I}(u)du + \int_t^x (x-t)\phi_{p,I}(u)du \\
&= \int_x^1 (u-t)\phi_{p,I}(u)du - \int_x^1 (x-t)\phi_{p,I}(u)du \\
&= \int_x^1 (u-x)\phi_{p,I}(u)du \le 0.
\end{aligned}$$

Assume that for some $x_0 \in (0,1)$, $\int_0^{x_0}(x_0-u)\phi_p(u)du < 0$. If $x_0 \in (0,t)$, then g_D is a linear polynomial in a neighborhood of x_0 and so is g. If $x_0 \in (t,1)$, then by the above reasoning, we have $\int_{x_0}^1 (u-x_0)\phi_{p,I}(u)du < 0$. Thus, g_I is a linear polynomial in a neighborhood of x_0 and so is g. If $x_0 = t$, in view of the continuity of $\int_0^x (x-u)\phi_p(u)du$ for $x \in [0,1]$,

$$\int_0^x (x-u)\phi_p(u)du < 0,\ x \in (t-\delta_1, t], \quad \text{for some} \quad \delta_1 > 0.$$

By the characterization of best nonincreasing convex L_p approximation, we fine that $g'_-(t^-) = g'_{D-}(t^-) = 0$ and g is a linear polynomial on $(t-\delta_1, t]$. In addition, since (ii) holds, $\int_t^1 (x-t)\phi_{p,I}(x)dx < 0$. Similarly, $g'_+(t^-) = g'_{I+}(t^+) = 0$, and g is a linear polynomial on $[t, t+\delta_2)$ for some $\delta_2 > 0$. Hence, $0 = g'_-(t^-) \le g'(t) \le g'_+(t^+) = 0$, and thus $g'(t)$ exists and vanishes. Therefore g is a constant on $(t-\delta_1, t+\delta_2)$.

The conditions that we verify guarantee that g is the best convex L_p approximation to f on $[0,1]$.

For $p = 1$, we have the following similar result:

THEOREM 4.5. *Let $f \in L_1[0,1]$ and $t \in (0,1)$. Assume $g_D \in M_D(0,t)$ is a best nonincreasing convex L_1 approximation to f on $[0,t]$ and $g_I \in M_I(t,1)$ is a best nondecreasing convex L_1 approximation to f on $[t,1]$. Define*

$$g(x) = \begin{cases} g_D(x) & x \in [0,t] \\ g_I(x) & x \in [t,1], \end{cases}$$

Let $\Phi(g_D)$ be the set of $\phi \in L_\infty[0,t]$ with $\|\phi\|_\infty = 1$, and $\int_0^t \phi(f-g_D) = \|f-g_D\|_1$, satisfying conditions (i)-(v) of Corollary 4.1. Let $\Phi(g_I)$ be the set of $\phi \in L_\infty[t,1]$ with $\|\phi\|_\infty = 1$, and $\int_t^1 \phi(f-g_I) = \|f-g_I\|_1$, satisfying conditions (i)-(v) of Corollary 4.2. Then, g is a best convex L_1 approximation to f on $[0,1]$ if and only if *(i)* $g_D(t) = g_I(t)$,

(ii) there exist $\phi_D \in \Phi(g_D)$ *and* $\phi_I \in \Phi(g_I)$ *such that*

$$\int_0^t (t-x)\phi_D(x)dx = \int_t^1 (x-t)\phi_I(x)dx.$$

PROOF: Let

$$\phi(x) = \begin{cases} \phi_D(x) & x \in [0,t] \\ \phi_I(x) & x \in (t,1]. \end{cases}$$

Then, $\|\phi\|_\infty = 1$ and

$$\int_0^1 \phi(f-g) = \int_0^t \phi_D(f-g_D) + \int_t^1 \phi_I(f-g_I) = \|f-g\|_1.$$

The rest of this proof is similar to the proof of Theorem 4.4.

Chapter 5: Best L_p Approximation by Multiply Constrained Splines for $1 \leq p < \infty$

5.1. Introduction

In this chapter, we consider best L_p approximation by multiply constrained splines for $1 \leq p < \infty$, i.e. best L_p approximation to an L_p function by (m, n)-convex splines.

Given a partition Δ of $[0,1]$, with $\Delta : 0 = x_0 < x_1 < ... < x_{k+1} = 1$. Let $S_n^k(\Delta)$ denote the space of polynomial splines of degree $n-1$ with k simple knots at $x_1, ..., x_k$. As in Chapter 4, let $K_{m,n}^p$ be the closed convex cone of all (m, n)-convex functions in $L_p[0,1]$. Define

$$S_{m,n}^{k,p}(\Delta) = S_n^k(\Delta) \cap K_{m,n}^p. \tag{5.1.1}$$

$S_{m,n}^{0,p}(\Delta)$ is the set of (m, n)-convex polynomials of degree $n - 1$.

Given $f \in L_p[0,1]$, $s^* \in S_{m,n}^{k,p}(\Delta)$ is called *a best (m, n)-convex spline L_p approximation* to f if

$$\| f - s^* \|_p = \inf\{\| f - s \|_p : s \in S_{m,n}^{k,p}(\Delta)\}. \tag{5.1.2}$$

For $1 \leq p < \infty$, the existence of a best approximation to $f \in L_p[0,1]$ from $S_{m,n}^{k,p}(\Delta)$ follows from the fact that $S_{m,n}^{k,p}(\Delta)$ is a finite dimensional closed subset of L_p. For $1 < p < \infty$, unicity follows from the fact that L_p is strictly convex. For $p = 1$, unicity needs further investigation.

In section 5.2, the characterizations of best (m,n)-convex spline L_p approximations for $1 \leq p < \infty$ will be presented. In section 5.3, we investigate the uniqueness of a best (m,n)-convex spline L_1 approximation. In section 5.4, we discuss some applications in best L_p approximation by n-convex splines of degree $n-1$ and $(n-1)$-convex polynomials of degree $n-1$.

5.2. CHARACTERIZATION OF L_p APPROXIMATIONS FOR $1 \leq p < \infty$

By Theorem A, if K_p is a convex cone in L_p for $1 \leq p < \infty$, it is known that

(i) $s_p^* \in K_p$ is a best L_p approximation to $f \in L_p$ for $1 < p < \infty$ if and only if

$$\int_0^1 s_p^* \phi_p = 0, \tag{5.2.1}$$

and

$$\int_0^1 s\phi_p \leq 0, \quad \text{for all} \quad s \in K_p, \tag{5.2.2}$$

where $\phi_p = (f - s_p^*)|f - s_p^*|^{p-2}$; and

(ii) $s_1^* \in K_p$ is a best L_1 approximation to $f \in L_1$ if and only if there exists a $\phi_1 \in L_\infty$ with $\| \phi_1 \|_\infty = 1$ and $\int_0^1 \phi_1(f - s_1^*) = \| f - s_1^* \|_1$, satisfying (5.2.1) and (5.2.2) with $p = 1$.

With the above duality in mind, we have the following characterization of best L_p approximation from $S^{k,p}_{m,n}(\Delta)$ to $f \in L_p$ for $1 \le p < \infty$.

THEOREM 5.1 (CHARACTERIZATION). *For $1 \le p < \infty$, let $f \in L_p[0,1]$ and let $s_p^* \in S^{k,p}_{m,n}(\Delta)$.*

(a) For $1 < p < \infty$, let $\phi_p = (f - s_p^*)|f - s_p^*|^{p-2}$, and

$$H_{p,i}(x) = \{1/(i-1)!\} \int_0^x (x-t)^{i-1} \phi_p(t)dt, \quad x \in [0,1], \quad i \in N_n. \tag{5.2.3}$$

Then, s_p^* is the best L_p approximation from $S^{k,p}_{m,n}(\Delta)$ to f if and only if

(i) $H_{p,i}(1) = 0, \quad i \in N_m;$

(ii) $(-1)^m H_{p,i}(1) \le 0, \quad i \in N_{m,n};$

(iii) $(-1)^m H_{p,n}(x_j) \le 0, \quad j \in N_k;$

(iv) if $(-1)^m H_{p,i}(1) < 0$, for some $i \in N_{m,n}$, then $s_p^{*(i-1)}(1) = 0$;

(v) if $(-1)^m H_{p,n}(x_j) < 0$, for some $j \in N_k$, then $s_p^{*(n-1)}(x_j^-) = s_p^{*(n-1)}(x_j^+)$.

(b) For $p = 1$, s_1^* is a best L_1 approximation from $S^{k,1}_{m,n}(\Delta)$ to f if and only if there exists a $\phi_1 \in L_\infty$ with $\| \phi_1 \|_\infty = 1$ and $\int_0^1 \phi_1(f - s_1^*) = \| f - s_1^* \|_1$, satisfying the conditions (i)-(v) of part (a) with $p = 1$. We shall call ϕ_1 an associated functional of s_1^*.

PROOF: (a) This proof will depend on the duality theorem stated before this theorem. Since $S^{k,p}_{m,n}(\Delta)$ is a closed convex cone in L_p, by the duality, s_p^* is the best approximation to f from $S^{k,p}_{m,n}(\Delta)$ if and only if

$$\int_0^1 s_p^* \phi_p = 0, \tag{5.2.4}$$

and

$$\int_0^1 s\phi_p \leq 0, \quad \text{for all} \quad s \in S^{k,p}_{m,n}(\Delta). \tag{5.2.5}$$

(Necessity) First, note that $(1-x)^{i-1}/(i-1)!$, $-(1-x)^{i-1}/(i-1)! \in S^{k,p}_{m,n}(\Delta)$ for $i = 1, 2, ..., m$. By substituting these functions into inequality (5.2.5), we find

$$\int_0^1 \{(1-x)^{i-1}/(i-1)!\}\phi_p(x)dx = 0, \quad i = 1, 2, ..., m.$$

This proves (i).

Next, since $(-1)^m(1-x)^{i-1}/(i-1)! \in S^{k,p}_{m,n}(\Delta)$, $i = m+1, ..., n$, by using (5.2.5) once again, we obtain (ii). Similarly, in (5.2.5), let $s = (-1)^m(x_j - x)_+^{n-1}/(n-1)!$, $j = 1, 2, ..., k$, respectively, and we have

$$\int_0^{x_j} \{(-1)^m(x_j - x)^{n-1}/(n-1)!\}\phi_p(x)dx \leq 0, \quad j = 1, 2, ..., k.$$

Now, by integration by parts and by using (i),

$$\begin{aligned}
&\int_0^1 s_p^*(x)\phi_p(x)dx \\
&= \int_0^1 (-1)^m H_{p,m}(x) s_p^{*(m)}(x)dx \\
&= \sum_{i=m}^{n-1} (-1)^i H_{p,i+1}(1) s_p^{*(i)}(1) + \sum_{j+1}^{k} (-1)^n H_{p,n}(x_j)[s_p^{*(n-1)}(x_j^+) - s_p^{*(n-1)}(x_j^-)]
\end{aligned}$$

Combining the above equation with (5.2.4) gives

$$\sum_{i=m}^{n-1}(-1)^i H_{p,i+1}(1)s_p^{*(i)}(1)+\sum_{j+1}^{k}(-1)^n H_{p,n}(x_j)[s_p^{*(n-1)}(x_j^+)-s_p^{*(n-1)}(x_j^-)]=0. \tag{5.2.6}$$

Since $s_p^* \in K_{m,n}$, $(-1)^{i-m}s_p^{*(i)}(1) \geq 0$, and

$$(-1)^{n-m}[s_p^{*(n-1)}(x_j^+)-s_p^{*(n-1)}(x_j^-)] \geq 0.$$

It follows from (ii)and (iii) that each term in (5.2.6) is nonpositive. Hence,

$$(-1)^m H_{p,i+1}(1)s_p^{*(i)}(1)=0, \quad i=m,m+1,...,n-1, \tag{5.2.7}$$

and

$$(-1)^m H_{p,n}(x_j)[s_p^{*(n-1)}(x_j^+)-s_p^{*(n-1)}(x_j^-)]=0, \quad j=1,2,...,k. \tag{5.2.8}$$

(5.2.7) implies (iv) and (5.2.8) implies (v).

Sufficiency. If $s_p^* \in S_{m,n}^{k,p}(\Delta)$ satisfying conditions (i)-(v), then by integration by parts, it is easy to verify that (5.2.4)and (5.2.5) hold. Therefore, s_p^* is the best approximation to f from $S_{m,n}^{k,p}(\Delta)$.

(b) The proof is similar to (a). Thus we omit the details.

In order to derive some structural properties of best approximation, we introduce some additional notation. For $1 \leq p < \infty$ and $\phi_p \in (L_p)^*$, define $H_{p,i}$ as in (5.2.3) and

$$I(\phi_p) = \{i \in N_{m,n} : (-1)^m H_{p,i}(1) < 0\}, \tag{5.2.9}$$

and

$$J(\phi_p) = \{j \in N_k : (-1)^m H_{p,n}(x_j) < 0\}. \tag{5.2.10}$$

THEOREM 5.2. *Let $1 \leq p < \infty$ and let $f \in L_p[0,1]$.*

(a) For $1 < p < \infty$, $s_p^* \in S_{m,n}^{k,p}(\Delta)$ is the best L_p approximation to f from $S_{m,n}^{k,p}(\Delta)$ if and only if s_p^* is the solution of the following spline approximation problem:

$$min\{\| f - s \|_p : s \in S_n^k(\Delta)\}, \tag{5.2.11}$$

subject to the interpolation constraints:

$$s^{(i)}(1) = 0, \quad i \in I(\phi_p), \tag{5.2.12}$$

and

$$s^{(n-1)}(x_j^-) = s^{(n-1)}(x_j^+), \quad j \in J(\phi_p), \tag{5.2.13}$$

where $\phi_p = (f - s_p^*)|f - s_p^*|^{p-2}$.

(b) For $p = 1$, $s_1^* \in S_{m,n}^{k,1}(\Delta)$ is a best L_1 approximation to f from $S_{m,n}^{k,1}(\Delta)$ if and only if there is a $\phi_1 \in L_\infty$, with $\|\phi_1\|_\infty = 1$ and $\int_0^1 \phi_1(f - s_1^*) = \|f - s_1^*\|_1$ such that s_1^* is a solution of (5.2.11) subject to (5.2.12) and (5.2.13) with $p = 1$.

PROOF: (a) Let

$$S_n^{k*}(\Delta) = \{s \in S_n^k(\Delta) : \quad s^{(i)}(1) = 0, \, i \in I(\phi_p);$$
$$s^{(n-1)}(x_j^-) = s^{(n-1)}(x_j^+), \, j \in J(\phi_p)\}.$$

Then, problem (5.2.11)-(5.2.13) is equivalent to the following problem:

$$\min\{\| f - s \|_p \colon s \in S_n^{k*}(\Delta)\}. \tag{5.2.14}$$

Observe that $S_n^{k*}(\Delta)$ is a finite dimensional subspace of $S_n^k(\Delta)$ and that $S_n^{k*}(\Delta)$ has a basis $\{(1-x)^{i-1}, i \in N_n - I(\phi_p), (x_j - x)_+^{n-1}, j \in N_k - J(\phi_p)\}$. By the characterization of best L_p approximation to f from a finite dimentional subspace of L_p, $s_p^* \in S_n^{k*}(\Delta)$ is a best approximation to f from $S_n^{k*}(\Delta)$ if and only if it satisfies the conditions

$$\int_0^1 (1-t)^{i-1}\phi_p(t)dt = 0, \quad i \in N_n - I(\phi_p), \tag{5.2.15}$$

and

$$\int_0^1 (x_j - t)_+^{n-1}\phi_p(t)dt = 0, \quad j \in N_k - J(\phi_p). \tag{5.2.16}$$

Now, by Theorem 5.1, s_p^* is a best L_p approximation to f from $S_{m,n}^{k,p}(\Delta)$ if and only if conditions (i)-(v) of Theorem 5.1 are satisfied. Hence, it follows from the definitions of $I(\phi_p)$ and $J(\phi_p)$ and from (5.2.15) and (5.2.16) that s_p^* is a best L_p approximation to f from $S_{m,n}^{k,p}(\Delta)$ if and only if s_p^* is a solution of problem (5.2.11)-(5.2.13).

(b) The proof of (b) is similar to that of (a).

5.3. Uniqueness of L_1 approximation

To investigate the uniqueness of best L_1 approximation to a continuous function from the convex set $S^{k,1}_{m,n}(\Delta)$, we need the following definition which is introduced by Strauss in [49].

DEFINITION 5.1: Let $V = span\{v_1, ..., v_n\}$ be a subspace of $C[a,b]$ such that every function v in V has only a finite number of separated zeros. We say that the subspace V satisfies condition A, if for every nonzero v in V and every finite subset $Z_1 = \{t_1, ..., t_r\}$ of $Z(v) \cap (a,b)$, there exists a nonzero w in V such that

(a) $(-1)^i w(x) \geq 0$, for $x \in [t_{i-1}, t_i]$, $i = 1, ..., r+1$, where $t_0 = a$, $t_{r+1} = b$;

(b) if v vanishes on an open subset of $[a,b]$, then so does w.

If V satisfies condition A, then V is called an A-space.

LEMMA 5.1. *Let $f \in C[0,1]$. Let $I \subseteq \{m, m+1, ..., n-1\}$ with $I = \{i_q\}_{q=1}^{\mu}$ satisfying*

$$M_{i-1} + n + k - \mu \geq i, \quad \text{for} \quad i = 1, 2, ..., n, \tag{5.3.1}$$

where M_i counts the number of terms in $\{i_1, ..., i_\mu\}$ less then or equal to i. Then, for any partition Δ,

$$S^k_n(\Delta, I) = \{s \in S^k_n(\Delta) : s^{(i)}(1) = 0, i \in I\}, \tag{5.3.2}$$

is an A-space.

PROOF: The proof follows directly from Theorem 3.2 of [49].

From the proof of Theorem 5.2, we have the following stronger result:

LEMMA 5.2. *Let $f \in L_1[0,1]$, let $s_1^* \in S_{m,n}^{k,1}(\Delta)$ be a best approximation to f from $S_{m,n}^{k,1}(\Delta)$ and let $\phi_1 \in L_\infty$ be an associated functional of s_1^*. Then, s_1^* is a solution of the following spline approximation problem:*

$$min\{\|f - s\|_1 : s \in S_n^k(\Delta)\}$$

subject to the interpolation constraints:

$$s^{(i)}(1) = 0, i \in I(\phi_1),$$

and

$$s^{(n-1)}(x_j^-) = s^{(n-1)}(x_j^+), j \in J(\phi_1).$$

Now, we can prove the uniqueness of best L_1 approximation to $f \in C[0,1]$ from $S_{m,n}^{k,1}(\Delta)$.

THEOREM 5.3. *Let $f \in C[0,1]$, let $s^* \in S_{m,n}^{k,1}(\Delta)$ be a best L_1 approximation to f from $S_{m,n}^{k,1}(\Delta)$ and let ϕ^* be an associated functional of s^*. Assume $I(\phi^*)$ satisfies condition (5.3.1) with μ being the number of indexes in $I(\phi^*)$ and M_i counting the number of terms in $I(\phi^*)$ less than or equal to i. Then s^* is the unique best L_1 approximation to f from $S_{m,n}^{k,1}(\Delta)$.*

PROOF: Since s_1^* is a best L_1 approximation to f from $S_{m,n}^{k,1}(\Delta)$, by the duality theorem, there exists a $\phi^* \in L_\infty$ with $\|\phi^*\|_\infty = 1$ and $\int_0^1 \phi^*(f - s^*) = \|f - s^*\|_1$, satisfying (5.2.1) and (5.2.2). Then, for all $s \in S_{m,n}^{k,1}(\Delta)$,

$$\begin{aligned} \| f - s^* \|_1 &= \int_0^1 \phi^*(f - s^*) \\ &= \int_0^1 \phi^*(f - s) + \int_0^1 \phi^* s \\ &\leq \| f - s \|_1 + \int_0^1 \phi^* s. \end{aligned} \tag{5.3.3}$$

By Lemma 5.2, s^* is a best L_1 approximation to f from $S_n^{k*}(\Delta)$, with $I(\phi^*)$ and $J(\phi^*)$. Let us define a new partition by $\Delta' = \Delta - \{x_j : j \in J(\phi^*)\}$. Then,

$$S_n^{k*}(\Delta) = S_n^{k-l}(\Delta', I(\phi^*)),$$

where l counts the number of indexes in $J(\phi^*)$. By Lemma 5.1, $S_n^{k-l}(\Delta', I(\phi^*))$ is an A-space, and so is $S_n^{k*}(\Delta)$.

If s_0 is another best L_1 approximation to f from $S_{m,n}^{k,1}(\Delta)$, then, by (5.3.3), $\int_0^1 s_0\phi^* \geq 0$. Hence,

$$\int_0^1 s_0\phi^* = 0, \tag{5.3.4}$$

and from (5.3.3) it follows that

$$\int_0^1 \phi^*(f - s_0) = \| f - s_0 \|_1 . \tag{5.3.5}$$

Define

$$H_i^*(x) = \{1/(i-1)!\} \int_0^x (x-t)^{i-1}\phi^*(t)dt, \quad x \in (0,1),\ i = 1, 2, ..., n. \tag{5.3.6}$$

By using (5.3.4) and integration by parts, we have

$$\sum_{i=m}^{n-1}(-1)^i H_{i+1}^*(1)s_0^{(i)}(1) + \sum_{j=1}^{k}(-1)^n H_n^*(x_j)[s_0^{(n-1)}(x_j^+) - s_0^{(n-1)}(x_j^-)] = 0. \quad (5.3.7)$$

Since s^* is a best L_1 approximation to f from $S_{m,n}^{k,1}(\Delta)$, $(-1)^m H_i^*(1) \leq 0$, for $i = m+1, ..., n$, and $(-1)^m H_n^*(x_j) \leq 0$, for $j = 1, 2, ..., k$. Thus,

$$(-1)^m H_i^*(1)s_0^{(i-1)}(1) = 0, \quad \text{for} \quad i = m+1, ..., n,$$

and

$$(-1)^m H_n^*(x_j)[s_0^{(n-1)}(x_j^+) - s_0^{(n-1)}(x_j^-)] = 0, \quad \text{for} \quad j = 1, 2, ..., k.$$

Therefore, if $(-1)^m H_i^*(1) < 0$, for some $i \in N_{m,n}$, then $s_0^{(i-1)}(1) = 0$. If $(-1)^m H_n^*(x_j) < 0$, for some $j \in N_k$, then $s_0^{(n-1)}(x_j^+) = s_0^{(n-1)}(x_j^-)$. In view of (5.3.5), ϕ^* is an associated functional of s_0, a best L_1 approximation to f from $S_{m,n}^{k,1}(\Delta)$. Furthermore, Lemma 5.2 implies that s_0 is also a best L_1 approximation to f from $S_n^{k*}(\Delta)$, with $I(\phi^*)$ and $J(\phi^*)$. However, $S_n^{k*}(\Delta)$ is an A-space. Hence, $s^* = s_0$. This prove the theorem.

COROLLARY 5.1. *Let $f \in C[0,1]$ and $k \geq n$. Then the best L_1 approximation to f from $S_{m,n}^{k,1}(\Delta)$ is unique.*

PROOF: Let s^* be a best L_1 approximation to f from $S_{m,n}^{k,1}(\Delta)$ and ϕ^* be an associated functional of s^*. Then $I(\phi^*)$ satisfies condition (5.3.1). By Theorem 5.3, s^* is the unique best L_1 approximation to f from $S_{m,n}^{k,1}(\Delta)$.

5.4. Applications

In this section we apply the general results obtained in the previous sections to best L_p approximations from $S_{n,n}^{k,p}(\Delta)$, the set of n-convex splines of degree $n-1$, and from the set of $(n-1)$-convex polynomials of degree $n-1$.

COROLLARY 5.2. *For $1 \le p < \infty$, let $f \in L_p[0,1]$ and let $s_p^* \in S_{n,n}^{k,p}(\Delta)$.*

(a) For $1 < p < \infty$, s_p^* is the best L_p approximation to f from $S_{n,n}^{k,p}(\Delta)$ if and only if

(i) $H_{p,i}(1) = 0, \quad i = 1,2,...,n,$

(ii) $(-1)^n H_{p,n}(x_j) \le 0, \quad j = 1,2,...,k,$

(iii) if $(-1)^n H_{p,n}(x_j) < 0$, for some $j \in N_k$, then $s_p^{*(n-1)}(x_j^-) = s_p^{*(n-1)}(x_j^+)$.

(b) For $p = 1$, s_1^* is a best L_1 approximation to f from $S_{n,n}^{k,1}(\Delta)$ if and only if there exists a $\phi_1 \in L_\infty$ with $\| \phi_1 \|_\infty = 1$ and $\int_0^1 \phi_1 (f - s_1^*) = \| f - s_1^* \|_1$, satisfying the conditions (i)-(iii) of part (a) with $p = 1$. In addition, if $f \in C[0,1]$, then best L_1 approximation to f from $S_{n,n}^{k,1}(\Delta)$ is unique.

PROOF: (a) and the first sentence of (b) are direct consequences of Theorem 5.1. Next, let s^* be a best L_1 approximation to f from $S_{n,n}^{k,1}(\Delta)$. Then $I(s^*)$ is empty. Thus, $M_i = 0$, $\mu = 0$, and condition (5.3.1) is automatically satisfied. By Theorem 5.3, s^* is the unique best L_1 approximation to f from $S_{n,n}^{k,1}(\Delta)$.

There is some interesting relationship between best n-convex L_p approximation and best L_p approximation by n-convex splines of degree $n-1$. Let $K_{n,p}$ denote the set of n-convex functions in $L_p[0,1]$.

COROLLARY 5.3. *Let $f \in C[0,1]$. For $1 \le p < \infty$, let $g_p^* \in K_{n,p}$ such that $f \ne g_p^*$ a.e. in [0,1] and $f - g_p^*$ has a finite munber of sign changes at $t_1 < ... < t_N$ in (0,1). Let*

$$\phi_p = \begin{cases} (f-g_p^*)|f-g_p^*|^{p-2} & \text{for } 1 < p < \infty \\ sgn(f-g_1^*) & \text{for } p = 1 \end{cases}$$

and define $H_{p,n}$ as in (5.2.3). If g_p^* is a best L_p approximation to f from $K_{n,p}$, then g_p^* is a best L_p approximation to f from $S_{n,n}^{r,p}(\Delta')$, where $\Delta' : 0 < y_1 < ... < y_r < 1$, and the y_i's are the distinct zeros of $H_{p,n}$ in $(0,1)$.

PROOF: It follows from Corollary 3.1 that g_p^* is a spline of degree $n-1$ with simple knots at $y_1, ..., y_r$, the distinct zeros of $H_{p,n}$ in $(0,1)$. Moreover, Corollary 3.4 implies that g_p^* is a best L_p approximation to f from $S_n^r(\Delta')$. However, $g_p^* \in K_{n,p}$, thus, $g_p^* \in S_{n,n}^{r,p}(\Delta')$. Hence, g_p^* is a best approximation to f from $S_{n,n}^{r,p}(\Delta')$.

COROLLARY 5.4. *Best L_1 approximation to $f \in C[0,1]$ by $(n-1)$-convex polynomials of degree $n-1$ is unique.*

PROOF: The proof follows from Lemma 5.1 and Theorem 5.3.

Chapter 6: A Duality Approach to Best Convex Uniform Approximation

6.1. Introduction

We now consider best convex uniform approximation in the space $C[a,b]$. Let K be the set of convex functions defined on $[a,b]$. As usual, a function $g^* \in K$ is said to be *a best convex uniform approximation* to $f \in C[a,b]$, if

$$\| f - g^* \|_\infty = \inf\{\| f - g \|_\infty : g \in K\}. \tag{6.1.1}$$

The existence of a best convex uniform approximation to a bounded function was demonstrated in [56], where an algorithm for the computation of a best convex uniform approximation by means of linear programming was also presented. The characterization of alternant-type is a special case of a result announced in [6] and proved in [69]. We shall establish a duality theorem which establishes an error representation of the best convex uniform approximation and use this duality result to obtain some bounds for the error of best convex uniform approximation, to give an alternative proof to the characterization of best convex uniform approximation, and to characterize the set of linear negative alternants. We also define a "function interval" (similar to that defined in [58] for monotone approximation) which we show is a necessary condition for best convex uniform approximation.

A similar duality approach is used in Chapter 7 and Chapter 8 to investigate best quasi-convex uniform approximation and best piecewise monotone uniform approximation.

6.2. DUALITY

Define

$$S = \{(x, y; \lambda) : \ x, y \in [a, b], \ 0 \le \lambda \le 1\}. \tag{6.2.1}$$

S is a compact set in R^3. For $f \in C[a, b]$, define the function F on S by

$$F(x, y, \lambda) = (-1/2)[\lambda f(x) - f(\lambda x + (1-\lambda)y) + (1-\lambda)f(y)]. \tag{6.2.2}$$

Let

$$\delta = \delta(f) = \sup\{F(x, y, \lambda) : (x, y; \lambda) \in S\}. \tag{6.2.3}$$

δ is a measure of the convexity of the function f. We shall see in Lemma 6.1 that $\delta = 0$ is equivalent to f being convex. Let

$$\Delta(f) = \{(x, y; \lambda) \in S : F(x, y, \lambda) = \delta\}. \tag{6.2.4}$$

Since f is continuous on $[a,b]$, F is continuous on S. Thus, F assumes its maximum on S, and therefore, $\Delta(f)$ is nonempty. For $f \in C[a,b]$, define *the greatest convex minorant* or *convex envelop* of f by

$$env f(t) = \sup\{g(t) : g \in K \text{ and } f \geq g \text{ on } [a,b]\}, t \in [a,b], \tag{6.2.5}$$

where $f \geq g$ on $[a,b]$ means that $f(s) \geq g(s)$ for all $s \in [a,b]$. We remark that $env f$ is the largest continuous convex function that does not exceed f at any point in $[a,b]$ (see [41]).

LEMMA 6.1. *Let $f \in C[a,b]$. Then $\delta = 0$ if and only if f is convex.*

PROOF: If f is convex, then for all $(x,y;\lambda) \in S$, $F(x,y,\lambda) \leq 0$. Hence, $\delta = 0$. Conversely, if f is not convex, then there exists $(x,y;\lambda) \in S$ with $x \neq y$ and $0 < \lambda < 1$ such that $F(x,y,\lambda) > 0$. Thus, $\delta > 0$.

LEMMA 6.2. *Let $f \in C[a,b] - K$. If $(x,y;\lambda) \in \Delta(f)$, then $x \neq y$ and $0 < \lambda < 1$.*

PROOF: Assume to the contrary that one of the following statements is true: $x = y$, $\lambda = 0$ or $\lambda = 1$. Thus $F(x,y,\lambda) = 0$. Since $(x,y;\lambda) \in \Delta(f)$, $\delta = F(x,y,\lambda) = 0$. By Lemma 6.1, f is convex. This contradict the hypothesis.

The following lemma was basically proved in [56]:

LEMMA 6.3. *Let $f \in C[a,b]$. Then, $f(a) = env f(a)$ and $f(b) = env f(b)$.*

Now we can establish a duality theorem showing that $\delta(f)$ is the error of best approximation.

THEOREM 6.1 (DUALITY). *Let $f \in C[a,b]$. Then,*

$$inf\{\| f - g \|_\infty : g \in K\} = \delta(f). \tag{6.2.6}$$

PROOF: For any $(x, y; \lambda) \in S$ and all $g \in K$,

$$\lambda g(x) - g(\lambda x + (1-\lambda)y) + (1-\lambda)g(y) \geq 0,$$

and thus

$$F(x,y,\lambda) \leq F(x,y,\lambda) + (1/2)[\lambda g(x) - g(\lambda x + (1-\lambda)y) + (1-\lambda)g(y)] \leq \| f - g \|_\infty .$$

Conseqently, $\delta(f) \leq \inf\{\| f - g \|_\infty : g \in K\}$.

To complete this proof, let

$$\overline{g}(t) = envf(t) + \delta(f), \quad \text{for all} \quad t \in [a,b]. \tag{6.2.7}$$

Since $envf \leq f$, on $[a,b]$, we have $\overline{g}(t) \leq f(t) + \delta(f)$, for all $t \in [a,b]$. Assume that there exists an $x_0 \in (a,b)$ such that

$$f(x_0) - \delta(f) > \overline{g}(x_0) = envf(x_0) + \delta(f). \tag{6.2.8}$$

By virtue of the continuity of $f - envf$, there exists some open interval $I \subset [a,b]$ such that

$$f(t) - envf(t) > 2\delta(f), \quad \text{for all} \quad t \in I.$$

Lemma 6.3 then implies that there exist $x_1, x_2 \in [a,b]$ with $I \subseteq (x_1, x_2)$ such that

$$f(x_1) = envf(x_1), \quad f(x_2) = envf(x_2),$$

and

$$f(t) - envf(t) > 0 \ \text{ for all } \ t \in (x_1, x_2).$$

By a similar reasoning as in [56], we can show that $envf$ is linear on (x_1, x_2). Therefore, for some $\lambda_0 \in (0,1)$, $x_0 = \lambda_0 x_1 + (1-\lambda_0)x_2$ and

$$envf(\lambda_0 x_1 + (1-\lambda_0)x_2) = \lambda_0 f(x_1) + (1-\lambda_0) f(x_2).$$

It follows that

$$\begin{aligned} & f(\lambda_0 x_1 + (1-\lambda_0)x_2) - envf(\lambda_0 x_1 + (1-\lambda_0)x_2) \\ & = \lambda_0 f(x_1) + f(\lambda_0 x_1 + (1-\lambda_0)x_2) - (1-\lambda_0) f(x_2) \\ & > 2\delta(f). \end{aligned}$$

This last inequality contradicts the definition of $\delta(f)$. This contradiction implies that (6.2.8) cannot hold. Thus

$$\overline{g}(t) \geq f(t) - \delta(f), \quad \text{for all } \ t \in [a,b].$$

Hence,

$$f(t) - \delta(f) \leq \overline{g}(t) \leq f(t) + \delta(f), \quad \text{for all } \ t \in [a,b].$$

Since $\overline{g} \in K$, we have established equation (6.2.6).

COROLLARY 6.1. *Let $f \in C[a,b]$. Then, $\bar{g} = env f + \delta(f)$ is a best convex approximation to f.*

THEOREM 6.2.

(i) Let $f \in C^1[a,b]$. Then,

$$\delta(f) \le [(b-a)/8] \, sup\{f'(x) - f'(y) : a \le x \le y \le b\}. \tag{6.2.9}$$

(ii) Let $f \in C^2[a,b]$. Then,

$$\delta(f) \le [(b-a)^2/16] \, sup\,\{[-f''(x)]_+ : x \in [a,b]\}, \tag{6.2.10}$$

where

$$[a]_+ = \begin{cases} 0 & \text{if } \ a \le 0 \\ a & \text{if } \ a > 0. \end{cases}$$

PROOF: (i) Note that for $(x, y; \lambda) \in S$,

$$F(x,y,\lambda) = (1/2)\{\lambda[f(\lambda x + (1-\lambda)y) - f(x)] + (1-\lambda)[f(\lambda x + (1-\lambda)y) - f(y)]\}.$$

Since $f \in C^1[a,b]$, for some $t_1 \in [x, \lambda x + (1-\lambda)y]$ and $t_2 \in [\lambda x + (1-\lambda)y, y]$,

$$F(x,y,\lambda) = (1/2)\lambda(1-\lambda)(y-x)[f'(t_1) - f'(t_2)].$$

Therefore,

$$\delta(f) \le [(b-a)/8] \, \text{sup}\{f'(x) - f'(y) : a \le x \le y \le b\}.$$

(ii) If $f \in C^2[a,b]$ and $(x,y;\lambda) \in S$, then, for some $t \in [x,y]$,

$$\begin{aligned} F(x,y,\lambda) &= (-1/2)\lambda(1-\lambda)(y-x)^2[x, \lambda x + (1-\lambda)y, y]f \\ &= (-1/4)\lambda(1-\lambda)(y-x)^2 f''(t), \end{aligned}$$

where $[t_1, t_2, t_3]f$ denotes the second divided difference of f at t_1, t_2, t_3. Hence, inequality (6.2.10) follows.

As another application of Theorem 6.1, we provide an alternative proof of the characterization of best convex approximation to a continuous function, which was announced in [6] and proved in [69].

CHARACTERIZATION THEOREM. *Let $f \in C[a,b] - K$. $g^* \in K$ is a best convex approximation to f if and only if there exist $x < y$ in $[a,b]$ and $\lambda \in (0.1)$ such that g^* is linear on $[x,y]$ and satisfies*

$$f(x) - g^*(x) = f(y) - g^*(y) = - \| f - g^* \|_\infty,$$

and

$$f(\lambda x + (1-\lambda)y) - g^*(\lambda x + (1-\lambda)y) = \| f - g^* \|_\infty .$$

PROOF: (Necessity) By the hypothesis and Theorem 6.1, $\| f - g^* \|_\infty = \delta(f)$. In view of the continuity of f, $\Delta(f)$ is nonempty. Assume $(x,y;\lambda) \in \Delta(f)$. Then by Lemma 6.2, $x < y$ and $0 < \lambda < 1$. Since $g^* \in K$, the following inequality holds:

$$G(x,y,\lambda) \equiv (1/2)[\lambda g^*(x) - g^*(\lambda x + (1-\lambda)y) + (1-\lambda)g^*(y)] \geq 0.$$

If $G(x,y,\lambda) > 0$, then

$$\delta(f) = F(x,y,\lambda) < F(x,y,\lambda) + G(x,y,\lambda)$$

$$\leq (1/2)[\lambda \parallel f - g^* \parallel_\infty + \parallel f - g^* \parallel_\infty + (1-\lambda) \parallel f - g^* \parallel_\infty]$$

$$= \parallel f - g^* \parallel_\infty .$$

This contradicts Theorem 6.1. Thus $G(x,y,\lambda) = 0$. It follows from this equation and the convexity of g^* that $g^*$is linear on $[x,y]$. Therefore,

$$\delta(f) = F(x,y,\lambda) + G(x,y,\lambda)$$

$$= (1/2)\{\lambda[g^*(x) - f(x)] + [f(\lambda x + (1-\lambda)y) - g^*(\lambda x + (1-\lambda)y)]$$

$$+ (1-\lambda)[g^*(y) - f(y)]\}.$$

However, from Theorem 6.1, we have $-\delta(f) \leq f(t) - g^*(t) \leq \delta(f)$, for all $t \in [a,b]$.

If $g^*(x) - f(x) < \delta(f)$, then

$$(1-\lambda/2)\delta(f)$$

$$< (1/2)\{[f(\lambda x + (1-\lambda)y) - g^*(\lambda x + (1-\lambda)y)] + (1-\lambda)[g^*(y) - f(y)]\}$$

$$\leq (1-\lambda/2) \parallel f - g^* \parallel_\infty,$$

and thus $\delta(f) < \parallel f - g^* \parallel_\infty$, which is a contradiction. This contradiction implies that $g^*(x) - f(x) = \delta(f)$. Similarly, we can show that $g^*(y) - f(y) = \delta(f)$ and $f(\lambda x + (1-\lambda)y) - g^*(\lambda x + (1-\lambda)y) = \delta(f)$. These three equations and Theorem 6.1 establish the necessity of this characterization.

(Sufficiency) From the assumptions we have

$$\delta(f) \geq F(x,y,\lambda)$$

$$= \parallel f - g^* \parallel_\infty + (1/2)[-\lambda g^*(x) + g^*(\lambda x + (1-\lambda)y) - (1-\lambda)g^*(y)]$$

$$= \parallel f - g^* \parallel_\infty,$$

where the last equality holds because of the linearity of g^* on $[x,y]$. Therefore, by Theorem 6.1, g^* is a best convex approximation to f on $[a,b]$.

6.3. Some Properties of Best Convex Uniform Approximations

In this section, we characterize the set of linear negative alternants of $f - g^*$, where g^* is a best convex approximation to $f \in C[a,b]$, and identify two functions which are respectively a lower bound and an upper bound of any best approximation to f.

For a real-valued function h defined on $[a,b]$, $a \leq x_1 < x_2 < x_3 \leq b$ is said to be a *negative alternant* of h, if $-h(x_1) = h(x_2) = -h(x_3) = \| h \|_\infty$. For $f \in C[a,b] - K$ and $g \in K$, define the set of *linear negative alternants* of $f - g$ by

$$A(f-g) = \{(x,y;\lambda) \in S : g \text{ is linear on } [x,y] \text{ and } x < \lambda x + (1-\lambda)y < y \text{ is a negative alternant of } f-g\}. \tag{6.3.1}$$

The following theorem characterizes the set of linear negative alternants of $f - g^*$, where g^* is a best convex approximation to f.

Theorem 6.3. *Let $f \in C[a,b] - K$ and let g^* be a best convex approximation to f on $[a,b]$. Then,*

$$A(f - g^*) = \Delta(f). \tag{6.3.2}$$

Proof: Let $(x,y;\lambda) \in \Delta(f)$. By a similar reasoning as in the proof of the characterization of best convex approximation, we find $(x,y;\lambda) \in A(f - g^*)$. This gives

$\Delta(f) \subseteq A(f-g^*)$. Conversely, assume $(x,y;\lambda) \in A(f-g^*)$. Then, g^* is linear on $[x,y]$ and satisfies

$$f(x) - g^*(x) = f(y) - g^*(y) = -\| f - g^* \|_\infty = -\delta(f).$$

and

$$f(\lambda x + (1-\lambda)y) - g^*(\lambda x + (1-\lambda)y) = \| f - g^* \|_\infty = \delta(f).$$

Hence,

$$\begin{aligned}\delta(f) &\geq F(x,y,\lambda) \\ &= (1/2)\{-\lambda[g^*(x) - \delta(f)] + [g^*(\lambda x + (1-\lambda)y) - \delta(f)] \\ &\qquad - (1-\lambda)[g^*(y) - \delta(f)]\} \\ &= \delta(f).\end{aligned}$$

This implies that $F(x,y,\lambda) = \delta(f)$, and thus $(x,y;\lambda) \in \Delta(f)$. Accordingly, (6.3.2) holds.

COROLLARY 6.2. *Let $f \in C[a,b] - K$ and let g^* be a best convex approximation to f on $[a,b]$. Then, for all $(x,y;\lambda) \in \Delta(f)$ with $x < y$, g^* is linear on $[x,y]$ and*

$$g^*(\mu x + (1-\mu)y) = \mu f(x) + (1-\mu)f(y) + \delta(f), \quad \mu \in [0,1].$$

PROOF: For $(x,y;\lambda) \in \Delta(f)$, by Theorem 6.3, $(x,y;\lambda) \in A(f-g^*)$. Hence, g^* is linear on $[x,y]$, and $g^*(x) = f(x) + \delta(f)$ and $g^*(y) = f(y) + \delta(f)$. Therefore, by linear interpolation, for all $\mu \in [0,1]$,

$$\begin{aligned} &g^*(\mu x + (1-\mu)y) \\ &= g^*(x)[\mu x + (1-\mu)y - y]/(x-y) + g^*(y)[\mu x + (1-\mu)y - x]/(y-x) \\ &= f(x)\mu + f(y)(1-\mu) + \delta(f). \end{aligned}$$

Now we identify two functions which bound any best approximation. For $(x, y; \lambda) \in \Delta(f)$, denote the linear interpolant to $(x, f(x)+\delta(f))$ and $(y, f(y)+\delta(f))$ on $[a,b]$ by

$$l[x,y](t) = f(x)(t-y)/(x-y) + f(y)(t-x)/(y-x) + \delta(f), \quad t \in [a,b].$$

Let

$$L = \{l[x,y] : (x,y;\lambda) \in \Delta(f)\}, \tag{6.3.3}$$

and

$$G = \{envf - \delta(f)\} \cup L. \tag{6.3.4}$$

Define

$$\underline{g}(t) = \sup\{g(t) : g \in G\}, \quad t \in [a,b]. \tag{6.3.5}$$

It is easy to verify that $\underline{g}$ is a convex function on $[a,b]$ and if f is convex then $\underline{g} = f$. The next theorem show that this convex function is a lower bound of the best approximations to f.

THEOREM 6.4. *Let $f \in C[a,b]$. If $g^* \in K$ is a best convex approximation to f, then*

$$\underline{g}(t) \leq g^*(t) \leq \overline{g}(t), \quad \text{for all} \quad t \in [a,b], \tag{6.3.6}$$

where $\overline{g}$ was defined in (6.2.7).

PROOF: In [56], it has been proved that

$$g^*(t) \leq envf(t) + \| f - g^* \|_\infty .$$

By replacing $\| f - g^* \|_\infty$ by $\delta(f)$, we obtain the upper bound. To show the lower bound, assume to the contrary that there exists some $z \in [a,b]$ such that $\underline{g}(z) > g^*(z)$. Define

$$P = \cup\{[x,y] : (x,y;\lambda) \in \Delta(f)\}. \tag{6.3.7}$$

By the definition of $\underline{g}$,

$$\underline{g}(t) = g^*(t), \quad \text{for all} \quad t \in P.$$

Hence z is not in P. If $\underline{g}(z) = envf(z) - \delta(f)$, then

$$f(z) - \delta(f) \geq envf(z) - \delta(f) > g^*(z),$$

which contradicts the hypothesis that g^* is a best convex approximation to f. Therefore, there exists $(x,y;\lambda) \in \Delta(f)$, such that $l[x,y](z) > g^*(z)$. This contradicts the convexity of g^*. It follows that $\underline{g}(t) \leq g^*(t)$, for all $t \in [a,b]$.

As was shown in Corollary 6.1, $\bar{g}$ is a best convex approximation to f. Hence it is the greatest best convex approximation to f. However, $\underline{g}$ need not be a best convex approximation to f.

It is shown in [58] that if f is continuous but not nondecreasing on $[a, b]$, then there exists a best monotone approximation to f that is in C^∞. However, an analogous statement is not true for best convex approximation. To see this, let us consider the following example: Assume

$$f(x) = \begin{cases} -6x + 1 & 0 \le x \le 1/8 \\ 4x - 1/4 & 1/8 < x \le 1/4 \\ -3x + 3/2 & 1/4 < x \le 1/2 \\ 3x - 3/2 & 1/2 < x \le 3/4 \\ -4x + 15/4 & 3/4 < x \le 7/8 \\ 6x - 5 & 7/8 < x \le 1. \end{cases}$$

Then f is continuous but is not convex on $[0, 1]$. $\delta(f) = 5/16$ and

$$\Delta(f) = \{(1/8, 1/2; 1/2), (1/2, 7/8; 1/2)\}.$$

Hence, every best convex approximation has a knot at $x = 1/2$, and thus is not differentiable at $1/2$.

Chapter 7: Best Quasi-convex Uniform Approximation

7.1. Introduction

A function $g \in B$ is said to be *quasi-convex* [41] if

$$g(x) \leq \max\{g(s), g(t)\}, \quad \text{for all } 0 \leq s \leq x \leq t \leq 1.$$

Let $K \subset B$ denote the set of all quasi-convex functions on $[0,1]$.

Ubhaya [63] has proved that g is quasi-convex if and only if there exists a point $p \in [0,1]$, such that either

(i) g is nonincreeasing on $[0,p)$ and is nondecreasing on $[p,1]$, or

(ii) g is nonincreasing on $[0,p]$ and is nondecreasing on $(p,1]$.

We shall call the point p (in either (i) or (ii)) a knot of g. Let K_p denote the set of functions in K which have a knot at p. Then, $K = \cup\{K_p : p \in [0,1]\}$. In general, the set of all the knots of a quasi-convex function is a closed subinterval of $[0,1]$.

The problem of best quasi-convex approximation is to find a $g^* \in K$, such that

$$\| f - g^* \|_\infty = \inf\{\| f - g \|_\infty : g \in K\}. \qquad (7.1.1)$$

This problem is considered in [63], where a sufficient condition for a best quasi-convex approximation to a bounded function is obtained, and some structural properties of best approximation are established. Algorithms for the computation of a best discrete quasi-convex approximation are presented in [9, 57].

Given $f \in C[0,1]$, let

$$G = G(f) = \{g^* \in K : \| f - g^* \|_\infty = \inf\{\| f - g \|_\infty : g \in K\}\}, \tag{7.1.2}$$

be the set of best quasi-convex approximations to f, and let

$$P^* = \{p \in [0,1] : p \text{ is a knot for some } g^* \in G(f)\}. \tag{7.1.3}$$

We shall call P^* the *set of optimal knots.*

In this chapter, we shall characterize both the best quasi-convex approximations and the optimal knots. In addition, we shall prove that a best quasi-convex approximation is unique if and only if f is quasi-convex.

7.2. Preliminaries

Similar to the developments in [58], we define two functionals δ_l and δ_r, which we shall use to obtain the error of best quasi-convex approximation. For $f \in C[0,1]$ and $p \in [0,1]$, let

$$\delta_l(p) = \sup\{[f(y) - f(x)]/2 : 0 \leq x \leq y \leq p\}, \tag{7.2.1}$$

and

$$\delta_r(p) = \sup\{[f(x) - f(y)]/2 : p < x \leq y \leq 1\}. \qquad (7.2.2)$$

We remark that δ_l is a measure of the "decreasingness" of f on $[0,p]$ and δ_r is a measure of the "increasingness" of f on $(p,1]$. Moreover we define

$$\delta(p) = \max\{\delta_l(p), \delta_r(p)\}. \qquad (7.2.3)$$

Denote the minimum value of $\delta(p)$ on $[0,1]$ by

$$\delta^* = \inf\{\delta(p) : 0 \leq p \leq 1\}. \qquad (7.2.4)$$

Let

$$P = \{p \in [0,1] : \delta(p) = \delta^*\} \qquad (7.2.5)$$

be the set of minima for $\delta(p)$, and let

$$S = \{s \in [0,1] : f(s) = m\} \qquad (7.2.6)$$

be the set of minima for f, where $m = \inf\{f(x) : 0 \leq x \leq 1\}$. Denote the convex hull of S by $[s_l, s_r]$. Then,

$$s_l = \inf S, \quad \text{and} \quad s_r = \sup S. \qquad (7.2.7)$$

In addition, define

$$\eta_l = \inf\{x \in [0, s_l] : f(t) \leq m + 2\delta^*, \quad \text{for all} \quad t \in [x, s_l]\}, \qquad (7.2.8)$$

and

$$\eta_r = \sup\{x \in [s_r, 1] : f(t) \leq m + 2\delta^*, \quad \text{for all} \quad t \in [s_r, x]\}. \tag{7.2.9}$$

Thus, $[s_l, s_r] \subseteq [\eta_l, \eta_r]$. We shall prove that $P = [\eta_l, \eta_r]$, and that $P = P^*$, the set of optimal knots.

Next, let $f \in B$. For each $p \in [0,1]$, similar to the definitions of u_p^- and v_p^- in [8] with θ_p^- replaced by δ^* we define the two functions:

$$\underline{g}_p(x) = \begin{cases} \sup\{f(t) : t \in [x,p]\} - \delta^* & x \in [0,p] \\ \sup\{f(t) : t \in (p,x]\} - \delta^* & x \in (p,1] \end{cases} \tag{7.2.10}$$

and

$$\overline{g}_p(x) = \begin{cases} \inf\{f(t) : t \in [0,x]\} + \delta^* & x \in [0,p] \\ \inf\{f(t) : t \in (x,1]\} + \delta^* & x \in (p,1]. \end{cases} \tag{7.2.11}$$

LEMMA 7.1. *Let $f \in C[0,1]$. Then,*

(i) $|\Delta\delta_l(p)| \leq (1/2)\omega_f(|\Delta p|)$, and $|\Delta\delta_r(p)| \leq (1/2)\omega_f(|\Delta p|)$, where ω_f denotes the modulus of continuity of f. Thus, δ_l and δ_r are continuous functions;

(ii) $\delta^* = 0$ if and only if $f \in K$;

(iii) $S \subset P$.

PROOF: (i) If $\Delta p > 0$ then,

$$\delta_l(p + \Delta p) \leq \delta_l(p) + \sup\{[f(y) - f(x)]/2 : p \leq x \leq y \leq p + |\Delta p|\},$$

and If $\Delta p < 0$ then,

$$\delta_l(p) \leq \delta_l(p - |\Delta p|) + \sup\{[f(y) - f(x)]/2 : p - |\Delta p| \leq x \leq y \leq p\}.$$

It follows that

$$\Delta\delta_l(p) \le \sup\{[f(y) - f(x)]/2 : 0 \le y - x \le |\Delta p|\} = (1/2)\omega_f(|\Delta p|).$$

Similarly, we may show the second inequality of (i).

(ii) First let $\delta^* = 0$. By (i) δ_l and δ_r are continuous and thus so is $\delta(p)$. Hence, there exists a $p_0 \in [0,1]$, such that $\delta(p_0) = \delta^* = 0$. Thus, $\delta_l(p_0) = \delta_r(p_0) = 0$, since δ_l and δ_r are both nonnegative functions. Consequently, by the definitions of δ_l and δ_r, f is nonincreasing on $[0, p_0]$, and nondecreasing on $(p_0, 1]$. Thus, $f \in K$. Conversely, assume that $f \in K$. Then there exists a $p_0 \in [0,1]$ such that $f \in K_{p_0}$. Therefore, $\delta_l(p_0) = \delta_r(p_0) = 0$, which implies that $\delta(p_0) = 0$. Hence, $\delta^* = 0$.

(iii) It is sufficient to show that if $s \in S$, then,

$$\delta_l(s) \le \max\{\delta_l(p), \delta_r(p)\} \quad \text{for all } \; p \in [0,1] \tag{7.2.12}$$

and

$$\delta_r(s) \le \max\{\delta_l(p), \delta_r(p)\} \quad \text{for all } \; p \in [0,1] \tag{7.2.13}$$

The proofs of (7.2.12) and (7.2.13) are similar. Thus we shall only present the proof of (7.2.12).

If $s = 0$, then, since $\delta_l(0) = 0$ and since δ_l and δ_r are both nonnegative functions, (7.2.12) holds. If $s \in (0,1]$, we consider two cases. First assume that $p \ge s$. Then $\delta_l(s) \le \delta_l(p)$ and thus (7.2.12) holds. Next, assume that $p < s$, and $\delta_l(p) < \delta_r(s)$.

$f \in C[0,1]$ implies that $2\delta_l(s) = f(y_1) - f(x_1)$ for some $x_1 \le y_1$ in $[0,s]$. It follows that $2\delta_l(p) < f(y_1) - f(x_1)$ and $p < y_1$. Hence,

$$2\delta_l(s) \le f(y_1) - f(s) \le \sup\{[f(x) - f(y)] : p \le x \le y \le s\} \le 2\delta_r(p).$$

Therefore, (7.2.12) holds.

LEMMA 7.2. *$\underline{g}_p$ and $\overline{g}_p$ as defined by (7.2.10) and (7.2.11) have the following properties:*

(i) $\underline{g}_p, \overline{g}_p \in K_p$ for all $p \in [0,1]$;

(ii) if $f \in C[0,1]$ then

(a) $\underline{g}_p \in C[0,1]$ for all $p \in [0,1]$,

(b) $\overline{g}_p \in C[0,1]$ if and only if $p \in [s_l, s_r]$,

(c) if $p \in [s_l, s_r]$, then $\overline{g}_p(x) = \overline{g}_{s_l}(x)$ for all $x \in [0,1]$,

(d) if $p \in [0,1]$, then $\overline{g}_p(x) \le \overline{g}_{s_l}(x)$ for all $x \in [0,1]$.

PROOF: (i) follows from the definitions (7.2.10) and (7.2.11).

(ii) (a) For all $p \in [0,1]$, (7.2.10) implies that $\underline{g}_p$ is continuous at any $x \ne p$. Next, to prove the continuity of $\underline{g}_p$ at $x = p$, we observe that

$$\underline{g}_p(p^-) = \lim_{\epsilon \to 0} \sup\{f(t) : t \in [p-\epsilon, p]\} - \delta^* = f(p) - \delta^*,$$

and

$$\underline{g}_p(p^+) = \lim_{\epsilon \to 0} \sup\{f(t) : t \in (p, p+\epsilon]\} - \delta^* = f(p) - \delta^*.$$

Since $f \in C[0,1]$. Thus, $\underline{g}_p(p^-) = \underline{g}_p(p^+) = \underline{g}_p(p)$, and (a) is proved.

(b) Similarly, for all $p \in [0,1]$, $\overline{g}_p$ is continuous where $x \neq p$. Next, if $x = p$ and $p \in [s_l, s_r]$, then

$$\overline{g}_p(p^-) = \lim_{\epsilon \to 0} \inf\{f(t) : t \in [0, p-\epsilon]\} + \delta^* = f(s_l) + \delta^*,$$

and

$$\overline{g}_p(p^+) = \lim_{\epsilon \to 0} \inf\{f(t) : t \in [p+\epsilon, 1]\} + \delta^* = f(s_r) + \delta^*.$$

Hence, $\overline{g}_p(p^-) = \overline{g}_p(p) = \overline{g}_p(p^+)$.

Conversely, suppose that $p \notin [s_l, s_r]$. If $p < s_l$, then

$$\begin{aligned} \overline{g}_p(p^-) &= \lim_{\epsilon \to 0} \inf\{f(t) : t \in [0, p-\epsilon]\} + \delta^* > f(s_l) + \delta^* \\ &= \lim_{\epsilon \to 0} \inf\{f(t) : t \in [p+\epsilon, 1]\} + \delta^* = \overline{g}_p(p^+). \end{aligned}$$

While if $p > s_r$ then

$$\begin{aligned} \overline{g}_p(p^-) &= \lim_{\epsilon \to 0} \inf\{f(t) : t \in [0, p-\epsilon]\} + \delta^* = f(s_l) + \delta^* \\ &< \lim_{\epsilon \to 0} \inf\{f(t) : t \in [p+\epsilon, 1]\} + \delta^* = \overline{g}_p(p^+). \end{aligned}$$

(c) Let $p \in [s_l, s_r]$. For $x \in [s_l, p]$,

$$\overline{g}_p(x) = \inf\{f(t) : t \in [0, x]\} + \delta^* = f(s_l) + \delta^* = m + \delta^*,$$

and for $x \in (p, s_r]$,

$$\overline{g}_p(x) = \inf\{f(t) : t \in [x, 1]\} + \delta^* = f(s_r) + \delta^* = m + \delta^*,$$

and for $x \notin [s_l, s_r]$, $\overline{g}_p(x) = \overline{g}_{s_l}(x)$. Thus, $\overline{g}_p = \overline{g}_{s_l}$.

(d) Assume that $p \notin [s_l, s_r]$. If $p < s_l$, then $\overline{g}_p(x) = \overline{g}_{s_l}(x)$, for all $x \in [0,p] \cup [s_l, 1]$, and

$$\overline{g}_p(x) = \inf\{f(t) : t \in [x,1]\} + \delta^* = f(s_r) + \delta^*$$

$$< \inf\{f(t) : t \in [0,x]\} + \delta^* = \overline{g}_{s_l}(x) \quad \text{for all} \quad x \in (p, s_l).$$

If $p > s_r$, then $\overline{g}_p(x) = \overline{g}_{s_l}(x)$, for all $x \in [0, s_r] \cup [p, 1]$, and

$$\overline{g}_p(x) = \inf\{f(t) : t \in [0,x]\} + \delta^* = f(s_r) + \delta^*$$

$$< \inf\{f(t) : t \in [x,1]\} + \delta^* = \overline{g}_{s_l}(x) \quad \text{for all} \quad x \in (s_r, p).$$

Thus, by (c) if $p \in [0,1]$, then $\overline{g}_p(x) \leq \overline{g}_{s_l}(x)$ for all $x \in [0,1]$.

THEOREM 7.1. *Let $f \in C[0,1]$, and let P be the set of minimum points for δ. Then, $P = [\eta_l, \eta_r]$, where η_l and η_r are defined by (7.2.8) and (7.2.9) respectively.*

PROOF: Assume that $x_0 \in [\eta_l, \eta_r]$. We consider three cases.

Case 1: $x_0 \in [\eta_l, s_l]$. Then, $\delta_l(x_0) \leq \delta_l(s_l)$. However, since $s_l \in S \subset P$,

$$\delta_r(x_0) = \max\{\sup\{[f(x) - f(y)]/2 : x_0 < x \leq y \leq s_l\},$$

$$\sup\{[f(x) - f(y)]/2 : s_l \leq x \leq y \leq 1\}\}$$

$$= \max\{\sup\{f(x)/2 : x_0 \leq x \leq s_l\} - f(s_l)/2, \delta_r(s_l)\}$$

$$\leq \delta^*.$$

Case 2: $x_0 \in (s_l, s_r)$. Then,

$$\sup\{[f(y) - f(x)]/2 : x_l \leq x, y \leq s_r\} \leq \delta^*.$$

Since $s_l \in P$,

$$\delta_l(x_0) = \max\{\delta_l(s_l), \sup\{[f(y) - f(x)]/2 : s_l \leq x \leq y \leq x_0\}\} \leq \delta^*,$$

and

$$\delta_r(x_0) = \max\{\delta_l(s_r), \sup\{[f(x) - f(y)]/2 : x_0 \leq x \leq y \leq s_r\}\} \leq \delta^*.$$

Case 3: $x_0 \in [s_r, \eta_r]$. Then , $\delta_r(x_0) \leq \delta_r(s_r) \leq \delta^*$. Also, since $s_r \in P$,

$$\begin{aligned}\delta_l(x_0) &= \max\{\delta_l(s_r), \sup\{[f(y) - f(x)]/2 : s_r \leq x \leq y \leq x_0\}\} \\ &= \max\{\delta_l(s_r), \sup\{f(x)/2 : s_r \leq x \leq x_0\} - f(s_r)/2\} \\ &\leq \delta^*.\end{aligned}$$

Combining all three cases, $\delta(x_0) = \max\{\delta_l(x_0), \delta_r(x_0)\} \leq \delta^*$, for $x_0 \in [\eta_l, \eta_r]$. Hence, $x_0 \in P$, and thus, $[\eta_l, \eta_r] \subseteq P$.

Next, assume that $x_0 \notin [\eta_l, \eta_r]$. If $x_0 < \eta_l$, then by the definition of η_l, there exists a $t_0 \in [x_0, s_l]$ such that $(1/2)f(t_0) > (1/2)m + \delta^*$. Hence,

$$\begin{aligned}\delta_r(x_0) &\geq \sup\{[f(x) - f(y)]/2 : x_0 \leq x \leq y \leq s_l\} \\ &\geq (1/2)[f(t_0) - f(s_l)] \\ &> \delta^*.\end{aligned}$$

This implies that $x_0 \notin P$. If $x_0 > \eta_r$, then by the definition of η_r, there exists a $t_0 \in [s_r, x_0]$ such that $(1/2)f(t_0) > (1/2)m + \delta^*$. Hence,

$$\begin{aligned}\delta_r(x_0) &\geq \sup\{[f(y) - f(x)]/2 : s_r \leq x \leq y \leq x_0\} \\ &\geq (1/2)[f(t_0) - f(s_r)] \\ &> \delta^*.\end{aligned}$$

which implies that $x_0 \notin P$. Thus, $P \subseteq [\eta_l, \eta_r]$.

7.3. Duality

In this section we prove that δ^* is the error of best approximation and for $p \in [\eta_l, \eta_r]$, $\underline{g}_p$ and $\overline{g}_p$ are both best quasi-convex approximations to $f \in C[0,1]$.

LEMMA 7.3. *Let $f \in C[0,1]$ and $p \in [\eta_l, \eta_r]$. Then,*

$$\|f - \underline{g}_p\|_\infty \leq \delta^* \text{ and } \|f - \overline{g}_p\|_\infty \leq \delta^*.$$

PROOF: The proofs of these two inequalities are similar. Thus, we present only the proof of the second.

If $x \in [0,p]$ then $\overline{g}_p(x) \leq f(x) + \delta^*$. Also, for each $\epsilon > 0$, there exists a $t \in [0,x]$ such that $\overline{g}_p(x) > f(t) + \delta^* - \epsilon$. Since $p \in P$, $\delta(p) = \max\{\delta_l(p), \delta_r(p)\} = \delta^*$, and thus $\delta^* \geq [f(x) - f(t)]/2$. Hence,

$$\overline{g}_p(x) > f(t) + \delta^* - \epsilon \geq f(x) - \delta^* - \epsilon.$$

Consequently, if $x \in [0,p]$, then $|f(x) - \overline{g}_p(x)| \leq \delta^*$. Similarly, we can show that if $x \in (p,1]$, then $|f(x) - \overline{g}_p(x)| \leq \delta^*$. Thus, $\|f - \overline{g}_p\|_\infty \leq \delta^*$.

The following theorem shows that δ^* is the measure of best quasi-convex approximation to $f \in C[0,1]$.

THEOREM 7.2 (DUALITY). *Let $f \in C[0,1]$. Then, $inf\{\|f - g\|_\infty : g \in K\} = \delta^*$, with δ^* as defined by (7.2.4).*

PROOF: For each $g \in K$, there exists a $p \in [0,1]$ such that $g \in K_p$. Hence, for $0 \le x \le y \le p$ (or $0 \le x \le y < p$),

$$\begin{aligned} f(y) - f(x) &\le f(y) - f(x) + g(x) - g(y) \\ &\le |f(y) - g(y)| + |f(x) - g(x)| \\ &\le 2\|f - g\|_\infty, \end{aligned}$$

and for $p < x \le y \le 1$ (or $p \le x \le y \le 1$),

$$\begin{aligned} f(x) - f(y) &\le f(x) - f(y) + g(y) - g(x) \\ &\le |f(y) - g(y)| + |f(x) - g(x)| \\ &\le 2\|f - g\|_\infty. \end{aligned}$$

It follows that $\delta_l(p) \le \|f - g\|_\infty$ and $\delta_r(p) \le \|f - g\|_\infty$. Therefore, for each $g \in K$,

$$\|f - g\|_\infty \ge \max\{\delta_l(p), \delta_r(p)\} = \delta(p) \ge \delta^*,$$

and thus $\inf\{\|f-g\|_\infty : g \in K\} \ge \delta^*$. By Lemma 7.3, we also have $\|f - \overline{g}_p\|_\infty \le \delta^*$, and by Lemma 7.2, $\overline{g}_p \in K_p \subset K$. Consequently, $\inf\{\|f - g\|_\infty : g \in K\} = \delta^*$.

Theorem 7.2 can be extended to bounded f by using Theorem 4.2 of [63] and (A) of Theorem 1 of [58].

COROLLARY 7.1. *If $f \in C[0,1]$ and $p \in P = [\eta_l, \eta_r]$, then*

$$\|f - \underline{g}_p\|_\infty = \|f - \overline{g}_p\|_\infty = \delta^*.$$

Therefore, $\underline{g}_p$ and $\overline{g}_p$ are both best approximations to f, and $P \subseteq P^*$.

We now characterize P^*, the set of optimal knots.

LEMMA 7.4. *If g is a best quasi-convex approximation to $f \in C[0,1]$, and p is a knot for g, then $p \in P = [\eta_l, \eta_r]$. Thus, $P^* \subseteq P$.*

PROOF: Assume that $p \notin P$, then by the definition of P either $\delta_l(p) > \delta^*$ or $\delta_r(p) > \delta^*$.

If $\delta_l(p) > \delta^*$, then there exists an $x_1 < y_1$ in $[0,p]$ such that

$$(1/2)[f(y_1) - f(x_1)] > \delta^*.$$

Since g is a best approximation, it follows from Theorem 7.2 that

$$-\delta^* \leq g(x_1) - f(x_1) \leq \delta^*.$$

Hence,

$$\begin{aligned} g(y_1) - f(y_1) &\leq g(x_1) - f(y_1) \\ &= g(x_1) - f(x_1) + f(x_1) - f(y_1) \\ &< \delta^* - 2\delta^* = -\delta^*. \end{aligned}$$

Similarly, if $\delta_r(p) > \delta^*$ then there exist $x_2 < y_2$ in $(p,1]$ such that

$$(1/2)[f(x_2) - f(y_2)] > \delta^*,$$

and as above, $g(y_2) - f(y_2) > \delta^*$. Consequently, g is not a best quasi-convex approximation to f. This contradiction implies that $p \in P$.

Combining Corollary 7.1 and Lemma 7.4 we have the following:

THEOREM 7.3. *If $f \in C[0,1]$ then, $P^* = P$, where P^* is the set of optimal knots and $P = [\eta_l, \eta_r]$ is the set of minimum points for δ.*

7.5. THE CHARACTERIZATION OF THE BEST APPROXIMATIONS

In this section we present a characterization of best quasi-convex approximations to $f \in C[0,1]$.

LEMMA 7.5. *Let $f \in C[0,1]$ and let g be a best quasi-convex approximation to f. Then, there exists a $p \in [\eta_l, \eta_r]$ such that*

$$\underline{g}_p(x) \leq g(x), \quad \text{for all} \quad x \in [0,1].$$

PROOF: Assume, to the contrary, that there exists an $x_0 \in [0,1]$ such that

$$g(x_0) < \underline{g}_p(x_0), \quad \text{for all} \quad p \in [\eta_l, \eta_r].$$

If $x_0 \in [0, \eta_l]$, then

$$g(x_0) < \underline{g}_{\eta_l}(x_0) = \sup\{f(t) : t \in [x_0, \eta_l]\} - \delta^*.$$

Hence, there exists a $t_0 \in [x_0, \eta_l]$ such that $g(x_0) < f(t_0) - \delta^*$. By Lemma 7.4, if we let p_0 be a knot of g, then $p_0 \in [\eta_l, \eta_r]$. Thus,

$$g(t_0) \leq g(x_0) < f(t_0) - \delta^*.$$

If $x_0 \in (\eta_l, \eta_r)$, then $\underline{g}_{x_0}(x_0) = f(x_0) - \delta^*$, and hence, $g(x_0) < f(x_0) - \delta^*$. If $x_0 \in [\eta_l, 1]$, then

$$g(x_0) < \underline{g}_{\eta_r}(x_0) = \sup\{f(t) : t \in [\eta_r, x_0]\} - \delta^*.$$

Hence, there exists a $t_0 \in [\eta_r, x_0]$ such that $g(x_0) < f(t_0) - \delta^*$. Thus, there exists a $t_0 \in [0,1]$ such that $g(t_0) < f(t_0) - \delta^*$. Hence, g cannot be a best approximation to f, which is a contradiction.

THEOREM 7.4 (CHARACTERIZATION). *Let $f \in C[0,1]$. Then, $g \in K$ is a best quasi-convex uniform approximation to f on $[0,1]$ if and only if there exists a $p \in [\eta_l, \eta_r]$ such that*

$$\underline{g}_p(x) \leq g(x) \leq \overline{g}_{s_r}(x), \quad \textit{for all} \quad x \in [0,1]. \tag{7.5.1}$$

PROOF: (Necessity) Let g be a best approximation to f from K. The first inequality follows from Lemma 7.5. It remains to show that $g(x) \leq \overline{g}_{s_r}(x)$, for all $x \in [0,1]$.

For $t \in [0,1]$, $-\delta^* \leq f(t) - g(t) \leq \delta^*$. By the definition of $\overline{g}_{s_r}$, for $x \in [0, s_r]$ and for all $\epsilon > 0$, there exists a $t \in [0,x]$ satisfying

$$\overline{g}_{s_r}(x) > f(t) + \delta^* - \epsilon. \tag{7.5.2}$$

Also, for $x \in (s_r, 1]$ and for all $\epsilon > 0$ there exists a $t \in [x, 1]$ satisfying (7.5.2). Let p_0 be a knot for g. If $p_0 \leq s_r$, then $g(x) \leq g(t)$ for $0 \leq t \leq x \leq p_0$, (or

$0 \le t \le x < p_0$), and moreover

$$g(x) \le g(t) \le f(t) + \delta^* < \overline{g}_{s_r}(x) + \epsilon, \quad \text{for} \quad x \in [0, p_0], \text{ (or } \quad x \in [0, p_0)).$$

It follows that $g(x) \le \overline{g}_{s_r}(x)$ for $x \in [0, p_0]$, (or $x \in [0, p_0)$). Also, $g(x) \le g(t)$ for $s_r < x \le t \le 1$, (or $s_r \le x \le t \le 1$), and

$$g(x) \le g(t) \le f(t) + \delta^* < \overline{g}_{s_r}(x) + \epsilon, \quad \text{for} \quad x \in (s_r, 1], \text{ (or } \quad x \in [s_r, 1]).$$

Thus,

$$g(x) \le \overline{g}_{s_r}(x), \quad \text{for} \quad x \in (s_r, 1], \text{ (or } \quad x \in [s_r, 1]).$$

In either case, $g(s_r^+) \le \overline{g}_{s_r}(s_r^+)$. Hence for $x \in (p_0, s_r]$, (or $[p_0, s_r)$), by Lemma 7.2,

$$g(x) \le g(s_r^+) \le \overline{g}_{s_r}(s_r^+) = \overline{g}_{s_r}(s_r) \le \overline{g}_{s_r}(x).$$

Therefore, if $p_0 \le s_r$ then

$$g(x) \le \overline{g}_{s_r}(x), \quad \text{for all} \quad x \in [0, 1]. \tag{7.5.3}$$

If $p_0 > s_r$, then we can similarly prove (7.5.3).

(Sufficiency) If $g \in K$ and there exists a $p \in [\eta_l, \eta_r]$ such that (7.5.1) holds, then by Corollary 7.1, $\|f - \underline{g}_p\|_\infty = \|f - \overline{g}_{s_r}\|_\infty = \delta^*$. Thus, $\|f - g\|_\infty = \delta^*$, and g is a best approximation to f.

The following corollary gives the structure of G, the set of best approximations:

COROLLARY 7.2. *Let $f \in C[0,1]$, then*

$$G = \bigcup_{p \in [\eta_l, \eta_r]} \left\{ g^* \in K : \underline{g}_p(x) \le g^*(x) \le \overline{g}_{s_r}(x), \ \text{for all} \ x \in [0,1] \right\}. \qquad (7.5.4)$$

THEOREM 7.5. *Let $f \in C[0,1]$. Then f has a unique best quasi-convex uniform approximation if and only if f is quasi-convex.*

PROOF: If $f \in K$ then f is its own unique best approximation from K. Next, assume that G has a unique element. Then by Corollary 7.2 for all $p \in [\eta_l, \eta_r]$, $\underline{g}_p(x) = \overline{g}_{s_r}(x)$, for all $x \in [0,1]$. In particular, we find that $\underline{g}_{s_r}(s_r) = \overline{g}_{s_r}(s_r)$. Hence, by the definitions of $\underline{g}_{s_r}$ and $\overline{g}_{s_r}$, $f(s_r) - \delta^* = f(s_r) + \delta^*$. Hence, $\delta^* = 0$, and by Lemma 7.1, $f \in K$.

Theorem 7.5 can also be derived from Theorem 5.1 of [63].

Chapter 8: Best Piecewise Monotone Uniform Approximation

8.1. Introduction

For any integer $n \geq 1$, let

$$\Omega_n = \{p = (p_0, p_1, ..., p_n) \in R^{n+1} : 0 = p_0 \leq p_1 \leq ... \leq p_n = 1\}. \tag{8.1.1}$$

Then, Ω_n is compact in R^{n+1}.

Given a $p \in \Omega_n$, let $I_i = [p_{i-1}, p_i)$, for $i = 1, 2, ..., n-1$, and $I_n = [p_{n-1}, p_n]$.

Let

$$\begin{aligned} A(p) = \{g \in B : g & \text{ is nondecreasing on } I_{2j-1}, \quad j = 1, 2, ..., [(n+1)/2], \\ \text{and } g & \text{ is nonincreasing on } I_{2j}, \quad j = 1, 2, ..., [n/2]\}. \end{aligned} \tag{8.1.2}$$

We call $A(p)$ the set of *n-piecewise monotone functions with knot vector* p. Some functions in $A(p)$ have more than one knot vector. In general, the set of all knot vectors for a given function $g \in A(p)$ is a convex subset of Ω_n. Next, let

$$A_n = \cup\{A(p) : p \in \Omega_n\}. \tag{8.1.3}$$

A_n is called the set of *n-piecewise monotone functions.* When no ambiguity arises, we call A_n the set of *piecewise monotone functions.*

For a fixed n, $g^* \in A_n$ is said to be *a best piecewise monotone uniform approximationto* $f \in C[0,1]$, if

$$\| f - g^* \|_\infty = \inf \{\| f - g \|_\infty \colon g \in A_n\}. \tag{8.1.4}$$

For $n = 1$, this problem reduces to best monotone uniform approximation, which is investigated in [58, 59], and for $n = 2$, this reduces to the problem of the best quasi-convex uniform approximation studied in [63] and Chapter 7. In this chapter, we consider the problem for $n \geq 1$.

DEFINITION 8.1: For $f \in C[0,1]$, let

$$P_n^* = \{p \in \Omega_n : \inf\{\| f - g \|_\infty \colon g \in A(p)\} = \inf\{\| f - g \|_\infty \colon g \in A_n\}\}. \tag{8.1.5}$$

P_n^* is called *the set of best knot vectors* for piecewise monotone approximation to f.

Equivalently, if we let A_n^* denote the set of all best approximations to f from A_n, then $P_n^* = \{p \in \Omega_n : A(p) \cap A_n^* \neq \phi\}$.

We shall present characterizations of both the set of best approximations, A_n^* and the set of best knot vectors, P_n^*. Our approach is to find a representation of the error of best approximation. Then, by employing this representation, we characterize A_n^* and P_n^*, and prove the existence and nonuniqueness of a best approximation.

8.2. Preliminaries

Ubhaya [58], showed that if $f \in C[a,b]$, then

$$\delta \equiv \sup\{[f(x) - f(y)]/2 : a \leq x \leq y \leq b\}$$

provides a measure of best uniform nondecreasing approximation to f. We shall use a similar development to obtain a measure of best uniform piecewise monotone approximation.

DEFINITION 2: For $f \in C[0,1]$, $0 \leq x \leq y \leq 1$, and $k = 0,1,2,...,n-1$, let

$$F_k(x,y) = \begin{cases} [f(x) - f(y)]/2, & \text{if } k \text{ is even,} \\ [f(y) - f(x)]/2, & \text{if } k \text{ is odd.} \end{cases} \tag{2.1}$$

For $0 \leq \alpha \leq \beta \leq 1$, let

$$d_k(\alpha,\beta) = \sup\{F_k(x,y) : \alpha \leq x \leq y \leq \beta\}. \tag{8.2.2}$$

For $p \in \Omega_n$, let

$$\delta_n(p) = \max\{d_k(p_k, p_{k+1}),\ k = 0,1,...,n-1\}. \tag{8.2.3}$$

Define

$$\delta_n^* = \inf\{\delta_n(p) : p \in \Omega_n\}, \tag{8.2.4}$$

and

$$P_n = \{p \in \Omega_n : \delta_n(p) = \delta_n^*\}. \tag{8.2.5}$$

We remark that $d_k(\alpha,\beta)$ is a measure of the "decreasingness" of f on (α,β), if k is even, and a measure of the "increasingness" of f on (α,β), if k is odd. P_n is the set of all minima for the function $\delta_n(p)$.

LEMMA 8.1. *Let $f \in C[0,1]$. Then*

(i) $|d_k(t_1+\Delta t_1, t_2+\Delta t_2) - d_k(t_1,t_2)| \le (1/2)[\omega_f(|\Delta t_1|) + \omega_f(|\Delta t_2|)]$,

where ω_f denotes the modulus of continuity of f;

(ii) $\delta_n(p) \in C(\Omega_n)$;

(iii) P_n is nonempty;

(iv) $\delta_n^* = 0$ *if and only if* $f \in A_n$.

PROOF: (i) We only present the proof for even k. Consider the following four cases:

Case 1: $\Delta t_1 < 0$ and $\Delta t_2 > 0$. Since $d_k(t_1+\Delta t_1, t_2+\Delta t_2) \ge d_k(t_1,t_2)$,

and

$$\begin{aligned} d_k(t_1+\Delta t_1, t_2+\Delta t_2) \le &\sup\{[f(x)-f(y)]/2 : t_1 - |\Delta t_1| \le x \le y \le t_1\} \\ &+ \sup\{[f(x)-f(y)]/2 : t_1 \le x \le y \le t_2\} \\ &+ \sup\{[f(x)-f(y)]/2 : t_2 \le x \le y \le t_2 + |\Delta t_2|\}, \end{aligned}$$

it follows that

$$\begin{aligned} &|d_k(t_1+\Delta t_1, t_2+\Delta t_2) - d_k(t_1,t_2)| \\ &\le \sup\{[f(x)-f(y)]/2 : 0 \le y-x \le |\Delta t_1|\} \\ &\quad + \sup\{[f(x)-f(y)]/2 : 0 \le y-x \le |\Delta t_2|\} \\ &= (1/2)[\omega_f(|\Delta t_1|) + \omega_f(|\Delta t_2|)]. \end{aligned}$$

Case 2: $\Delta t_1 < 0$ and $\Delta t_2 < 0$. Since

$$\begin{aligned}
&d_k(t_1 + \Delta t_1, t_2 + \Delta t_2) \\
&\leq \sup\{[f(x) - f(y)]/2 : t_1 - |\Delta t_1| \leq x \leq y \leq t_1\} \\
&\quad + \sup\{[f(x) - f(y)]/2 : t_1 \leq x \leq y \leq t_2 - |\Delta t_2|\} \\
&\leq \sup\{[f(x) - f(y)]/2 : 0 \leq y - x \leq |\Delta t_1|\} + d_k(t_1, t_2),
\end{aligned}$$

and

$$\begin{aligned}
d_k(t_1, t_2) &\leq \sup\{[f(x) - f(y)]/2 : t_1 \leq x \leq y \leq t_2 - |\Delta t_2|\} \\
&\quad + \sup\{[f(x) - f(y)]/2 : t_2 - |\Delta t_2| \leq x \leq y \leq t_2\} \\
&\leq d_k(t_1 + \Delta t_1, t_2 + \Delta t_2) \\
&\quad + \sup\{[f(x) - f(y)]/2 : 0 \leq y - x \leq |\Delta t_1|\},
\end{aligned}$$

(i) holds in this case.

Case 3: $\Delta t_1 > 0$ and $\Delta t_2 > 0$. Let $t_1' = t_1 + \Delta t_1$, $t_2' = t_2 + \Delta t_2$, $t_1' + \Delta t_1' = t_1$ and $t_2' + \Delta t_2' = t_2$. Then $\Delta t_1' = -\Delta t_1 < 0$ and $\Delta t_2' = -\Delta t_2 < 0$. This reduces to case 2.

Case 4: $\Delta t_1 > 0$ and $\Delta t_2 < 0$. Let $t_1' = t_1 + \Delta t_1$, $t_2' = t_2 + \Delta t_2$, $t_1' + \Delta t_1' = t_1$ and $t_2' + \Delta t_2' = t_2$. Then $\Delta t_1' = -\Delta t_1 < 0$ and $\Delta t_2' = -\Delta t_2 > 0$. This reduces to case 1.

(ii) From (i), we find that $d_k(t_1, t_2)$ is continuous on the compact set $\{(t_1, t_2) \in R^2 : 0 \leq t_1 \leq t_2 \leq 1\}$. Hence, $\delta_n(p)$ is continuous on Ω_n.

(iii) In view of the continuity of $\delta_n(p)$, P_n is nonempty since Ω_n is compact.

(iv) Assume $\delta_n^* = 0$. By (ii), there exists a $p \in \Omega_n$ such that $\delta_n(p) = 0$. Thus,

$d_k(p_k, p_{k+1}) = 0$, for $k = 0, 1, ..., n-1$ and $f \in A(p) \subset A_n$. Conversely, assume $f \in A_n$. Then for some $p \in \Omega_n, f \in A(p)$. It follows that $\delta_n(p) = 0$, and thus $\delta_n^* = 0$.

8.3. Duality and Existence

We shall prove that the error of the best approximation to f is δ_n^*. One consequence of this is the existence of the best approximation from A_n.

THEOREM 8.1 (DUALITY). *Let $f \in C[0,1]$. Then*

$$inf\{\| f - g \|_\infty : g \in A_n\} = \delta_n^*. \tag{8.3.1}$$

The proof of Theorem 8.1 follows from the next lemma. We first define two approximations $\underline{g}_p$ and $\overline{g}_p$, which we shall prove are best approximations to f from A_n whenever $p \in P_n$.

DEFINITION 8.3: For $f \in C$ and $p \in \Omega_n$, define $\underline{g}_p$, $\overline{g}_p$ in A_n by

$$\underline{g}_p(x) = \begin{cases} \sup\{f(t) : p_{2i-2} \le t \le x\} - \delta_n^*, & \text{for } x \in I_{2i-1},\ i = 1, 2, ..., [(n+1)/2] \\ \sup\{f(t) : x \le t \le p_{2i}\} - \delta_n^*, & \text{for } x \in I_{2i},\ i = 1, 2, ..., [n/2], \end{cases} \tag{8.3.2}$$

and

$$\overline{g}_p(x) = \begin{cases} \inf\{f(t) : x \le t \le p_{2i-1}\} + \delta_n^*, & \text{for } x \in I_{2i-1},\ i = 1, 2, ..., [(n+1)/2] \\ \inf\{f(t) : p_{2i-1} \le t \le x\} + \delta_n^*, & \text{for } x \in I_{2i},\ i = 1, 2, ..., [n/2]. \end{cases} \tag{8.3.3}$$

DEFINITION 8.4: For $f \in C[0,1]$, define the set of alternant local extremal points of f by

$$Q_n = \{p \in \Omega_n : f(p_{2i-1}) = max\{f(x) : x \in [p_{2i-2}, p_{2i}]\},\ i = 1, 2, ..., [(n+1)/2];$$

$$\text{and} \quad f(p_{2i}) = min\{f(x) : x \in [p_{2i-1}, p_{2i+1}]\},\ i = 1, 2, ..., [n/2]\}. \tag{8.3.4}$$

LEMMA 8.2. *Let $f \in C[0,1]$. Then*

(i) $\underline{g}_p, \overline{g}_p \in C[0,1]$, for $p \in Q_n$;

(ii) $\| f - \underline{g}_p \|_\infty \leq \delta_n^*$, and $\| f - \overline{g}_p \|_\infty \leq \delta_n^*$, for $p \in P_n$.

PROOF OF LEMMA 8.2: (i) Note both $\underline{g}_p$ and $\overline{g}_p$ are continuous at $x \neq p_i$, $i = 1, 2, ..., n-1$. In addition, for $p \in Q_n$, we have $\overline{g}_p(p_{2i-1}^-) = f(p_{2i-1}) + \delta_n^*$, and $\overline{g}_p(p_{2i-1}^+) = f(p_{2i-1}) + \delta_n^*$, for $i = 1, 2, ..., [(n+1)/2]$. Hence, $\overline{g}_p \in C[0,1]$. Similarly we can show $\underline{g}_p \in C[0,1]$.

(ii) We present only the proof of the first inequality, since the proof of the second is similar. By the definition of $\underline{g}_p$, if $x \in I_{2i-1}$, $i = 1, 2, ..., [(n+1)/2]$, $\underline{g}_p(x) \geq f(x) - \delta_n^*$, and for any $\epsilon > 0$, there exists $t \in [p_{2i-1}, x]$ such that $\underline{g}_p(x) \leq f(t) - \delta_n^* + \epsilon$. Since $p \in P_n$, we have $[f(t) - f(x)]/2 \leq \delta_n^*$. This implies that

$$\underline{g}_p(x) \leq f(t) - \delta_n^* + \epsilon \leq f(x) + \delta_n^* + \epsilon.$$

Hence,

$$|\underline{g}_p(x) - f(x)| \leq \delta_n^*, \quad x \in I_{2i-1}, \quad i = 1, 2 ..., [(n+1)/2].$$

Similarly, we prove that

$$|\underline{g}_p(x) - f(x)| \leq \delta_n^*, \quad x \in I_{2i}, \quad i = 1, 2..., [n/2].$$

Therefore, $\| f - \underline{g}_p \|_\infty \leq \delta_n^*$.

PROOF OF THEOREM 8.1: For each $g \in A_n$, $g \in A(p)$ for some $p \in \Omega_n$. Hence, $g(y) - g(x) \geq 0$, for $x \leq y$ in I_{2i-1}, $i = 1, 2, ..., [(n+1)/2]$, and $g(x) - g(y) \geq 0$, for $x \leq y$ in I_{2i}, $i = 1, 2, ..., [n/2]$. It follows that for $x \leq y$ in I_{2i-1},

$$f(x) - f(y) \leq f(x) - f(y) + g(y) - g(x) \leq 2 \| f - g \|_\infty,$$

and for $x \leq y$ in I_{2i},

$$f(y) - f(x) \leq f(y) - f(x) + g(x) - g(y) \leq 2 \| f - g \|_\infty .$$

Consequently, $d_k(p_k, p_{k+1}) \leq \| f - g \|_\infty$, $k = 0, 1, ..., n-1$, and therefore, $\delta_n^* \leq \| f - g \|_\infty$, for all $g \in A_n$.

On the other hand, by Lemma 8.2, we also have $\| f - \underline{g}_p \|_\infty \leq \delta_n^*$, for $p \in P_n$. Therefore, $\inf\{\| f - g \|_\infty : g \in A_n\} = \| f - \underline{g}_p \|_\infty = \delta_n^*$.

COROLLARY 8.1. *If $f \in C[0,1]$ and $p \in P_n$, then $\underline{g}_p$ and $\overline{g}_p$ are best approximations to f from A_n. Hence, $P_n \subseteq P_n^*$.*

8.4. CHARACTERIZATION

In this section, we characterize both the best approximations from A_n and P_n^*, the set of knot vectors of the best approximations.

THEOREM 8.2 (CHARACTERIZATION). *Let $f \in C[0,1]$ and $g \in A_n$. Then, g is a best uniform piecewise monotone approximation to f if and only if there exists a $p \in P_n$ such that*

$$\underline{g}_p(x) \leq g(x) \leq \overline{g}_p(x), \qquad x \in [0,1]. \tag{8.4.1}$$

To prove this characterization, we need the following lemma:

LEMMA 8.3. *Let $f \in C[0,1]$. Then, $P_n^* \subseteq P_n$.*

For $0 \leq \alpha \leq \beta \leq 1$, we define the following notation:

$$m(\alpha, \beta) = \min\{f(x) : \alpha \leq x \leq \beta\},$$

and

$$M(\alpha, \beta) = \max\{f(x) : \alpha \leq x \leq \beta\}.$$

PROOF OF LEMMA 8.3: Assume to the contrary that g is a best approximation to f with a knot $p \notin P_n$. Then, for some index $i \in \{0, 1, ..., n-1\}$, $d_i(p_i, p_{i+1}) > \delta_n^*$. There exist two points $x < y$ in I_{i+1} such that $F_i(x,y) = d_i(p_i, p_{i+1})$. Thus, for even i, $f(x) - f(y) > 2\delta_n^*$. By the hypotheses, we find

$$-\delta_n^* \leq g(x) - f(x) \leq \delta_n^*.$$

Hence,

$$g(y) - f(y) \geq g(x) - f(y) = [g(x) - f(x)] + [f(x) - f(y)] > \delta_n^*.$$

Similarly, for odd i, we have $g(x) - f(x) > \delta_n^*$. This contradiction implies $p \in P_n$.

PROOF OF THEOREM 8.2: (Necessity) Let $g \in A(q)$ be a best approximation to f from A_n. By Lemma 8.3, $q \in P_n$. Assume to the contrary that there exists an $x_0 \in [0,1]$ such that for all $p \in P_n$, $g(x_0) > \overline{g}_p(x_0)$ or $g(x_0) < \underline{g}_p(x_0)$. Let $I_i' = [q_{i-1}, q_i)$, for $i = 1, 2, ..., n-1$, and $I_n' = [q_{n-1}, q_n]$. If $x_0 \in I_{2i-1}'$, then we have $t_m \in [x_0, q_{2i-1}]$ and $t_M \in [q_{2i-1}, x_0]$ such that $f(t_m) = m(x_0, q_{2i-1})$ and $f(t_M) = M(q_{2i-2}, x_0)$. If $g(x_0) > \overline{g}_p(x_0)$, then

$$g(t_m) \geq g(x_0) > \overline{g}_q(x_0) = m(x_0, q_{2i-1}) = f(t_m) + \delta_n^*.$$

If $g(x_0) < \underline{g}_p(x_0)$, then

$$g(t_M) \leq g(x_0) < \underline{g}_q(x_0) = M(q_{2i-2}, x_0) = f(t_M) - \delta_n^*.$$

Similarly, if $x_0 \in I_{2i}'$, we arrive at a contradiction.

(Sufficiency) If $g \in A_n$ and there exists a $p \in P_n$ such that inequality (8.4.1) holds, then by Corollary 8.1, we have $\| f - g \|_\infty = \delta_n^*$. Thus, g is a best approximation to f from A_n.

The following theorem follows from Lemma 8.3 and Corollary 8.1. Recall that P_n^* is the set of best knot vectors.

THEOREM 8.3. *Let $f \in C[0,1]$. Then, $P_n^* = P_n$.*

COROLLARY 8.2. *Let $f \in C[0,1]$. Then,*

$$A_n^* = \bigcup_{p \in P_n} \left\{ g \in A_n : \underline{g}_p(x) \le g(x) \le \overline{g}_p(x),\ x \in [0,1] \right\}. \tag{8.4.2}$$

8.5. NONUNIQUENESS OF BEST APPROXIMATION

THEOREM 8.4. *Let $f \in C[0,1]$. Then, the best approximation to f from A_n is unique if and only if $f \in A_n$.*

The proof of Theorem 8.4 depends on Lemma 8.4 and Theorem 8.5 which follow.

LEMMA 8.4. *Let $f \in C[0,1]$ and $p \in P_n$.*

(i) Let k be an integer in $[1, n-1]$ and $p_k^{(1)} \in [p_{k-1}, p_{k+1}]$ such that for odd k, $f(p_k^{(1)}) = M(p_{k-1}, p_{k+1})$, and for even k, $f(p_k^{(1)}) = m(p_{k-1}, p_{k+1})$. Then $d_{k-1}(p_{k-1}, p_k^{(1)}) \le \delta_n^*$ and $d_k(p_k^{(1)}, p_{k+1}) \le \delta_n^*$.

(ii) Let k be an integer in $[1, n-1]$ and $p_k^{(1)} \in [p_k, p_{k+1}]$ such that

(a) for odd k, $f(p_k^{(1)}) = M(p_{k-1}, p_{k+1})$, $m(p_k, p_k^{(1)}) < f(p_{k-1}) = m(p_{k-2}, p_k)$,
and $p_{k-1}^{(1)} = inf\{s \in [p_k, p_k^{(1)}] : f(s) = m(p_k, p_k^{(1)})\}$;

(b) for even k, $f(p_k^{(1)}) = m(p_{k-1}, p_{k+1})$, $M(p_k, p_k^{(1)}) < f(p_{k-1}) = M(p_{k-2}, p_k)$,
and $p_{k-1}^{(1)} = inf\{s \in [p_k, p_k^{(1)}] : f(s) = M(p_k, p_k^{(1)})\}$.

Then, $d_{k-1}(p_{k-1}^{(1)}, p_k^{(1)}) \leq \delta_n^*$, and $d_{k-2}(p_{k-2}, p_{k-1}^{(1)}) \leq \delta_n^*$.

PROOF OF LEMMA 8.4: (i) We present only the proof for k odd. If $p_k^{(1)} = p_k$, the proof is trivial. If $p_k^{(1)} < p_k$, then

$$d_{k-1}(p_{k-1}, p_k^{(1)}) \leq d_{k-1}(p_{k-1}, p_k) \leq \delta_n^*.$$

Assume $d_k(p_k^{(1)}, p_{k+1}) > \delta_n^*$. Then, there exist two points $x_1 < y_1$ in $[p_k^{(1)}, p_{k+1}]$ such that $f(y_1) - f(x_1) > 2\delta_n^*$. If $p_k^{(1)} \leq x_1 \leq p_k$,

$$d_{k-1}(p_{k-1}, p_k) \geq [f(p_k^{(1)}) - f(x_1)]/2 \geq [f(y_1) - f(x_1)]/2 > \delta_n^*,$$

and if $p_k < x_1 \leq p_{k+1}$,

$$d_k(p_k, p_{k+1}) \geq [f(y_1) - f(x_1)]/2 > \delta_n^*,$$

which is a contradiction. The case $p_k^{(1)} > p_k$ can be handled similarly to obtain a contradiction.

(ii) We prove this result only for odd k. Since $p_{k-1}^{(1)} \in [p_k, p_k^{(1)}]$, by (i),

$$d_{k-1}(p_{k-1}^{(1)}, p_k^{(1)}) \leq d_{k-1}(p_{k-1}, p_k^{(1)}) \leq \delta_n^*.$$

On the other hand,

$$d_{k-2}(p_{k-2}, p_{k-1}^{(1)}) \leq max\{d_{k-2}(p_{k-2}, p_{k-1}), d_{k-2}(p_{k-1}, p_{k-1}^{(1)}),$$
$$sup\{[f(y) - f(x)]/2 : p_{k-1} \leq y \leq p_{k-1}^{(1)}, p_{k-2} \leq x \leq p_{k-1}\}\}.$$

But

$$d_{k-2}(p_{k-1}, p_{k-1}^{(1)}) \leq M(p_{k-1}, p_{k-1}^{(1)}) - f(p_{k-1}^{(1)})]/2$$

$$\leq [f(p_k^{(1)}) - f(p_{k-1}^{(1)})]/2$$

$$\leq d_{k-2}(p_k, p_{k+1})$$

$$\leq \delta_n^*,$$

and

$$sup\{[f(y) - f(x)]/2 : p_{k-1} \leq y \leq p_{k-1}^{(1)}, p_{k-2} \leq x \leq p_{k-1}\}$$

$$\leq [M(p_{k-1}, p_{k-1}^{(1)}) - f(p_{k-1})]/2$$

$$\leq [M(p_{k-1}, p_{k-1}^{(1)}) - f(p_{k-1}^{(1)})]/2 \leq \delta_n^*.$$

Hence,

$$d_{k-2}(p_{k-2}, p_{k-1}^{(1)}) \leq \delta_n^*.$$

The proof of Theorem 8.4 also depends on the following theorem, which is used in Chapter 9 to develop an algorithm to find a best knot vector p^*, and a corresponding best approximation g^*.

THEOREM 8.5. *Let $f \in C[0,1]$. Then, $P_n \cap Q_n$ is nonempty.*

PROOF OF THEOREM 8.5: By Lemma 8.1, P_n is nonempty. Assume $p \in P_n$. Let k be the smallest index in $\{1, 2, ..., n-1\}$ such that $f(p_k)$ does not assume its local maximum for odd k or local minimum for even k, on $[p_{k-1}, p_{k+1}]$. If k is odd, find $p_k^{(1)} \in [p_{k-1}, p_{k+1}]$ such that $f(p_k^{(1)}) = M(p_{k-1}, p_{k+1})$ and replace p_k by $p_k^{(1)}$. If $f(p_{k-1}) = m(p_{k-2}, p_k^{(1)})$, then let $p_i^{(1)} = p_i$, $i = 1, 2, ..., k-1$. Otherwise we deduce

$p_k < p_k^{(1)}$ and $m(p_k, p_k^{(1)}) < f(p_{k-1})$, and let

$$p_{k-1}^{(1)} = \inf\{s \in [p_k, p_k^{(1)}] : f(s) = m(p_k, p_k^{(1)})\}.$$

Now, by Lemma 8.4, $d_k(p_{k-2}, p_{k-1}^{(1)}) \leq \delta_n^*$, $d_{k-1}(p_{k-1}^{(1)}, p_k^{(1)}) \leq \delta_n^*$, and $d_k(p_k^{(1)}, p_{k+1}) \leq \delta_n^*$. Also, $f(p_{k-1}^{(1)}) = m(p_{k-2}, p_k^{(1)})$ and $f(p_k^{(1)}) = M(p_{k-1}^{(1)}, p_{k+1})$. If $f(p_{k-2}) = M(p_{k-3}, p_{k-1}^{(1)})$ with $k-2 \geq 1$, then let $p_i^{(1)} = p_i, i = 1, 2, ..., k-2$. Otherwise we deduce $p_{k-1} < p_{k-1}^{(1)}$ and $M(p_{k-1}, p_{k-1}^{(1)}) > f(p_{k-2})$, and let

$$p_{k-2}^{(1)} = \inf\{s \in [p_{k-1}, p_{k-1}^{(1)}] : f(s) = M(p_{k-1}, p_{k-1}^{(1)})\}.$$

Thus, $d_{k-3}(p_{k-3}, p_{k-2}^{(1)}) \leq \delta_n^*$, $d_{k-2}(p_{k-2}^{(1)}, p_{k-1}^{(1)}) \leq \delta_n^*$, $d_{k-1}(p_{k-1}^{(1)}, p_k^{(1)}) \leq \delta_n^*$, and $d_k(p_k^{(1)}, p_{k+1}) \leq \delta_n^*$, with $f(p_{k-2}^{(1)}) = M(p_{k-3}, p_{k-1}^{(1)})$, $f(p_{k-1}^{(1)}) = m(p_{k-2}^{(1)}, p_k^{(1)})$, and $f(p_k^{(1)}) = M(p_{k-1}^{(1)}, p_{k+1})$.

By repeating this procedure, we shall obtain $p_{k-2}^{(1)}, ..., p_2^{(1)}, p_1^{(1)}$ such that

(i) $(p_0^{(1)}, p_1^{(1)}, ..., p_k^{(1)}, p_{k+1}, ..., p_n) \in P_n$, with $p_0^{(1)} = p_0$,

(ii) $f(p_i^{(1)}) = M(p_{i-1}^{(1)}, p_{i+1}^{(1)})$, $i = 1, 3, ..., k-2$,

$$f(p_i^{(1)}) = m(p_{i-1}^{(1)}, p_{i+1}^{(1)}),\ i = 2, 4, ..., k-1,$$

and

$$f(p_k^{(1)}) = M(p_{k-1}^{(1)}, p_{k+1}).$$

Let $p_i^{(1)} = p_i$, $i = k+1, ..., n$. Then, $p^{(1)} \in P_n$. Apply the same procedure to $p^{(1)}$ and obtain $p^{(2)}$. In at most n iterations, we shall obtain $p^* \in P_n \cap Q_n$.

If k is even, we can define a similar construction to find $p^* \in P_n \cap Q_n$.

COROLLARY 8.3. *Let $f \in C[0,1]$ and, $p \in P_n \cap Q_n$. Then, $\underline{g}_p$ and $\overline{g}_p$ are continuous best approximations to f. Thus, there always exist continuous best approximations to a continuous function.*

PROOF OF THEOREM 8.4: If $f \in A_n$, then f is its own unique best approximation from A_n.

Next, assume that f has a unique best piecewise monotone approximation. By Corollary 8.2 and Theorem 8.5, for $p \in P_n \cap Q_n$, $\underline{g}_p = \overline{g}_p$. Hence,

$$\underline{g}_p(p_1) = M(p_1, p_2) - \delta_n^* = f(p_1) - \delta_n^*,$$

and

$$\overline{g}_p(p_1) = f(p_1) + \delta_n^*.$$

It follows that $f(p_1) - \delta_n^* = f(p_1) + \delta_n^*$. This implies that $\delta_n^* = 0$. By Lemma 8.1, $f \in A_n$.

Chapter 9: The Computation of A Best Piecewise Monotone Uniform Approximation

9.1. PRELIMINARIES

The existence, error representation, characterization and nonuniqueness of a best piecewise monotone uniform approximation to a continuous function are proved in Chapter 8. Ubhaya in [59] gave an algorithm for the computation of a best monotone approximation to a continuous function on $[a, b]$. Once we have an algorithm to obtain a best knot vector, we can compute a best piecewise monotone approximation by employing the algorithm given by Ubhaya in each subinterval. Hence, it is sufficient to establish an algorithm to compute a best knot vector. In this chapter, we shall characterize the set of best knot vectors of the approximations and present an algorithm to compute a best knot vector.

In this chapter, when we use the notation defined in Chapter 8, we shall not repeat the corresponding definition.

Let $f \in C[0,1]$. For $0 \leq \alpha \leq \beta \leq 1$, let

$$e(\alpha,\beta) = \sup\{\mid f(x) - f(y) \mid /2 : x,y \in [\alpha,\beta]\}. \tag{9.1.1}$$

Let

$$E_n = \{p = (p_0, p_1, ..., p_n) \in \Omega_n : e(p_{i-1}, p_i) > \delta_n(p), \quad i = 1,2,...,n\}, \tag{9.1.2}$$

where Ω_n is defined by (8.1.1).

LEMMA 9.1. *Let $f \in C[0,1]$ and let $p = (p_0, p_1, ..., p_n) \in E_n \cap Q_n$. Define $q_0 = p_0$, $q_n = p_n$ and for $k = 1,2,...,n-1$, let*

$$q_k = \inf\{x \in [q_{k-1}, p_k] : f(x) = f(p_k) \text{ and } e(x,p_k) \leq \delta_n(p)\}. \tag{9.1.3}$$

Then,

(i) $\delta_n(q) = \delta_n(p)$,

(ii) $q \in E_n \cap Q_n$,

(iii) if $x \in [q_k, q_{k-1})$ *such that* $f(x) = f(q_{k+1})$, *then* $e(x, p_{k+1}) > \delta_n(p)$,

$k = 1,2,...,n-2$.

PROOF: (i) First, we claim that $q_i \leq p_i < q_{i+1}$, $i = 1,2,...,n-2$. If for some $j \in \{1,2,...,n-1\}$, $p_j \geq q_{j+1}$, by the definition of q, $e(q_{j+1}, p_{j+1}) \leq \delta_n(p)$, and thus, $e(p_j, p_{j+1}) \leq e(q_{j+1}, p_{j+1}) \leq \delta_n(p)$, which is a contradiction.

For simplicity in the reasoning of this proof, we introduce the following notation:

$$d_k(t_1,t_2,t_3) \equiv \sup\{F_k(x,y) : t_1 \leq x \leq t_2 \leq y \leq t_3\}. \tag{9.1.4}$$

For $k = 1, 2, ..., n-1$, we have,

$$d_k(q_k, q_{k+1}) \leq \max\{d_k(q_k, p_k), d_k(p_k, q_{k+1}), d_k(q_k, p_k, q_{k+1})\}.$$

Observing $d_k(q_k, p_k) \leq e(q_k, p_k) \leq \delta_n(p)$, $d_k(p_k, q_{k+1}) \leq d_k(p_k, p_{k+1}) \leq \delta_n(p)$, and

$$d_k(q_k, p_k, q_{k+1}) = \begin{cases} [M(q_k, p_k) - f(p_k)]/2 & \text{if } k \text{ is even} \\ [f(p_k) - m(q_k, p_k)]/2 & \text{if } k \text{ is odd} \end{cases} = e(q_k, p_k) \leq \delta_n(p),$$

we deduce that $d_k(q_k, q_{k+1}) \leq \delta_n(p)$, for $k = 1, 2, ..., n-1$. Also, for $k = 0$,

$$d_0(q_0, q_1) = d_0(p_0, q_1) \leq d_0(p_0, p_1) \leq \delta_n(p).$$

Hence, $\delta_n(q) \leq \delta_n(p)$.

Now, assume that for some $k \in \{0, 1, 2, ..., n-1\}$, $\delta_n(p) = d_k(p_k, p_{k+1})$. Then, there exist $x_1 < y_1$ in $[p_k, p_{k+1}]$ such that $F_k(x_1, y_1) = \delta_n(p)$. If $x_1, y_1 \in [p_k, q_{k+1}]$ then $F_k(x_1, y_1) = d_k(q_k, q_{k+1})$ and thus $\delta_n(q) = \delta_n(p)$ in this case. Otherwise, $q_{k+1} < p_{k+1}$ and there exists $x' \in [q_{k+1}, p_{k+1}]$ such that $F_k(q_{k+1}, x') = \delta_n(p)$. If $x_1 \in [p_k, q_{k+1})$ and $y_1 \in (q_{k+1}, p_{k+1}]$, then,

$$F_k(x_1, q_{k+1}) \geq F_k(x_1, y_1) = \delta_n(p) \geq d_k(p_k, p_{k+1}) \geq F_k(x_1, q_{k+1}).$$

Hence, $F_k(x_1, q_{k+1}) = \delta_n(p)$, which is the first case. By virtue of $f(q_{k+1}) = f(p_{k+1})$, we have

$$\begin{aligned} d_{k+1}(q_{k+1}, q_{k+2}) &\geq F_{k+1}(x', p_{k+1}) = F_k(q_{k+1}, x') \\ &= \delta_n(p) \geq \delta_n(q) \geq d_{k+1}(q_{k+1}, q_{k+2}). \end{aligned}$$

Thus, $\delta_n(p) = \delta_n(q)$, in this case.

(ii) Since $p \in Q_n$ and $f(q_i) = f(p_i), i = 1, 2, ..., n-1$, we have $q \in Q_n$. By (i), $q \in E_n$. Hence, $q \in E_n \cap Q_n$.

(iii) This is an immediate consequence of the definition of q.

9.2. Characterization of the Set of Best Knot Vectors

In this section, we characterize the set of best knot vectors under the assumption that $E_n \cap Q_n$ is nonempty. In the next section we shall prove for $n \geq 3$ that $E_n \cap Q_n$ is nonempty if and only if the approximation is nondegenerate.

TEOREM 9.1. *Let $f \in C[0,1]$. Then, $E_n \cap Q_n \subseteq P_n = P_n^*$.*

PROOF: It is proved in Chapter 8 that $P_n = P_n^*$. We show that $E_n \cap Q_n \subseteq P_n$. If $E_n \cap Q_n$ is empty then the proof is trivial. Assume that $E_n \cap Q_n$ is nonempty. By Lemma 9.1, we can define $q = (q_0, q_1, ..., q_n) \in E_n \cap Q_n$ and $\delta_n(q) = \delta_n(p)$. By Theorem 8.5, $P_n \cap Q_n \neq \phi$. Hence, it is sufficient to show that $\delta_n(q) \leq \delta_n(v)$ for all $v = (v_0, v_1, ..., v_n) \in Q_n$. Suppose for some $j \in \{0, 1, ..., n-1\}$, $d_j(q_j, q_{j+1}) = \delta_n(q)$. Then, there exist $x_1 < y_1$ in (q_j, q_{j+1}) such that $F_j(x_1, y_1) = \delta_n(q)$.

Case 1. There is no point of $v_1, v_2, ..., v_{n-1}$ lying in $[q_j, q_{j+1}]$. Then there must be an integer $k \in \{0, 1, ..., n-1\}$ such that $(v_k, v_{k+1}) \supset [q_j, q_{j+1}]$. If $k+j$ is odd, then $d_k(v_k, v_{k+1}) \geq e(q_j, q_{j+1}) > \delta_n(q)$. If $k+j$ is even and $q_{j-1} \geq v_k$, then $d_k(v_k, v_{k+1}) \geq e(q_{j-1}, q_j) > \delta_n(q)$. If k and j are both even and $q_{j-1} < v_k$, then

$f(v_k) \leq f(x)$ for all $x \in [v_k, v_{k+1}]$, and thus $f(v_k) \leq f(q_j)$. Since p, $q \in Q_n$ and $q_{j-1} < v_k \leq q_j < q_{j+1} \leq v_{k+1}$, $f(v_k) = f(q_j)$. By Lemma 9.1, $e(v_k, q_j) > \delta_n(q)$. Hence, $d_k(v_k, v_{k+1}) \geq e(v_k, q_j) > \delta_n(q)$. If k and j are both odd and $q_{j-1} < v_k$, we can similarly show $f(v_k) = f(q_j)$, and thus $d_k(v_k, v_{k+1}) \geq e(v_k, q_j) > \delta_n(q)$. Consequently, in this case we have $\delta_n(v) > \delta_n(q)$.

Case 2. There is exactly one point of $v_1, ..., v_{n-1}$ lying in $[q_j, q_{j+1}]$. Assume $v_k \in [q_j, q_{j+1}]$. If $v_k \in [q_j, x_1)$ then

$$d_k(v_k, v_{k+1}) \geq \begin{cases} F_j(x_1, y_1) & \text{if } k+j \text{ is even} \\ F_j(y_1, q_{j+1}) & \text{if } k+j \text{ is odd} \end{cases} \geq F_j(x_1, y_1) = \delta_n(q).$$

If $v_k \in [x_1, y_1]$, then $v_{k-1} < q_j < x_1 \leq v_k \leq y_1 < q_{j+1} < v_{k+1}$. Thus,

$$d_{k-1}(v_{k-1}, v_k) \geq F_{j+1}(q_j, x_1) \geq F_j(x_1, y_1) = \delta_n(q), \quad \text{for } j+k \text{ even,.}$$

and

$$d_k(v_k, v_{k+1}) \geq F_{j+1}(y_1, q_{j+1}) \geq F_j(x_1, y_1) = \delta_n(q), \quad \text{for } j+k \text{ odd.}$$

If $v_k \in (y_1, q_{j+1})$, then,

$$d_{k-1}(v_{k-1}, v_k) \geq \begin{cases} F_j(x_1, y_1) & \text{if } k+j \text{ is odd} \\ F_{j+1}(q_j, x_1) & \text{if } k+j \text{ is even} \end{cases} \geq F_j(x_1, y_1) = \delta_n(q).$$

Hence, $\delta_n(v) \geq \delta_n(q)$ in this case.

Case 3. There are exactly two points, say v_k and v_{k+1} of $\{v_1, ..., v_{n-1}\}$ lying in $[q_j, q_{j+1}]$. If $k \neq j$ then there is another interval $[q_i, q_{i+1}]$ contained in (v_l, v_{l+1})

for some l, which is case 1, since the proof of case 1 does not use the fact that $d_j(q_j, q_{j+1}) = \delta_n(q)$. Let $k = j$. If $v_j \in [q_j, x_1)$ and $v_{j+1} \in (y_1, q_{j+1}]$ then

$$d_j(v_j, v_{j+1}) \geq F_j(x_1, y_1) = \delta_n(q).$$

If $v_j \in [x_1, q_{j+1}]$ or $v_{j+1} \in [q_j, y_1]$, then, respectively,

$$d_{j-1}(v_{j-1}, v_j) \geq F_{j-1}(q_j, x_1) \geq F_j(x_1, y_1) = \delta_n(q),$$

or

$$d_{j+1}(v_{j+1}, v_{j+2}) \geq F_{j+1}(y_1, q_{j+1}) \geq F_j(x_1, y_1) = \delta_n(q).$$

Case 4. There are more than two points of $v_1, .., v_{n-1}$ lying in $[q_j, q_{j+1}]$. Then there is another interval $[q_i, q_{i+1}]$ contained in (v_l, v_{l+1}) for some l. This reduces to case 1.

For all cases, $\delta_n(v) \geq \delta_n(q)$ for all $v \in Q_n$. Hence, $E_n \cap Q_n \subseteq P_n$.

We now use Theorem 9.1 to characterize the set of best knot vectors of the approximation to a continuous function. Let $p^* \in E_n \cap Q_n$, and define

$$\eta_l^{(j)}(p^*) = \inf\{x \in [0, p_j^*] : \text{ for all } t \in [x, p_j^*],\ F_j(t, p_j^*) \leq \delta_n^*\}, \tag{9.2.1}$$

$$\eta_r^{(j)}(p^*) = \sup\{x \in [p_j^*, 1] : \text{ for all } t \in [p_j^*, x],\ F_j(p_j^*, t) \geq -\delta_n^*\}, \tag{9.2.2}$$

and

$$P(p^*) = \{p \in \Omega_n : p_i \in [\eta_l^{(i)}(p^*), \eta_r^{(i)}(p^*)],\ i = 1, 2, ..., n-1\}. \tag{9.2.3}$$

THEOREM 9.2. *Let $f \in C[0,1]$. If $E_n \cap Q_n$ is nonempty, Then,*

$$P_n = \cup\{P(p^*) : p^* \in E_n \cap Q_n\}. \tag{9.2.4}$$

PROOF: Assume $p = (p_0, p_1, ..., p_n) \in \cup\{P(p^*) : p^* \in E_n \cap Q_n\}$. Then, there exists some $p^* \in E_n \cap Q_n$ such that $p \in P(p^*)$. By Theorem 9.1, $p^* \in P_n$.

If $p_{k+1} \in [\eta_l^{(k+1)}(p^*), p_{k+1}^*]$, then, for $p_k^* \leq p_k$,

$$d_k(p_k, p_{k+1}) \leq d_k(p_k^*, p_{k+1}^*) \leq \delta_n^*,$$

and for $p_k < p_k^*$,

$$\begin{aligned} d_k(p_k, p_{k+1}) &\leq d_k(p_k, p_{k+1}^*) \\ &\leq \max\{d_k(p_k, p_k^*), d_k(p_k^*, p_{k+1}^*), d_k(p_k, p_k^*, p_{k+1}^*)\} \\ &\leq \begin{cases} \max\{\delta_n^*, [M(p_k, p_k^*) - f(p_k^*]/2\} & \text{if } k \text{ is even} \\ \max\{\delta_n^*, [f(p_k^*) - m(p_k, p_k^*)]/2\} & \text{if } k \text{ is odd} \end{cases} \\ &\leq \delta_n^*. \end{aligned}$$

If $p_{k+1} \in [p_{k+1}^*, \eta_r^{(k+1)}(p^*)]$, then, for $p_k^* \leq p_k$,

$$\begin{aligned} d_k(p_k, p_{k+1}) &\leq d_k(p_k^*, p_{k+1}) \\ &\leq \max\{d_k(p_k^*, p_{k+1}^*), d_k(p_{k+1}^*, p_{k+1}), d_k(p_k^*, p_{k+1}^*, p_{k+1})\} \\ &\leq \delta_n^*. \end{aligned}$$

and for $p_k < p_k^*$,

$$\begin{aligned} d_k(p_k, p_{k+1}) \leq \max\{&d_k(p_k, p_k^*), d_k(p_k^*, p_{k+1}^*), d_k(p_{k+1}^*, p_{k+1}), d_k(p_k, p_k^*, p_{k+1}^*), \\ &\sup\{F_k(x,y), p_k \leq x \leq p_k^*, p_{k+1}^* \leq y \leq p_{k+1}\}, d_k(p_k^*, p_{k+1}^*, p_{k+1})\}\} \\ \leq &\begin{cases} \max\{\delta_n^*, 2\delta_n^* - [f(p_{k+1}^*) - f(p_k^*)]/2\} & \text{if } k \text{ is even} \\ \max\{\delta_n^*, 2\delta_n^* + [f(p_{k+1}^*) - f(p_k^*)]/2\} & \text{if } k \text{ is odd} \end{cases} \\ \leq &\ \delta_n^*. \end{aligned}$$

Hence, $p \in P_n$. It follows that $\cup\{P(p^*) : p^* \in E_n \cap Q_n\} \subseteq P_n$.

Next, we assume that $p \notin \cup\{P(p^*) : p^* \in E_n \cap Q_n\}$. Then, $p \notin P(p^*)$ for every $p^* \in E_n \cap Q_n$, and thus for every $p^* \in E_n \cap Q_n$, $p_j \notin [\eta_l^{(j)}(p^*), \eta_r^{(j)}(p^*)]$ for some $j \in \{1, 2, ..., n-1\}$. First, consider the case where j is even. If $p_j < \eta_l^{(j)}(p^*)$, then by the definition of $\eta_l^{(j)}$, there exists $t \in [p_j, p_j^*]$ such that $f(t) - f(p_j^*) > 2\delta_n^*$. Hence, for $p_{j+1} \geq p_j^*$ $d_j(p_j, p_{j+1}) \leq [f(t) - f(p_j^*)]/2 > \delta_n^*$, and for $p_{j+1} < p_j^*$, there exist indices $k \leq j$ and i odd, such that the interval $[p_{k+i}, p_{k+i+1}]$ contains the interval $[p_k^*, p_{k+1}^*]$ and such that

$$d_{k+i}(p_{k+i}, p_{k+i+1}) \geq d_{k+i}(p_k^*, p_{k+1}^*) = e(p_k^*, p_{k+1}^*) > \delta_n^*.$$

Hence, $\delta_n(p) > \delta_n^*$ in this case. If $p_j > \eta_r^{(j)}(p^*)$, we can similarly prove $\delta_n(p) > \delta_n^*$. The case where j is odd can be handled similarly to show $\delta_n(p) > \delta_n^*$. Therefore, $p \notin P_n$. This implies that $P_n \subseteq \cup\{P(p^*) : p^* \epsilon E_n \cap Q_n\}$. Hence, we have proved the theorem.

9.3. Nondegenerate Approximation

Definition 9.1: Let $f \in C[0,1] - A_n$. Best n-piecewise monotone approximation problem is said to be *nondegenerate* if $0 < \delta_n^* < \delta_{n-1}^*$, and in this case, δ_n^* is also said to be *nondegenerate*. Otherwise the approximation problem and δ_n^* are both called *degenerate*. Let

$$E_n^* = \{p = (p_0, p_1, ..., p_n) \in \Omega_n : e(p_i, p_{i+1}) > \delta_n^*, \ i = 0, 1, ..., n-1\}. \qquad (9.3.1)$$

Clearly, if $f \in C[0,1]$, $\delta_n^* \geq \delta_{n+1}^*$.

THEOREM 9.3. *Let $f \in C[0,1] - A_n$. Then the best approximation problem is nondegenerate if and only if $P_n \cap Q_n \subseteq E_n^*$.*

PROOF: For $n = 1$, this result is trivial and for $n = 2$, it is also true (see Chapter 7). In this proof, we assume $n \geq 3$.

(Necessity) Let $p \in P_n \cap Q_n$ and for some $k \in \{0, 1, ..., n-1\}$, $e(p_k, p_{k+1}) \geq \delta_n^*$. Define $p^* = (p_0^*, ..., p_{n-2}^*) \in \Omega_{n-2}$ by $p_i^* = p_i$, $i = 0, 1, ..., k-1$, and $p_i^* = p_{i+2}$, $i = k, k+1, ..., n-2$. Then, p^* satisfies the following conditions $d_i(p_i^*, p_{i+1}^*) \geq \delta_n^*$, for $i = 0, 1, ..., k-2$, and $d_i(p_i^*, p_{i+1}^*) = d_i(p_{i+2}, p_{i+3}) = d_{i+2}(p_{i+2}, p_{i+3}) \leq \delta_n^*$, for $i = k$, $k+1, ..., n-3$. In addition,

$$\begin{aligned} d_{k-1}(p_{k-1}^*, p_k^*) &= d_{k-1}(p_{k-1}, p_{k-2}) \\ &\leq \max\{d_{k-1}(p_{k-1}, p_k), d_{k-1}(p_k, p_{k+1}), d_{k-1}(p_{k+1}, p_{k+2}), \\ &\qquad d_{k-1}(p_{k-1}, p_k, p_{k+1}), d_{k-1}(p_k, p_{k+1}, p_{k+2}), \\ &\qquad \sup\{F_{k-1}(x,y) : p_{k-1} \leq x \leq p_k, p_{k+1} \leq y \leq p_{k+2}\}\} \\ &\geq \begin{cases} \max\{\delta_n^*, [f(p_k) - f(p_{k+1})]/2\} & \text{if } k-1 \text{ is even} \\ \max\{\delta_n^*, [f(p_{k+1} - f(p_k)]/2\} & \text{if } k-1 \text{ is odd} \end{cases} \\ &\leq \delta_n^*, \end{aligned}$$

It follows that $\delta_{n-2}^* \leq \delta_n^*$. But, $\delta_{n-2}^* \geq \delta_{n-1}^* \geq \delta_n^*$. Hence, $\delta_{n-2}^* = \delta_{n-1}^* = \delta_n^*$, which is a contradiction.

(Sufficiency) Let $0 < \delta_n^* = \delta_{n-1}^*$. Assume $(q_0, q_1, ..., q_{n-1}) \in P_{n-1} \cap Q_{n-1}$.

Define $p_i^* = q_i$, $i = 0, 1, ..., n-1$, and $p_n^* = q_{n-1}$. Then,

$$\delta_{n-1}^* = \delta_{n-1}(q_0, q_1, ..., q_{n-1}) = \delta_n(p_0^*, p_1^*, ..., p_n^*) = \delta_n^*.$$

Hence, $p^* = (p_0^*, p_1^*, ..., p_n^*) \in P_n \cap Q_n$. However, since $e(p_{n-1}^*, p_n^*) = 0 < \delta_n^*$, we find that $(p_0^*, p_1^*, ..., p_n^*) \notin E_n^*$.

COROLLARY 9.1. *Let $f \in C[0,1] - A_n$, $p \in Q_n$, and $0 < \delta_n^* < \delta_{n-1}^*$. Then, $p \in P_n$ if and only if $p \in E_n$.*

PROOF: The sufficient condition follows from Theorem 9.1. To prove the necessity, assume $p \in P_n$. Since δ_n^* is nondegenerate and $p \in P_n \cap Q_n$, by Theorem 9.3, $p \in E_n^*$. Thus, $\delta_n(p) = \delta_n^*$ and $e(p_i, p_{i+1}) > \delta_n^* = \delta_n(p)$, $i = 0, 1, ..., n-1$. If follows that $p \in E_n$.

The next corollary is an immediate consequence of Theorem 9.3.

COROLLARY 9.2. *Let $f \in C[0,1] - A_n$ with $n \geq 3$. Suppose $p \in P_n \cap Q_n$ and for some $k \in \{0, 1, ..., n-1\}$, $e(p_k, p_{k+1}) \leq \delta_n^*$. Then, $(p_0, ..., p_{k-1}, p_{k+2}, ..., p_n) \in P_{n-2}$ and $\delta_{n-2}^* = \delta_{n-1}^* = \delta_n^*$.*

COROLLARY 9.3. *Let $f \in C[0,1] - A_n$. If $\delta_{n-1}^* = \delta_n^*$, then $\delta_1^* = \delta_2^* = ... = \delta_n^*$.*

PROOF: Since $\delta_{n-1}^* = \delta_n^*$, by Theorem 9.3, $P_n \cap Q_n$ is not contained in E_n^*. Thus, there exists a $p \in P_n \cap Q_n$ but $p \notin E_n^*$. Let $p = (p_0, p_1, ..., p_n)$ be such a knot vector.

Then, for some $k \in \{0, 1, ..., n-1\}$, $e(p_k, p_{k+1}) \leq \delta_n^*$. Hence, $\delta_{n-2}^* = \delta_{n-1}^* = \delta_n^*$ by Corollary 9.2. Similarly, $\delta_{n-2}^* = \delta_{n-1}^*$ implies $\delta_{n-3}^* = \delta_{n-2}^* = \delta_{n-1}^*$. By repeating this procedure, we obtain $\delta_1^* = ... = \delta_n^*$.

THEOREM 9.4. *Let $f \in C[0,1] - A_n$. Then the best approximation problem is nondegenerate if and only if $E_n \cap Q_n$ is nonempty.*

PROOF: For the case that $n \leq 2$, see Chapter 7. We assume that $n \geq 3$. By Theorem 8.5, $P_n \cap Q_n$ is nonempty. If δ_n^* is nondegenerate, then $p_n \cap Q_n \subseteq E_n^*$. Let $p = (p_0, p_1, ..., p_n) \in P_n \cap Q_n$. Thus, $e(p_i, p_{i+1}) > \delta_n^* = \delta_n(p)$, for $i = 0, 1, ..., n-1$. Hence, $p \in E_n$. This implies $E_n \cap Q_n$ is nonempty. Conversely, let $E_n \cap Q_n$ be nonempty. Assume $p \in E_n \cap Q_n$. By Theorem 9.1, $p \in P_n$. Hence, $\delta_n(p) = \delta_n^*$. Thus $e(p_i, p_{i+1}) > \delta_n(p) = \delta_n^*$, for $i = 0, 1, ..., n-1$. Since $n \geq 3$,

$$\delta_1^* = \sup\{[f(x) - f(y)]/2 : 0 \leq x \leq y \leq 1\} \geq e(p_1, p_2) > \delta_n^*.$$

By Corollary 9.3, $\delta_{n-1}^* > \delta_n^*$. Noting $f \notin A_n$, $\delta_n^* > 0$. Therefore, δ_n^* is nondegenerate.

Combining Theorem 9.2 and Theorem 9.4 gives the following

THEOREM 9.5. *Let $f \in C[0,1] - M_n$. If the best approximaltion problem is nondegenerate, then*

$$P_n = \cup\{P(p^*) : p^* \epsilon E_n \cap Q_n\}. \tag{9.3.3}$$

9.4. An Algorithm For Computation

In Chapter 8, it is proved that if $p^* = (p_0^*, p_1^*, ..., p_n^*) \in P_n$ then, $\underline{g}_{p^*}(x)$ and $\overline{g}_{p^*}(x)$, as defined in (8.3.2) and(8.3.3) respectively, are two best approximations to f. Hence, the computation of $\underline{g}_{p^*}$ and $\overline{g}_{p^*}$ follows from the computation of a best knot vector $p^* \in P_n$ and δ_n^*. The algorithm presented in this section computes p^* and δ_n^* simultaneously. Before describing the algorithm, we extend Lemma 9.1 and Theorem 9.1. Let

$$E_n^0 = \{p = (p_0, p_1, ..., p_n) \in \Omega_n : e(p_{i-1}, p_i) \geq \delta_n(p), i = 1, 2, ..., n\}. \qquad (9.4.1)$$

LEMMA 9.1'. *Let $f \in C[0,1]$ and $p = (p_0, p_1, ..., p_n) \in E_n^0 \cap Q_n$. Define $q_0 = p_0$, $q_n = p_n$ and $q_k = \inf\{x \in [q_{k-1}, p_k] : f(x) = f(p_k) \text{ and } e(x, p_k) < \delta_n(p)\}$, for $k = 1, 2, .., n-1$. Then,*

(i) $\delta_n(q) = \delta_n(p)$;

(ii) $q \in E_n^0 \cap Q_n$;

(iii) if for some $x \in [q_k, q_{k+1})$, $f(x) = f(q_{k+1})$, *then* $e(x, p_{k+1}) \geq \delta_n(p)$, *for* $k = 1, ..., n-2$.

PROOF: First, we prove that $q_i \leq p_i \leq q_{i+1}$, for $i = 1, 2, ..., n-2$. Assume, to the contrary, that for some integer $j \in \{1, 2, ..., n-1\}$, $p_j > q_{j+1}$. If q_{j+1} satisfies $f(q_{j+1}) = f(p_{j+1})$ and $e(q_{j+1}, p_{j+1}) < \delta_n(p)$, then we arrive at a contradition that $e(p_j, p_{j+1}) \leq e(q_{j+1}, p_{j+1}) < \delta_n(p)$. If q_{j+1} satisfies $f(q_{j+1}) = f(p_{j+1})$ and

$e(p_j, p_{j+1}) = \delta(p)$, then there is a sequence $\{t_m\}_{m=1}^{\infty}$ with $t_m \in [q_j, p_{j+1}]$ satisfying $f(t_m) = f(p_{j+1})$, $e(t+m, p_{j+1}) < \delta_n(p)$, and $t_m \downarrow q_{j+1}$, as $m \to \infty$. Since $q_{j+1} < p_j$, there exists a sufficiently large integer M such that $t_M < p_j$. Therefore $e(p_j, p_{j+1}) \leq e(t_M, p_{j+1}) < \delta_n(p)$. We arrive at the same contradiction. Hence, $q_i \leq p_i \leq q_{i+1}$, $i = 1, 2, ..., n-2$. With this condition, the rest of this proof is similar to the proof of Lemma 9.1. Thus we omit the details.

Similar to Theorem 9.1, we have the following

THEOREM 9.1'. *Let* $f \in C[0,1]$. *Then,* $E_n^0 \cap Q_n \subseteq P_n$.

Assume $p = (p_0, p_1, ..., p_n) \in Q_n$. Let

$$W_p\{(x,y) : F_i(x,y) = \delta_n(p),\ p_i < x < y < p_{i+1}\}. \tag{9.4.2}$$

If $(x_1, y_1) \in W_p$, then for some j, $p_{j-1} < x_1 < y_1 < p_j$ and $F_{j-1}(x_1, y_1) = \delta_n(p)$. Assume for some integer $i \in [1, n]$, $e(p_{k-1}, p_k) \geq \delta_n(p)$, $k = 1, 2, ..., i-1$, and $e(p_{i-1}, p_i) < \delta_n(p)$. Clearly, $i \neq j$. If $j < i$, let

$$p_l^{(1)} = \begin{cases} p_l & l = 0, 1, ..., j-1, \\ x_1 & l = j, \\ y_1 & l = j+1, \\ p_{l-2} & l = j+2, ..., i, \\ p_l & l = i+1, ..., n. \end{cases} \tag{9.4.3}$$

If $i < j$, let

$$p_l^{(1)} = \begin{cases} p_l & l = 0, 1, ..., i-2, \\ p_{l+2} & l = i-1, i, ..., j-3, \\ x_1 & l = j-2, \\ y_1 & l = j-1, \\ p_l & l = j, j+1, ..., n. \end{cases} \tag{9.4.4}$$

THEOREM 9.5. *Let $f \in C[0,1]$ and $p \in Q_n$ satisfies the above assumptions. Then,* $p^{(1)} = (p_0^{(1)}, p_1^{(1)}, ..., p_n^{(1)})$ *satisfies*

(i) $p^{(1)} \in Q_n$;

(ii) $\delta_n(p) \geq \delta_n(p^{(1)})$;

(iii) *if* $j < i$ *then,* $e(p_{k-1}^{(1)}, p_k^{(1)}) \geq \delta_n(p^{(1)})$, $k = 1, 2, ..., i+1$; *if* $i < j$ *then,* $e(p_{k-1}^{(1)}, p_k^{(1)}) \geq \delta_n(p^{(1)})$, $k = 1, 2, ..., i-1,\ j-2,\ j-1,\ j$.

PROOF: (i) For even j, if there exists some $x' \in [p_{j-1}, y_1]$ such that $f(x') < f(x_1)$, then,

$$d_{j-1}(p_{j-1}, p_j) \geq f(y_1) - f(x') > f(y_1) - f(x_1) = \delta_n(p),$$

which is a contradiction. Hence $f(x_1) = m(p_{j-1}, y_1)$. Similarly, $f(y_1) = M(x_1, p_j)$. For odd j we similarly have $f(x_1) = M(p_{j-1}, y_1)$ and $f(y_1) = m(x_1, p_j)$. In view of $p \in Q_n$ and $e(p_{i-1}, p_i) < \delta_n(p)$, $p^{(1)} \in Q_n$.

(ii) Since

$$\begin{aligned} d_{i-2}(p_{i-2}, p_{i+1}) \leq \max\{ & d_{i-2}(p_{i-2}, p_{i-1}), d_{i-2}(p_{i-1}, p_i), d_{i-2}(p_i, p_{i+1}), \\ & d_{i-2}(p_{i-2}, p_{i-1}, p_i), d_{i-2}(p_{i-1}, p_i, p_{i+1}), \\ & \sup\{F_{i-2}(x,y) : p_{i-2} \leq p_{i-1}, p_i \leq y \leq p_{i+1}\}\} \\ \leq & \begin{cases} \max\{\delta_n(p), [f(p_{i-1}) - f(p_i)]/2\}, & \text{if } i \text{ is even} \\ \max\{\delta_n(p), [f(p_i) - f(p_{i-1})]/2\}, & \text{if } i \text{ is odd} \end{cases} \\ \leq & \ \delta_n(p), \end{aligned}$$

$d_{j-1}(p_{j-1}, x_1) \leq d_{j-1}(p_{j-1}, p_j) \leq \delta_n(p)$, $d_j(x_1, y_1) \leq \mid f(y_1) - f(x_1) \mid /2 = \delta_n(p)$, and $d_{j+1}(y_1, p_j) \leq d_{j-1}(p_{j-1}, p_j) \leq \delta_n(p)$, by the definition of $p^{(1)}$, we have

$\delta_n(p^{(1)}) \leq \delta_n(p)$.

(iii) For even j, we have $f(p_{j-1}) \geq f(y_1)$ and $f(p_j) \leq f(x_1)$. Hence,

$$e(p_{j-1}, x_1) \geq [f(p_{j-1}) - f(x_1)]/2 \geq [f(y_1) - f(x_1)]/2 \geq \delta_n(p^{(1)}),$$

$$e(x_1, y_1) \geq [f(y_1) - f(x_1)]/2 \geq \delta_n(p^{(1)}),$$

and

$$e(y_1, p_j) \geq [f(y_1) - f(p_j)]/2 \geq [f(y_1) - f(x_1)]/2 \geq \delta_n(p^{(1)}).$$

For odd j, we similarly prove that $e(p_{j-1}, x_1) \geq \delta_n(p^{(1)})$, $e(x_1, y_1) \geq \delta(p^{(1)})$, and $e(y_1, p_j) \geq \delta_n(p^{(1)})$. If follows that if $j < i$, then,

$$e(p_{k-1}^{(1)}, p_k^{(1)}) = e(p_{k-1}, p_k) \geq \delta_n(p) \geq \delta_n(p^{(1)}), \quad k = 1, 2, ..., j-1,$$

$$e(p_{j-1}^{(1)}, p_j^{(1)}) = e(p_{j-1}, x_1) \geq \delta_n(p^{(1)}),$$

$$e(p_j^{(1)}, p_{j+1}^{(1)}) = e(x_1, y_1) \geq \delta_n(p^{(1)}),$$

$$e(p_{j+1}^{(1)}, p_{j+2}^{(1)}) = e(y_1, p_j) \geq \delta_n(p^{(1)}),$$

$$e(p_{k-1}^{(1)}, p_k^{(1)}) = e(p_{k-3}, p_{k-2}) \geq \delta_n(p) \geq \delta_n(p^{(1)}), \quad k = j+3, ..., i,$$

$$e(p_i^{(1)}, p_{i+1}^{(1)}) = e(p_{i-2}, p_{i+1}) \geq e(p_{i-2}, p_{i-1}) \geq \delta_n(p) \geq \delta_n(p^{(1)});$$

and if $i < j$ then,

$$e(p_{k-1}^{(1)}, p_k^{(1)}) = e(p_{k-1}, p_k) \geq \delta_n(p) \geq \delta_n(p^{(1)}), \quad k = 1, 2, ..., i-2,$$

$$e(p_{i-2}^{(1)},p_{i-1}^{(1)}) = e(p_{i-2},p_{i+1}) \geq e(p_{i-2},p_{i-1}) \geq \delta_n(p) \geq \delta_n(p^{(1)}),$$

$$e(p_{j-3}^{(1)},p_{j-2}^{(1)}) = e(p_{j-1},x_1) \geq \delta_n(p) \geq \delta_n(p^{(1)}),$$

$$e(p_{j-2}^{(1)},p_{j-1}^{(1)}) = e(x_1,y_1) \geq \delta_n(p) \geq \delta_n(p^{(1)}),$$

$$e(p_{j-1}^{(1)},p_{j}^{(1)}) = e(y_1,p_j) \geq \delta_n(p) \geq \delta_n(p^{(1)}).$$

We have proved this theorem.

Theorem 9.5 allows us to establish the following algorithm to compute a best knot vector and the error of best approximation.

ALGORITHM:

Step 1: Find $p^{(0)} = (p_0^{(0)}, p_1^{(0)}, ..., p_n^{(0)}) \in Q_n$. Calculate $\delta_n(p^{(0)})$ and $e(p_{i-1}^{(0)}, p_i^{(0)})$, $i = 1,2,...,n$.

Step 2: For $k \leq n$, if $e(p_{i-1}^{(k-1)}, p_i^{(k-1)}) \geq \delta_n(p^{(k-1)})$, $i = 1,2,...n$, then, goto Step 4. Otherwise assume for some j, $F_{j-1}(x_1^{(k-1)}, y_1^{(k-1)}) = \delta_n(p^{(k-1)})$, where $p_{j-1}^{(k-1)} < x_1^{(k-1)} < y_1^{(k-1)} < p_j^{(k-1)}$, $e(p_{l-1}^{(k-1)}, p_l^{(k-1)}) \geq \delta_n(p^{(k-1)})$, for $l = 1,2,...,i \leq n-1$, and $e(p_i^{(k-1)}, p_{i+1}^{(k-1)}) < \delta_n(p^{(k-1)})$. If $j < i$, let

$$p_l^{(k)} = \begin{cases} p_l^{(k-1)} & \ell = 0,1,...,j-1, \\ x_1^{(k-1)} & \ell = j, \\ y_1^{(k-1)} & \ell = j+1, \\ p_{l-2}^{(k-1)} & \ell = j+2,...,i, \\ p_l^{(k-1)} & \ell = i+1,...,n. \end{cases}$$

If $i < j$, let

$$p_j^{(k)} = \begin{cases} p_l^{(k-1)} & \ell = 0, 1, \ldots, i-2, \\ p_{l+2}^{(k-1)} & \ell = i-1, i, \ldots, j-3, \\ x_1^{(k-1)} & \ell = j-2, \\ y_1^{(k-1)} & \ell = j-1, \\ p_l^{(k-1)} & \ell = j, j+1, \ldots, n. \end{cases}$$

Step 3: Let $k := k+1$ and goto Step 2.

Step 4: Let $p^* = p^{(k)}$ and $\delta_n^* = \delta_n(p^{(k)})$. Stop.

This algorithm enables us to obtain a best knot vector p^* and δ_n^*. Then, we can define $\underline{g}_{p^*}$ and $\bar{g}_{p^*}$, which were proved to be best approximations to f from A_n in Chapter 8.

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