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Boundary Value Problems in Elasticity and Thermoelasticity

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BOUNDARY VALUE PROBLEMS IN ELASTICITY AND THERMOELASTICITY

by

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DEDICATION

To my wife, Betty, for all her patience

and

Gordon Melrose for his friendship

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ABSTRACT

Boundary Value Problems In Elasticity and Thermoelasticity

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In this dissertation the author solves a series of mixed boundary value problems arising from crack problems in elasticity and thermoelasticity. Using integral transform techniques and separation of variables appropriately, it is shown that the solutions can be found by solving a corresponding set of triple or dual integral equations in some instances, while in others the solutions of triple or dual series relations are required. These in turn reduce to various singular integral equations which are solved in closed form, in two cases, or by numerical methods. The stress intensity factors at the crack tips, the physical parameters of interest, are found and the results are recorded in both tabular and graphical form.

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Chapter 1

Introduction

The purpose of this dissertation is to determine the distribution of stress associated with various crack configurations which occur in elastic bodies with particular profiles. The main portion of the text is concerned with evaluating the thermal stresses set up in heated elastic solids - restricted here to be infinite, two-dimensional, homogeneous and isotropic - containing cracks. The presence of the cracks, which are assumed to be thermally insulated and traction free, will block the flow of heat thereby inducing a local intensification of the temperature gradient and hence a concentration of the stress field at the crack tips. The situation described here will cause the crack surfaces to slide over one another in opposite directions but in the same plane. This type of opening is called a mode II opening. Other types of crack deformation are known as mode I–when the crack surfaces tend to separate symmetrically under the action of an internal pressure and mode III–when the movement of the crack surfaces is associated with an out of plane tearing motion. It follows naturally that each of the three types of movement is associated with a stress field in the immediate vicinity of the crack tips. The physical parameters of interest are the stress intensity factors at the crack tips. These quantities, which measure the strength of the stress field singularities at the crack tips, can be used to predict crack growth. Imposing the assumptions of small displacements and a linear stress-strain law gives rise to the linear theory of thermoelasticity used in this discussion. The reader is directed to the works of [1] and [2] which substantiate the foundations of this theory.

The strategy for solving problems of this type is to follow two consecutive steps: (1) find the undisturbed thermoelastic field in the region without any cracks and (2) obtain the perturbations of this field due to the presence of the cracks. Using linear superposition, the second step can be separated into appropriate thermal and isothermal problems. Employing integral transform techniques and separation of variables to solve these component problems, it is shown that the stress intensity factors are dependent on the solutions to certain singular integral equations. In one case the solution of the singular integral equation is found in closed form, while the others are solved using numerical techniques.

In chapter 2, the basic relations and equations of linear thermoelasticity are combined to show that the temperature and displacements are solutions of the Duhamel-Neumann equations. A special form due to Sneddon [3], in which the solutions are written in terms of arbitrary harmonic functions, is highlighted.

Simple heat flow problems are discussed in chapter 3 which correspond to solving step (1). These solutions will be referred to in subsequent chapters.

Recent endeavours motivating the work presented here have been made by Florence and Goodier [4] who discussed the thermal stresses in an infinite plate containing an ovaloid hole under a constant temperature gradient. Tweed and Melrose [5] investigated the disturbance of a uniform heat flow by two collinear cracks in an infinite plate. In this paper, the authors were able to obtain exact expressions for the stress intensity factors in terms of elliptic integrals.

A natural extension of these two cases is to determine the state of stress in an infinite plate when a uniform heat flow is disturbed by a collinear array of cracks. Chapter 4 is concerned with this particular problem.

The inclusion of boundaries to reduce the infinite nature of the aforementioned situations accounts for the contents of the next three chapters.

Chapter 5 deals with the case of a single crack parallel to the edges of a thermoelastic strip which disturbs a uniform heat flow. This problem was first tackled by Konishi and Atsumi [6]. They were able to reduce the problem to a pair of linked Fredholm integral equations with kernels involving integrals of products of Bessel functions. Unfortunately this paper contains a significant algebraic error, in the calculation of the second kernel, which invalidates their results.

The arrangement of two collinear cracks and a collinear array of cracks parallel to the edges of a thermoelastic strip determine the material presented in chapters 6 and 7 respectively.

In chapter 8, the author supplements the previous thermal problems with the solution of a mixed boundary value problem occuring in orthotropic fracture. In particular, the state of stress is sought when a cracked rectangular orthotropic block is subjected to an out of plane shear. In collaboration with his advisor Dr. Melrose, two articles pertaining to this problem have been published [7], [8]. Early investigations of this problem [9], [10] and [11] reduced the problem to the solution of a Fredholm integral equation which has to be performed numerically. The alternative approach presented here reduces the problem to the solution of a singular integral equation which belongs to a class that can be solved exactly.

The discussion is concluded in the appendix with a summary of the numerical procedure employed to solve the singular integral equations appearing in chapters 5, 6, and 7. The method is due to Ioakimidis and Theocaris [12] and for a glimpse of the underlying theory a text by Kopal [13] may prove useful.

The numbering system used in the text is a standard one. Equations are numbered $(m.n)$ where m is the section number and n refers to the n^{th} equation in the section. Equations are referenced by $(k.m.n)$ where k is the number of the chapter in which the equation occurs.

Chapter 2

Steady State Equations of Linear Thermoelasticity

2.1 Introduction

Many common engineering materials have the property that they undergo very small changes of shape when subjected to the temperatures and forces they normally encounter. They all, to a certain extent, possess the property of elasticity: that is, the deformation of a body disappears with the removal of the temperatures and forces producing the deformation. For the theory developed here these agents are restricted to lie within certain limits. To further characterize the type of solid under investigation it is assumed to be homogeneous and isotropic. An isotropic body is one having no preferred directions so that the thermal and mechanical properties are the same in all directions. Additionally, we assume there are no external forces being applied and no heat sources present within the material. The thermomechanical behaviour of a solid with these characteristics can be described most simply by the linear theory of thermoelasticity.

We are now in a position to state the basic relations and equations governing the thermal and mechanical responses of a thermoelastic solid as described above. In some equations Einstein's summation notation is used for brevity.

2.2 Basic Relations and Equations

The temperature at the point (X, Y, Z) will be denoted by

$$T(X,Y,Z) = T_0 + \Theta(X,Y,Z) \tag{2.1}$$

where T_0 is the temperature of a body in a state of zero stress and strain. The temperature differences between points of a body will result in a flow of heat where the heat flux $\boldsymbol{Q}$ is proportional to the temperature gradient and is given by Fourier's Law

$$\boldsymbol{Q} = -\kappa \nabla T \tag{2.2}$$

where κ, the thermal conductivity of the material, is a constant due to the assumption of homogeneity.

In the absence of heat sources inside a body we obtain, from (2.2.2) and conservation of energy, the equation of heat conduction

$$\nabla^2 T = 0 \tag{2.3}$$

where ∇^2 is the Laplacian operator in rectangular coordinates.

Let $U_I\,(I = X, Y, Z)$ denote the components of the displacement vector U, $\varepsilon_{IJ}\,(I, J = X, Y, Z)$ are the components of the strain tensor while the components of the stress tensor are given by $\sigma_{IJ}\,(I, J = X, Y, Z)$.

The strain tensor is given in terms of the displacement vector via the relations

$$\varepsilon_{IJ} = \frac{1}{2}(U_{I,J} + U_{J,I}) \tag{2.4}$$

Clearly, any displacement U gives a unique strain tensor. However, an arbitrary ε_{IJ} may not be described by a unique U. To guarantee this, the components of the strain tensor must satisfy the following compatibility equations

$$\varepsilon_{IJ,KL} + \varepsilon_{KL,IJ} - \varepsilon_{IK,JL} - \varepsilon_{JL,IK} = 0 \tag{2.5}$$

where $I, J, K, L \in \{X, Y, Z\}$.

The relations between the components of strain, the components of stress, and the temperature are a generalization of Hooke's Law and result in the thermomechanical constitutive equations

$$\varepsilon_{IJ} = \frac{(1+\nu)}{E}\left[\sigma_{IJ} - \frac{\nu}{(1+\nu)}\delta_{IJ}\sigma_{KK}\right] + \alpha\Theta\delta_{IJ} \tag{2.6}$$

or

$$\sigma_{IJ} = \frac{E}{(1+\nu)}\left[\varepsilon_{IJ} + \frac{\nu}{(1-2\nu)}\left\{\delta_{IJ}\varepsilon_{KK} - \frac{(1+\nu)}{\nu}\alpha\Theta\delta_{IJ}\right\}\right] \tag{2.7}$$

where α: the coefficient of linear expansion

ν: Poisson's ratio

E: Young's modulus

are material dependent constants and δ_{IJ} is the Kronecker delta symbol defined by

$$\delta_{IJ} = \begin{cases} 0 \ , & \text{if } I \neq J \\ 1 \ , & \text{if } I = J \end{cases} \tag{2.8}$$

Furthermore, in the absence of external forces we obtain the equilibrium equations

$$\sigma_{IJ,J} = 0 \quad \text{and} \quad \sigma_{IJ} = \sigma_{JI} \tag{2.9}$$

where $I, J \in \{X, Y, Z\}$.

On substituting (2.2.7) into (2.2.9) and using the strain-displacement equations (2.2.4) along with the heat conduction equation (2.2.3) we obtain for $\left(0 < \nu < \frac{1}{2}\right)$

$$\nabla^2 U + \frac{1}{1-2\nu}\nabla(\nabla \cdot U) = \frac{2\alpha(1+\nu)}{1-2\nu}\nabla\Theta \tag{2.10}$$

$$\text{and} \qquad \nabla^2\Theta = 0 \tag{2.11}$$

the Duhamel-Neumann Equations.

In principle (2.2.11) can be solved for Θ and the result substituted into (2.2.10). The solution of (2.2.10) may then be written in the form

$$\boldsymbol{U} = \boldsymbol{U}_p + \boldsymbol{U}_c \tag{2.12}$$

where $\boldsymbol{U}_p$ is any particular integral and $\boldsymbol{U}_c$ is the complementary function.

The complementary function is the general solution to the homogeneous partial differential equations

$$\nabla^2 \boldsymbol{U} + \frac{1}{1-2\nu}\nabla(\nabla\cdot\boldsymbol{U}) = 0 \tag{2.13}$$

and is shown to be of the form [14],

$$\boldsymbol{U} = \nabla(\chi + \boldsymbol{X}\cdot\boldsymbol{\Psi}) - 4(1-\nu)\boldsymbol{\Psi} \tag{2.14}$$

where χ is an arbitrary harmonic scalar function, $\boldsymbol{\Psi}$ is an arbitrary harmonic vector function and $\boldsymbol{X} = (X, Y, Z)$. At first glance there seem to be four unknown functions in the representation (2.2.14). It has been shown [15], that at most three of these functions are independent.

A particular solution of (2.2.10) and (2.2.11) is due to Sneddon [3], and has the form

$$\boldsymbol{U} = Y\nabla\psi - \psi\boldsymbol{j} \tag{2.15}$$

and $$\Theta = \frac{(1-2\nu)}{\alpha(1+\nu)}\frac{\partial\psi}{\partial Y} \tag{2.16}$$

where ψ is any harmonic scalar function. This particular solution is chosen specifically to solve problems with boundary conditions on $Y = 0$.

Combining (2.2.14), (2.2.15) and (2.2.16) results in Sneddon's general solution to the Duhamel-Neumann Equations

$$U = \nabla(\chi + X \cdot \Psi) - 4(1-\nu)\Psi + Y\nabla\psi - \psi j \tag{2.17}$$

$$\text{and} \qquad \Theta = \frac{(1-2\nu)}{\alpha(1+\nu)} \frac{\partial \psi}{\partial Y} \tag{2.18}$$

where $\nabla^2\chi = 0, \nabla^2\psi = 0$ and $\nabla^2\Psi = 0$.

At this juncture, we shall introduce some new notation and display a dimensionless form of Sneddon's general solution. Let L be a characteristic length in the problem under consideration and p_0 a constant with dimension of stress. Introduce the dimensionless variables

$$x_i = (x, y, z) = (X/L, Y/L, Z/L) \tag{2.19}$$

the dimensionless displacements

$$u_i = (u, v, w) = \frac{E}{p_0 L(1+\nu)}(U_X, U_Y, U_Z) \tag{2.20}$$

the dimensionless stresses and strains

$$s_{ij} = \frac{\sigma_{IJ}}{p_0} \ , \ e_{ij} = \frac{E}{p_0(1+\nu)}\varepsilon_{IJ} \tag{2.21}$$

and a scaled temperature

$$\theta(x,y,z) = \Theta(X,Y,Z)/T_0 \tag{2.22}$$

Additionally, we define the constants

$$\beta^2 = \frac{2(1-\nu)}{(1-2\nu)} \quad \text{and} \quad \delta = \frac{E\alpha\beta^2 T_0}{2(1-\nu)p_0} \tag{2.23}$$

In terms of these quantities our field equations become,

Equilibrium

$$s_{ij,j} = 0 \quad \text{and} \quad s_{ij} = s_{ji} \tag{2.24}$$

Stress-Strain

$$s_{ij} = e_{ij} + \delta_{ij}\left[\frac{1}{2}(\beta^2 - 2)e_{kk} - \delta\theta\right] \tag{2.25}$$

Duhamel-Neumann Equations

$$\begin{aligned} &\nabla^2 \boldsymbol{u} + (\beta^2 - 1)\nabla(\nabla \cdot \boldsymbol{u}) = 2\delta\nabla\theta \\ &\text{and} \qquad \nabla^2\theta = 0 \end{aligned} \tag{2.26}$$

Sneddon's General Solution

$$\begin{aligned} &\boldsymbol{u} = \nabla(\chi + \boldsymbol{X}\cdot\boldsymbol{\Psi}) - 4(1-\nu)\boldsymbol{\Psi} + y\nabla\psi - \psi\boldsymbol{j} \\ &\text{and} \qquad \theta = \delta^{-1}\frac{\partial\psi}{\partial y} \end{aligned} \tag{2.27}$$

where $\nabla^2\chi = 0, \nabla^2\psi = 0$ and $\nabla^2\boldsymbol{\Psi} = 0$

2.3 Plane Strain Equations

If a cylindrical body with generators parallel to the z-axis is subjected to loads independent of z, then all cross-sections are in the same condition. The components of displacement and heat flux in the z-direction vanish while the additional components remain constant along the z-direction. Such a body is said to be in a state of plane strain.

The plane strain equations of thermoelasticity are easily obtained from the three-dimensional equations in section two by setting

$$w = 0 \text{ and } \frac{\partial}{\partial z} \equiv 0 \tag{3.1}$$

This being the case we obtain the field equations below

Displacement

$$\boldsymbol{u} = u(x,y)\boldsymbol{i} + v(x,y)\boldsymbol{j} \tag{3.2}$$

Temperature

$$T = T_0[1 + \theta(x,y)] \tag{3.3}$$

Strain

$$\begin{aligned} e_{xx} &= \frac{\partial u}{\partial x} \ , \ e_{yy} = \frac{\partial v}{\partial y} \ , \ e_{zz} = 0 \\ e_{xy} &= \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \ , \ e_{xz} = e_{zy} = 0 \end{aligned} \tag{3.4}$$

Equilibrium

$$\begin{aligned} \frac{\partial s_{xx}}{\partial x} &+ \frac{\partial s_{xy}}{\partial y} = 0 \\ \frac{\partial s_{yx}}{\partial x} &+ \frac{\partial s_{yy}}{\partial y} = 0 \\ s_{xy} &= s_{yx} \end{aligned} \tag{3.5}$$

Stress-Strain

$$\begin{aligned} s_{xx} &= e_{xx} + \tfrac{1}{2}(\beta^2 - 2)[e_{xx} + e_{yy}] - \delta\theta \\ s_{yy} &= e_{yy} + \tfrac{1}{2}(\beta^2 - 2)[e_{xx} + e_{yy}] - \delta\theta \\ s_{zz} &= \tfrac{1}{2}(\beta^2 - 2)[e_{xx} + e_{yy}] - \delta\theta \\ s_{xy} &= e_{xy} \;,\; s_{xz} = s_{yz} = 0 \end{aligned} \tag{3.6}$$

Duhamel-Neumann Equations

$$\begin{aligned} &\nabla^2 u + (\beta^2 - 1)\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] = 2\delta\frac{\partial \theta}{\partial x} \\ &\nabla^2 v + (\beta^2 - 1)\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] = 2\delta\frac{\partial \theta}{\partial y} \\ &\nabla^2 \theta = 0 \\ &\text{where } \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \end{aligned} \tag{3.7}$$

Sneddon's Solution

$$\begin{aligned} &u = \frac{\partial}{\partial x}(\chi + \boldsymbol{x}\cdot\boldsymbol{\Psi}) - 4(1-\nu)\Psi_1 + y\frac{\partial \psi}{\partial x} \\ &v = \frac{\partial}{\partial y}(\chi + \boldsymbol{x}\cdot\boldsymbol{\Psi}) - 4(1-\nu)\Psi_2 + y\frac{\partial \psi}{\partial y} - \psi \\ &\text{and } \theta = \delta^{-1}\frac{\partial \psi}{\partial y} \end{aligned} \tag{3.8}$$

where χ, ψ are two-dimensional harmonic functions and $\boldsymbol{\Psi} = (\Psi_1, \Psi_2)$ is a two-dimensional harmonic vector function.

Let us now consider a special case of (2.3.8) in which the functions $\chi, \psi, \boldsymbol{\Psi}$ are replaced by $\chi+\phi, \psi$ and $(\beta^2-1)\frac{\partial\phi}{\partial y}\boldsymbol{j}$ respectively. The solution (2.3.8) becomes

$$
\begin{aligned}
u &= \frac{\partial\chi}{\partial x} + \frac{\partial\phi}{\partial x} + (\beta^2-1)y\frac{\partial^2\phi}{\partial x\partial y} + y\frac{\partial\psi}{\partial x} \\
v &= \frac{\partial\chi}{\partial y} - \beta^2\frac{\partial\phi}{\partial y} + (\beta^2-1)y\frac{\partial^2\phi}{\partial y^2} + y\frac{\partial\psi}{\partial y} - \psi \\
&\text{and } \theta = \delta^{-1}\frac{\partial\psi}{\partial y} \\
&\text{where } \nabla^2\chi = 0,\ \nabla^2\phi = 0 \quad \text{and} \quad \nabla^2\psi = 0
\end{aligned}
\tag{3.9}
$$

In addition the stress-strain equations (2.3.6) yield

$$
\begin{aligned}
s_{xx} &= -\frac{\partial^2\chi}{\partial y^2} - (\beta^2-1)\frac{\partial^2\phi}{\partial y^2} - (\beta^2-1)y\frac{\partial^3\phi}{\partial y^3} - y\frac{\partial^2\psi}{\partial y^2} - \frac{\partial\psi}{\partial y} \\
s_{yy} &= \frac{\partial^2\chi}{\partial y^2} - (\beta^2-1)\frac{\partial^2\phi}{\partial y^2} + (\beta^2-1)y\frac{\partial^3\phi}{\partial y^3} + y\frac{\partial^2\psi}{\partial y^2} - \frac{\partial\psi}{\partial y} \\
s_{xy} &= \frac{\partial^2\chi}{\partial x\partial y} + (\beta^2-1)y\frac{\partial^3\phi}{\partial x\partial y^2} + y\frac{\partial^2\psi}{\partial x\partial y}
\end{aligned}
\tag{3.10}
$$

By choosing the harmonic functions χ, ϕ and ψ appropriately, many interesting solutions can be found.

Chapter 3

Solutions To Simple Heat Flows

3.1 Introduction

When evaluating the thermal stresses set up in a heated solid containing cracks, it is usual to begin by solving the heat conduction problem in the region without any cracks. To this end we consider three simple heat conduction problems whose solutions will be referred to in subsequent chapters.

3.2 Problem A

Consider a uniform heat flow in the y-direction given by

$$Q = Q_0 j \tag{2.1}$$

in the strip defined by the inequalities

$$0 < x < \pi \ ; \ -\infty < y < \infty \tag{2.2}$$

with wall conditions as shown in Figure 1.

To solve this problem we appeal directly to Fourier's Law (2.2.2) and the Duhamel-Neumann equations (2.3.7). Writing the temperature in the form (2.3.3) we obtain the equation

$$Q_0 = \frac{-\kappa T_0}{L}\frac{\partial \theta}{\partial y}(x,y) \tag{2.3}$$

Integrating (3.2.3) yields the temperature function

$$\theta(x,y) = -\frac{Q_0 L}{\kappa T_0} y \tag{2.4}$$

Assuming the displacements are independent of x we obtain from (2.3.7) and (2.2.23) the pair of ordinary differential equations

$$\frac{d^2u}{dy^2} = 0 \ , \ \frac{d^2v}{dy^2} = \frac{-E\alpha Q_0 L}{\kappa(1-\nu)p_0} \tag{2.5}$$

and hence, to within a rigid body motion, the displacements are

$$u(x,y) = 0 \ , \ v(x,y) = \frac{-E\alpha Q_0 L}{2\kappa(1-\nu)p_0} y^2 \tag{2.6}$$

Employing (2.3.4) and (2.3.6) results in the stresses

$$\begin{aligned} s_{xy}(x,y) &= s_{yy}(x,y) = 0 \\ s_{xx}(x,y) &= \frac{E\alpha Q_0 L}{\kappa(1-\nu)p_0} y \end{aligned} \tag{2.7}$$

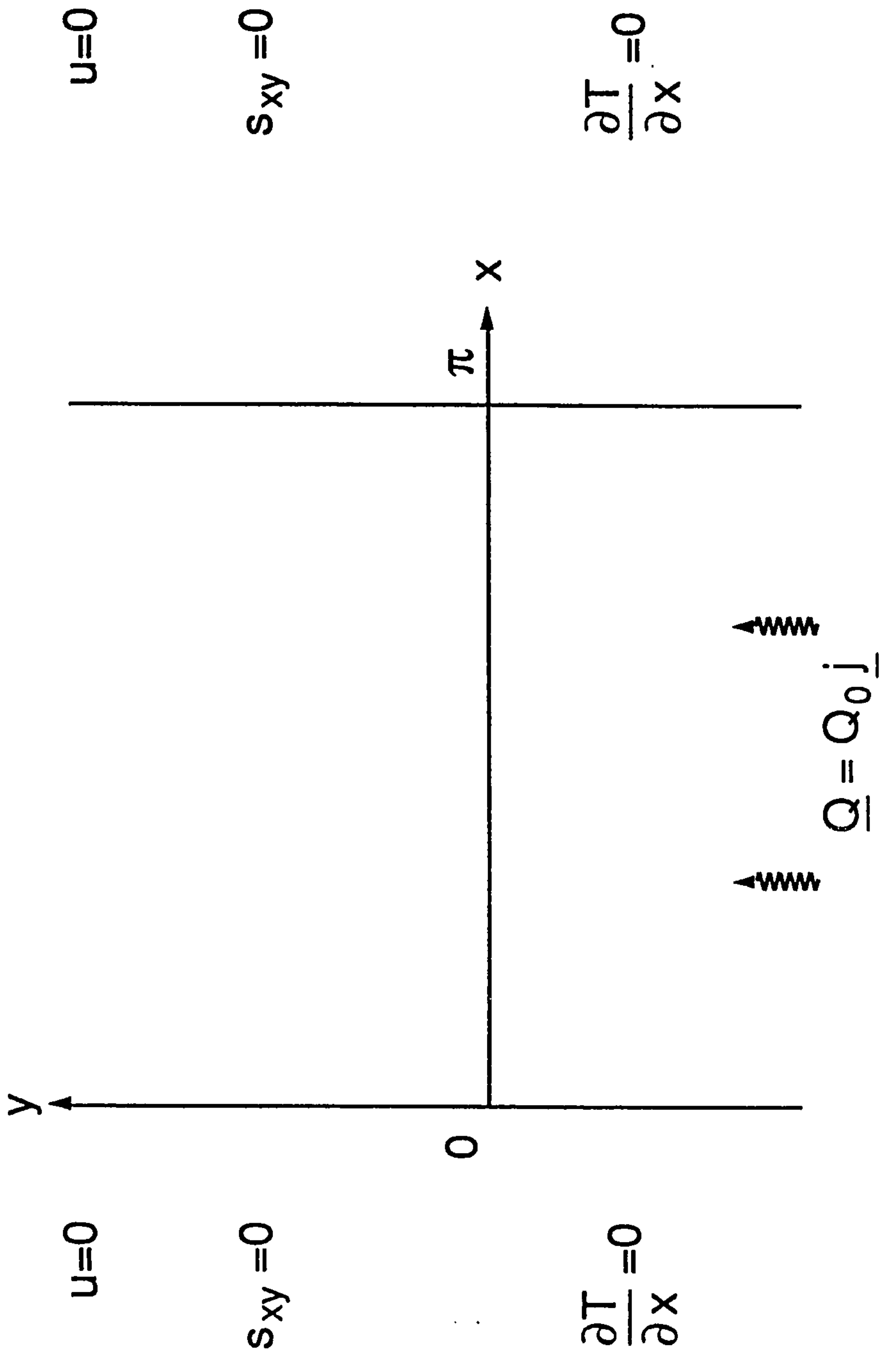

Figure 1. Boundary conditions for problem A.

3.3 Problem B

Consider the strip determined by the inequalities

$$-\infty < x < \infty \ ; \ -h < y < h \tag{3.1}$$

with wall conditions as shown in Figure 2. The different temperatures on the edges $y = -h$ and $y = h$ ($T_1 > T_2$) cause a heat flow in the y-direction of strength

$$Q_0 = \frac{(T_1 - T_2)\kappa}{2hL} \tag{3.2}$$

with corresponding temperature function

$$\theta(x,y) = \left(\frac{T_1 + T_2 - 2T_0}{2T_0}\right) - \left(\frac{T_1 - T_2}{2T_0}\right)\frac{y}{h} \tag{3.3}$$

Notice that for this configuration

$$s_{xx}(x,y) = s_{yy}(x,y) = s_{xy}(x,y) = 0 \tag{3.4}$$

Using (2.3.4) and (2.3.6) we can find the displacements due to the even and odd parts of the temperature function and superimpose the solutions to find

$$u(x,y) = M\left[\frac{T_1 + T_2 - 2T_0}{T_1 - T_2}hx - xy\right] \tag{3.5}$$

$$v(x,y) = M\left[\frac{T_1 + T_2 - 2T_0}{T_1 - T_2}hy + \frac{1}{2}(x^2 - y^2)\right] \tag{3.6}$$

where $$M = \frac{E\alpha(T_1 - T_2)}{2hp_0} \tag{3.7}$$

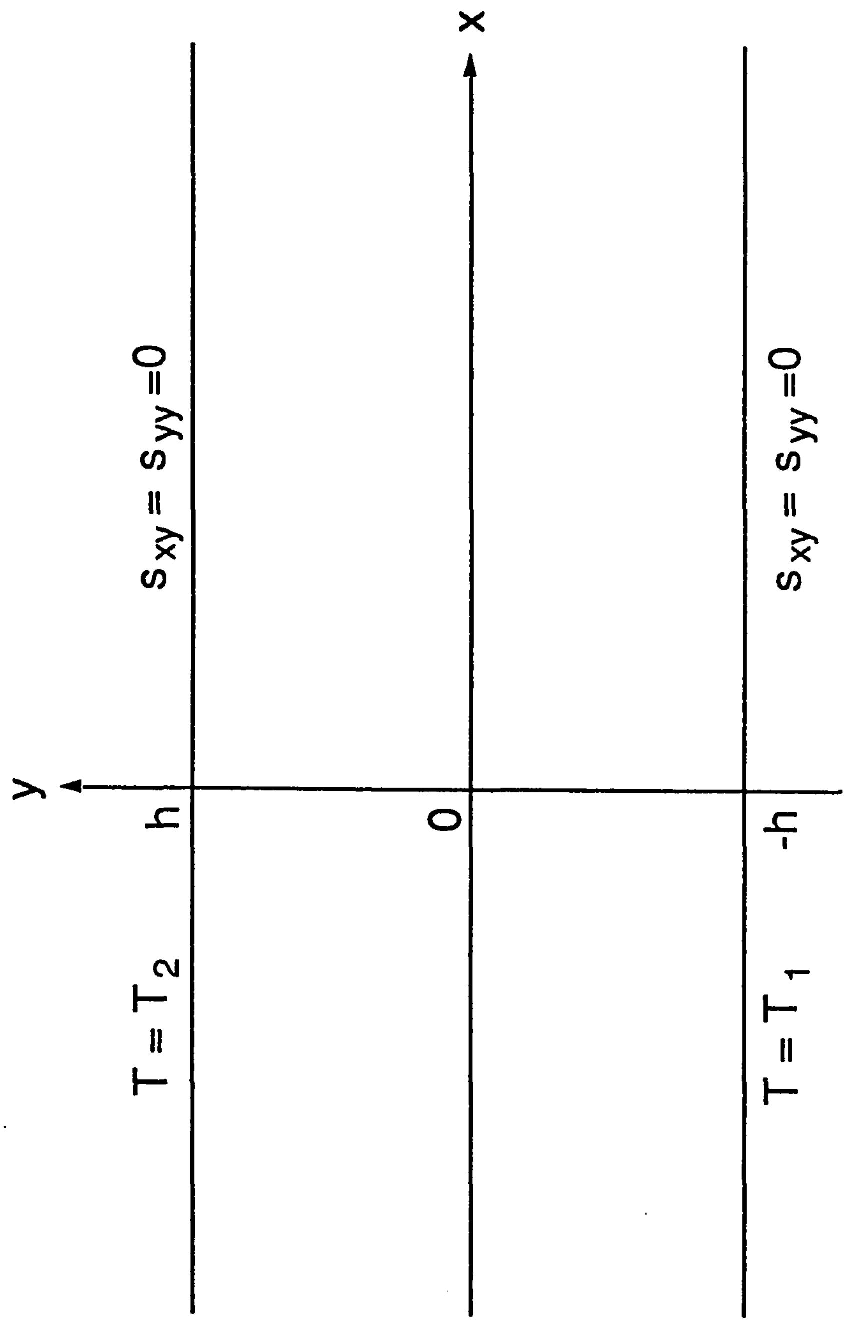

Figure 2. Boundary conditions for problem B.

3.4 Problem C

Consider the rectangle determined by the inequalities

$$0 < x < \pi \ ; \ -h < y < h \tag{4.1}$$

with wall condition as shown in Figure 3. The different temperatures on the edges $y = -h$ and $y = h$ cause a heat flow given by (3.3.2) and temperature function by (3.3.3).

The displacements are assumed to be independent of x. From equations (2.3.4), (2.3.6), (2.3.7) and (2.2.23) we obtain the field

$$\begin{aligned} u(x,y) &= s_{xy}(x,y) = s_{yy}(x,y) = 0 \\ v(x,y) &= -\frac{E\alpha(T_1 - T_2)}{4h(1-\nu)p_0} y^2 \\ s_{xx}(x,y) &= \frac{E\alpha(T_1 - T_2)}{2h(1-\nu)p_0} y \end{aligned} \tag{4.2}$$

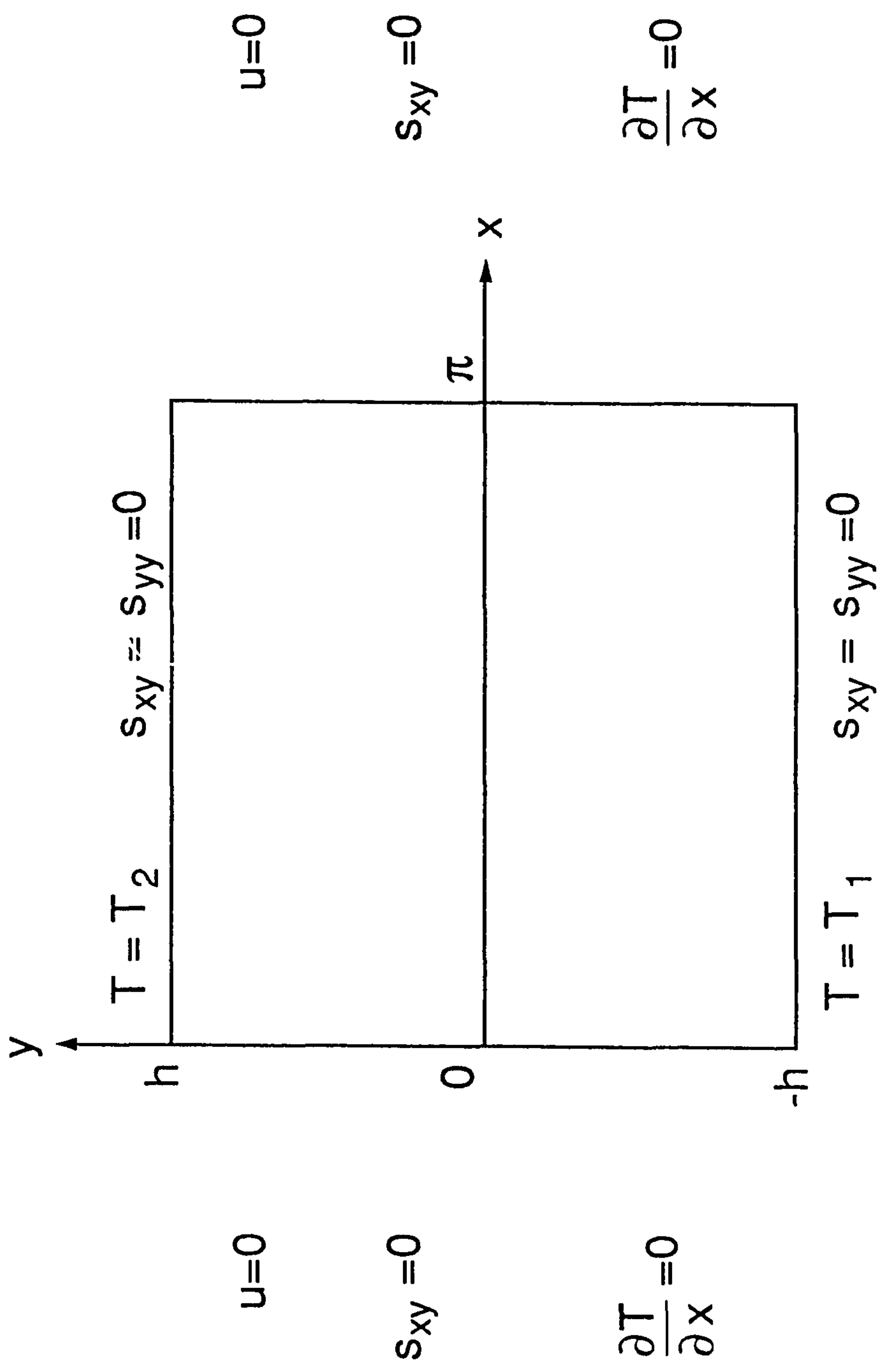

Figure 3. Boundary conditions for problem C.

Chapter 4

Uniform Heat Flow Past A Collinear Array of Griffith Cracks

4.1 Introduction

The first problem to be considered is the determination of the thermal stresses when a collinear array of Griffith cracks disturb a uniform heat flow in an infinite, two-dimensional, linear, isotropic, homogeneous thermoelastic solid under plane strain conditions. The solid occupies the (X, Y) plane while the cracks, which are thermally insulated and traction free, are assumed to form a periodic array defined by the inequalities

$$0 < A < | X - 2NP | < B < P \quad ; \quad N = 0, \pm 1, \ldots \tag{1.1}$$

on $Y = 0$. It is to be assumed that the undisturbed heat flow is given by

$$\boldsymbol{Q} = Q_0 \boldsymbol{j} \tag{1.2}$$

Extensive use of separation of variables yields suitable representations for the harmonic functions in Sneddon's solution (2.3.9), (2.3.10) which leads to triple series relations whose solutions are known.

4.2 Resolution Into Problems 1 and 2

To use the dimensionless form of Sneddon's solution (2.3.9), (2.3.10) the following quantities are introduced

$$L = \frac{P}{\pi} \quad \text{and} \quad \mathrm{p}_0 = \frac{\mathrm{E}\alpha \mathrm{Q}_0 \mathrm{L}}{2\kappa(1-\nu)} \tag{2.1}$$

which have the appropriate dimensions of length and stress respectively. In addition, the dimensionless parameters

$$a = \frac{A}{L} \quad \text{and} \quad b = \frac{B}{L} \tag{2.2}$$

are defined.

In the absence of cracks and by virtue of symmetry, it is clear that the thermoelastic field is the solution to problem A of chapter 3. Scaling this solution using equations (4.2.1) gives the field below.

$$\begin{aligned}
\theta^0(x,y) &= -\frac{Q_0 L}{\kappa T_0} y \\
v^0(x,y) &= -y^2 \\
s^0_{xx}(x,y) &= 2y
\end{aligned} \tag{2.3}$$

and

$$u^0(x,y) = s^0_{xy}(x,y) = s^0_{yy}(x,y) = 0$$

The presence of the cracks perturb this field thereby yielding a new field

$$\begin{aligned}
\theta(x,y) &= \theta^0(x,y) + \theta^p(x,y) \\
v(x,y) &= v^0(x,y) + v^p(x,y) \\
s_{xx}(x,y) &= s^0_{xx}(x,y) + s^p_{xx}(x,y) \\
u(x,y) &= u^p(x,y)\,,\; s_{xy}(x,y) = s^p_{xy}(x,y), \\
s_{yy}(x,y) &= s^p_{yy}(x,y)
\end{aligned} \tag{2.4}$$

where the perturbations, denoted by the superscript p, can be found by solving the following mixed boundary value problem.

Problem Solve the dimensionless, plane strain equations of linear thermoelasticity in the region $0 < x < \pi$, $0 < y < \infty$ subject to the conditions:

1. At infinity $\theta^p, s^p_{ij} \sim O(1)$ and $u^p, v^p \sim O(y)$
2. $u^p(0,y) = u^p(\pi,y) = s^p_{xy}(0,y) = s^p_{xy}(\pi,y) = 0 \qquad (y > 0)$
3. $s^p_{yy}(x,0) = 0 \qquad (0 < x < \pi)$
4. $u^p(x,0) = 0 \qquad (0 < x < a) \cup (b < x < \pi)$

   $s^p_{xy}(x,0) = 0 \qquad (a < x < b)$
5. $\dfrac{\partial \theta^p}{\partial x}(0,y) = \dfrac{\partial \theta^p}{\partial x}(\pi,y) = 0 \qquad (y > 0)$
6. $\theta^p(x,0) = 0 \qquad (0 < x < a) \cup (b < x < \pi)$

   $\dfrac{\partial \theta^p}{\partial y}(x,y) = M \qquad (a < x < b)$

where $$M = \frac{Q_0 L}{\kappa T_0}$$

The solution of this problem is obtained by superimposing the solutions of appropriate thermal and isothermal boundary value problems, which will be referred to by problem 1 and problem 2 respectively.

Problem 1 is found by removing the zero shear stress condition at the crack site. A thermal field can be found that satisfies all boundary conditions but will give rise to a thermal load at the crack site. Problem 2 is a relatively simple isothermal problem chosen to cancel this contribution, thereby leaving the crack traction free as desired. This is a standard technique which will be employed in subsequent chapters. These two problems are stated now.

Problem 1 Find a solution to the dimensionless, plane strain equations of linear thermoelasticity in the region $0 < x < \pi$, $y > 0$ subject to the conditions:

1. At infinity $\theta^1, s^1_{ij} \sim O(1)$ and $u^1, v^1 \sim O(y)$
2. $u^1(0,y) = u^1(\pi,y) = s^1_{xy}(0,y) = s^1_{xy}(\pi,y) = 0 \qquad (y > 0)$
3. $s^1_{yy}(x,0) = 0 \qquad (0 < x < \pi)$
4. $u^1(x,0) = 0 \qquad (0 < x < \pi)$
5. $\dfrac{\partial \theta^1}{\partial x}(0,y) = \dfrac{\partial \theta^1}{\partial x}(\pi,y) = 0 \qquad (y > 0)$
6. $\theta^1(x,0) = 0 \qquad (0 < x < a) \cup (b < x < \pi)$

   $\dfrac{\partial \theta^1}{\partial x}(x,0) = M \qquad (a < x < b)$

Calculate $\quad s^1_{xy}(x,0) = f(x) \qquad (a < x < b)$

Problem 2 Solve the dimensionless plane strain equations of linear isothermal elasticity in the region $0 < x < \pi$, $y > 0$ subject to the conditions:

1. At infinity $s_{ij}^2 \sim O(y^{-1})$ and $u^2, v^2 \sim O(1)$
2. $u^2(0,y) = u^2(\pi,y) = s_{xy}^2(0,y) = s_{xy}^2(\pi,y) = 0 \quad (y > 0)$
3. $s_{yy}^2(x,0) = 0 \quad (0 < x < \pi)$
4. $u^2(x,0) = 0 \quad (0 < x < a) \cup (b < x < \pi)$

   $s_{xy}^2(x,0) = -f(x) \quad (a < x < b)$

4.3 The Solution of Problem 1

In the special case of Sneddon's solution (2.3.9), (2.3.10) set $\chi = -\phi$ and $\psi = -\beta^2 \dfrac{\partial \phi}{\partial y}$ where

$$\frac{\partial \phi}{\partial y}(x,y) = A_0 y - \sum_{n=1}^{\infty} n^{-2} A_n e^{-ny} \cos nx \tag{3.1}$$

then

$$u^1(x,y) = -y \sum_{n=1}^{\infty} n^{-1} A_n e^{-ny} \sin nx$$

$$\theta^1(x,y) = -M \left[A_0 + \sum_{n=1}^{\infty} n^{-1} A_n e^{-ny} \cos nx \right] \tag{3.2}$$

$$s_{xy}^1(x,y) = -\frac{\partial}{\partial y} \left[y \sum_{n=1}^{\infty} n^{-1} A_n e^{-ny} \sin nx \right]$$

and

$$s^1_{yy}(x,y) = y\sum_{n=1}^{\infty} A_n e^{-ny}\cos nx$$

Clearly all of the conditions in problem 1 will be satisfied provided the $\{A_n\}_0^\infty$ solve the following triple series relations

$$\begin{aligned} G_1(x) &= A_0 + \sum_{n=1}^{\infty} n^{-1}A_n\cos nx &=& \; 0 \;\; (0<x<a)\cup(b<x<\pi) \\ F_1(x) &= \sum_{n=1}^{\infty} A_n\cos nx &=& \; 1 \;\; (a<x<b) \end{aligned} \tag{3.3}$$

The triple series (4.3.3) have been solved by Parihar [16], from which it follows that by setting

$$A_0 = -\frac{1}{\pi}\int_a^b tp(t)dt \tag{3.4}$$

$$A_n = -\frac{2}{\pi}\int_a^b p(t)\sin nt dt \quad (n \geq 1) \tag{3.5}$$

we obtain

$$G_1(x) = \begin{cases} -\int_a^b p(t)dt & (0<x<a) \\ -\int_x^b p(t)dt & (a<x<b) \\ 0 & (b<x<\pi) \end{cases} \tag{3.6}$$

where the following series identity has been used [17]

$$t + 2\sum_{n=1}^{\infty} n^{-1}\sin nt\cos nx = \pi H(t-x) \quad (0<x,t<\pi) \tag{3.7}$$

The conditions on $G_1(x)$ stated in (4.3.3) can now be satisfied provided

$$\int_a^b p(t)dt = 0 \tag{3.8}$$

Additionally, we find that

$$\theta^1(x,0) = -MH[(b-x)(x-a)]\int_a^x p(t)dt \tag{3.9}$$

where $H(u)$ is the Heaviside step function.

Furthermore, using the identity [17]

$$\sum_{n=1}^{\infty} n^{-1}\sin nt \sin nx = \frac{1}{2}\log\left|\frac{\sin\frac{1}{2}(x+t)}{\sin\frac{1}{2}(x-t)}\right| \quad (0 < x, t < \pi) \tag{3.10}$$

it can be shown that

$$F_1(x) = \frac{1}{\pi}\int_a^b p(t)\frac{\sin t}{\cos t - \cos x}dt \tag{3.11}$$

It is apparent that (4.3.3) will be solved if $p(t)$ is given by the singular integral equation

$$\frac{1}{\pi}\int_a^b p(t)\frac{\sin t}{\cos t - \cos x}dt = 1 \quad (a < x < b) \tag{3.12}$$

subject to (4.3.8).

A simple change of variables reduces (4.3.12) to the finite Hilbert transform [18],

$$\frac{1}{\pi}\int_{\xi_1}^{\xi_2} \frac{R(\xi)}{\xi-\eta}d\xi = S(\eta) \qquad (\xi_1 < \eta < \xi_2) \tag{3.13}$$

the solution of which can be written in each of the three forms:

$$R(\eta) = \frac{C_1}{\sqrt{(\xi_2-\eta)(\eta-\xi_1)}} - \frac{1}{\pi\sqrt{(\xi_2-\eta)(\eta-\xi_1)}}\int_{\xi_1}^{\xi_2}\frac{\sqrt{(\xi_2-\xi)(\xi-\xi_1)}S(\xi)}{\xi-\eta}d\xi \tag{3.14}$$

or

$$R(\eta) = \frac{C_2}{\sqrt{(\xi_2 - \eta)(\eta - \xi_1)}} - \frac{1}{\pi}\sqrt{\frac{\xi_2 - \eta}{\eta - \xi_1}} \int_{\xi_1}^{\xi_2} \sqrt{\frac{\xi - \xi_1}{\xi_2 - \xi}} \frac{S(\xi)}{\xi - \eta} d\xi \tag{3.15}$$

or

$$R(\eta) = \frac{C_3}{\sqrt{(\xi_2 - \eta)(\eta - \xi_1)}} - \frac{1}{\pi}\sqrt{\frac{\eta - \xi_1}{\xi_2 - \eta}} \int_{\xi_1}^{\xi_2} \sqrt{\frac{\xi_2 - \xi}{\xi - \xi_1}} \frac{S(\xi)}{\xi - \eta} d\xi \tag{3.16}$$

where C_1, C_2, C_3 are arbitrary constants. Using the first of these representations along with the condition (4.3.8) we obtain the solution

$$p(t) = \frac{P(t)}{\Delta(t)} \tag{3.17}$$

with

$$P(t) = (1 + \cos t) - (1 + \cos b)\Pi(\alpha^2, k')/K' \tag{3.18}$$

where $K' = K(k')$ and Π are associated complete elliptic integrals of the first and third kinds respectively, with parameters

$$k = \frac{\tan\frac{a}{2}}{\tan\frac{b}{2}} \quad \text{and} \quad \alpha^2 = \frac{\cos a - \cos b}{1 + \cos a} \tag{3.19}$$

and

$$\Delta(t) = \{(\cos a - \cos t)(\cos t - \cos b)\}^{1/2} \tag{3.20}$$

It can also be shown that

$$F_1(x) = \begin{cases} 1 - P(x)/\Delta_1(x) & (0 < x < a) \\ 1 & (a < x < b) \\ 1 + P(x)/\Delta_1(x) & (b < x < \pi) \end{cases} \tag{3.21}$$

where

$$\Delta_1(x) = \{(\cos a - \cos x)(\cos b - \cos x)\}^{1/2} \tag{3.22}$$

To obtain $s^1_{xy}(x,0)$ for $(a < x < b)$ note that

$$\frac{\partial}{\partial x} s^1_{xy}(x,0) = \frac{\partial^3 \phi}{\partial y^3}(x,0) = -F_1(x) \tag{3.23}$$

and so

$$s^1_{xy}(x,0) = -x + \frac{\pi F(\frac{b}{2}, k')}{K'} = f(x) \tag{3.24}$$

where $F(\theta, k)$ is the incomplete elliptic integral of the first kind.

The following integrals [19] were found useful in obtaining (4.3.18) and (4.3.24)

$$\int_a^b \frac{dt}{\Delta(t)} = gK' \tag{3.25}$$

$$\int_a^b \frac{\cos t dt}{\Delta(t)} = g\{(1 + \cos b)\Pi(\alpha^2, k') - K'\} \tag{3.26}$$

$$\int_0^a \frac{dt}{\Delta_1(t)} = gK \tag{3.27}$$

$$\int_0^a \frac{\cos t dt}{\Delta_1(t)} = g\{(\cos a - \cos b)\Pi(\gamma^2, k) + K \cos b\} \tag{3.28}$$

where

$$g = \frac{2}{\sqrt{(1 + \cos a)(1 - \cos b)}} \tag{3.29}$$

and

$$\gamma^2 = \frac{1-\cos a}{1-\cos b} \tag{3.30}$$

4.4 Isotherms

At this point the effect of the cracks on the temperature field is shown. By substituting (4.3.4) and (4.3.5) into (4.3.2) it can be shown that the perturbation of the temperature has the form

$$\theta^1(x,y) = \frac{M}{\pi}\int_a^b p(t)\{t + J(t+x) + J(t-x)\}dt \tag{4.1}$$

where

$$J(x) = \tan^{-1}\left\{\frac{\sin x}{e^y - \cos x}\right\} \tag{4.2}$$

with $p(t)$ defined by (4.3.17).

Using (4.3.9) and (4.3.17) we find the temperature on the crack to be

$$\theta^1(x,0) = Mg\left\{\Pi(\eta,\alpha^2,k') - \frac{\Pi(\alpha^2,k')F(\eta,k')}{K'}\right\} \tag{4.3}$$

where $\Pi(\eta,\alpha^2,k')$ is the incomplete elliptic integral of the third kind with

$$\sin^2\eta = \frac{(1+\cos a)(\cos x - \cos b)}{(1+\cos x)(\cos a - \cos b)} \tag{4.4}$$

and g defined previously by (4.3.29).

Again, the following integrals [19] were useful in obtaining (4.4.3)

$$\int_x^b \frac{\cos t dt}{\Delta(t)} = g\{(1+\cos b)\Pi(\eta,\alpha^2,k') - F(\eta,k')\} \tag{4.5}$$

$$\int_x^b \frac{dt}{\Delta(t)} = gF(\eta,k') \tag{4.6}$$

Evaluating (4.4.1) and (4.4.3) enables the level curves

$$\frac{\theta^1(x,y)}{M} = \text{constant} \tag{4.7}$$

to be plotted using the software package Surfer [20]. The results of such calculations are recorded in Figure 4.

4.5 The Solution of Problem 2

In the special case of Sneddon's solution (2.3.9), (2.3.10) set $\psi = 0$ and $\chi = (\beta^2 - 1)\phi$ where

$$\phi(x,y) = (\beta^2-1)^{-1}\sum_{n=1}^{\infty} n^{-2}B_n e^{-ny}\cos nx \tag{5.1}$$

then

$$\begin{aligned} u^2(x,y) &= \sum_{n=1}^{\infty} n^{-1}B_n e^{-ny}\left\{ny - \frac{\beta^2}{\beta^2-1}\right\}\sin nx \\ s^2_{xy}(x,y) &= \frac{\partial}{\partial y}\left[y\sum_{n=1}^{\infty} B_n e^{-ny}\sin nx\right] \end{aligned} \tag{5.2}$$

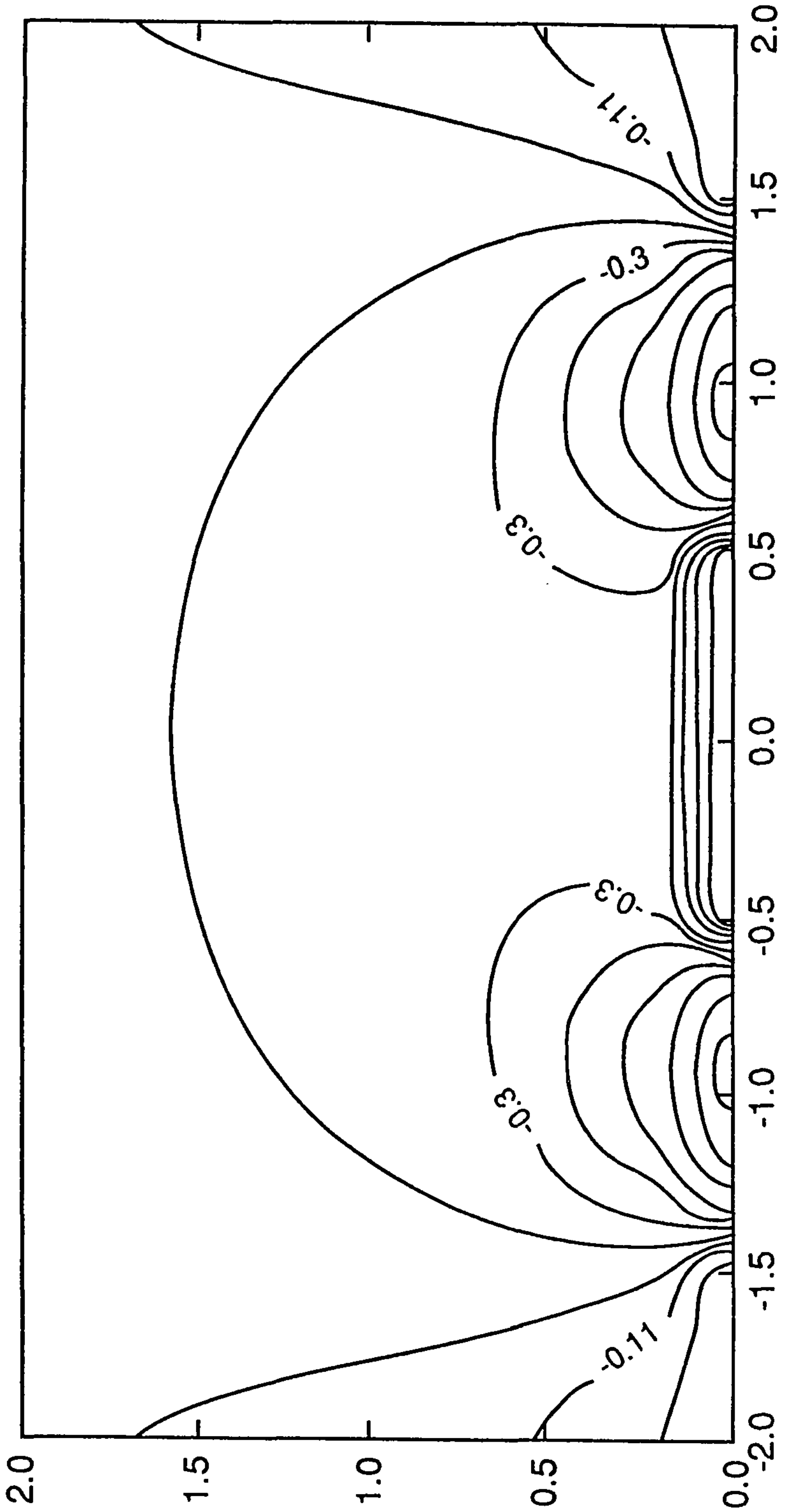

Figure 4. Isotherms associated with a uniform heat flow disturbed by an array of Griffith cracks.

and

$$s^2_{yy}(x,y) = -y\sum_{n=1}^{\infty} nB_n e^{-ny}\cos nx$$

As before, all conditions of problem 2 are satisfied with the exception of the mixed boundary condition 4. This condition will be satisfied provided the $\{B_n\}_1^\infty$ solve the triple series relations

$$\begin{aligned} G_2(x) &= \sum_{n=1}^{\infty} n^{-1}B_n \sin nx = 0 \quad (0 < x < a) \cup (b < x < \pi) \\ F_2(x) &= \sum_{n=1}^{\infty} B_n \sin nx = -f(x) \quad (a < x < b) \end{aligned} \tag{5.3}$$

The triple series (4.5.3) have also been studied by Parihar [16] who shows that

$$B_n = \frac{2}{\pi}\int_a^b q(t)\cos nt\,dt \qquad (n \geq 1) \tag{5.4}$$

From this substitution, $G_2(x)$ takes the form

$$G_2(x) = \begin{cases} -\dfrac{x}{\pi}\displaystyle\int_a^b q(t)dt & (0 < x < a) \\ \displaystyle\int_a^x q(t)dt - \dfrac{x}{\pi}\int_a^b q(t)dt & (a < x < b) \\ \dfrac{(\pi - x)}{\pi}\displaystyle\int_a^b q(t)dt & (b < x < \pi) \end{cases} \tag{5.5}$$

again using the identity (4.3.7). To satisfy (4.5.3) it is required that

$$\int_a^b q(t)dt = 0 \tag{5.6}$$

Furthermore, using the series identity [17]

$$\sum_{n=1}^{\infty} n^{-1} \cos nx \cos nt = \frac{1}{2} \log \mid 2(\cos x - \cos t) \mid \qquad (0 < x, t < \pi) \tag{5.7}$$

it can be shown that

$$F_2(x) = -\frac{1}{\pi} \int_a^b q(t) \frac{\sin x}{\cos t - \cos x} dt \tag{5.8}$$

It is now apparent that (4.5.3) will be solved if $q(t)$ is given by the singular integral equation

$$\frac{1}{\pi} \int_a^b q(t) \frac{\sin x}{\cos t - \cos x} dt = f(x) \qquad (a < x < b) \tag{5.9}$$

subject to (4.5.6).

The solution of (4.5.9) subject to (4.5.6) is found to be

$$q(t) = \frac{-\sin t}{\pi \Delta(t)} \int_a^b \frac{\Delta(x) f(x)}{\cos t - \cos x} dx \tag{5.10}$$

and so

$$u^2(x,0) = -2(1-\nu) H[(b-x)(x-a)] \int_a^x q(t) dt \tag{5.11}$$

where $H(u)$ is the Heaviside function and

$$s_{xy}^2(x,0) = \begin{cases} \dfrac{\sin x}{\pi \Delta_1(x)} \displaystyle\int_a^b \frac{\Delta(y) f(y)}{\cos x - \cos y} dy & (0 < x < a) \\ -f(x) & (a < x < b) \\ \dfrac{-\sin x}{\pi \Delta_1(x)} \displaystyle\int_a^b \frac{\Delta(y) f(y)}{\cos x - \cos y} dy & (b < x < \pi) \end{cases} \tag{5.12}$$

with $\Delta(x)$ and $\Delta_1(x)$ defined by (4.3.20) and (4.3.22) respectively.

4.6 Stress Intensity Factors

The stress intensity factors at the crack tips $(A,0)$ and $(B,0)$ are defined respectively by

$$k_{II}(A) = \lim_{X\to A^-}\sqrt{2(A-X)}\,\sigma_{XY}(X,0) = p_0\sqrt{L}\lim_{x\to a^-}\sqrt{2(a-x)}\,s_{xy}(x,0) \tag{6.1}$$

and

$$k_{II}(B) = \lim_{X\to B^+}\sqrt{2(X-B)}\,\sigma_{XY}(X,0) = p_0\sqrt{L}\lim_{x\to b^+}\sqrt{2(x-b)}\,s_{xy}(x,0) \tag{6.2}$$

On using (4.5.12) and scaling with respect to

$$k_0 = \frac{p_0\sqrt{2L}}{8}(b-a)^{3/2} \tag{6.3}$$

which is the stress intensity factor for a single crack in an infinite sheet subject to a constant temperature gradient [4], we find that

$$\frac{k_{II}(A)}{k_0} = \frac{8}{\pi(b-a)}\left[\frac{\sin a}{(b-a)(\cos a-\cos b)}\right]^{1/2}\int_a^b \frac{\Delta(y)f(y)}{\cos a-\cos y}dy \tag{6.4}$$

and

$$\frac{k_{II}(B)}{k_0} = \frac{-8}{\pi(b-a)}\left[\frac{\sin b}{(b-a)(\cos a-\cos b)}\right]^{1/2}\int_a^b \frac{\Delta(y)f(y)}{\cos b-\cos y}dy \tag{6.5}$$

Finally, on substituting (4.3.24) into (4.6.4) and (4.6.5) we obtain

$$\frac{k_{II}(A)}{k_0} = \frac{8}{(b-a)^{3/2}}\left(\frac{\sin a}{\cos a - \cos b}\right)^{1/2}\left\{C_1 - \frac{1}{\pi}\int_a^b \left\{\frac{\cos y - \cos b}{\cos a - \cos y}\right\}^{1/2} y dy\right\} \tag{6.6}$$

and

$$\frac{k_{II}(B)}{k_0} = \frac{8}{(b-a)^{3/2}}\left(\frac{\sin b}{\cos a - \cos b}\right)^{1/2}\left\{C_2 - \frac{1}{\pi}\int_a^b \left\{\frac{\cos a - \cos y}{\cos y - \cos b}\right\}^{1/2} y dy\right\} \tag{6.7}$$

where

$$C_1 = 2F\left(\frac{b}{2}, k'\right) Z\left(\frac{b}{2}, k'\right) \tag{6.8}$$

and

$$C_2 = 2F\left(\frac{b}{2}, k'\right) Z\left(\frac{\pi - a}{2}, k'\right) \tag{6.9}$$

and $Z(\theta, k)$ is Jacobi's Zeta function.

Table 1 shows the values of $k_{II}(A)/k_0$ and $-k_{II}(B)/k_0$ for a small crack as a tends to zero and b fixed. The asymptotics for this case give $k_{II}(A)/k_0$ tending to zero and $-k_{II}(B)/k_0$ tending to $2\sqrt{2}$.

Table 2 shows the variation of $k_{II}(A)/k_0$ and $-k_{II}(B)/k_0$ as crack length increases with the midpoint of the crack (denoted by c) held fixed. As expected, for a small crack and c near $\frac{\pi}{2}$, both $k_{II}(A)/k_0$ and $-k_{II}(B)/k_0$ tend to one.

a	b	$k_{II}(A)/k_0$	$-k_{II}(B)/k_0$
10^{-6}	0.1	$.755\times10^{-2}$	2.390
10^{-8}	0.1	$.792\times10^{-3}$	2.505
10^{-10}	0.1	$.813\times10^{-4}$	2.572
10^{-12}	0.1	$.827\times10^{-5}$	2.616
10^{-14}	0.1	$.837\times10^{-6}$	2.645

Table 1. Values of $k_{II}(A)/k_0$ and $-k_{II}(B)/k_0$ when b is fixed and a approaches zero.

a	b	c	$k_{II}(A)/k_0$	$-k_{II}(B)/k_0$
1.4	1.6	1.5	.996	1.004
1.3	1.7	1.5	.993	1.007
1.2	1.8	1.5	.989	1.011
0.6	0.8	0.7	.940	1.060
0.4	1.0	0.7	.810	1.190
0.2	1.2	0.7	.633	1.370
0.1	1.3	0.7	.490	1.520

Table 2. Values of $k_{II}(A)/k_0$ and $-k_{II}(B)/k_0$ when crack mid-point is held fixed and crack length varies.

Chapter 5

The Linear Thermoelastic Problem For A Strip With a Line Crack Parallel To Its Edges

5.1 Introduction

The problem to be considered is that of finding the stress intensity factors of a line crack in a linear, isotropic, thermoelastic strip whose surfaces are traction free but held at different temperatures. The strip, hereafter referred to by S, occupies the region $-\infty < X < \infty, -H < Y < H$ and is assumed to deform under plane strain conditions while the crack, $-C < X < C, Y = 0$ is thermally insulated and traction free. The surfaces of S, $Y = -H$ and $Y = H$, are held at fixed temperatures denoted by T_1 and T_2 respectively $(T_1 > T_2)$.

The solution presented here utilizes integral transform techniques to represent the harmonic functions appearing in Sneddon's solution (2.3.9), (2.3.10). The problem then reduces to a singular integral equation which is solved numerically.

The appropriate stress intensity factor is found to depend on this numerical solution and the results are shown in graphical form.

5.2 Resolution into Problems 1 and 2

To use the dimensionless form of Sneddon's solution (2.3.9), (2.3.10) the following quantities are introduced

$$L = C \qquad \text{and} \qquad p_0 = \frac{E\alpha(T_1 - T_2)C}{4H(1-\nu)} \tag{2.1}$$

which have dimensions of length and stress respectively. In addition the dimensionless parameter

$$h = H/C \tag{2.2}$$

is defined.

It is readily shown that, in the absence of the crack, the thermoelastic field is the solution to problem B of chapter 3 and is given by

$$\begin{aligned} T(x,y) &= T_0[1 + \theta^0(x,y)] \\ \theta^0(x,y) &= \left(\frac{T_1 + T_2 - 2T_0}{2T_0}\right) - \left(\frac{T_1 - T_2}{2T_0}\right)\frac{y}{h} \\ u^0(x,y) &= 2(1-\nu)\left[\frac{T_1 + T_2 - 2T_0}{T_1 - T_2}hx - xy\right] \\ v^0(x,y) &= 2(1-\nu)\left[\frac{T_1 + T_2 - 2T_0}{T_1 - T_2}hy + \frac{1}{2}(x^2 - y^2)\right] \end{aligned} \tag{2.3}$$

and

$$s^0_{xx}(x,y) = s^0_{xy}(x,y) = s^0_{yy}(x,y) = 0$$

The presence of the crack disturbs this field thereby yielding a new field

$$\begin{aligned} \theta(x,y) &= \theta^0(x,y) + \theta^p(x,y) \\ u(x,y) &= u^0(x,y) + u^p(x,y) \\ v(x,y) &= v^0(x,y) + v^p(x,y) \\ \text{and} \quad s_{ij}(x,y) &= s_{ij}^p(x,y) \end{aligned} \tag{2.4}$$

where by virtue of symmetry, the perturbations are obtained by solving the following mixed boundary value problem.

Problem Solve the dimensionless, plane strain equations of linear thermoelasticity in the strip $0 < x < \infty, 0 < y < h$ subject to the conditions:

1. At infinity $\theta^p, s_{ij}^p \sim O(x^{-1})$ and $u^p, v^p \sim O(\ln x)$
2. $u^p(0,y) = s_{xy}^p(0,y) = 0$ $\quad (0 < y < h)$
3. $s_{xy}^p(x,h) = s_{yy}^p(x,h) = 0$ $\quad (0 < x < \infty)$
4. $s_{yy}^p(x,0) = 0$ $\quad (0 < x < \infty)$
5. $s_{xy}^p(x,0) = 0$ $\quad (0 < x < 1)$

   $u^p(x,0) = 0$ $\quad (1 < x < \infty)$
6. $\dfrac{\partial \theta^p}{\partial x}(0,y) = 0$ $\quad (0 < y < h)$
7. $\theta^p(x,h) = 0$ $\quad (0 < x < \infty)$
8. $\dfrac{\partial \theta^p}{\partial y}(x,0) = \dfrac{T_1 - T_2}{2T_0 h}$ $\quad (0 < x < 1)$

   $\theta^p(x,0) = 0$ $\quad (1 < x < \infty)$

The solution of this problem is obtained by superimposing the solutions of the following two problems:

Problem 1 Solve the dimensionless, plane strain equations of linear thermoelasticity in the strip $0 < x < \infty$, $0 < y < h$ subject to the conditions:

1. At infinity $\theta^1, s^1_{ij} \sim O(x^{-1})$ and $u^1, v^1 \sim O(\ln x)$
2. $u^1(0,y) = s^1_{xy}(0,y) = 0$ $\qquad (0 < y < h)$
3. $s^1_{xy}(x,h) = s^1_{yy}(x,h) = 0$ $\qquad (0 < x < \infty)$
4. $s^1_{yy}(x,0) = 0$ $\qquad (0 < x < \infty)$
5. $u^1(x,0) = 0$ $\qquad (0 < x < \infty)$
6. $\dfrac{\partial \theta^1}{\partial x}(0,y) = 0$ $\qquad (0 < y < h)$
7. $\theta^1(x,h) = 0$ $\qquad (0 < x < \infty)$
8. $\dfrac{\partial \theta^1}{\partial y}(x,0) = \dfrac{T_1 - T_2}{2T_0 h}$ $\qquad (0 < x < 1)$

   $\theta^1(x,0) = 0$ $\qquad (1 < x < \infty)$

Calculate $\quad s^1_{xy}(x,0) = f(x) \qquad (0 < x < 1)$

Problem 2 Solve the dimensionless, plane strain equations of linear isothermal elasticity in the strip $0 < x < \infty$, $0 < y < h$ subject to the conditions:

1. At infinity $s^2_{ij} \sim O(x^{-2})$ and $u^2, v^2 \sim O(x^{-1})$
2. $u^2(0,y) = s^2_{xy}(0,y) = 0$ $\qquad (0 < y < h)$
3. $s^2_{xy}(x,h) = s^2_{yy}(x,h) = 0$ $\qquad (0 < x < \infty)$
4. $s^2_{yy}(x,0) = 0$ $\qquad (0 < x < \infty)$
5. $s^2_{xy}(x,0) = -f(x)$ $\qquad (0 < x < 1)$

   $u^2(x,0) = 0$ $\qquad (1 < x < \infty)$

5.3 The Solution of Problem 1

Using the special case of Sneddon's general solution (2.3.9), (2.3.10) let

$$\begin{aligned}\chi(x,y) &= \mathcal{F}_c\left[\frac{2hA(\xi)}{\xi^2 D(\xi h)}\left\{\mathrm{cth}\xi h - \xi h - \frac{1}{\xi h}\right\}\mathrm{sh}\xi y;\ \xi \to x\right] \\ &- \mathcal{F}_c\left[\frac{A(\xi)\mathrm{sh}\xi(h-y)}{\xi^3 \mathrm{sh}\xi h};\ \xi \to x\right] \qquad (3.1)\end{aligned}$$

$$\begin{aligned}\phi(x,y) &= -\frac{2}{\beta^2-1}\mathcal{F}_c\left[\frac{A(\xi)\mathrm{cth}\xi h}{\xi^3 D(\xi h)}\left\{\mathrm{cth}\xi h - \xi h - \frac{1}{\xi h}\right\}\mathrm{sh}\xi y;\ \xi \to x\right] \\ &- \frac{1}{h(\beta^2-1)}\mathcal{F}_c\left[\frac{A(\xi)\mathrm{sh}\xi y}{\xi^4 \mathrm{sh}^2\xi h};\ \xi \to x\right]\end{aligned}$$

$$+ \ \mathcal{F}_c\left[\frac{A(\xi)\text{sh}\xi(h-y)}{\xi^3\text{sh}\xi h}\,;\,\xi\rightarrow x\right] \tag{3.2}$$

and

$$\psi(x,y)=\beta^2\mathcal{F}_c\left[\frac{A(\xi)\text{ch}\xi(h-y)}{\xi^2\text{sh}\xi h}\,;\,\xi\rightarrow x\right] \tag{3.3}$$

where $\mathcal{F}_c$ is the Fourier Cosine Transform defined by

$$\mathcal{F}_c\left[g(\xi)\,;\,\xi\rightarrow x\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty g(\xi)\cos x\xi d\xi \tag{3.4}$$

and

$$D(\rho)=\text{sh}\,2\rho-2\rho \tag{3.5}$$

All the conditions of problem 1 are satisfied if $A(\xi)$ is given by the dual integral equations

$$\begin{aligned} F_1(x) &= \mathcal{F}_c[A(\xi)\text{cth}\xi h\,;\,x]=1 \qquad (0<x<1) \\ G_1(x) &= \mathcal{F}_c[\xi^{-1}A(\xi)\,;\,x]=0 \qquad (1<x<\infty) \end{aligned} \tag{3.6}$$

Let

$$A(\xi)=\sqrt{\frac{2}{\pi}}\int_0^1 p(t)\sin\xi t dt \tag{3.7}$$

then

$$G_1(x)=H(1-x)\int_x^1 p(t)dt \qquad (x>0) \tag{3.8}$$

where $H(x)$ is the Heaviside step function. Additionally

$$F_1(x)=\frac{1}{h}\int_0^1\frac{p(t)\text{sh}\frac{\pi t}{h}dt}{\text{ch}\frac{\pi t}{h}-\text{ch}\frac{\pi x}{h}} \tag{3.9}$$

and hence $p(t)$ must satisfy the singular integral equation

$$\frac{1}{h}\int_0^1 \frac{p(t)\mathrm{sh}\frac{\pi t}{h}dt}{\mathrm{ch}\frac{\pi t}{h}-\mathrm{ch}\frac{\pi x}{h}} = 1 \qquad (0 < x < 1) \tag{3.10}$$

with subsidiary condition

$$p(0) = 0 \tag{3.11}$$

which is due to zero flux across $x = 0$.

A simple change of variables reduces (5.3.10) to the finite Hilbert transform [18], and hence yields

$$p(t) = \left(\frac{\mathrm{ch}\frac{\pi t}{h}-1}{\mathrm{ch}\frac{\pi}{h}-\mathrm{ch}\frac{\pi t}{h}}\right)^{\frac{1}{2}} \tag{3.12}$$

It follows at once that

$$\theta^1(x,0) = -\frac{T_1-T_2}{2\pi T_0}H(1-x)\cos^{-1}\left\{2\frac{\mathrm{ch}^2\frac{\pi x}{2h}}{\mathrm{ch}^2\frac{\pi}{2h}}-1\right\} \tag{3.13}$$

and also that

$$\frac{\partial\theta^1}{\partial y}(x,0) = \frac{T_1-T_2}{2hT_0}\left\{1-H(x-1)\left(\frac{\mathrm{ch}\frac{\pi x}{h}-1}{\mathrm{ch}\frac{\pi x}{h}-\mathrm{ch}\frac{\pi}{h}}\right)^{1/2}\right\} \tag{3.14}$$

Next, by virtue of (2.3.10) and (5.3.1) it can be shown that

$$f(x) = -\mathcal{F}_s\left[\frac{A(\xi)}{\xi}\{1-M(\xi h)\};\ \xi\rightarrow x\right] \tag{3.15}$$

where

$$M(\rho) = \frac{1-2\rho+2\rho^2-e^{-2\rho}}{\mathrm{sh}\,2\rho-2\rho} \tag{3.16}$$

Therefore, on making use of (5.3.7), (5.3.8) and (5.3.12)

$$f(x) = \frac{1}{\pi}\int_{-1}^{1} G_1(t)K(t-x)dt \tag{3.17}$$

where

$$K(x) = \frac{1}{x} - \frac{1}{h}\int_0^\infty M(\rho)\sin\frac{\rho x}{h}d\rho \tag{3.18}$$

5.4 Isotherms

The effects of the crack can again be visualised. Using (2.3.9) along with (5.3.3) enables the perturbation of the temperature field to be written as

$$\theta^1(x,y) = -\frac{\beta^2}{\delta}\mathcal{F}_c\left[\frac{A(\xi)\mathrm{sh}\xi(h-y)}{\xi\mathrm{sh}\xi h}\quad;\quad \xi \to x\right] \tag{4.1}$$

Combining (5.3.7), (5.3.8) and (5.4.1) with symmetry reveals

$$\theta^1(x,y) = -\frac{\beta^2 \sin\frac{\pi(h-y)}{h}}{\delta 2h}\int_{-1}^{1} G_1(t)S(t+x,y)dt \tag{4.2}$$

where

$$S(t,y) = \frac{1}{\mathrm{ch}\frac{\pi t}{h} + \cos\frac{\pi(h-y)}{h}} \tag{4.3}$$

Evaluating (5.4.2) and (5.3.13) allows the level curves

$$\frac{\theta^1(x,y)}{\frac{\beta^2}{\delta}} = \text{constant} \tag{4.4}$$

to be plotted using the software package Surfer [20]. They are shown in Figure 5 for $H/C = 1$.

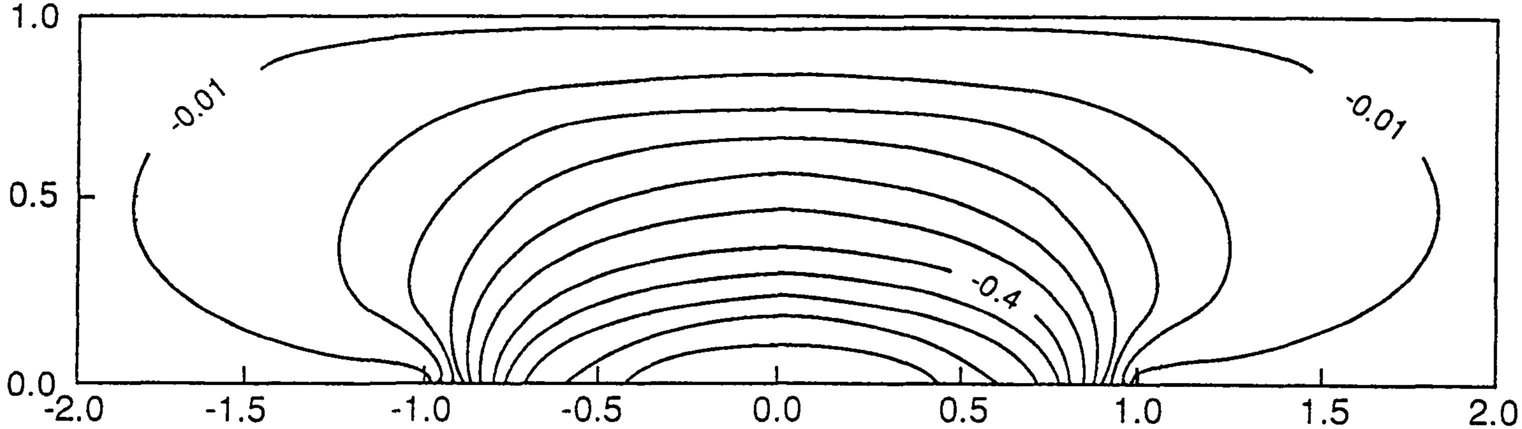

Figure 5. Isotherms associated with a uniform heat flow disturbed by a single crack parallel to the edges of a strip.

5.5 The Solution of Problem 2.

In the special case of Sneddon's solution (2.3.9), (2.3.10) let $\psi = 0$,

$$\chi(x,y) = \mathcal{F}_c\left[\frac{B(\xi)}{\xi^2}\left\{\mathrm{ch}\xi y + \frac{2(\xi^2h^2 - \mathrm{sh}^2\xi h)}{D(\xi h)}\mathrm{sh}\xi y\right\};\xi \to x\right] \tag{5.1}$$

and

$$\phi(x,y) = \frac{1}{\beta^2 - 1}\mathcal{F}_c\left[\frac{B(\xi)}{\xi^2}\left\{\mathrm{ch}\xi y - \frac{2\mathrm{sh}^2\xi h}{D(\xi h)}\mathrm{sh}\xi y\right\};\xi \to x\right] \tag{5.2}$$

then all the conditions of problem 2 are satisfied if $B(\xi)$ is given by the dual integral equations

$$\begin{aligned} F_2(x) &= \mathcal{F}_s[B(\xi)\{1 - M(\xi h)\};x] = -f(x) \qquad (0 < x < 1) \\ G_2(x) &= \mathcal{F}_s[\xi^{-1}B(\xi);x] = 0 \qquad (1 < x < \infty) \end{aligned} \tag{5.3}$$

Let

$$B(\xi) = \sqrt{\frac{2}{\pi}}\int_0^1 \frac{q(t)}{\sqrt{1-t^2}}(1 - \cos\xi t)dt \tag{5.4}$$

where

$$\int_0^1 \frac{q(t)}{\sqrt{1-t^2}}dt = 0 \tag{5.5}$$

then

$$G_2(x) = H(1-x)\int_x^1 \frac{q(t)}{\sqrt{1-t^2}}dt \qquad (0 \le x < \infty) \tag{5.6}$$

and

$$F_2(x) = \frac{1}{\pi}\int_0^1 \frac{q(t)}{\sqrt{1-t^2}}\{K(t-x) - K(t+x)\}dt \tag{5.7}$$

where $K(x)$ is given by (5.3.18). It follows at once that $q(t)$ is given by the singular integral equation

$$\frac{1}{\pi}\int_0^1 \frac{q(t)}{\sqrt{1-t^2}}\{K(t-x)-K(t+x)\}dt = -f(x) \qquad (0<x<1) \tag{5.8}$$

with subsidiary condition (5.5.5). By virtue of (5.3.17) and the symmetry of the problem, equation (5.5.8) and its subsidiary condition (5.5.5) may be written in the form

$$\frac{1}{\pi}\int_{-1}^1 \frac{P(t)}{\sqrt{1-t^2}}K(t-x)dt \;=\; 0 \qquad (-1<x<1) \tag{5.9}$$

$$\int_{-1}^1 \frac{P(t)}{\sqrt{1-t^2}}dt \;=\; C_0 \tag{5.10}$$

where

$$P(t) = q(t) + G_1(t)\sqrt{1-t^2} \tag{5.11}$$

and

$$C_0 = 2\int_0^1 G_1(t)dt \tag{5.12}$$

5.6 The Stress Intensity Factor

The mode II stress intensity factor at the tip $(C,0)$ is defined by

$$k_{II}(C) = -\lim_{X\to C^-}\sqrt{2(C-X)}\frac{E}{2(1-\nu^2)}\frac{\partial U_X^{(2)}}{\partial X}(X,0) \tag{6.1}$$

It is easily shown that

$$k_{II}(C) = -p_0\sqrt{C}P(1) \tag{6.2}$$

It is well known [4], that when a crack of length 2C in an infinite thermoelastic solid disturbs a uniform heat flux of strength $(T_1 - T_2)\kappa/2H$ perpendicular to its length, the stress intensity factor is given by

$$k_0 = -\frac{1}{2}p_0\sqrt{C} \tag{6.3}$$

The scaled stress intensity factor is given by

$$\frac{k_{II}(C)}{k_0} = 2P(1) \tag{6.4}$$

Solving equations (5.5.9) and (5.5.10) by the method outlined in the appendix, $P(1)$ can be found. The results of such a computation are shown in Figure 6 which illustrates the way in which $k_{II}(C)/k_0$ varies with H/C. As expected, $k_{II}(C)/k_0$ tends to one as H tends to infinity. As the strip width is reduced the decreasing values of the stress intensity factor imply that the sliding of the crack surfaces becomes more pronounced.

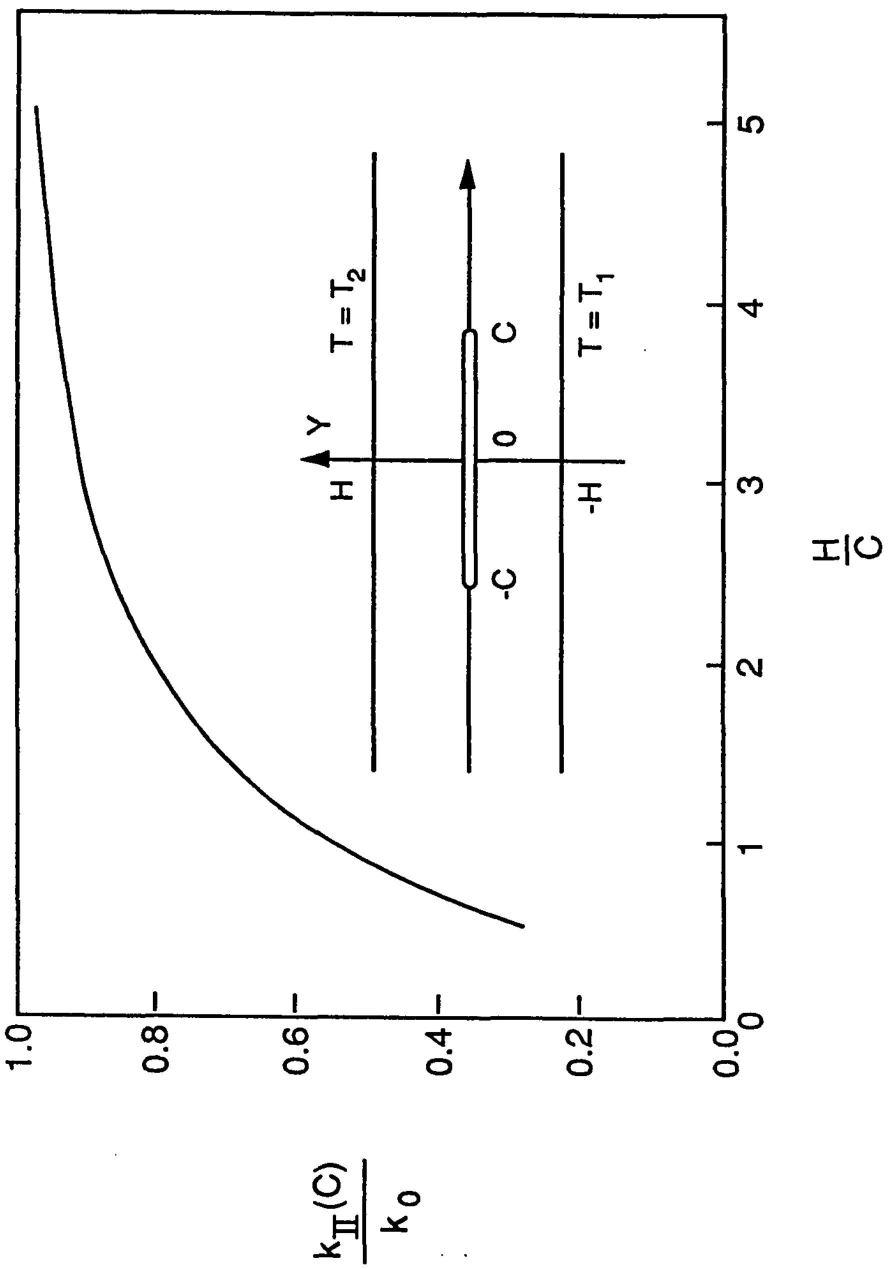

Figure 6. Values of $k_{II}(C)/k_0$ vs. H/C

Chapter 6

The Linear Thermoelastic Problem For A Strip With Two Collinear Cracks Parallel To Its Edges

6.1 Introduction

The problem of interest in this chapter is the evaluation of the thermal stresses associated with a pair of collinear cracks in the linear, isotropic, thermoelastic strip S defined in the previous chapter. The cracks, which are thermally insulated and traction free, are given by the inequalities

$$0 < B < | X | < C \ ; \ Y = 0 \tag{1.1}$$

Integral transform techniques are employed to determine the harmonic functions appearing in Sneddon's solution (2.3.9), (2.3.10). The problem then reduces to the solution of a singular integral equation which is found numerically. The appropriate stress intensity factors are given in terms of the numerical solution and results are shown in graphical form.

6.2 Resolution into Problems 1 and 2

To use the dimensionless form of Sneddon's solution (2.3.9), (2.3.10) the following quantities are introduced

$$L = C \quad \text{and} \quad p_0 = \frac{E\alpha(T_1 - T_2)C}{4H(1-\nu)} \tag{2.1}$$

which have dimensions of length and stress respectively. In addition, the dimensionless parameters

$$b = \frac{B}{C} \quad \text{and} \quad h = \frac{H}{C} \tag{2.2}$$

are defined.

In the absence of the cracks the thermoelastic field is the solution of problem B in chapter 3 and is given by

$$\begin{aligned}
T(x,y) &= T_0[1+\theta^0(x,y)] \\
\theta^0(x,y) &= \left(\frac{T_1+T_2-2T_0}{2T_0}\right) - \left(\frac{T_1-T_2}{2T_0}\right)\frac{y}{h} \\
u^0(x,y) &= 2(1-\nu)\left[\frac{T_1+T_2-2T_0}{T_1-T_2}hx - xy\right] \\
v^0(x,y) &= 2(1-\nu)\left[\frac{T_1+T_2-2T_0}{T_1-T_2}hy + \frac{1}{2}(x^2-y^2)\right] \\
s^0_{xx}(x,y) &= s^0_{xy}(x,y) \;=\; s^0_{yy}(x,y) \;=\; 0
\end{aligned} \tag{2.3}$$

The presence of the cracks disturb this field thereby yielding a new field

$$\begin{aligned} \theta(x,y) &= \theta^0(x,y) + \theta^p(x,y) \\ u(x,y) &= u^0(x,y) + u^p(x,y) \\ v(x,y) &= v^0(x,y) + v^p(x,y) \\ s_{ij}(x,y) &= s_{ij}^p(x,y) \end{aligned} \tag{2.4}$$

where, by virtue of symmetry, the perturbations are obtained by solving the following mixed boundary value problem.

Problem Solve the dimensionless, plane strain equations of linear thermoelasticity in the strip $0 < x < \infty, 0 < y < h$ subject to the conditions:

1. At infinity $\theta^p, s_{ij}^p \sim O(x^{-1})$ and $u^p, v^p \sim O(\ln x)$
2. $u^p(0,y) = s_{xy}^p(0,y) = 0$ $\quad (0 < y < h)$
3. $s_{xy}^p(x,h) = s_{yy}^p(x,h) = 0$ $\quad (0 < x < \infty)$
4. $s_{yy}^p(x,0) = 0$ $\quad (0 < x < \infty)$
5. $s_{xy}^p(x,0) = 0$ $\quad (b < x < 1)$

   $u^p(x,0) = 0$ $\quad (0 < x < b) \cup (1 < x < \infty)$
6. $\dfrac{\partial \theta^p}{\partial x}(0,y) = 0$ $\quad (0 < y < h)$
7. $\theta^p(x,h) = 0$ $\quad (0 < x < \infty)$
8. $\dfrac{\partial \theta^p}{\partial y}(x,0) = \dfrac{T_1 - T_2}{2hT_0}$ $\quad (b < x < 1)$

   $\theta^p(x,0) = 0$ $\quad (0 < x < b) \cup (1 < x < \infty)$

This problem is solved by superimposing the solutions of the following two problems:

Problem 1 Solve the dimensionless, plane strain equations of linear thermoelasticity in the strip $0 < x < \infty, 0 < y < h$ subject to the conditions:

1. At infinity $\theta^1, s^1_{ij} \sim O(x^{-1})$ and $u^1, v^1 \sim O(\ln x)$
2. $u^1(0,y) = s^1_{xy}(0,y) = 0$ $\quad (0 < y < h)$
3. $s^1_{xy}(x,h) = s^1_{yy}(x,h) = 0$ $\quad (0 < x < \infty)$
4. $u^1(x,0) = s^1_{yy}(x,0) = 0$ $\quad (0 < x < \infty)$
5. $\dfrac{\partial \theta^1}{\partial x}(0,y) = 0$ $\quad (0 < y < h)$
6. $\theta^1(x,h) = 0$ $\quad (0 < x < \infty)$
7. $\theta^1(x,0) = 0$ $\quad (0 < x < b) \cup (1 < x < \infty)$

   $\dfrac{\partial \theta^1}{\partial y}(x,0) = \dfrac{T_1 - T_2}{2hT_0}$ $\quad (b < x < 1)$

Calculate $\quad s^1_{xy}(x,0) = f(x) \quad (b < x < 1)$

Problem 2 Solve the dimensionless, plane strain equations of linear isothermal elasticity in the strip $0 < x < \infty, 0 < y < h$ subject to the conditions:

1. At infinity $s_{ij}^2 \sim O(x^{-2})$ and $u^2, v^2 \sim O(x^{-1})$
2. $u^2(0, y) = s_{xy}^2(0, y) = 0 \qquad (0 < y < h)$
3. $s_{xy}^2(x, h) = s_{yy}^2(x, h) = 0 \qquad (0 < x < \infty)$
4. $s_{yy}^2(x, 0) = 0 \qquad (0 < x < \infty)$
5. $u^2(x, 0) = 0 \qquad (0 < x < b) \cup (1 < x < \infty)$

   $s_{xy}^2(x, 0) = -f(x) \qquad (b < x < 1)$

6.3 The Solution of Problem 1

In the special case of Sneddon's general solution (2.3.9), (2.3.10) let

$$\begin{aligned}\chi(x,y) &= \mathcal{F}_c\left[\frac{2hA(\xi)}{\xi^2 D(\xi h)}\left\{\mathrm{cth}\xi h - \xi h - \frac{1}{\xi h}\right\}\mathrm{sh}\xi y\,;\,\xi\to x\right] \\ &- \mathcal{F}_c\left[\frac{A(\xi)\mathrm{sh}\xi(h-y)}{\xi^3\mathrm{sh}\xi h}\,;\,\xi\to x\right]\end{aligned} \tag{3.1}$$

$$\begin{aligned}\phi(x,y) &= -\frac{2}{\beta^2-1}\mathcal{F}_c\left[\frac{A(\xi)\mathrm{cth}\xi h}{\xi^3 D(\xi h)}\left\{\mathrm{cth}\xi h - \xi h - \frac{1}{\xi h}\right\}\mathrm{sh}\xi y\,;\,\xi\to x\right] \\ &- \frac{1}{h(\beta^2-1)}\mathcal{F}_c\left[\frac{A(\xi)\mathrm{sh}\xi y}{\xi^4\mathrm{sh}\xi h}\,;\,\xi\to x\right]\end{aligned} \tag{3.2}$$

$$+ \ \mathcal{F}_c\left[\frac{A(\xi)\mathrm{sh}\xi(h-y)}{\xi^3\mathrm{sh}\xi h}\ ;\ \xi\to x\right]$$

and

$$\psi(x,y) = \beta^2\mathcal{F}_c\left[\frac{A(\xi)\mathrm{ch}\xi(h-y)}{\xi^2\mathrm{sh}\xi h}\ ;\ \xi\to x\right] \tag{3.3}$$

where $\mathcal{F}_c$ is the Fourier cosine transform defined by (5.3.4) and

$$D(\rho) = \mathrm{sh}2\rho - 2\rho \tag{3.4}$$

Clearly, all conditions of problem 1 are satisfied if $A(\xi)$ is given by the triple integral equations

$$\begin{aligned} G_1(x) &= \mathcal{F}_c[\xi^{-1}A(\xi);\xi\to x] = 0 \qquad (0<x<b)\cup(1<x<\infty) \\ F_1(x) &= \mathcal{F}_c[A(\xi)\mathrm{cth}\xi h;\xi\to x] = 1 \qquad (b<x<1) \end{aligned} \tag{3.5}$$

Let

$$A(\xi) = \sqrt{\frac{2}{\pi}}\int_b^1 p(t)\sin\xi t\,dt \tag{3.6}$$

with

$$\int_b^1 p(t)dt = 0 \tag{3.7}$$

then

$$G_1(x) = -H[(1-x)(x-b)]\int_b^x p(t)dt \qquad (x>0) \tag{3.8}$$

where $H(u)$ is the Heaviside step function. Additionally,

$$F_1(x) = \frac{1}{h}\int_b^1 p(t)\frac{\mathrm{sh}\frac{\pi t}{h}}{\mathrm{ch}\frac{\pi t}{h}-\mathrm{ch}\frac{\pi x}{h}}dt \tag{3.9}$$

and so $p(t)$ must solve the singular integral equation

$$\frac{1}{h}\int_b^1 p(t)\frac{\mathrm{sh}\frac{\pi t}{h}}{\mathrm{ch}\frac{\pi t}{h}-\mathrm{ch}\frac{\pi x}{h}}dt = 1 \qquad (b < x < 1) \tag{3.10}$$

subject to the condition (6.3.7).

By an obvious change of variables (6.3.10) reduces to the finite Hilbert transform [18], and hence yields the solution

$$p(t) = P(t)/\Delta(t) \tag{3.11}$$

where

$$\Delta(t) = \{(\mathrm{ch}\frac{\pi}{h} - \mathrm{ch}\frac{\pi t}{h})(\mathrm{ch}\frac{\pi t}{h} - \mathrm{ch}\frac{\pi b}{h})\}^{1/2} \tag{3.12}$$

and

$$P(t) = \left(\mathrm{ch}\frac{\pi t}{h} - 1\right) - \pi\mathrm{ch}\frac{\pi b}{2h}\mathrm{sh}\frac{\pi}{2h}\frac{\Lambda_0(\gamma, k')}{K'} \tag{3.13}$$

where $K' = K(k')$ is the associated complete elliptic integral of the first kind and $\Lambda_0(\gamma, k')$ is Heumann's Lambda function with parameters

$$k = \frac{\mathrm{th}\frac{\pi b}{2h}}{\mathrm{th}\frac{\pi}{2h}} \qquad \text{and} \qquad \sin\gamma = \mathrm{th}\frac{\pi}{2h} \tag{3.14}$$

It follows at once that for $x > 0$

$$\theta^1(x,0) = -\frac{(T_1 - T_2)}{2\pi K' T_0} H[(1-x)(x-b)]$$

$$\left[\pi\Lambda_0(\gamma, k')F(\phi, k) - \frac{\mathrm{sh}^2\frac{\pi b}{2h} K'\Pi(\phi, \alpha^2, k')}{\mathrm{ch}\frac{\pi b}{2h}\mathrm{sh}\frac{\pi}{2h}}\right] \tag{3.15}$$

where $F(\phi, k')$ and $\Pi(\phi, \alpha^2, k')$ are incomplete elliptic integrals of the first and third kinds respectively. The parameters α^2 and ϕ are given by

$$k'^2 < \alpha^2 = \frac{\mathrm{ch}\frac{\pi}{h} - \mathrm{ch}\frac{\pi b}{h}}{\mathrm{ch}\frac{\pi}{h} - 1} < 1 \tag{3.16}$$

and

$$\sin\phi = \frac{\mathrm{ch}\frac{\pi}{2h}}{\mathrm{ch}\frac{\pi x}{2h}} \left(\frac{\mathrm{ch}\frac{\pi x}{h} - \mathrm{ch}\frac{\pi b}{h}}{\mathrm{ch}\frac{\pi}{h} - \mathrm{ch}\frac{\pi b}{h}}\right)^{1/2} \tag{3.17}$$

Furthermore, we discover that

$$\frac{\partial\theta^1}{\partial y}(x,0) = \frac{T_1 - T_2}{2hT_0} \begin{cases} 1 + \dfrac{P(x)}{\Delta_1(x)} & (0 < x < b) \\ 1 & (b < x < 1) \\ 1 - \dfrac{P(x)}{\Delta_1(x)} & (1 < x < \infty) \end{cases} \tag{3.18}$$

where

$$\Delta_1(x) = \{(\mathrm{ch}\frac{\pi x}{h} - \mathrm{ch}\frac{\pi}{h})(\mathrm{ch}\frac{\pi x}{h} - \mathrm{ch}\frac{\pi b}{h})\}^{1/2} \tag{3.19}$$

Next, by virtue of (2.3.10), (6.3.1) and some algebra

$$f(x) = -\mathcal{F}_s\left[\frac{A(\xi)}{\xi}\{1 - M(\xi h)\}\ ;\ \xi \to x\right] \tag{3.20}$$

where

$$M(\rho) = \frac{1 - 2\rho + 2\rho^2 - e^{-2\rho}}{\mathrm{sh}2\rho - 2\rho} \tag{3.21}$$

Therefore, on making use of (6.3.6) and (6.3.8) it is apparent that

$$f(x) = \frac{1}{\pi} \int_b^1 G_1(t) \{K(t - x) - K(t + x)\} \, dt \qquad (b < x < 1) \tag{3.22}$$

where

$$K(x) = \frac{1}{x} - \frac{1}{h} \int_0^\infty M(\rho) \sin \frac{\rho}{h} d\rho \tag{3.23}$$

6.4 Isotherms

Using (2.3.9) and (6.3.3), the disturbed temperature field can be written as

$$\theta^1(x, y) = \frac{\beta^2 \sin \frac{\pi(h-y)}{h}}{\delta 2h} \int_b^1 G_1(t)\{S(t + x, y) + S(t - x, y)\}dt \tag{4.1}$$

where $S(t, y)$ is given by (5.4.3).

Evaluating (6.4.1) and (6.3.15) allows the level curves

$$\frac{\theta^1(x, y)}{\frac{\beta^2}{\delta}} = \text{constant} \tag{4.2}$$

to be plotted using Surfer [20]. The effect of a pair of cracks in a strip with $\frac{H}{C} = 1$ can be seen in Figure 7.

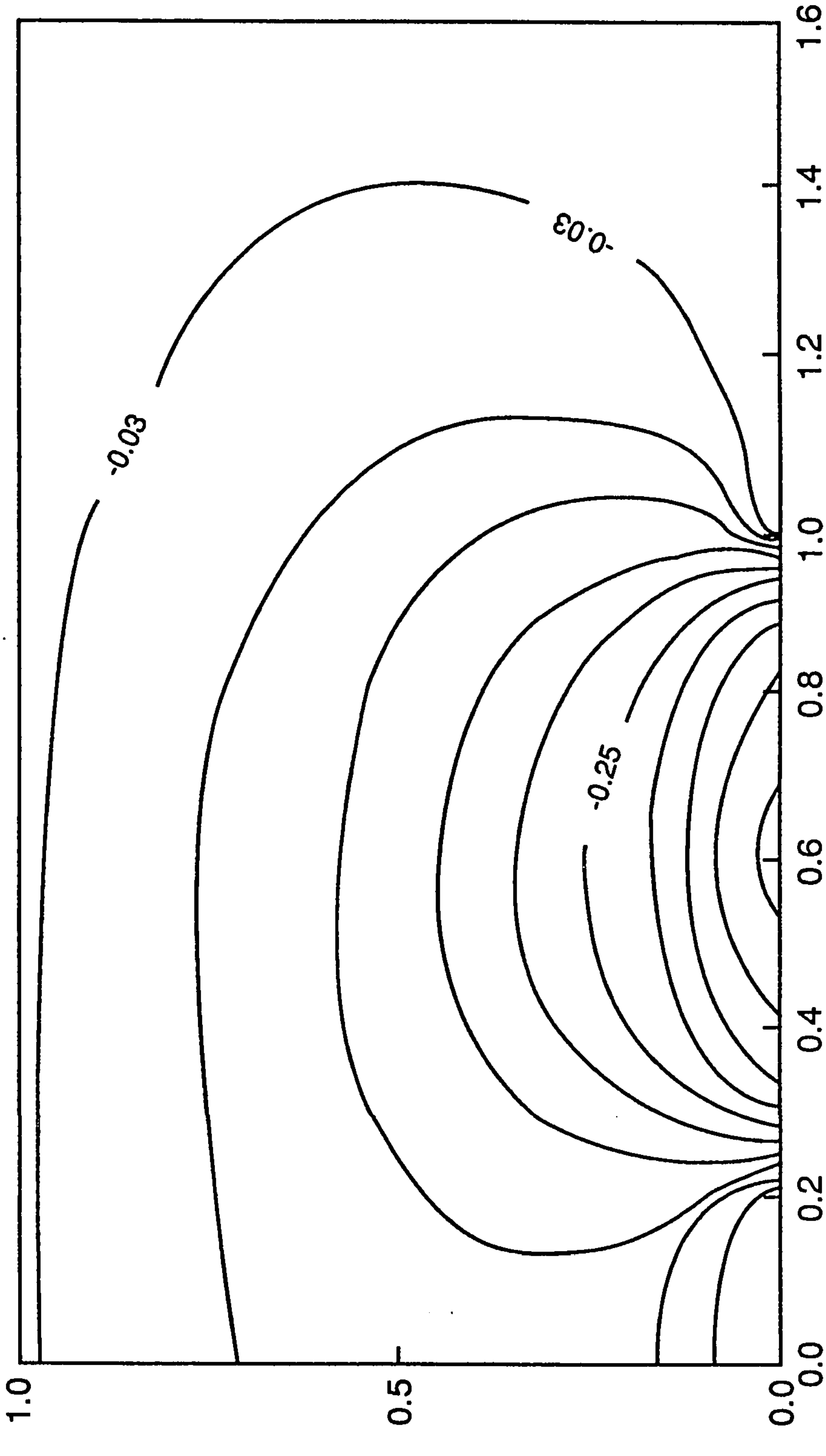

Figure 7. Isotherms associated with a uniform heat flow disturbed by a pair of cracks parallel to the edges of a strip.

6.5 The Solution of Problem 2

In the special case of Sneddon's solution (2.3.9), (2.3.10) let $\psi = 0$

$$\chi(x,y) = \mathcal{F}_c\left[\frac{B(\xi)}{\xi^2}\left\{\mathrm{ch}\xi y + \frac{2(\xi^2h^2 - \mathrm{sh}^2\xi h)}{D(\xi h)}\mathrm{sh}\xi y\right\}\,;\,\xi \to x\right] \tag{5.1}$$

and

$$\phi(x,y) = (\beta^2-1)^{-1}\mathcal{F}_c\left[\frac{B(\xi)}{\xi^2}\left\{\mathrm{ch}\xi y - \frac{2\mathrm{sh}^2\xi h}{D(\xi h)}\mathrm{sh}\xi y\right\}\,;\,\xi \to x\right] \tag{5.2}$$

then all conditions of problem 2 are satisfied if $B(\xi)$ is given by the triple integral equations

$$\begin{aligned} G_2(x) &= \mathcal{F}_s[\xi^{-1}B(\xi)\,;\,x] = 0 \qquad (0 < x < b) \cup (1 < x < \infty) \\ F_2(x) &= \mathcal{F}_s[B(\xi)\{1 - M(\xi h)\}\,;\,x] = 1 \qquad (b < x < 1) \end{aligned} \tag{5.3}$$

Let

$$B(\xi) = \sqrt{\frac{2}{\pi}}\int_b^1 \frac{q(t)}{\sqrt{(1-t)(t-b)}}(1-\cos\xi t)dt \tag{5.4}$$

where

$$\int_b^1 \frac{q(t)}{\sqrt{(1-t)(t-b)}}dt = 0 \tag{5.5}$$

then

$$G_2(x) = -H[(1-x)(x-b)]\int_b^x \frac{q(t)}{\sqrt{(1-t)(t-b)}}dt \tag{5.6}$$

and

$$F_2(x) = \frac{1}{\pi}\int_b^1 \frac{q(t)}{\sqrt{(1-t)(t-b)}}\{K(t-x) - K(t+x)\}dt \tag{5.7}$$

where $K(x)$ is given by (6.3.23). It follows at once that $q(t)$ must solve the singular integral equation

$$\frac{1}{\pi}\int_b^1 \frac{q(t)}{\sqrt{(1-t)(t-b)}}\{K(t-x) - K(t+x)\}dt = -f(x) \qquad (b < x < 1) \tag{5.8}$$

subject to (6.5.5). Using (6.3.22) we may rewrite (6.5.8) to obtain the alternate form

$$\frac{1}{\pi}\int_b^1 \frac{Q(t)}{\sqrt{(1-t)(t-b)}}\{K(t-x) - K(t+x)\}dt = 0 \quad (b < x < 1) \tag{5.9}$$

$$\int_b^1 \frac{Q(t)}{\sqrt{(1-t)(t-b)}}dt = C_0 \tag{5.10}$$

where

$$Q(t) \;=\; q(t) + \sqrt{(1-t)(t-b)}G_1(t) \tag{5.11}$$

$$C_0 \;=\; \int_b^1 G_1(t)dt \tag{5.12}$$

6.6 The Stress Intensity Factors

The mode II stress intensity factors at the tips $(B,0)$ and $(C,0)$ are defined by

$$k_{II}(B) = \lim_{X\to B+} \sqrt{2(X-B)}\frac{E}{2(1-\nu^2)}\frac{\partial U_X^2}{\partial X}(X,0) \tag{6.1}$$

and

$$k_{II}(C) = -\lim_{X \to C^-} \sqrt{2(C-X)} \frac{E}{2(1-\nu^2)} \frac{\partial U_X^2}{\partial X}(X,0) \tag{6.2}$$

It is easily shown that

$$k_{II}(B) = p_0\sqrt{2C}(1-b)^{-1/2}Q(b) \tag{6.3}$$

and

$$k_{II}(C) = -p_0\sqrt{2C}(1-b)^{-1/2}Q(1) \tag{6.4}$$

It is well known [4], that when a crack of length $C - B$ in an infinite thermoelastic solid disturbs a uniform flux of strength $(T_1 - T_2)\kappa/2H$, the stress intensity factor is given by

$$k_0 = \frac{p_0\sqrt{2C}(1-b)^{3/2}}{8} \tag{6.5}$$

We therefore see that

$$\frac{k_{II}(B)}{k_0} = \frac{8Q(b)}{(1-b)^2} \tag{6.6}$$

and

$$\frac{k_{II}(C)}{k_0} = \frac{-8Q(1)}{(1-b)^2} \tag{6.7}$$

Solving equations (6.5.9) and (6.5.10) using the method outlined in the appendix enables us to compute the stress intensity factors for varying strip width and crack length.

The results are recorded in Figures 8 and 9 which show how $k_{II}(B)/k_0$ and $-k_{II}(C)/k_0$ vary with B/C and H/C. As expected, both $k_{II}(B)/k_0$ and $-k_{II}(C)/k_0$ tend to one as $H \to \infty$ and $B \to C$, while $-k_{II}(C)/k_0$ tends to $2\sqrt{2}$ and $k_{II}(B)/k_0$ tends to zero as $B \to 0$ and $H \to \infty$. As the strip width is reduced the results also indicate that it becomes more likely for the cracks to spread at the tips $(\pm C, 0)$ rather than $(\pm B, 0)$.

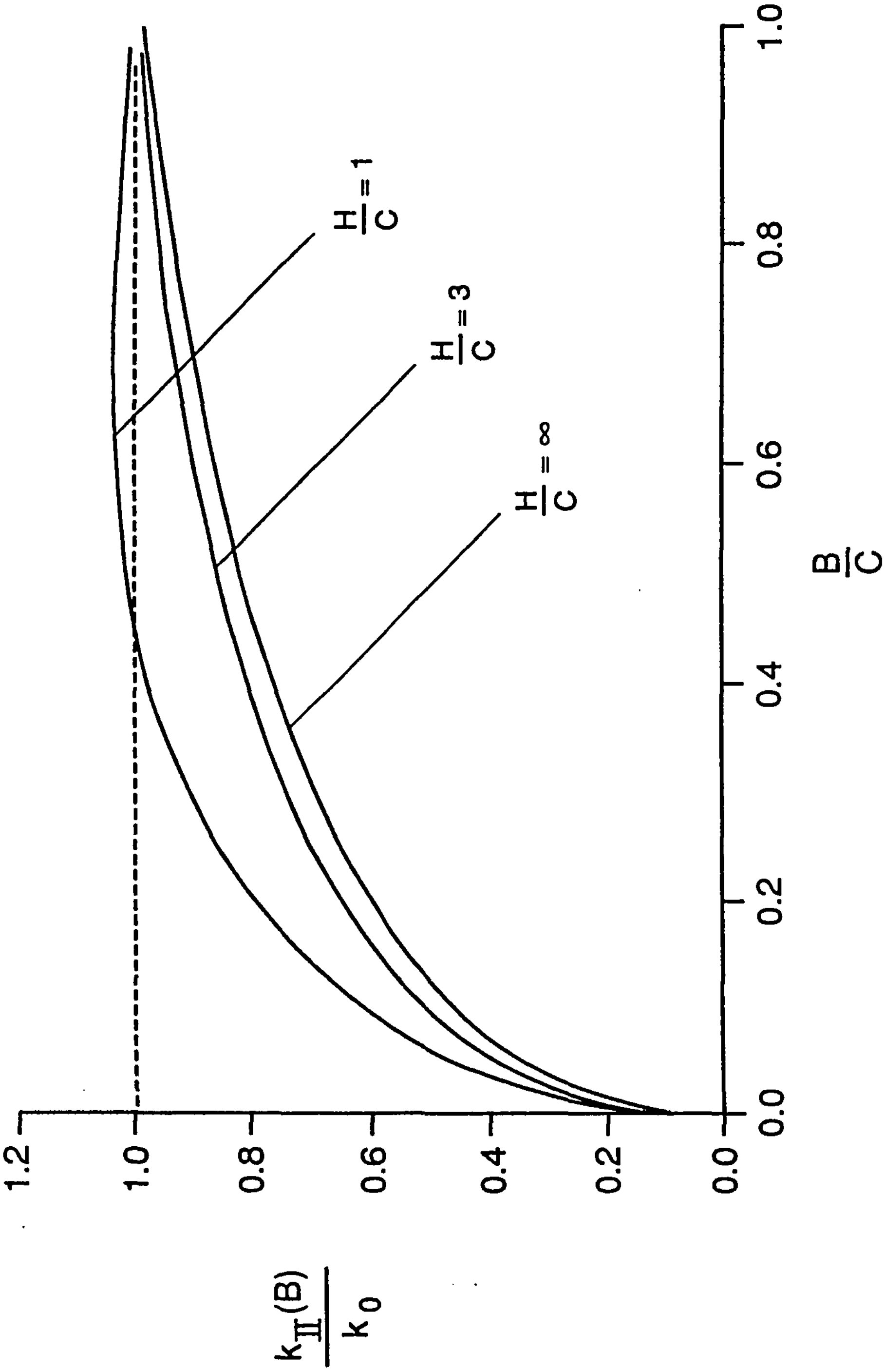

Figure 8. Values of $k_{II}(B)/k_0$ compared with B/C and H/C

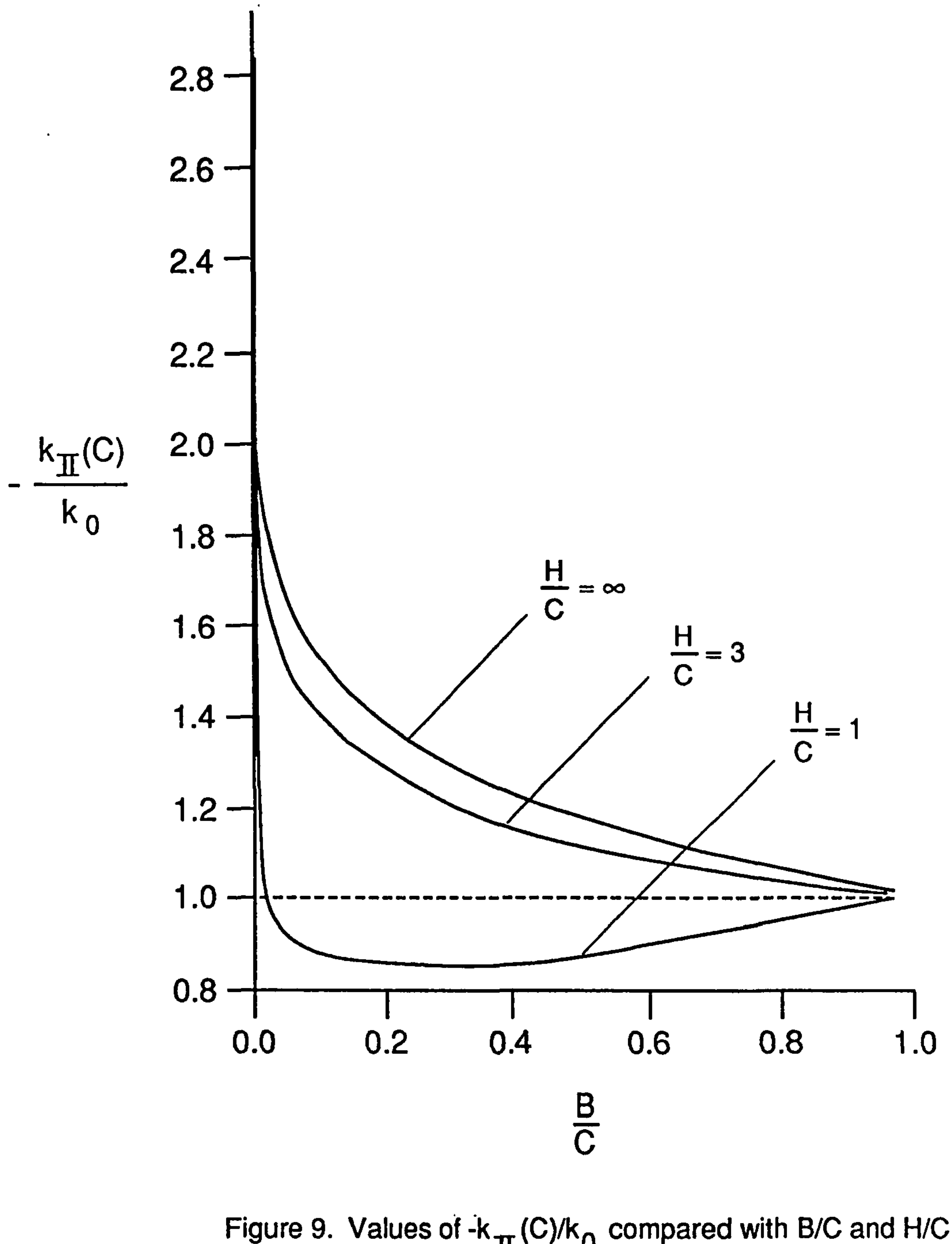

Figure 9. Values of $-k_{II}(C)/k_0$ compared with B/C and H/C

Chapter 7

The Linear Thermoelastic Problem For A Strip With A Collinear Array of Cracks Parallel To Its Edges

7.1 Introduction

The final thermal problem to be discussed is that of determining the stress intensity factors when a collinear array of Griffith cracks disturb a uniform heat flow in the strip S. In this case the cracks, which are thermally insulated and traction free, are assumed to form a periodic array defined by

$0 < | X - 2NP | < B < P \;\; ; \;\; Y = 0$, where $N = 0, \pm1, \pm2, \ldots$.

Separation of variables yields suitable representations for the harmonic functions in Sneddon's solution (2.3.9), (2.3.10) which leads to dual series relations whose solutions are known.

7.2 Resolution into Problems 1 and 2

To use the dimensionless form of Sneddon's solution (2.3.9), (2.3.10) the following quantities are introduced,

$$L = P/\pi \qquad \text{and} \qquad p_0 = \frac{E\alpha(T_1 - T_2)P}{4H\pi(1-\nu)} \tag{2.1}$$

which have dimensions of length and stress respectively. In addition, the dimensionless parameters

$$b = B/L \qquad \text{and} \qquad h = H/L \tag{2.2}$$

are defined.

By symmetry the undisturbed thermoelastic field is required in the region $0 < x < \pi, -h < y < h$. This is obtained by problem C of chapter 3 and yields

$$\begin{aligned} T(x,y) &= T_0[1+\theta^0(x,y)] \\ \theta^0(x,y) &= \left(\frac{T_1+T_2-2T_0}{2T_0}\right) - \left(\frac{T_1-T_2}{2T_0}\right)\frac{y}{h} \\ v^0(x,y) &= -y^2 \ , \ s^0_{xx}(x,y) = 2y \\ \text{and } u^0(x,y) &= s^0_{xy}(x,y) = s^0_{yy}(x,y) = 0 \end{aligned} \tag{2.3}$$

The presence of the cracks disturbs this field thereby yielding a new field

$$\begin{aligned}
\theta(x,y) &= \theta^0(x,y) + \theta^p(x,y) \\
v(x,y) &= v^0(x,y) + v^p(x,y) \\
s_{xx}(x,y) &= s^0_{xx}(x,y) + s^p_{xx}(x,y) \\
u(x,y) &= u^p(x,y), \quad s_{xy}(x,y) = s^p_{xy}(x,y) \\
s_{yy}(x,y) &= s^p_{yy}(x,y)
\end{aligned} \tag{2.4}$$

where, again by symmetry, the perturbations are obtained by solving the following mixed boundary value problem.

Problem Solve the dimensionless, plane strain equations of linear thermoelasticity in the region $0 < x < \pi, 0 < y < h$ subject to the conditions:

1. $u^p(0,y) = u^p(\pi,y) = s^p_{xy}(0,y) = s^p_{xy}(\pi,y) = 0 \qquad (0 < y < h)$
2. $s^p_{xy}(x,h) = s^p_{yy}(x,h) = 0 \qquad (0 < x < \pi)$
3. $s^p_{yy}(x,0) = 0 \qquad (0 < x < \pi)$
4. $s^p_{xy}(x,0) = 0 \qquad (0 < x < b)$

   $u^p(x,0) = 0 \qquad (b < x < \pi)$
5. $\dfrac{\partial \theta^p}{\partial x}(0,y) = \dfrac{\partial \theta^p}{\partial x}(\pi,y) = 0 \qquad (0 < y < h)$
6. $\theta^p(x,h) = 0 \qquad (0 < x < \pi)$
7. $\dfrac{\partial \theta^p}{\partial y}(x,0) = \dfrac{(T_1 - T_2)}{2hT_0} \qquad (0 < x < b)$

   $\theta^p(x,0) = 0 \qquad (b < x < \pi)$

The solution to this problem is obtained by superimposing the solutions to the following two problems.

Problem 1 Solve the dimensionless, plane strain equations of linear thermoelasticity in the region $0 < x < \pi, 0 < y < h$ subject to the conditions:

1. $u^1(0,y) = u^1(\pi,y) = s^1_{xy}(0,y) = s^1_{xy}(\pi,y) = 0 \qquad (0 < y < h)$
2. $s^1_{xy}(x,h) = s^1_{yy}(x,h) = 0 \qquad (0 < x < \pi)$
3. $u^1(x,0) = s^1_{yy}(x,0) = 0 \qquad (0 < x < \pi)$
4. $\dfrac{\partial\theta^1}{\partial x}(0,y) = \dfrac{\partial\theta^1}{\partial x}(\pi,y) = 0 \qquad (0 < y < h)$
5. $\theta^1(x,h) = 0 \qquad (0 < x < \pi)$
6. $\dfrac{\partial\theta^1}{\partial y}(x,0) = \dfrac{(T_1 - T_2)}{2hT_0} \qquad (0 < x < b)$

   $\theta^1(x,0) = 0 \qquad (b < x < \pi)$

Calculate $\quad s^1_{xy}(x,0) = f(x) \qquad (0 < x < b)$

Problem 2 Solve the dimensionless, plane strain equations of linear isothermal elasticity in the region $0 < x < \pi, 0 < y < h$ subject to the conditions:

1. $u^2(0,y) = u^2(\pi,y) = s^2_{xy}(0,y) = s^2_{xy}(\pi,y) = 0 \qquad (0 < y < h)$
2. $s^2_{xy}(x,h) = s^2_{yy}(x,h) = 0 \qquad (0 < x < \pi)$
3. $s^2_{yy}(x,0) = 0 \qquad (0 < x < \pi)$
4. $s^2_{xy}(x,0) = -f(x) \qquad (0 < x < b)$

   $u^2(x,0) = 0 \qquad (b < x < \pi)$

7.3 The Solution of Problem 1

In the special case of Sneddon's general solution (2.3.9), (2.3.10) the harmonic functions are chosen to be

$$\begin{aligned}\phi(x,y) &= \frac{A_0x^2}{2}\left(\frac{y}{h}-1\right)+\frac{A_0y^2}{2}\left(1-\frac{y}{3h}\right)\\ &+ \sum_{n=1}^{\infty} n^{-3}A_n\mathrm{ch}ny\cos nx\\ &+ \sum_{n=1}^{\infty}\frac{2n^{-3}A_n}{D(nh)}\left[\frac{\beta^2(nh\,\mathrm{cth}(nh)-1)}{\beta^2-1}-\mathrm{sh}^2nh\right]\mathrm{sh}ny\cos nx \qquad (3.1)\end{aligned}$$

$$\begin{aligned}\chi(x,y) &= \frac{A_0}{2}\left(x^2-y^2\right)-\sum_{n=1}^{\infty} n^{-3}A_n\mathrm{ch}ny\cos nx\\ &+ \sum_{n=1}^{\infty}\frac{2n^{-3}A_n}{D(nh)}\left[\mathrm{sh}^2nh-(nh)^2\right]\mathrm{sh}ny\cos nx \qquad (3.2)\end{aligned}$$

and

$$\psi(x,y) = -\beta^2\left\{A_0y+\frac{A_0}{2h}(x^2-y^2)-\sum_{n=1}^{\infty}n^{-2}A_n\frac{\mathrm{ch}n(h-y)}{\mathrm{sh}nh}\cos nx\right\} \qquad (3.3)$$

where

$$D(nh) = \mathrm{sh}2nh-2nh \qquad (3.4)$$

All conditions of problem 1 will be satisfied provided the $\{A_n\}_0^\infty$ solve the following dual equations

$$F_1(x) = h^{-1}A_0 + \sum_{n=1}^{\infty} A_n \mathrm{cth} nh \cos nx = 1 \qquad (0 < x < b)$$

(3.5)

$$G_1(x) = A_0 + \sum_{n=1}^{\infty} n^{-1} A_n \cos nx = 0 \qquad (b < x < \pi)$$

The equations in (7.3.5) are solved [21] by letting

$$A_0 = -\frac{1}{\pi}\int_0^b tp(t)dt \tag{3.6}$$

$$A_n = -\frac{2}{\pi}\int_0^b p(t)\sin nt dt \quad (n \geq 1) \tag{3.7}$$

from which it follows that

$$G_1(x) = -H(b-x)\int_x^b p(t)dt \qquad (0 < x < \pi) \tag{3.8}$$

where $H(u)$ is the Heaviside step function. Furthermore, on using the series identity

$$2\sum_{n=1}^{\infty} \mathrm{cth(nh)} \sin nt \cos nx = \frac{\partial}{\partial x}\log\left|\frac{\Theta_1\left(\frac{x+t}{2}, q\right)}{\Theta_1\left(\frac{x-t}{2}, q\right)}\right| \qquad (0 < x, t < \pi) \tag{3.9}$$

$$= \frac{2K}{\pi} Z\left(\frac{Kt}{\pi}\right) + \frac{\frac{2K}{\pi}\mathrm{sn}\frac{Kt}{\pi}\mathrm{cn}\frac{Kt}{\pi}\mathrm{dn}\frac{Kt}{\pi}}{\mathrm{sn}^2\frac{Kt}{\pi} - \mathrm{sn}^2\frac{Kx}{\pi}} \tag{3.10}$$

where $\Theta_1(z, q)$ is Jacobi's Theta function of the first kind and $Z(z)$ is Jacobi's Zeta function with

$$K = \frac{\pi}{2}\left[1 + 2\sum_{n=1}^{\infty} q^{n^2}\right]^2 \quad ; \quad q = e^{-h} \tag{3.11}$$

the complete elliptic integral of the first kind with modulus

$$k = \frac{2\pi}{K}\left[\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2}\right]^2 \tag{3.12}$$

we obtain

$$F_1(x) = -\frac{1}{\pi}\int_0^b p(t)\left\{\frac{t}{h} + \frac{2K}{\pi}Z\left(\frac{Kt}{\pi}\right) + \frac{\frac{2K}{\pi}\mathrm{sn}\frac{Kt}{\pi}\mathrm{cn}\frac{Kt}{\pi}\mathrm{dn}\frac{Kt}{\pi}}{\mathrm{sn}^2\frac{Kt}{\pi} - \mathrm{sn}^2\frac{Kx}{\pi}}\right\}dt \tag{3.13}$$

where sn, cn, dn are Jacobian elliptic functions [22].

Hence $p(t)$ must solve the singular integral equation

$$\frac{1}{\pi}\int_0^b p(t)\frac{\frac{2K}{\pi}\mathrm{sn}\frac{Kt}{\pi}\mathrm{cn}\frac{Kt}{\pi}\mathrm{dn}\frac{Kt}{\pi}}{\mathrm{sn}^2\frac{Kt}{\pi} - \mathrm{sn}^2\frac{Kx}{\pi}}dt = B_1 - 1 \qquad (0 < x < b) \tag{3.14}$$

with subsidiary condition, due to zero heat flow across $x = 0$

$$p(0) = 0 \tag{3.15}$$

where

$$B_1 = -\frac{1}{\pi}\int_0^b p(t)\left\{\frac{t}{h} + \frac{2K}{\pi}Z\left(\frac{Kt}{\pi}\right)\right\}dt \tag{3.16}$$

The solution of (7.3.14) subject to (7.3.15) is found to be

$$p(t) = -\frac{(B_1 - 1)}{\pi\Delta(t)}\int_0^b \frac{\Delta(x)\frac{2K}{\pi}\mathrm{sn}\frac{Kx}{\pi}\mathrm{cn}\frac{Kx}{\pi}\mathrm{dn}\frac{Kx}{\pi}}{\mathrm{sn}^2\frac{Kx}{\pi} - \mathrm{sn}^2\frac{Kt}{\pi}}dx \tag{3.17}$$

where

$$\Delta(x) = \frac{\left(\mathrm{sn}^2\frac{Kb}{\pi} - \mathrm{sn}^2\frac{Kx}{\pi}\right)^{1/2}}{\mathrm{sn}\frac{Kx}{\pi}} \tag{3.18}$$

Notice that for $0 \leq t < b$

$$p(t) = \frac{(B_1 - 1)\mathrm{sn}\frac{Kt}{\pi}}{(\mathrm{sn}^2\frac{Kb}{\pi} - \mathrm{sn}^2\frac{Kt}{\pi})^{1/2}} \tag{3.19}$$

Substituting (7.3.19) into (7.3.16) it can be shown that

$$B_1 - 1 = -(1 + I_1)^{-1} \tag{3.20}$$

where

$$I_1 = \frac{1}{\pi}\int_0^b \frac{\left\{\frac{t}{h} + \frac{2K}{\pi}Z\left(\frac{Kt}{\pi}\right)\right\} dt}{\Delta(t)} \tag{3.21}$$

It follows at once that

$$\theta^1(x,0) = \frac{(T_1 - T_2)\pi(B_1 - 1)F(\phi, \bar{k})}{2hT_0K\mathrm{dn}\frac{Kb}{\pi}} \qquad (0 < x < b) \tag{3.22}$$

where $F(\phi, \bar{k})$ is the incomplete elliptic integral of the first kind with parameters

$$\bar{k} = \frac{k'}{\mathrm{dn}\dfrac{Kb}{\pi}} \tag{3.23}$$

$$\sin\phi = \frac{\left(\mathrm{sn}^2\dfrac{Kb}{\pi} - \mathrm{sn}^2\dfrac{Kx}{\pi}\right)^{1/2}}{\mathrm{cn}\dfrac{Kx}{\pi}} \tag{3.24}$$

The temperature distribution (7.3.22) is shown in Figure 10 for $b = 1$.

Additionally,

$$\frac{\partial\theta^1}{\partial y}(x,0) = \frac{(T_1 - T_2)}{2hT_0}\begin{cases} 1 & (0 < x < b) \\ 1 + \dfrac{B_1 - 1}{\Delta_1(x)} & (b < x < \pi) \end{cases} \tag{3.25}$$

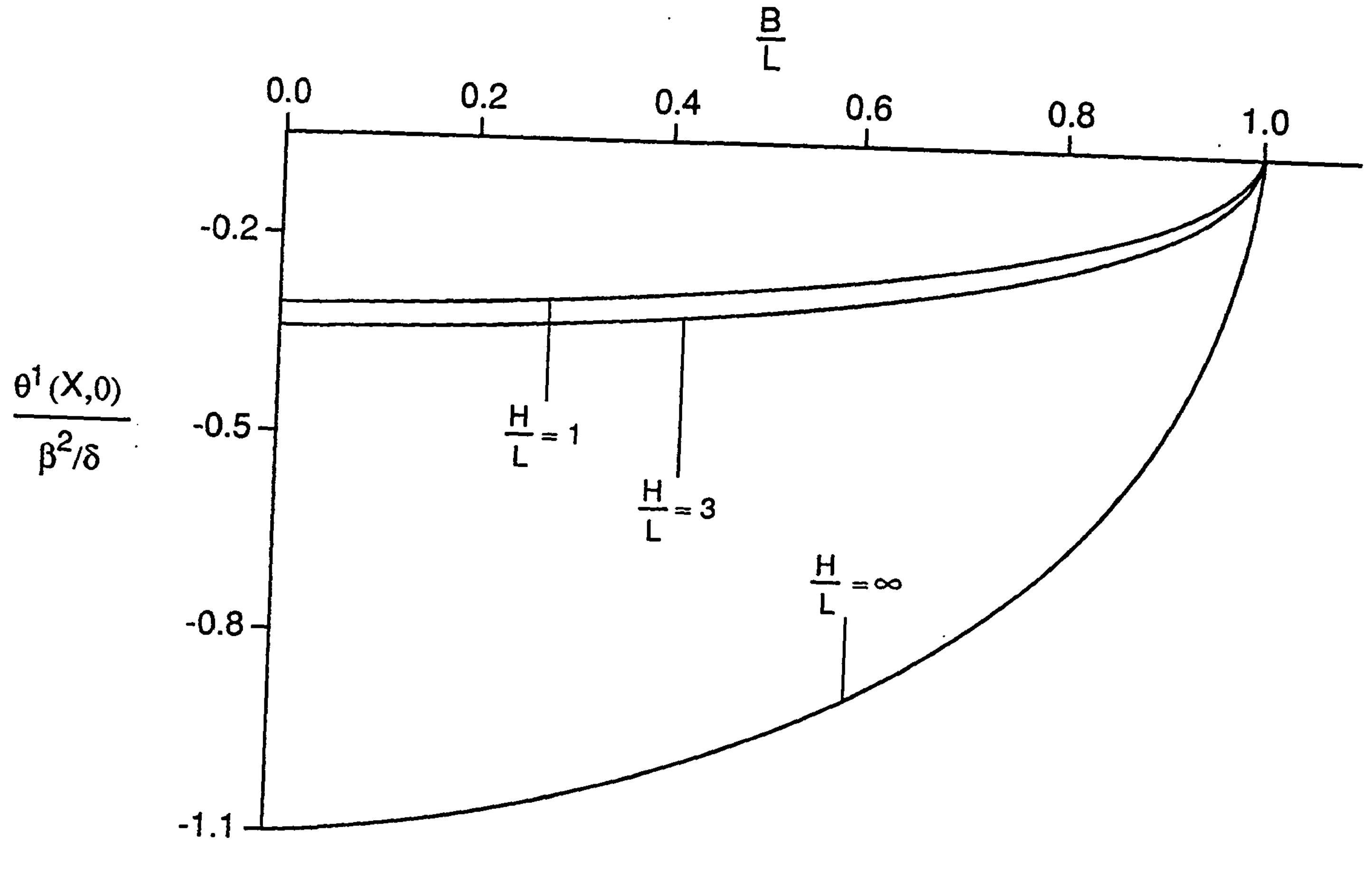

Figure 10. Temperature distribution on an array of cracks parallel to the edges of a strip.

with

$$\Delta_1(x) = \frac{\left(\mathrm{sn}^2\frac{Kx}{\pi} - \mathrm{sn}^2\frac{Kb}{\pi}\right)^{1/2}}{\mathrm{sn}\frac{Kx}{\pi}} \tag{3.26}$$

Next, by virtue of (2.3.10) and (7.3.2) it can be shown that

$$f(x) = -\sum_{n=1}^{\infty} n^{-1}A_n[1 - M(nh)]\sin nx \tag{3.27}$$

where

$$M(\rho) = \frac{1 - 2\rho + 2\rho^2 - e^{-2\rho}}{D(\rho)} \tag{3.28}$$

Employing equations (7.3.7) and (7.3.8) yield

$$f(x) = \frac{1}{\pi}\int_{-b}^{b} G_1(t)J(t-x)dt \tag{3.29}$$

where

$$J(x) = \frac{1}{2}\cot\frac{x}{2} - \sum_{n=1}^{\infty} M(nh)\sin nx \tag{3.30}$$

7.4 The Solution of Problem 2

In the special case of Sneddon's general solution (2.3.9), (2.3.10) the harmonic functions are chosen to be

$$\phi(x,y) = (\beta^2 - 1)^{-1}\sum_{n=1}^{\infty} n^{-2}B_n\left\{\mathrm{ch}ny - \frac{2\mathrm{sh}^2 nh}{D(nh)}\mathrm{sh}ny\right\}\cos nx \tag{4.1}$$

and

$$\chi(x,y) = \sum_{n=1}^{\infty} n^{-2}B_n\left\{\mathrm{ch}ny - \frac{2(\mathrm{sh}^2 nh - n^2h^2)}{D(nh)}\mathrm{sh}ny\right\}\cos nx \tag{4.2}$$

while

$$\psi(x,y)=0. \tag{4.3}$$

All conditions of problem 2 are satisfied provided the $\{B_n\}_1^\infty$ solve the following dual equations

$$F_2(x) = \sum_{n=1}^{\infty} B_n[1-M(nh)]\sin nx = -f(x) \qquad (0<x<b) \tag{4.4}$$

$$G_2(x) = \sum_{n=1}^{\infty} n^{-1}B_n \sin nx = 0 \qquad (b<x<\pi) \tag{4.5}$$

where $M(\rho)$ is given by equation (7.3.28).

Let

$$B_n = \frac{2}{\pi}\int_0^b \frac{q(t)}{\sqrt{b^2-t^2}}\cos nt\,dt \qquad (n\geq 1) \tag{4.6}$$

then

$$G_2(x) = H(b-x)\int_0^x \frac{q(t)}{\sqrt{b^2-t^2}}dt \qquad (0<x<\pi) \tag{4.7}$$

provided that

$$\int_0^b \frac{q(t)}{\sqrt{b^2-t^2}}dt = 0 \tag{4.8}$$

Additionally,

$$F_2(x) = -\frac{1}{\pi}\int_0^b \frac{q(t)}{\sqrt{b^2-t^2}}\{J(t-x)-J(t+x)\}\,dt \tag{4.9}$$

where $J(x)$ is given by (7.3.30). It follows at once that $q(t)$ is the solution to the singular integral equation.

$$\frac{1}{\pi}\int_0^b \frac{q(t)}{\sqrt{b^2-t^2}}\{J(t-x)-J(t+x)\}\,dt = f(x) \qquad (0<x<b) \tag{4.10}$$

with subsidiary condition (7.4.8).

Using (7.3.29) and appealing to the symmetry of the problem we can write equation (7.4.10) and its subsidiary condition (7.4.8) in the form

$$\frac{1}{\pi}\int_{-b}^{b}\frac{Q(t)}{\sqrt{b^2-t^2}}J(t-x)dt=0 \qquad |x|<b \tag{4.11}$$

$$\int_{-b}^{b}\frac{Q(t)dt}{\sqrt{b^2-t^2}}=C_0 \tag{4.12}$$

where

$$Q(t)=q(t)-\sqrt{b^2-t^2}\ G_1(t) \tag{4.13}$$

and

$$C_0=-2\int_0^b G_1(t)dt \tag{4.14}$$

The solution of (7.4.11) and (7.4.12) is achieved using the method outlined in the appendix.

7.5 The Stress Intensity Factor

The mode II stress intensity factor at the crack tip $(B,0)$ is defined by

$$k_{II}(B)=-\lim_{X\to B^-}\sqrt{2(B-X)}\frac{E}{2(1-\nu^2)}\frac{\partial U_X^{(2)}}{\partial X}(X,0) \tag{5.1}$$

It is easily shown that

$$k_{II}(B)=p_o\sqrt{L}\frac{Q(b)}{\sqrt{b}} \tag{5.2}$$

It is well known [4] that when a crack of length $2B$ in an infinite thermoelastic solid disturbs a uniform heat flow of strength $(T_1 - T_2)\kappa/2H$, the stress intensity factor is given by

$$k_0 = -\frac{1}{2} p_0 \sqrt{L} b^{3/2} \tag{5.3}$$

The scaled stress intensity factor is found to be

$$\frac{k_{II}(B)}{k_0} = -\frac{2Q(b)}{b^2} \tag{5.4}$$

The values of $k_{II}(B)/k_0$ as it varies with both $\frac{H}{L}$ and $\frac{B}{L}$ are recorded in Figure 11. As expected, $k_{II}(B)/k_0$ tends towards one as $\frac{B}{L}$ tends to zero.

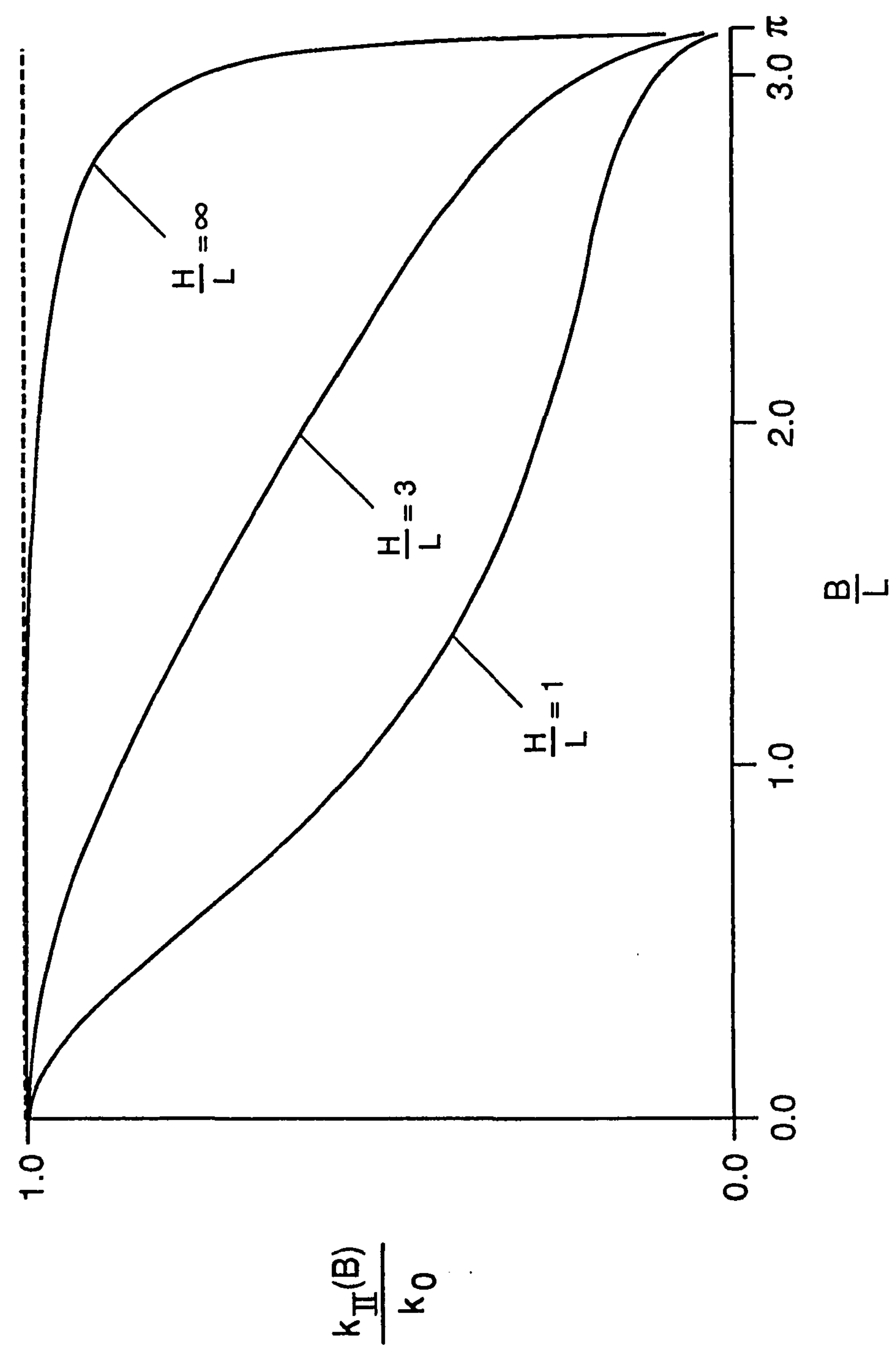

Figure 11. Values of $k_{II}(B)/k_0$ compared with B/L and H/L

Chapter 8

Orthotropic Fracture

8.1 Introduction

In this chapter we deviate from the theme of linear thermoelasticity developed for homogeneous and isotropic materials.

In particular, we turn our attention to a problem in linear isothermal elasticity in which the material has strong directional properties. Specifically, the material has reflectional symmetry with respect to three mutually orthogonal planes (usually taken to coincide with the coordinate planes). Such a body is said to be orthotropic, and to a good approximation, wood is an example of an orthotropic body.

The following problem initiated my interest in boundary value problems and resulted in two publications with my advisor [7], [8].

The problem to be considered is the out of plane shear of a cracked rectangular orthotropic block, which is described by the inequalities

$$0 \leq x \leq \pi \;\; ; \;\; -\lambda\pi \leq y \leq \lambda\pi \;\; ; \;\; -\infty < z < \infty \qquad (1.1)$$

The solid contains a crack located on $y = 0$ at $a \leq x \leq b$ and the crack runs the length of the solid in the z-direction. We deal with both the internal and edge crack problems which are specified by the inequalities

case (i) internal crack

$$0 < a \leq x \leq b < \pi \;\; ; \;\; y = 0 \qquad (1.2)$$

case (ii) edge crack

$$0 < a \leq x \leq b = \pi \;\; ; \;\; y = 0 \qquad (1.3)$$

On the faces $y = \pm\lambda\pi$ we assume there are arbitrary shear stresses $\pm\tau(x)$ respectively.

The solution of each case is obtained by stating the problem in terms of the displacement. The solution of the resulting mixed boundary value problem leads to in case (i) a triple trigonometric series relation and in case (ii) a dual trigonometric series relation. In either case they are shown to be equivalent to singular integral equations whose solutions are known exactly. In the case of a constant shear, the problem simplifies greatly and numerical results are given for the stress intensity factors at the crack tips. In this case the stress intensity factors are associated with the tearing mode–mode III.

8.2 Formulation of the Problem

For out of plane shear it is well known that the only non-zero displacement is

$$u_z = w(x,y) \tag{2.1}$$

and for an orthotropic body the non-zero stresses are given by

$$\sigma_{xz} = G_{13}\frac{\partial w}{\partial x} \quad , \quad \sigma_{yz} = G_{23}\frac{\partial w}{\partial y} \tag{2.2}$$

where G_{13}, G_{23} are the shear moduli of the orthotropic material. Substituting (8.2.2) into the non-trivial equilibrium equation

$$\frac{\partial}{\partial x}\sigma_{xz} + \frac{\partial}{\partial y}\sigma_{yz} = 0 \tag{2.3}$$

yields the partial differential equation

$$\beta^2\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \quad , \quad \beta^2 = \frac{G_{13}}{G_{23}} \tag{2.4}$$

Using symmetry the problem is reduced to solving the following mixed boundary value problem

P.D.E. $\beta^2 \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \qquad (0 < x < \pi \ , \ 0 < y < \lambda\pi)$

B.C. 1. $\frac{\partial w}{\partial x}(0,y) = \frac{\partial w}{\partial x}(\pi,y) = 0 \qquad (0 < y < \lambda\pi)$

2. $\frac{\partial w}{\partial y}(x,\lambda\pi) = \frac{\tau(x)}{G_{23}} \qquad (0 < x < \pi)$

case (i)

3(i) $w(x,0) = 0 \qquad (0 < x < a)$

$\frac{\partial w}{\partial y}(x,0) = 0 \qquad (a < x < b)$

$w(x,0) = 0 \qquad (b < x < \pi)$

case (ii)

3(ii) $w(x,0) = 0 \qquad (0 < x < a)$

$\frac{\partial w}{\partial y}(x,0) = 0 \qquad (a < x < \pi)$

$\lim_{x \to \pi^-} \frac{\partial w}{\partial x}(x,0) = 0$

It is easily verified that by writing

$$w(x,y) = \frac{1}{G_{23}}[\bar{w}(x,y) + \phi(x,y)] \tag{2.5}$$

where

$$\bar{w}(x,y) = \frac{1}{2}B_0 y + \sum_{n=1}^{\infty} \frac{B_n \mathrm{sh}(n\beta y)}{n\beta \mathrm{ch}(n\beta\lambda\pi)} \cos nx \tag{2.6}$$

with

$$B_n = \frac{2}{\pi}\int_0^{\pi} \tau(x) \cos nx \, dx \tag{2.7}$$

then $\phi(x,y)$ must solve the following mixed boundary value problem:

P.D.E. $\beta^2\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$ $(0<x<\pi,\ 0<y<\lambda\pi)$

B.C. 1. $\frac{\partial\phi}{\partial x}(0,y)=\frac{\partial\phi}{\partial x}(\pi,y)=0$ $(0<y<\lambda\pi)$

2. $\frac{\partial\phi}{\partial y}(x,\lambda\pi)=0$ $(0<x<\pi)$

case (i)

3(i) $\phi(x,0)=0$ $(0<x<a)$

$\frac{\partial\phi}{\partial y}=-f(x)$ $(a<x<b)$

$\phi(x,0)=0$ $(b<x<\pi)$

case (ii)

3(ii) $\phi(x,0)=0$ $(0<x<a)$

$\frac{\partial\phi}{\partial y}(x,0)=-f(x)$ $(a<x<\pi)$

$\lim_{x\to\pi^-}\frac{\partial\phi}{\partial x}(x,0)=0$

where

$$f(x)=\frac{\partial\bar{w}}{\partial y}(x,0)=\frac{1}{2}B_0+\sum_{n=1}^{\infty}B_n\text{sech}(n\beta\lambda\pi)\cos nx \tag{2.8}$$

8.3 Determination of the Perturbation ϕ

Clearly,

$$\phi(x,y)=\frac{A_0}{\beta}+\sum_{n=1}^{\infty}\frac{A_n\cosh n\beta(\lambda\pi-y)}{n\beta\cosh(n\beta\lambda\pi)}\cos nx \tag{3.1}$$

satisfies the p.d.e. and the first two boundary conditions. Furthermore,

$$\phi(x,0)=\beta^{-1}G(x)=\beta^{-1}\{A_0+\sum_{n=1}^{\infty}n^{-1}A_n\cos nx\} \tag{3.2}$$

and

$$\frac{\partial \phi}{\partial y}(x,0) = -F(x) = -\sum_{n=1}^{\infty} A_n \operatorname{th}(n\beta\lambda\pi)\cos nx \qquad (3.3)$$

Thus, boundary condition 3 reduces to

case (i) the triple series

$$\begin{aligned} G(x) &= A_0 + \sum_{n=1}^{\infty} n^{-1} A_n \cos nx = 0 && (0 < x < a) \\ F(x) &= \sum_{n=1}^{\infty} A_n \operatorname{th}(n\beta\lambda\pi)\cos nx = f(x) && (a < x < b) \\ G(x) &= A_0 + \sum_{n=1}^{\infty} n^{-1} A_n \cos nx = 0 && (b < x < \pi) \end{aligned} \qquad (3.4)$$

case (ii) the dual series

$$\begin{aligned} G(x) &= A_0 + \sum_{n=1}^{\infty} n^{-1} A_n \cos nx = 0 && (0 < x < a) \\ F(x) &= \sum_{n=1}^{\infty} A_n \operatorname{th}(n\beta\lambda\pi)\cos nx = f(x) && (a < x < \pi) \\ & \lim_{x\to\pi^-} \frac{dG(x)}{dx} = 0 \end{aligned} \qquad (3.5)$$

The solution of the triple series relation is given in [21] and is outlined here. By choosing

$$A_0 = -\frac{1}{\pi}\int_a^b tp(t)dt \qquad (3.6)$$

$$A_n = -\frac{2}{\pi}\int_a^b p(t)\sin ntdt \qquad (n \geq 1) \qquad (3.7)$$

we find that

$$\phi(x,0) = \beta^{-1}G(x) = \beta^{-1}H[(b-x)(x-a)]\int_a^x p(t)dt \qquad (3.8)$$

provided

$$\int_a^b p(t)dt = 0 \tag{3.9}$$

where $H(u)$ is the Heaviside step function. Additionally,

$$F(x) = \frac{1}{\pi}\int_a^b M(x,t)p(t)dt \quad (0 < x < \pi) \tag{3.10}$$

where

$$M(x,t) = -\frac{\partial}{\partial x}\log\left|\frac{tn(\frac{Kx}{\pi}) + tn(\frac{Kt}{\pi})}{tn(\frac{Kx}{\pi}) - tn(\frac{Kt}{\pi})}\right| \tag{3.11}$$

and $tn(u) \equiv tn(u,k)$ is a Jacobian elliptic function [22].

The series identity

$$2\sum_{n=1}^{\infty} \mathrm{th}(n\beta\lambda\pi)n^{-1}\sin nt \sin nx = \log\left|\frac{\mathrm{tn}(\frac{Kx}{\pi}) + \mathrm{tn}(\frac{Kt}{\pi})}{\mathrm{tn}(\frac{Kx}{\pi}) - \mathrm{tn}(\frac{Kt}{\pi})}\right| \tag{3.12}$$

where $0 < x,t < \pi$, $\lambda\beta > 0$ was used to find equation (8.3.11).

It is now evident that the triple series will be solved provided $p(t)$ is given by the following integral equation

$$\frac{1}{\pi}\int_a^b p(t)\frac{\partial}{\partial x}\log\left|\frac{tn(\frac{Kx}{\pi}) + tn(\frac{Kt}{\pi})}{tn(\frac{Kx}{\pi}) - tn(\frac{Kt}{\pi})}\right|dt = -f(x) \quad (a < x < b) \tag{3.13}$$

subject to (8.3.9)

From [21] it is shown that

$$p(t) = \frac{K\mathrm{dc}(\frac{Kt}{\pi})F_1(t)}{\pi\mathrm{cn}(\frac{Kt}{\pi})\Delta(t)} \tag{3.14}$$

and

$$\frac{\partial\phi}{\partial y}(x,0) = -F(x) = \begin{cases} \dfrac{K\mathrm{dc}(\frac{Kx}{\pi})F_1(x)}{\pi\mathrm{cn}(\frac{Kx}{\pi})\Delta_1(x)} & (0 < x < a) \\ -f(x) & (a < x < b) \\ \dfrac{-K\mathrm{dc}(\frac{Kx}{\pi})F_1(x)}{\pi\mathrm{cn}(\frac{Kx}{\pi})\Delta_1(x)} & (b < x < \pi) \end{cases} \tag{3.15}$$

where

$$F_1(t) = C + \frac{2}{\pi}\int_a^b \frac{\Delta(x)f(x)\mathrm{tn}(\frac{Kx}{\pi})}{\mathrm{tn}^2(\frac{Kx}{\pi}) - \mathrm{tn}^2(\frac{Kt}{\pi})}dx \tag{3.16}$$

$$C = \frac{-2K\mathrm{tn}(\frac{Kb}{\pi})}{\pi^3 K_1'}\int_a^b \frac{\mathrm{dc}(\frac{Kt}{\pi})dt}{\mathrm{cn}(\frac{Kt}{\pi})\Delta(t)}\int_a^b \frac{\Delta(x)f(x)\mathrm{tn}(\frac{Kx}{\pi})}{\mathrm{tn}^2(\frac{Kx}{\pi}) - \mathrm{tn}^2(\frac{Kt}{\pi})}dx \tag{3.17}$$

$$\Delta(x) = \left\{\left[\mathrm{tn}^2(\frac{Kb}{\pi}) - \mathrm{tn}^2(\frac{Kx}{\pi})\right]\left[\mathrm{tn}^2(\frac{Kx}{\pi}) - \mathrm{tn}^2(\frac{Ka}{\pi})\right]\right\}^{1/2} \tag{3.18}$$

$$\Delta_1(x) = \left\{\left[\mathrm{tn}^2(\frac{Kb}{\pi}) - \mathrm{tn}^2(\frac{Kx}{\pi})\right]\left[\mathrm{tn}^2(\frac{Ka}{\pi}) - \mathrm{tn}^2(\frac{Kx}{\pi})\right]\right\}^{1/2} \tag{3.19}$$

$$\text{and} \quad K_1' = K(k_1') \tag{3.20}$$

$$\text{with} \quad k_1 = \frac{\mathrm{tn}(\frac{Ka}{\pi})}{\mathrm{tn}(\frac{Kb}{\pi})} \tag{3.21}$$

The solution of the dual series relation (8.3.5) is found in a similar manner. Here choose

$$A_0 = -\frac{1}{\pi}\int_a^{\pi}(t-\pi)p(t)dt \tag{3.22}$$

$$A_n = -\frac{2}{\pi}\int_a^{\pi} p(t)\sin ntdt \qquad (n \geq 1) \tag{3.23}$$

whereby

$$\phi(x,0) = \beta^{-1}G(x) = \beta^{-1}H(x-a)\int_a^x p(t)dt \tag{3.24}$$

and

$$\frac{\partial\phi}{\partial y}(x,0) = -F(x) \tag{3.25}$$

with

$$F(x) = \frac{1}{\pi}\int_a^{\pi} M(x,t)p(t)dt \qquad (0 < x < \pi) \tag{3.26}$$

and $M(x,t)$ given by (8.3.11).

Using (8.3.26) and (8.3.11) it is clear that solving the dual series is equivalent to solving the singular integral equation

$$\frac{1}{\pi}\int_a^{\pi} p(t)\frac{\partial}{\partial x}\log\left|\frac{tn(\frac{Kx}{\pi}) + tn(\frac{Kt}{\pi})}{tn(\frac{Kx}{\pi}) - tn(\frac{Kt}{\pi})}\right|dt = -f(x) \quad (a < x < \pi) \tag{3.27}$$

subject to

$$\lim_{x\to\pi^-} p(x) = 0 \tag{3.28}$$

Equation (8.3.27) subject to (8.3.28) is of a form investigated by Tricomi [18] and has solution

$$p(t) = -\frac{dn(\frac{Kt}{\pi})}{\pi\delta(t)} \int_a^\pi \frac{\delta(x) f(x) \frac{2K}{\pi} \mathrm{sn}^2(\frac{Kx}{\pi}) \mathrm{cn}(\frac{Kx}{\pi})}{\mathrm{sn}^2(\frac{Kx}{\pi}) - \mathrm{sn}^2(\frac{Kt}{\pi})} dx \tag{3.29}$$

where

$$\delta(x) = \frac{\sqrt{\mathrm{sn}^2(\frac{Kx}{\pi}) - \mathrm{sn}^2(\frac{Ka}{\pi})}}{\mathrm{cn}(\frac{Kx}{\pi})} \tag{3.30}$$

Substituting (8.3.29) into (8.3.26)

$$\frac{\partial \phi}{\partial y}(x,0) = \begin{cases} \dfrac{\mathrm{dn}(\frac{Kx}{\pi})}{\pi\delta_1(x)} \displaystyle\int_a^\pi \frac{\delta(u) f(u) \frac{2K}{\pi} \mathrm{sn}(\frac{Ku}{\pi}) \mathrm{cn}(\frac{Ku}{\pi})}{\mathrm{sn}^2(\frac{Ku}{\pi}) - \mathrm{sn}^2(\frac{Kx}{\pi})} du & (0 < x < a) \\ \\ -f(x) & (a < x < \pi) \end{cases} \tag{3.31}$$

and

$$\delta_1(x) = \frac{\sqrt{\mathrm{sn}^2(\frac{Ka}{\pi}) - \mathrm{sn}^2(\frac{Kx}{\pi})}}{\mathrm{cn}(\frac{Ka}{\pi})} \tag{3.32}$$

8.4 Stress Intensity Factors

The stress intensity factors k_a and k_b are given by the formulae

$$k_a = \lim_{x \to a^-} \sqrt{2(a-x)}\sigma_{yz}(x,0) \tag{4.1}$$

$$k_b = \lim_{x \to b^+} \sqrt{2(x-b)}\sigma_{yz}(x,0) \tag{4.2}$$

Utilizing equations (8.2.2) and (8.2.5) these equations become

$$k_a = \lim_{x \to a^-} \sqrt{2(a-x)} \frac{\partial \phi}{\partial y}(x,0) \tag{4.3}$$

$$k_b = \lim_{x \to b^+} \sqrt{2(x-b)} \frac{\partial \phi}{\partial y}(x,0) \tag{4.4}$$

case (i)

Substituting (8.3.15) into (8.4.3) and (8.4.4)

$$k_a = F_1(a) \left[\frac{K \mathrm{dn}(\frac{Ka}{\pi})}{\pi \{ \mathrm{tn}^2(\frac{Kb}{\pi}) - \mathrm{tn}^2(\frac{Ka}{\pi}) \} \mathrm{sn}(\frac{Ka}{\pi}) \mathrm{cn}(\frac{Ka}{\pi})} \right]^{1/2} \tag{4.5}$$

$$k_b = -F_1(b) \left[\frac{K \mathrm{dn}(\frac{Kb}{\pi})}{\pi \{ \mathrm{tn}^2(\frac{Kb}{\pi}) - \mathrm{tn}^2(\frac{Ka}{\pi}) \} \mathrm{sn}(\frac{Kb}{\pi}) \mathrm{cn}(\frac{Kb}{\pi})} \right]^{1/2} \tag{4.6}$$

case (ii)

Substitute (8.3.31) into (8.4.3) to obtain

$$k_a = \frac{2}{\pi} \left[\frac{K \mathrm{cn}(\frac{Ka}{\pi}) \mathrm{dn}(\frac{Ka}{\pi})}{\pi \mathrm{sn}(\frac{Ka}{\pi})} \right]^{1/2} \int_a^{\pi} \frac{f(y) \mathrm{sn}(\frac{Ky}{\pi}) dy}{\sqrt{\mathrm{sn}^2(\frac{Ky}{\pi}) - \mathrm{sn}^2(\frac{Ka}{\pi})}} \tag{4.7}$$

8.5 Constant Shear

For the special case of a constant shear the problems simplify substantially. First, observe that

$$\bar{w}(x,y) = \tau y \tag{5.1}$$

and hence

$$f(x) = \tau \tag{5.2}$$

This allows simplification of the integral terms in equations (8.4.5), (8.4.6) and (8.4.7).

Indeed,

case (i)

$$\begin{aligned} \frac{F_1(a)}{\tau} &= \frac{2K_2'}{K\mathrm{dn}(\frac{Ka}{\pi})}\left[\frac{\mathrm{cn}(\frac{Ka}{\pi})}{\mathrm{cn}(\frac{Kb}{\pi})} - \frac{\mathrm{cn}(\frac{Kb}{\pi})}{\mathrm{cn}(\frac{Ka}{\pi})} - tn(\frac{Kb}{\pi})Z(\beta, k_1')\right] \\ &\quad - \frac{2K_2'\mathrm{tn}(\frac{Kb}{\pi})}{\pi K_1'}\int_a^b Z(\delta, k_2')dt \end{aligned} \tag{5.3}$$

and

$$\frac{F_1(b)}{\tau} = -\frac{2K_2'\mathrm{tn}(\frac{Kb}{\pi})}{K\mathrm{dn}(\frac{Ka}{\pi})}Z(\beta, k_1') - \frac{2K_2'\mathrm{tn}(\frac{Kb}{\pi})}{\pi K_1'}\int_a^b Z(\delta, k_2')dt \tag{5.4}$$

with

$$k_2 = \mathrm{dn}(\frac{Kb}{\pi})/\mathrm{dn}(\frac{Ka}{\pi}) \tag{5.5}$$

$$\sin\beta = \mathrm{sn}(\frac{Kb}{\pi}) \tag{5.6}$$

and

$$\sin^2\delta = \frac{\mathrm{sn}^2(\frac{Kt}{\pi}) - \mathrm{sn}^2(\frac{Ka}{\pi})}{\mathrm{sn}^2(\frac{Kb}{\pi}) - \mathrm{sn}^2(\frac{Ka}{\pi})} \tag{5.7}$$

and $Z(\theta, k)$ is Jacobi's Zeta function.

Figures 12 and 13 give the scaled stress intensities versus aspect ratio for various values of $(\lambda\beta)$. Here $k_0 = \tau\sqrt{d}$ with $d = (b-a)/2$ and c denotes the midpoint of the crack. In particular, we have recorded results for a central crack $(c = \pi/2, k = k_a = k_b)$ and off-center crack $(c = \pi/4)$ which shows the effect of orthotropy.

case (ii)

$$k_a = 2\tau K_3' \left[\frac{\mathrm{cn}(\frac{Ka}{\pi})}{\pi K \mathrm{sn}(\frac{Ka}{\pi})\mathrm{dn}(\frac{Ka}{\pi})}\right]^{1/2} \tag{5.8}$$

where

$$K_3' = K(k_3') \tag{5.9}$$

and

$$k_3 = \frac{k'}{\mathrm{dn}(\frac{Ka}{\pi})} \tag{5.10}$$

Figure 14 gives values of the scaled stress intensity versus aspect ratio for various values of $(\lambda\beta)$ where $k_o = \tau\sqrt{\pi - a}$. Observe that for a constant shear our problem is equivalent to the case of a central crack investigated in case (i) and our results agree accordingly.

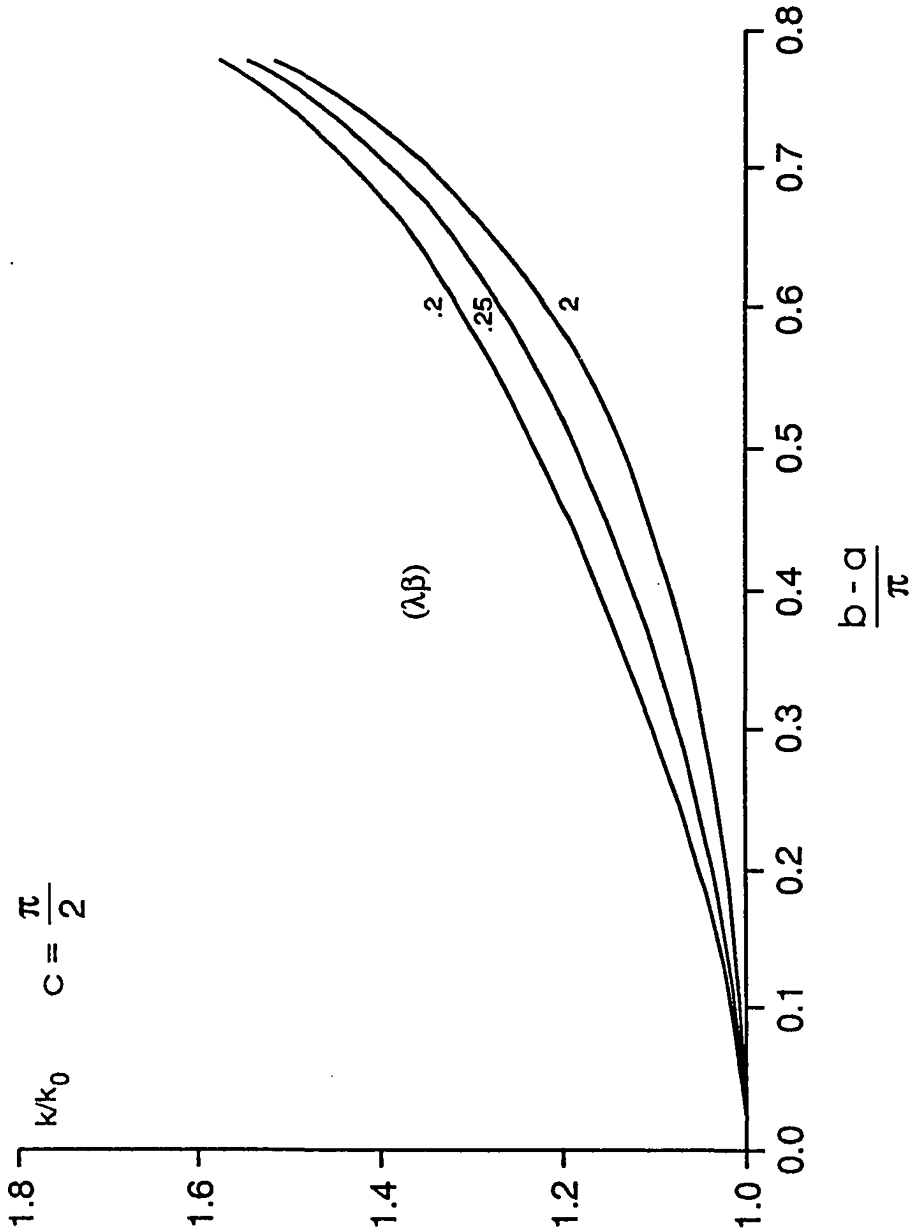

Figure 12. Stress intensity vs aspect ratio for various values of ($\lambda\beta$)

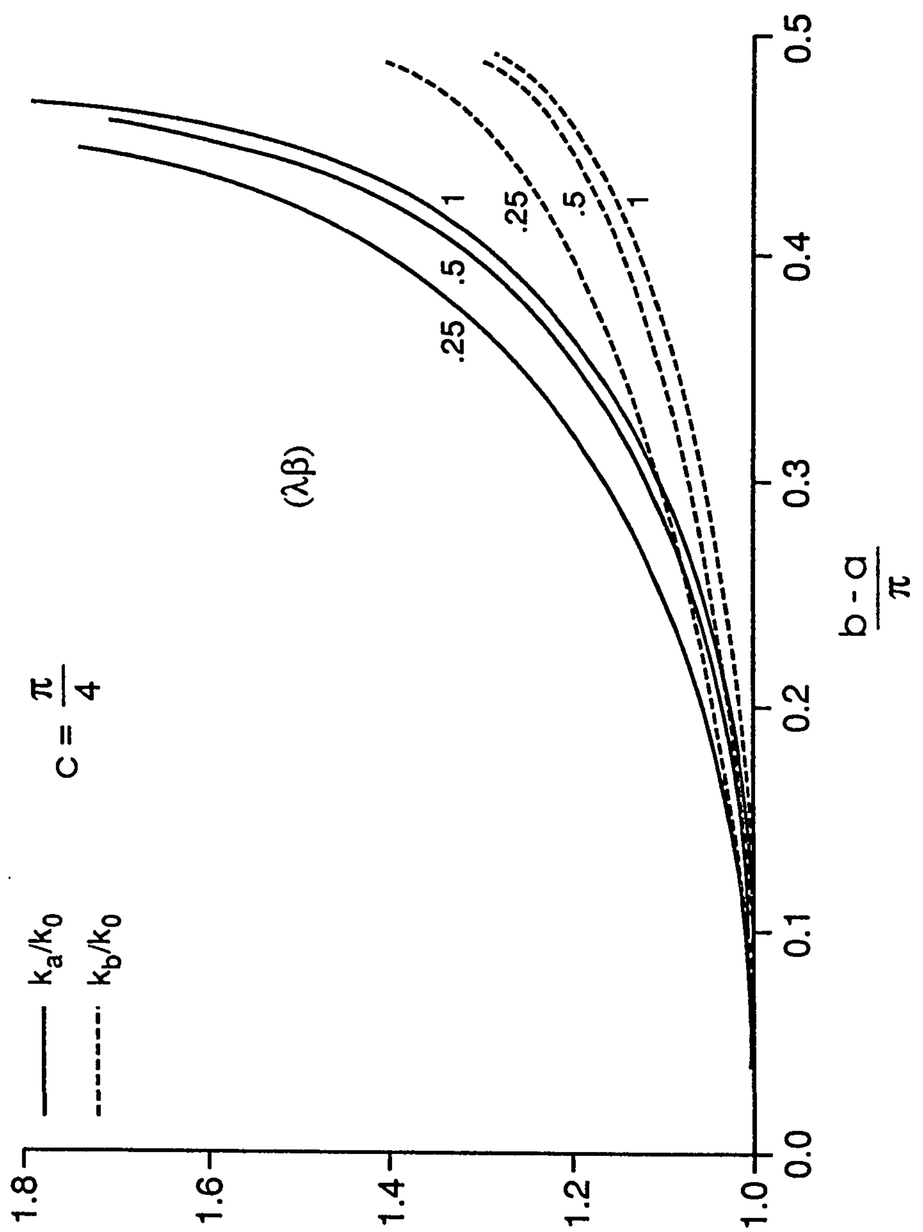

Figure 13. Stress intensity vs aspect ratio for various values of (λβ)

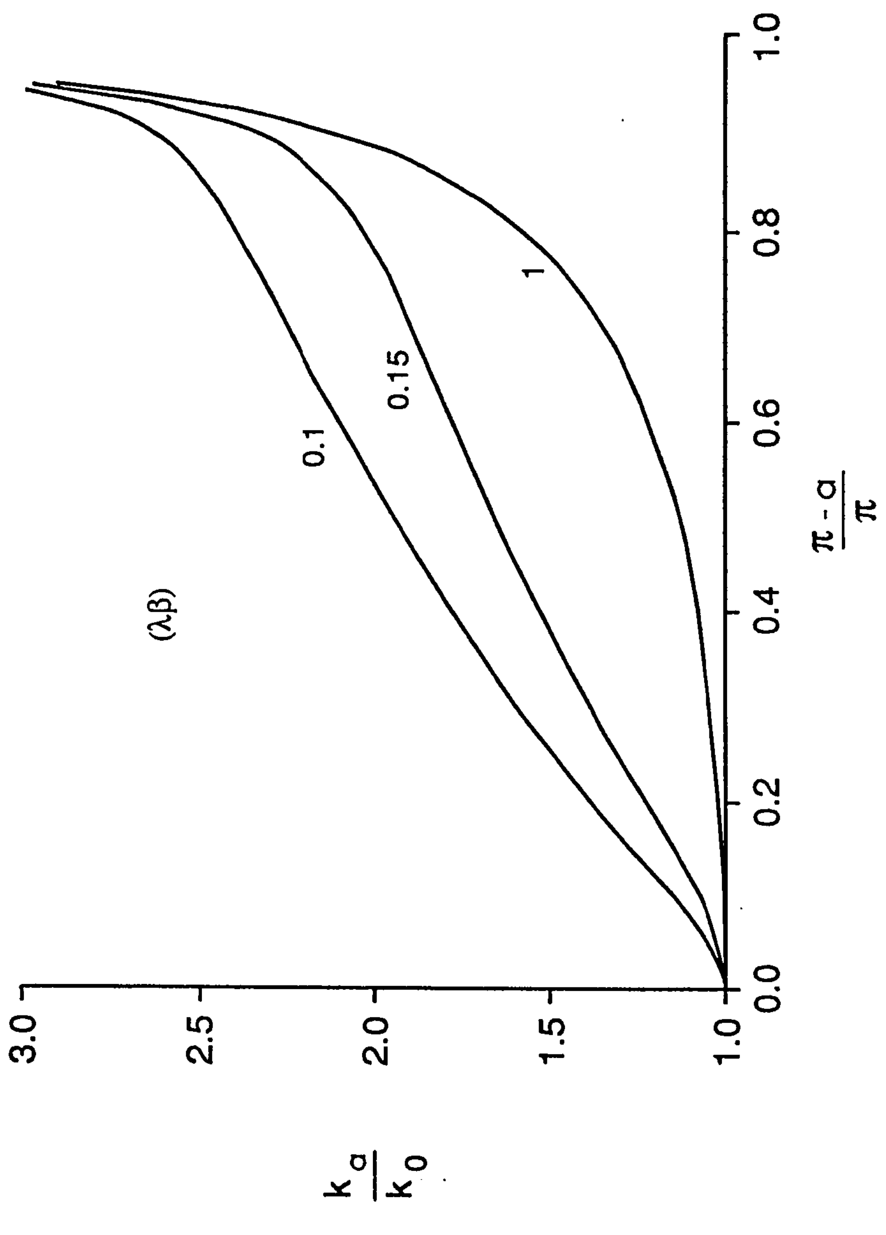

Figure 14. Scaled stress intensity vs. aspect ratio for various values of (λβ)

Appendix A

Numerical Procedure

The thermoelastic problems investigated in this dissertation require the numerical solution to certain singular integral equations with subsidiary conditions. The procedure employed to complete this task will be summarized now.

The singular integral equations assume the form

$$\frac{1}{\pi}\int_{\alpha}^{\beta} w(t)\phi(t)K(x,t)dt = 0 \qquad (\alpha < x < \beta) \tag{A.1}$$

where the unknown function is $\phi(t)$ with weight function

$$w(t) = \{(\beta - t)(t - \alpha)\}^{1/2} \tag{A.2}$$

and kernel $K(x,t)$ which, in this discussion, can be characterized by the equation

$$K(x,t) = \begin{cases} \dfrac{1}{t-x} + k_1(x,t) & \text{, chapter 5} \\ \dfrac{1}{t-x} + k_2(x,t) & \text{, chapter 6} \\ \dfrac{1}{2}\cot\left(\dfrac{t-x}{2}\right) + k_3(x,t) & \text{, chapter 7} \end{cases} \tag{A.3}$$

Each $k_i(x,t)(i=1,2,3)$ is regular in the integration interval $[\alpha,\beta]$.

For the three cases indicated by (A.3) there is a subsidiary condition of (A.1) which has the form

$$\int_\alpha^\beta w(t)\phi(t)dt = C_i \qquad (i=1,2,3) \tag{A.4}$$

where C_i is a constant.

A method widely used to solve equations of the form (A.1) and (A.4) is due to Erdogan and Gupta [23]. Using their method the equations (A.1) and (A.4) are reduced to a system of linear algebraic equations whose solutions provides values of the unknown function $\phi(t)$ at discrete points $t_k(k=1,2,...,N)$ within the integration interval $[\alpha,\beta]$. However, as already shown in chapter 5, 6, and 7, the stress intensity factors at the crack tips are proportional to the values $\phi(\alpha)$ and $\phi(\beta)$, which must therefore be evaluated from the values $\phi(t_k)$ using an extrapolation formula. An additional error is introduced due to the extrapolation procedure.

A suitable method to calculate $\phi(\alpha)$ and $\phi(\beta)$ avoiding any extrapolation error is due to Ioakimidis and Theocaris [11] who modified the procedure of Erdogan and Gupta [23]. Using their method, the equations (A.1) and (A.4) are replaced by the linear algebraic system

$$\frac{1}{\pi}\sum_{k=1}^{N} H_k\phi(t_k)K(x_j,t_k) = 0 \ , \ 1 \le j \le N-1 \tag{A.5}$$

$$\sum_{k=1}^{N} H_k \phi(t_k) = C \tag{A.6}$$

with weights

$$H_k = \frac{\pi}{N-1} \begin{cases} \frac{1}{2} & , \quad k=1, N \\ 1 & , \quad k=2, 3, \ldots, N-1 \end{cases} \tag{A.7}$$

the abscissae given by

$$t_k = \left(\frac{\beta+\alpha}{2}\right) \cos\left(\frac{k-1}{N-1}\right)\pi + \left(\frac{\beta-\alpha}{2}\right) \quad , \quad 1 \leq k \leq N \tag{A.8}$$

and collocation points

$$x_j = \left(\frac{\beta+\alpha}{2}\right) \cos\left(\frac{j-\frac{1}{2}}{N-1}\right)\pi + \left(\frac{\beta-\alpha}{2}\right) \quad , \quad 1 \leq j \leq N-1 \tag{A.9}$$

The abscissae and collocation points are not arbitrary but satisfy the equations

$$U_{N-2}(t_k) = 0 \qquad k = 2, \ldots, N-1 \tag{A.10}$$

$$T_{N-1}(x_j) = 0 \qquad j = 1, \ldots, N-1 \tag{A.11}$$

where T and U are the Chebyshev polynomials of the first and second kind on $[\alpha, \beta]$ respectively.

It must be noted that the kernels $k_i(x,t)(i = 1,2,3)$ involve complicated integral terms which were evaluated using subroutines from Quadpack-A Subroutine Package for Automatic Integration [24].

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