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Stability Analysis of Closed Surge Tanks By Phase-Plane Method

Mostafa A. Sabbah
Old Dominion University

STABILITY ANALYSIS OF CLOSED

SURGE TANKS BY PHASE-PLANE METHOD

by

Mostafa A. Sabbah

B.S.C.E. June 1958, Alexandria University
Alexandria, Egypt
Diploma in Coastal Engineering, September 1962
Delft University, Delft, The Netherlands
M.A.C.E. January 1971, Purdue University
West Lafayette, Indiana, U.S.A.

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Approved by:

M. Hanif Chaudhry (Chairman)

A. Osman Akan

John M. Kuhlman

ABSTRACT

The governing equations describing water level oscillations in a closed surge tank with compressed air at the top of the tank are a set of nonlinear ordinary differential equations if the hydraulic system is analyzed as a lumped system. These oscillations are stable or unstable depending on the parameters of the plant and the type and magnitude of the disturbance.

The present available stability criterion has been developed by linearizing the governing equations and is, therefore, valid only for small disturbances.

In the research reported herein, the governing equations are normalized to reduce the number of parameters from nine to four and the stability of oscillations is studied by using the phase plane method which allows inclusion of nonlinear terms in the analysis and, therefore, would be valid for small and large disturbances.

Four cases of turbine flow demand are investigated. These are: constant discharge, constant gate opening, constant power and constant power combined with constant gate opening. Singularities of the governing equations are determined and analyzed in each case and stability criteria are developed. For illustration purposes,

Driva Hydroelectric Power Plant System in Norway is analyzed and phase portraits are presented.

These investigations show that the oscillations are always stable for the case of constant discharge and of constant gate opening. For the constant power case, oscillations are stable only if the system parameters satisfy certain criteria. For a combination of constant power and constant gate opening, it has been found that stability criteria for constant power are valid for heads greater than rated head and for constant gate opening for heads less than the rated head.

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TABLE OF CONTENTS

LIST OF FIGURES

NOMENCLATURE

$a_1 - a_4$	=	Constants;
A_s	=	Horizontal area of surge tank, in m^2;
A_t	=	Cross-sectional area of tunnel, in m^2;
g	=	Acceleration due to gravity, in m/s^2;
h_f	=	Head losses in tunnel, in m;
H_g	=	Gross head=upstream reservoir level-tail-water level, in m;
k	=	Coefficient of tunnel head loss, $h_f = k\ Q\,\lvert Q\rvert$;
L	=	Tunnel length, in m;
n	=	Exponent in the polytropic gas equation;
p	=	Gauge pressure of enclosed air, in m;
p_a	=	Atmospheric pressure, in m;
q	=	Normalized turbine flow;
Q	=	Tunnel flow, in m^3/s;
Q_{tur}	=	Turbine flow in m^3/s;
t	=	Time, in s;
T	=	Period of oscillations, in s;
$\forall$	=	Volume of enclosed air, in m^3;
x	=	Normalized discharge;
x_s	=	x- coordinate of singular point;
y	=	Normalized tank-water-surface level;
y_s	=	y-coordinate of singular point;
z	=	Water-surface level in the tank below the upstream reservoir level, in m;

Z	=	Amplitude of oscillations, in m.

<u>Subscripts</u>

"o"	=	Initial steady-state values;
"s"	=	Singular point

<u>Greek Letters</u>

λ_1, λ_2	=	Characteristics roots

CHAPTER 1

LITERATURE REVIEW

1.1 Open Surge Tanks

Surge tanks have been used to control the undesirable transient. In hydroelectric power plants, they have been used to reduce the amplitude of pressure fluctuations by reflecting the incoming pressure waves and to improve the regulating characteristics of a hydraulic turbine. An upstream surge tank in a hydropower plant acts as a storage for excess water during load reduction and it provides water during load acceptance. Therefore, the water is accelerated or decelerated in the tunnel slowly, and the amplitude of the pressure fluctuations in the system is reduced.

Figure 1.1 shows a schematic of a hydroelectric power plant system. When the flow in the tunnel is steady, the water surface in the tank is lower in elevation than that in the upstream reservoir due to friction loss. Load changes produce disturbances in the hydraulic system resulting in two types of oscillation: long period, e.g., oscillation of water surface in the surge tank and short period, e.g. water hammer. These oscillations are stable or unstable depending upon the parameters of the

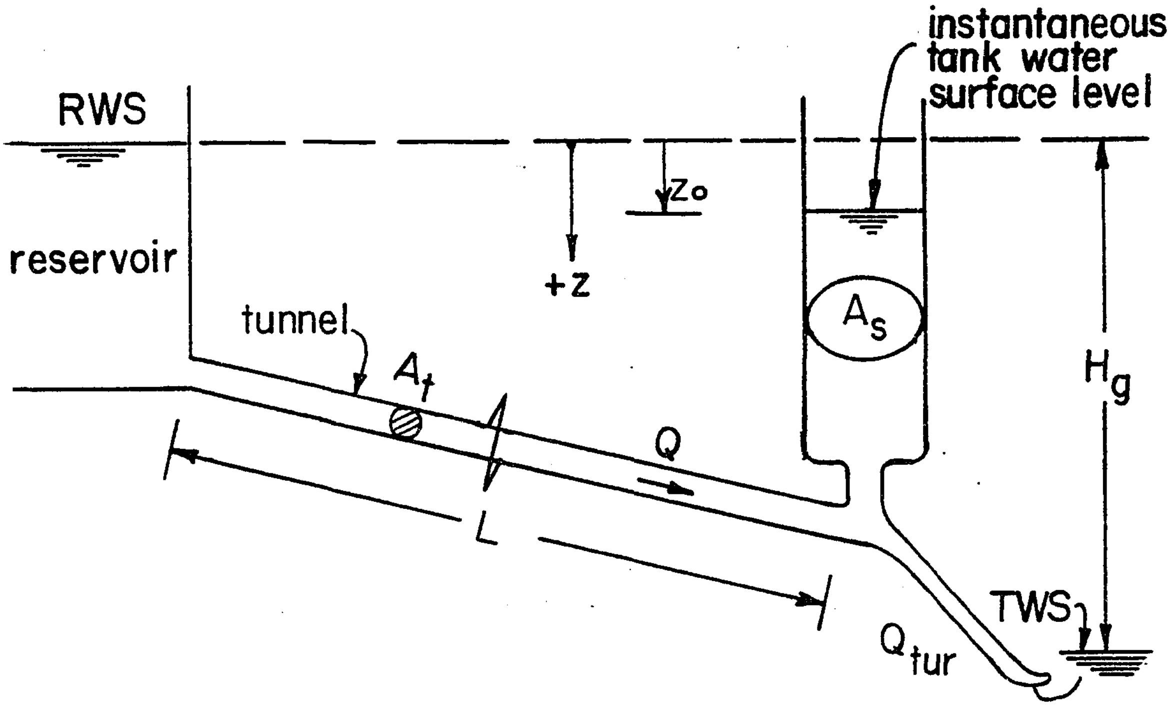


FIG. 1.1 SCHEMATIC DIAGRAM OF HYDROELECTRIC PLANT WITH AN OPEN SURGE TANK

plant and the type and magnitude of the disturbance.

Oscillations are said to be stable if they dampen to the final steady state in a reasonable time [1] and unstable if their magnitude increases with time.

While designing a surge tank, it is necessary to determine the maximum and minimum elevation of the water surface in the tank, rate of damping of the oscillations and their frequencies. In addition to oscillatory instability discussed above, a condition called tank drainage has to be avoided. In case of a sudden increase in power demand and improperly designed surge tank, the water in the tunnel does not accelerate fast enough to meet the turbine demand. Therefore, the surge tank supplies the required flow resulting in a continuous drop in the tank water level until it drains.

Surge tank stability and design have been studied by a number of investigators. Because of the presence of nonlinear terms in the differential equation describing the water level oscillation in the tank following a change in power demand, a closed-form solution is not available. Therefore, they are analyzed by linearizing them and numerical methods are used to integrate them.

The dynamic and continuity equations for a simple surge tank may be written as [2]:

$$\frac{L}{gA_t} \frac{dQ}{dt} + z + kQ^2 = 0 \qquad (1.1)$$

$$A_s \frac{dz}{dt} + Q - Q_{tur} = 0 \qquad (1.2)$$

in which

Q = tunnel discharge;

Q_{tur} = turbine discharge;

A_t = tunnel cross-sectional area;

A_s = horizontal cross-sectional area of the surge tank;

z = instantaneous water level in the tank;

k = coefficient of head losses, h_f, in the tunnel ($h_f = kQ^2$);

g = acceleration due to gravity;

t = time; and

L = tunnel length.

For the case of a simple surge tank, Thoma [14] using Eqs. 1.1 and 1.2 and assuming very small oscillations and a constant power condition, developed his classical stability criterion. Thoma's analysis is based on the following assumptions:

1. The turbine governor maintains a constant power output;
2. The surge tank oscillations are small;
3. The turbine efficiency is constant;
4. Pressure losses in the penstock are negligible;

5. Velocity head in the tunnel is neglected;
6. Head losses in the tunnel are determined from the steady flow formula;
7. Power station is isolated, i.e., it is not connected with another station; and
8. Highly sensitive governor which reacts immediately.

Based on the above assumptions, Thoma demonstrated that the oscillations are unstable if the tank area is less than a minimum. This minimum area is called Thoma's area or Thoma's condition for oscillations of perpetually constant amplitude. The Thoma's formula for the tank area

$$A_{Th} = \frac{V_o^2}{2g} \frac{LA_t}{kV_o^2 H_o} \qquad (1.3)$$

or

$$A_s = NA_{Th} \qquad (1.4)$$

in which,

A_{Th} = the critical surge tank area by Thoma;
V_o = steady state velocity in tunnel;
g = acceleration of gravity;
L = length of tunnel;
A_t = tunnel cross-sectional area;
k = coefficient of tunnel head loss
H_o = net head; and
N = stability factor

If N = 1.0, then the surge tank area is the critical area and oscillation is perpetual with constant amplitude.

If N is less than 1.0, oscillations are unstable and if N is greater than 1.0, oscillations are stable.

Several investigators, such as Jaeger [3], Paynter [4], Marris [7], Ruus [6] and Chaudhry [2], have undertaken studies to determine the validity of Thoma's Formula; and to investigate effect of large oscillations on stability as discussed in the following paragraphs.

Jaeger [3] expanded analytically Thoma's Formula to allow for large oscillations by introducing the ratio Z^*/H_o, where Z^* is the maximum surge neglecting friction in tunnel. He developed a formula:

$$A_s = \frac{v_o}{2g} \frac{LA_t}{kv_o^2 H_o} \quad (1 + 0.482 \frac{Z^*}{H_o})$$

which gives surge tank area larger than Thoma's area. This formula may be simplified as

$$A_s = A_{Th} \ (1 + 0.5 \frac{Z^*}{H_o}) \qquad (1.5)$$

in which

$$Z^* = v_o \sqrt{LA_t/gA_s}$$

Jaeger stated that when the surges are large, additional losses occur at the base of the surge shaft where the water is diverted from the tunnel into the shaft, or from the shaft into the tunnel. These losses have stabilizing effect. Jaeger's analytical approach to the problem has been confirmed by Frank [9] who used a graphical solution.

Paynter [4] analyzed the phenomena of large oscillations by means of an analog computer and presented his formula:

$$\frac{kv_o^2}{H_o} = 2\,\frac{kv_o^2}{Z^*}\left(1 - \frac{kv_o^2}{H_o}\right) \qquad (1.6)$$

To account for large oscillations, Marris [5], employing phase plane method analyzed the stability of a surge tank without neglecting the nonlinear terms in the governing equation. His non-dimensional form of the equation of motion for a simple surge tank operating under the condition of constant hydraulic power is shown to have two singular points: One of them accounts for small displacement phenomena and the other accounts for the occurrence of drainage due to insufficient power being available at the turbine. Marris investigated the stability of the two singular points by means of Liapunoff's theorem (29) and concluded that the occurrence of drainage is due to insufficient tank size and due to conduit friction which prohibits acceleration to the flow velocity demanded by the turbine.

Ruus [6], using a digital computer, studied the stability of the simple surge tank for the case of large oscillations. In his analysis, he considered constant power output as well as constant gate opening. Other assumptions by Thoma were kept the same.

In case of a constant power condition, it is assumed that the turbine gates can be opened to any value to maintain constant power. However, on actual installations once gates reach their fully open position, they cannot be opened more. Therefore, a constant power will only be maintained if the net head on the turbine is more than or equal to the rated head.

In his analysis, Ruus made two approximations with respect to the relationship between the turbine discharge and the net head. Above the rated head, the head-discharge relationship is represented by qH_n = constant. Below the rated head, two approximations are presented:

1. A parabola expressed by the equation qH_n = constant; and
2. A straight line which is represented by the equation:

$$q = q_r \left(1 + k \frac{H_n - H_r}{H_r}\right)$$

in which: q = turbine discharge; q_r = rated discharge; H_n = rated head; and k = numerical constant equals to 0.62.

The actual head-discharge curve for the range below the rated head is supposed to be located between the above two curves.

Due to these approximations, the results are mainly applicable for preliminary calculations of the stability of a simple surge tank.

In general, Ruus [6] shows that the relative rated head, H_r/H_g (H_g = gross head), has a substantial effect on the stability of a surge tank, and that for the values of the head below the rated head, the difference in head-discharge relationships has a relatively minor effect on the stability of the tank.

The computer analyses results by Ruus agree very closely with those derived by Frank [9] by graphical methods for a constant power. The restriction on the turbine gate opening under heads lower than the rated head causes rapid damping of large oscillations in the surge tank. Ruus concluded that small oscillations rather than large ones are critical to surge tank stability, and should be considered in determining the necessary tank area. Therefore, increasing the surge tank area is not required for damping of large oscillations.

Marris [7] and Sideriades [8] used the phase plane method to study the nonlinear differential equation for a simple surge tank for the case of constant power. Marris obtained the solution curves of the equation near each of its singularities and investigated the types of instability to be expected. He demonstrated that Thoma's criterion does not hold for large oscillations.

Chaudhry and Ruus [2], also utilized the phase plane technique to study the stability oscillations for the following cases:

a. Constant flow,

b. Constant gate opening,

c. Constant power, and

d. Constant power combined with full gate opening.

They derived the following conclusions from their studies:

1. Stable oscillations always occur in the cases of constant flow and constant gate opening. This result was obtained earlier by investigators using other analysis approaches.
2. For the case of an ideal governor, which insures constant power but can open the gates only to their specified maximum limit, the phase plane is divided into two parts:
   a. The region in which power can be maintained constant; and
   b. The region of maximum gate opening in which power cannot be maintained constant.

   The solution trajectories in the region of constant power correspond to the stable oscillations if $A_s > A_{Th}$, and to the unstable oscillations if $A_s < A_{Th}$. The solution trajectories in the latter region are always stable. Hence, the oscillations, large or small, are stable if $A_s > A_{Th}$. For $A_s < A_{Th}$, the solution trajectories in the phase plane correspond to stable oscillations in the region defined in (b) and to unstable oscillations in the region defined in (a). Due to these stabilizing and destabilizing effects, a solution trajectory corresponding to perpetual oscillations is obtained, which in the phase-plane terminology is called a limit cycle. The region enclosed by the limit cycle depends upon the stabilizing effect of the constant gate opening and upon the destabilizing influence of the governor. The oscillations inside the limit cycle are unstable, and their amplitude increases until it is equal to that of the limit cycle. The oscillations outside the limit cycle are stable, and their amplitude decreases until it is equal to that of the limit cycle.

3. The danger of drainage of the surge tank for the case of constant power combined with constant gate opening is considerably less than that indicated by the stability analysis assuming constant power only.

1.2 Closed Surge Tanks

Surge tanks (air chambers, air vessels, air bottles, etc.) have been used as devices for controlling surges in pumping plants [11,12] for about 50 years.

Because of recent advances in technology [16], it is economically feasible nowadays to excavate the headrace tunnel of a hydropower development at slopes as much as one in eight. Thus a straight inclined tunnel may be used from the intake to the powerhouse. In such a case, it becomes attractive to provide a surge tank with closed top [1] instead of a conventional surge tank in which the shaft is excavated to the free surface. Such surge tanks were used about 60-70 years ago in small power plants in the

U.S.A., but were discontinued because of problems with governing stability [10]. These problems were most probably due to the governor and not due to the surge tank. However, major credit goes to the Norwegian engineers who have used these tanks at several large power plants, eight of which are operating satisfactorily.

Svee [16] investigated the stability of closed surge tanks. His solution for the governing differential equations was based upon the theory of small oscillations, therefore, the nonlinear terms were neglected.

Figure 1.2 shows schematic of a hydroelectric power plant with a closed surge tank. A chamber partly filled with compressed air has been used replacing the conventional open surge tank. In the case of an instantaneous load change in the power plant, the mass of water will start to discharge into the chamber raising the air pressure. The growing excess pressure, together with the friction drag along the tunnel walls will exert a steadily growing retarding force on the flowing water until the water in the chamber will flow back. The water level in the chamber will then oscillate with damped motions around a new equilibrium.

Equations describing the water level oscillations in a closed or open surge tank are nonlinear. The stability criteria for open surge tank has been developed without neglecting the nonlinear terms. However, because of the very recent introduction of the closed surge tanks for application in hydroelectric power plants, no criteria has

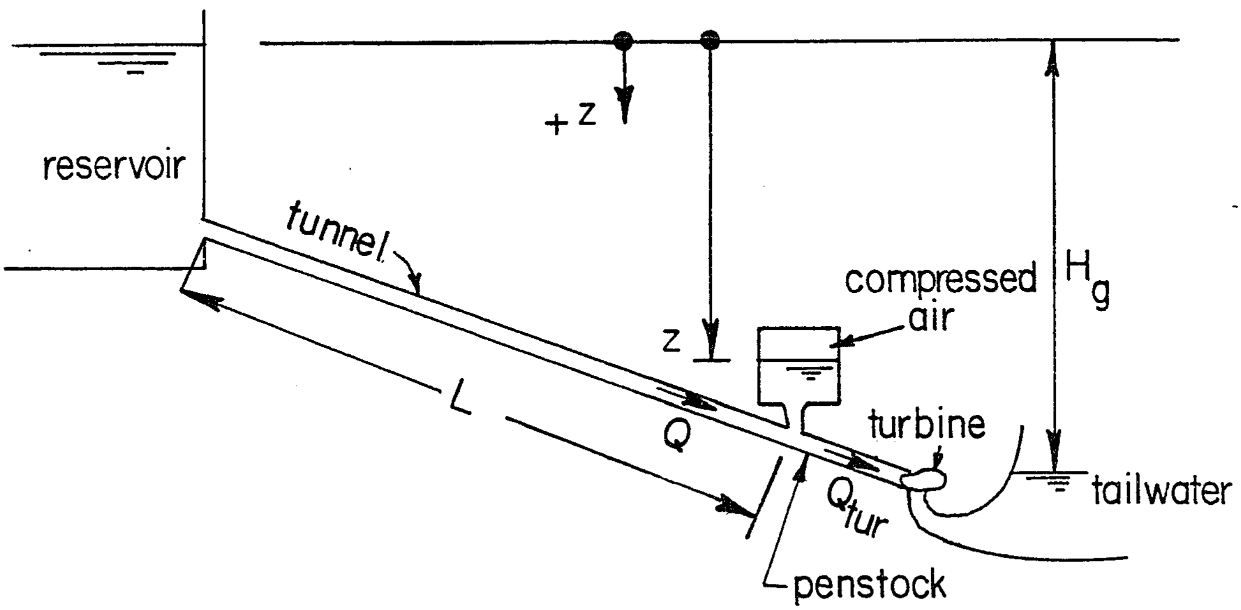


FIG. 1.2 SCHEMATIC DIAGRAM OF A HYDROELECTRIC POWER PLANT WITH A CLOSED SURGE TANK

yet been developed in which nonlinear terms in the governing equations are included. The presently available stability criteria have been developed by applying the theory of small oscillations [16], and by making the same assumptions as for the analysis of stability of an open surge chamber.

To investigate the stability, Svee [16] assumed a small disturbance imposed on the hydraulic system in steady conditions, an approach similar to that used by Thoma [14]. According to this approach it is imagined that a water layer of thickness Δz is placed on the water surface in the surge tank at stationary conditions. The effect of such an equilibrium disturbance is then investigated. The basic equations were derived based on the following assumptions:

1. Turbine governor maintains constant power in all phases of oscillations.
2. The inertia of the water in the surge chamber is neglected in the direction normal to the tunnel axis.
3. The water mass in both the shaft and the chamber is neglected.
4. No time lag exists between a water level alteration in the surge chamber and its effect on the acceleration of the tunnel water mass.
5. Velocity is constant across the tunnel area.
6. Expansion and contraction of the enclosed air is mathematically modelled during the transient state

conditions by the polytropic equation for a perfect gas, i.e., $P\forall^n$ = constant, in which P = instantaneous pressure of air cushion and $\forall$ = instantaneous volume of air cushion.

7. The water mass, dm, that enters the tunnel from the shaft during oscillation has no velocity component in the direction of the tunnel before entering the tunnel.

8. The Δ values are very small, but finite deviations from the respective stationary values. Small terms of second or higher order are neglected.

The following equations are also used:

$$v = v_o + \Delta v$$

$$z = z_o + \Delta z$$

$$q = q_o + \Delta q$$

$$\eta = \eta_o + \Delta\eta$$

$$p = p_o + \Delta p$$

$$\forall = \forall_o + \Delta\forall = \forall_o + A_s\ \Delta z$$

Based on the above assumptions and equations and the basic equations, Svee derived the following three equations, viz, Dynamic, Continuity and Regulation (constant power) equations:

$$\frac{L}{g}\frac{d(\Delta v)}{dt} = A\Delta z - 2\alpha v_o \Delta v + \frac{v_o A_s}{gA_t}\frac{d(\Delta z)}{dt} \qquad (1.7)$$

$$\Delta q = A_s \frac{d(\Delta z)}{dt} + A_t\ \Delta v \qquad (1.8)$$

$$E\,\Delta q = A\,\frac{q_o}{\xi_o}\,\Delta z - \frac{q_o v_o}{\xi_o g}\,\Delta v \qquad (1.9)$$

in which A, E and ξ_o are constants. By eliminating Δv and Δq in the above equations, a second order differential equation in Δz with constant coefficients will result from which Svee derived the following stability criterion:

$$(A_s)_{cr} = A_{Th}\,(1 + \frac{nP_o}{a_o}) \qquad (1.10)$$

in which,

$(A_s)_{cr}$ = critical area of the closed surge tank;

n = the exponent in the polytropic gas equation;

a_o = distance between the surge chamber roof and the water level in a surge chamber with vertical walls and a horizontal roof $(= \frac{\forall_o}{A_s})$

Accordingly, if the critical area of a surge chamber with an enclosed compressed air cushion is to be determined, the critical area of an open surge tank should be calculated first, then the critical area of the enclosed surge tank is given by Eq. 1.10. This procedure is only possible when determining critical areas for small oscillations.

Eight high pressure head power plants with the air cushion design have been constructed in Norway. The problem of air leakage from the chamber is one major concern. The amount of air escaping from the chamber appears to be the only uncertain factor as far as the practical aspects of the closed surge tank solution. The air loss was found to increase with increased turbulence in the chamber [17]. No cavitation has been developed in the turbines, and it was found that the presence of the air chamber reduced cavitation risks.

Mathematical modelling of the expansion and contraction of air in the closed chamber during the transient state conditions has been under dispute. The debate is about applying the polytropic equation for a perfect gas, i.e., $p\forall^n$=constant. For isothermal behavior, n=1.0 and for purely adiabatic behavior n=1.4.

An average value of n=1.2 has been widely used and has been found to yield satisfactory results when compared with the field analysis. However, Graze [18, 19] who developed a rational heat transfer equation has severely criticized the use of polytropic gas equation calling its use theoretically incorrect and stating that n=1.2 does not necessarily give conservative results.

To compare the two approaches, the computed results for Driva power plant were compared. These results are for a flow reduction of 30 m^3/s to zero. The air pressure variation computed by the rational heat transfer equation was taken from Ref. 20 while the results for the polytropic gas equation were computed for n=1.4 [31]. It has been found that there is close agreement between the results of the two approaches [31].

As discussed above, the stability criteria and numerical methods have been developed for open surge tanks, for both small and large oscillations. However, available stability criteria for closed surge tank have been developed by linearizing the governing equations. This criteria, developed by Svee [16], is therefore applicable only for small oscillations.

In the research reported herein, the stability of oscillations is studied by using the phase plane method in which nonlinear terms of the governing equation are retained. This method, which has been successfully applied [2, 7, 8] in investigating stability of a simple surge tank, is used particularly in connection with non-linear differential equations. All the singular points of the governing equations will be determined and analyzed to establish stability criteria for different possible cases of turbine flow demand. For illustration purposes, Driva hydroelectric power plant system [17] will be used.

CHAPTER 2

PHASE PLANE METHOD

2.1 INTRODUCTION

The basic problems to be studied in a nonlinear system are:

1. To determine the equilibrium conditions of the system and to analyze their stability.
2. To investigate the transition of the system from one stable state of motion to another.
3. To investigate the dependence of the solution curves on the system parameters.

One technique for studying these problems is to represent the motion of the nonlinear system in phase plane.

2.2 ELEMENT OF THE METHOD

Let us consider a differential equation:

$$\frac{d^2x}{dt^2} + f\left(x, \frac{dx}{dt}\right) = 0 \qquad (2.1)$$

in which $f\left(x, \frac{dx}{dt}\right)$ may be a nonlinear function of x and $\frac{dx}{dt}$. Let $v = y = \frac{dx}{dt}$, then, Eq. 2.1 can be rewritten in the form of two first order differential equations as follows:

$$\frac{dy}{dt} = -f(x,y) \qquad (2.2)$$

$$\frac{dx}{dt} = y$$

Equation 2.2 is a special case of a more general nonlinear system of the form:

$$\frac{dy}{dt} = Q(x,y)$$
$$\frac{dx}{dt} = P(x,y) \qquad (2.3)$$

If the functions P(x,y) and Q(x,y) are independent of time, then system 2.3 is said to be autonomous. If x,y are orthogonal cartesian coordinates, where y = dx/dt represents the rate of change of coordinate x, then to each time t_i (i.e., one state of the system) there corresponds a point in the xy-plane. Conversely, to every point of the xy-plane where the functions P and Q are defined there corresponds one and only one state of the system. Therefore, the xy-plane is called the state plane or the phase plane. The point in the phase plane which corresponds to a given state of the system is called the representative point.

When the state of the system changes to a new state there will be a new representative point. To a continuing change in the state of the system there corresponds a motion of the representative point in the phase plane. The curve described by the representative point provides a solution x(t) and y(t) of the system of Eq. 2.3. This curve or solution is called a solution curve or trajectory. Thus, a solution curve or trajectory depends on the position and the velocity of the system.

By a complete trajectory of a dynamical system, it means a curve which is composed of all representative points the system passes through in the course of time. The speed of the representative point in the phase plane is the state speed v, or the phase velocity where:

$$V = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} = \sqrt{P^2(x,y) + Q^2(x,y)} \qquad (2.4)$$

When the velocity $(\frac{dx}{dt})$ and the acceleration $(\frac{dy}{dt} = \frac{d^2x}{dt^2})$ are simultaneously equal to zero, the system will be in an equilibrium state. According to Eq. 2.4 the state speed will have to be zero at those points for which the following conditions are simultaneously satisfied:

$$P(x,y) = 0$$
$$Q(x,y) = 0 \qquad (2.5)$$

By eliminating dt from Eq. 2.3, then

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}, \quad P(x,y) \neq 0 \qquad (2.6)$$

which is the differential equation for the trajectories. If in Eq. 2.6, $P = 0$, $Q \neq 0$, we may interchange and write $\frac{dx}{dy} = \frac{P}{Q}$, in which case the integral curves (trajectories) are given in the form $x = \alpha y$.

It should be noted that Eqs. 2.3 and 2.6 are equivalent, i.e., they have the same integral curves with the difference that Eq. 2.6 gives a geometrical curve without any reference to what happens in time, whereas Eq. 2.3 in addition tells how this curve is described in time and direction by the representative point. The solution of the differential equation 2.6 can always be expressed in the form: [26]

$$F(x,y) = \text{Constant} \qquad (2.7)$$

If the value of the constant is held fixed, Eq. 2.7 is the equation of a curve in the phase plane. This curve may be connected or it may consist of several disconnected components. Such a curve is called an integral curve of Eq. 2.6 [26]. Each integral curve is composed of one or more trajectories.

Though every point of the plane for which the state speed is not zero, i.e., the functions $P(x,y)$ and $Q(x,y)$ are not both zero, there will pass one and only one trajectory or integral curve [26,27]. Eq. 2.6 may be thought of as a "flow" in the phase plane defined by the velocity-vector field. The direction of motion at each point being specified by this vector, it follows that the integral curves which are tangent at every point to this vector are completely determined by this requirement. Since $P(x,y)$ and $Q(x,y)$ satisfy Cauchy's - Lipschitz conditions for the uniqueness of the solution of 2.6 [26,27] two integral curves will not have a common point, i.e., two trajectories will not cross each other.

The phase velocity or the state speed will be zero at those points of the phase plane for which condition 2.5 is satisfied and, where the slope of a solution curve is not defined. These points are stationary points of the flow. The integral curve passing through a stationary point consists only of the point itself. Such points are called singular points and can be interpreted as the equilibrium states of the dynamical system in question.

At any point x,y for which P and Q do not vanish simultaneously is called an ordinary point of Eq. 2.6.

Thus, a singularity is always a point of equilibrium since both $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are zero. However, the resulting equilibrium may be stable or unstable.

To know the behavior of the solution in the entire phase plane we must know the type of the singularity and in particular the behavior of the solution or the trajectories near the singular points. Therefore, it is necessary to investigate first the nature of the states of equilibrium of a system.

2.3 TYPE OF SINGULAR POINTS AND STABILITY

Lyapunov [29] defines stability of an equilibrium state in a dynamical system as follows: Given an arbitrary ε - neighborhood of the equilibrium state, one can always find a corresponding $\delta\varepsilon$ - neighborhood such that a representative point located at the initial moment in the latter neighborhood never reaches the boundary of the ε - neighborhood. In other words, an equilibrium condition is said to be stable if upon slight disturbance, the departure from this condition remains small. For example, if the coordinates of the singular points of Eq. 2.6 are x_s, y_s; then it is required to have a solution that is valid in the neighborhood of this singular point x_s, y_s to examine the stability of this equilibrium condition.

A singular point belongs to one of the following types:

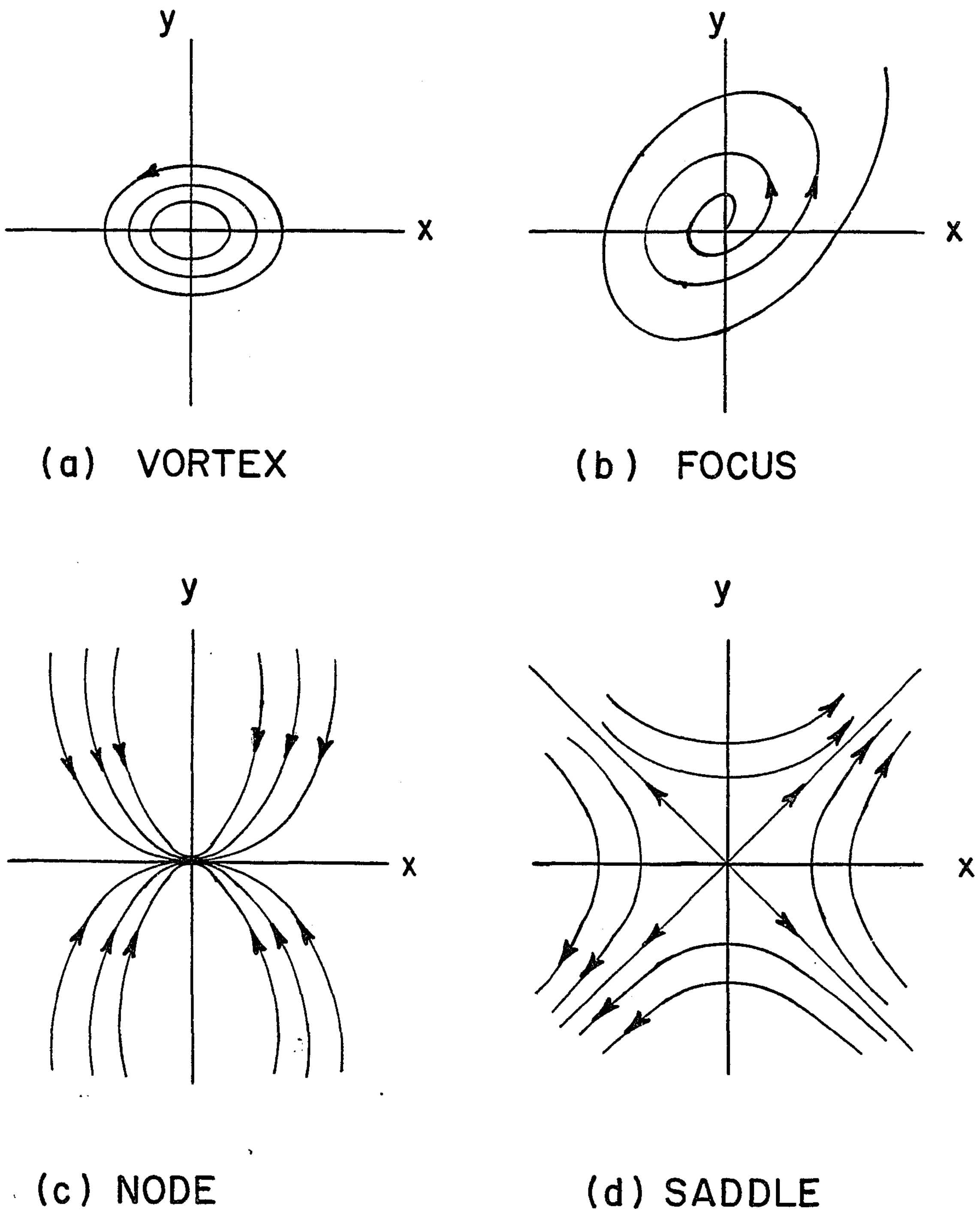


FIG. 2.1 TYPES OF SINGULAR POINTS

1. A vortex is a singular point surrounded by closed trajectories or integral curves as shown in Fig. 2.1a.
2. A focus is a singular point with spiral-like trajectories, Fig. 2.1b.
3. A node is a singular point where the integral curves have the same slope, Fig. 2.1c.
4. A saddle is a singular point with hyperbola-like trajectories, Fig. 2.1d.

According to the above definitions, a vortex is a stable singular point, i.e., stable equilibrium condition, while a saddle point represents an unstable condition. A focal point and a nodal point may be either stable or unstable depending on whether the trajectories are directed toward or away from the singular point. Point is stable if trajectories are directed toward the singular point and unstable if trajectories are directed away from the singular point.

To study the nature of solutions in the neighborhood of a singularity, let $x = x_s + u$, $y = y_s + v$, where u and v are small variations from the singular point. Expand the functions P and Q in Eq. 2.6 into Taylor's series about x_s, y_s;

$$Q(x,y) = Q(x_s, y_s) + \left.\frac{\partial Q}{\partial x}\right|_s u + \left.\frac{\partial Q}{\partial y}\right|_s v + \left.\frac{\partial^2 Q}{\partial x^2}\right|_s u^2 + \left.\frac{\partial^2 Q}{\partial y}\right|_s v^2 + \cdots \tag{2.8}$$

$$P(x,y) = P(x_s, y_s) + \frac{\partial P}{\partial x}\bigg|_s u + \frac{\partial P}{\partial y}\bigg|_s v + \frac{\partial^2 p}{\partial x^2}\bigg|_s u^2 + \frac{\partial^2 p}{\partial y}\bigg|_s v^2 + \cdots \tag{2.9}$$

Since by definition of a singular point $Q(x_s, y_s) = P(x_s, y_s) = 0$; therefore Eqs. 2.8 and 2.9 may be written as

$$Q(x,y) = c'u + d'v + c''u^2 + d''v^2 + \cdots \tag{2.10}$$

$$P(x,y) = a'u + b'v + a''u^2 + b''v^2 + \cdots \tag{2.11}$$

and Eq. 2.6 becomes:

$$\frac{dy}{dx} = \frac{dv}{du} = \frac{c'u + d'v + c''u^2 + d''v^2 + \cdots}{a'u + b'v + a''u^2 + b''v^2 + \cdots} \tag{2.12}$$

where a', b', c', d', a'', b'', c'', and d'' are real constants which represent the values of the first and second derivatives of the functions P and Q evaluated at the singular point x_s, y_s.

Since we are interested in investigating the nature of solutions in the neighborhood of the singular point, the linear terms in u,v are important in determining this nature and the second order terms can be neglected. The type of singularity depends only upon the linear terms, provided these terms are present if nonlinear terms are present [30]. This is the case of a simple singularity.

In other words, if $a'' \neq 0$ in Eq. 2.12, the singularity is a simple one if $a' \neq 0$. This condition applies in both numerator and denominator and for both terms including u and v.

Therefore, with a simple singularity the solution in its neighborhood will depend on Eq. 2.12 with only the linear terms i.e.,

$$\frac{dy}{dx} = \frac{dv}{du} = \frac{c'u + d'v}{a'u + b'v} \tag{2.13}$$

Eq. 2.13 is equivalent to the following two simultaneous first-order equations:

$$\begin{aligned} \frac{du}{dt} &= a'u + b'v \\ \frac{dv}{dt} &= c'u + d'v \end{aligned} \tag{2.14}$$

The characteristic equation for Eq. 2.14 is expressed as

$$\lambda^2 - (a' + d')\,\lambda + (a'd' - c'b') = 0 \tag{2.15}$$

with the roots:

$$\begin{aligned} \lambda_1 &= \frac{(a' + d') + \sqrt{(a' + d')^2 + 4(a'd' - c'b')}}{2} \\ \lambda_2 &= \frac{(a' + d') - \sqrt{(a' + d')^2 + 4(a'd' - c'b')}}{2} \end{aligned} \tag{2.16}$$

The general solution of the equation in the form of Eq. 2.13

$$\begin{aligned} u &= Ae^{\lambda_1 t} + B\,e^{\lambda_2 t} \\ v &= Ce^{\lambda_1 t} + D\,e^{\lambda_2 t} \end{aligned} \tag{2.17}$$

in which A, B, C and D are constants.

Therefore, from the above solution, it is obvious that the integral curves and the type and the character of the singular points are determined by the roots of the characteristic equation. The roots as defined in Eq. 2.16 depend on the values of the coefficients in Eq. 2.14.

Accordingly, if $\dot{a} + \dot{b} \neq 0$, $\dot{a}\dot{d} - \dot{c}\dot{b} \neq 0$ and $(\dot{a} + \dot{d})^2 \neq (\dot{a}\dot{d} - \dot{c}\dot{b})$. The following types of equilibrium states can exist [26]:

1. If λ_1 and λ_2 are real and negative, singularity is a stable node.
2. If λ_1 and λ_2 are real and positive, singularity is an unstable node.
3. If λ_1 and λ_2 are real and with different signs, singularity is a saddle point;
4. If λ_1 and λ_2 are complex conjugates with Real $|\lambda| > 0$, singularity is an unstable focus.
5. If λ_1 and λ_2 are complex conjugates with Real $|\lambda| < 0$, singularity is a stable focus.

Thus, Eq. 2.14 can be used [26] to determine the character and stability of the equilibrium state of the nonlinear system represented by Eq. 2.3. If the real parts of both roots of the characteristic equation are zero, or if one root is zero and the other is negative, Eq. 2.14 cannot be used to determine the stability of the equilibrium state.

The above cases of the equilibrium states can be displayed in a diagram called p-q plane as shown in Fig. 2.2. First assume that

$$p = -(\dot{a} + \dot{d})$$
$$q = \dot{a}\dot{d} - \dot{c}\dot{b}$$

Thus, the characteristic Eq. 2.15 can be rewritten as

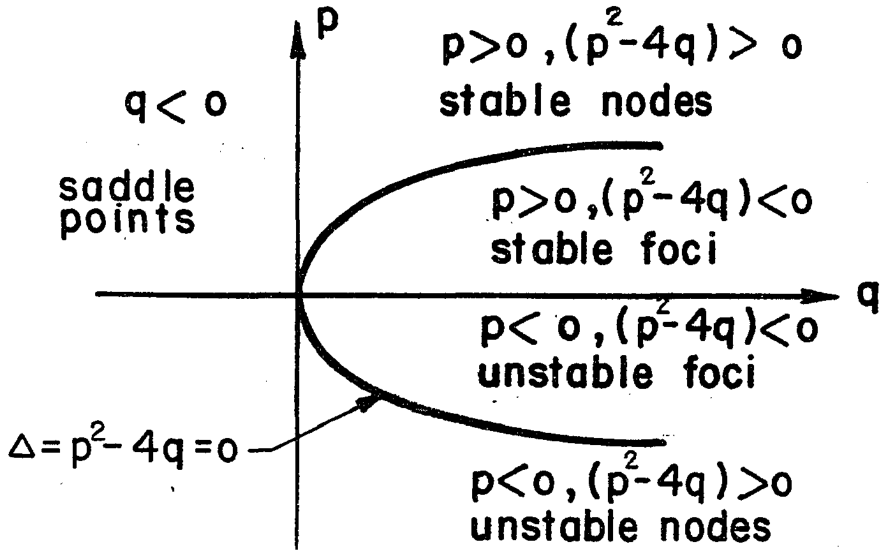


FIG. 2.2 CASES OF EQUILIBRIUM STATES IN A p-q PLANE

$$\lambda^2 + p\lambda + q = 0 \tag{2.18}$$

with its roots

$$\lambda_1, \lambda_2 = \frac{-p \pm \sqrt{p^2 - 4q}}{2} \tag{2.19}$$

For the part in the q-p plane where $q < 0$, the singular points will be saddle points, and the equilibrium state will be unstable. When $q > 0$ and $p > 0$, this corresponds to stable nodes and stable foci.

In the plane where $q > 0$ and $p < 0$, the singular points will be unstable nodes or unstable foci. If $\Delta = p^2 - 4q$, this curve will separate foci from nodes.

2.4 METHOD OF CONSTRUCTING THE TRAJECTORIES

Method of Isocline:

For a nonlinear dynamical system represented by an equation similar to Eq. 2.3, and where analytical solution is difficult or impossible, considerable insight into the qualitative aspects of the solution as well as some quantitative information can be obtained by a study of the integral curves plotted on the phase plane.

The basic graphical method is known as the method of isocline which is explained as follows:

Let the dynamical system be governed by

$$\frac{dx}{dt} = P(x,y)$$

$$\frac{dy}{dt} = Q(x,y)$$

So that the integral curves are

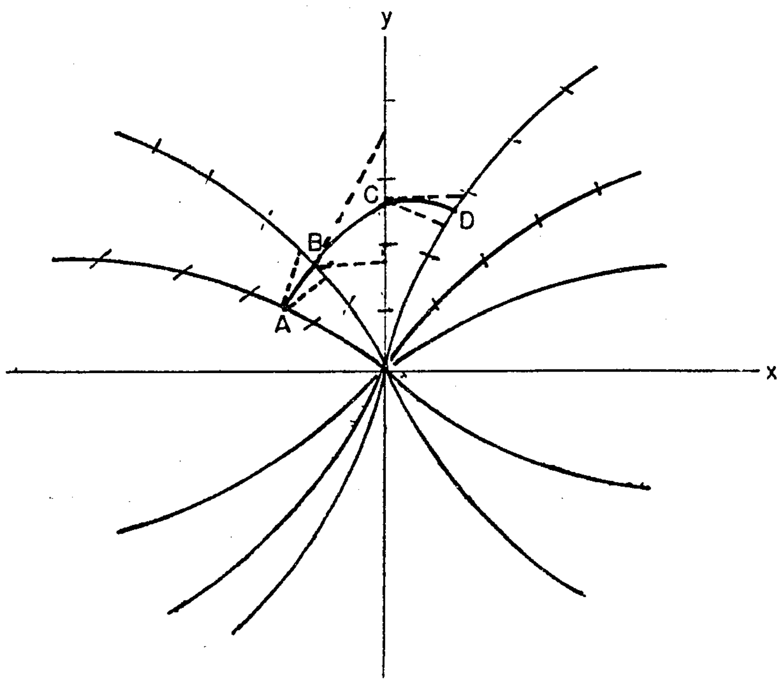


FIG. 2.3 METHOD OF CONSTRUCTING TRAJECTORIES

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}$$

where the function P and Q may be nonlinear functions of the variables x and y.

If it is assumed that

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)} = F\ (x,y) = \text{Constant} = c \tag{2.20}$$

then this equation describes a curve which passes through points in the x-y plane for which the slopes dy/dx of the integral curves are the same. This curve is called an isocline. At the singularity, the slope of the integral curves is not defined since the value of dy/dx is indeterminate and isoclines will intersect at this point. More isoclines can be plotted assuming different values for c, so that

$$F\ (x,y) = c_1,\ c_2,\ c_3,\ \ldots \tag{2.21}$$

On each curve F (x,y) = c, small line segments having the same slope dy/dx = c can be drawn. These slopes define the directional field of tangents to the integral curves in a certain region of the plane. Thus, the isocline method makes it possible to show graphically the general behavior of the solution especially in the case of nonlinear differential equation where the solution cannot be found explicitly.

We start at a point (x_s, y_s) or at an arbitrary point A on the isocline for which $c = c_1$, and pass through it a line with slope c_1. Then we pass a line with slope c_2

through the same point. Divide the distance between the points of intersection of these lines and the isocline corresponding to $c = c_2$ into two equal parts and denote the midpoint by B. The line segment AB is taken as an arc of the integral curve between these two isoclines. For more exact construction of this part of the integral curve it is necessary to choose values of c_1 and c_2 which are closer to each other. The construction is then continued in a similar way, starting with the point B.

2.5 CONSERVATIVE SYSTEMS

Assume we have a dynamical system which is represented by the following nonlinear differential equation:

$$\frac{d^2x}{dt^2} = f(x) \tag{2.22}$$

As an equivalent to system to Eq. 2.22, put

$$\frac{dx}{dt} = y \tag{2.23}$$

and then

$$\frac{d^2x}{dt^2} = \frac{dy}{dt} = f(x) \tag{2.24}$$

Thus, the integral curves will be represented by

$$\frac{dy}{dx} = \frac{f(x)}{y} \tag{2.25}$$

or

$$ydy = f(x)\ dx$$

By integration it follows that

$$\tfrac{1}{2}\, y^2 - \int f(x)\ dx = h \tag{2.26}$$

in which h is the constant of integration.

If this assumed dynamical system represents a mechanical system, then $g(x) = -\int f(x)dx$ can be considered as potential energy, and $\frac{1}{2} y^2$ can be considered as kinetic energy with mass equal to unity. Thus, equation 2.26 can be written in the form

$$\tfrac{1}{2} y^2 + g(x) = h$$

or

$$\tfrac{1}{2} y^2 = h - g(x) \qquad (2.27)$$

where the constant h represents the total energy. This type of system is known to be a conservative system.

From Eq. 2.25, integral curves which have vertical tangents will intersect the x-axis, and from Eq. 2.27 the integral curves will be symmetric relative to the x-axis. Also, from Eq. 2.25 the integral curves have a horizontal tangent at the point x_i which are the roots of $f(x) = 0$, where $y \neq 0$ at these points. Singular points will exist if these two equations are simultaneously satisfied, i.e.,

$$y = 0 \text{ and } f(x) = 0$$

According to Eq. 2.27 it follows that for all x for which $h - g(x) < 0$ there will be no real integral curves.

For a given value of h Eq. 2.27 represents a real trajectory in the phase plane as long as $h - g(x) > 0$.

Assume that $z = g(x)$, then on the x-z plane construct the function $z = g(x)$ and the line $z = h$ (i.e., the difference $h - g(x)$) as shown in Fig. 2.4a.

It is obvious that real motions occur for the values of x which are located to the left of the point of inter-

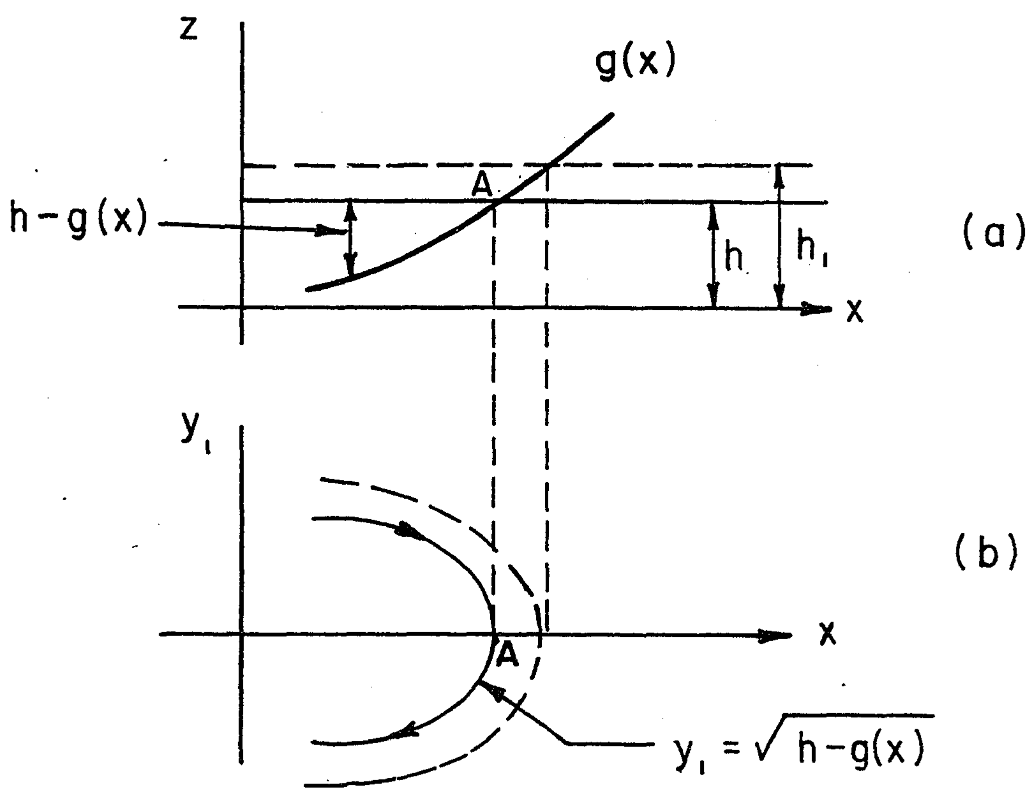


FIG. 2.4 a : x − z plane (z = g(x) and the line z = h)

FIG. 2.4 b : PHASE PLANE x , y = $y/\sqrt{2}$

FIG. 2.4 CONSERVATIVE SYSTEM

section of the line $z = h$ and the curve $z = g(x)$.

According to Eq. 2.27,

$$y/\sqrt{2} = \pm \sqrt{h - g(x)} \tag{2.28}$$

Therefore, on the phase plane (x, y_1) the integral curve will have the form shown in Fig. 2.4b. If the energy constant changes from h to h_1, a similar integral curve can be drawn as shown by the broken lines.

If the potential energy is a minimum, i.e., $z = g(x)$ has a minimum value, integral curves will be as shown in Fig. 2.5.

If $h = h_o$, the integral curve degenerates to a point where the state speed is zero and the point is therefore a singularity. Closed integral curves surrounding the singular point represent all values of $h > h_o$ and correspond to periodic motion in this system. This central singular point is stable in the sense of Lyapunov [26,29].

If the potential energy has a maximum value $h = h_o = g(x)$, the integral curves consist of four trajectories called separatrices which meet at point A as shown in Fig. 2.6b. Other trajectories will not pass through the point A. If $h > h_o$ the trajectories are above and below the point A. If $h < h_o$ trajectories will be to the right and to the left of the point which is classified as a saddle point.

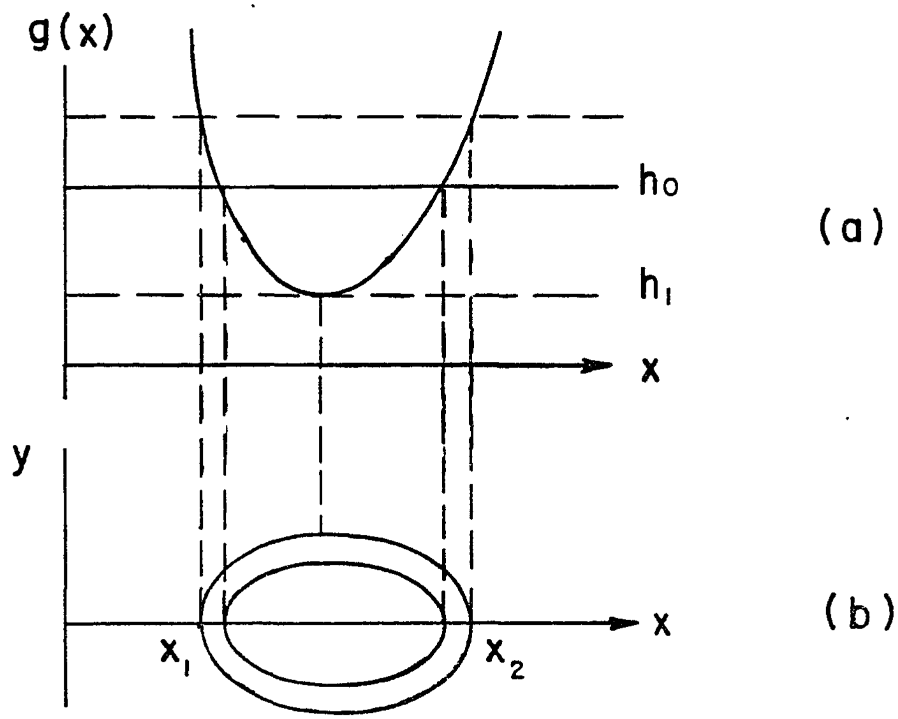


FIG. 2.5 PERIODIC MOTION

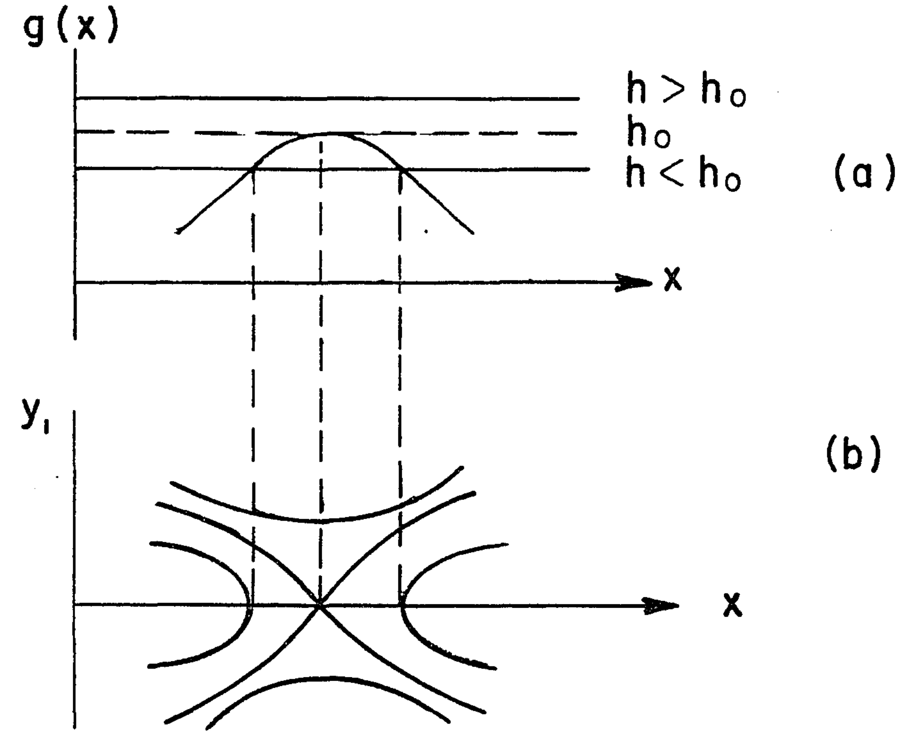


FIG. 2.6 SEPARATRICES

Assuming that $z = g(x)$ is an analytic function for all values of x and that h is a fixed value, the possible motions on the phase plane will be as follows [26]:

1. The curve $z = g(x)$ does not intersect the line $z = h$ anywhere. There will be no motion anywhere on the phase plane if the line $z = h$ lies below the curve $z = g(x)$. This is due to the fact that y is imaginary. As a result, motion is not possible with this value of total energy h. If the line $z = h$ lies above the curve $z = g(x)$, Eq. 2.27 shows that two symmetrical branches of the integral curve will exist on the phase plane. The representative point will approach either $x = \infty$ or $x = -\infty$. Such trajectories are called divergent.
2. The line $z = h$ intersects the curve $z = g(x)$ but is nowhere tangent to it. For all values of x for which $g(x) > h$ motion does not exist. For all x for which $g(x) < h$ trajectories will be either closed arcs corresponding to periodic motions or infinite arcs corresponding to divergent motions.
3. The line $z = h$ has several points of tangency with the curve $z = g(x)$. Possible trajectories are of the following forms:
   a) Isolated points, in the neighborhood of which there are no trajectories for the given h. When h increases, there will be a closed tra-

jectory and when it decreases, there will be imaginary trajectories.

b) Simple closed trajectories which correspond to periodic motions or self-intersecting trajectories which correspond to separatrices. These points are where the line z = h is tangent to the curve z = g(x) at its maxima. By increasing h we get an integral curve which surrounds the whole separatrix; by decreasing h we get closed trajectories as shown in Fig. 2.7.

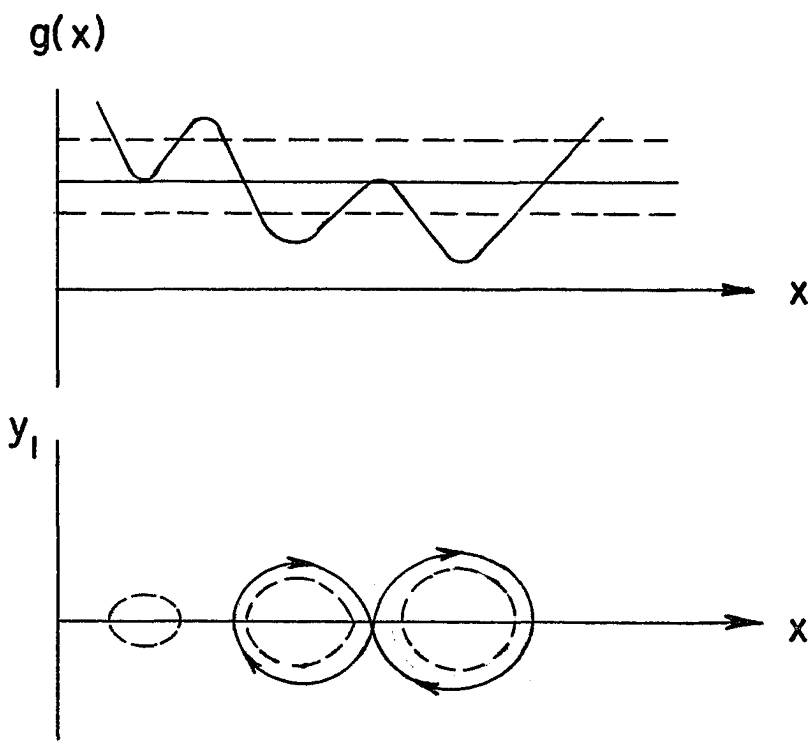


FIG. 2.7 POSSIBLE TRAJECTORIES

c) Trajectories which pass to infinity. These trajectories are also separatrices because with variations in h we get basically different trajectories.

2.6 INTEGRAL CURVES FOR A PENDULUM

To illustrate the methodology of the phase plane technique and the construction of the integral curves (trajectories), the motion of a simple pendulum will be investigated.

The motion of a pendulum is described by the equation

$$\frac{d^2\phi}{dt^2} + k^2 \sin\phi = 0 \tag{2.28}$$

in which,

k = constant = g/L;

ϕ = the angle of the pendulum with the vertical;

L = the length of the pendulum; and

g = acceleration of gravity.

It is assumed that Eq. 2.28 represents a frictionless system.

Let

$$w = \frac{d\phi}{dt},$$

then we can write

$$\begin{aligned} \frac{dw}{dt} &= -k^2 \sin\phi \\ \frac{d\phi}{dt} &= w \end{aligned} \tag{2.29}$$

From the preceeding discussion, if $z = g(\phi)$ then

$$g(\phi) = \int^{\phi} k^2 \sin\phi \, d\phi$$
$$= -k^2 \cos\phi$$

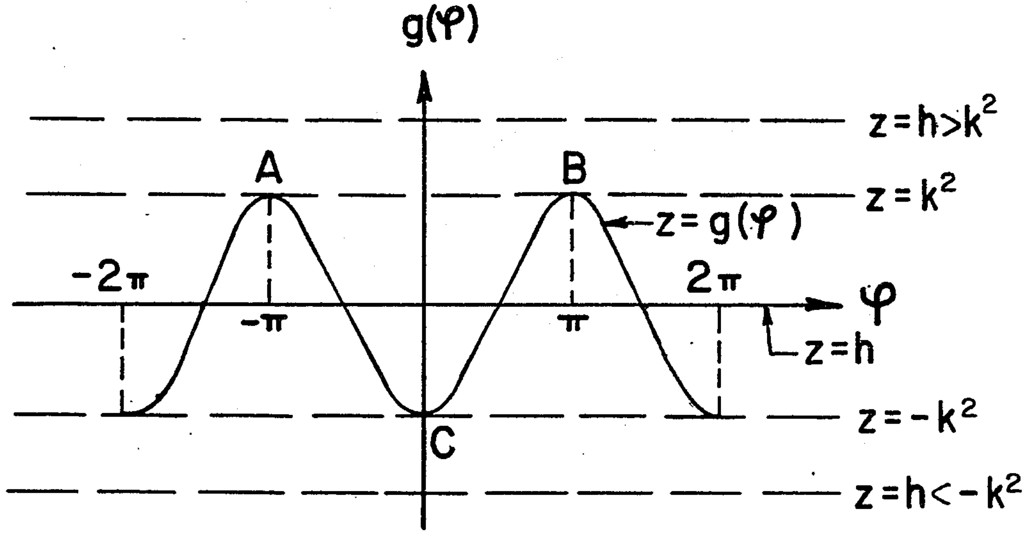
According to Eq. 2.27, the equation for the integral curves can be written as

$$\tfrac{1}{2} w^2 = h + k^2 \cos\phi \qquad (2.30)$$

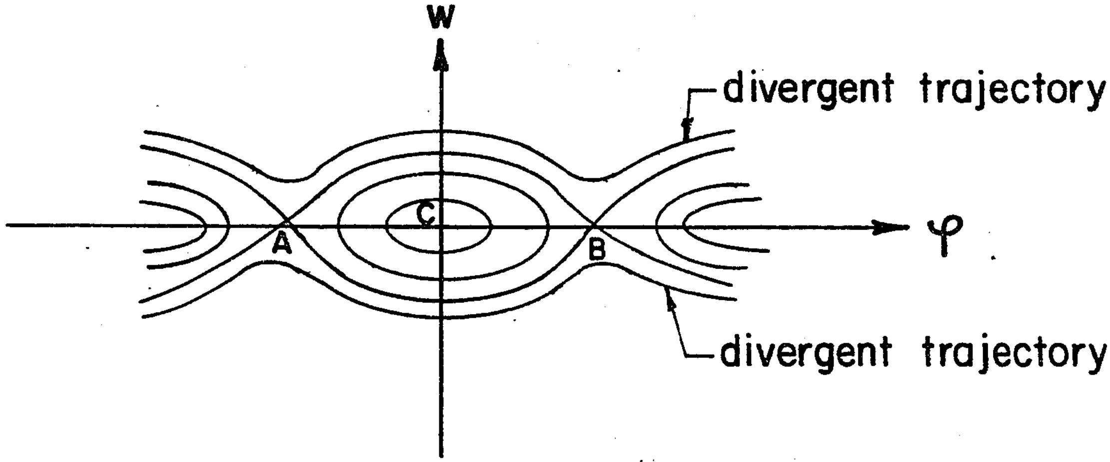
The function $z = g(x) = -k^2 \cos\phi$ is shown in Fig. 2.8. It has minimum values at $\phi = 0, \pm 2\Pi, \pm 4\Pi, \ldots$ These minima correspond to integral curves degenerated to points in the phase plane where the phase velocities are zeros, i.e., correspond to singular points called centers. Considering Lyapunovs definition of stability, these singular points are stable, surrounded by closed trajectories which correspond to <u>periodic motions</u>. The function $Z = g(\phi)$, has maxima for the values $\phi = \pm \Pi, \pm 3\Pi, \pm 5\Pi, \ldots$ On the phase plane, they correspond to unstable equilibrium states which are represented by saddle points.

From the previous discussion (Figs. 2.5, 2.6 and 2.7) and from the character of the trajectories for any fixed value of the total energy 'h', the motions of the pendulum can be explained on the phase plane for different possible cases as follows:

1. If the line $z = h$ on the ϕ-w plane intersect the curve $z = g(\phi)$ so that in Eq. 2.30 $k^2 > h > -k^2$, the trajectories are closed, surrounding singular points which are centers. These trajectories represent the periodic oscillation of the pendulum,



(a) Function $z = k^2 \cos\varphi$



(b) Trajectories

FIG. 2.8 TRAJECTORIES FOR A SIMPLE PENDULUM i.e., the pendulum cannot reach the upper position (unstable position at $\phi = \Pi$.)

2. If the line $z = h = k^2$, i.e., the line is tangent to the curve $Z = g(\phi)$ at the points A and B on the ϕ-w plane, this will give rise to two saddle points at A and B on the phase plane (Fig. 2.8). These trajectories are called separatrices.

3. If the line $z = h = -k^2$, i.e., line is tangent to the curve $z = g(\phi)$ at the point c in the ϕ-z plane. This curve is similar to that in Fig. 2.7 where the closed trajectory shrinks to one point which is a center.

4. If the line $z = h < -k^2$, i.e., the line lies below the curve $z = g(\phi)$ and does not intersect the curve anywhere. In this case the trajectory which shrunk to a center will disappear and real motions of the pendulum do not exist.

5. If the line $z = h > k^2$, i.e., the line lies above the curve $z = g(\phi)$ and does not intersect with it anywhere. There will be two symmetrical branches of the integral curve which are divergent. Since the ϕ coordinate is periodic, the motion of the representative point along the trajectories corresponds to a clockwise or counterclockwise rotation of the pendulum about its suspension point. In other words, the pendulum rotates all the time in the same direction and its velocity w keeps the

same sign but fluctuates in values between the minimum at $\phi = 0, 2\Pi, 4\Pi, \ldots$ and maximum at $\phi = \Pi, 3\Pi, 5\Pi, \ldots$

The physical significance of this analysis can be illustrated if we consider a pendulum that has been acted on by an external moment. If the moment is small, the pendulum oscillates about the new position of equilibrium as long as ϕ is less than $\Pi/2$. If the moment is large enough, the pendulum may swing over its upper unstable position of equilibrium and a rotary motion will result. If, however, the moment was such that the kinetic energy gained by the pendulum is equal to its potential energy, the pendulum will rotate to its upper position and will stay at this position.

CHAPTER 3

GOVERNING EQUATIONS FOR A CLOSED SURGE TANK SYSTEM

3.1 INTRODUCTION

Fig. 3.1 shows the schematic of a closed surge tank system. A rapid gate opening or closure due to increase or decrease in power demand will cause flow variations which in turn results in the oscillations of the water surface in the closed chamber. When the mass of water starts to discharge into the chamber, the air pressure will gradually increase which in addition to wall friction will exert an increasing retarding force on the flowing water until the water in the chamber will flow back.

To derive the governing equations (dynamic and continuity equations), the following assumptions are made:

1. The inertia of the liquid in the surge tank is small relative to that of the liquid in the tunnel and therefore can be neglected.
2. No time lag exists between the water level change in the surge chamber and its effect on the acceleration of the water mass in the tunnel.
3. The head losses in the system during the transient state can be computed by using the steady-state formula for the corresponding flow velocities, i.e.,

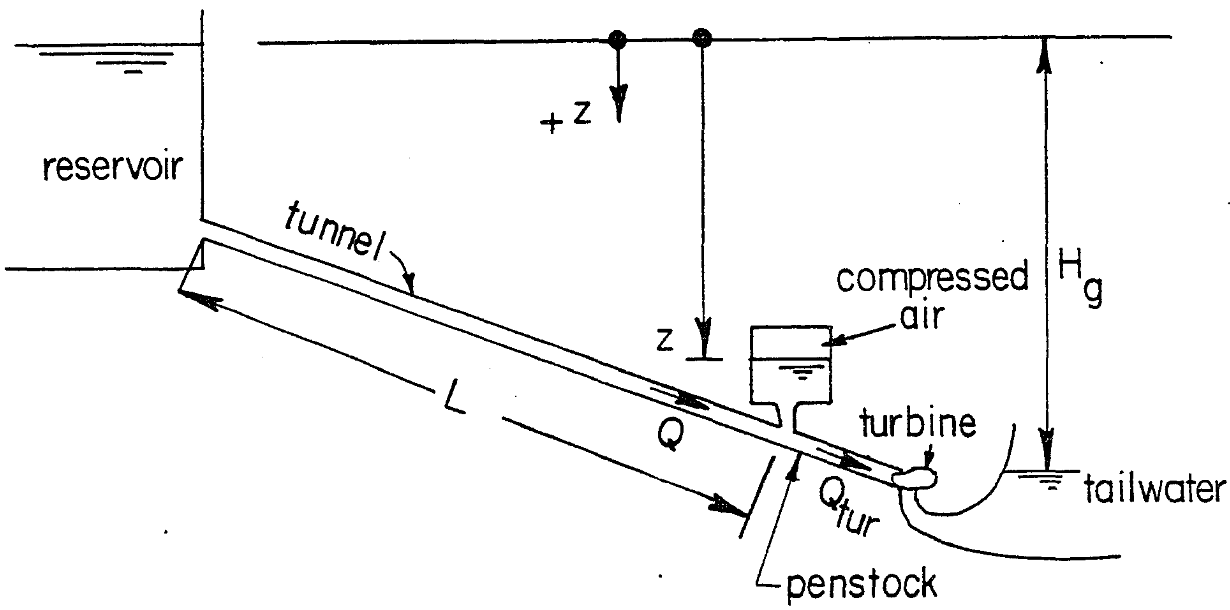


FIG. 3.1 SCHEMATIC DIAGRAM OF A HYDRAULIC SYSTEM WITH AN ENCLOSED SURGE TANK

if h_f is the instantaneous head loss in the tunnel, then h_f can be expressed by

$$h_f = k\,Q^2 \tag{3.1}$$

in which, Q represents the discharge through the tunnel.

4. Liquid is incompressible and tunnel walls are rigid.
5. Tunnel cross-section is constant. If this is not the case, then the actual tunnel cross-section may be replaced by an equivalent-area tunnel, i.e.,

$$A_e = L_e / \sum_{i=1}^{N} \frac{L_i}{A_i}$$

in which subscript e refers to equivalent tunnel and N = total number of cross sections.

6. Velocity distribution over the tunnel cross-section is uniform.
7. The minor losses and the velocity head are usually small compared to the head losses due to friction and, therefore, are neglected.
8. Air expansion and contraction in the chamber follow a polytropic law for perfect gasses, i.e.,

$$(p + p_a)\,\forall^n = (p_o + p_a)\,\forall_o^{\,n}$$

in which,

p = pressure of enclosed air;

p_a = atmospheric pressure $(p_a << p)$;

$\forall$ = air volume in the chamber;

n = the exponent in the polytropic gas equation;

p_o = absolute pressure of enclosed air at the steady state; and

$\mathcal{V}_o$ = air volume in the chamber at the steady state.

Since atmospheric pressure is very small relative to the pressure of the enclosed air in the chamber*, therefore, it will be neglected and the polytropic law for perfect gas becomes

$$p\,\mathcal{V}^n = p_o\,\mathcal{V}_o^{\,n} \tag{3.2}$$

3.2 DYNAMIC EQUATION

A freebody diagram of the tunnel having constant cross sectional area is shown in Fig. 3.2. Forces acting on the liquid are

$$F_1 = \gamma H_r A_t \tag{3.3}$$

$$F_2 = \gamma A_t (H_r + L \sin\theta - z + P) \tag{3.4}$$

$$F_3 = \gamma h_f A_t \tag{3.5}$$

$$F_4 = \gamma A_t\ L \sin\theta \tag{3.6}$$

Thus, the resultant force acting on the liquid in the tunnel is

$$\begin{aligned} \sum F_i &= \gamma H_r A_t + \gamma A_t\ L \sin\theta \\ &\quad - \gamma A_t (H_r + L \sin\theta - z + P) \\ &\quad - \gamma A_t h_f \end{aligned} \tag{3.7}$$

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* In Driva Hydroelectric Power Plant the ratio of atmospheric air pressure to the pressure of the enclosed air is less than 2%.

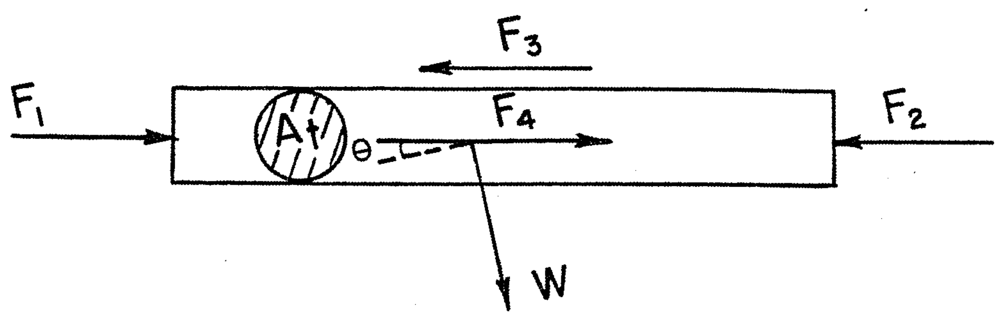


FIG. 3.2 FREEBODY DIAGRAM

in which

A_t = cross-sectional area of the tunnel;

H_r = water surface height in the reservoir;

L = tunnel length;

z = water level in the surge tank below the reservoir level (positive downward);

γ = specific weight of water;

θ = tunnel slope from the horizontal; and

p = pressure of enclosed air.

The mass of the liquid element = $\frac{\gamma}{g} A_t L$ in which g = acceleration due to gravity. According to Newton's Second Law of Motion

$$F = \frac{d(mv)}{dt} = m\,\frac{dv}{dt}$$
$$= m\,\frac{d(Q/A_t)}{dt} \qquad (3.8)$$
$$= \frac{\gamma}{g} A_t L\,\frac{d(Q/A_t)}{dt}$$

From Eqs. 3.7 and 3.8, it follows that

$$\frac{\gamma}{g} A_t L\,\frac{d(Q/A_t)}{dt} = \gamma H_r A_t + \gamma A_t\,L\sin\theta$$
$$- \gamma A_t (H_r + L\sin\theta - z + P)$$
$$- \gamma A_t h_f$$

which upon simplications becomes

$$\frac{L}{gA_t}\,\frac{dQ}{dt} = + z - P - h_f \qquad (3.9)$$

in which $h_f = kQ|Q|$ = the instantaneous frictional head in the tunnel, and Q is the tunnel discharge. To account for the reverse flow, expression for h_f was written as $kQ|Q|$.

If the air pressure in the tank would become zero, the dynamic equation (Eq. 3.9) would become identical to the dynamic equation for an open surge tank.

3.3 CONTINUITY EQUATION

Based on the principle of conservation of mass, the continuity equation at the junction of the tank and the tunnel (Fig. 3.1) may be written as

$$Q = Q_{tur} - A_s \frac{dz}{dt}$$

or

$$\frac{dz}{dt} = - \frac{1}{A_s} (Q - Q_{tur}) \qquad (3.10)$$

in which

Q_{tur} = turbine flow;

A_s = cross-sectional area of the surge tanks; and

t = time in seconds.

In both Eqs. 3.9 and 3.10, we do not have spatial variations, i.e., variations with respect to x, and the tunnel discharge and the water surface elevation in the chamber are only expressed as a function of time. This is due to the assumption that the liquid and the tunnel are rigid. Accordingly, Eqs. 3.9 and 3.10 are a set of ordinary differential equations. However, due to the presence of the frictional term $kQ|Q|$, Eq. 3.9 is nonlinear.

3.4 OSCILLATIONS AMPLITUDE FOR A FRICTIONLESS SYSTEM

Assume that the surge tank system is frictionless, i.e., k in Eq. 3.9 is zero. Also, assume that the initial steady-state turbine flow Q_o is instantaneously reduced to zero at $t = 0$. Thus

$$\text{at } t < 0 \text{ , } Q_{tur} = Q_o \tag{3.11}$$

$$\text{and at } t \geq 0 \text{ , } Q_{tur} = 0 \tag{3.12}$$

Therefore, for a frictionless system, Eq. 3.1 becomes

$$\frac{dQ}{dt} = \frac{gA_t}{L} (z - P) \tag{3.13}$$

and from initial condition Eq. 3.12, Eq. 3.10 would become

$$\frac{dz}{dt} = - \frac{Q}{A_s} \tag{3.14}$$

Differentiating Eq. 3.14 with respect to t, we obtain

$$\frac{d^2z}{dt^2} = - \frac{1}{A_s} \frac{dQ}{dt} \tag{3.15}$$

Substituting Eq. 3.13 into Eq. 3.15, we get

$$\frac{d^2z}{dt^2} = - \frac{gA_t}{LA_s} (z - P) \tag{3.16}$$

If $P = 0$, Eq. 3.16 becomes

$$\frac{d^2z}{dt^2} + \frac{gA_t}{LA_s} z = 0 \tag{3.17}$$

Let $\frac{gA_t}{LA_s} = c$.

The general solution for the homogeneous equation 3.17 is

$$z = A \cos\sqrt{c}\ t + B \sin\sqrt{c}\ t \tag{3.18}$$

Using the initial condition at $t = 0$, $z = 0$, in Eq. 3.18, we obtain $A = 0$ and Eq. 3.18 becomes

$$z = B \sin\sqrt{c}\ t \tag{3.19}$$

Now at $t = 0$, $\frac{dz}{dt} = - \frac{Q_o}{A_s}$, from Eq. 3.19

$$\frac{dz}{dt} = B\sqrt{c}$$

or

$$-\frac{Q_o}{A_s} = B\sqrt{c}$$

and therefore,

$$B = -\frac{Q_o}{A_s\sqrt{c}} \qquad (3.20)$$

Substituting this into Eq. 3.19, we obtain

$$z = -\frac{Q_o}{A_s\sqrt{c}} \sin\sqrt{c}\ t \qquad (3.21)$$

By substituting the value of c, we get the general solution

$$z = -\frac{Q_o}{A_s \frac{gA_T}{LA_s}} \sin\sqrt{\frac{gA_t}{LA_s}}\ t$$

or

$$z = -\frac{Q_o}{\sqrt{\frac{gA_tA_s}{L}}} \sin\sqrt{\frac{gA_t}{LA_s}}\ t \qquad (3.22)$$

z will be maximum if

$$\sin\sqrt{\frac{gA_t}{LA_s}} = 1 .$$

Therefore,

$$Z = Q_o\sqrt{\frac{L}{gA_tA_s}} \qquad (3.23)$$

where Z = the maximum amplitude of oscillation in an open surge tank.

3.5 PERIOD OF OSCILLATION

Eq. 3.23 represents periodic oscillations with the period of oscillation

$$T = 2\Pi \sqrt{LA_s/gA_t} \tag{3.24}$$

3.6 NORMALIZATION OF THE GOVERNING EQUATIONS

In order to minimize the number of parameters in the governing equations which describe the oscillation of the water level in a closed surge tank, the dynamic and continuity equations will be normalized. The two equations, Eqs. 3.9 and 3.10 are

$$\frac{dQ}{dt} = \frac{gA_t}{L} (z - P - h_f)$$

$$\frac{dz}{dt} = - \frac{1}{A_s} (Q - Q_{tur})$$

Let

$$y = z/Z$$

$$x = Q/Q_o \tag{3.25}$$

$$q = Q_{tur}/Q_o$$

and

$$\tau = 2\Pi t/T$$

in which

Z = maximum amplitude of the water surface oscillation in a frictionless open-surge tank system, and

Q_o = initial steady state discharge

From Eq. 3.25, it follows that

$$z = Zy$$

$$Q = Q_o x \tag{3.26}$$

$$Q_{tur} = Q_o q$$

$$t = \frac{T}{2\Pi} \tau$$

Substituting this in the dynamic equation and simplifying

$$\frac{dQ}{dt} = \frac{d(Q_o x)}{d(\frac{T}{2\Pi}\tau)} = \frac{Q_o}{T/2\Pi}\frac{dx}{d\tau}$$

Therefore,

$$\frac{Q_o}{T/2\Pi}\frac{dx}{dt} = \frac{gA_t}{L}(Zy - p - kQ_o^2\, x|x|)$$

or

$$\frac{dx}{d\tau} = \frac{gA_t T}{2\Pi L Q_o}(Zy - P - kQ_o^2\, x|x|) \tag{3.27}$$

Since we have

$$Z = Q_o\sqrt{\frac{L}{gA_t A_s}}$$

and

$$T = 2\Pi\sqrt{LA_s/gA_t}$$

and by substituting them in Eq. 3.27 and simplifying, we obtain

$$\frac{dx}{d\tau} = y - \frac{P}{Z} - \frac{kQ_o^2}{Z}\, x|x| \tag{3.28}$$

According to Fig. 3.3

$$\forall = \forall_o - A_s(z_o - z)$$

since

$$p\forall^n = p_o\,\forall_o^{\,n} \tag{3.29}$$

Dividing Eq. 3.29 by $\forall_o^{\,n}$

$$p = \frac{p_o}{\left[\dfrac{V_o - A_s(z_o - z)}{V_o}\right]^n} = \frac{p_o}{[1 - A_s(z_o - z)/\forall_o]^n} \tag{3.30}$$

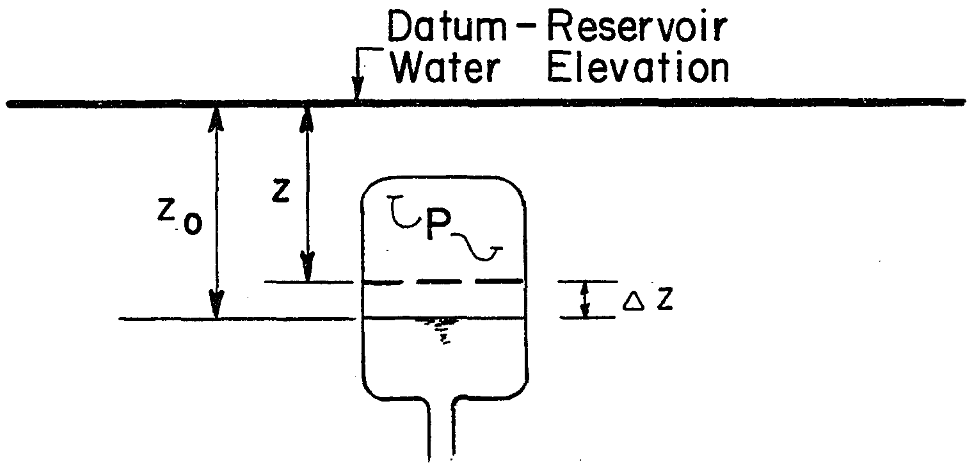


FIG. 3.3 SCHEMATIC DIAGRAM SHOWING THE RELATION BETWEEN z & ¥ IN A CLOSED TANK

By expanding the term $[1 - A_s(z_o - z)/\forall_o]^n$

$$p = p_o[1 - A_s(z_o - z)/\forall_o]^{-n}$$

$$= p_o[1 + \frac{n(z_o - z)A_s}{\forall_o} + \cdots] \tag{3.31}$$

Since $(z_o - z)A_s = \Delta\forall$ and $\Delta\forall/\forall_o << 1.0$, therefore Eq. 3.31 can be reduced to

$$p = p_o[1 + \frac{n(z_o - z)A_s}{\forall_o}] \tag{3.32}$$

Divide Eq. 3.32 by Z and substitute for z = Zy

$$\frac{p}{Z} = \frac{p_o}{Z}[1 + \frac{n(z_o - Zy)A_s}{\forall_o}]$$

$$= \frac{p_o}{Z} + \frac{np_oz_oA_s}{Z\ \forall_o} - \frac{np_oZA_sy}{Z\ \forall_o}$$

and then, we get

$$\frac{p}{Z} = (a_1 - a_2y) \tag{3.33}$$

in which

$$a_1 = \frac{p_o}{Z}\ (1 + \frac{nz_oA_s}{\forall_o}) \tag{3.34}$$

and

$$a_2 = \frac{np_oA_s}{\forall_o} \tag{3.35}$$

Let

$$a_3 = kQ_o^2/Z = h_{f_o}/Z \tag{3.36}$$

Substituting Eqs. 3.33 and 3.36 into the dynamic equation (Eq. 3.28), we finally obtain

$$\frac{dx}{d\tau} = -a_1 + (1 + a_2)y - a_3x^2 \tag{3.37}$$

Now, substitute Eq. 3.26 into the continuity equation (Eq. 3.10) and proceed in the same manner by substituting for T and Z and simplifying, Eq. 3.10 becomes

$$\frac{dy}{d\tau} = -x + q \qquad (3.38)$$

To summarize, the dynamic and the continuity equations in the non-dimensional form are:

$$\frac{dx}{d\tau} = -a_1 + (1 + a_2)y - a_3x^2$$

$$\frac{dy}{d\tau} = -x + q$$

in which the constants a_1, a_2 and a_3 are previously defined (Eqs. 3.34, 3.35 and 3.36) in terms of the initial steady-state parameters p_o, z_o, Q_o and $\forall_o$, the area of the surge tank A_s, the friction coefficient k, the maximum amplitude of oscillations of an open surge tank Z and the exponent in the polytropic gas equation n.

The term q in the continuity equation will be defined according to the demand condition. This is discussed in the following section.

3.7 FLOW CONDITIONS

Four cases of turbine flow demand for simple surge tank have been investigated by Chaudhry and Ruus [2]. These cases are:

a) Constant Flow: Practically, there is no way of keeping the turbine discharge constant during the surge condition. In the case of a very high head installation where oscillations in the surge tank are a small fraction compared to the static head, an approximation of a constant flow is possible and accurate results can be achieved from relatively simple analysis. Therefore, in this case

$$Q_{tur} = \text{constant}$$

and since, in this case, $Q_{tur} = Q_o$ therefore,

$$q = 1 . \tag{3.39}$$

b) Constant Gate Opening: This case occurs during downsurges when the gates have been opened to the maximum position while the governor is regulating for constant power or when the plant is under manual control after a change in load demand or when the governor is inoperative due to a malfunction.

Under this condition the turbine discharge Q_{tur} can be approximated as follows: At the steady state and from Fig. 3.4,

$$H_n = H_g - (z - P)$$

$$= H_g - h_{f_o}$$

Therefore,

$$Q_{tur} = \frac{H_g - (z - p)}{H_g - h_{f_o}} Q_o$$

Then, by substituting for z = Zy and dividing by Z, we obtain

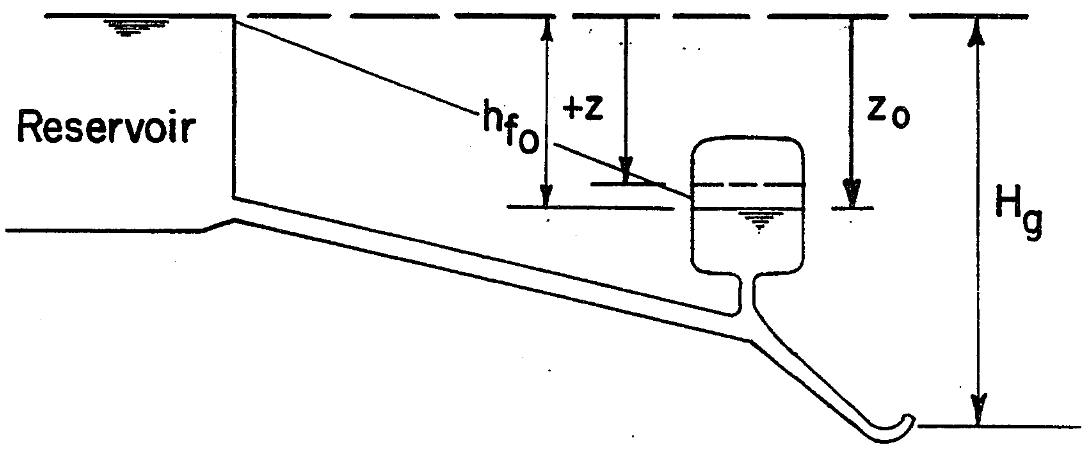


FIG. 3.4 SCHEMATIC DIAGRAM OF A HYDRAULIC SYSTEM AT THE STEADY STATE

$$q = \frac{Q_{tur}}{Q_o}$$

$$= \frac{H_g - Zy + P}{H_g - h_{f_o}} \qquad (3.40)$$

$$= \frac{H_g/Z - y + P/Z}{H_g/Z - h_{f_o}/Z}$$

Substituting from Eqs. 3.33 and 3.36, Eq. 3.40 becomes

$$q = \frac{a_4 - y + (a_1 - a_2)y}{a_4 - a_3}$$

$$= \frac{a_1 + a_4 - (1 + a_2)y}{a_4 - a_3} \qquad (3.41)$$

in which

$$a_4 = H_g/Z \qquad (3.42)$$

c) Constant Power:

In this case the turbine governor regulates the flow in such a manner that maintains constant power input if the turbine efficiency is to be considered constant. Following a load increase the governor opens the gates, additional water is withdrawn from the surge tank and the water level is lowered. Accordingly, the net head on the turbine is reduced and therefore, the governor has to open the gates further to maintain a constant power. No restriction on the turbine-gate opening is assumed, which implies that the turbine discharge can be increased to any required amount to keep the power constant.

It has been found that in an open surge tank system where the turbine is governed automatically for a constant hydraulic power, unstable oscillations are possible [1,2].

The governor action of a turbine is represented by the equation

$$P_{w_o} = \eta_o Q_o H_{n_o} = \text{constant} \qquad (3.43)$$

since

$$H_{n_o} = H_{g_o} - h_{f_o}$$

$$= P_o + (H_{g_o} - z_o)$$

Therefore, by substituting for H_{n_o} into Eq. 3.43, we obtain

$$P_{w_o} = \eta_o Q_o [P_o + (H_g - z_o)]$$

$$= \eta_o Q_o [H_{g_o} - h_{f_o}] \qquad (3.44)$$

and

$$P_w = \eta Q_{tur} [P + (H_g - z)] \qquad (3.45)$$

Thus, from Eqs. 3.44 and 3.45, Eq. 3.43 becomes

$$\eta_o Q_o (H_{g_o} - h_{f_o}) = \eta Q_{tur} [P + (H_g - z)]$$

If it is assumed that the efficiency of the turbine 'η' does not change with changes in water level in the tank or in the gate opening

$$q = \frac{Q_{tur}}{Q_o} = \frac{H_{g_o} - h_{f_o}}{P + (H_{g_o} - z)}$$

Dividing the numerator and the denominator of the above expression for q by Z, substituting for z = Zy and simplifying, we obtain

$$q = \frac{Q_{tur}}{Q_o} = \frac{H_{g_o}/Z - h_{f_o}/Z}{P/Z + H_{g_o}/z - y}$$

$$= \frac{}{(a_1 - a_2 y) + a_4 - y}$$

or

$$q = \frac{a_4 - a_3}{a_1 + a_4 - (1 + a_2)y} \qquad (3.46)$$

in which, a_1, a_2, a_3 and a_4 are constants and as defined previously by Eqs. 3.34, 3.35, 3.36 and 3.42.

d) <u>Constant Power Combined With Constant Gate Opening</u>: Analysis of this case depends upon the relation between the turbine discharge and the rated head. This will be discussed in Chapter 4.

CHAPTER 4

PHASE PLANE ANALYSIS

4.1 INTRODUCTION

The differential equations governing the oscillations of the water level in a surge tank are nonlinear. The usual approach to solve these equations has been to linearize them by neglecting all the nonlinear terms or by linearizing the nonlinear terms. The stability of open surge tank has been studied without neglecting the nonlinear terms. Marris [7] and Sideriadas [8] used the phase plane method to investigate the stability of a simple surge tank for the case of constant power. Chaudhry and Ruus [2], also, utilized the phase plane technique to investigate the stability of oscillations for four cases: constant flow, constant gate opening, constant power, and constant power combined with constant gate opening.

Most of the tank stability investigations have been for open surge tanks. Because of the recent introduction of closed surge tanks for application in hydroelectric power plants, very little work has been done to establish a stability criterion for a closed surge chamber.

The main objective of this chapter is to develop a stability criterion for a closed surge tank system by using non-linear analysis so that it would be valid for both small and large oscillations. The stability of these tanks will be investigated by the phase plane method which allows inclusion of nonlinear terms in the analysis. This method has been described in Chapter 2. This method is particularly used in connection with non-linear differential equations, since a closed form solution is not available.

Singularities of the governing equations will be determined and analyzed to develop the stability criteria for four cases of turbine flow demand which has been investigated for open surge tank by Chaudhry and Ruus [2].

4.2 SINGULARITIES

A. Constant Flow

As derived in Chapter 3, the normalized governing equations are

$$\frac{dx}{d\tau} = -a_1 + (1+a_2)y - a_3x^2 \quad (3.37)$$

$$\frac{dy}{d\tau} = -x + q \quad (3.38)$$

in which

$$q = \frac{Q_{tur}}{Q_o} = 1.0$$

By eliminating τ, Eqs. 3.37 and 3.38 reduce to

$$\frac{dy}{dx} = \frac{-x + q}{-a_1 + (1+a_2)y - a_3x^2} \quad (4.1)$$

A point (x_s, y_s), where

$$-x_s + q = 0 \quad (4.2)$$

and

$$-a_1 + (1+a_2)y_s - a_3x_s^2 = 0 \quad (4.3)$$

is an equilibrium condition since $dx/d\tau=0$ and $dy/d\tau=0$

at this point. Mathematically this point is called a singular point as has been indicated in Chapter 2.

Therefore, to determine the singularities, the two equations 4.2 and 4.3 should be solved for x_s and y_s. From equation 4.2 we get

$$x_s = q = 1 \ . \tag{4.4}$$

Substituting Eq. 4.4 in Eq. 4.3, and solving for y_o, we obtain $-a_1 + (1+a_2)y_s - a_3 = 0$. Then,

$$y_s = \frac{a_1 + a_3}{1 + a_2} \tag{4.5}$$

Therefore, for the case of constant flow condition, there exists one singularity with the following coordinates:

$$x_s = 1.0$$

$$y_s = \frac{a_1 + a_3}{1 + a_2} \tag{4.6}$$

An equilibrium condition is said to be <u>stable</u> if, upon slight disturbance, the departure from this condition remains small. Otherwise, the equilibrium condition is said to be <u>unstable</u>. To determine whether this singular point is stable or not, it is required to find a solution which is valid in the neighborhood of this singular point. If such a solution shows that upon being displaced to a new position in this neighborhood x and y will approach the singular point (x_s,y_s), then the singular point is stable. If x and y will recede further from the singularity x_s,y_s, then the singularity is unstable.

Therefore, to study this singularity and its neighborhood, let

$$u = x - x_s$$

$$v = y - y_s$$

and

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}$$

in which

$$Q(x,y) = -x + 1.0 \qquad (4.7)$$

$$P(x,y) = -a_1 + (1 + a_2)y - a_3x^2 \qquad (4.8)$$

By expanding the two functions Q and P into Taylor's series about the point (x_s, y_s), we get

$$Q(x,y) = Q(x_s, y_s) + \left.\frac{\partial Q}{\partial x}\right|_s (x - x_s) + \left.\frac{\partial Q}{\partial y}\right|_s (y - y_s) + \cdots \qquad (4.9)$$

and

$$P(x,y) = P(x_s, y_s) + \left.\frac{\partial P}{\partial x}\right|_s (x - x_s) + \left.\frac{\partial P}{\partial y}\right|_s (y - y_s) + \cdots \qquad (4.10)$$

Since by definition $Q(x_s, y_s) = P(x_s, y_s) = 0$, and by taking the partial derivatives of the two functions Q and P, and neglecting higher power of u and v since u and v are small, Eqs. 4.9 and 4.10 become

$$Q(x,y) = -u$$

and

$$P(x,y) = -2a_3x_su + (1 + a_2)v$$

and accordingly,

$$\frac{dy}{dx} = \frac{dv}{du} = \frac{-u}{-2a_3x_su + (1 + a_2)v} \quad (4.11)$$

The general form of Eq. 4.11 as defined in Chapter 2 is:

$$\frac{dy}{dx} = \frac{c'u + d'v + c''u^2 + d''v^2 + \cdots}{a'u + b'v + a''u^2 + b''v^2 + \cdots} \quad (4.12)$$

In the case of a simple singularity, where both linear and higher power terms in u and v are present in the denominator or in the numerator, or both, the higher-power terms can be neglected because their effect on the solution in the neighborhood of the singularity is small as compared to that of the linear terms. In the case of non-simple singularity where the linear terms are missing, the higher-power terms cannot be neglected.

Therefore, Eq. 4.12 can be rewritten as

$$\frac{dy}{dx} = \frac{dv}{du} = \frac{c'u + d'v}{a'u + b'v} \quad (4.13)$$

This is equivalent to the two equations

$$\frac{dv}{dt} = c'u + d'v \quad (4.14)$$

and

$$\frac{du}{dt} = a'u + b'v \quad (4.15)$$

The characteristic roots of the two equations as determined in Chapter 2 are:

$$\lambda_1, \lambda_2 = \tfrac{1}{2}[(a' + d') \pm \sqrt{(a' + d')^2 + 4(b'c' - a'd')}] \qquad (4.16)$$

The two equations 4.11 and 4.13 are similar. By comparison, we find out that in the case of a constant flow condition

$$a' = -2a_3x_s$$

$$b' = (1 + a_2)$$

$$c' = -1.0$$

$$d' = 0$$

By substituting the values of a', b', c' and d' into Eq. 4.16, the characteristic roots become

$$\lambda_1, \lambda_2 = \tfrac{1}{2}[(-2a_3) \pm \sqrt{(-2a_3)^2 + 4[(1 + a_2)(-1)}$$

$$= [-a_3 \pm \sqrt{4a_3^2 - 4(1 + a_2)}] \qquad (4.17)$$

Now, according to the criteria established in Chapter 2, singularity is <u>stable</u> if one of the following conditions are satisfied:

1. λ_1 and λ_2 are real and negative.
2. λ_1 and λ_2 are complex conjugates with negative real parts.

Singularity is <u>unstable</u> if one of the following conditions is satisfied:

1. λ_1 and λ_2 are real and have positive signs.
2. λ_1 and λ_2 are real and with different signs.
3. λ_1 and λ_2 are complex conjugates with positive real parts.

If $a_3^2 > (1 + a_2)$, both roots are real and have the same negative sign. In such case, singularity is a stable node. If $a_3^2 < (1 + a_2)$, the roots are complex conjugates with negative real parts. The singularity is, therefore, a stable focus.

Eqs. 3.34, 3.35 and 3.36 in Chapter 3 define the constants a_1, a_2 and a_3 as follows

$$a_1 = \frac{p_o}{Z}\left(1 + \frac{nz_oA_s}{\forall_o}\right)$$

$$a_2 = \frac{nP_oA_s}{\forall_o}$$

and

$$a_3 = h_{f_o}/Z$$

From the definition of a_3, the first condition for stable node implies that $h_{f_o} > Z$.

Comparison between the two conditions for stable node and stable focus shows that the condition for stable focus requires less friction than the condition for stable node.

From Eq. 4.17, if $4(1 + a_2) < 0$, λ_1 and λ_2 are real but with opposite signs. In this case singularity is a saddle. However, since a_2 is always positive, therefore, this condition is impossible.

In general, if

$$(h_{f_o}/Z)^2 > \left(1 + \frac{nP_oA_s}{\forall_o}\right) \qquad (4.18)$$

we get a stable node.

If

$$(h_{f_o}/Z)^2 < (1 + \frac{nP_oA_s}{\forall_o}) \qquad (4.19)$$

we get a stable focus. A saddle is impossible in this case as previously discussed.

B. Constant Gate Opening

This mode of operation occurs if the governor has opened the wicket gate to full open position while trying to maintain constant power, or when the turbine is under manual control after a change in load.

The turbine characteristics in reference 15 show that the flow through a Francis turbine operating at full gate and at constant speed is approximately proportional to the net head. Hence, neglecting penstock losses and velocity head, q can be defined as shown in Chapter 3 (Eq. 3.41),

$$q = \frac{Q_{tur}}{Q_o} = \frac{a + a_4 - (1 + a_2)y}{a_4 - a_3}$$

where a_1, a_2, a_3 and a_4 are constants and as previously defined.

The normalized governing equations are

$$\frac{dx}{d\tau} = -a_1 + (1 + a_2)y - a_3x^2 \qquad (3.37)$$

$$\frac{dy}{d\tau} = -x + q \qquad (3.38)$$

Substituting the value for q in Eq. 3.38, we get

$$\frac{dy}{d\tau} = -x + \frac{a_1 + a_4 - (1 + a_2)y}{a_4 - a_3}$$

By eliminating $d\tau$ from Eqs. 3.37 and 3.38, we obtain

$$\frac{dy}{dx} = \frac{-x + \dfrac{a_1 + a_4 - (1 + a_2)y}{a_4 - a_3}}{-a_1 + (1 + a_2)y - a_3x^2} \tag{4.20}$$

To determine the singularities, we have to solve the following equations:

$$-x_s + \frac{a_1 + a_4 - (1 + a_2)y_s}{a_4 - a_3} = 0 \tag{4.21}$$

and

$$-a_1 + (1 + a_2)y_s - a_3x_s^2 = 0 \tag{4.22}$$

Let

$$(a_1 + a_4) = A$$

$$(a_4 - a_3) = B \tag{4.23}$$

and

$$(1 + a_2) = C$$

Substituting into Eq. 4.21 and 4.22,

$$-x_s + \frac{A - cy_s}{B} = 0 \tag{4.24}$$

$$-a_1 + cy_s - a_3x_s^2 = 0 \tag{4.25}$$

Simplifying Eq. 4.24, we get

$$-Bx_s + A - cy_s = 0$$

from which

$$y_s = \frac{A - Bx_s}{c} \tag{4.26}$$

From Eq. 4.25 we get

$$y_s = \frac{a_1 + a_3x_s^2}{c} \tag{4.27}$$

Therefore, according to Eqs. 4.26 and 4.27 and by simplifying, we obtain

$$a_3 x_s^{\,2} + Bx_s - A + a_1 = 0 \qquad (4.28)$$

substituting the values for A and B into Eq. 4.28

$$a_3 x_s^{\,2} + (a_4 - a_3)x_s - a_1 - a_4 + a_1 = 0$$

Thus,

$$x_s^{\,2} + (\frac{a_4}{a_3} - 1)\; x_s - \frac{a_4}{a_3} = 0 \qquad (4.29)$$

The solution of this quadratic equation is

$$x_{s1},\ x_{s2} = \frac{-(\frac{a_4}{a_3} - 1) \pm \sqrt{(\frac{a_4}{a_3} - 1)^2 + 4\,\frac{a_4}{a_3}}}{2}$$

By expanding the term $(\frac{a_4}{a_3} - 1)^2$ under the square root and simplifying the solution for the two roots, we obtain,

$$x_{s1} = 1.0$$

and (4.30)

$$x_{s2} = -\,\frac{a_4}{a_3}$$

Substituting x_{s1} and x_{s2} into Eq. 4.27, we obtain

$$y_{s1} = \frac{a_1 + a_3}{1 + a_2} \qquad (4.31)$$

$$y_{s2} = \frac{a_1 + a_3(-\,\frac{a_4}{a_3})^2}{1 + a_2}$$

$$= \frac{a_1 + a_4^2/a_3}{1 + a_2}$$

$$= \frac{a_1 a_3 + a_4^2}{a_3(1+a_2)} \qquad (4.32)$$

Accordingly, for this condition of constant gate opening, Eq. 4.20 has two singular points

$$\left[1.0 \; , \; \frac{a_1 + a_3}{1 + a_2} \right]$$

and

$$\left[-\frac{a_4}{a_3} \; , \; \frac{a_1 a_3 + a_4^2}{1 + a_2} \right]$$

Following the same procedure as previously outlined,

$$Q(x,y) = -x + \frac{a_1 + a_4 - (1 + a_2)y}{a_4 - a_3} \qquad (4.33)$$

and

$$P(x,y) = -a_1 + (1 + a_2)y - a_3 x^2 \qquad (4.34)$$

By expanding the two functions Q and P around x_s, y_s, considering $Q(x_s, y_s) = P(x_s, y_s) = 0$ and substituting $u = x - x_s$, $v = y - y_s$, therefore

$$Q(x,y) = -u + \frac{-(1 + a_2)}{a_4 - a_3} v + \cdots \qquad (4.35)$$

and

$$P(x,y) = -2a_3 x_s u + (1 + a_2)v + \cdots \qquad (4.36)$$

Then

$$\frac{dy}{dx} = \frac{-u - \dfrac{1 + a_2}{a_4 - a_3} v}{-2a_3 x_s u + (1 + a_2)v} \qquad (4.37)$$

When Eq. 4.37 is compared with the general form (Eq. 4.13), one obtains

$$
\begin{aligned}
\dot{a}_1 &= -2a_3 x_{s1} = -2a_3 \\
\dot{a}_2 &= -2a_3 x_{s2} = 2a_4 \\
\dot{b} &= 1 + a_2 \\
\dot{c} &= -1.0 \\
\dot{d} &= -\frac{1 + a_2}{a_4 - a_3}
\end{aligned}
\qquad (4.39)
$$

The two singular points are analyzed as follows:

1. First Singularity: $[1.\ ,\ \frac{a_1 + a_3}{1 + a_2}]$

Substituting Eq. 4.39 into Eq. 4.16, the characteristic roots become

$$
\lambda_1, \lambda_2 = \left[\tfrac{1}{2} \left\{ -\left(2a_3 + \frac{1+a_2}{a_4-a_3}\right) \pm \sqrt{\left(2a_3 + \frac{1+a_2}{a_4-a_3}\right)^2 - 4\left[(1+a_2) + 2a_3\left(\frac{1+a_2}{a_4-a_3}\right)\right]} \right\} \right. \qquad (4.40)
$$

The characteristic roots λ_1 and λ_2 are real with a negative sign if the following condition is satisfied:

$$
\left[2a_3 + \frac{1+a_2}{a_4-a_3}\right]^2 > 4\left[(1+a_2) + 2a_3\left(\frac{1+a_2}{a_4-a_3}\right)\right]
$$

Expanding the left hand side of the inequality and simplifying we obtain

$$
\left(2a_3 - \frac{1+a_2}{a_4-a_3}\right)^2 > 4(1+a_2) \qquad (4.41)
$$

If the inequality 4.41 is satisfied, the singularity is a stable node.

If this inequality is not satisfied, then the roots are complex conjugates with negative real parts and hence the singularity is stable focus.

If the system is assumed to be frictionless, i.e., $a_3 = h_{f_o}/Z = 0$, then the characteristic roots become

$$\lambda_1, \lambda_2 = \tfrac{1}{2}\left[-\frac{1+a_2}{a_4} \pm \sqrt{\left(\frac{1+a_2}{a_4}\right)^2 - 4(1+a_2)}\right] \quad (4.42)$$

Singularity is a stable node if

$$\left(\frac{1+a_2}{a_4}\right)^2 > 4(1+a_2)$$

or by simplifying, the condition for a stable mode becomes

$$1+a_2 > 4a_4^2$$

If $(1+a_2) < 4a_4^2$, singularity is a stable focus.

Therefore, it can be concluded that the first singularity is always stable. If $a_2 = 0$, i.e., $p_o = 0$ (open surge tank) the same results hold. This confirms the results obtained by Chaudhry and Ruus [10] who investigated the stability of open surge tank system by the phase plane method technique.

2. Second Singularity: $\left[-\frac{a_4}{a_3}, \frac{a_1 a_3 + a_4^2}{a_3(1+a_2)}\right]$

The characteristic roots, in this case are

$$\lambda_1, \lambda_2 = \frac{1}{2}\left\{2a_4 - \frac{1+2_a}{a_4-a_3} \pm \sqrt{\left(2a_4 - \frac{1+a_2}{a_4-a_3}\right)^2 - 4\left[(1+a_2) - 2a_4\left(\frac{1+a_2}{a_4-a_3}\right)\right]}\right\}$$

(4.43)

Singularity is stable node or focus if the following condition is satisfied:

$$2a_4 < \frac{1+a_2}{a_4-a_3} \tag{4.44}$$

and unstable node or focus if

$$2a_4 > \frac{1+a_2}{a_4-a_3} \tag{4.45}$$

However, since $x_{s2} < 0$ and since Eqs. 3.37 and 3.38 are not valid for $x < 0$, therefore, this singularity is virtual.

C. <u>Constant Power</u>

The normalized governing equations are

$$\frac{dx}{d\tau} = -a_1 + (1+a_2)y - a_3x^2 \tag{3.37}$$

$$\frac{dy}{d\tau} = -x + q \tag{3.38}$$

and for the constant power condition

$$q = \frac{a_4 - a_3}{a_1 + a_4 - (1+a_2)y} \tag{3.46}$$

By substituting Eq. 3.46 into Eq. 3.38 and eliminating $d\tau$, we obtain

$$\frac{dy}{dx} = \frac{-x + \dfrac{a_4 - a_3}{a_1+a_4 - (1+a_2)y}}{-a_1 + (1+a_2)y - a_3x^2} \qquad (4.46)$$

To determine the singularities, the following two equations should be simultaneously solved:

$$-x_s + \frac{a_4 - a_3}{a_1 + a_4 - (1+a_2)y_s} = 0 \qquad (4.47)$$

and

$$-a_1 + (1 + a_2)y_s - a_3x_s^2 = 0 \qquad (4.48)$$

Let

$$a_4 - a_3 = A$$

$$a_1 + a_4 = B \qquad (4.49)$$

and

$$1 + a_2 = C$$

From Eqs. 4.47 and 4.49, we obtain

$$x_s = \frac{A}{B - cy_s} \qquad (4.50)$$

Substituting Eq. 4.49 into Eq. 4.48 and simplifying, y_s can be defined as follows:

$$y_s = \frac{a_1}{c} + \frac{a_3}{c} x_s^2 \qquad (4.51)$$

From Eq. 4.50, we obtain

$$y_s = \frac{B}{c} - \frac{A}{cx_s} \qquad (4.52)$$

It follows from Eqs. 4.49, 4.51 and 4.52 and after simplifying that

$$a_3(x_s^3-1) - a_4(x_s-1) = 0 \qquad (4.53)$$

Eq. 4.53 shows that $x_{s1} = 1.0$ is one solution of the equation

$$a_3 x_s^3 - a_4 x_s + a_4 - a_3 = 0 \qquad (4.54)$$

Dividing Eq. 4.54 by (x_s-1), we obtain

$$x_s^2 + x_s + (1 - \frac{a_4}{a_3}) = 0$$

from which the two remaining roots are determined. The two roots are

$$x_{s2}, s_3 = \tfrac{1}{2}\left[-1 \pm \sqrt{1-4\ (1 - \frac{a_4}{a_3})}\right] \qquad (4.55)$$

By substituting $x_{s1} = 1$ and Eq. 4.55 in Eq. 4.51, it follows that

$$y_{s1} = \frac{a_1 + a_3}{1 + a_2}$$

$$y_{s2,s3} = \frac{a_1 + a_{3/4}\left[-1 \pm \sqrt{1-4\ 1 - \frac{a_4}{a_3}}\right]^2}{1 + a_2} \qquad (4.56)$$

Accordingly, for the case of constant power condition, there are three singularities (x_{s1}, y_{s1}), (x_{s2}, y_{s2}) and (x_{s3}, y_{s3}).

To determine the type of singularities, we follow the same procedure outlined in the two cases of constant flow and constant gate opening.

The two functions $Q(x,y)$ and $P(x,y)$ are

$$Q(x,y) = -x + \frac{a_4 - a_3}{a_1 + a_4 - (1+a_2)y} \qquad (4.57)$$

$$P(x,y) = -a_1 + (1+a_2)y - a_3x^2 \qquad (4.58)$$

Expanding P and Q around x_s, y_s and assuming that $u = x - x_s$ and $v = y - y_s$ and considering that $P(x_s,y_s) = Q(x_s,y_s) = 0$, then

$$\frac{dy}{dx} = \frac{dv}{du} = \frac{-u + \dfrac{(1+a_2)(a_4-a_3)}{[a_1+a_4 - (1+a_2)y_s]^2}\, v}{-2a_3x_su + (1+a_2)v} \qquad (4.59)$$

From Eq. 4.59 and by comparing it with Eq. 4.13, it follows that

$$\begin{aligned} a' &= -2a_3x_s \\ b' &= (1+a_2) \\ c' &= -1.0 \\ d' &= \frac{(1+a_2)(a_4-a_3)}{[a_1+a_4 - (1+a_2)y_s]^2} \end{aligned} \qquad (4.60)$$

in which x_s and y_s stand for the three singularity coordinates.

1. <u>First Singularity</u>: (x_{s1},y_s)

The characteristic roots for this singularity are

$$\lambda_1,\lambda_2 = \tfrac{1}{2}[-A \pm \sqrt{A^2-4B}] \qquad (4.61)$$

in which

$$A = 2a_3 - (1+a_2)/(a_4-a_3)$$

$$B = (1+a_2) - 2a_3(1+a_2)/(a_4-a_3)$$

If $A^2 > 4B$ and $4B > 0$, both roots are real and therefore the singularity is a node. If $A^2 < 4B$, both roots are complex conjugates and the singularity is a focus. If $4B < 0$, i.e.,

$$4[(1+a_2) - 2a_3(1+a_2)/(a_4-a_3)] < 0 ,$$

both roots are real but with opposite signs and therefore singularity is a saddle. By simplifying this inequality, it reduces to

$$2a_3 > (a_4-a_3) \qquad (4.62)$$

Substituting expressions for the constants a_3 and a_4 and simplifying inequality 4.62 becomes $h_{f_o} > \frac{1}{3}H_g$. In other words, for a saddle point to exist a large head loss has to exist. However, such a large head loss is not economically feasible and accordingly this case is of academic interest only.

It follows from Eq. 4.61 that the node or focus will be stable if

$$2a_3 > \frac{1+a_2}{a_4-a_3} \qquad (4.63)$$

By substituting expressions for a_2, a_3, a_4 and Z, the condition for stable node or focus becomes

$$A_s > \frac{Q_o^2\ L}{2gA_t h_{f_o}(H_g-h_{f_o})}\left(1 + \frac{nP_oA_s}{\forall_o}\right) \qquad (4.64)$$

For perpetual oscillations

$$(A_s)_{cr} = \frac{Q_o^2\ L}{2gA_t h_{f_o}(H_g-h_{f_o})}\left(1 + \frac{nP_oA_s}{\forall_o}\right) \qquad (4.65)$$

For an open surge tank, i.e., $p_o = 0$, Eq. 4.65 is reduced to the expression for the well-known Thoma area

$$(A_s^{open})_{cr} = A_{Thoma} = \frac{Q_o^2\, L}{2gA_t h_{f_o}(H_g - h_{f_o})} \qquad (4.66)$$

Let a_o = distance between the roof of the tank and the initial-steady-state tank water level, i.e., $a_o = \forall_o/A_s$. Substituting this into Eq. 4.65, the following expression for stable oscillations is obtained

$$(A_s)_{cr} = (A_s^{open})_{cr}\,(1 + \frac{np_o}{a_o}) \qquad (4.67)$$

which is the same expression as obtained by Svee [16] by using a linearized analysis.

2. <u>Second Singularity</u>: (x_{s2}, y_{s2})

$$x_{s2} = \tfrac{1}{2}\left[-1 + \sqrt{-3 + \frac{4a_4}{a_3}}\right]$$

$$y_{s2} = \frac{a_1 + a_{3/4}\left[-1 + \sqrt{1-4(1 - \frac{a_4}{a_3})}\right]^2}{1 + a_2} \qquad (4.68)$$

Substituting Eq. 4.68 into Eq. 4.60 and replacing a_1, a_2, a_3 and a_4 by their expressions, we obtain

$$\begin{aligned} \dot{a} &= -2a_3 x_{s2} \\ &= \frac{h_{f_o}}{Z}\left[1 - \sqrt{-3 + \frac{4H_g}{h_{f_o}}}\right] \end{aligned} \qquad (4.69)$$

and

$$d' = \frac{(1 + a_2)(a_4 - a_3)}{\left\{a_1 + a_4 - (1+a_2)\dfrac{a_1 + a_{3/4}\left[-1 + \sqrt{1-4\ 1 - \dfrac{a_4}{a_3}}\right]^2}{1 + a_2}\right\}^2}$$

$$= \frac{\left[1 + \dfrac{nP_oA_s}{\mathcal{V}_o}\right]\left[H_g - h_{f_o}\right]/Z}{\left\{\dfrac{H_g}{Z} - \dfrac{h_{f_o}}{4Z}\left[-1 + \sqrt{-3 + \dfrac{4H_g}{h_{f_o}}}\right]^2\right\}^2} \tag{4.70}$$

$$= \frac{4\left[1 + \dfrac{nP_oA_s}{\mathcal{V}_o}\right]\left[H_g - h_{f_o}\right]Z}{\left\{H_g - h_{f_o}\left[1 + \sqrt{-3 + \dfrac{4H_g}{h_{f_o}}}\right]\right\}^2}$$

b' and c' remain the same as in Eq. 4.60.

This singularity will be a stable node if

$$(a' + d') < 0$$

and (4.71)

$$-(a' + d')^2 < 4(b'c' - a'd') < 0$$

Singularity will be a stable focus if

$$(a' + d') < 0$$

and (4.72)

$$-(a' + d')^2 > 4(b'c' - a'd') < 0$$

If $4(b'c' - a'd') > 0$, singularity will be a saddle.

Eq. 4.69 shows that a' is a real number since $H_g > h_{f_o}$. However, a' will be negative if $4H_g/h_{f_o} > 1.0$ and positive if $4H_g/h_{f_o} < 1.0$. Since $H_g >> h_{f_o}$, the latter case is practically impossible and therefore a' will always be negative.

Eq. 4.70 shows that $\dot{d}$ is always positive. Accordingly, from the first condition of inequalities 4.71 and 4.72, singularity will be a stable focus or node if $\dot{d} < 0$ or if $|\dot{d}| < |\dot{a}|$.

3. Third Singularity

$$x_{s3} = \tfrac{1}{2}\left[-1 - \sqrt{-3 + \frac{4a_4}{a_3}}\right]$$

$$y_{s3} = \frac{a_1 + a_{3/4}\left[-1 - \sqrt{-3 + \frac{4a_4}{a_3}}\right]^2}{1 + a_2} \tag{4.73}$$

Following the same procedure in the preceding singularity, we obtain

$$\dot{a} = h_{f_o}/Z\left[1 + \sqrt{-3 + \frac{4H_g}{h_{f_o}}}\right] \tag{4.74}$$

and

$$\dot{d} = \frac{4\left(1 + \frac{nP_oA_s}{\forall_o}\right)(H_g - h_{f_o})Z}{\left[H_g - h_{f_o}\left\{1 - \sqrt{-3 + \frac{4H_g}{h_{f_o}}}\right\}\right]^2} \tag{4.75}$$

Since in this case $\dot{a}$ and $\dot{d}$ are both always positive, therefore $(\dot{a} + \dot{d}) > 0$ and the condition of stability for a node or a focus is not satisfied.

This singularity will be a saddle if $4(\dot{b}\dot{c} - \dot{a}\dot{d}) > 0$. For typical installations, it will be found in Chapter 5 that this singularity is a saddle.

D. Constant Power Combined With Constant Gate Opening

In the case of constant power, it has been assumed that the turbine gates can be opened to any value to maintain constant power. In reality, however, there is a restriction on the turbine gate opening. The gates cannot be opened beyond their fully open position and therefore the turbine discharge q cannot be increased indefinitely to maintain constant power as the level in the surge tank falls.

Fig. 4.1 shows the relation between the turbine discharge and the rated head. For net heads greater than the rated head, i.e., for $y > -y_o$ (in which y_o = final steady state water level in the tank) the governor operates the gates in such a manner that turbine discharge corresponding to constant power is obtained. For net heads less than the rated head, i.e., $y < -y_o$ the governor keeps the gate open at the maximum value and the discharge through the gate is determined by the maximum gate characteristics (Fig. 4.1). In this case the discharge is less than that required for constant power, and therefore, power output cannot be maintained constant and the oscillations should be analyzed based on the constant gate opening flow condition.

Accordingly, the phase plane is divided into two regions:

1. constant power region for $y > -y_o$, and
2. constant gate opening for $y < -y_o$.

There are five singular points: two in the constant gate opening condition and three in the constant

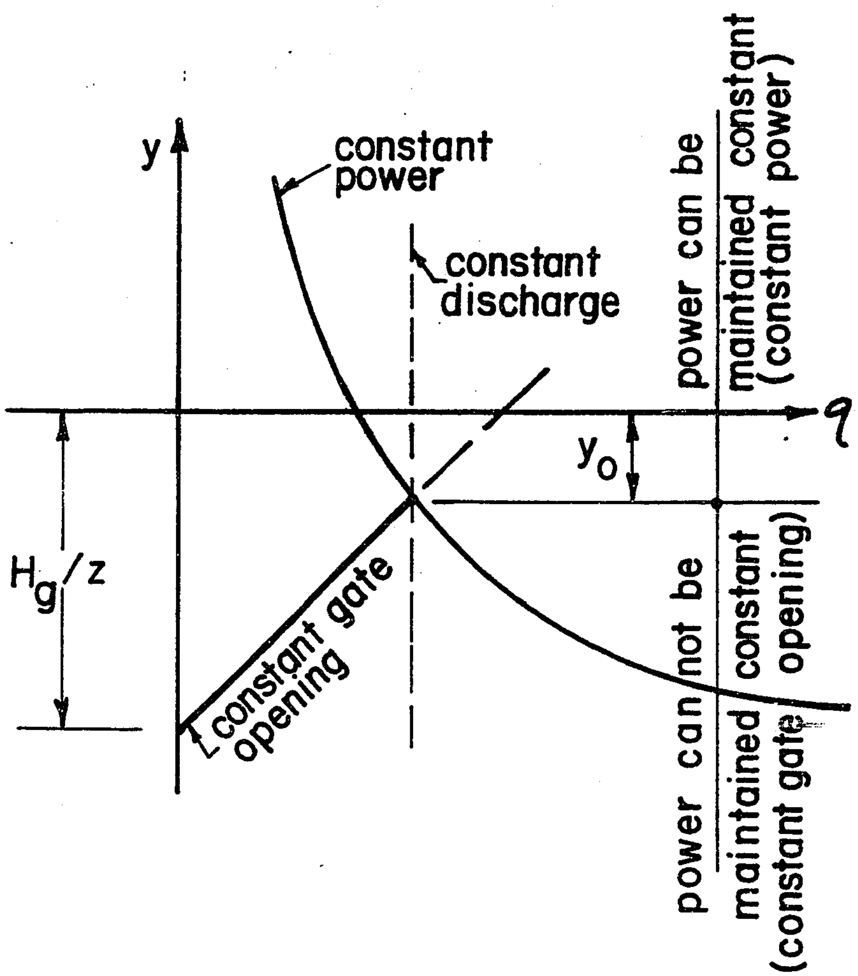


FIG. 4.1 FLOW DEMAND CHARACTERISTICS

power condition. The singular point at $(1., \frac{a_1+a_3}{1+a_2})$ which is common to both the constant gate opening and constant power regions is called a compound singularity. The five singular points have been previously analyzed for the two conditions of constant gate opening and constant power.

For $y < -y_o$ (region of maximum gate opening), the compound singular point $(1., \frac{a_1+a_3}{1+a_2})$ is always stable. For $y > -y_o$ (region of constant power), it may be stable or unstable depending upon whether Eq. 4.65 is satisfied or not. In other words, if the surge tank area A_s is greater than $(A_s)_{cr}$, the oscillations are stable whether they are large or small. However, if A_s is less than $(A_s)_{cr}$, the oscillations may be stable, unstable or of constant magnitude depending upon the stabilizing action of the gate and the point from which the trajectory emanates. The case of oscillations with constant magnitude is represented in phase plane by the limit cycle. The trajectories emanating inside the limit cycle are unstable and their amplitude increases until it is equal to that of the limit cycle. The oscillations outside the limit cycle are stable and their amplitude decreases until it is equal to that of the limit cycle.

CHAPTER 5

PHASE PLANE ANALYSIS FOR DRIVA HYDROELECTRIC POWER PLANT

5.1 INTRODUCTION

The system to be analyzed is the surge tank at the Driva hydroelectric power plant in Norway (17). In this system a closed surge tank, instead of conventional, has been used. Fig. 5.1

The main parameters of this system are as follows:

L = Tunnel length = 18,800 m

A_t = Tunnel section = 20.5 m^2

A_s = Surge Tank Cross-Sectional Area = 780 m^2

$\forall_o$ = Air Volume at the initial Steady State = 5,000 m^3

P_o = Compressed Air Pressure at the Initial State = 386 m

Q_o = Turbine discharge at the initial state = 30 m^3/s

and $Z_o = h_{f_o} + P_o = 408$ m.

Using Maning's formula to determine h_{f_o}, assuming n = .028,

$$Q_o = \frac{1}{n} AR^{2/3} S^{\frac{1}{2}}$$

Since, $\frac{\Pi}{4} D^2 = 20.5$, therefore, D = 5.11 m and R = 1.28. Substituting for A_s, R, S and n in Maning's formula,

$$30 = \frac{1}{.028} (20.5) \left(\frac{5.11}{4}\right)^{2/3} \left(\frac{h_{f_o}}{18,800}\right)$$

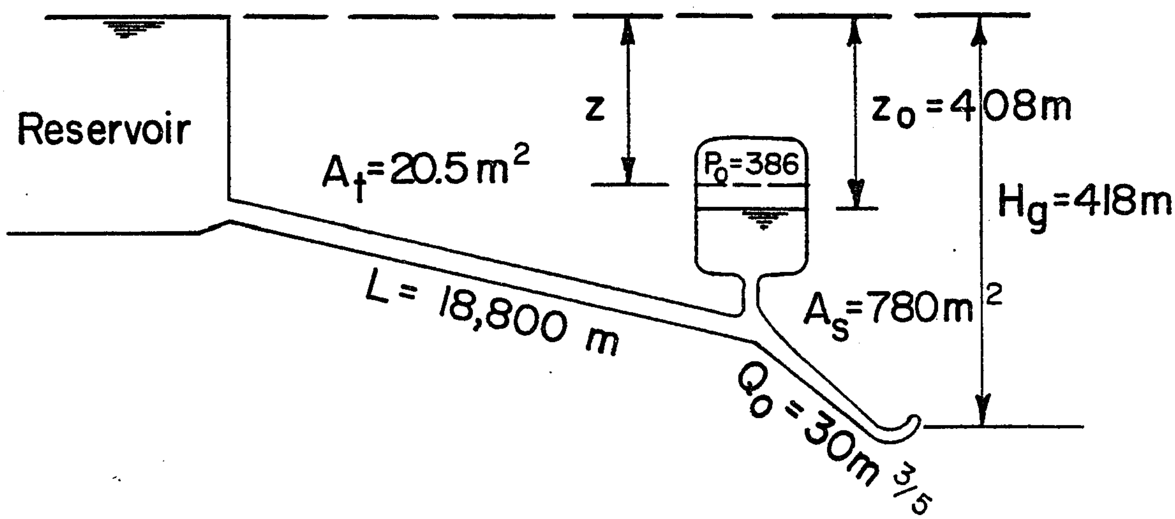


FIG. 5.1 SCHEMATIC DIAGRAM FOR THE HYDRAULIC SYSTEM OF DRIVA POWER PLANT SYSTEM

Therefore, $h_{f_o} = 22$ m. Since, $408 = 22 + P_o$, therefore, $P_o = 386$ m. From Eq. 3.23

$$Z = Q_o \sqrt{\frac{L}{gA_tA_s}}$$

$$= 30 \sqrt{\frac{18,800}{9.81(20.5)\ (780)}} = 10.4 \text{ m}$$

5.2 COMPUTATIONS OF THE CONSTANTS a_1, a_2, a_3 and a_4

From Eqs. 3.34, 3.35, 3.36 and 3.42, and assuming n in the polytropic gas equation is 1.4,

$$a_1 = P_o/Z\ (1 + \frac{nz_oA_s}{\forall_o})$$

$$= \frac{386}{10.4}\ (1 + \frac{1.4(408)780}{5000}) = \underline{3344}$$

$$a_2 = nP_oA_s/\forall_o$$

$$= \frac{1.4(386)780}{5000} = \underline{84.3}$$

$$a_3 = h_{f_o}/Z$$

$$= \frac{22}{10.4} = \underline{2.1}$$

and

$$a_4 = H_g/Z$$

$$= \frac{418}{10.4} = \underline{40}$$

5.3 PHASE PLANE ANALYSIS

A. Constant Flow:

In this case there is one singular point. Its coordinates are:

$$x_s = 1.0$$

$$y_s = \frac{a_1 + a_3}{1 + a_2}$$

$$= \frac{3344 + 84.3}{1 + 84.3} = 39.2$$

From Eq. 4.16, the characteristic roots are:

$$\lambda_1, \lambda_2 = \tfrac{1}{2}[(a' + d') \pm \sqrt{(a' + d')^2 + 4(b'c' - a'd')}]$$

in which

$$a' = -2a_3 = -2(2.1) = -4.2$$

$$b' = (1 + a_2) = 1 + 84.3 = 85.3$$

$$c' = -1.0$$

and

$$d = 0$$

Substituting these numerical values in the equation for the characteristic roots, we obtain:

$$\lambda_1, \lambda_2 = \tfrac{1}{2}[-4.20 \pm \sqrt{(4.20)^2 - 4(85.3)}]$$

which indicates that both roots are complex conjugates with negative real parts. Therefore, singularity is a <u>stable focus</u>.

Since in this flow condition $d' = 0$ and c' is negative, therefore Eq. 4.16 becomes

$$\lambda_1, \lambda_2 = \tfrac{1}{2}[-a' \pm \sqrt{(a)^2 - 4(b'c')}]$$

Phase portraits will be shown in Chapter 6 to illustrate the cases of stable focus and node for different flow conditions.

B. Constant Gate Opening:

In Chapter 4, it has been found that this flow condition has two singular points: $(1., \frac{a_1 + a_3}{1 + a_2})$ and $(-\frac{a_4}{a_3}, \frac{a_1 a_3 + a_4{}^2}{1 + a_2})$. By substituting the numerical values for the constants a_1 through a_4, the coordinates of the two singular points become (1., 39.2) and (-2.1, 39.4).

1. First Singularity: (1., 39.2)

From Eq. 4.39, the parameters of the characteristic roots are:

$$a' = -2a_3 x_s = -4.2$$

$$b' = 1 + a_2 = 85.3$$

$$c' = -1.0$$

and

$$d' = -\frac{1+a_2}{a_4 - a_3} = -2.25$$

and the characteristic roots become

$$\lambda_1, \lambda_2 = \tfrac{1}{2}[-(6.45) \pm \sqrt{(-6.45)^2 - 379}]$$

which indicates that both roots are complex conjugates with negative real parts. Therefore, singularity is a stable focus.

2. Second Singularity: (-2.1, 39.4)

From Eq. 4.39, the parameters of the characteristic roots are the same as for the first singularity except for a' which becomes 8.82.

The condition for a stable focus or a stable node implies that $(a' + d')$ should be less than zero. In this case $a' + d'$ is positive. Accordingly, this singular point is not stable. However, since $x_o < 0$ and since Eqs. 3.37 and 3.38 are not valid for $x < 0$, therefore, this singularity is virtual.

C. Constant Power:

From the preceding analysis in Chapter 4 of this condition, it has been found that there are three singular points: (x_{s1}, y_{s1}), (x_{s2}, y_{s2}) and (x_{s3}, y_{s3}). By substituting the numerical values for the constants a_1, a_2, a_3 and a_4 into Eqs. 4.55 and 4.56, the coordinates of the three singular points become: (1., 39.2), (3.78, 39.2) and (-4.78, 39.14).

1. First Singularity: (1, 39.2)

From Eq. 4.60, the parameters of the characteristic roots equation can be determined as follows:

$$a' = 2a_3x_s = -4.2$$

$$b' = (1+a_2) = 85.3$$

$$c' = -1.0$$

$$d' = \frac{(1 + a_2)\ (a_4 - a_3)}{[a_1 + a_4 - (1+a_2)y_s]^2} = 2.0$$

and the characteristic roots become

$$\lambda_1, \lambda_2 = \tfrac{1}{2}\{(-2.2) \pm \sqrt{(-22)^2 + 4[(-85.3) - (4.2)(2)]}\}$$

from which it is obvious that λ_1 and λ_2 are complex conjugates and have negative real parts. Accordingly, singularity is a stable focus.

2. Second Singularity: (3.78, 39.2)

From Eq. 4.68, the parameters of the equation of the characteristic roots are: $a' = -15.9$, $b' = 85.3$, $c' = -1.0$ and $d' = 2.5$. Substituting these parameters in Eq. 4.16, we obtain

$$\lambda_1, \lambda_2 = \tfrac{1}{2}[(-13.4) \pm \sqrt{-2.65}]$$

Therefore, the roots are complex conjugates with negative real parts. Accordingly, the singular point is a stable focus.

3. Third Singularity: (-4.78, 39.14)

From Eqs. 4.73, 4.74 and 4.75, we obtain $a' = 20.1$, $b' = 85.3$, $c' = -1.0$ and $d' = 1.57$. Since the condition of stable focus or node ($a' + d' < 0$) is not satisfied, therefore singularity is not stable. Singularity in this case is a saddle. However, since Eqs. 3.37 and 3.38 are not valid for $x < 0$, therefore, this singularity is virtual.

D. Constant Power Combined With Constant Gate Opening:

In this case there are five singular points: two in the constant gate opening region and three in the constant power region as has been discussed in Chapter 4. The five singular points have been analyzed for each condition separately.

If the net head is greater than the rated head, i.e., $y > -y_o$, the governor operates the gates in such a manner that turbine discharge corresponding to constant power is obtained. The analysis in this case is similar to that of the constant power condition in which we have found that three singular points exist. The first singular point (1, 39.2) which is common to both constant power and constant gate opening found to be a stable focus. The second singularity (3.78, 39.2) is, also, found to be a stable focus. The third singularity is a saddle, however, it is virtual since Eqs. 3.37 and 3.38 are not valid for $x < 0$.

CHAPTER 6

PHASE PORTRAITS

6.1 Introduction

The phase portraits for the four flow conditions analyzed in the preceding chapters may be plotted by the method of isoclines, details of which are given in Chapter 2. To obtain the phase portraits, the isoclines are first plotted and the solution trajectories are then drawn.

Phase portraits for Driva hydroelectric power plant, Norway, have been plotted using the actual values of the parameters of the power plant. However, to illustrate the effect of high friction on a system, phase portrait for each flow condition has been plotted assuming a high friction value h_{f_o}.

6.2 Phase Portraits for Constant Flow

A. $h_{f_o}/Z = 2.1$ and $h_{f_o}/Hg = .05$:

The equation of the integral curves has been defined in Chapter 4, Eq. 4.1 as

$$\frac{dy}{dx} = \frac{-x + q}{-a_1 + (1 + a_2)\, y - a_3\, x^2}$$

Let
$$\frac{-x + q}{-a_1 + (1 + a_2)\, y - a_3\, x^2} = m \qquad (6.1)$$

in which m is a constant.

Substituting the values for a_1, a_2, a_3 and q and rearranging, Eq. 6.1 can be rewritten in the form:

$$y = \frac{-x + 1}{85.3m} + 39.2 + .02x^2$$

in which the term in x^2 can be neglected so that

$$y = \frac{-x + 1}{85.3m} + 39.2 \qquad (6.2)$$

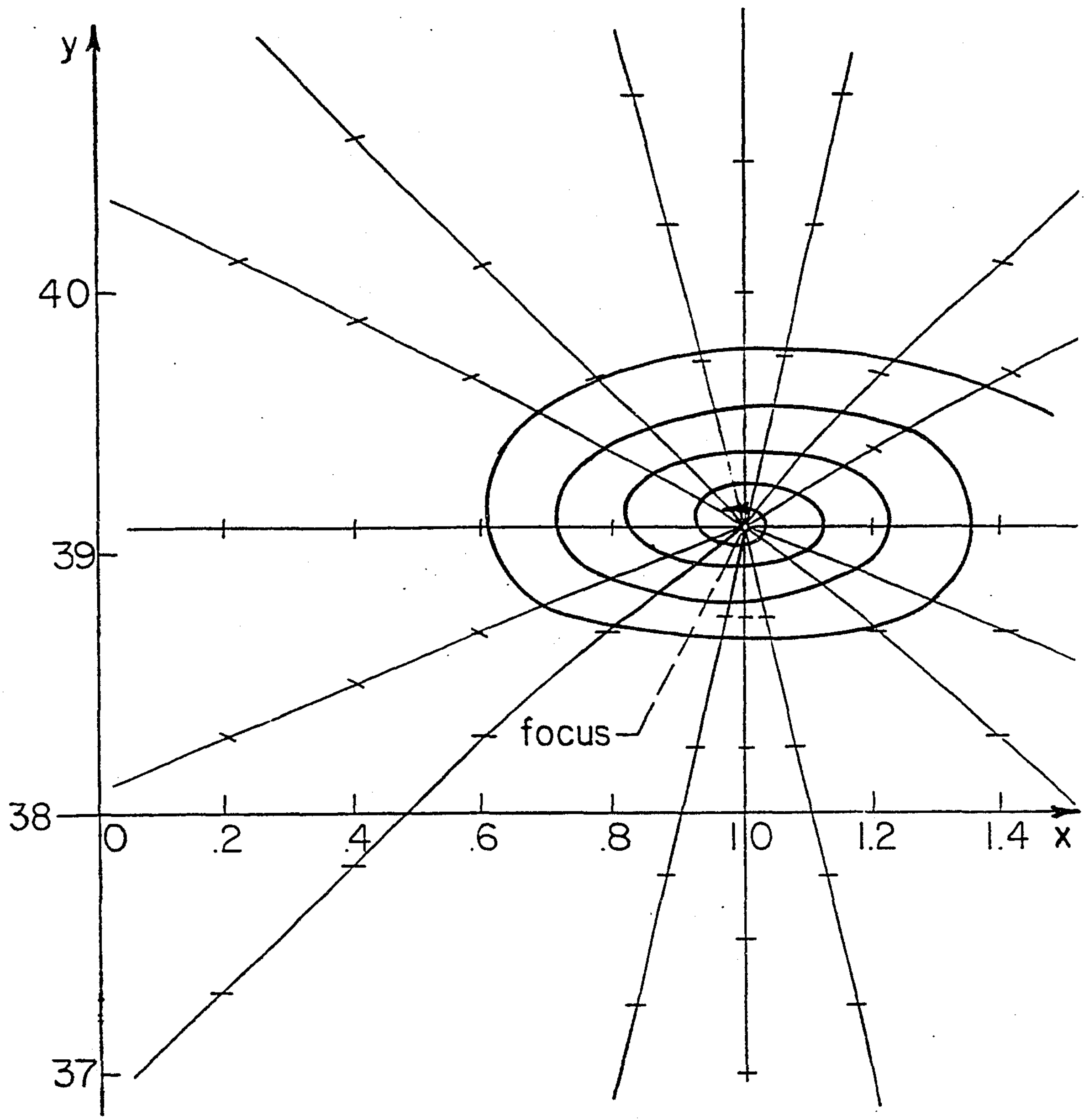
For each assumed value of m, Eq. 6.2 describes a curve (isocline) which passes through points in the x-y plane for which the slopes dy/dx of the integral curves are the same. At the singularity (x_s, y_s) the slope of the integral curves is not defined since the value of dy/dx is indeterminate and isoclines will intersect at this point. Assuming different values for m, more isoclines can be plotted. On each isocline F (x,y) = m, small line segments with the same slope dy/dx = m have been drawn in Figure 6.1. These slopes define the directional field of tangents to the integral curves in a certain region of the plane.

Figure 6.1 shows the phase portrait for the actual existing friction of the Driva power plant.

B. $h_{f_o}/Z = 9.6$ and $h_{f_o}/Hg = .24$:

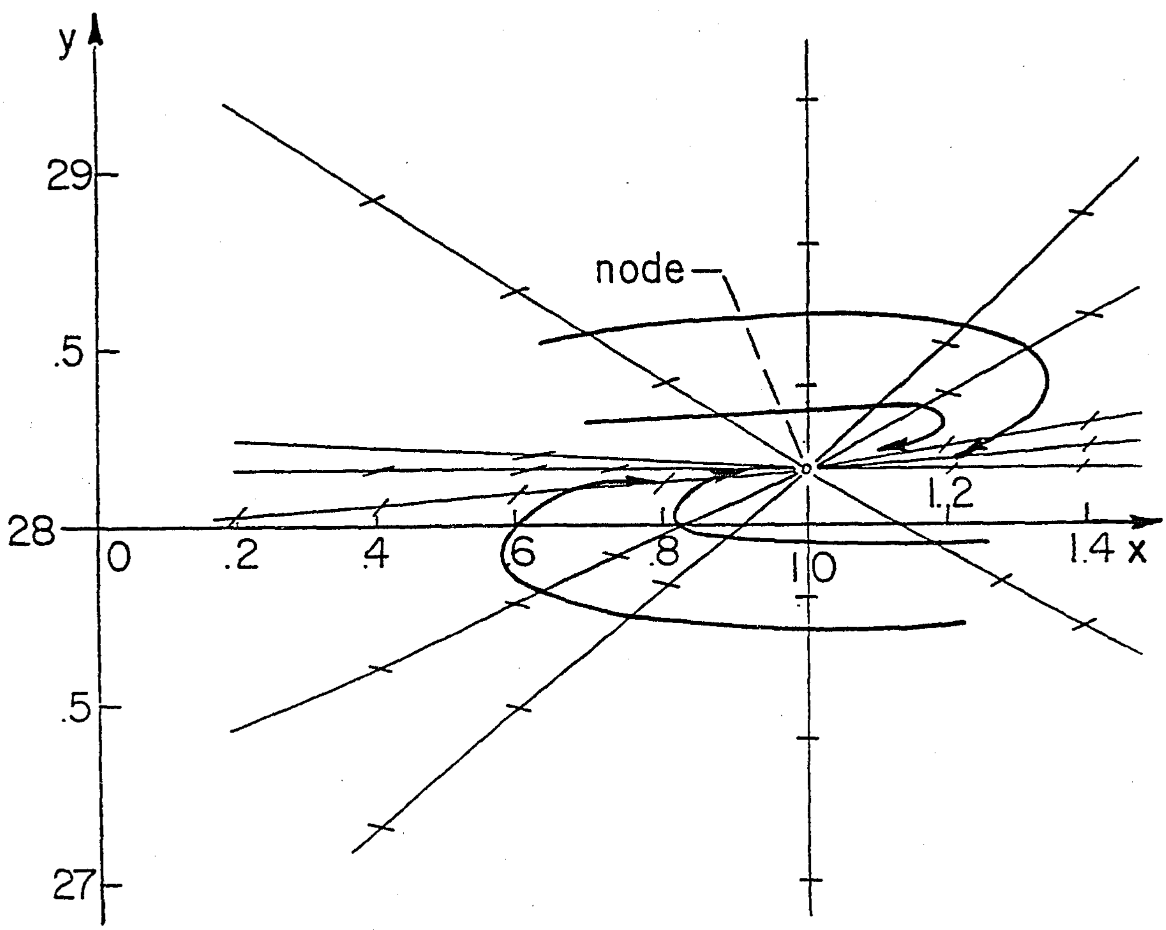
The actual friction loss h_{f_o} has been found in Chapter 5 to be 22 m. If a high friction is assumed, singularity will become a stable node instead of a stable focus.

Assume h_{f_o} is increased to 100 meters and z_o remained the same. Since $z_o = 408 = h_{f_o} + p_o$, therefore, p_o



(a) $h_{f_o}/Z = 2.1$, $h_{f_o}/H_g = .05$

FIG. 6.1 PHASE PORTRAIT FOR CONSTANT FLOW



(b) $h_{f_0}/Z = 9.6$, $h_{f_0} / H_g = .24$

FIG. 6.2 PHASE PORTRAIT FOR CONSTANT FLOW will become equal to 308 meters. The constants a_1, a_2 and a_3 are expressed in terms of p_o. Consequently their new values are determined as follows:

$$a_1 = p_o/Z \ (1 + \frac{z_o A_s}{V_o}) \qquad = 1914$$

$$a_2 = \frac{np_o A_s}{V_o} \qquad = 67.3$$

$$a_3 = h_{f_o}/Z \qquad = 9.6$$

By substituting for a_1, a_2, a_3 and q into Eq. 6.1 and rearranging, we obtain

$$y = \frac{-x + 1}{68.3m} + 28.02 + 0.14\,x^2 \qquad (6.3)$$

At $x_s = 1.$, Eq. 6.3 gives

$$y_s = 28.16$$

Therefore the new coordinates of the singular point are (1., 28.16).

Figure 6.2 shows the phase portrait for the constant flow condition but with high friction as it has been discussed in Chapter 4 and has been indicated by the inequality 4.18.

6.3 Phase Portraits for Constant Gate Opening

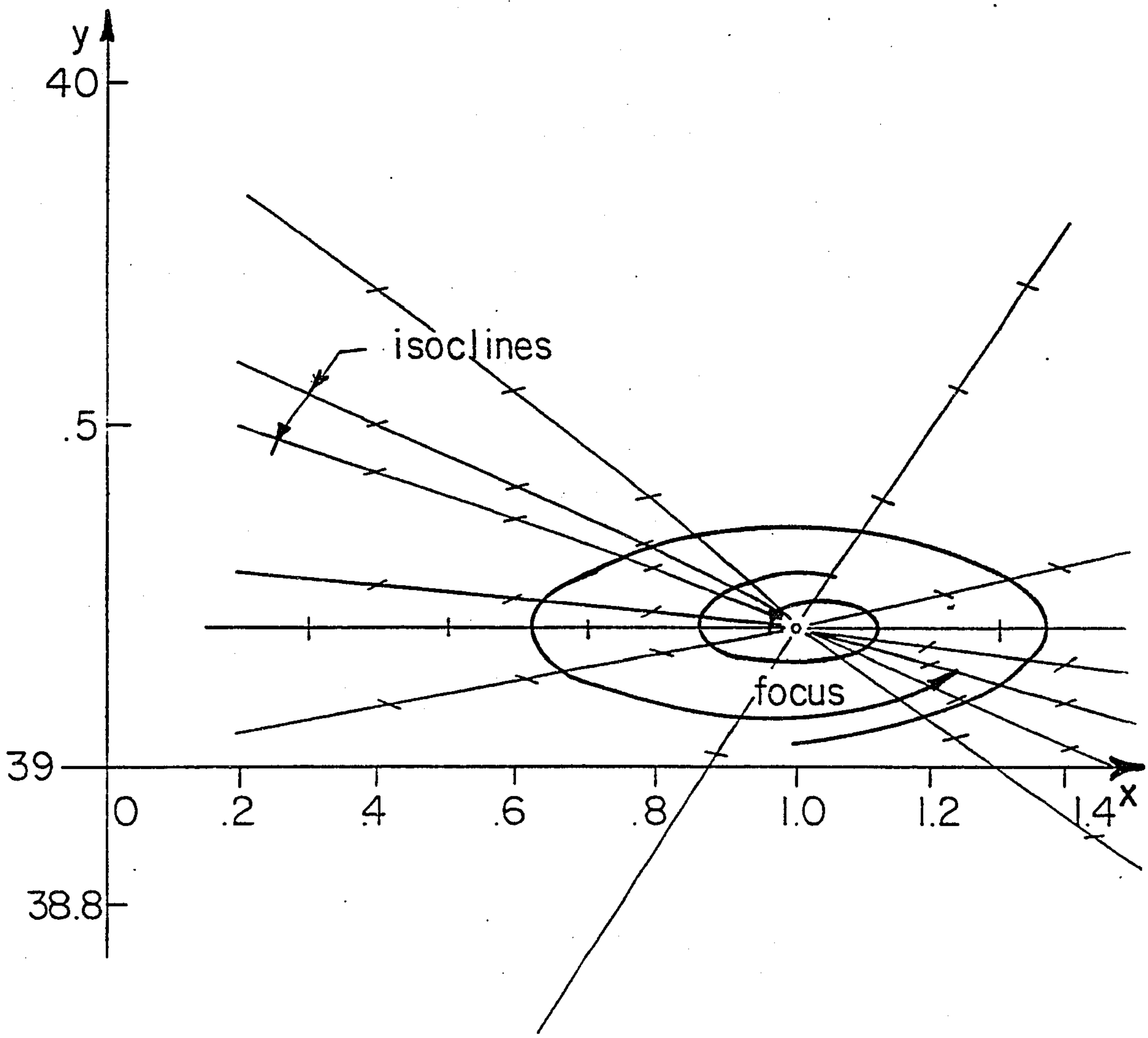
A. For $h_{f_o}/Z = 2.1$ and $h_{f_o}/H_g = .05$:

Let dy/dx in Eq. 4.20, Chapter 4 equals to m. Therefore,

$$\frac{-x + \dfrac{a_1 + a_4 - (1 + a_2)\,y}{a_4 - a_3}}{-a_1 + (1 + a_2)y - a_3 x^2} = m \qquad (6.4)$$

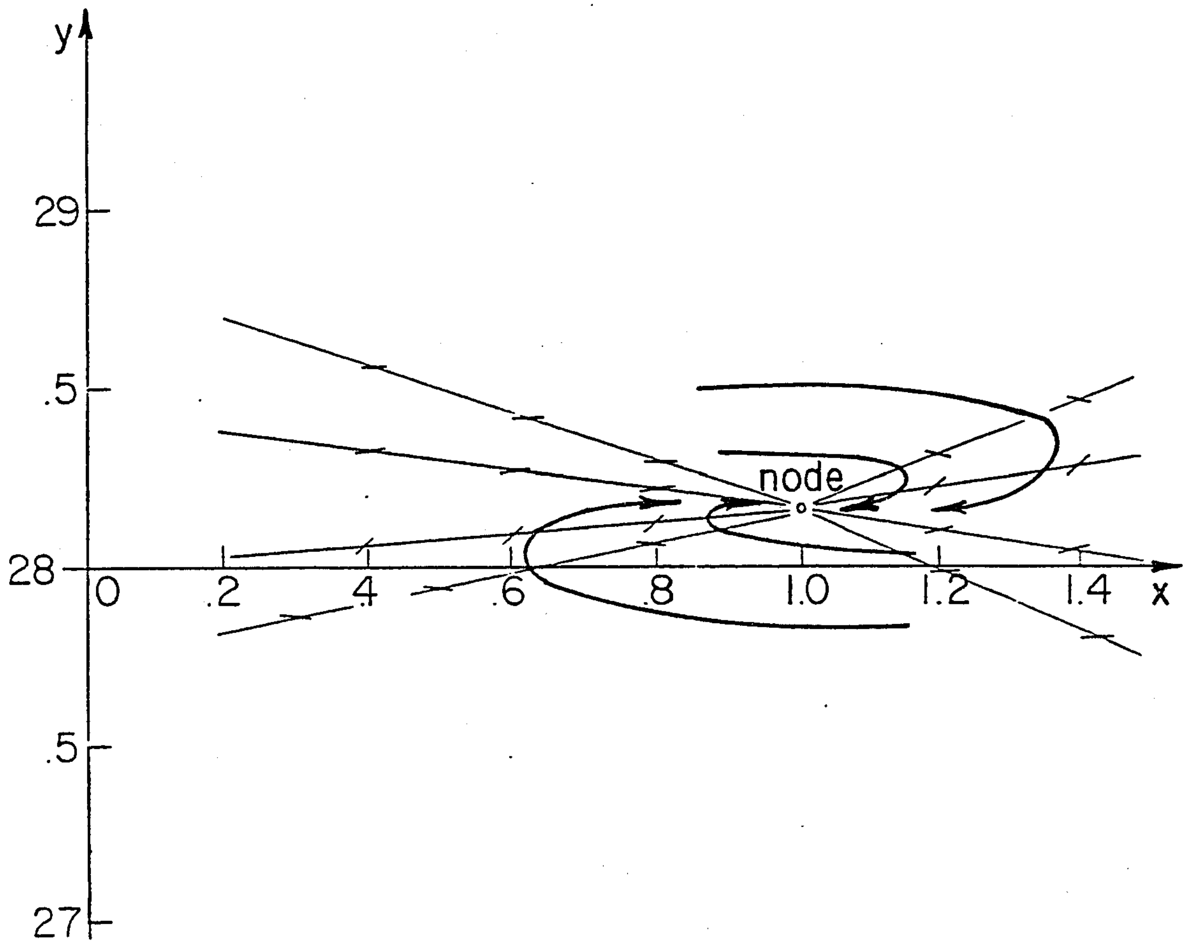
By substituting expressions for a_1 through a_4 into Eq. 6.3 and rearranging, we obtain

$$y = \frac{1}{2.25 + 85.3\,m}\,[2.1mx^2 - x + 89.3 + 3344m] \quad (6.5)$$



(a) $h_{f_0}/Z = 2.1$, $h_{f_0}/H_g = .05$

FIG. 6.3 PHASE PORTRAIT FOR CONSTANT GATE OPENING



(b) $^{h}f_{o}/Z = 9.6$, $^{h}f_{o}/H_{g} = .24$

FIG. 6.4 PHASE PORTRAIT FOR CONSTANT GATE OPENING

Figure 6.3 shows the phase portrait for the actual parameters of the power plant system.

B. For $h_{f_o}/Z = 9.6$ and $h_{f_o}/H_g = .24$:

Assuming h_{f_o} is increased to 100 meters and z_o remained the same, p_o becomes equal to 308 meters and a_1, a_2 and a_3 will be the same as determined in the preceding article. Substituting these values in Eq. 6.4, we obtain the new isoclines equation

$$y = \frac{1}{2.25 + 85.3m}\left[9.62mx^2 - x + 64.3 + 1914m\right] \quad (6.6)$$

Figure 6.4 shows the phase portrait for the condition of constant gate opening with high friction.

6.4 Phase Portraits for Constant Power

A. For $h_{f_o}/Z = 2.1$ and $h_{f_o}/H_g = .05$

The isoclines are $dy/dx = m =$ constant so that from Eq. 4.46 we will have

$$\frac{- x + \dfrac{a_4 - a_3}{a_1 + a_4 - (1 + a_2)y}}{- a_1 + (1 + a_2)\, y - a_3x^2} = m \quad (6.7)$$

Substituting the values for a_1, a_2, a_3 and a_4 and simplifying, Eq. 6.7 can be rewritten in the form

$$y^2 - By - C = 0 \quad (6.8)$$

in which

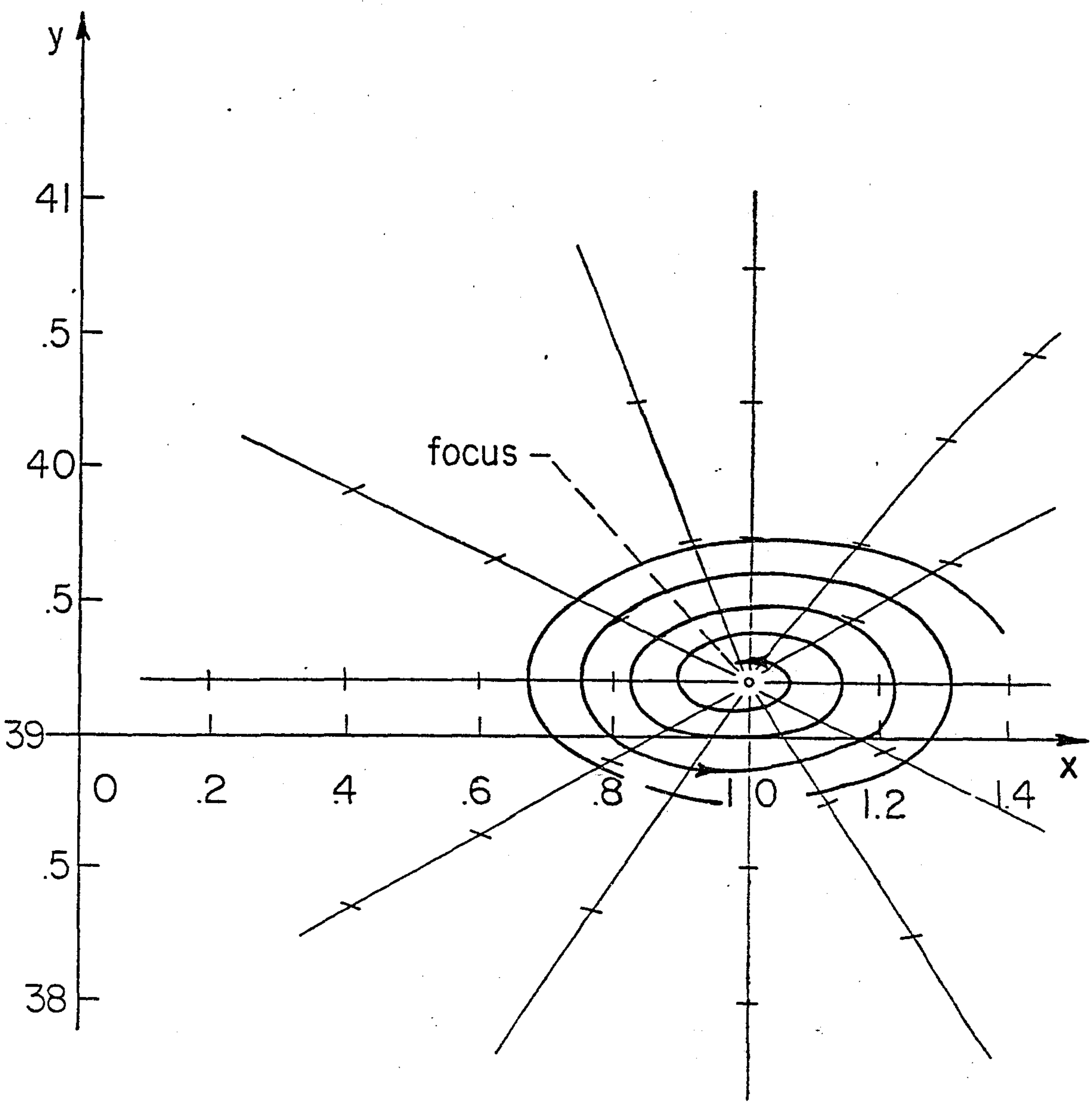
$$B = (78.87 + \frac{2.1}{85.3}\ x^2 - \frac{1}{85.3m}\ x)$$

and $C = (-1555.24 - .977\ x^2 + .465\ \frac{x}{m} - .005/m)$

The solution of the quadratic Eq. 6.8 is

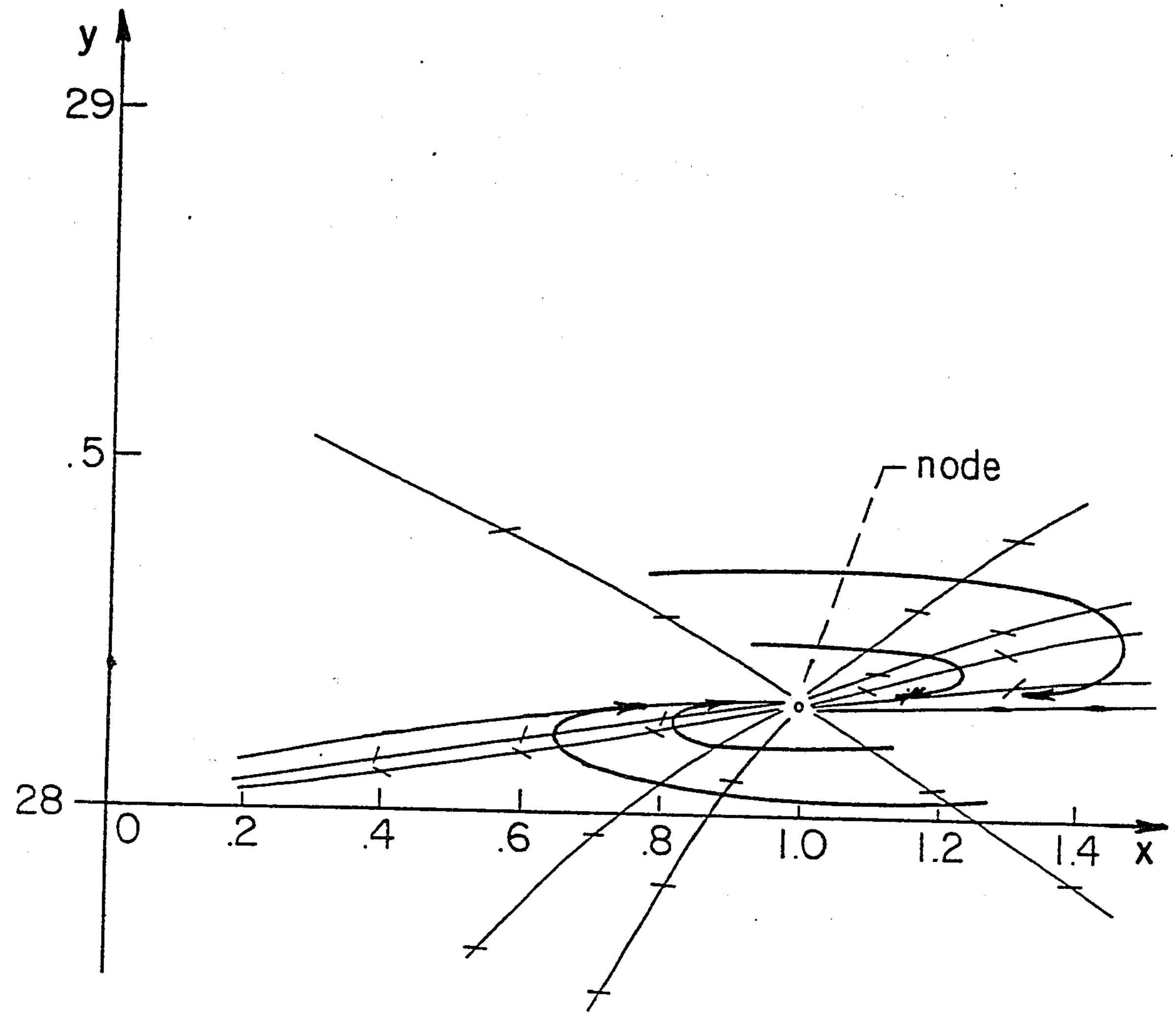
$$y = \tfrac{1}{2}\ (B \pm \sqrt{B^2 + 4C}) \quad (6.9)$$

The phase portrait for this condition is shown in Figure



(a) $h f_0 / Z = 2.1$, $h f_0 / H_g = .05$

FIG. 6.5 PHASE PORTRAIT FOR CONSTANT POWER



(b) $^{h}f_o/Z = 9.6$, $^{h}f_o/H_g = .24$

FIG. 6.6 PHASE PORTRAIT FOR CONSTANT POWER

6.5. The singular point is shown to be a stable focus.

B. For $h_{f_o}/Z = 9.6$ and $h_{f_o}/H_g = .24$

Following the same procedure outlined in the two previous flow conditions, Eq. 6.7 yields a quadratic equation similar to Eq. 6.8 but with different values for the parameters B and C. In this case B and C are defined as follows:

$$B = (56.63 + .141\ x^2 - .0146\ \frac{x}{m}),\ \text{and}$$

$$C = (-801.72 - 4.03x^2 + .419\ \frac{x}{m} - \frac{.0065}{m})$$

Under this flow condition and with the assumption of a system with high friction, the singular point will change from a stable focus to a stable node as shown in the phase portrait, Figure 6.6.

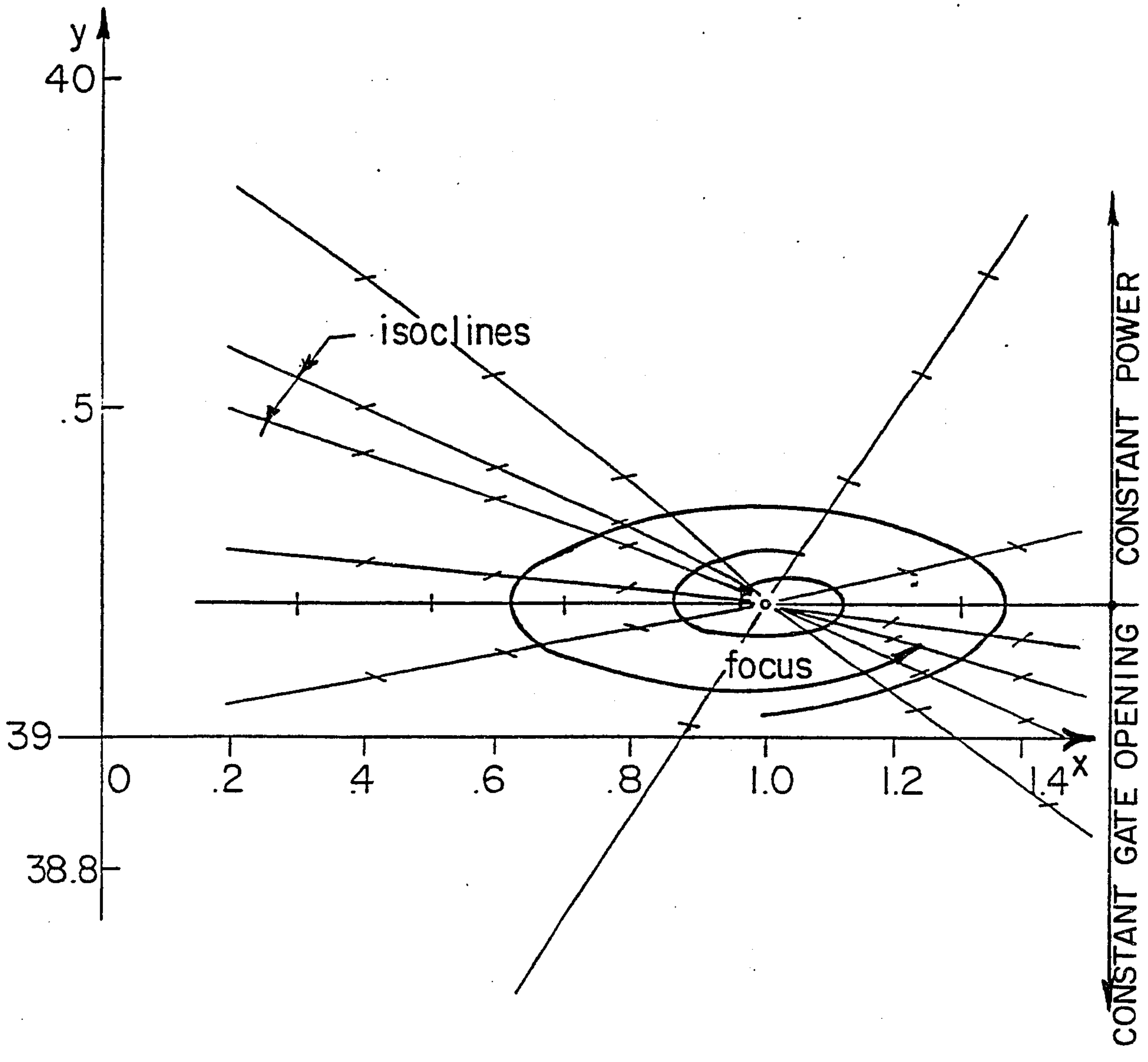
6.5 Phase Portraits for Constant Power Combined with Constant Gate Opening

For this combined governing case, the phase plane is divided into two regions:

1. Constant power region where $y > -y_o$ (see Figure 4.1), and
2. Constant gate opening region where $y < -y_o$.

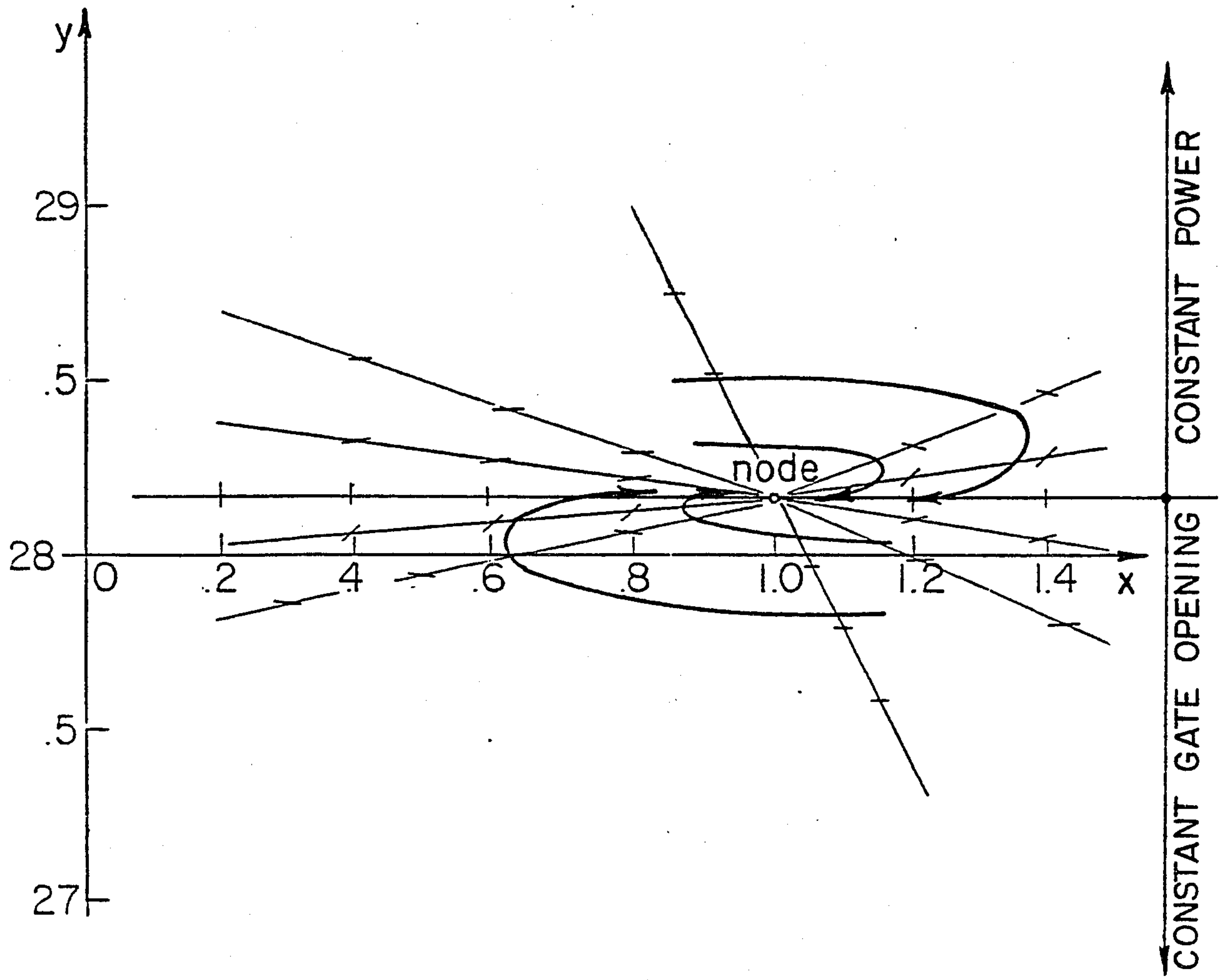
The singular point at $(1., \frac{a_1 + a_3}{1 + a_x})$ is common to both the constant gate opening and constant power conditions and is called a compound singularity.

Phase portraits for this case are shown in Figures 6.7 and 6.8.



(a) $h f_0 / Z = 2.1$, $h f_0 / H_g = .05$

FIG. 6.7 PHASE PORTRAIT FOR CONSTANT POWER COMBINED WITH CONSTANT GATE OPENING



(b) $^{h}f_o/Z = 9.6$, $^{h}f_o/H_g = .24$

FIG. 6.8 PHASE PORTRAIT FOR CONSTANT POWER

SUMMARY AND CONCLUSIONS

The governing equation describing the water level oscillations in a closed surge tank with compressed air at the top have been derived. These equations consist of a set of nonlinear ordinary differential equation with nine parameters. To reduce the number of parameters from nine to four, the governing equations have been normalized.

The usual approach to solve these equations has been to linearize them by neglecting or linearizing the nonlinear terms.

The stability of oscillations in these closed surge tanks is investigated by the phase-plane method which allows inclusion of nonlinear terms in the analysis. The following four cases of turbine flow demands were considered: Constant discharge, constant gate opening, constant power and constant power combined with constant gate opening. All singularities have been analyzed for each case and stability criteria have been developed.

The following conclusions can be drawn from the preceding analyses:

1. Oscillations are always stable in the case of constant discharge.

2. In the case of constant gate opening, there are two singular points: The first singular point which is

of practical importance is either a stable focus or a stable node. It is a stable node if $(1 + a_2) > 4\, a_4^2$ and a stable focus if $(1 + a_2) < 4\, a_4^2$. Therefore, this singularity is always stable. The second singularity is virtual since the governing equations are not valid for $x < 0$.

3. For the constant power case, there are three singular points. The first and the second singular points are found to be either stable nodes or stable foci if the parameters of the hydraulic system satisfy the following equation:

$$(A_s)_{cr} = \frac{Q_o^2\, L}{2g\, A_t\, h_{f_o}\, (H_g - h_{f_o})}\, \left(1 + \frac{nP_o A_s}{\forall_o}\right)$$

In addition, unstable oscillations are possible if h_{f_o} is greater than $\frac{1}{3} H_g$. However, such a large head loss is not economically feasible and accordingly this case is of academic interest only. The third singular point has been found to be a saddle. However, it is a virtual singularity since the governing equations are not valid for $x < 0$.

4. In the case of constant power combined with constant gate opening, there are five singular points, two in the constant gate opening region and three in the constant power region. The first singularity $(1., \frac{a_1+a_3}{1 + a_2})$ for both conditions is common. Stability analysis for this case depends upon whether the turbine gate has reached their fully open position or not. This can be determined

from the relation between the turbine discharge and the rated head. For heads greater than the rated head, stability analysis will be the same as that for constant power, i.e., two stable singular points and one virtual. For heads less than the rated head, stability analysis will be the same as that for constant gate opening, i.e., one stable singular point and one virtual singular point. For $y < -y_o$ (region of maximum gate opening) the common singular point which is called a compound singularity is always stable. For $y > -y_o$ (region of constant power), it may be stable or unstable depending upon whether the stability criterion (Eq. 4.65) is satisfied or not. In other words, if the surge tank area A_s is greater than $(A_s)_{cr}$, the oscillations are stable whether they are large or small. However, if A_s is less than $(A_s)_{cr}$, the oscillations may be stable, unstable or of constant magnitude depending upon the stabilizing action of the gate and the point from which the trajectory emanates. If $A_s < (A_s)_{cr}$ and $y < -y_o$, the solution trajectories in the phase plane show stable oscillations. However, if $A_s < (A_s)_{cr}$ and $y > -y_o$, stability criterion is not satisfied and oscillations are unstable due to these stabilizing and destabilizing effects, a solution trajectory corresponding to perpetual oscillations is obtained and in the phase plane terminology is called a limit cycle. The trajectories emanating inside the limit cycle are unstable and their amplitude increases until it is equal to that of the limit cycle. The oscillations outside the limit cycle are stable and their amplitude decreases until it is equal to that of the limit cycle.

5. Results obtained from the analysis of the first singularity for the case of constant power agree with the results obtained by Svee by linearizing the governing equations. The phase plane method which is used herein provides stability criteria not only for one singularity, but for all possible singularities of the governing equations representing each of the four flow demand conditions. In other words, this technique investigates the transition of the nonlinear system from one stable state of motion to another.

6. The critical area of the closed surge tank depends on the initial air pressure in the chamber and the distance from the water elevation surface to the top of the tank. For fixed initial air pressure the larger this distance, the smaller the critical area. This distance would be determined based on the topography of the site.

Since the studies herein are for one closed surge tank in a hydroelectric power plant, it would be interesting to investigate the effect of a series of closed surge tanks on the stability of the system.

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