Winter 1993

Theoretical Results Supporting the Use of Passive Damping as Augmentation to the Active Control of Flexible Structures

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THEORETICAL RESULTS SUPPORTING THE USE OF PASSIVE DAMPING AS AUGMENTATION TO THE ACTIVE CONTROL OF FLEXIBLE STRUCTURES

by

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M.S.-Physics June 1971, Old Dominion University

A Dissertation submitted to the Faculty of Old Dominion University in Partial Fulfillment of the Requirement for the Degree of

DOCTOR OF PHILOSOPHY

MECHANICAL ENGINEERING

OLD DOMINION UNIVERSITY
December, 1993

Approved by:

Thomas E. Alberts (Director)

ABSTRACT

THEORETICAL RESULTS SUPPORTING THE USE OF PASSIVE DAMPING AS AUGMENTATION TO THE ACTIVE CONTROL OF FLEXIBLE STRUCTURES

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One challenge of modern control technology is how to control a flexible structure with accuracy, speed, and economy of effort. Controlling a structure with many degrees of freedom by purely active means implies the implementation of inordinate sensors and actuators and creates the need for numerous calculations that must be done instantly. Experiments have shown that practical structures under active control alone can suffer instabilities due to modal vibrations beyond the bandwidth of the active controller. Furthermore, if there is a high degree of model uncertainty, instabilities can be produced by inputs of modal vibrations not occurring in the system model. The use of passive damping to stabilize those vibrations beyond the domain of the active controller and to help reduce the effects of model uncertainty has been shown to be critical to enabling control of flexible structures.

The question remains as to how passive damping should best be implemented to aid active control. The same amount of damping (by weight) can be applied in different ways - some

ways may satisfy performance constraints, while others may not. Part I of this thesis deals with the effects of damping on control. The system to be controlled is defined by its linear matrix differential equation. The system is under the influence of a disturbance and a set of control forces. A performance index is defined, after which are derived closed-form expressions for the optimal feedback gains and the optimal value of the performance index. A modern passive damping technique is applied to a beam, and the cost function is optimized subject to the appropriate constraints. The benefits of the damping are demonstrated in the performance, the displacement output, and in the economic savings.

Part II of this thesis pursues the effects of passive damping on plant model reduction in modal coordinates. Prevailing closed-form expressions in this field assume light damping and widespread natural frequencies. A formula is derived based upon general constant-ratio damping and general spectrum of natural frequencies. Conclusions are drawn, and numerical examples demonstrate the effects of this new formula on model reduction as the modal damping ratio is varied.

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ACKNOWLEDGEMENTS

I want to express my gratitude to the people that made this undertaking a reality. First, I thank my wife, Debbie, who typed the original version of this monstrosity of symbols, for her hard work and patience during this six year trial. Next, I thank my mother, Marjorie, for her moral support and good counsel. My special thanks go to my advisor, Dr. Thomas Alberts, who, despite his relative youth, is excellent both as a research director and as a teacher. My appreciation goes to my colleagues at Tidewater Community College, Aubrey Hartman and George Fowler, who were extremely helpful in getting this document on the computer, and my division chairman Dr. Thomas Hubbard, who gave me sound advice going into the oral exam and defense. Finally, I would like to thank all of the other O. D. U. professors and administrators including my remaining panel members, Drs. Huang, Mei and McRee, and the graduate program directors Drs. Baysal and Britcher for their patience and cooperation with an older than average graduate student.

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LIST OF SYMBOLS

A - system matrix

B - spatial input influence matrix

C - spatial output influence matrix

D - spatial damping matrix

E - total system energy

F - control input force

G - modal position feedback matrix

H - modal velocity feedback matrix

I - identity matrix

J - performance index

K - system stiffness matrix

L - length of experimental beam

M - system inertia matrix

N - index of highest retained mode in model

Q - quadratic performance matrix

T - orthonormal coordinate transformation matrix

U - disturbance magnitude matrix

X - matrix relating damping vector to modal expansion coefficient vector

Z - matrix of modal damping ratios

b - modal input influence vector

c - spatial output coefficient vector

$\mathbf{d}$ - damping vector

$\mathbf{f}$ - modal control input vector

g_i - ith modal position feedback coefficient

h_i - ith modal velocity feedback coefficient

m - mass

n - number of system degrees of freedom

$\mathbf{p}$ - second-order modal position output coefficient vector

$\mathbf{q}$ - spatial position vector

r - index of highest retained mode in model

$\mathbf{r}$ - second-order modal velocity output coefficient vector

s - Laplace transform variable

t - time

$\mathbf{u}$ - spatial disturbance input vector

$\mathbf{v}$ - spatial velocity vector

$\mathbf{x}$ - spatial position vector (or general state vector)

$\mathbf{y}$ - system output vector

α - effort weighting coefficient

$\boldsymbol{\alpha}$ - modal damping expansion coefficient vector

$\boldsymbol{\beta}$ - modal disturbance input influence matrix

β_i - ith modal disturbance input influence vector

δ - impulse function

ε - maximum allowable modelling error

ζ_i - ith modal damping ratio

η - modal position vector

θ - relative self-adjointness correlation angle

λ_i - ith first-order modal eigenvalue

μ_j - magnitude of jth modal input disturbance

ρ - linear mass density

σ_i - real coordinate of ith complex pole

ϕ - modal damping perturbation parameter

ω_i - ith modal frequency

Γ - first-order modal position output coefficient vector

P - first-order modal velocity output coefficient vector

CHAPTER ONE

INTRODUCTION

I. Introduction to Damping and Control

The need to make some modern solid structures (robots, satellites, etc.) lighter weight in order to improve their economic feasibility has led to a new challenge in the control of such structures. The old philosophy of bang-bang (minimum time) control has given way to control strategies that must account for the flexible motions of the structures involved. This has led to control systems that are more complicated dimensionally and technologically.

New problems are attendant with these modern control systems. The decision must be made as to how many actuators and sensors are to be employed to effectively control a structure and yet fall within the economic and computation time constraints that prevail. Furthermore, a system cannot be controlled accurately if it is not modelled accurately. A continous structure has, in theory, an infinite number of modal degrees of freedom.

Assuming the use of discrete sensors and actuators, a finite approximation model of some sort must be made for the system. This leads inevitably to modelling error which can degrade the effectiveness of the control system whose design is based on the model. For this reason, the use of passive mechanical damping would seem to be a welcome addition for two reasons - it adds the element of robustness to a system that is only marginally stable, and it can damp out vibrations that are beyond the practical range of existing control actuators.

Provided we accept the notion of adding passive damping, keeping in mind that this adds somewhat to the inertia of the structure to be controlled, this leads us to the inevitable question of how to apply the damping to best achieve performance objectives. For a given weight, there can be a best way to implement passive damping so as to minimize a standard quadratic index involving kinetic energy, elastic potential energy, and control effort. Variations in the design of available passive damping implementations supplies the variability needed about which to produce an optimal design.

Survey of Pertinent Research on Damping and Control

A. General Systems Theory

Lord Rayleigh [1] was the first to show that an undamped system obeying a second-order linear differential equation possessed normal modes - oscillations in which each point of the structure moved in phase. Rayleigh showed that a sufficient condition for the existence of these modes was that the damping matrix be, in general, a linear combination of the mass and stiffness matrices.

Caughey [2,3] generalized Rayleigh's work with two landmark papers from the early 1960's. He wrote the system equation as $I\ddot{q} + A\dot{q} + Bq = 0$ where the inertia matrix has been transformed into the identity matrix I. He argued that since A and B are symmetric and positive definite, it is always possible to find a transformation that will simultaneously diagonalize A and B. That tranformation may or may not leave the inertia matrix diagonal, depending on the relationship between A and B. In the first paper, Caughey produced a formula relating A and B that gave sufficient conditions for the total uncoupling of the system equations. In the second paper he produced the necessary and sufficient conditions relating A and B to insure the existence of classical normal modes.

With reference to normal modes, Hughes [4] argued that while a structure can be thought of as continuous for mathematical purposes, the reality is that it is a composition of a finite number of molecules and thus has only a finite number of modal vibrations. Likewise, he argued that modal truncation is an accurate method of approximation only if the system is relatively simple, such as a rod or a beam. More complicated structures require the careful selection of modes for accurate modelling.

In a lengthy paper [5], Meirovitch displays some illuminating insights into the relationship between modelling a continous structure with partial differential equations and thus obtaining continuous (and infinitely many) mode shapes, and with finite element methods obtaining finitely many discretized eigenvectors and eigenfrequencies. He cites what he calls the inclusion principle [6] in which the first N eigenvalues of the Nth-order model are bracketed by the N+1 eigenvalues of the (N+1)th-order model. Moreover, he argues that the computed eigenvalues of any order approach the actual eigenvalues of the continuous structure as N tends to infinity. However, since N is finite, he points out that the lower eigenvalues are accurately represented, but those at the upper end of the spectrum tend to be wildly in error. Hence, no discretized model can yield a totally accurate representation of a distributed structure.

B. General Control Theory

In an early work [7], Balas discusses the problem of controlling a large dimensionsal system with a much smaller dimensional controller. He points out how limitations due to on-board computer capability combined with modelling errors make it impossible to control a large number of the structure's elastic modes. For this reason, he argues, control must be restricted to a few critical modes. He defines the effects of the uncontrolled and unmodelled modes (residual modes) as "spillover."

Balas states that all modal controllers have the potential to generate instabilities unless observation spillover can be eliminated. He recommends prefiltering the sensor data with narrow bandpass filters which can be implemented with phase-locked loops. The PLL's should be tuned to controlled frequencies. He applies his ideas to a simply-supported beam with an active controller using a linear feedback control law, a state observer, and the sensor prefilter.

Meirovitch [5] disputes Balas contention [7] that observation spillover necessarily produces instability. He argues that as long as the residual modes are included in the observer dynamics and the observer gains are chosen properly, observation spillover cannot destabilize the system. In this same paper, Meirovitch argues that a control system that uses modal filters is superior to one that uses Luenberger observers because, due to the orthogonality property, the filters screen out inputs from the higher uncontrolled modes whether they are known or not. It is argued that modal filters will require only simple on-line operations so that fast filtering is assured.

He then presents the main thesis in this work - independent modal space control (IMSC). In this method, actuators chosen can be distributed or discrete. Each modal control depends only on that mode's coordinate and velocity.

For this reason, there can be no control spillover. Controls are designed in modal space, and then these controls are transformed back to Cartesian space for implementation. The modal filters provide the coordinates and velocities needed for feedback. The one major weakness with this method is the requirement that there must be an actuator for each mode being controlled. Despite this weakness, IMSC was chosen for the control system used in this thesis.

S.R. Vadali [8] considers the problem of controlling the large-angle maneuvers of a structure regarded as a central rigid body with long flexible booms. He defines a linear control law based solely on the hub deflection angle and its angular velocity, and shows by way of a Liapunov function that the large-angle maneuvers under this control law are asymptotically stable.

D. Franke [9,10,11] considers the problem of estimating and bounding the effects of modelling error in the design of finite controllers for infinite-dimensional linear feedback control systems. First, he uses controls in finite first-order modal space designed to place the first N eigenvalues at specified locations in the complex plane. Because the actual system is distributed and not discrete, modelling error causes the eigenvalues to shift from the design locations. He estimates the amount of shifting using the theory of Gershgorin. The actual eigenvalues lie within "disks" which are circles centered at the design locations and have predictable radii. This method of analysis therefore provides sufficient conditions for stability.

Meirovitch and Silverberg [12] utilize the IMSC method of control to design a globally optimal control of a self-adjoint distributed parameter system. A quadratic index is defined for the distributed parameter system in terms of its total kinetic plus potential energy added to its weighted distributed control effort. This index is then minimized as a function of the feedback control gain coefficients. These control gains are found to be functions of the open-loop natural frequencies and the weighting factor from the quadratic index. These optimal modal controls correspond to the optimal distributed control in Cartesian space.

C. Control With Damping

In an early work [13], B. Anderson and J. Moore consider the optimal control of a system with a prescribed degree of stability. They introduce the classic form of solution to the optimal control problem by Athans, Falb and Kalman [14,15,16] using the matrix Riccatti equation. It is noted that while system poles are in the left half of the s-plane, their distance from the imaginary axis is not known. They therefore set out to find an optimal control system with a prescribed minimum degree of stability. For this purpose, they define a parametrized quadratic index which has the mathematical effect of placing all of the resulting poles of the optimally controlled system to the left of the given parameter. Observations are then noted that sensitivity to plant-parameter variation, phase shift and tolerance range to non-linearities are better under the new scheme than under the old. The obvious disadvantage of requiring greater control effort is correctly noted.

C. Greene and G. Stein [17] discuss the weaknesses of designing controls for distributed structures by assuming that inherent structural damping is zero. They point out that not only must a controllable structure have some inherent damping, but the damping must be provided by certain mechanisms if the system is to be stable. The ideal mechanisms they define are

1. $$\zeta_i = \frac{c}{\omega_i} \quad , \quad i = 1, 2, \ldots$$

2. $$\zeta_i = c \quad , \quad i = 1, 2, \ldots$$

The first mechanism is representative of deflection rate dependent viscous damping discussed in [18]. This gives the same constant damping coefficient c for each mode (uniform damping). The second mechanism, constant ratio damping, is representative of structural (hysteretic) damping under cyclic excitation, also explained in [18]. Greene and Stein argue that typical pole-placement methods, involving the feedback of rate-dependent outputs, y, to the inputs, u, with a negative definite symmetric gain matrix, K, selected large enough to achieve the desired damping for an initially assumed undamped structure overlook the problem of control loop phase variations greater than $\pi/2$ radians such as when sensor and actuator dyanamics are included. They argue that a gain crossover frequency, ω_c , must be reached beyond which $|g(j\omega)| < 1$ for all $\omega > \omega_c$. They concluded that this condition can be satisfied by damping mechanism #2 but cannot be satisfied by damping mechanism #1. They demonstrate their ideas with a beam example damped with each of the given mechanisms.

Plunkett and Lee [19] experimented with certain visco-elastic constraining layers and observed that a relationship exists between the length of the constraining layer segments and the amount of damping provided by the treatment. The graph of modal damping ratio vs. modal frequency shows that under these treatments, damping ratio increases very quickly with frequency then reaches a maximum value at a certain frequency and then gently declines with higher frequencies. Changing the segment length, in affect, changes the frequency at which the damping will be maximum.

H. Ashley [20] observed the generally recognized fact that it is unlikely that all flexible modes can be stabilized without structural damping except in the case of the impractical arrangement of sensors and actuators in large space structures (LSS). This will be equally true for spacecraft, as there is no surrounding environment to provide a means of external energy dissipation. The following three damping mechanisms are considered:

1. Inherent material damping
2. Damping at interconnections
3. Damping furnished by dashpots and viscoelastic layers

He considers the third mechanism impractical from a weight limitation viewpoint. Except in the section on scale effects, only the first listed mechanism is studied. His work shows that damping ratios and natural frequencies decrease as the length of a characteristic dimension L increases among a family of geometrically similar structures of the same materials. He concludes that the function dependence of ζ_n, the damping ratio of the nth mode, will vary with L according to $\zeta_n = L^m$ where $-1 < m < 0$. To show that structural damping can be varied, he cites the well-known Debye formula [21] which predicts a damping peak when the driving frequency is given by $\omega = \tau^{-1}$, where τ is a characteristic time that can be controlled by choice of material. Aluminum and magnesium are considered materials that hold the greatest promise among common aerospace alloys, due to their high thermal expansivities (α).

Finally, Ashley considers composite materials for LSS. An equation for the average loss factor, $\bar{\eta}$, a constant proportionial to ζ, shows that very high expansive ratios

α_2/α_1 may yield greater damping than for either constituent material separately. He concludes reluctantly that "cases will arise where a combination of active control (at the lower end of the frequency spectrum) and artifical means such as viscoelastic inserts will prove necessary for satisfactory performance."

R. Gehling [22] considers the problem of the pointing and retargeting requirements of the LSS. He cites the results of previous experiments [23,24] in which high modal density at low frequencies indicate serious difficulties in the practical application of control laws arising from sensor/actuator dynamics, spillover, and model inaccuracies. He applies a low authority control approach involving the design of wideband control algorithms which generally provide high robust damping to several structural modes in a representative LSS called Representative System Article (RSA). The RSA is a representative space structure derived from a survey which included consideration of both military and civilian system concepts and disturbances affecting such systems. They are generally large systems possessing high modal density at relatively low frequencies. The ten most significant flexible modes were selected for the purposes of this study. A modal viscous damping of 0.2% (considered typical of LSS) was assumed for the structure. Serious stability problems were shown to exist in simple attitude or slow maneuvering in the absence of flexible modal control. The following approaches to control the flexible modes were considered:

1) active control alone
2) passive damping alone
3) an integrated active-passive approach

In the purely active system, six collocated sensors and actuators were needed to satisfy time requirements. This system suffered from control spillover to the extent that some modes not targeted for control became overdamped while several controlled modes possessed insufficient damping values. Passive control alone using a damping treatment with $\zeta = 5\%$ (considered feasible), could not provide a sufficiently short settling time. However, integrated active and passive controls in which only two sensor/actuator pairs were used, satisfied settling performance criteria.

L. Silverberg [25], in a follow-up work to his earlier paper on global optimal control [12], shows that uniform damping in which the damping constant is the same for each mode, provides a close approximation to the solution of the global optimal control problem. Uniform damping control is characterized by

a) a uniform decay rate for each mode,
b) controlled oscillation frequencies that are the same as the uncontrolled natural frequencies,
c) closed-loop modes of vibration that are identical to the uncontrolled natural modes,

d) a control law independent of structural stiffness, and

e) control forces proportional to mass density.

The control system has feedback gains that are derived assuming that $\alpha < \omega_n$ where $1/2\alpha^2$ = effort weighting coefficient in the quadratic functional, and ω_n = frequency of the nth mode. Silverberg shows that α is also the decay rate for each mode. Hence, the decay rate is the same for each mode. In the light of works by Greene and Stein [17] on the one hand, and Ashley [20] on the other, it is not likely that uniform damping will be more than just a mathematical fascination to control designers.

Meirovitch and Norris [26] take up the problem of nonproportional damping in which the damping matrix is not a linear combination of the mass and stiffness matrices. It is pointed out that undamped structures are self-adjoint, meaning that their motions are the linear superposition of the individual independent modes. General forms of damping tend to destroy this self-adjointness property and couple the modes. The problem is considered in which a pole-placement control law (by IMSC) is applied to the assumed undamped design structure. These controls are then applied to a structure with slight amounts of general viscous damping, that is, the damping matrix is assumed fully populated. They show that perturbations in the eigenvalues of the controlled system are sensitive only to the diagonal values of the damping matrix, those values which do not induce non-self-adjointness. They further argue that if the damping matrix is diagonally dominant, then the perturbations in the pole-placements of the design model are second-order effects. Their ideas are demonstrated on an 8-mode model using strain-rate damping in a simply-supported beam. The results show that the presence of small damping in the actual distributed structure does not affect the control system performance significantly when the poles are placed in the self-adjoint design model by the IMSC method.

Pan, Rao, and Venkayya [27,28] attempt to design an active LQR control system for a large flexible structure and then replace it with a passive control system consisting of springs and dashpot-type dampers that best approximate the pole locations of the original system. They found that the so-called unified passive damping design (UPD) approximates the LQR controlled system well only in the higher modes. To provide a better overall approximation to original design, they integrate the passive damping design with an active control system designed to give better compensation for the first few modes. This system is referred to as interacting substructure decentralized control (ISDC). The researchers note a significant savings in performance of the hybrid controller compared to either the original LQR controller or the ISDC controller when used alone. There seems to be no concern, however, for any constraints that might be placed on dampers and stiffness that would affect the feasibility of the passive components of the system.

Alberts [29] considers the hybrid active/passive flexible manipulator with a payload mass at the tip. For damping purposes, he uses the viscoelastic layer treatment of Plunkett and Lee [19]. A four state design model including the rigid body and one flexible mode is assumed. Two simulated plant models are used - one with three flexible modes and the other with six. An LQR type control system is designed to penalize the error in the tip position. The steady-state Riccatti equation is used to determine the four state feedback gains. The results show that the problems of control and observation spillover associated with the unmodelled modes are more acute when the system is undamped than when it is damped, particularly when the performance weighting favors active control.

Van Flotow and Vos [30] take for granted that passive damping will be required to stabilize an undamped infinite dimensionsal flexible structure. They take up the problem of just how much damping is needed to guarantee closed-loop stability if the structural dynamics are known only with a given level of certainty. Mode shape uncertainty is considered a critical factor since it is mode shapes that determine the locations of plant zeros. They claim that this uncertainty can lead to transfer function phase uncertainties of 180° or greater for dislocated sensor/actuator pairs. For the purposes of their solution to the problem, they assume that the plant poles are spread widely, and thus can be considered as isolated. They assert that the amount of damping required to gain stabilize the system depends on

1. gain roll-off of the loop,
2. spectral separation between the modal natural frequency and the control bandwidth, and
3. modal participation (residue).

They claim that notch compensation used to invert structural dynamics is acceptable provided we assume perfect plant modelling. With uncertainty present, inexact matchups between poles and compensator zeros can create instability when low levels of damping are present. The authors develop a formula to determine how much passive damping (ζ) is sufficient when uncertainty in plant frequency ($\delta\omega$) is much less than the modal frequency or the modal separation.

They used the same formula to find the minimum permissible damping when a close ($\delta\omega$) pole-zero combination exists in the plant dynamics but the sequence is uncertain. The authors then note that even for carefully identified lightly damped structures , a minimum passive damping level of 1 to 4 percent is needed to permit robust plant inversion in feedback control.

II. Introduction to Damping and Model Reduction

One of great problems with controlling a multi-dimensional system is how best to model the system so that a control design based on this model will fit within performance and economics criteria. One popular method to this end is to determine how much disturbance energy is contributed to each flexible mode of the system and then to eliminate from the model those modes in which the contributions are small. To accomplish this last objective formulas are required which determine just how much energy is contributed to each mode. Current literature in this field assumes light damping and widely-spaced resonances in order to use available closed-form solutions.

Another method of reduction that is popular involves the use of balanced coordinates. This is a state realization in which the magnitude of the effects of control on a given state is the same as the magnitude of the effects of that same state on the system output. When the system is written in these coordinates, elimination is done on the basis of states with the smallest singular values (second-order eigenvalues).

The use of damping to reduce the model size of a system is a wide-open issue. The advantage of producing robustness and controlling vibrations beyond the normal range of active control in a system is clear, but whether a damping method or a state realization exists in which damping favors the elimination of one state over another is not clearly known.

Survey of Pertinent Research on Model Reduction

K. Warwick [31] considers the case of approximating a high order plant in the frequency domain with a lower order transfer fucnction. His approach is based on determining the error between system and model responses, which is found in the form of an error polynomial, the coefficients of which tend to zero as the model response tends to that of the system. A cost function is defined, which is equivalent to a weighted sum of error polynomial coefficients. By minimizing the specified cost function, the best model, in the sense of the definition made, can be found. Different cost function weightings reflect whether one is more interested in approximation in the near time or in the steady state following a change in input.

Much of the open-loop model reduction theory presented in the literature is due to Skelton et al. He introduces the notion of model reduction of a plant by working in modal coordinates and defining a quadratic cost function that is the sum of the energies associated with the flexible motions and the control effort. The modal states retained in his reduced-order model correspond to those that contribute the most to the cost function. In an early paper [32], Skelton and Gregory develop some general formulas for modal costs in terms of system modal parameters. A ranking of the modes according to modal cost allows the design of feedback controllers which control only those modes that are believed most critical according to quadratic cost criteria.

In another paper [33], Skelton defines a modelling error index (ME) which is the ratio of the total costs of the truncated modes to the total costs of all the systems modes. Truncated modes are selected to make this ME as small as possible. Skelton and Hughes [34] develop more specific formulas for the modal costs in terms of modal parameters ω_i and ζ_i when damping is considered light, i.e., $0 < \zeta_i < 1$. A model quality index (MQI) is defined in this paper. It is defined as MQI = 1 - ME. Modes are selected using component costs that maximize the MQI. This open-loop truncation criterion is used to reduce models to a size that is feasible for optimal control calculations in off-line computers. Skelton, Hughes, and Hablani [35] apply this reduction technique to a high-order finite element model of a large platform-type structure. Evident from this work is the importance of not only modal parameters but also control-related weightings and disturbance intensities. Furthermore, they show that different groups of modes should be selected for different control tasks. Noisy actuators are shown to necessitate more control software than benign environmental disturbances, and shape control requires higher order models than does attitude control.

In a new idea for reducing system models, Moore [36] shows how under certain conditions a state realization exists in which a controllability gramian and an observability gramian are equal and diagonal. This realization is referred to as balanced, and the eigenvalues of the balanced gramian are called singular values. Model reduction is accomplished by excluding those states with the smallest singular values. These will represent the states that are least controllable and least observable.

Jonckheare [37] shows that a SISO system of flexible modes will have balanced state coordinates that approach the modal coordinates as the damping ratio approaches zero. Furthermore, he develops simple formulas giving the asymptotic singular values as functions of the modal parameters.

Gregory [38] uses Moore's singular values to pursue Skelton's objective of finding the most significant modes for the design model of a system. He derives expressions for singular values in terms of modal parameters, and disturbance, and observability coefficients. Two different singular values are definable in terms of the characteristics of a single mode. He then assumes light damping which causes the balanced coordinates to asymptotically approach the modal coordinates [37] which then produces, in affect, a lone singular value per mode. These modal singular values are then used to select modes for modelling in the manner of Moore [36]. An application of Moore's modelling index, which is defined the same way as Skelton's, except he uses singular values instead of modal costs, gives a quantitative measure of his light damping approximation. The index is shown to improve as damping approaches zero and natural frequencies become more diverse.

Skelton and Kabamba [39,40] demonstrate with an ill-conditioned example the degree to which it is erroneous to use singular values as the sole criterion for model reduction in the L^2 sense. They find that the contribution of the ith state to the L^2 magnitude of the impulse response is given by $\gamma_i v_i^2$, where γ_i is the Hankel (Moore) singular value and $v_i = \left(b_i^T b_i\right)^{\frac{1}{2}} = \left(c_i^T c_i\right)^{\frac{1}{2}}$, the so-called balanced gains of the system. This means that a state truncated from the system model on the basis of its singular value might still contribute substantially to the output due to its balanced gain.

Gawronski and Williams [41] discuss the conditions under which a model reduction will result in a small reduction error which is defined in the same manner as Skelton. To do this, they make use of modal correlations derived from closed-form expressions for the gramians and show that modal and balanced reductions give very different results for the typical LSS in the case of densely-spaced resonances. Examples show that reductions using balanced coordinates generally give better results than reductions using modal coordinates.

III. Summary and Dissertation Outline

In this dissertation we attempt to accomplish two main objectives. The model of a general damped linear dynamical system is introduced. Our first goal is to find, of all the possible ways to damp the system, the design that minimizes a quadratic cost function of energy and effort. Formulas for the optimal feedback gains will be derived as well as a closed-form expression for the cost function.

This cost function, which depends on the system parameters and the disturbability coefficients of the system, will be minimized with respect to the damping parameter ζ subject to any applicable constraints. The theory will then be applied to a simple structure that we wish to control. Different damping methods will be used to demonstrate how the optimal damping design may be the difference between economic feasibility and infeasibility.

The second main goal of this work is to decide if passive damping can be used to reduce the model dimensionality of a linear dynamical system. To this end, we attempt to derive a closed-form expression for a quadratic cost function when the system is subject to a disturbance and measurements are made using general output. The formula derived will assume ζ = constant for all modes but is otherwise general in magnitude. Natural frequencies may be generally spaced as well, so that we are attempting to improve on the current literature, in which authors generally produce closed-form expressions by first assuming light damping and diverse resonances. We then endeavor to derive whatever conclusions can be made as we apply the result to plant reduction via modal costs.

To achieve these main goals the thesis organization will proceed as follows:

In chapter two, we investigate system self-adjointness and derive the expression for the modal damping ratio in terms of the modal frequency for a linear, self-adjoint dynamical system. The traditional formula for Rayleigh damping will be shown to be a special case of this formula. A criterion will be established and demonstrated for measuring relative self-adjointness using least squares estimations in conjunction with a derived formula for system damping.

In chapter three, we attempt to do the following things:

A) Derive expressions for optimal feedback gains for a damped linear system using <u>open-loop</u> energy and effort

B) Derive expressions for optimal feedback gains for a damped linear system using <u>closed-loop</u> energy and effort

C) Derive the expressions for the cost functions associated with a damped linear system assuming closed-loop and open-loop energy and effort

D) Show that the cost functions for open-loop and closed-loop energies merge when the modal frequencies are much greater than the effort weighting coefficient

E) Demonstrate that a passive damping design based on minimizing the closed-loop cost function is insensitive to truncation of the higher modes for reasonable values of the effort weighting coefficient

In chapter four, we apply the cost function for the closed-loop energy plus effort case to the design of an optimal damping treatment for a Bernoulli-Euler beam. Graphs of the cost function will indicate the superiority of the optimal damping design over other damping designs.

In chapter five, we will demonstrate the following points:

A) Show that the number of modes needed to model a lightly viscously-damped linear system is independent of the constant damping ratio

B) Derive a closed-form solution to determine the number of modes needed to accurately model a system in which modal damping is proportional to modal frequency

C) Derive the formula for the cost function associated with a damped, linear system in which the damping ratio is constant (same for each mode) but general in magnitude

D) Use the formula from (C) to derive an expression for a commonly used model reduction index

E) Use the formula in (D) to investigate circumstances under which the index changes as damping ratio increases

In chapter six, we present a summary of the results and accomplishments in this dissertation.

IV. Contributions

A) Derivation of the general expression for the modal damping ratio as a function of modal frequency for a damped linear dynamical system possessing classical normal modes

B) Definition of a criterion for measuring degree of self-adjointness using least squares estimation on a newly derived formula for system damping

C) Determination of the formulas for the optimal feedback gains and the closed-loop quadratic cost function for a damped linear dynamical system

D) Implementation of the closed-form cost function from (C) to optimize the damping design of an ordinary continuous structure subject to a disturbance

E) Demonstration that light constant damping cannot reduce the model dimensionality of a system, but progressive damping can

F) Derivation of the cost function of a damped linear dynamical system when the damping is constant but general in magnitude and natural frequencies are generally distributed

G) Demonstration that the model reduction index can decrease under certain circumstances as damping is increased

CHAPTER TWO

DAMPING AND SELF-ADJOINTNESS

In this chapter we will analyze the concepts of modal coordinates and system self-adjointness and see how the latter is a necessary condition for the former. Caughey's famous generalization of Rayleigh's formula for proportional damping will be taken one step further. It will be shown that when viscous damping is present in a system, independent modal coordinates will exist if and only if the damping matrix obeys a precise relationship to the stiffness and mass matrices. A new formula will then be derived relating modal damping ratios to modal frequencies. Rayleigh's famous formulas for proportional damping will be shown to be special cases of these more general relationships. The implications of this derivation will spread over into the succeeding chapters of this text in which modal coordinates are assumed for purposes of analysis. With these new generalizations, it is somewhat simpler for an arbitrary linear system to possess classical normal modes than it would be under the Rayleigh criterion. Furthermore, by using least squares analysis, it will be shown that these new relationships can be used to determine the relative degree to which an arbitrary linear dynamical system with viscous damping will approximate a linear system possessing uncoupled modal coordinates.

It is a well-known fact that an undamped system possesses classical normal modes- motions in which the various parts of the system vibrate in the same phase. When damping is present, however, this property of self-adjointness is generally violated, and classical normal modes do not exist.

To understand this concept, we refer to the equations for continuous structures from which our matrix equations are the discrete representations. Assume we have an undamped flexible structure whose equation of motion is given by

$$w_{tt}(x,t) + Lw(x,t) = 0 \tag{2.1}$$

where

$w(x,t)$ = the deflection at some position x on the structure at time t, and

L = linear stiffness operator of spatial variables only

It is well known that structural stiffness operators have property of self-adjointness. Consider two functions $u(x)$ and $v(x)$ in the domain of operator L. These are two possible functions on which the operator is defined to act. The operator L is self-adjoint if

$$\langle Lu, v\rangle = \langle u, Lv\rangle \tag{2.2}$$

where the notation $\langle .,. \rangle$ indicates inner product. For such an operator, the solution $w(x,t)$ considered as a function of x for a fixed value of t can be written as

$$w(x,t) = \sum_{i=1}^{\infty} \eta_i(t)\phi_i(x) \qquad (2.3)$$

where $\phi_i(x)$ is the ith eigenfunction of L.

The Fourier coefficient $\eta_i(t)$ is given by

$$\eta_i(t) = \int_D w(x,t)\phi_i(x)\,dD \qquad (2.4)$$

where we have integrated over the domain D of the structure. Multiplying equation (2.1) by $\phi_i(x)$ and integrating over D yields

$$\int_D w_{tt}(x,t)\phi_i(x)\,dD + \int_D Lw(x,t)\phi_i(x)\,dD = 0 \qquad (2.5)$$

which means that

$$\frac{\partial^2}{\partial t^2}\langle w,\phi_i\rangle + \langle Lw,\phi_i\rangle = 0 \qquad (2.6)$$

but

$$\langle Lw,\phi_i\rangle = \langle w, L\phi_i\rangle \qquad (2.7)$$

because L is self-adjoint, and

$$L\phi_i = \lambda_i\phi_i \qquad (2.8)$$

because ϕ_i is an eigenfunction of L. Substituting equations (2.7) and (2.8) into equation (2.6), we get

$$\frac{\partial^2}{\partial t^2}\langle w, \phi_i\rangle + \lambda_i\langle w, \phi_i\rangle = 0$$

and from equation (2.4) this means that

$$\ddot{\eta}_i(t) + \lambda_i\eta_i(t) = 0 \qquad (2.9)$$

Equation (2.9) is the equation of motion of this *i*th mode in modal coordinates for this undamped structure.

When damping is introduced to the system, the system equations may or may not separate into uncoupled modal equations. Consider the equation of a viscously damped structure

$$w_{tt}(\boldsymbol{x}, t) + L_1 w_t(\boldsymbol{x}, t) + L_2 w(\boldsymbol{x}, t) = 0$$

Clearly , this system will admit normal modes only if L_1 and L_2 are self-adjoint with the same eigenfunctions. When this is the case, modal equations are of the form

$$\ddot{\eta}_i(t) + \lambda_n^{(1)}\dot{\eta}_i(t) + \lambda_n^{(2)}\eta_i(t) = 0 \qquad i=1,2,\ldots n$$

where $\lambda_n^{(1)}$ and $\lambda_n^{(2)}$ are the respective eigenvalues of L_1 and L_2.

For a vibrating system with damping matrix Z and eigenfrequency matrix $\boldsymbol{\Omega}$, the modal equations of the discretized model are

$$\ddot{\boldsymbol{\eta}} + 2Z\boldsymbol{\Omega}\dot{\boldsymbol{\eta}} + \boldsymbol{\Omega}^2\boldsymbol{\eta} = 0$$

Rayleigh [1] demonstrated some sufficient conditions for self-adjointness. He showed that a damped system would uncouple provided that the damping matrix is a linear combination of the symmetric inertia and stiffness matrices. Consider the dynamical system

$$M\ddot{\boldsymbol{x}} + D\dot{\boldsymbol{x}} + K\boldsymbol{x} = 0$$

There is always [42] a non-singular transformation T that simultaneously produces

$$T^{-1}MT = I \quad \text{and} \quad T^{-1}KT = \boldsymbol{\Omega}^2 \qquad \textbf{(2.10)}$$

where

I is an n x n identity matrix, and

$\boldsymbol{\Omega}$ is an n x n diagonal matrix of frequencies.

Let $\boldsymbol{x} = T\boldsymbol{\eta}$, so that $MT\ddot{\boldsymbol{\eta}} + DT\dot{\boldsymbol{\eta}} + KT\boldsymbol{\eta} = 0$.

Operate on the left with T^{-1} and get

$$(T^{-1}MT)\ddot{\boldsymbol{\eta}} + (T^{-1}DT)\dot{\boldsymbol{\eta}} + (T^{-1}KT)\boldsymbol{\eta} = 0$$

or

$$\ddot{\boldsymbol{\eta}} + (T^{-1}DT)\dot{\boldsymbol{\eta}} + \boldsymbol{\Omega}^2\boldsymbol{\eta} = 0$$

This system will uncouple if and only if $T^{-1}DT$ is diagonal. Rayleigh considered a damping matrix of the form

$$D = \boldsymbol{\alpha}_0 M + \boldsymbol{\alpha}_1 K \qquad \textbf{(2.11)}$$

where $\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1$ are arbitrary constants. In this case

$$T^{-1}DT = T^{-1}(\alpha_0 M + \alpha_1 K)T$$

$$= \alpha_0 (T^{-1}MT) + \alpha_1 (T^{-1}KT)$$

$$= \alpha_0 I + \alpha_1 \Omega^2$$

which is a diagonal matrix.

In 1960 Caughey [2] produced more general sufficient conditions than these, and in 1965 he [3] established the necessary and sufficient conditions for a system to uncouple into modal equations. He started with the following system:

$$\ddot{\boldsymbol{q}} + A\dot{\boldsymbol{q}} + B\boldsymbol{q} = 0$$

He proved that the necessary and sufficient conditions for this system to possess uncoupled modes is that

$$A = \sum_{i=0}^{n-1} \alpha_i B^i \qquad (2.12)$$

In order to utilize this theorem for the present case, we start with

$$M\ddot{\boldsymbol{x}} + D\dot{\boldsymbol{x}} + K\boldsymbol{x} = 0 \qquad (2.13)$$

Now let $\boldsymbol{x} = M^{-\frac{1}{2}}\boldsymbol{y}$ so that

$$M(M^{-\frac{1}{2}}\ddot{\boldsymbol{y}}) + D(M^{-\frac{1}{2}}\dot{\boldsymbol{y}}) + K(M^{-\frac{1}{2}}\boldsymbol{y}) = 0$$

Multiply on the left by $M^{-\frac{1}{2}}$.

$$M^{-\frac{1}{2}}M^{\frac{1}{2}}\ddot{y} + M^{-\frac{1}{2}}DM^{-\frac{1}{2}}\dot{y} + M^{-\frac{1}{2}}KM^{-\frac{1}{2}}y = 0$$

or

$$\ddot{y} + (M^{-\frac{1}{2}}DM^{-\frac{1}{2}})\dot{y} + (M^{-\frac{1}{2}}KM^{-\frac{1}{2}})y = 0$$

Now apply Caughey's theorem.

$$M^{-\frac{1}{2}}DM^{-\frac{1}{2}} = \sum_{i=0}^{n-1}\alpha_i(M^{-\frac{1}{2}}KM^{-\frac{1}{2}})^i$$

$$= \alpha_0 I + \alpha_1(M^{-\frac{1}{2}}KM^{-\frac{1}{2}})$$

$$+ \alpha_2(M^{-\frac{1}{2}}KM^{-\frac{1}{2}})(M^{-\frac{1}{2}}KM^{-\frac{1}{2}}) + \ldots + \alpha_{n-1}(M^{-\frac{1}{2}}KM^{-\frac{1}{2}})^{n-1}$$

The $(j+1)$th term of this sum is

$$= \alpha_j(m^{-\frac{1}{2}}KM^{-\frac{1}{2}})(M^{-\frac{1}{2}}KM^{-\frac{1}{2}}) \ldots (M^{-\frac{1}{2}}KM^{-\frac{1}{2}})$$

$$= \alpha_j M^{-\frac{1}{2}}(KM^{-1})^{j-1}KM^{-\frac{1}{2}}$$

We see that

$$D_j = M^{\frac{1}{2}}[\alpha_j(M^{-\frac{1}{2}}(KM^{-1})^{j-1}KM^{-\frac{1}{2}}]M^{\frac{1}{2}}$$
$$= \alpha_j(KM^{-1})^{j-1}K$$

and therefore

$$D = \alpha_0 M + \alpha_1 K + \alpha_2(KM^{-1}K) + \alpha_3(KM^{-1})^2K + \ldots + \alpha_{n-1}(KM^{-1})^{n-1}K \qquad (2.14)$$

Equation (2.14) represents <u>the most general form of the viscous damping matrix for a system possessing uncoupled</u>

normal modes. Now apply the tranformation T that simultaneously diagonalizes M and K such that $T^{-1}MT = I$ and $T^{-1}KT = \Omega$, so that $K = T\Omega^2 T^{-1}$.

$$T^{-1}DT = \alpha_0 T^{-1}MT + \alpha_1 T^{-1}KT + \alpha_2 T^{-1}(KM^{-1}K)T$$
$$+\alpha_3 T^{-1}(KM^{-1})^2 KT + \ldots + \alpha_{n-1} T^{-1}(KM^{-1})^{n-1}KT$$

The $(j+1)$th term of this sum is

$$= \alpha_j T^{-1}(KM^{-1}KM^{-1}\ldots M^{-1}K)T$$

$$= \alpha_j T^{-1}(T\Omega^2 T^{-1})M^{-1}(T\Omega^2 T^{-1})M^{-1}\ldots M^{-1}(T\Omega^2 T^{-1})T$$

$$= \alpha_j (T^{-1}T)\Omega^2 (T^{-1}M^{-1}T)\Omega^2 (T^{-1}M^{-1}T)\ldots(T^{-1}M^{-1}T)\Omega^2 (T^{-1}T)$$

but

$$T^{-1}M^{-1}T = (T^{-1}MT)^{-1} = I^{-1} = I$$

therefore the $(j+1)$th term reduces to

$$\alpha_j \cdot I \cdot \Omega^2 \cdot I \cdot \Omega^2 \cdot I \ldots I \cdot \Omega^2 \cdot I$$

$$= \alpha_j \Omega^{2j}$$

Therefore

$$T^{-1}DT = \alpha_0 I + \alpha_1 \Omega^2 + \alpha_2 \Omega^n + \ldots + \alpha_{n-1}\Omega^{2n-2}$$

which is diagonal.

Define $T^{-1}DT = 2Z\Omega$, where

$$Z = diag(\zeta_1, \zeta_2, \ldots \zeta_n) \text{ and } \Omega = diag(\omega_1, \omega_2, \ldots \omega_n)$$

so that

$$Z = \frac{1}{2}(\alpha_0 \Omega^{-1} + \alpha_1 \Omega^1 + \alpha_2 \Omega^3 + \dots + \alpha_{n-1} \Omega^{2n-3}) \qquad \textbf{(2.15)}$$

from which we get

$$\zeta_i = \frac{1}{2}(\alpha_0 \omega_i^{-1} + \alpha_1 \omega_i^1 + \alpha_2 \omega_i^3 + \dots + \alpha_{n-1} \omega^{2n-3}) \qquad \textbf{(2.16)}$$

Equation (2.16) is an expression for the ith modal damping ratio of a self-adjoint system in terms of the ith modal frequency. When a linear system has a damping matrix satisfying equation (2.14), it's modal damping ratios are related to it's modal frequencies by equation (2.16).

Rayleigh Damping

We get the type of damping considered by Lord Rayleigh by choosing

$$\alpha_2 = \alpha_3 = \alpha_4 \dots = \alpha_{n-1} = 0$$

so that

$$D = \alpha_0 M + \alpha_1 K \qquad \textbf{(2.17)}$$

and

$$\zeta_i = \frac{1}{2}[\alpha_0 \omega_i^{-1} + \alpha_1 \omega_i]$$

Thus Rayleigh damping is now seen as a special case of the more general damping for normal modes as prescribed by Caughey.

Example Problem

An example of a damped system is presented by Meirovitch [43]. He demonstrates how the system damping obeys Caughey's sufficiency theorem [2]. We will show that the damping obeys the results of our new derivations.

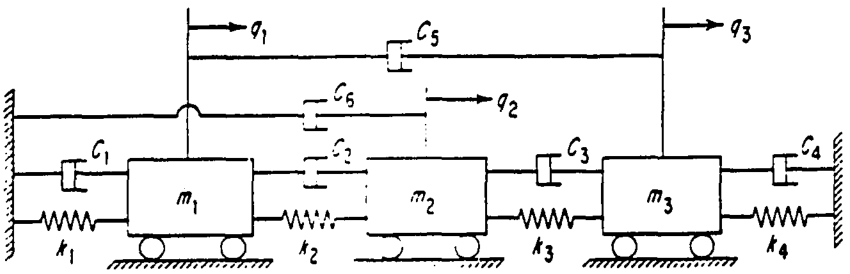


Figure 2.1 - Dynamical System Possessing Normal Modes

Choose:

$m_1 = m_2 = m$ and $m_3 = 2m$

$k_1 = k_2 = k_3 = k$ and $k_4 = 2k$

$d_1 = 2.9316d,\quad d_2 = 0.3747d,\quad d_3 = 0.4079d$

$d_4 = 5.9128d,\quad d_5 = 0.0581d,\quad d_6 = 2.5511d$

where m, k, and d are constants. The $M, K,$ and D matrices are

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$K = \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3+k_4 \end{bmatrix} = k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

$$D = \begin{bmatrix} d_1+d_2+d_5 & -d_2 & -d_5 \\ -d_2 & d_2+d_3+d_6 & -d_3 \\ -d_5 & -d_3 & d_3+d_4+d_5 \end{bmatrix} = d \begin{bmatrix} 3.3644 & -0.3747 & -0.0581 \\ -0.3747 & 3.3337 & -0.4079 \\ -0.0581 & -0.4079 & 6.3788 \end{bmatrix}$$

The eigenvalue problem associated with the undamped system is written

$$KT = TM\lambda, \text{ where } \lambda = \Omega^2 .$$

This transformation produces the following system equations:

$$\ddot{\eta} + 2Z\Omega\dot{\eta} + \Omega^2\eta = 0$$

where

$$\Omega = \sqrt{\frac{k}{m}} \begin{bmatrix} 0.8120 & 0 & 0 \\ 0 & 1.2957 & 0 \\ 0 & 0 & 1.7781 \end{bmatrix}$$

To discover why in fact normal modes result in this case, consider the third-order expansion of equation (2.13):

$$Z = \frac{d}{m} \begin{bmatrix} 1.7321 & 0 & 0 \\ 0 & 1.2718 & 0 \\ 0 & 0 & 1.0632 \end{bmatrix}$$

$$\alpha_0 M + \alpha_1 K + \alpha_2 (KM^{-1}K)$$

with

$$\alpha = \begin{pmatrix} 2.4387\dfrac{d}{m} \\ 0.6077\dfrac{d}{k} \\ -0.0580\dfrac{dm}{k^2} \end{pmatrix}$$

In this case

$$\alpha_0 M + \alpha_1 K + \alpha_2 (KM^{-1}K) = d\begin{pmatrix} 3.3641 & -0.3757 & -0.0580 \\ -0.3757 & 3.3351 & -0.4047 \\ -0.0580 & -0.4047 & 6.3815 \end{pmatrix}$$

which compares very closely (within roundoff error) to the damping matrix D. We note that the damping is not of the Rayleigh type (2.17). At first glance, it might appear that the expansion above might always give the damping matrix for correctly chosen values of α_i. This is not correct. The damping matrix has $(n^2 + n)/2$ independent elements, whereas we are only free to chose n values of α_i. This points out the simple fact that only those systems whose damping matrix satisfies the expansion formula (2.14) for n values of α_i will possess classical normal modes.

Degree of Self-Adjointness

Generally speaking, a damped structure will not be perfectly self-adjoint. For this reason, equation (2.14) will not be satisfied exactly by matrices D, K, and M. However, it may be that a structure will almost satisfy this equation. To measure this degree of self-adjointness (or decoupleability), we consider the equation (2.14) to be a system of $(n^2 + n)/2$ equations in n variables. To see how this is possible, we write the damping matrix as

$$D = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & & & \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{bmatrix}$$

Because the damping matrix is symmetric, we define the following vector $\boldsymbol{d}$ based on its lower triangular elements

$$\boldsymbol{d} = [d_{11}\ d_{21}\ \ldots\ d_{n1}\ d_{22}\ d_{32}\ \ldots\ d_{n2}\ d_{33}\ \ldots\ d_{n3} \ldots\ d_{nn}]^T$$

Consider the right hand side of equation (2.14). M and K are symmetric matrices. While it is true that the product of two symmetic matrices is not necessarily symmetric, when a symmetric matrix is pre and post multiplied by a matrix and its transpose, respectively, the product of the three matrices is symmetric [42]. The matrix product in the third term in equation (2.14) is $KM^{-1}K$ which must therefore be symmetric. The matrix product in the fourth term is $K(M^{-1}KM^{-1})K$, which is a symmetric matrix pre and post multiplied by a matrix and its transpose, respectively, and is therefore symmetric. Clearly, this argument can be extended to prove that each matrix product on the right in equation (2.14) is symmetric. For this reason, let the ith term of that expression be given by

$$(KM^{-1})^{i-1}K = S_i = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ s_{21} & s_{22} & \cdots & s_{2n} \\ \vdots & & & \\ s_{n1} & s_{n2} & \cdots & s_{nn} \end{bmatrix}$$

Since the matrix S_i is symmetric, we define a vector $\boldsymbol{s}_i$

corresponding to S_l as

$$\mathbf{s}_l = [s_{11}\ s_{21}\ \dots\ s_{n1}\ s_{22}\ s_{32}\ \dots\ s_{n2}\ s_{33}\ \dots\ s_{n3}\ \dots\ s_{nn}]^T$$

This is now a classic situation for using multiple regression (m.s. estimation) and correlation theory. For this purpose, we define the matrix of column vectors

$$X = [\mathbf{s}_0\ \mathbf{s}_1\ \dots\ \mathbf{s}_{n-1}]$$

Equation (2.14) may be cast in the form

$$\mathbf{d} = X\boldsymbol{\alpha} \tag{2.18}$$

where

$\boldsymbol{\alpha} \in \Re^n$, is a vector of unknown constants.

$\mathbf{d} \in \Re^{\frac{n^2+n}{2}}$, is a vector of known values.

$X \in \Re^{\frac{n^2+n}{2}}$, is a matrix of known values.

We use least squares estimation to determine values of $\boldsymbol{\alpha}$. Of course, if the system is self-adjoint, we need only to consider the first n rows of $\mathbf{d}$ and X, and we get $\boldsymbol{\alpha}$ by matrix inversion. In the present case, we seek to minimize

$$(\mathbf{d} - X\boldsymbol{\alpha})^T(\mathbf{d} - X\boldsymbol{\alpha})$$

$$= \mathbf{d}^T\mathbf{d} - 2\mathbf{d}^T(X\boldsymbol{\alpha}) + \boldsymbol{\alpha}^T X^T X\boldsymbol{\alpha}$$

Take the partial derivative with respect to $\boldsymbol{\alpha}$, and set the resulting expression equal to zero for a minimum.

$$0 - 2X^T\boldsymbol{d} + 2X^TX\boldsymbol{\alpha} = 0$$

from which we get

$$\hat{\boldsymbol{\alpha}} = (X^TX)^{-1}X^T\boldsymbol{d} \qquad (2.19)$$

where $\hat{\boldsymbol{\alpha}}$ is the least squares estimate of $\boldsymbol{\alpha}$. The corresponding $\boldsymbol{d}$ for this estimate of $\boldsymbol{\alpha}$ is given by

$$\hat{\boldsymbol{d}} = X\hat{\boldsymbol{\alpha}} \qquad (2.20)$$

To determine how close our system is to being self-adjoint, we find the correlation between $\boldsymbol{d}$ and $\hat{\boldsymbol{d}}$. From the theory of estimation, the correlation is given by

$$r = \frac{\boldsymbol{d}^T\hat{\boldsymbol{d}}}{\sqrt{(\boldsymbol{d}^T\boldsymbol{d})\,(\hat{\boldsymbol{d}}^T\hat{\boldsymbol{d}})}} \qquad (2.21)$$

and

$$\theta = \cos^{-1}(r)$$

where θ is the angle between $\boldsymbol{d}$ and $\hat{\boldsymbol{d}}$ in n-space.

The last formula can be used as a relative measure of self-adjointness. Clearly, for a perfectly self-adjoint system,

$$\hat{\boldsymbol{d}}_i = \boldsymbol{d}_i \quad \text{for all } i, \qquad r = 1 \quad \text{and} \quad \theta = 0^0.$$

<u>Example Problem</u>

Consider the following example from Hauser [44].

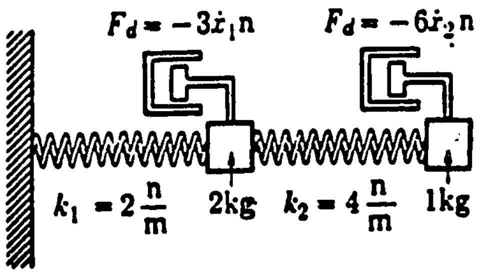


Figure 2.2 - Dynamical System With Coupled Modes

The system equation is

$$\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}\ddot{x}_1\\\ddot{x}_2\end{bmatrix} + \begin{bmatrix}3 & 0\\0 & 6\end{bmatrix}\begin{bmatrix}\dot{x}_1\\\dot{x}_2\end{bmatrix} + \begin{bmatrix}6 & -4\\-4 & 4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = 0$$

From equation (2.14), we have

$$D = \alpha_0 M + \alpha_1 K$$

because the system is 2-dimensional. This gives

$$\begin{bmatrix}3 & 0\\0 & 6\end{bmatrix} = \alpha_0\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix} + \alpha_1\begin{bmatrix}6 & -4\\-4 & 4\end{bmatrix}$$

leading us to the equations

$$3 = 2\alpha_0 + 6x_1$$

$$0 = -4\alpha_1$$

$$6 = \alpha_0 + 4\alpha_1$$

or in matrix form

$$\begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 0 & -4 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}$$

$$\boldsymbol{d} = X \cdot \boldsymbol{\alpha}$$

Using (2.19) and (2.20), we find

$$\hat{\boldsymbol{\alpha}} = (X^T X)^{-1} X^T \boldsymbol{d} = \begin{bmatrix} 1.7143 \\ 0.2143 \end{bmatrix}$$

$$\hat{\boldsymbol{d}} = X \hat{\boldsymbol{\alpha}} = \begin{bmatrix} 4.7143 \\ -0.8571 \\ 2.5714 \end{bmatrix}$$

Now we apply (2.21) to get

$$r = \frac{\boldsymbol{d}^T \hat{\boldsymbol{d}}}{\sqrt{(\boldsymbol{d}^T \boldsymbol{d})\,(\hat{\boldsymbol{d}}^T \hat{\boldsymbol{d}})}} = 0.8106$$

and

$$\theta = \cos^{-1} r = 35.8^0$$

which means that the system is strongly non-self-adjoint. To understand why, we determine the matrices of eigenvectors and eigenvalues:

$$T = \begin{bmatrix} 0.7662 & -0.6426 \\ 0.6426 & 0.7662 \end{bmatrix}$$

$$\Omega = \begin{bmatrix} 0.7923 & 0 \\ 0 & 2.5423 \end{bmatrix}$$

which lead to the damping matrices

$$T^{-1}DT = \begin{bmatrix} 3.3583 & 2.2156 \\ 2.2156 & 4.1417 \end{bmatrix}$$

and

$$Z = \frac{1}{2}(T^{-1}DT)\,\Omega^{-1} = \begin{bmatrix} 2.1194 & 0.4389 \\ 1.3983 & 0.8203 \end{bmatrix}$$

each showing the presence of large coupling terms off the diagonal.

Concluding Remarks

In this chapter we have derived the most general form for the damping matrix (eqns. 2.14 and 2.16) in a linear dynamical system that possesses classical normal modes. We now see that there will many more systems possessing uncoupled modal coordinates than those whose damping matrix just happens to satisfy the traditional Rayleigh criteria. Furthermore, a measure of the degree to which a general linear system fails to be self-adjoint has been derived (eqn. 2.21) using least squares estimation in system coordinates. These results, coupled with the research of Meirovitch [26] demonstrating that systems with small damping terms off the main diagonal may be adequately controlled as if they were self-adjoint, have made it practical to use modal coordinates in the analysis and control of viscously damped linear systems and without the fear of a great loss in generality.

CHAPTER THREE

OPTIMAL ACTIVE/PASSIVE CONTROL DESIGN

It has been noted that it is now widely accepted that some form of passive damping is necessary to control the flexible motions of large space structures. The intentional addition of designed-in passive damping can enhance the performance of active controllers while easing the burden of active control and providing a crucially needed margin of stability. In the interest of achieving the desired performance, while maintaining reasonable dollar costs, weight, and reliability, it is desirable to design control measures, active and/or passive, as integral components of the overall system, rather than as a afterthought. The problem of designing a system in which active and passive control measures interact in an optimal fashion has received little attention to date. In this chapter a design methodology for optimal active and passive control of a flexible structure is developed. For this purpose, Independent Modal Space Control (IMSC) will be employed as the active control measure. IMSC was selected because of its simplicity and the intuitive feel that it lends to the active control design problem.

In initial experiments with the JPL precision truss, Fanson et. al. [45] found that simple controllers successfully attenuated lower frequency modes but inadvertently destabilized higher modes. Viscoelastic passive damping treatments can be designed to perform well in a prescribed frequency range and are particularly well suited for the control of higher modes of vibration. Moreover, active control of higher modes is seldom required for purposes of, for example, shape control. Rather, enhanced damping of the higher modes is the desired property. Active control measures are normally viewed as being well suited only for lower frequency modes due to actuator bandwidth and computational speed limitations. Considered together, these facts suggest that a synergistic approach employing active control for lower frequency modes and passive control for higher frequency modes deserves serious consideration.

Optimal Active/Passive Design Methodology

Consider a linear or linearized system represented in the following standard second-order form:

$$M\ddot{\mathbf{x}} + D\dot{\mathbf{x}} + K\mathbf{x} = \mathbf{F} \tag{3.1}$$

where

$\mathbf{F}$ = vector of control input forces

M = the system mass/inertia matrix

D = a damping matrix

K = a stiffness matrix

In general, this system represents a set of n coupled differential equations, where n is the number of vibration modes included in the model. Assuming D satisfies the restrictions from equation (2.14), such a system will be self-adjoint. The self-adjointness property implies that equation (3.1) can be transformed into decoupled independent modal coordinates $\boldsymbol{\eta}$ such that

$$\ddot{\boldsymbol{\eta}} + 2Z\boldsymbol{\Omega}\dot{\boldsymbol{\eta}} + \boldsymbol{\Omega}^2\boldsymbol{\eta} = \boldsymbol{f} \tag{3.2}$$

where

$\boldsymbol{\Omega}$ = a diagonal matrix of modal frequencies ω_i

Z = a diagonal matrix of modal damping ratios ζ_i

In this case each of the modes can be controlled independently using the modal control law:

$$\boldsymbol{f} = -G\boldsymbol{\eta} - H\dot{\boldsymbol{\eta}} \tag{3.3}$$

When each element f_i of the modal control vector is designed to depend only upon the corresponding ith modal displacement and velocity,

$$f_i = f_i(\eta_i, \dot{\eta}_i) \qquad i = 1,2,3,\ldots n$$

where n = represents the number of actively controlled modes, then the closed-loop system remains decoupled, and the control is referred to as Independent Modal Space Control (IMSC). The IMSC method has several desirable characteristics [5,12,25,26] including simplicity in design and implementation and relatively good robustness properties. IMSC is well suited for this initial investigation because it leads to a simple formulation for the determination of optimal control gains and consequently facilitates the process of determining an optimal modal damping distribution.

For design of an optimal passive/active control system, we define the performance index:

$$J = \int_0^\infty \left[E(t) + \frac{1}{4\alpha^2} \boldsymbol{F}^T M^{-1} \boldsymbol{F} \right] dt$$

where

$$E(t) = \frac{1}{2}\dot{\boldsymbol{x}}^T M \dot{\boldsymbol{x}} + \frac{1}{2}\boldsymbol{x}^T K \boldsymbol{x}$$

$E(t)$ = total mechanical energy

M = the system mass/inertia matrix

$\boldsymbol{F}$ = the control input vector

α = the control effort penalty factor

If we transform to the modal coordinates, we have for $\boldsymbol{x} = T\boldsymbol{\eta}$

$$E(t) = \frac{1}{2}\dot{\boldsymbol{x}}^T M \dot{\boldsymbol{x}} + \frac{1}{2}\boldsymbol{x}^T K \boldsymbol{x}$$

$$= \frac{1}{2}\dot{\boldsymbol{\eta}}^T (T^T M T) \dot{\boldsymbol{\eta}} + \frac{1}{2}\boldsymbol{\eta}^T (T^T K T) \boldsymbol{\eta}$$

$$= \frac{1}{2}\dot{\boldsymbol{\eta}}^T \dot{\boldsymbol{\eta}} + \frac{1}{2}\boldsymbol{\eta}^T \Omega^2 \boldsymbol{\eta}$$

where we have used properties of equations (2.10)

Referring to equation (3.1), and recall the process for converting this equation to modal coordinates, it therefore follows that the converted control force is

$$\boldsymbol{f} = T^{-1}M^{-\frac{1}{2}}\boldsymbol{F}, \quad \text{so that} \quad \boldsymbol{F} = M^{\frac{1}{2}}T\boldsymbol{f}.$$

$$\frac{1}{4\alpha^2}(\boldsymbol{F}^T M^{-1}\boldsymbol{F}) = \frac{1}{4\alpha^2}\left[\boldsymbol{f}^T T^T (M^{\frac{1}{2}})^T \cdot M^{-1} \cdot M^{\frac{1}{2}} T\boldsymbol{f}\right]$$

$$= \frac{1}{4\alpha^2}\left[\boldsymbol{f}^T T^T (M^{\frac{1}{2}} M^{-1} M^{\frac{1}{2}}) T\boldsymbol{f}\right]$$

$$= \frac{1}{4\alpha^2}[\boldsymbol{f}^T T^T T\boldsymbol{f}]$$

$$= \frac{1}{4\alpha^2}\boldsymbol{f}^T\boldsymbol{f}$$

since M is symmetric, and T is orthonormal. The transformed performance index is therefore

$$J = \int_0^\infty \left[\dot{\boldsymbol{\eta}}^T\dot{\boldsymbol{\eta}} + \boldsymbol{\eta}^T\Omega^2\boldsymbol{\eta} + \frac{1}{2}\alpha^2\boldsymbol{f}^T\boldsymbol{f}\right]dt$$

Now let $\boldsymbol{f} = G\boldsymbol{\eta} - H\dot{\boldsymbol{\eta}}$, and we get

$$\boldsymbol{f}^T\boldsymbol{f} = (-\boldsymbol{\eta}^T G^T - \dot{\boldsymbol{\eta}}^T H^T)(-G\boldsymbol{\eta} - H\dot{\boldsymbol{\eta}})$$

$$= \boldsymbol{\eta}^T G^2\boldsymbol{\eta} + \boldsymbol{\eta}^T G^T H\dot{\boldsymbol{\eta}} + \dot{\boldsymbol{\eta}}^T H^T G\boldsymbol{\eta} + \dot{\boldsymbol{\eta}}^T H^2\dot{\boldsymbol{\eta}}$$

Since G and H are diagonal in IMSC, the mixed middle terms involve the following integral:

$$\int_0^\infty \dot{\eta}_i\eta_i\,dt = \frac{1}{2}\int_0^\infty \frac{d}{dt}(\eta_i^2)\,dt$$

$$= \frac{1}{2}\eta_i^2\Big|_0^\infty = 0$$

since we assume that the system is stable and initially unexcited. This means that

$$f^T f = \sum_{i=1}^{n} (g_i^2 \eta_i^2 + h_i^2 \dot{\eta}_i^2)$$

and

$$J = \int_0^\infty \left[\sum_{i=1}^{n} (\eta_i^2 + \omega_i^2 \dot{\eta}_i^2) + \frac{1}{2\alpha^2} \sum_{i=1}^{n} (g_i^2 \eta_i^2 + h_i^2 \dot{\eta}_i^2) \right] dt$$

$$= \sum_{i=1}^{n} \left[\int_0^\infty \dot{\eta}_i^2 dt + \omega_i^2 \int_0^\infty \eta_i^2 dt \right] + \frac{1}{2\alpha^2} \sum_{i=1}^{n} \left[g_i^2 \int_0^\infty \eta_i^2 dt + h_i^2 \int_0^\infty \dot{\eta}_i^2 dt \right] \qquad \textbf{(3.4)}$$

$$= \sum_{i=1}^{n} \left[\left(\omega_i^2 + \frac{g_i^2}{2\alpha^2} \right) \int_0^\infty \eta_i^2 dt + \left(1 + \frac{h_i^2}{2\alpha^2} \right) \int_0^\infty \dot{\eta}^2 dt \right]$$

The evaluations of the modal state time integrals depend on whether we consider open-loop or closed-loop energy and the type of disturbance input to the system.

Assume we disturb the original system with a general impulse input, i.e.,

$$u_j(t) = \mu_j \delta(t) \qquad j = 1, 2 \ldots m$$

In modal coordinates, this disturbance becomes

$$T^{-1} M^{-\frac{1}{2}} \mathbf{u}$$

and the *i*th component of this disturbance (effect on mode *i*) is given by

$$\boldsymbol{b}_i^{*}\boldsymbol{u} \qquad i = 1, 2, \ldots n$$

where $\boldsymbol{b}_i^{*}$ is the ith row of $T^{-1}M^{-\frac{1}{2}}$.

The open-loop system is:

$$\ddot{\eta}_i + 2\zeta_i\omega_i\dot{\eta}_i + \omega_i^2\eta_i = \boldsymbol{b}_i^{*}\boldsymbol{u}$$

and the closed-loop system becomes:

$$\ddot{\eta}_i + 2\zeta_i\omega_i\dot{\eta}_i + \omega_i^2\eta_i = \boldsymbol{b}_i^{*}\boldsymbol{u} - g_i\eta_i - h_i\dot{\eta}_i \tag{3.5}$$

<u>Evaluation of Integrals</u>

If we assume the modes remain underdamped, which is typical of controlled structures, the $\zeta_i < 1$ for all i, then we have the following derivation:

$$\ddot{\eta}_i + 2\zeta_i\omega_i\dot{\eta}_i + \omega_i^2\eta_i = \left(\sum_{j=1}^{m} b_{ij}\mu_j\right)\zeta(t)$$

Taking Laplace transforms, get

$$(s^2 + 2\zeta_i\omega_i s + \omega_i^2)\,\eta_i(s) = C_i$$

where $$C_i = \left(\sum_{j=1}^{m} b_{ij}\mu_j\right)$$

and zero initial conditions on $\eta_i(t)$ and $\dot{\eta}_i(t)$ have been assumed.

$$\eta_i(s) = \frac{C_i}{s^2 + 2\zeta_i\omega_i s + \omega_i^2}$$

$$= \frac{C_i}{(s + \zeta_i\omega_i)^2 + \omega_i^2(1-\zeta_i^2)}$$

which inverts as

$$\eta_i(t) = \frac{C_i}{\omega_i\sqrt{1-\zeta_i^2}} e^{-\zeta_i\omega_i t} \sin(\omega_i\sqrt{1-\zeta_i^2}\,t)$$

Let $B_i = \frac{C_i}{\omega_i\sqrt{1 - \zeta_i^2}}$ and $\omega_{di} = \omega_i\sqrt{1 - \zeta_i^2}$

then $\eta_i(t) = B_i e^{-\zeta_i\omega_i t}\sin(\omega_{di} t)$

Using tables of integrals, we get

$$\int_0^\infty \eta_i^2 dt = \frac{B_i^2}{4}\left[\frac{1}{\zeta_i\omega_i} - \frac{\zeta_i\omega_i}{\zeta_i^2\omega_i^2 + \omega_{di}^2}\right]$$

$$= \frac{B_i^2}{4\omega_i}\left[\frac{1}{\zeta_i} - \zeta_i\right]$$

$$= \frac{C_i^2}{4\omega_i^3(1-\zeta_i^2)}\cdot\frac{1-\zeta_i^2}{\zeta_i}$$

$$= \frac{C_i^2}{4\omega_i^3\zeta_i}$$

$$\int_0^\infty \dot{\eta}_i^2 dt = \frac{1}{4} B_i^2 \omega_i \left(\frac{1}{\zeta_i} - \zeta_i \right)$$

$$= \frac{1}{4} \left(\frac{C_i^2 \omega_i}{\omega_i^2 (1-\zeta_i^2)} \right) \left(\frac{1-\zeta_i^2}{\zeta_i} \right)$$

$$= \frac{C_i^2}{4\zeta_i \omega_i}$$

$$A_i = C_i^2 = \boldsymbol{b}_i^* U \boldsymbol{b}_i$$

where

$$U_{ij} = \mu_i^2 \text{ for } i = j, \text{ and } U_{ij} = 0 \text{ otherwise.}$$

Therefore, if the system is open-loop

$$\int_0^\infty \eta_i^2 dt = \frac{\boldsymbol{b}_i^* U \boldsymbol{b}_i}{4\omega_i^3 \zeta_i} \qquad (3.6)$$

$$\int_0^\infty \dot{\eta}_i^2 dt = \frac{\boldsymbol{b}_i^* U \boldsymbol{b}_i}{4\zeta_i \omega_i}$$

For the closed-loop system

$$\ddot{\eta}_i + 2\zeta_i \omega_i \dot{\eta}_i + \omega_i^2 \eta_i = \boldsymbol{b}_i^* \boldsymbol{u} - g_i n_i - h_i \dot{\eta}_i$$

so that

$$\ddot{\eta}_i + (2\zeta_i \omega_i + h_i) \dot{\eta}_i + (\omega_i^2 + g_i) \eta_i = \boldsymbol{b}_i^* \boldsymbol{u}$$

Let

$$2\zeta_i\omega_i + h_i = 2\zeta_i'\omega_i'$$

$$\omega_i^2 + g_i = \omega_i'^2$$

Under these circumstances

$$\int_0^\infty \eta_i^2 dt = \frac{\mathbf{b}_i^* U \mathbf{b}_i}{4\zeta'_i\omega'_i} = \frac{\mathbf{b}_i^* U \mathbf{b}_i}{4\zeta_i\omega_i + 2h_i}$$

$$\int_0^\infty \dot{\eta}_i^2 dt = \frac{\mathbf{b}_i^* U \mathbf{b}_i}{4\omega_i'^3\zeta'_i} = \frac{\mathbf{b}_i^* U \mathbf{b}_i}{(4\zeta_i\omega_i + 2h_i)(\omega_i^2 + g_i)} \tag{3.7}$$

Open-loop Energy plus Effort

If we substitute equations (3.7) into equation (3.4), we obtain

$$J = \sum_{i=1}^{n}\left(1 + \frac{h_i^2}{2\alpha^2}\right)\left(\frac{\mathbf{b}_i^* U \mathbf{b}_i}{4\zeta_i\omega_i + 2h_i}\right)$$

$$+ \sum_{i=1}^{n}\left(\omega_i^2 + \frac{g_i^2}{2\alpha^2}\right)\left(\frac{\mathbf{b}_i^* U \mathbf{b}_i}{(4\zeta_i\omega_i + 2h_i)(\omega_i^2 + g_i)}\right)$$

$$= \frac{1}{2}\sum_{i=1}^{n}\left[1 + \frac{h_i}{2\alpha^2} + \frac{\omega_i^2 + g_i^2/2\alpha^2}{\omega_i^2 + g_i}\right]\left[\frac{\mathbf{b}_i^* U \mathbf{b}_i}{2\zeta_i\omega_i + h_i}\right]$$

This is the closed-form expression of the cost function assuming open-loop energy and effort. We may now calculate the optimal feedback gains by differentiating J with respect to g_i and h_i and setting the derivatives equal to zero.

$$\frac{\partial J}{\partial g_i} = \frac{\left(\omega_i^2 + g_i\right) \cdot \left(\frac{2g_i}{2\alpha^2}\right) - \left(\omega_i^2 + g_i^2/2\alpha^2\right) \cdot 1}{\left(\omega_i^2 + g_i\right)^2}$$

$$\cdot \frac{\boldsymbol{b}_i^* U \boldsymbol{b}_i}{4\zeta_i\omega_i + 2h_i} = 0$$

from which we derive

$$(\omega_i^2 + g_i) \cdot \frac{2g_i}{2\alpha^2} - (\omega_i^2 + g_i^2/2\alpha^2) = 0 \qquad \textbf{(3.8)}$$

$$\frac{g_i^2}{2\alpha^2} + \frac{2\omega_i^2 g_i}{2\alpha^2} - \omega_i^2 = 0$$

or

$$g_i^2 + 2\omega_i^2 g_i - 2\alpha^2\omega_i^2 = 0$$

$$g_i^{opt} = \frac{-2\omega_i^2 \pm \sqrt{4\omega_i^4 + 8\alpha^2\omega_i^2}}{2}$$

Since $g_i \geq 0$, we have

$$g_i^{opt} = -\omega_i^2 + \omega_i\sqrt{\omega_i^2 + 2\alpha^2} \qquad \textbf{(3.9)}$$

$$\frac{\partial J}{\partial h_i} = \frac{2h_i}{2\alpha^2} \cdot \frac{\boldsymbol{b}_i^* U \boldsymbol{b}_i}{2\zeta_i\omega + h_i}$$

$$+ \left[1 + \frac{h_i^2}{2\alpha^2} + \frac{\omega_i^2 + g_i^2/2\alpha^2}{\omega_i^2 + g_i}\right] \cdot \frac{-\boldsymbol{b}_i^* U \boldsymbol{b}_i}{(2\zeta_i\omega_i + h_i)^2}$$

so that

$$\frac{2h_i}{2\alpha^2}(2\zeta_i\omega_i + h_i) - \left(1 + \frac{h_i^2}{2\alpha^2} + \frac{\omega_i^2 + g_i^2/2\alpha^2}{\omega_i^2 g_i}\right) = 0 \tag{3.10}$$

If we use equation (3.8) in this calculation, we see that

$$\frac{\omega_i^2 + g_i^2/2\alpha^2}{\omega_i^2 + g_i} = \frac{g_i}{\alpha^2}$$

so that

$$\frac{h_i}{\alpha^2}(2\zeta_i\omega_i + h_i) - \left(1 + \frac{h_i^2}{2\alpha^2} + \frac{g_i}{\alpha^2}\right) = 0 \tag{3.11}$$

$$2h_i(2\zeta_i\omega_i + h_i) - (2\alpha^2 + h_i^2 + 2g_i) = 0$$

which results in

$$h_i^2 + 4\zeta_i\omega_i h_i - (2\alpha^2 + 2g_i) = 0$$

so that

$$h_i^{opt} = \frac{-4\zeta_i\omega_i \pm \sqrt{16\zeta_i^2\omega_i^2 + 8\alpha^2 + 8g_i}}{2}$$

but $h_i \geq 0$, and get g_i from equation (3.9), so that

$$h_i^{opt} = -2\zeta_i\omega_i + \sqrt{4\zeta_i^2\omega_i^2 + 2\alpha^2 - 2\omega_i^2 + 2\omega_i\sqrt{\omega_i^2 + 2\alpha^2}} \tag{3.12}$$

Equations (3.9) and (3.12) provide us with the uncoupled optimal feedback gains assuming open-loop energy and effort.

Meirovitch and Silverberg [12] perform a similar calculation in finding the globally optimal control of an undamped distributed parameter system using IMSC. If we let $\zeta_i = 0$ in equations (3.9) amd (3.12), we get identical results to theirs.

Closed-loop Energy Plus Effort

We now perform the same calculations but using closed-loop energy and effort. In this case we make the substitutions:

$$\omega^2 \rightarrow (\omega')^2 = \omega^2 + g_i$$

so that equation (3.4) becomes

$$J = \sum_{i=1}^{n}\left[\left(\omega_i'^2 + \frac{g_i^2}{2\alpha^2}\right)\int_0^\infty \eta_i^2 dt + \left(1 + \frac{h_i^2}{2\alpha^2}\right)\int_0^\infty \dot{\eta}_i^2 dt\right]$$

$$= \sum_{i=1}^{n}\left[\left(\omega_i^2 + g_i + \frac{g_i^2}{2\alpha^2}\right)\int_0^\infty \eta_i^2 dt + \left(1 + \frac{h_i^2}{2\alpha^2}\right)\int_0^\infty \dot{\eta}_i^2 dt\right]$$

$$= \frac{1}{2}\sum_{i=1}^{n}\left[1 + \frac{h_i^2}{2\alpha^2} + \frac{\omega_i^2 + g_i + g_i^2/2\alpha^2}{\omega_i^2 + g_i}\right]\frac{\boldsymbol{b}_i^* U \boldsymbol{b}_i}{2\zeta_i\omega_i + h_i}$$

$$= \frac{1}{2}\sum_{i=1}^{n}\left[2 + \frac{h_i}{2\alpha^2} + \frac{g_i^2/2\alpha^2}{\omega_i^2 + g_i}\right]\frac{\boldsymbol{b}_i^* U \boldsymbol{b}_i}{2\zeta_i\omega_i + h_i} \qquad \textbf{(3.13)}$$

Now we calculate the feedback gains.

$$\frac{\partial J}{\partial g_i} = \frac{(\omega_i^2 + g_i)\left(\frac{2g_i}{2\alpha^2}\right) - \frac{g_i^2}{2\alpha^2}\cdot 1}{(\omega_i^2 + g_i)^2} \cdot \frac{\boldsymbol{b}_i^* U \boldsymbol{b}_i}{2\zeta_i\omega_i + h_i} = 0$$

which produces

$$(\omega_i^2 + g_i)\left(\frac{2g_i}{2\alpha^2}\right) - \frac{g_i^2}{2\alpha^2} = 0$$

$$g_i^2 + 2\omega_i^2 g_i = 0$$

$$g_i(g_i + 2\omega_i^2) = 0$$

and since $g_i \geq 0$, then

$$g_i^{opt} = 0. \qquad \textbf{(3.14)}$$

This last equation has the interesting implication that the closed-loop system stiffness and natural frequencies are the same as those of the open-loop under optimal control.

$$\frac{\partial J}{\partial h_i} = \frac{2h_i}{2\alpha^2}\cdot\frac{\boldsymbol{b}_i^* U \boldsymbol{b}_i}{2\zeta_i\omega_i + h_i} + \left[2 + \frac{h_i^2}{2\alpha^2} + \frac{g_i^2/2\alpha^2}{\omega_i^2 + g_i}\right]\cdot\frac{-\boldsymbol{b}_i^* U \boldsymbol{b}_i}{(2\zeta_i\omega_i + h_i)^2}$$

$$= 0$$

which gives us

$$2h_i(2\zeta_i\omega_i + h_i) - \left(4\alpha^2 + h_i^2 + \frac{g_i^2}{\omega_i^2 + g_i}\right) = 0 \qquad \textbf{(3.15)}$$

leading to

$$h_i^2 + 4\zeta_i\omega_i h_i - 4\alpha^2 = 0$$

where we have used equation (3.14).

Solving, we get

$$h_i^{opt} = -2\zeta_i\omega_i + 2\sqrt{\zeta_i^2\omega_i^2 + \alpha^2} \tag{3.16}$$

Equations (3.14) and (3.16) give us the optimal feedback gains when cost is composed of closed-loop energy and effort.

<u>Evaluation of J-optimal</u>

First, we consider open-loop energy and effort.

$$J = \frac{1}{2}\sum_{i=1}^{n}\left(1 + \frac{h_i^2}{2\alpha^2} + \frac{\omega_i^2 + g_i^2/2\alpha^2}{\omega_i^2 + g_i}\right)\left(\frac{\mathbf{b}_i^*U\mathbf{b}_i}{2\zeta_i\omega_i + h_i}\right)$$

If we now use equation (3.10), we can get an expression for the optimal cost.

$$J^{opt} = \frac{1}{2}\sum_{i=1}^{n}\frac{h_i}{\alpha^2}(2\zeta_i\omega_i + h_i)\left(\frac{\mathbf{b}_i^*U\mathbf{b}_i}{2\zeta_i\omega_i + h_i}\right)$$

$$= \frac{1}{2\alpha^2}\sum_{i=1}^{n}h_i^{opt}A_i$$

so that

$$J^{opt} = \frac{1}{2\alpha^2}\sum_{i=1}^{n}\left[-2\zeta_i\omega_i + \sqrt{4\zeta_i^2\omega_i^2 + 2\alpha^2 - 2\omega_i^2 + 2\omega_i\sqrt{\omega_i^2 + 2\alpha^2}}\right]A_i$$

(3.17)

which is the optimal cost when cost is composed of open-loop energy and effort. Now perform the same evaluation using closed-loop energy and effort.

$$J^{opt} = \frac{1}{2}\sum_{i=1}^{n}\left[2 + \frac{h_i^2}{2\alpha^2} + \frac{g_i/2\alpha^2}{\omega_i^2 + g_i}\right]\frac{\mathbf{b}_i^* U \mathbf{b}_i}{2\zeta_i\omega_i + h_i}$$

We find J^{opt} using equation (3.15).

$$J^{opt} = \frac{1}{4\alpha^2}\sum_{i=1}^{n} 2h_i(2\zeta_i\omega_i + h_i)\cdot\frac{\mathbf{b}_i^* U \mathbf{b}_i}{2\zeta_i\omega_i + h_i}$$

$$= \frac{1}{2\alpha^2}\sum_{i=1}^{n} h_i^{opt}\cdot A_i$$

so that

$$J^{opt} = \frac{1}{\alpha^2}\sum_{i=1}^{n}\left[-\zeta_i\omega_i + \sqrt{\zeta_i^2\omega_i^2 + \alpha^2}\right]A_i \qquad \textbf{(3.18)}$$

which is the optimal cost when it is composed of closed-loop energy and effort.

Comparison of Cost Functions

A comparison of the cost functions corresponding to open-loop and closed-loop energies amounts to a comparison of $\left(h_i^{opt}\right)_{open}$ versus $\left(h_i^{opt}\right)_{closed}$, that is,

$$-2\zeta_i\omega_i + \sqrt{4\zeta_i^2\omega_i^2 + 2\alpha^2 - 2\omega_i^2 + 2\omega_i\sqrt{\omega_i^2 + 2\alpha^2}}$$

versus

$$-2\zeta_i\omega_i + \sqrt{4\zeta_i^2\omega_i^2 + 4\alpha^2}$$

which reduces to a comparison between

$$2\alpha^2 - 2\omega_i^2 + 2\omega_i\sqrt{\omega_i + 2\alpha^2} \text{ and } 4\alpha^2.$$

This leads to a comparison between

$$-2\omega_i^2 + 2\omega_i\sqrt{\omega_i^2 + 2\alpha^2} \text{ and } 2\alpha^2.$$

It is not hard to show that

$$-2\omega_i^2 + 2\omega_i\sqrt{\omega_i^2 + 2\alpha^2} \quad < \quad 2\alpha^2 \text{ for all } \omega_i.$$

Start with

$$4\omega_i^4 + 8\alpha^2\omega_i^2 < 4\omega_i^4 + 8\alpha^2\omega_i^2 + 4\alpha^4$$

$$4\omega_i^2(\omega_i^2 + 2\alpha^2) < 4(\omega_i^4 + 2\alpha^2\omega_i^2 + \alpha^4)$$

Take the principal square root to each side.

$$2\omega_i\sqrt{\omega_i^2 + 2\alpha^2} < 2(\omega_i^2 + \alpha^2)$$

or

$$-2\omega_i^2 + 2\omega_i\sqrt{\omega_i^2 + 2\alpha^2} \quad < \quad 2\alpha^2 \text{ for all } \omega_i.$$

which means that

$$(J^{opt})_{open} \; < \; (J^{opt})_{closed} \text{ for all } \omega_i.$$

Consider now the effect at high frequencies - frequencies such that $\omega_i \gg \alpha$.

For this case

$$\sqrt{\omega_i^2 + 2\alpha^2} = \omega_i\sqrt{1 + \frac{2\alpha^2}{\omega_i^2}}$$

$$= \omega_i\left(1 + \frac{\alpha^2}{\omega_i}\right)$$

$$= \omega_i + \frac{\alpha^2}{\omega_i}$$

$$\left(h_i^{opt}\right)_{open} \approx -2\zeta_i\omega_i + \sqrt{4\zeta_i^2\omega_i^2 + 2\alpha^2 - 2\omega_i^2 + 2\omega_i^2 + 2\alpha^2}$$

$$= -2\zeta_i\omega_i + \sqrt{4\zeta_i^2\omega_i^2 + 4\alpha^2}$$

$$= \left(h_i^{opt}\right)_{closed}$$

The costs therefore approach each other asymptotically as $\omega_i \to \infty$. We note at $\omega_i = 0$, we have

$$\left(h_i^{opt}\right)_{closed} = \sqrt{4\alpha^2} = 2\alpha,$$

$$\left(h_i^{opt}\right)_{open} = \sqrt{2\alpha^2} = \alpha\sqrt{2}$$

This means that the closed-loop cost function will tend to penalize lower modes to a greater degree than the open-loop function, but otherwise the functions measure about the same quantity. Fig.(3.1) will demonstrate this fact.

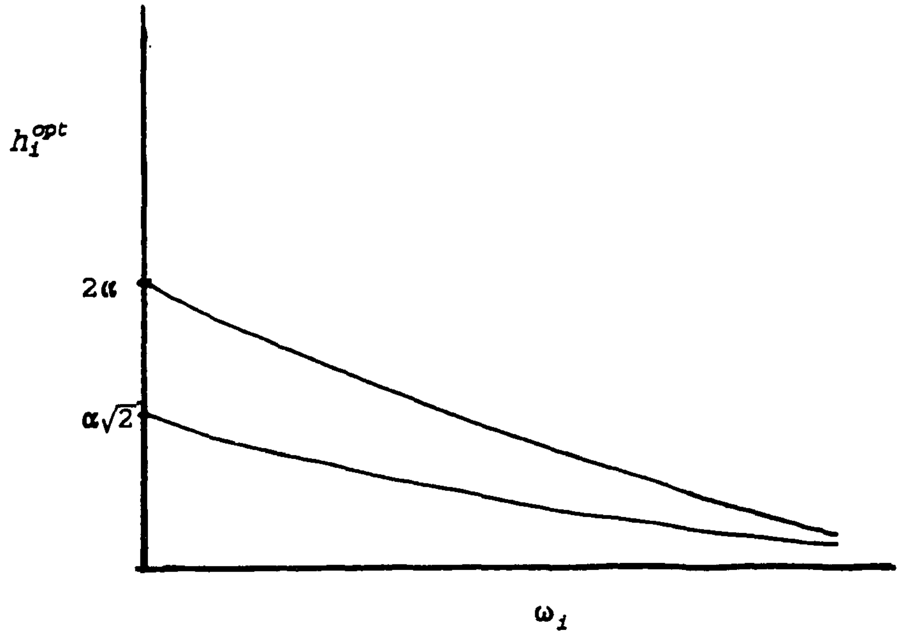


Figure 3.1 - Graph of h_1^{opt} vs. Frequency

The closed-loop cost function was chosen for the experimental example in the next chapter for four reasons:

1. The open-loop and closed-loop functions measure essentially the same quantity.
2. The closed-loop function is more representative of the controlled system, which is closed-loop.
3. The closed-loop system penalizes to a slightly greater extent the lower modes, which is normally where most of energy in a system is concentrated.
4. The closed-loop optimal control does not change the natural frequencies of the original open-loop system.

Pole Shifting Under Optimal Control

For the closed-loop system

$$h_i^{opt} = -2\zeta_i\omega_i + 2\sqrt{\zeta_i^2\omega_i^2 + \alpha^2}$$

The system equation under optimal control is

$$\ddot{\eta}_i + 2\zeta_i\omega_i\dot{\eta}_i + \omega_i^2\eta_i = -g_i^{opt}\eta_i - h_i^{opt}\dot{\eta}_i$$

or

$$\ddot{\eta}_i + (2\zeta_i\omega_i - 2\zeta_i\omega_i + 2\sqrt{\zeta_i^2\omega_i^2 + \alpha^2}) + \omega_i^2\eta_i = 0$$

where we have used equations (3.14) and (3.16). This reduces to

$$\ddot{\eta}_i + \left(2\sqrt{\zeta_i^2\omega_i^2 + \alpha^2}\right)\dot{\eta}_i + \omega_i^2\eta_i = 0 \qquad \textbf{(3.19)}$$

Get the poles of this equation using

$$s^2 + \left(2\sqrt{\zeta_i^2\omega_i^2 + \alpha^2}\right)s + \omega_i^2 = 0$$

$$s = \frac{-2\sqrt{\zeta_i^2\omega_i^2 + \alpha^2} \pm \sqrt{4\zeta_i^2\omega_i^2 + 4\alpha^2 - 4\omega_i^2}}{2}$$

$$s = -\sqrt{\zeta_i^2\omega_i^2 + \alpha^2} \pm j\sqrt{\omega_i^2 - \zeta_{i_2}\omega_i^2 - \alpha^2} \qquad \textbf{(3.20)}$$

where we have assumed as we did earlier that all the system's controlled modes stay underdamped. In equation (3.20) we see that the damping has increased, and the damped frequency has decreased. Neither of these effects is unexpected.

Determine the magnitude of s, which is the new modal frequency.

$$\omega_d^2 = s^2 = \zeta_i^2\omega_i^2 + \alpha^2 + \omega_i^2 - \zeta_i^2\omega_i^2 - \alpha^2$$

$$\omega_d^2 = \omega_i^2$$

so that the modal frequency has stayed the same, as we pointed out earlier. From these effects, it is clear that the close-loop poles have shifted slightly counter-clockwise(Fig. 3.2).

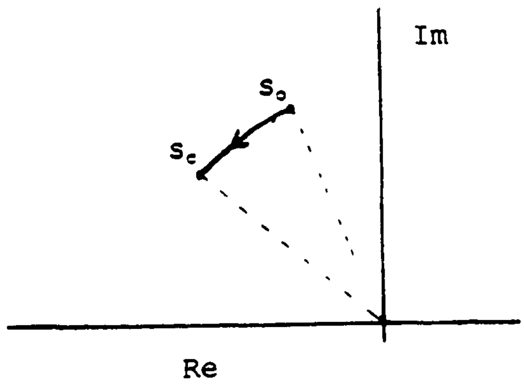


Fig. 3.2 - Graph Showing Pole Rotation Under Optimal Control

More specifically, the poles have shifted from

$$-\zeta_i\omega_i \pm j\omega_i\sqrt{1 - \zeta_i^2}$$

to

$$-\sqrt{\zeta_i^2\omega_i^2 + \alpha^2} \pm j\omega_i\sqrt{1 - \zeta_i^2 - \alpha^2/\omega_i^2}$$

We note that the poles have shifted leftward by the amount

$$-\zeta_i\omega_i + \sqrt{\zeta_i^2\omega_i^2 + \alpha^2}$$

Let $$f(\sigma_i) = \sqrt{\sigma_i^2 + \alpha^2} - \sigma_i \qquad \textbf{(3.21)}$$

where $\sigma_i = \zeta_i \omega_i$. Consider the following graph (Fig. 3.3).

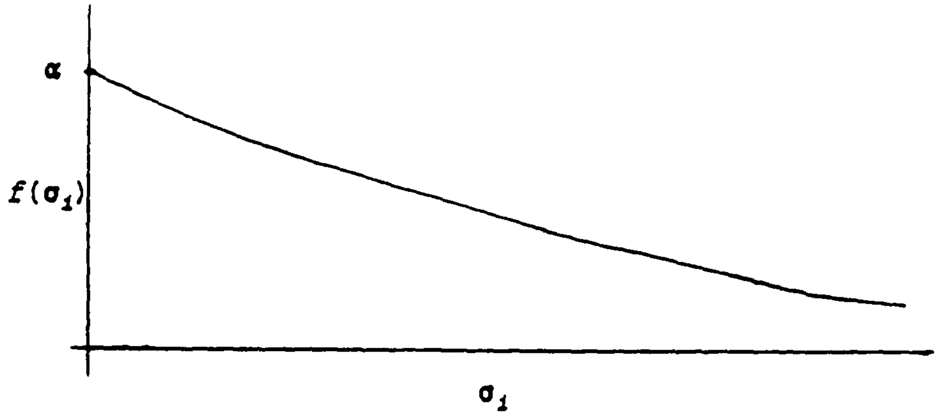


Fig. 3.3 - Graph of Leftward Pole Shift vs. Sigma

This graph shows that:

1. Poles with no passive damping are all shifted to the left by an amount α.
2. Poles with some initial passive damping are shifted to the left by an amount somewhat less that α, but such that the final damping constant > α with the change being inverse to σ_i.

The first point implies that a structure assumed initially undamped will best be controlled by <u>uniformly</u> damping its poles. Silverberg [25] first discovered this, but he did so assuming open-loop energy and only for frequencies for which $\omega_i \gg \alpha$. By using closed-loop energy, we find the last assumption is unnessary.

The second point shows that this form of control guarantees robustness, as all poles have damping constants that are least equal to α. Poles close to the y-axis are moved leftward by greatest amount, and those that are farthest away are moved by the least amount.

Optimization of J When Passive Damping is Considered a Variable

A time-invariant linear feedback system can be optimized in two ways. The first way is the familiar method of minimizing a quadratic functional with respect to the output feedback coefficients - the method we have used thus far in this chapter. The second way is to minimize the same functional with respect to the system parameters. A simple example of this latter optimization problem is presented in Ogata [46].

Example Problem

Consider the system shown below.

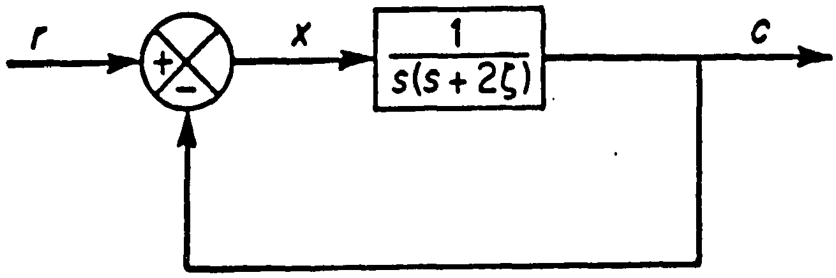


Figure 3.4 - Feedback System With Damping Parameter

We seek to minimize

$$J = \int_0^\infty \mathbf{x}^T(t)\,Q\mathbf{x}(t)\,dt$$

where

$$\mathbf{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & \alpha^2 \end{bmatrix}, \quad \alpha^2 > 0$$

$$x(0) = \dot{x}(0) = 0$$

The following system equation is easily obtained for the state displacement:

$$\ddot{x} + 2\zeta\dot{x} + x = 0 \qquad (t > 0)$$

We find that the system matrix is

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2\zeta \end{bmatrix}$$

Using the theory of Liapunov, we find that

$$J = \mathbf{x}^T(0)\,P\mathbf{x}(0)$$

where P is found using

$$A^T P + PA = -Q$$

We find that

$$P = \begin{bmatrix} \zeta + \dfrac{1+\alpha^2}{4\zeta} & \dfrac{1}{2} \\ \dfrac{1}{2} & \dfrac{1+\alpha^2}{4\zeta} \end{bmatrix} \quad \text{and} \quad J = \zeta + \frac{1+\alpha^2}{4\zeta}.$$

We note here that $J = J(\zeta)$, and now we optimize this function with respect to ζ.

$$\frac{\partial J}{\partial \zeta} = 1 - \frac{1+\alpha^2}{4\zeta^2} = 0$$

yielding
$$\zeta = \frac{\sqrt{1 + \alpha^2}}{2}.$$

Return now to the present case. Our performance index considered now to be a function of ζ_i is given by equation (3.18):

$$J(\zeta_i) = \frac{1}{\alpha^2}\sum_{i=1}^{n}\left[-\zeta_i\omega_i + \sqrt{\zeta_i^2\omega_i^2 + \alpha^2}\right](b_i^* U b_i)$$

If we now take the derivative with respect to ζ_i and set equal to zero, we have

$$\frac{\partial J}{\partial \zeta_i} = \frac{1}{\alpha^2}\left[-\omega_i + \frac{1}{2}(\zeta_i^2\omega_i^2 + \alpha^2)^{-\frac{1}{2}} \cdot 2\zeta_i\omega_i^2\right](b_i^* U b_i) = 0$$

This requires that

$$-\omega_i + \frac{1}{2}(\zeta_i^2\omega_i^2 + \alpha^2)^{-\frac{1}{2}} \cdot 2\zeta_i\omega_i^2 = 0$$

or

$$\omega_i\left[1 - \frac{\zeta_i\omega_i}{\sqrt{\zeta_i^2\omega_i^2 + \alpha^2}}\right] = 0$$

For $\omega_i \neq 0$ we need

$$\zeta_i\omega_i = \sqrt{\zeta_i^2\omega_i^2 + \alpha^2}$$

or

$$\sigma_i = \sqrt{\sigma_i^2 + \sigma^2}$$

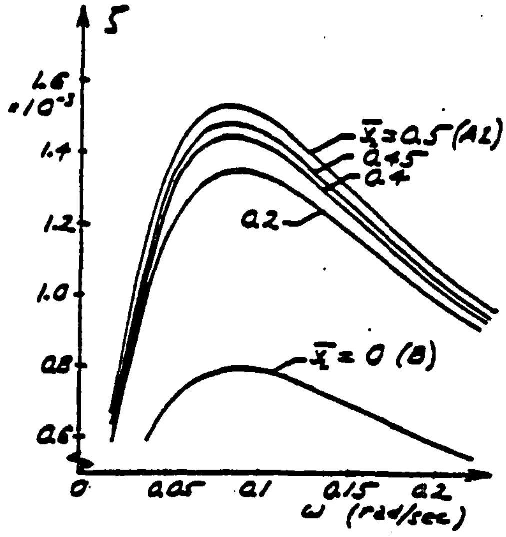
We saw earlier [Fig.3.3] that

$$\sqrt{\sigma_i^2 + \alpha^2} - \sigma_i \to 0 \quad as \quad \sigma_i \to \infty.$$

This implies that the performance is optimal when $\zeta_i \to \infty$ for all i. This result is not unexpected. J has an absolute minimum, in fact, $J = 0$ when $\zeta_i \to \infty$ for all i. Of course, such a system would have no practical utility. In a realistic system the values of ζ_i are constrained.

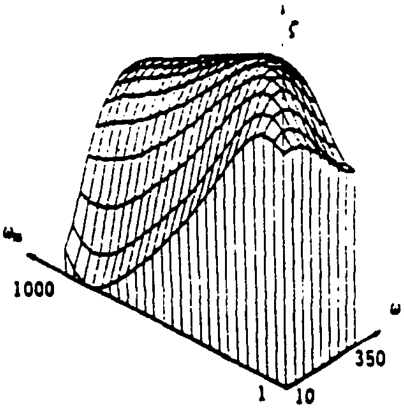
Optimizing $J(\zeta_i)$ Subject to Constraints

In a real structure the values of ζ_i are determined by characteristics of the structure - its material properties and geometric shape. Plunkett and Lee [19] developed theory on the use of visoelastic layers. Ashley [20] cites the theory of Debye [21], which deals with the variation of ζ_i with ω_i for various materials. Curves are exhibited showing the functional variation of ζ_i on ω_i in all of this research (Figs. 3.5, 3.6).



Damping ratios predicted in the vicinity of the Debye peak for vibration of the boron/aluminum "composite" beam

Fig. 3.5



Influence of optimization frequency upon damping distribution

Fig. 3.6

These curves of ζ_i vs. ω_i will serve as the constraints on ζ_i used in the optimization of J. In the next chapter we will apply the theory of this chapter to a Bernoulli-Euler beam damped according to the theory of Plunkett and Lee. A weight (thickness} limitation will provide the overall constraint on the selection of visoelastic damping layers. Within this weight limitation, we can select various damping designs leading to a field of damping curves ζ_i vs. ω_i. Figure 3.6 shows how the selection of a particular damping curve depends on the parameter ω_m, which is the frequency about which the damping has been maximized. The value of this parameter is a function of the segment size of the damping layer. We will optimize our system by selecting the damping design corresponding to the value of ω_m that minimizes J in equation (3.18).

Interdependence of Optimal Design and Control

It would be most fortunate if the optimal damping design produced by this technique would remain valid even if the type of active control were changed. In reality, there is an interdependence between the optimal damping design and the type of control utilized. A different type of control such as pole placement in IMSC or a general type of coupled control in any coordinates will shift the optimal control damping design curve somewhat. In order to understand this, we return to equation (3.13):

$$J = \frac{1}{2}\sum_{i=1}^{n}\left[2 + \frac{h_i^2}{2\alpha^2} + \frac{g_i^2/2\alpha^2}{\omega_i^2 + g_i}\right]\frac{A_i}{2\zeta_i\omega_i + h_i}$$

where $$\zeta_i = \zeta_i(\omega_i, \omega_m).$$

We note here that the modal damping ratios depend on the modal frequencies but also on the parameter ω_m, whose value determines the particular damping curve from which our damping design has been chosen.

Now differentiate J with respect to ω_m.

$$\frac{\partial J}{\partial \omega_m} = \frac{1}{2}\sum_{i=1}^{n}\left[2 + \frac{h_i^2}{2\alpha^2} + \frac{g_i^2/2\alpha^2}{\omega_i^2 + g_i}\right]\frac{-2\omega_i A_i \frac{\partial \zeta_i}{\partial \omega_m}}{(2\zeta_i\omega_i + h_i)^2} = 0$$

This last equation implies that

$$\sum_{i=1}^{n}\left[2 + \frac{h_i^2}{2\alpha^2} + \frac{g_i^2/2\alpha^2}{\omega_i^2 + g_i}\right]\frac{\omega_i A_i}{(2\zeta_i\omega_i + h_i)^2}\frac{\partial \zeta_i}{\partial \omega_m} = 0$$

with $A_i = \mathbf{b}_i^* U \mathbf{b}_i > 0$ for all i.

In order that the terms sum to zero, some of the partial derivatives in this sum must be positive and others negative. When a different type of control is used, we can expect the factors of the derivative terms in the sum above to vary somewhat from the optimal values. The net effect of this change in control will be to shift the value of ω_m by some amount. We note, however, that the feedback coefficients enter those factors in a way that prevents them from having a dramatic effect on these factors.

Modal Truncation of Model

The process presented here for designing optimal damping depends on the values of the each of the optimal feedback coefficients g_i^{opt} and h_i^{opt}. A common situation in everyday control design is to truncate the system model using only its most important components- for example, its first and second modes. Alberts [29] has shown that a simple structural system with many flexible modes can be adequately controlled using a two mode model with an optimal regulator if passive damping is used for augmentation.

The question that now arises: If the optimal damping is designed on the basis of the total system model, how close to optimal will this design remain if the model is modally trunctated for the purposes of active control? To answer this question we consider the equation used to solve for ω_m. The coefficient of the *i*th term is

$$\frac{\omega_i A_i \left[2 + \frac{h_i^2}{2\alpha^2} + \frac{g_i^2/2\alpha^2}{\omega_i^2 + g_i} \right]}{(2\zeta_i\omega_i + h_i)^2}$$

From equation (3.15), the quantity in brackets equals

$$\frac{h_i^{opt}}{\alpha^2}(2\zeta_i\omega_i + h_i)$$

The ith coefficient then becomes

$$\frac{1}{\alpha^2} \cdot \frac{h_i^{opt}\omega_i A_i}{2\zeta_i\omega_i + h_i^{opt}}$$

When a mode is truncated, this is equivalent to the assertion

$$g_i = 0, \; h_i = 0$$

for all of the truncated modes. The value of the *i*th coefficient for a truncated mode is

$$\frac{2\omega_i A_i}{4\zeta_i^2\omega_i^2} = \frac{A_i}{2\zeta_i^2\omega_i}$$

To determine the effects of the truncated modes, we determine the fractional change in the *i*th coefficient from the assumed value using optimal feedback coefficients to the actual value using coefficients of zero.

$$\frac{\Delta\left(\frac{\partial J}{\partial \omega_m}\right)_i}{\left(\frac{\partial J}{\partial \omega_m}\right)_i} = \frac{\frac{A_i}{2\zeta_i^2\omega_i} - \frac{h_i^{opt}\omega_i A_i}{\alpha^2(2\zeta_i\omega_i + h_i^{opt})}}{\frac{h_i^{opt}\omega_i A_i}{\alpha^2(2\zeta_i\omega_i + h_i^{opt})}} \tag{3.22}$$

$$= \frac{\alpha^2(2\zeta_i\omega_i + h_i^{opt}) - 2\zeta_i^2\omega_i^2 h_i^{opt}}{2\zeta_i^2\omega_i^2 h_i^{opt}}$$

Typically, the higher order modes are the ones that get truncated. For this reason, we assume $\zeta_i^2\omega_i^2 \gg \alpha^2$ for the truncated modes. In this case

$$h_i^{opt} = -2\zeta_i\omega_i + 2\sqrt{\zeta_i^2\omega_i^2 + \alpha^2}$$

$$= -2\zeta_i\omega_i + 2\zeta_i\omega_i\sqrt{1 + \frac{\alpha^2}{\zeta_i^2\omega_i^2}}$$

$$\approx -2\zeta_i\omega_i + 2\zeta_i\omega_i\left(1 + \frac{\alpha^2}{2\zeta_i^2\omega_i^2}\right)$$

$$= \frac{\alpha^2}{\zeta_i\omega_i}$$

Substitute into equation (3.22) and get

$$\frac{\Delta\left(\frac{\partial J}{\partial \omega_m}\right)_i}{\left(\frac{\partial J}{\partial \omega_m}\right)_i} = \frac{\alpha^2\left(2\zeta_i\omega_i + \frac{\alpha^2}{\zeta_i\omega_i}\right) - 2\zeta_i^2\omega_i^2\left(\frac{\alpha^2}{\zeta_i\omega_i}\right)}{2\zeta_i^2\omega_i^2\left(\frac{\alpha^2}{\zeta_i\omega_i}\right)}$$

$$= \frac{\alpha^2}{2\zeta_i^2\omega_i^2}$$

which, by our assumption, is << 1.

This means that the optimal passive damping design based on the complete system model should work well on a model in which the higher order modes have been truncated. That is, the optimal design should remain relatively insensitive to the truncation of the higher frequency modes. This is welcome news, since any useful discrete model of a continuous system will necessarily involve considerable coordinate truncation. For reasonable values of α, the optimal design based on the optimal control of all modes should remain nearly optimal when we use the same algorithm on only a small fraction of those modes. Finally, this result implies that the more we penalize active control (the lower the α), the closer our passive damping design will be to the optimal design regardless of the degree of modal truncation in our model.

CHAPTER FOUR

NUMERICAL EXAMPLE - SIMPLY-SUPPORTED BEAM

In this chapter we apply the theory of optimal passive design from chapter three to a simply supported Bernoulli - Euler beam. Let the ith mode of this system be represented by:

$$\ddot{\eta}_i + 2\zeta_i\omega_i\dot{\eta}_i + \omega_i^2\eta_i = \boldsymbol{b}_i^* u + f_i$$

To keep the example manageable, we consider only the first six modes. We assume that the system has a single impulse disturbance of magnitude μ located a distance r_c from one end of the beam, i.e., $u = \mu\delta t$, and a single output y located a distance r_o from the same end.

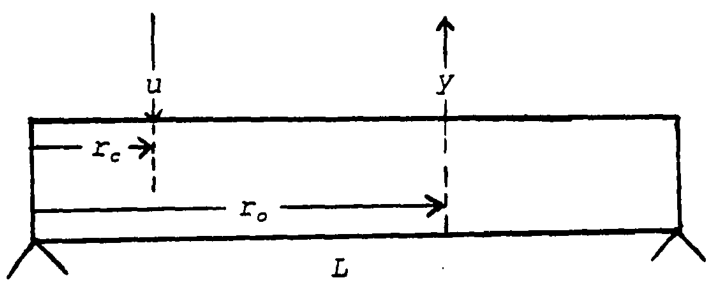


Fig. 4.1 - Diagram of Simply-Supported Bernoulli-Euler Beam

In this case, the modal influence coefficient for the disturbance is given by

$$b_i = \sqrt{\frac{2}{\rho L}}\sin\left(\frac{\pi r_c}{L}i\right)$$

where ρ = the mass per unit length,

L = the length of the beam,

and the A_i's are determined from

$$A_i = b_i^* U b_i$$

$$= \sqrt{\frac{2}{\rho L}}\sin\left(\frac{\pi r_c}{L}i\right) \cdot \mu^2 \cdot \sqrt{\frac{2}{\rho L}}\sin\left(\frac{\pi r_c}{L}i\right) \qquad \textbf{(4.1)}$$

$$A_i = \frac{2\mu^2}{\rho L}\sin^2\left(\frac{\pi r_c}{L}i\right)$$

For a simply supported beam, the modal frequencies are derived from

$$\omega_i = \sqrt{\frac{EI}{\rho}} \cdot \left(\frac{\pi}{L}i\right)^2 \qquad \textbf{(4.2)}$$

where E = the elastic modulus

I = the area moment of inertia

To simplify the example, we let each parameter equal unity ; i.e.,

$$E = \rho = L = I = \mu = 1$$

Under these circumstances, equations (4.1) and (4.2) render

$$\omega_i = \pi^2 i^2 \quad , \quad i = 1,2, \ldots 6$$

$$A_1 = 1.309 \ , \ A_2 = 1.809 \ , \ A_3 = 0.191 \ ,$$

$$A_4 = A_6 = 0.691 \ , \ A_5 = 2.0$$

To provide passive damping, we consider a constrained viscoelastic damping treatment using Scotchdamp ™ ISD 110 damping tape. By cutting the constraining layer into segments of length L_c, one can vary the frequency at which optimum damping is achieved. Such a treatment is illustrated in Fig. 4.2 on the next page. Plunkett and Lee [19] have developed a method by which the designer determines L_c by specifying an optimization frequency ω_m at which optimal passive damping is desired. Thus $L_c = L_c(\omega_m)$ and accordingly, for each i, $\zeta_i = \zeta_i(\omega_m)$. Beyond optimizing the damping at the prescribed frequency, the selection of ω_m influences the distribution of damping among a broad spectrum of frequencies. This is illustrated in Fig.(4.2).

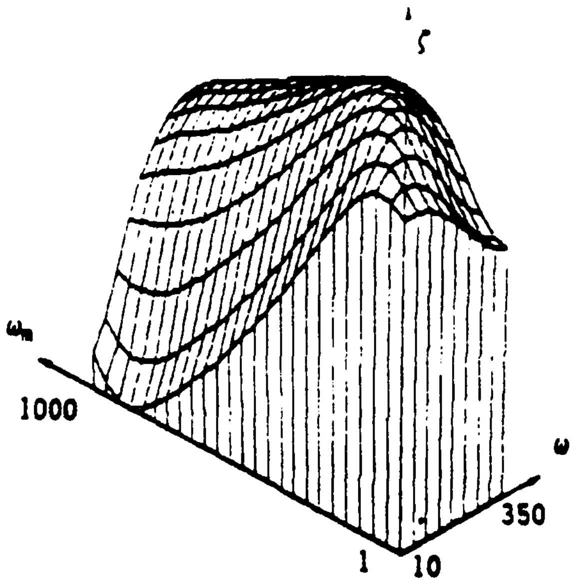


Figure 4.2 - Effects of Optimization Freq. on Zeta

For this example, we consider the damping optimization frequency ω_m the design parameter which is to be selected to minimize J. It is not clear, at the outset, which frequency ω_m will produce the best results.

Without going into details of computing the damping provided by a given constrained layer treatment, let it suffice to say that the damping produced is a complicated function of several dimensions and material properties. In particular, the frequency dependent loss factor η_G and shear modulus G properties that characterize the viscoelastic nature of the damping tape are empirically determined and tabulated.

The fact that these properties are not easily described in a functional form seems to eliminate any possibility of estabilishing a closed form solution for the optimal modal passive damping distribution.

For this example, a computational solution can be employed (Appendix-Program 1).

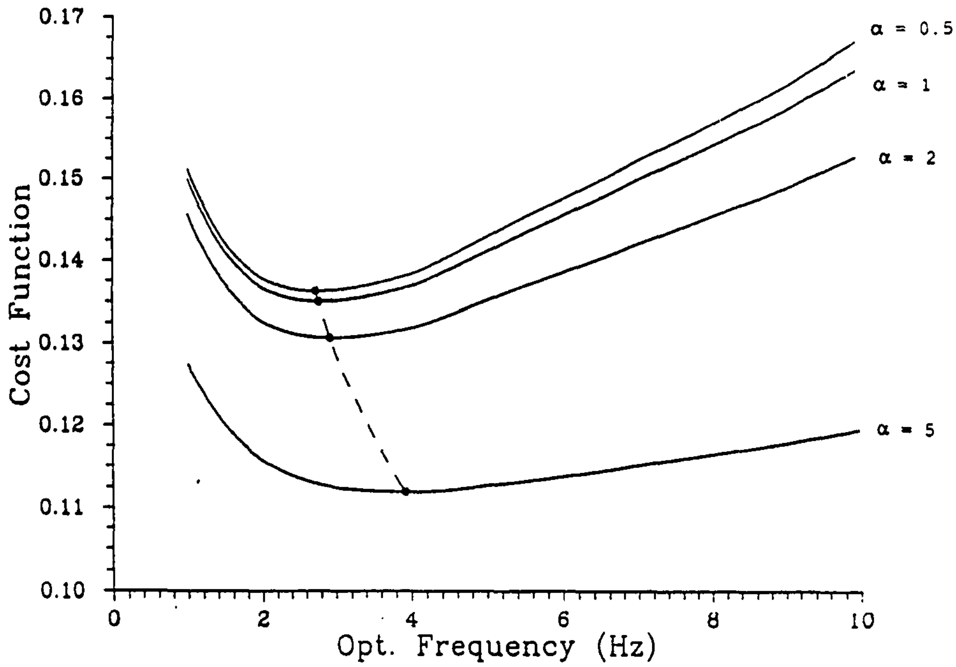


Figure 4.3 - Effects of ω_m on J as a Function of α

Fig.(4.3) illustrates the effect of ω_m on J for various values of α. The figure indicates that for small values of α, indicating a high penalty on the use of control effort, the optimization frequency has a strong influence upon the value of the performance index. In this example, with $\alpha = 0.5$, the optimization frequency of 2.73 HZ. produces the minimum J^{opt}.

It will be shown that the dominant effect of properly selecting ω_m in this case is reduced fuel consumption.

Results

As might be expected, the results of the optimization are manifested primarily in the performance integral, control effort, and system energy integral values. Here the performance index is as defined in equation (3.18), control of effort is defined as

$$J_{Eff} = \int_0^\infty \mathbf{f}^T \mathbf{f} dt$$

and the system energy integral, a measure of the system's total vibrational energy $E(t)$ integrated over time, is given by

$$J_{En} = \int_0^\infty E(T)\, dt$$

Figs. (4.4, 4.5, 4.6) graphically illustrate the resulting performance index, control effort, and the system energy integral versus α for various modal damping distributions. The sets of ζ values chosen correspond to an optimal set dependent on the value of α, two sub-optimal sets, and finally we set ζ equal to zero, corresponding to no damping. The sub-optimal sets of ζ chosen correspond to frequencies that are sub-optimal with reference to the corresponding α.

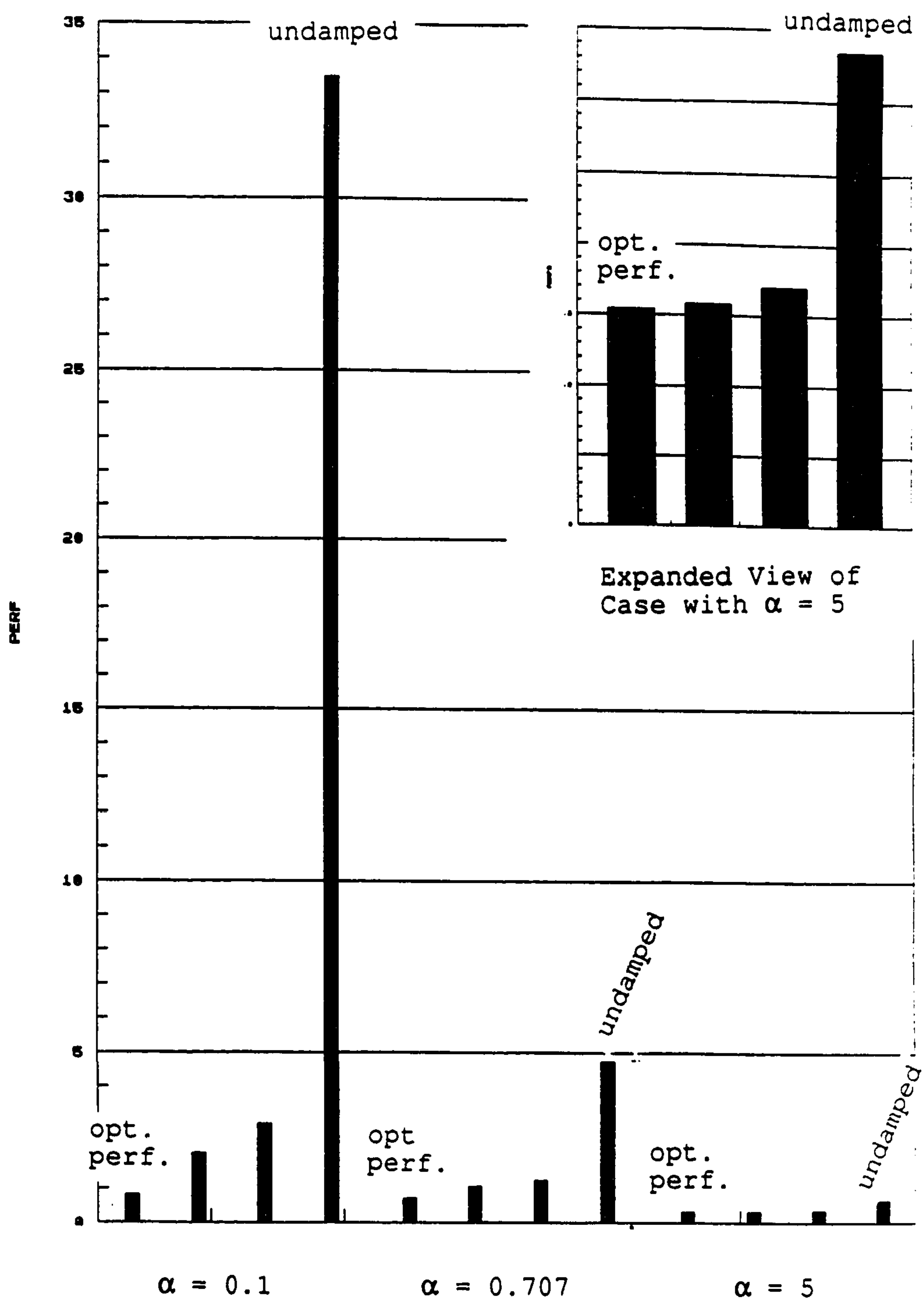


Figure 4.4 - Graphs of System Performance

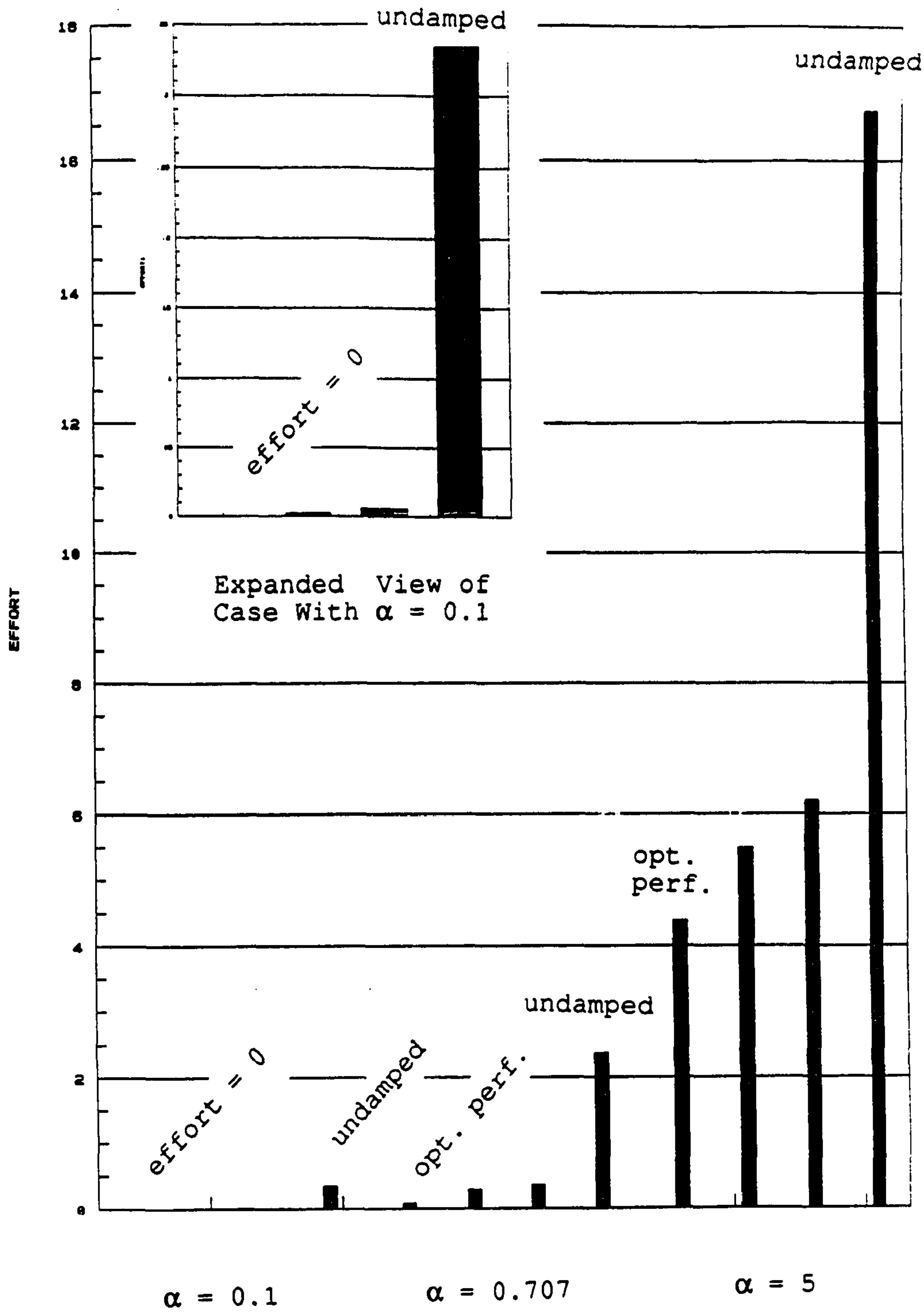


Figure 4.5 - Graphs of the Control Effort

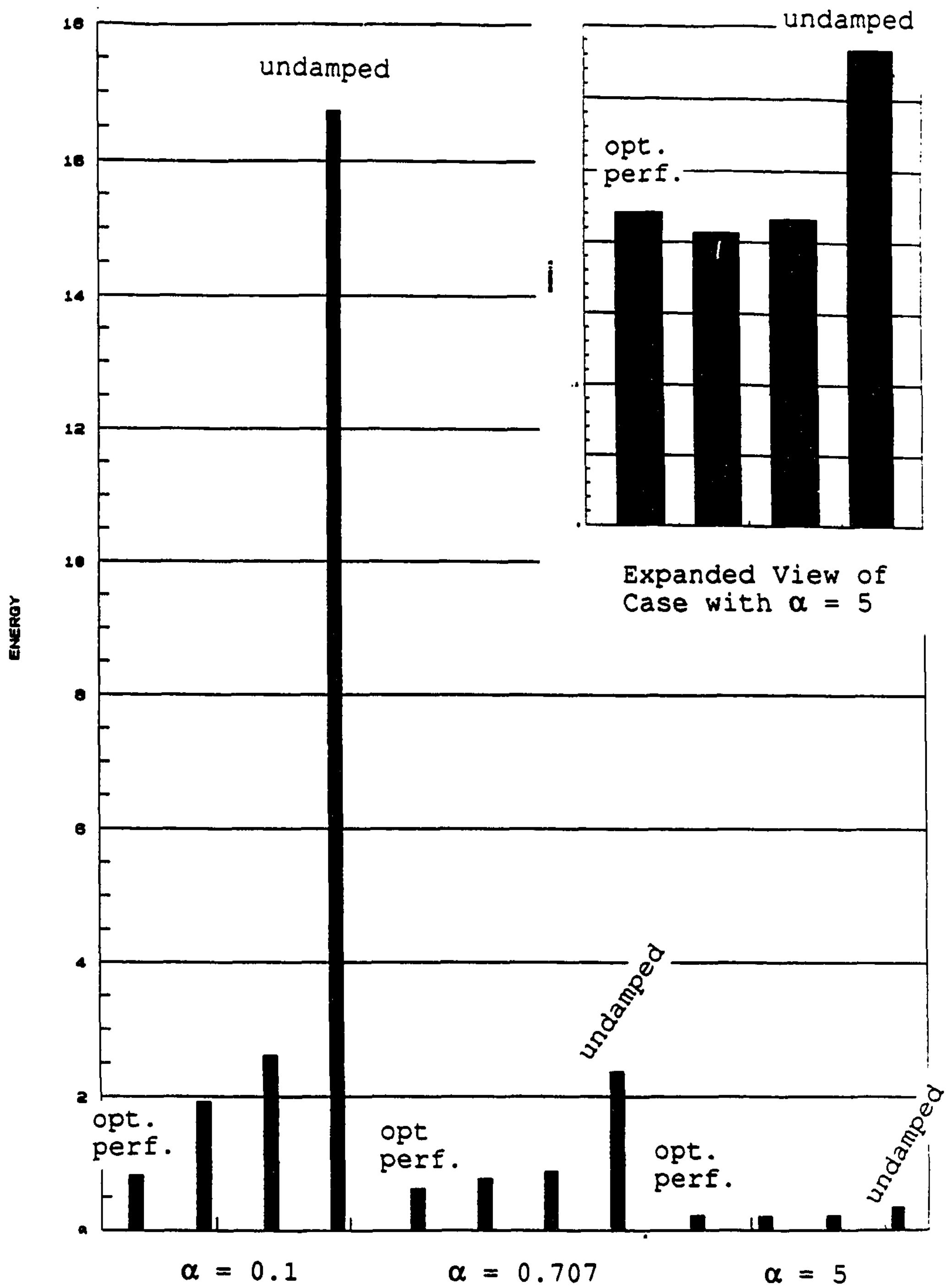


Figure 4.6 - Graphs of System Energy

A graph of the performance function versus optimal frequency (Fig.4.3) migrates as a function of α. The optimum frequencies for the given values of α are listed below:

Alpha	Optimal Frequency
0.5	2.73 HZ
1.0	2.80 HZ
5.0	3.90 HZ

Table - Optimal Distribution Frequency vs. Effort Weighting

The sub-optimal data corresponds to frequencies of 77 and 250 HZ. The graphs of performance (Fig. 4.4) show the expected high cost when no damping is present, particularly when α is small. This is attributable to the fact that as α gets smaller, there is a greater penalty associated with control effort. Therefore, as expected, the benefits of passive damping increase as more emphasis is placed on fuel consumption.

It should be noted that the results do not account for the fact that the damping treatment will slightly increase the weight of the structures over the undamped case. Typically, this weight increase can be kept well below five percent. Cases 1,2, and 3 would be equal in weight. Since fuel consumption increases in direct proportion to mass, the factor of added weight brought on by the viscoelastic layer will not appreciably affect the comparison between the damped cases and the undamped case (case 4). This is evident from the wide percentage differences between the performance of the damped and undamped cases. Even when the control effort is weakly penalized (α = 5), the cost ratio is about 2:1.

The comparison of the control effort (Fig. 4.5) also indicates a much higher effort when there is no damping than when damping exists. The effort in the cases involving optimal cost is always less than in the non-optimal cases. This is attributable to the fact that the greater proportion of the vibrational energy is concentrated in the lower modes, thus a damping treatment designed near the frequencies of the lower modes should result in reduced control effort. Predictably, the effort is near zero when α is small. With such a high penalty on control effort, system control depends almost entirely on passive damping.

The energy integral results (Fig. 4.6) exhibit similar trends with one exception - the case when α = 5. The energy in the case of optimal cost is about 7% more than in the best suboptimal case. This favorable difference in control effort is overshadowed by more than a 20% favorable difference in control effort.

The graphs of the impulse response (Fig. 4.7) show that when α is small, the optimal case has a signifcantly lower amplitude than in any of the sub-optimal cases. As α gets

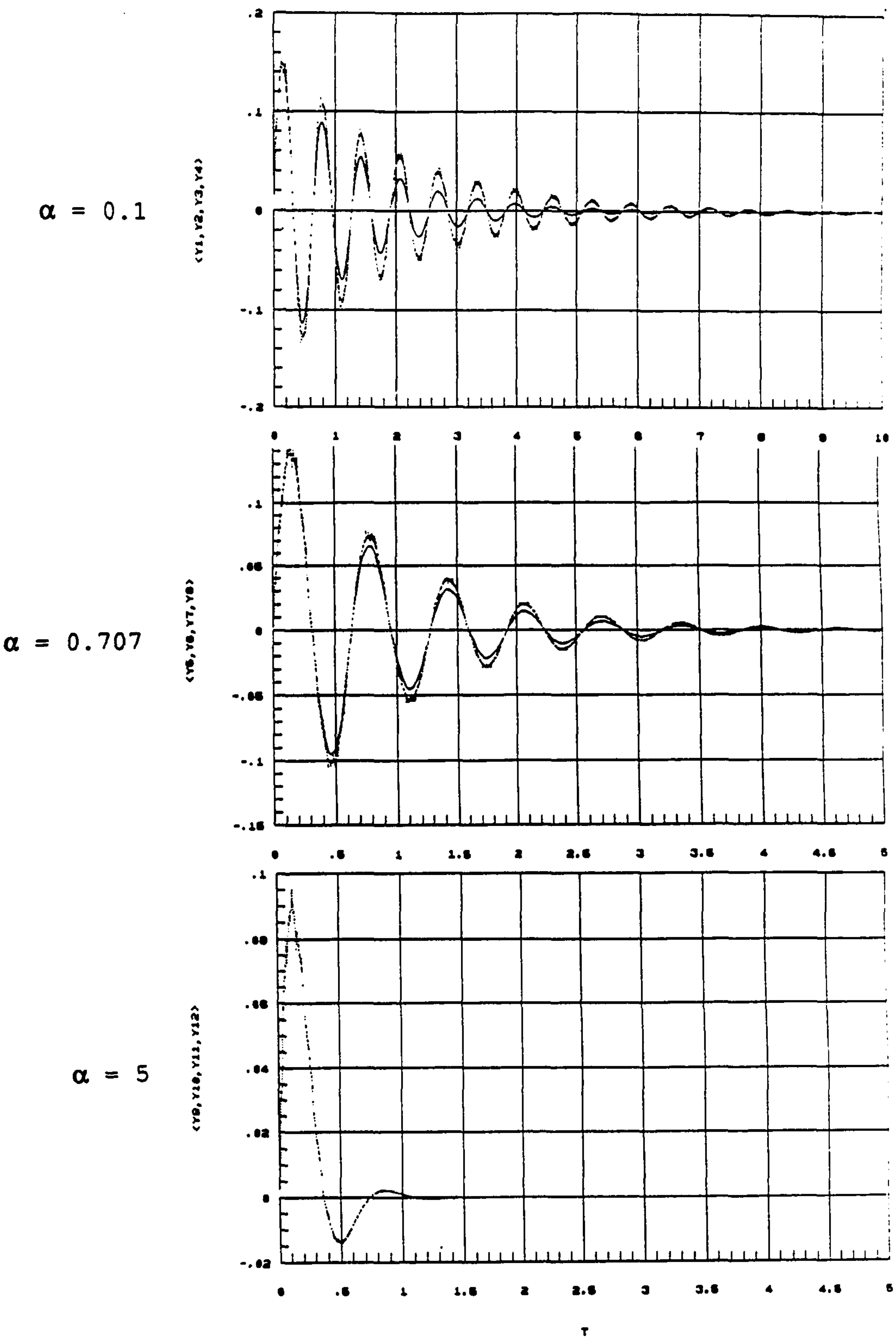


Figure 4.7 - Graphs of Impulse Response

larger, this difference in amplitude becomes less pronounced. When α = 5, the outputs are practically identical. The explanation for this effect comes from the modal equations of motion:

$$\ddot{\eta} + 2\zeta_i\omega_i\dot{\eta}_i + \omega_i^2\eta_i = -g_i\eta_i - h_i\dot{\eta}_i$$

where $g_i^{opt} = 0$, and $h_i^{opt} = -2\zeta_i\omega_i + 2\left(\zeta_i^2\omega_i^2 + \alpha^2\right)^{\frac{1}{2}}$.

Substitution of these optimal values into the equation of motion gives

$$\ddot{\eta}_i + 2\,(\zeta_i^2\omega_i^2 + \alpha^2)^{\frac{1}{2}}\dot{\eta}_i + \omega_i^2\eta_i = 0$$

For general inputs, the majority of the energy will be in the lower modes. Because the damping ratios are small, the coefficient of the velocity term will approach 2α (the value corresponding to no damping) as alpha gets larger and larger.

Self-Adjointness of the Damped Structure

As indicated in chapter two, practical structures do not exactly satisfy the self-adjointness criterion needed for decoupled modelling and control. Ideal structures with no damping or with the special classes of damping demonstrated in chapter two, are self-adjoint structures. Fortunately, many engineering structures come close enough to satisfying the self-adjointness to permit successful implementation of IMSC.

Meirovitch and Norris [26] showed that a small amount of non-proportional damping in a structure does not have a significant effect on IMSC system performance. At present, it is not clear whether viscoelastic damping intentionally introduced into a structure as described above will destroy the system's self-adjointness to the point of affecting IMSC performance and robustness.

Presumably, an optimal damping design based on modal coordinates will suffer somewhat if the damping is non-proportional. However, Meirovitch and Norris [26] show that a control design based on IMSC will suffer significantly only if the off-diagonal damping terms are at least as great in magnitude as the on-diagonal terms.

CHAPTER FIVE

PLANT MODEL REDUCTION - MODAL COORDINATES

A continuous structure represents a multi-degree of freedom system whose active control involves the use of many sensors and actuators, much computational real time, and the expenditure of requisite fuel. One way to partially relieve this control problem is to somehow reduce the scale of the system while still providing the necessary control of all its components.

One way to reduce the model of a plant is to transform the system spatial coordinates to a new set of coordinates in which at least some of these newly modeled state components can be thought of as insignificant and may thus be truncated from the system model. By controlling the remaining significant components, we may therefore adequately control the original plant.

Two popular sets of coordinates for this purpose are

1. modal coordinates
2. balanced coordinates

In this chapter we will transform our orignal system coordinates to modal coordinates and apply a model reduction criterion to determine which coordinates should remain in our model.

Quadratic Cost Function

A generally accepted criterion for state component truncation is a quadratic cost function. The importance of plant components is then decided based on their contribution to this function, which is essentially the total mechanical energy of the system. In modal coordinates, the individual contributions are referred to as modal costs. The idea is then to eliminate from the plant model those modes contributing the least to the total system.

Define the cost as

$$J = \sum_{i=1}^{m} \int_{0}^{\infty} \boldsymbol{y}^{i*}(t) Q \boldsymbol{y}^{i}(t)\, dt \tag{5.1}$$

where

$$\boldsymbol{y} = \sum_{i=1}^{n} \boldsymbol{c}_i x_i \quad , \quad x_i(0) = 0$$

and Q is a positive definite weighting matrix.

Model Reduction Index

A generally accepted index measuring the effectiveness of a component truncation based on cost is given by

$$I = \frac{\sum_{i=r+1}^{n} J_i}{\sum_{i=1}^{n} J_i} \tag{5.2}$$

where r = index of the last mode retained in the model.

System Dynamics

The system considered for this purpose is the usual linear matrix 2nd-order system given by

$$M\ddot{\mathbf{q}} + D\dot{\mathbf{q}} + K\mathbf{q} = B\mathbf{u}$$

If we assume the system is self-adjoint, then it may be transformed using $\mathbf{q} = T\boldsymbol{\eta}$ into the following set of modal equations:

$$\ddot{\boldsymbol{\eta}} + 2Z\boldsymbol{\Omega}\dot{\boldsymbol{\eta}} + \boldsymbol{\Omega}^2\boldsymbol{\eta} = T^T B\mathbf{u} = \boldsymbol{\beta}\mathbf{u}$$

where

$$\boldsymbol{\eta} \in R^{n \times 1} \quad , \quad \mathbf{B} \in C^{n \times m}$$

$$Z = diag[\zeta_1, \zeta_2, \ldots \zeta_n] \quad , \quad \boldsymbol{\Omega} = diag[\omega_1, \omega_2, \ldots \omega_n]$$

where ζ_i is the ith modal damping ratio, and ω_i is the ith modal frequency, and in these modal coordinates the system output is expressed as

$$\mathbf{y_p} = \sum_{i=1}^{n} \mathbf{p}_i \eta_i \quad , \quad \mathbf{y_r} = \sum_{i=1}^{n} \mathbf{r}_i \dot{\eta}_i$$

which we combine to form

$$y = \begin{bmatrix} y_p \\ y_r \end{bmatrix} = \sum_{i=1}^{n} \begin{bmatrix} p_i & 0 \\ 0 & r_i \end{bmatrix} \begin{bmatrix} \eta \\ \dot{\eta} \end{bmatrix}$$

We now define the following vectors:

$$\mathbf{P}_i = \begin{bmatrix} p_i \\ 0 \end{bmatrix}, \quad \mathbf{\Gamma}_i = \begin{bmatrix} 0 \\ r_i \end{bmatrix}$$

so that the output vector becomes

$$y_i = \sum_{i=1}^{n} (\mathbf{P}_i \eta_i + \mathbf{\Gamma}_i \dot{\eta}_i)$$

Conversion to State Space

Define the state vector as

$$\begin{bmatrix} \hat{x}_i \\ \hat{x}_{i+n} \end{bmatrix} = \begin{bmatrix} \eta_i \\ \dot{\eta}_i \end{bmatrix}$$

and the block diagonal system equations are

$$\frac{d}{dt} \begin{bmatrix} \hat{x}_i \\ \hat{x}_{i+n} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & 2\zeta_i \omega_i \end{bmatrix} \begin{bmatrix} \eta_i \\ \dot{\eta}_i \end{bmatrix} + \begin{bmatrix} 0 \\ \beta_i^* \end{bmatrix} u$$

A diagonalized state space representation of the form

$$\dot{x}_i = \lambda_i x_i + b_i^* u \quad , \quad y = \sum_{i=1}^{2n} c_i x_i$$

can be obtained using the transformation

$$\hat{x} = Rx$$

The eigenvalues of the system matrix are

$$\lambda_i = -\zeta_i\omega_i + j\omega_i\sqrt{1 - \zeta_i^2}\ ,$$
$$\lambda_{i+n} = -\zeta_i\omega_i - j\omega_i\sqrt{1 - \zeta_i^2} \qquad (5.3)$$

The corresponding eigenvectors form the transformation matrix

$$R_i = \begin{bmatrix} 1 & 1 \\ \lambda_i & \lambda_{i+n} \end{bmatrix}$$

but

$$\begin{bmatrix} b_i^* \\ b_{i+n}^* \end{bmatrix} = R_i^{-1} \begin{bmatrix} 0 \\ \beta_i^* \end{bmatrix}$$

so that

$$b_i^* = \frac{\beta_i^*}{2j\omega_i\sqrt{1 - \zeta_i^2}}$$
$$b_{i+n}^* = \frac{\beta_i^*}{-2j\omega_i\sqrt{1 - \zeta_i^2}} \qquad (5.4)$$

Likewise, the output coefficient vector becomes

$$c_i = \mathbf{P}_i + \Gamma_i\lambda_i \ , \quad c_{i+n} = \mathbf{P}_i + \Gamma_i\lambda_{i+n} \qquad (5.5)$$

Disturbance Input

To have any value generally, our results must be based on the use of some general (white noise) disturbance. Such an input is defined as

Skelton [34] shows that the cost equation (5.1) can be written

$$u_i(t) = \mu_i \delta(t) \qquad i = 1,2, \ldots m$$

$$J_{x_i} = [XC^* QC]_{ii} \tag{5.6}$$

where X, the covariance matrix, is solved from the Liapunov equation

$$0 = XA^* + AX + BUB^*$$

Solving for the ik-element, the following is obtained

$$X_{ik} = \frac{\boldsymbol{b}_i^* U \boldsymbol{b}_k}{\bar{\lambda}_k + \lambda_i}$$

where $\boldsymbol{b}_i$ and $\boldsymbol{b}_k$ are column vectors of B; λ_i and λ_k are the ith and kth eigenvalues of A.

<u>Derivation of Cost Formula - Light Damping</u>

Substitution of X_{ij} into cost function (5.6) yields

$$J_{x_i} = \sum_{k=1}^{2n} \frac{(-\boldsymbol{b}_i^* U \boldsymbol{b}_k)(\boldsymbol{c}_k^* Q \boldsymbol{c}_i)}{\bar{\lambda}_k + \lambda_i} \tag{5.7}$$

Skelton [34] derives a closed-form expression for this cost formula in terms of ζ_i and ω_i by making the following assumption:

$$\zeta_i < \frac{|\omega_i - \omega_k|}{2\omega_i} \qquad \forall k \neq i$$

When this assumption for light damping is made, the cost function reduces to

$$J_{x_i} = J_{x_{i+n}} = \frac{(b_i^* U b_i)(c_i^* Q c_i)}{2\zeta_i \omega_i}$$

Skelton then uses the transformations (5.4) and (5.5) together with

$$J_{\eta_i} = J_{x_i} + J_{x_{i+n}}$$

to derive the <u>2nd-order modal cost for light damping</u> as

$$J_{\eta_i} = \frac{(\beta_i^* U \beta_i)(p_i^* Q_p p_i + r_i^* Q_r r_i \omega_i^2)}{4\zeta_i \omega_i^3} \qquad (5.9)$$

where

J_{η_i} = cost attributable to the ith 2nd-order mode.

<u>A) Model Reduction Index - Light Constant Damping</u>

In his book on dynamics control, Skelton [47] applies the theory to a simple structure (beam) to determine the number of modes needed to model the system to within a certain accuracy. For this purpose, he assumes $\zeta = \bar{\zeta}$ = constant.

The model reduction index is used with $r = N$ and $n \to \infty$. His derivation leads to the formula

$$N \geq -1 + \left(\frac{5\epsilon\pi^6}{945}\right)^{-\frac{1}{5}}$$

where ϵ = maximum desirable error, which is the difference between the total cost function and the cost function assuming a model containing the first N modes.

The following table is produced

ϵ	N
0.1	1
0.01	1
0.001	2
0.0001	4
0.00001	7

Table 5.1 - Specified Error vs. No. of Modeled Modes Required

The important fact for us to note from this is that there seems to be no dependence between N and the value of $\bar{\zeta}$. This means that for sufficiently light constant damping, the value of the damping ratio has no influence on the relative distribution of energy among the modes.

B) Model Reduction Index - Light Progressive Damping

Suppose we perturb the system with additional damping provided by the term $\theta\omega_i$, so that

$$\zeta_i = \bar{\zeta} + \theta\omega_i \qquad (5.10)$$

where θ is a constant with

$$\zeta_i < \frac{|\omega_i - \omega_k|}{2\omega_i} \qquad \forall k \neq i$$

so that we again make the assumption of light damping. We apply the theory, as in the previous example, to a simply-supported beam of length L, uniform mass density ρ, an impulse force applied at position r_c, and an output deflection at r_o. For our purposes, we let $Q_p = 1$.

The system 2nd-order equations are

$$\ddot{\eta}_i + 2\zeta_i\omega_i\dot{\eta}_i + \omega_i^2\eta_i = \beta_i^* u$$

$$y = \sum_{i=1}^{n} p_i\eta_i$$

For this particular input

$$\beta_i^* = \sqrt{\frac{2}{\rho L}}\,\sin\left(\frac{\pi r_c}{L}i\right)$$

$$p_i = \sqrt{\frac{2}{\rho L}}\,\sin\left(\frac{\pi r_0}{L}i\right)$$

$$\omega_i = \sqrt{\frac{EI}{\rho}}\left(\frac{\pi}{L}i\right)^2$$

If we substitute these values into equation (5.9), we get

$$J = \sum_{i=1}^{\infty} J_{\eta_i}$$

$$= \sum_{i=1}^{\infty} \left[2\mu^2 \sin^2\left(\frac{\pi r_c}{L} i\right)\right] \frac{\left[\frac{2}{\rho L}\sin^2\left(\frac{\pi r_0}{L} i\right)\right]}{4\zeta_i \rho L \left[\sqrt{\frac{EI}{\rho}}\left(\frac{\pi}{L} i\right)^2\right]^3}$$

$$\leq \sum_{i=1}^{\infty} \frac{K}{i^6 \zeta_i} \qquad , \; K = \text{constant.} \qquad \textbf{(5.11)}$$

We can think of the terms in (5.11) as cost magnitudes. These magnitudes are the terms we will use in the model reduction index. If we now substitute the damping design (5.10) into (5.11), we get

$$J = \sum_{i=1}^{\infty} \frac{K}{i^6 (\overline{\zeta} + \theta \omega_i)}$$

but $\qquad \omega_i = c i^2, \quad$ where c = constant.

Specifying a maximum tolerable modelling error of ϵ, the model reduction index (5.2) becomes

$$I = \frac{\displaystyle\sum_{i=N+1}^{\infty} \frac{K}{i^6 (\overline{\zeta} + \theta c i^2)}}{\displaystyle\sum_{i=1}^{\infty} \frac{K}{i^6 (\overline{\zeta} + \theta c i^2)}} \leq \epsilon \qquad \textbf{(5.12)}$$

but

$$\sum_{i=1}^{\infty} \frac{1}{i^6(\overline{\zeta} + \theta c i^2)}$$

$$= \frac{1}{\overline{\zeta}} \sum_{i=1}^{\infty} \frac{1}{i^6\left(1 + \frac{\theta c}{\overline{\zeta}} i^2\right)} \tag{5.13}$$

Let $\frac{\theta c}{\overline{\zeta}} = \phi$, then (5.13) becomes

$$\frac{1}{\overline{\zeta}} \sum_{i=1}^{\infty} \frac{1}{i^6(1 + \phi i^2)}$$

and (5.12) can be written

$$\sum_{i=N+1}^{\infty} \frac{1}{i^6(1 + \phi i^2)} \leq \epsilon \sum_{i=1}^{\infty} \frac{1}{i^6(1 + \phi i^2)}$$

$$\sum_{i=1}^{\infty} \frac{1}{i^6(1 + \phi i^2)} \approx \frac{1}{1 + \phi}$$

and

$$\sum_{i=N+1}^{\infty} \frac{1}{i^6(1 + \phi i^2)} \leq \int_{i=1}^{\infty} \frac{1}{x^6(1 + \phi x^2)} dx$$

Using partial fractions

$$\frac{1}{x^6(1 + \phi x^2)} = \frac{Ax + B}{1 + \phi x^2} + \frac{Cx^5 + Dx^4 + Ex^3 + Fx^2 + Gx + H}{x^6}$$

$$1 = (Ax + B)x^6 + (Cx^5 + Dx^4 + Ex^3 + Fx^2 + Gx + H)(1 + \phi x^2)$$

Equating like coefficients, we have

$$x^7:\quad 0 = A + C\phi \qquad x^6:\quad 0 = B + D\phi$$

$$x^5:\quad 0 = C + E\phi \qquad x^4:\quad 0 = D + F\phi$$

$$x^3:\quad 0 = E + G\phi \qquad x^2:\quad 0 = F + H\phi$$

$$x^1:\quad 0 = G \qquad x^0:\quad 1 = H$$

so that

$$A = 0,\ B = -\phi^3,\ C = 0,\ D = \phi^2,$$

$$E = 0,\ F = -\phi,\ G = 0,\ \text{and}\ \ H = 1$$

Using these values,

$$\frac{1}{x^6(1 + \phi x^2)} = \frac{-\phi^3}{1 + \phi x^2} + \frac{\phi^2 x^4 - \phi x^2 + 1}{x^6}$$

Therefore

$$\int_{N+1}^{\infty} \frac{1}{x^6(1 + \phi x^2)}dx = \int_{N+1}^{\infty} \frac{-\phi^3 dx}{1 + \phi x^2}$$

$$+ \int_{N+1}^{\infty} (\phi^2 x^{-2} - \phi x^{-4} + x^{-6})dx$$

$$\int_{N+1}^{\infty} \frac{-\phi^3 dx}{1 + \phi x^2} = -\phi^2 \int_{N+1}^{\infty} \frac{dx}{\frac{1}{\phi} + x^2}$$

$$= -\phi^2[\sqrt{\phi}\tan^{-1}(\sqrt{\phi}x)]\Big|_{N+1}^{\infty}$$

$$= -\phi^{\frac{5}{2}}[\tan^{-1}\infty - \tan^{-1}\sqrt{\phi}(N+1)]$$

$$= \phi^{\frac{5}{2}}\left[\tan^{-1}\sqrt{\phi}(N+1) - \frac{\pi}{2}\right]$$

$$\int_{N+1}^{\infty} (\phi^2 x^{-2} - \phi x^{-4} + x^{-6}) dx$$

$$= \left(-\phi^2 x^{-1} + \frac{\phi}{3} x^{-3} - \frac{1}{5} x^{-5}\right)\Bigg|_{N+1}^{\infty}$$

$$= (0 + 0 + 0) - \left(-\frac{\phi^2}{N+1} + \frac{\phi}{3} \cdot \frac{1}{(N+1)^3} - \frac{1}{5(N+1)^5}\right)$$

$$= \frac{\phi}{N+1} - \frac{\phi}{3(N+1)^3} + \frac{1}{5(N+1)^5}$$

so that we require that

$$\phi^{\frac{5}{2}}\left[\tan^{-1}\sqrt{\phi}(N+1) - \frac{\pi}{2}\right] + \frac{\phi^2}{N+1} - \frac{\phi}{3(N+1)^3}$$

$$+ \frac{1}{5(N+1)^5} \leq \epsilon\left(\frac{1}{1+\phi}\right)$$

For $\epsilon = 10^{-6}$ the following table is generated:

ϕ	N
0	11
0.0001	11
0.001	11
0.01	10
0.1	8
1	6
10	5
∞	5

Table 5.2 - Perturbation vs. No. of Required Modes

Therefore, it appears that <u>if we could design damping to increase with frequency, significant model reduction results.</u>

C) Model Reduction Index - General Constant Damping

We now seek to establish a closed-form expression for the modal cost function when the damping ratios $\zeta_i = \zeta$ a constant but general in magnitude. For this purpose, we return to equations (5.3), (5.4), (5.5) and recall that for $1 \le i \le n$, we have

$$b_i = \frac{\beta_i}{-2j\omega_i\sqrt{1-\zeta_i^2}} \quad , \quad b_{i+n} = \frac{\beta_i}{2j\omega_i\sqrt{1-\zeta_i^2}}$$

so that

$$b_i^* = \frac{\beta_i^*}{2j\omega_i\sqrt{1-\zeta_i^2}} \quad , \quad b_{i+n}^* = \frac{\beta_i^*}{-2j\omega_i\sqrt{1-\zeta_i^2}}$$

Also, we have

$$c_i = \mathbf{P}_i + \mathbf{\Gamma}_i(-\zeta_i\omega_i + j\omega_i\sqrt{1-\zeta_i^2})$$

$$c_{i+n} = \mathbf{P}_i + \mathbf{\Gamma}_i(-\zeta_i\omega_i - j\omega_i\sqrt{1-\zeta_i^2})$$

so that

$$c_i^* = \mathbf{P}_i^* + \mathbf{\Gamma}_i^*(-\zeta_i\omega_i - j\omega_i\sqrt{1-\zeta_i^2})$$

$$c_{i+n}^* = \mathbf{P}_i^* + \mathbf{\Gamma}_i^*(-\zeta_i\omega_i + j\omega_i\sqrt{1-\zeta_i^2})$$

Now substitute these expressions for the input and output coefficients into equation (5.7).

For $1 \le i \le n$,

$$J_{x_i} = \sum_{k=1}^{2n} \frac{-(\mathbf{b}_i^* U \mathbf{b}_k)(\mathbf{c}_k^* Q \mathbf{c}_i)}{\overline{\lambda}_k + \lambda_i}$$

$$= \sum_{k=1}^{n} \frac{-(\mathbf{b}_i^* U \mathbf{b}_k)(\mathbf{c}_k^* Q \mathbf{c}_i)}{\overline{\lambda}_k + \lambda_i} + \sum_{k=n+1}^{2n} \frac{-(\mathbf{b}_i^* U \mathbf{b}_k)(\mathbf{c}_k^* Q \mathbf{c}_i)}{\overline{\lambda}_k + \lambda_i}$$

$$= \sum_{k=1}^{n} \frac{-(\mathbf{b}_i^* U \mathbf{b}_k)(\mathbf{c}_k^* Q \mathbf{c}_i)}{\overline{\lambda}_k + \lambda_i} + \sum_{k=1}^{n} \frac{-(\mathbf{b}_i^* U \mathbf{b}_{k+n})(\mathbf{c}_{k+n}^* Q \mathbf{c}_i)}{\overline{\lambda}_{k+n} + \lambda_i}$$

We note that for $1 \le i \le n$, we have

$$\overline{\lambda}_{k+n} = \lambda_k \text{ and } \overline{\lambda}_k = \lambda_{k+n}$$

so that for $1 \le i \le n$, we get

$$J_{x_i} = \sum_{k=1}^{n} \frac{-\left[\dfrac{\boldsymbol{\beta}_i^*}{2j\omega_i\sqrt{1-\zeta^2}} U \dfrac{\boldsymbol{\beta}_k}{-2j\omega_k\sqrt{1-\zeta^2}}\right]}{-\zeta\omega_k - j\omega_k\sqrt{1-\zeta^2} - \zeta\omega_i + j\omega_i\sqrt{1-\zeta^2}}$$

$$\cdot\left[\mathbf{P}_k^* + \boldsymbol{\Gamma}_k^*(-\zeta\omega_k - j\omega_k\sqrt{1-\zeta^2})\right] Q \left[\mathbf{P}_i + \boldsymbol{\Gamma}_i(-\zeta\omega_i + j\omega_i\sqrt{1-\zeta^2})\right]$$

$$+ \sum_{k=1}^{n} \frac{-\left[\dfrac{\boldsymbol{\beta}_i^*}{2j\omega_i\sqrt{1-\zeta^2}} U \dfrac{\boldsymbol{\beta}_k}{2j\omega_k\sqrt{1-\zeta^2}}\right]}{-\zeta\omega_k + j\omega_k\sqrt{1-\zeta^2} - \zeta\omega_i + j\omega_i\sqrt{1-\zeta^2}}$$

$$\cdot\left[\mathbf{P}_k^* + \boldsymbol{\Gamma}_k^*(-\zeta\omega_k + j\omega_k\sqrt{1-\zeta^2})\right] Q \left[\mathbf{P}_i + \boldsymbol{\Gamma}_i(-\zeta\omega_i + j\omega_i\sqrt{1-\zeta^2})\right]$$

$$J_{x_i} = \sum_{k=1}^{n} \Big[\frac{-\boldsymbol{\beta}_i^* U \boldsymbol{\beta}_k}{4\omega_i\omega_k(1-\zeta^2)\left[-\zeta\omega_i - \zeta\omega_k + j(\omega_i - \omega_k)\sqrt{1-\zeta^2}\right]}$$

$$\cdot \, [\mathbf{P}_k^* Q \mathbf{P}_i + \zeta^2\omega_i\omega_k(\boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i) - j\zeta\omega_i\omega_k\sqrt{1-\zeta^2}(\boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i)$$

$$+ j\zeta\omega_i\omega_k\sqrt{1-\zeta^2}(\boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i) + \omega_i\omega_k(1-\zeta^2)(\boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_k)]$$

$$+ \frac{-\boldsymbol{\beta}_i^* U \boldsymbol{\beta}_k}{-4\omega_i\omega_k(1-\zeta^2)\left[-\zeta\omega_i - \zeta\omega_k + j(\omega_i + \omega_k)\sqrt{1-\zeta^2}\right]}$$

$$\cdot \, [\mathbf{P}_k^* Q \mathbf{P}_i + \zeta^2\omega_i\omega_k(\boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i) - j\zeta\omega_i\omega_k\sqrt{1-\zeta^2}(\boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i)$$

$$-j\zeta\omega_i\omega_k\sqrt{1-\zeta^2}(\boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i) - \omega_i\omega_k(1-\zeta^2)(\boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i)]\Big]$$

where we note terms such as $\mathbf{P}_k^* Q \boldsymbol{\Gamma}_i$ and $\boldsymbol{\Gamma}_k^* Q \mathbf{P}_i$ must equal zero.

For example, $\mathbf{P}_k^* Q \boldsymbol{\Gamma}_i = \begin{bmatrix} \boldsymbol{p}_k^* & 0 \end{bmatrix} \begin{bmatrix} Q_p & 0 \\ 0 & Q_r \end{bmatrix} \begin{bmatrix} 0 \\ \boldsymbol{r}_i \end{bmatrix} = 0$

Now we must find $J_{x_{i+n}}$. For $1 \le i \le n$, we have from equation (5.7):

$$J_{x_{i+n}} = \sum_{k=1}^{2n} \frac{-(\boldsymbol{b}_{i+n}^* U \boldsymbol{b}_k)(\boldsymbol{c}_k^* Q \boldsymbol{c}_{i+n})}{\bar{\lambda}_k + \lambda_{i+n}}$$

$$= \sum_{k=1}^{n} \frac{-(\boldsymbol{b}_{i+n}^* U \boldsymbol{b}_k)(\boldsymbol{c}_k^* Q \boldsymbol{c}_{i+n})}{\bar{\lambda}_k + \lambda_{i+n}} + \sum_{k=n+1}^{2n} \frac{-(\boldsymbol{b}_{i+n}^* U \boldsymbol{b}_k)(\boldsymbol{c}_k^* Q \boldsymbol{c}_{i+n}}{\bar{\lambda}_k + \lambda_{i+n}}$$

$$= \sum_{k=1}^{n} \frac{-(b_{i+n}^{*} U b_k)(c_k^{*} Q c_{i+n})}{\overline{\lambda}_k + \lambda_{i+n}} + \sum_{k=1}^{n} \frac{-(b_{i+n}^{*} U b_{k+n})(c_{k+n}^{*} Q c_{i+n})}{\overline{\lambda}_{k+n} + \lambda_{i+n}}$$

$$= \sum_{k=1}^{n} - \frac{\left[\dfrac{\beta_i^{*}}{-2j\omega_i\sqrt{1-\zeta^2}} U \dfrac{\beta_k}{-2j\omega_k\sqrt{1-\zeta_2}}\right]}{-\zeta\omega_k - j\omega_k\sqrt{1-\zeta^2} - \zeta\omega_i - j\omega_i\sqrt{1-\zeta^2}}$$

$$\cdot \left[P_k^{*} + \Gamma_k^{*}(-\zeta\omega_k - j\omega_k\sqrt{1-\zeta^2}\right] Q \left[P_i + \Gamma_i(-\zeta\omega_i - j\omega_i\sqrt{1-\zeta^2})\right]$$

$$+ \sum_{k=1}^{n} \frac{-\left[\dfrac{\beta_i^{*}}{-2j\omega_i\sqrt{1-\zeta^2}} U \dfrac{\beta_k}{-2j\omega_k\sqrt{1-\zeta^2}}\right]}{-\zeta\omega_k + j\omega_k\sqrt{1-\zeta^2} - \zeta\omega_i - j\omega_i\sqrt{1-\zeta^2}}$$

$$\cdot \left[P_k^{*} + \Gamma_k^{*}(-\zeta\omega_k + j\omega_k\sqrt{1-\zeta^2})\right] Q \left[P_i + \Gamma_i(-\zeta\omega_i - j\omega_i\sqrt{1-\zeta^2})\right]$$

$$= \sum_{k=1}^{n} \frac{-\beta_i^{*} U \beta_k}{-4\omega_i\omega_k(1-\zeta^2)\left[-\zeta(\omega_i + \omega_k) - j(\omega_i + \omega_k)\sqrt{1-\zeta^2}\right]}$$

$$\cdot \; [P_k^{*} Q P_i + \zeta^2\omega_k\omega_i(\Gamma_k^{*} Q \Gamma_i) + j\zeta\omega_i\omega_k\sqrt{1-\zeta^2}(\Gamma_k^{*} Q \Gamma_i)$$

$$+ j\zeta\omega_i\omega_k\sqrt{1-\zeta^2}(\Gamma_k^{*} Q \Gamma_i) - \omega_i\omega_k(1-\zeta^2)(\Gamma_k^{*} Q \Gamma_i)]$$

$$\cdot \left[P_k^{*} + \Gamma_k^{*}(-\zeta\omega_k - j\omega_k\sqrt{1-\zeta^2}\right] Q \left[P_i + \Gamma_i(-\zeta\omega_i - j\omega_i\sqrt{1-\zeta^2})\right]$$

$$+ \sum_{k=1}^{n} \frac{-\left[\dfrac{\beta_i^{*}}{-2j\omega_i\sqrt{1-\zeta^2}} U \dfrac{\beta_k}{-2j\omega_k\sqrt{1-\zeta^2}}\right]}{-\zeta\omega_k + j\omega_k\sqrt{1-\zeta^2} - \zeta\omega_i - j\omega_i\sqrt{1-\zeta^2}}$$

$$\cdot\left[\mathbf{P}_k^* + \boldsymbol{\Gamma}_k^*(-\zeta\omega_k + j\omega_k\sqrt{1-\zeta^2})\right]Q\left[\mathbf{P}_i + \boldsymbol{\Gamma}_i(-\zeta\omega_i - j\omega_i\sqrt{1-\zeta^2})\right]$$

$$= \sum_{k=1}^{n} \frac{-\boldsymbol{\beta}_i^* U \boldsymbol{\beta}_k}{-4\omega_i\omega_k(1-\zeta^2)\left[-\zeta(\omega_i + \omega_k) - j(\omega_i + \omega_k)\sqrt{1-\zeta^2}\right]}$$

$$\cdot\ [\mathbf{P}_k^* Q \mathbf{P}_i + \zeta^2\omega_k\omega_i(\boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i) + j\zeta\omega_i\omega_k\sqrt{1-\zeta^2}(\boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i)$$

$$+ j\zeta\omega_i\omega_k\sqrt{1-\zeta^2}(\boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i) - \omega_i\omega_k(1-\zeta^2)(\boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i)]$$

$$+ \sum_{k=1}^{n} \frac{-\boldsymbol{\beta}_i^* U \boldsymbol{\beta}_k}{4\omega_i\omega_k(1-\zeta^2)\left[-\zeta(\omega_i + \omega_k) + j(\omega_k - \omega_i)\sqrt{1-\zeta^2}\right]}$$

$$\cdot\ [\mathbf{P}_k^* Q \mathbf{P}_i + \zeta^2\omega_k\omega_i(\boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i) + j\zeta\omega_i\omega_k\sqrt{1-\zeta^2}(\boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i)$$

$$- j\zeta\omega_i\omega_k\sqrt{1-\zeta^2}(\boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i) + \omega_i\omega_k(1-\zeta^2)(\boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i)]$$

$$J_{x_{i \cdot n}} = \sum_{k=1}^{n} [\frac{-\boldsymbol{\beta}_i^* U \boldsymbol{\beta}_k}{-4\omega_i\omega_k(1-\zeta^2)[-\zeta(\omega_i + \omega_k) - j(\omega_i + \omega_k)\sqrt{1-\zeta^2}}$$

$$\cdot\ [\ \mathbf{P}_k^* Q \mathbf{P}_i + \zeta^2\omega_i\omega_k(\boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}) + j\zeta\omega_i\omega_k\sqrt{1-\zeta^2}(\boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i)$$

$$+ j\zeta\omega_i\omega_k\sqrt{1-\zeta^2}(\boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i) - \omega_i\omega_k(1-\zeta^2)(\boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i)]$$

$$+ \frac{-\boldsymbol{\beta}_i^* U \boldsymbol{\beta}_k}{4\omega_i\omega_k(1-\zeta^2)\left[-\zeta(\omega_i + \omega_k) + j(\omega_k - \omega_i)\sqrt{1-\zeta^2}\right]}$$

$$\cdot\ [\mathbf{P}_k^* Q \mathbf{P}_i + \zeta^2\omega_i\omega_k(\boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i) + j\omega_i\omega_k\zeta\sqrt{1-\zeta^2}(\boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i)$$

$$-j\zeta\omega_i\omega_k\sqrt{1-\zeta^2}(\boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i) + \omega_i\omega_k(1-\zeta^2)(\boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_k)]\,]$$

The cost corresponding to the ith 2nd-order mode is given by

$$J_{\eta_i} = J_{x_i} + J_{x_{i+n}}$$

so that

$$J_{\eta_i} = \sum_{i=1}^{n} \Bigg[\frac{(-\boldsymbol{\beta}_i^* U \boldsymbol{\beta}_k)(\mathbf{P}_k^* Q \mathbf{P}_i + \omega_i \omega_k \boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i)}{4\omega_i \omega_i (1-\zeta^2)\left[-\zeta\omega_i - \zeta\omega_k + j(\omega_i - \omega_k)\sqrt{1-\zeta^2}\right]}$$

$$+ \frac{(-\boldsymbol{\beta}_i^* U \boldsymbol{\beta}_k)(\mathbf{P}_k^* Q \mathbf{P}_i + \omega_i \omega_k \boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i)}{4\omega_i \omega_i (1-\zeta^2)\left[-\zeta\omega_i - \zeta\omega_k - j(\omega_i - \omega_k)\sqrt{1-\zeta^2}\right]}$$

$$+ \frac{(-\boldsymbol{\beta}_i^* U \boldsymbol{\beta}_k)[\mathbf{P}_k^* Q \mathbf{P}_i + (2\zeta^2 - 1 - 2j\zeta\sqrt{1-\zeta^2})\omega_i \omega_k \boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i]}{-4\omega_i \omega_i (1-\zeta^2)\left[-\zeta\omega_i - \zeta\omega_k + j(\omega_i + \omega_k)\sqrt{1-\zeta^2}\right]}$$

$$+ \frac{(-\boldsymbol{\beta}_i^* U \boldsymbol{\beta}_k)[\mathbf{P}_k^* Q \mathbf{P}_i + (2\zeta^2 - 1 + 2j\zeta\sqrt{1-\zeta^2})\omega_i \omega_k \boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i]}{-4\omega_i \omega_i (1-\zeta^2)\left[-\zeta\omega_i - \zeta\omega_k - j(\omega_i + \omega_k)\sqrt{1-\zeta^2}\right]} \Bigg]$$

We note that the terms have been organized in complex conjugate pairs. We perform these sums in the following way

$$\frac{A + Bj}{C + Dj} + \frac{A - Bj}{C - Dj} = \frac{2AC + 2BD}{C^2 + D^2}$$

Therefore

$$J_{\eta_i} = \sum_{k=1}^{n} [\frac{-(\boldsymbol{\beta}_i^* U \boldsymbol{\beta}_k)(\mathbf{P}_k^* Q \mathbf{P}_i + \omega_i \omega_k \boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i)[-2\zeta(\omega_i + \omega_k)]}{4\omega_i\omega_k(1-\zeta^2)[\zeta^2(\omega_i + \omega_k)^2 + (\omega_i - \omega_k)^2(1-\zeta^2)]}$$

$$+ \frac{-\boldsymbol{\beta}_i U \boldsymbol{\beta}_k}{-4\omega_i\omega_k(1-\zeta^2)} [\frac{2[\mathbf{P}_k^* Q \mathbf{P}_i + (2\zeta^2-1)\omega_i\omega_k(\boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i)][-\zeta(\omega_i + \omega_k)]}{\zeta^2(\omega_i + \omega_k)^2 + (\omega_i + \omega_k)^2(1-\zeta^2)}$$

$$+ \frac{2[-2\zeta\sqrt{1-\zeta^2}\omega_i\omega_k(\boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i)(\omega_i + \omega_k)\sqrt{1-\zeta^2})]}{\zeta^2(\omega_i + \omega_k)^2 + (\omega_i + \omega_k)^2(1-\zeta^2)}]]$$

$$= \sum_{k=1}^{n} [\frac{-(\boldsymbol{\beta}_i^* U \boldsymbol{\beta}_k)(\mathbf{P}_k^* Q \mathbf{P}_i + \omega_i \omega_k \boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i)[-2\zeta(\omega_i + \omega_k)]}{4\omega_i\omega_k(1-\zeta^2)[\zeta^2(\omega_i + \omega_k)^2 + (\omega_i - \omega_k)^2(1-\zeta^2)}$$

$$+ \frac{-\boldsymbol{\beta}_i U \boldsymbol{\beta}_k}{-4\omega_i\omega_k(1-\zeta^2)} [\frac{-2[\mathbf{P}_i^* Q \mathbf{P}_i(\omega_i+\omega_k) - 2\zeta(2\zeta^2-1)\omega_i\omega_k(\omega_i+\omega_k)(\boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i)}{(\omega_i + \omega_k)^2}$$

$$+ \frac{2[-2\zeta\sqrt{1-\zeta^2}\omega_i\omega_k(\boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i)(\omega_i + \omega_k)\sqrt{1-\zeta^2})]}{(\omega_i + \omega_k)^2}]]$$

Simplifying the 2nd term, we get

$$\frac{\boldsymbol{\beta}_i U \boldsymbol{\beta}_k}{4\omega_i\omega_k(1-\zeta^2)} [\frac{-2\zeta(\omega_i+\omega_k)\mathbf{P}_k^* Q \mathbf{P}_i + (-4\zeta^3 + 2\zeta)\omega_i\omega_k(\omega_i+\omega_k)(\boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i)}{(\omega_i + \omega_k)^2}$$

$$+ \frac{(-4\zeta + 4\zeta^3)\omega_i\omega_k(\omega_i + \omega_k)(\boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i)}{(\omega_i + \omega_k)^2}]$$

$$= \frac{\boldsymbol{\beta}_i U \boldsymbol{\beta}_k}{4\omega_i\omega_k(1-\zeta^2)} \left[\frac{-2\zeta(\omega_i + \omega_k)(\mathbf{P}_k^* Q \mathbf{P}_i) - 2\zeta(\omega_i\omega_k)(\omega_i + \omega_k)(\boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i)}{(\omega_i + \omega_k)^2}\right]$$

$$= \frac{\boldsymbol{\beta}_i U \boldsymbol{\beta}_k}{4\omega_i\omega_k(1-\zeta^2)} \cdot \frac{-2\zeta(\mathbf{P}_k^* Q \mathbf{P}_i + \omega_i\omega_k \boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i)(\omega_i + \omega_k)}{(\omega_i + \omega_k)^2}$$

$$= \frac{\boldsymbol{\beta}_i U \boldsymbol{\beta}_k}{4\omega_i\omega_k(1-\zeta^2)} \cdot \frac{-2\zeta(\mathbf{P}_k^* Q \mathbf{P}_i + \omega_i\omega_k \boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i)}{\omega_i + \omega_k}$$

Simplifying the first term, we get

$$\frac{-(\boldsymbol{\beta}_i U \boldsymbol{\beta}_k)}{4\omega_i\omega_k(1-\zeta^2)} \cdot \frac{-2\zeta(\mathbf{P}_k^* Q \mathbf{P}_i + \omega_i\omega_k \boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i)(\omega_i + \omega_k)}{(\omega_i - \omega_k)^2 + 4\zeta^2\omega_i\omega_k}$$

Combining these terms, we get

$$J_{\eta_i} = 2\zeta \sum_{k=1}^{n} \frac{(\boldsymbol{\beta}_i U \boldsymbol{\beta}_k)(\mathbf{P}_k^* Q \mathbf{P}_i + \omega_i\omega_k \boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i)}{4\omega_i\omega_k(1-\zeta^2)} \cdot \left[\frac{(\omega_i + \omega_k)^2 - [(\omega_i - \omega_k)^2 + 4\zeta^2\omega_i\omega_k]}{(\omega_i + \omega_k)[(\omega_i - \omega_k)^2 + 4\zeta^2\omega_i\omega_k]}\right]$$

$$= 2\zeta \sum_{k=1}^{n} \frac{(\boldsymbol{\beta}_i U \boldsymbol{\beta}_k)(\mathbf{P}_k^* Q \mathbf{P}_i + \omega_i\omega_k \boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i)}{4\omega_i\omega_k(1-\zeta^2)} \cdot \left[\frac{4\omega_i\omega_k - 4\zeta^2\omega_i\omega_k}{(\omega_i + \omega_k)[(\omega_i - \omega_k)^2 + 4\zeta^2\omega_i\omega_k]}\right]$$

which means that the <u>ith 2nd-order modal cost</u> is

$$J_{\eta_i} = 2\zeta \sum_{k=1}^{n} \frac{(\boldsymbol{\beta}_i^* U \boldsymbol{\beta}_k)(\mathbf{P}_k^* Q \mathbf{P}_i + \omega_i\omega_k \boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i)}{(\omega_i + \omega_k)\left[(\omega_i - \omega_k)^2 + 4\zeta^2\omega_i\omega_k\right]} \qquad (5.14)$$

and therefore the <u>total modal cost</u> is

$$J = 2\zeta \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{(\boldsymbol{\beta}_i^* U \boldsymbol{\beta}_k)\,(\mathbf{P}_k^* Q \mathbf{P}_i + \omega_i \omega_k \boldsymbol{\Gamma}_k^* Q \boldsymbol{\Gamma}_i)}{(\omega_i + \omega_k)\left[(\omega_i - \omega_k)^2 + 4\zeta^2 \omega_i \omega_k\right]} \quad (5.15)$$

and now by making use of our earlier definitions, the total modal cost can written in the form

$$J = 2\zeta \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{(\boldsymbol{\beta}_i^* U \boldsymbol{\beta}_k)\,(\mathbf{p}_k^* Q_p \mathbf{p}_i + \omega_i \omega_k \mathbf{r}_k^* Q_r \mathbf{r}_i)}{(\omega_i + \omega_k)\left[(\omega_i - \omega_k)^2 + 4\zeta^2 \omega_i \omega_k\right]} \quad (5.16)$$

We emphasize that equation (5.16) is valid for constant damping that is general in magnitude. It is interesting to note that the expression contains a non-linearity which the corresponding formula for light damping did not possess. This means that for heavier damping, there is at least the hope that damping, even though constant throughout the modes, may yet produce a differential effect on the relative contributions of the different modes to the cost function.

Reduction to Formula for Light Damping

When damping is light, the only terms of any magnitude contributing to the cost function are those for which $i = k$. In this case $\omega_i = \omega_k$, and we get

$$J = 2\zeta \sum_{i=1}^{n} \frac{(\beta_i^* U \beta_i)(p_i^* Q_p p_i + \omega_i \omega_i r_i^* Q_r r_i)}{(2\omega_i)(4\zeta^2 \omega_i^2)}$$

$$= \sum_{i=1}^{n} \frac{(\beta_i^* U \beta_i)(p_i^* Q_p p_i + \omega_i^2 r_i^* Q_r r_i)}{4\zeta \omega_i^3}$$

which is Skelton's [34] formula for light damping when $\zeta_i = \zeta$ = constant.

Example Problem

We apply the theory to a simply-supported beam modeled with two damped modes. We shall assume a single input and a single output, and let $U = Q = 1$. In this situation, equation (5.16) becomes

$$J = \frac{2\zeta(\beta_1^* U \beta_1)(p_1^* Q_p p_1)}{8\zeta^2 \omega_1^3} + \frac{2\zeta(\beta_1^* U \beta_2)(p_2^* Q_p p_1)}{(\omega_1 + \omega_2)\left[(\omega_1 - \omega_2)^2 + 4\zeta^2 \omega_1 \omega_2\right]}$$

$$+ \frac{2\zeta(\beta_1^* U \beta_1)(p_1^* Q_p p_2)}{(\omega_2 + \omega_1)\left[(\omega_2 - \omega_1)^2 + 4\zeta^2 \omega_2 \omega_1\right]} + \frac{2\zeta(\beta_2^* U \beta_2)(p_2^* Q_p p_2)}{8\zeta^2 \omega_2^3}$$

$$= \frac{(\beta_1^* \beta_1)(p_1^* p_1)}{4\zeta \omega_1^3} + \frac{(\beta_2^* \beta_2)(p_2^* p_2)}{4\zeta \omega_2^3}$$

$$+ \frac{4\zeta(\beta_1^* \beta_2)(p_1^* p_2)}{(\omega_1 + \omega_2)\left[(\omega_1 - \omega_2)^2 + 4\zeta^2 \omega_1 \omega_2\right]}$$

We note that the first two terms are those diagonal terms corresponding to light damping. The third term is the cross term newly introduced by our theory. We will use several different values for damping and output locations as we apply the formulas. We assume the following values:

$$L = \pi, \quad d = \frac{2}{L}, \quad EI = d, \quad u = \text{torque}\,\delta(t), \quad r_u = 0,$$

$$\omega_i = \sqrt{\frac{EI}{d}\left(\frac{i\pi}{L}\right)^2} = i^2, \quad i = 1,2 \quad \text{and} \quad y = \sum_{i=1}^{2} p_i \eta_i .$$

From the theory of the beam, we have

$$\beta_i = \frac{\pi i}{L}\sqrt{\frac{2}{\rho L}}\cos\left(\frac{\pi r_u}{L} i\right) = i$$

$$p_i = \sqrt{\frac{2}{\rho L}}\sin\left(\frac{\pi r_0}{L} i\right) = \sin(r_0 i)$$

The following results were obtained:

ζ / r_0	0.45L	0.20L	0.33L
0.005	*1st*: 48.7808	*1st*: 17.2594	*1st*: 37.0559
	2nd: 3.0488	*2nd*: 1.0787	*2nd*: 2.3160
	cross: 0.0003	*cross*: 0.0005	*cross*: 0.0007
0.05	*1st*: 4.8781	*1st*: 1.7259	*1st* 3.7059
	2nd: 0.3049	*2nd*: 0.1079	*2nd*: 0.2316
	cross: 0.0021	*cross*: 0.0049	*cross*: 0.0067
0.5	*1st*: 0.4878	*1st*: 0.1726	*1st*: 0.3706
	2nd: 0.0305	*2nd*: 0.0108	*2nd*: 0.0232
	cross: 0.0188	*cross*: 0.0344	*cross*: 0.0464

Table 5.3 - Cost Function Values Vs. Damping Ratios

The data indicate that even for frequencies as relatively diverse as those assumed (1 Hz.and 4 Hz.), the cross term contribution to the total cost becomes evident as the damping achieves a value of 0.5. With more concentrated natural frequencies, one can expect the effects to be present at much lower values of the damping ratio.

Error System

The model reduction index defined as $I = J_e/J$ requires an expression for J_e, the modal costs associated with the reduction error. The output error is defined as

$$\boldsymbol{y_e} = \boldsymbol{y} - \boldsymbol{y_R}$$

where $\boldsymbol{y}$ = output of system

$\boldsymbol{y_R}$ = output of reduced model

We will now derive an expression for J_e, the costs associated with the output error. Consider the error system

$$\dot{\boldsymbol{x}}_e = \begin{pmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{x}}_R \end{pmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_R \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x}_R \end{bmatrix} + \begin{bmatrix} B \\ B_R \end{bmatrix} \boldsymbol{u}$$

$$= A_e \boldsymbol{x}_e + B_e \boldsymbol{u}$$

$$\boldsymbol{y}_e = \boldsymbol{y} - \boldsymbol{y}_R = \begin{bmatrix} C & -C_R \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x}_R \end{bmatrix}$$

and

$$0 = X_e A_e^* + A_e X_e + B_e B_e^*$$

We know that

$$A = diag[\lambda_1, \lambda_2, \dots \lambda_{2n}]$$

$$A_R = diag[\lambda_1, \lambda_2, \dots \lambda_{2r}]$$

so that

$$A_e = diag[\lambda_1, \lambda_2, \dots \lambda_{2n}, \lambda_1, \lambda_2, \dots \lambda_{2r}]$$

$$\boldsymbol{x_e} = [x_1, x_2 \dots x_{2n}, x_1, x_2, \dots x_{2r}]^T$$

$$B_e = [\boldsymbol{b_1}, \boldsymbol{b_2}, \dots \boldsymbol{b_{2n}}, \boldsymbol{b_1}, \boldsymbol{b_2}, \dots \boldsymbol{b_{2r}}]^T$$

$$C = [\boldsymbol{c_1}, \boldsymbol{c_2}, \dots \boldsymbol{c_n}, \boldsymbol{c_{n+1}}, \boldsymbol{c_{n+2}}, \dots \boldsymbol{c_{2n}}]$$

$$C_R = [\boldsymbol{c_1}, \boldsymbol{c_2} \dots \boldsymbol{c_r}, \boldsymbol{c_{n+1}}, \boldsymbol{C_{n+2}}, \dots \boldsymbol{c_{n+r}}]$$

so that

$$C_e = [\boldsymbol{c_1}, \boldsymbol{c_2}, \dots \boldsymbol{c_n}, \boldsymbol{c_{n+1}}, \boldsymbol{c_{n+2}}, \dots \boldsymbol{c_{2n}}, -\boldsymbol{c_1}, -\boldsymbol{c_2}, \dots -\boldsymbol{c_r}, -\boldsymbol{c_{n+1}}, -\boldsymbol{c_{n+2}}, \dots -\boldsymbol{c_{n+r}}]$$

which makes the error system output

$$\boldsymbol{y_e} = C_e \boldsymbol{x_e} = \boldsymbol{c_1} x_1 + \boldsymbol{c_2} x_2 + \dots + \boldsymbol{c_n} x_n$$

$$+ \boldsymbol{c_{n+1}} x_{n+1} + \boldsymbol{c_{n+2}} x_{n+2} + \dots + \boldsymbol{c_{2n}} x_{2n}$$

$$- c_1 x_1 - c_2 x_2 \dots c_r x_r - c_{n+1} x_{n+1}$$

$$- c_{n+2} x_{n+2} \dots - c_{n+r} x_{n+r}$$

$$= c_{r+1} x_{r+1} + c_{r+2} x_{r+2} + \dots + c_n x_n$$

$$+ c_{n+r+1} x_{n+r+1} + c_{n+r+2} x_{n+r+2} + \dots + c_{2n} x_{2n}$$

Therefore

$$y_e = [0,0,0,\dots 0, c_{r+1}, c_{r+2}, \dots c_n, 0,0,0,\dots 0, c_{n+r+1}, c_{n+r+2}, \dots c_{2n}] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \\ x_{r+1} \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_{n+r} \\ x_{n+r+1} \\ \vdots \\ x_{2n} \end{bmatrix}$$

(5.17)

but recall that

$$c_i = \mathbf{P}_i + \mathbf{\Gamma}_i(-\zeta_i\omega_i \pm j\omega_i\sqrt{1-\zeta^2})$$

Since $\mathbf{P}_i$ and $\mathbf{\Gamma}_i$ are independent, then

$$c_i = 0 \Rightarrow \begin{matrix} \mathbf{P}_i = 0 \\ \Gamma_i = 0 \end{matrix} \qquad (5.18)$$

If we apply the theorem on modal costs (ζ= constant.), we see that

$$J_e = 2\zeta \sum_{i=1}^{n+r} \sum_{k=1}^{n+r} \frac{(\boldsymbol{\beta}_i^* U \boldsymbol{\beta}_k)\left[\boldsymbol{p}_k^* Q_p \boldsymbol{p}_i + \omega_i \omega_k \boldsymbol{r}_k^* Q_r \boldsymbol{r}_i\right]}{(\omega_i + \omega_k)\left[(\omega_i - \omega_k)^2 + 4\zeta^2 \omega_i \omega_k\right]}$$

so that noting equations (5.17) and (5.18), we get

$$J_e = 2\zeta \sum_{i=r+1}^{n} \sum_{k=r+1}^{n} \frac{(\boldsymbol{\beta}_i^* U \boldsymbol{\beta}_k)\left[\boldsymbol{p}_k^* Q_p \boldsymbol{p}_i + \omega_i \omega_k \boldsymbol{r}_k^* Q_r \boldsymbol{r}_i\right]}{(\omega_i + \omega_k)\left[(\omega_i - \omega_k)^2 + 4\zeta^2 \omega_i \omega_k\right]}$$

and the model reduction index becomes

$$I = \frac{J_e}{J}$$

$$I = \frac{\displaystyle\sum_{i=r+1}^{n} \sum_{k=r+1}^{n} \frac{(\boldsymbol{\beta}_i^* U \boldsymbol{\beta}_k)\left[\boldsymbol{p}_k^* Q_p \boldsymbol{p}_i + \omega_i \omega_k \boldsymbol{r}_k^* Q_r \boldsymbol{r}_i\right]}{(\omega_i + \omega_k)\left[(\omega_i - \omega_k)^2 + 4\zeta^2 \omega_i \omega_k\right]}}{\displaystyle\sum_{i=1}^{n} \sum_{k=1}^{n} \frac{(\boldsymbol{\beta}_i^* U \boldsymbol{\beta}_k)\left[\boldsymbol{p}_k^* Q_p \boldsymbol{p}_i + \omega_i \omega_k \boldsymbol{r}_k^* Q_r \boldsymbol{r}_i\right]}{(\omega_i + \omega_k)\left[(\omega_i - \omega_k)^2 + 4\zeta^2 \omega_i \omega_k\right]}} \qquad (5.19)$$

<u>Graphs</u>

The model reduction index was applied to see if the non-linear term $4\zeta^2\omega_i\omega_k$ would yield some variation in the value of I as the magnitude of the damping ratio was changed. The input and output coefficients were all normalized to equal one, so that the effects produced were due only to changes in the damping ratio ζ.

As it turns out, the index does change, but the results are very mixed. Figs. 5.1 to 5.5 show graphs of the index versus the damping ratio ζ for various finite modal systems. The general trend in the graphs shows a decrease in the index with increasing ζ. In some cases the decrease is fairly substantial and rapid, but in others the decrease is very modest and occurs slowly. In still other cases, there is an increase up to about $\zeta = 0.2$ followed by a monotonic decrease. In one case shown, there is an actual increase in I as ζ increases.

One persistent pattern throughout the graphs indicates a monotonic decrease in ζ for the truncation of a single mode (Figs. 5.1(a,b). We now establish this fact formally.

<u>Theorem</u>

The model reduction index defined as $I = J_e/J$ decreases monotonically as ζ increases when $r = n-1$. It is assumed that $\omega_i \neq \omega_k$ for $i \neq k$.

Proof: Define

$$A_{ik} = (\boldsymbol{\beta}_i^* U \boldsymbol{\beta}_k)\,[\boldsymbol{p}_k^* Q \boldsymbol{p}_i + \omega_i \omega_k \boldsymbol{x}_k^* Q \boldsymbol{x}_i]$$

Therefore

$$I(\zeta) = \frac{\displaystyle\sum_{i=r+1}^{n}\sum_{k=r+1}^{n} \frac{A_{ik}}{(\omega_i + \omega_k)\left[(\omega_i - \omega_k)^2 + 4\zeta^2\omega_i\omega_k\right]}}{\displaystyle\sum_{i=1}^{n}\sum_{k=1}^{n} \frac{A_{ik}}{(\omega_i + \omega_k)^2\left[(\omega_i - \omega_k)^2 + 4\zeta^2\omega_i\omega_k\right]}}$$

Since $r = n-1$, this becomes

$$I(\zeta) = \frac{\dfrac{A_{nn}}{(2\omega_n)\left[(\omega_n - \omega_n)^2 + 4\zeta^2\omega_n^2\right]}}{\displaystyle\sum_{i=1}^{n}\sum_{k=1}^{n} \frac{A_{ik}}{(\omega_i + \omega_k)^2\left[(\omega_i - \omega_k)^2 + 4\zeta^2\omega_i\omega_k\right]}}$$

$$= \frac{\dfrac{A_{nn}}{8\zeta^2\omega_n^3}}{\displaystyle\sum_{i=1}^{n} \frac{A_{ii}}{8\zeta^2\omega_i^3} + \sum_{i=1}^{n}\sum_{\substack{k=1 \\ (k\neq i)}}^{n} \frac{A_{ik}}{(\omega_i + \omega_k)\left[(\omega_i - \omega_k)^2 + 4\zeta^2\omega_i\omega_k\right]}}$$

$$= \frac{\dfrac{A_{nn}}{8\omega_n^3}}{\displaystyle\sum_{i=1}^{n} \frac{A_{ii}}{8\omega_i^3} + \zeta^2\sum_{i=1}^{n}\sum_{\substack{k=1 \\ (k\neq i)}}^{n} \frac{A_{ik}}{(\omega_i + \omega_k)^2\left[(\omega_i - \omega_k)^2 + 4\zeta^2\omega_i\omega_k\right]}}$$

Examine the second term in the denominator.

$$\frac{d}{d\zeta}[2nd\ term] = \frac{(\omega_i + \omega_k)\ [(\omega_i - \omega_k)^2 + 4\omega_i\omega_k\zeta^2] \cdot 2A_{ik}\zeta}{D^2}$$

$$- \frac{A_{ik}\zeta^2 \cdot (\omega_i + \omega_k)\ (8\omega_i\omega_k\zeta)}{D^2}$$

$$= \frac{2\zeta(\omega_i + \omega_k)(\omega_i - \omega_k)^2 A_{ik}}{D^2}$$

and is > 0, since $\omega_i \neq \omega_k$ $(i \neq k)$. This means that I decreases monotonically with zeta.

Now we consider Figs. 5.2(a,b). These graphs show the variation in I for cases in which the modes are clustered, i.e., their separations are small compared to their magnitudes. Increased damping appears to cause the index to decrease very rapidly and then assume nearly a constant value. The rate of decline and limiting value of I appear to depend somewhat on the degree of system truncation and the degree of modal clustering.

In Figs. 5.3(a,b,c, and d), we see a series of graphs in which the index appears to increase to a certain value (approx. 0.2) and then starts decreasing. In these cases, the rate of decrease and the limiting value of I are dependent on the degree of system truncation.

Figs. 5.4(a,b) show some case groups in which the frequency ratios and degree of truncation are the same within each group. We note the similarities in the shapes and limiting values in the graphs within each group.

Figs. 5.5(a,b) show some cases in which at least some of the modal frequencies are widely separated from the others. The index seems to decrease very slowly and is almost invariant in cases in which the relative modal separation is great. One case pictured actually shows an increase in the index.

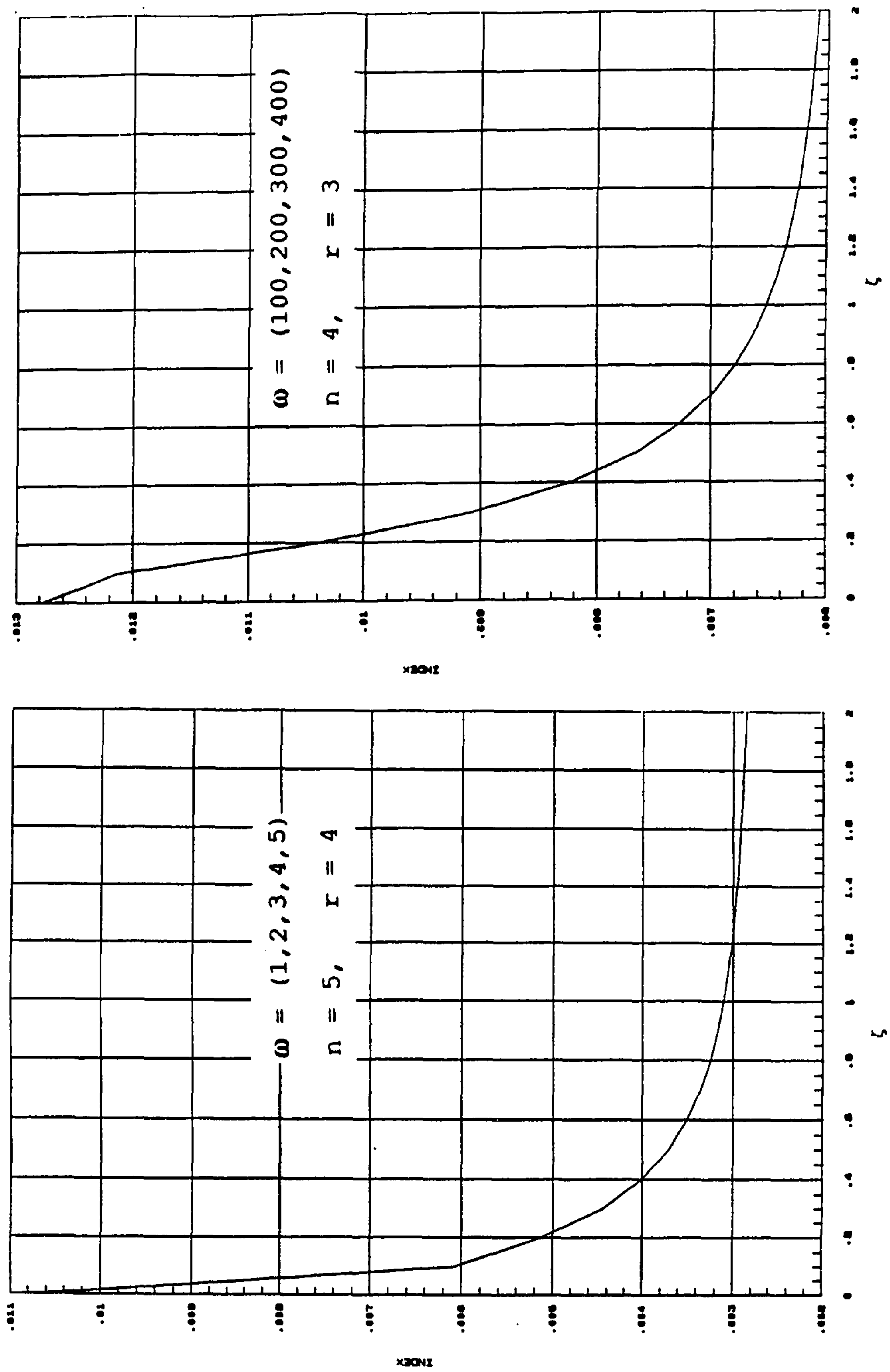


Figure 5.1(a) - Model Reduced by One Mode

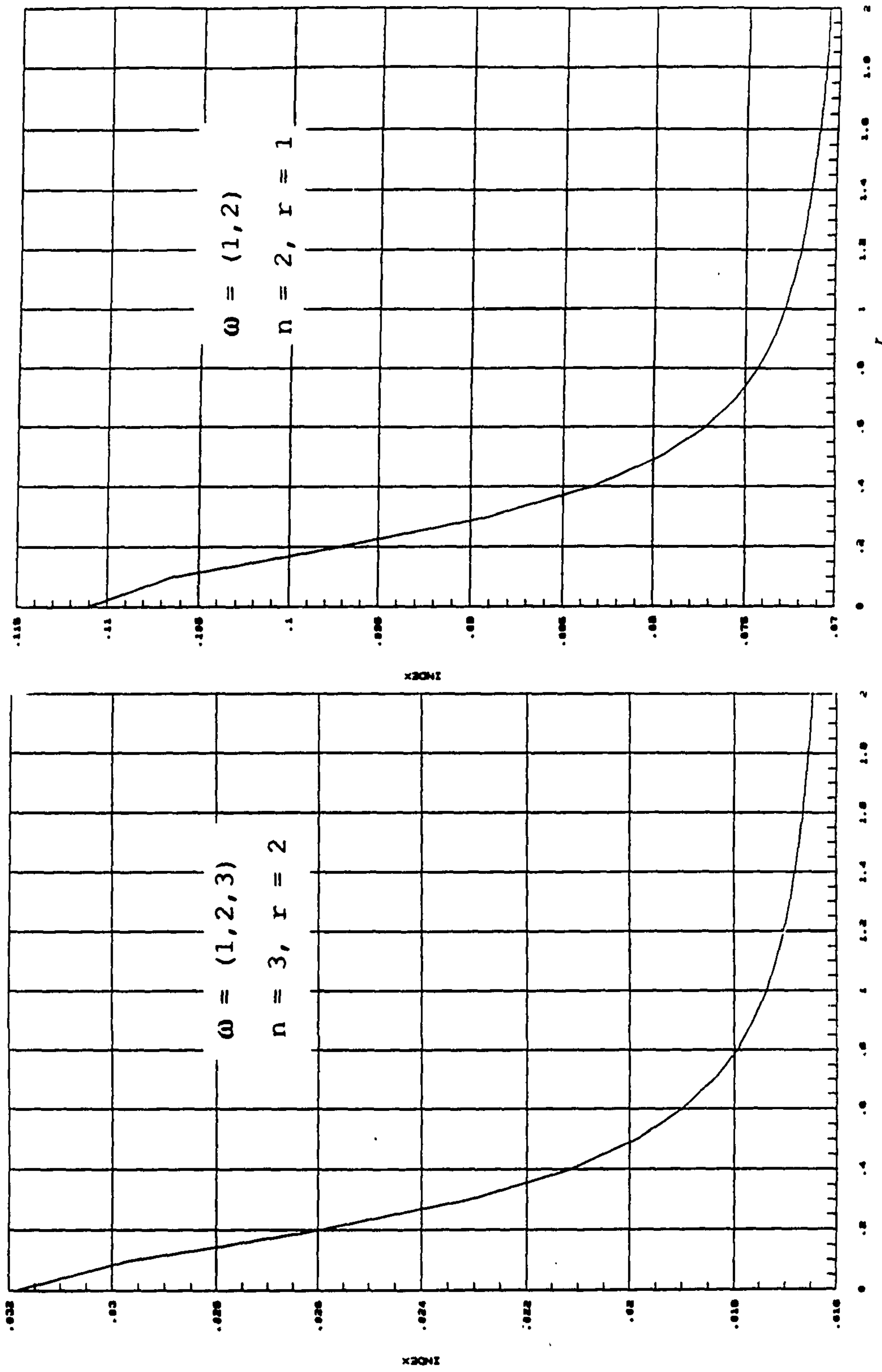


Figure 5.1(b) - Model Reduced by One Mode

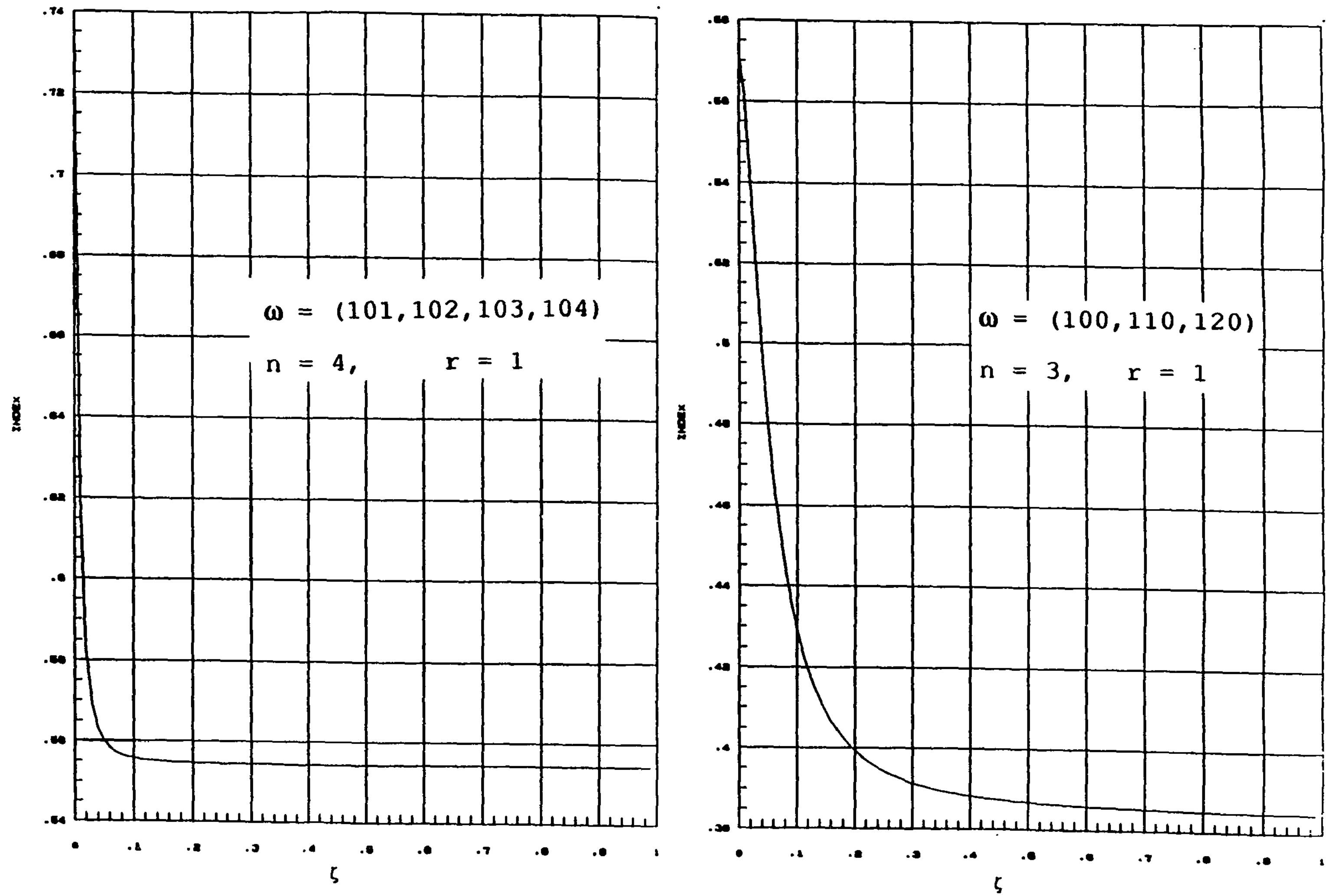


Figure 5.2(a) - Clustered Modes

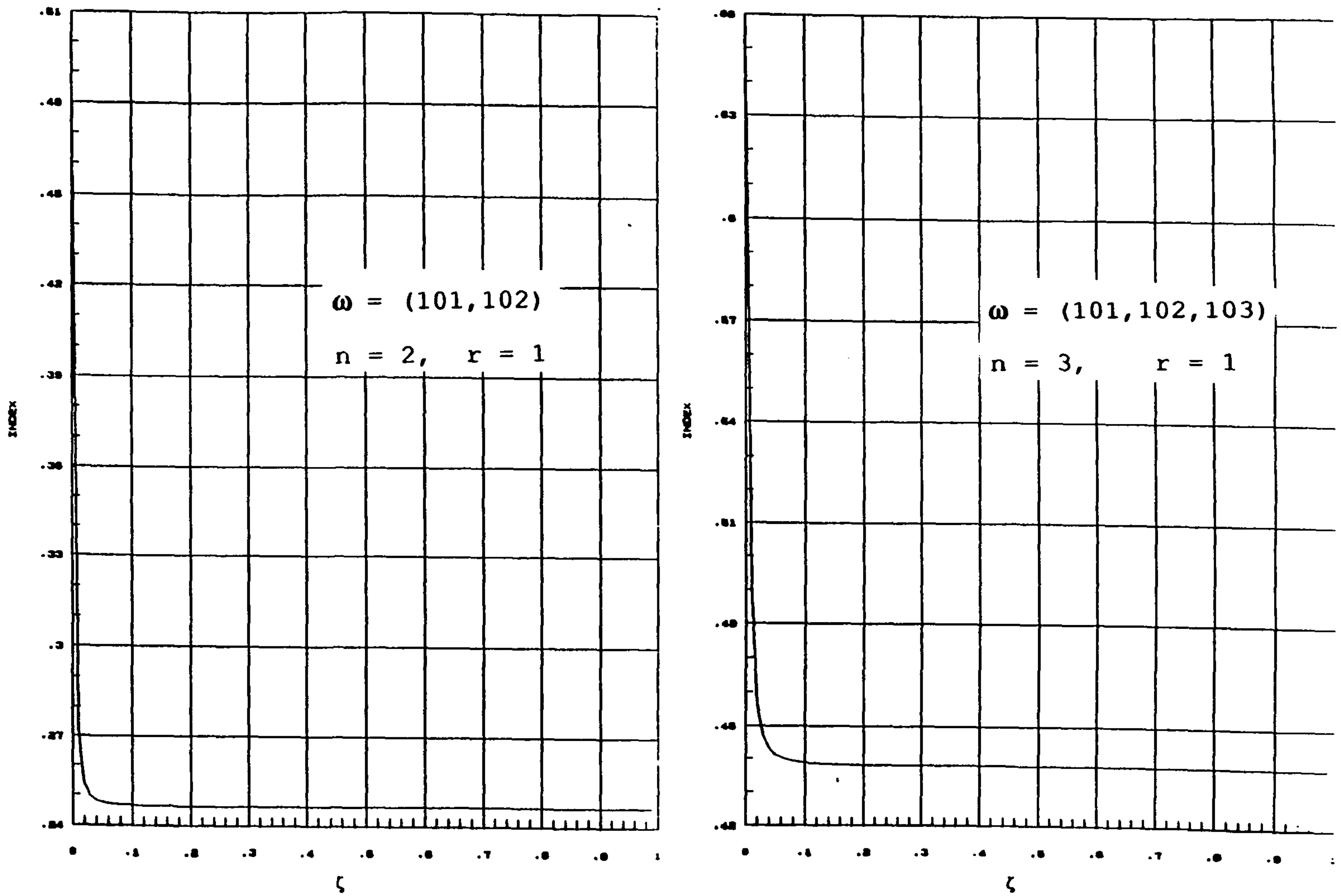


Figure 5.2(b) - Clustered Modes

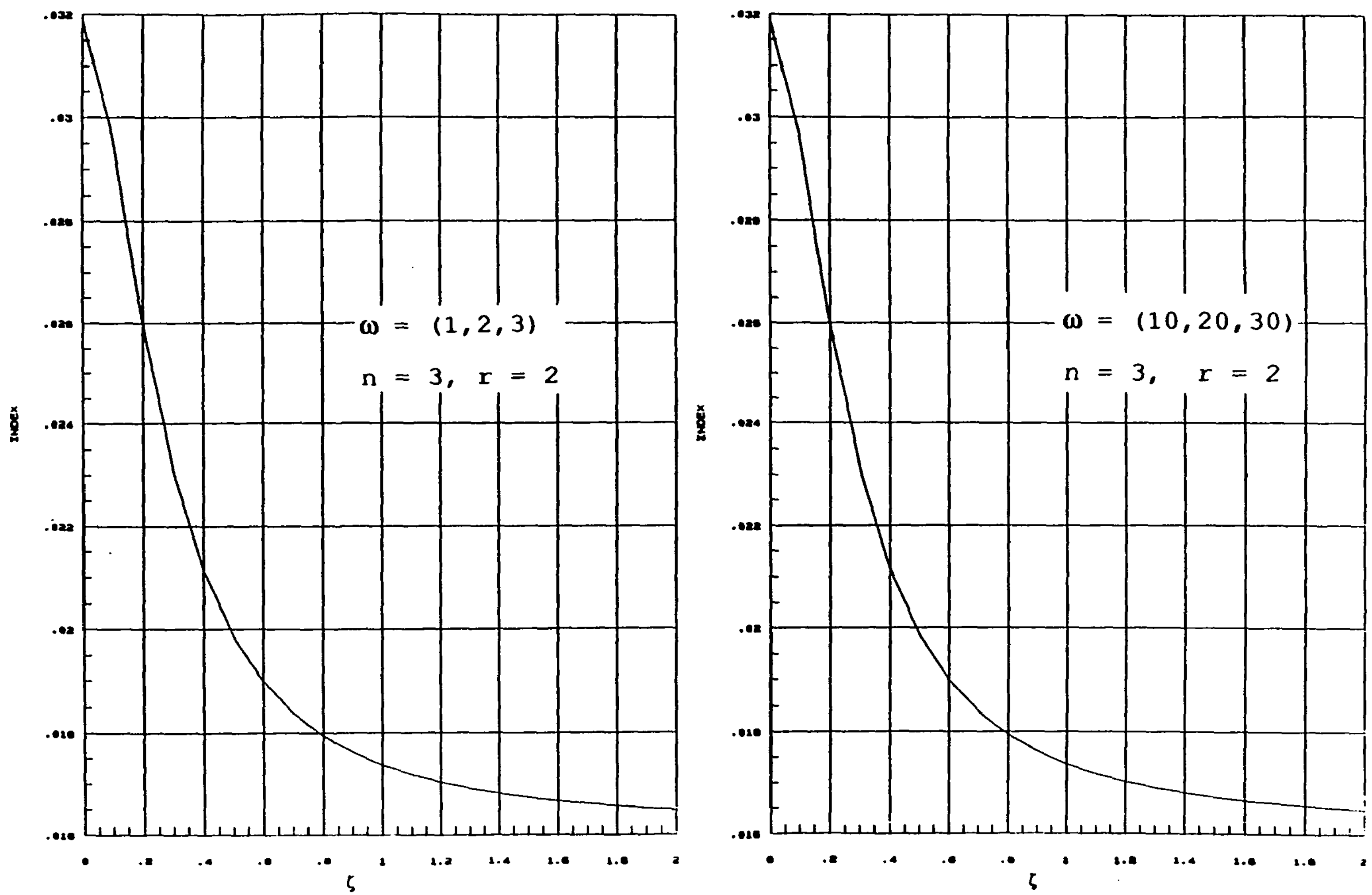


Figure 5.3(a) - Modal Frequencies in Same Ratio

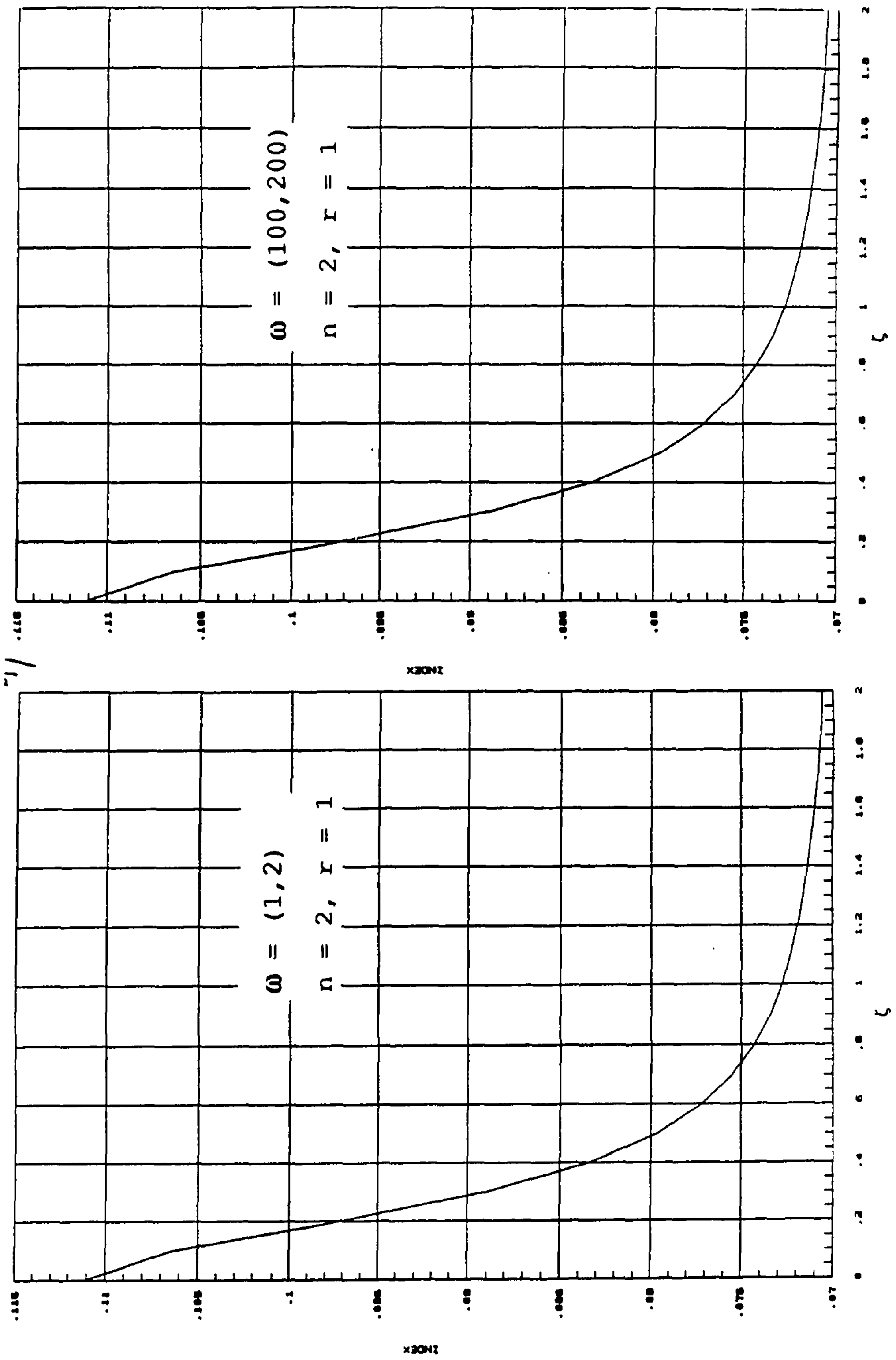


Figure 5.3(b) - Modal Frequencies in Same Ratio

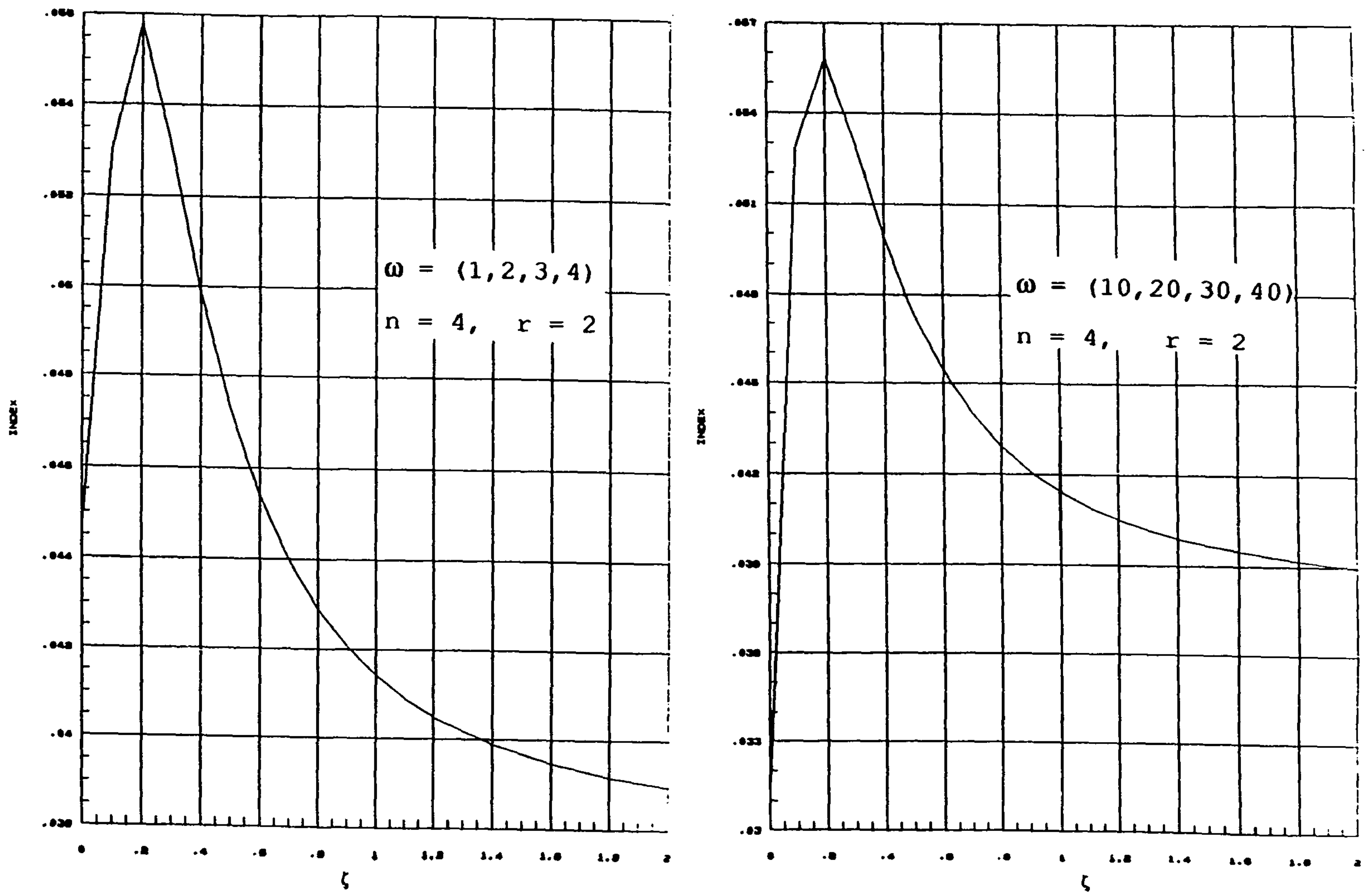


Figure 5.3(c) - Modal Frequencies in Same Ratio

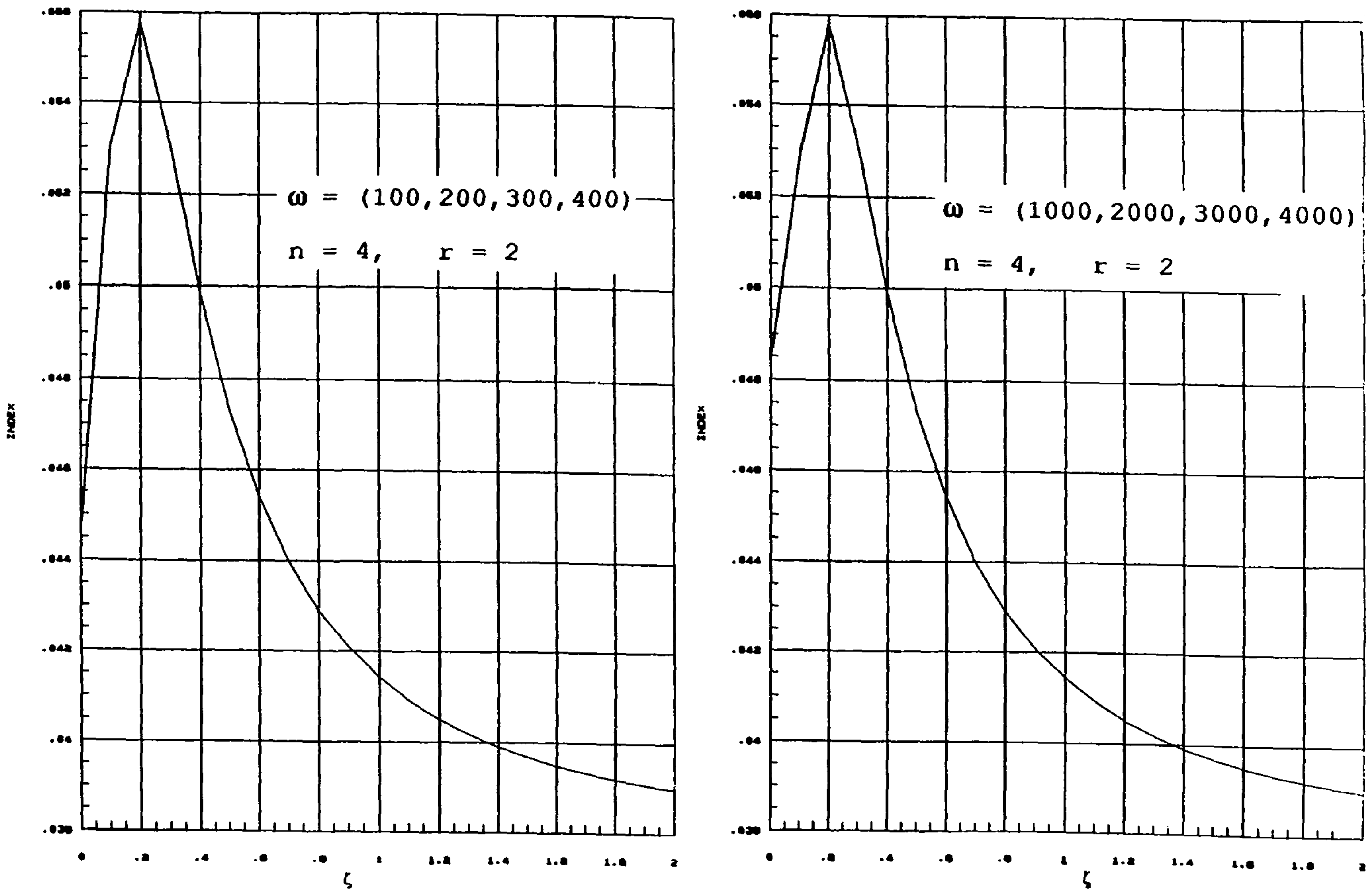


Figure 5.3(d) - Modal Frequencies in Same Ratio

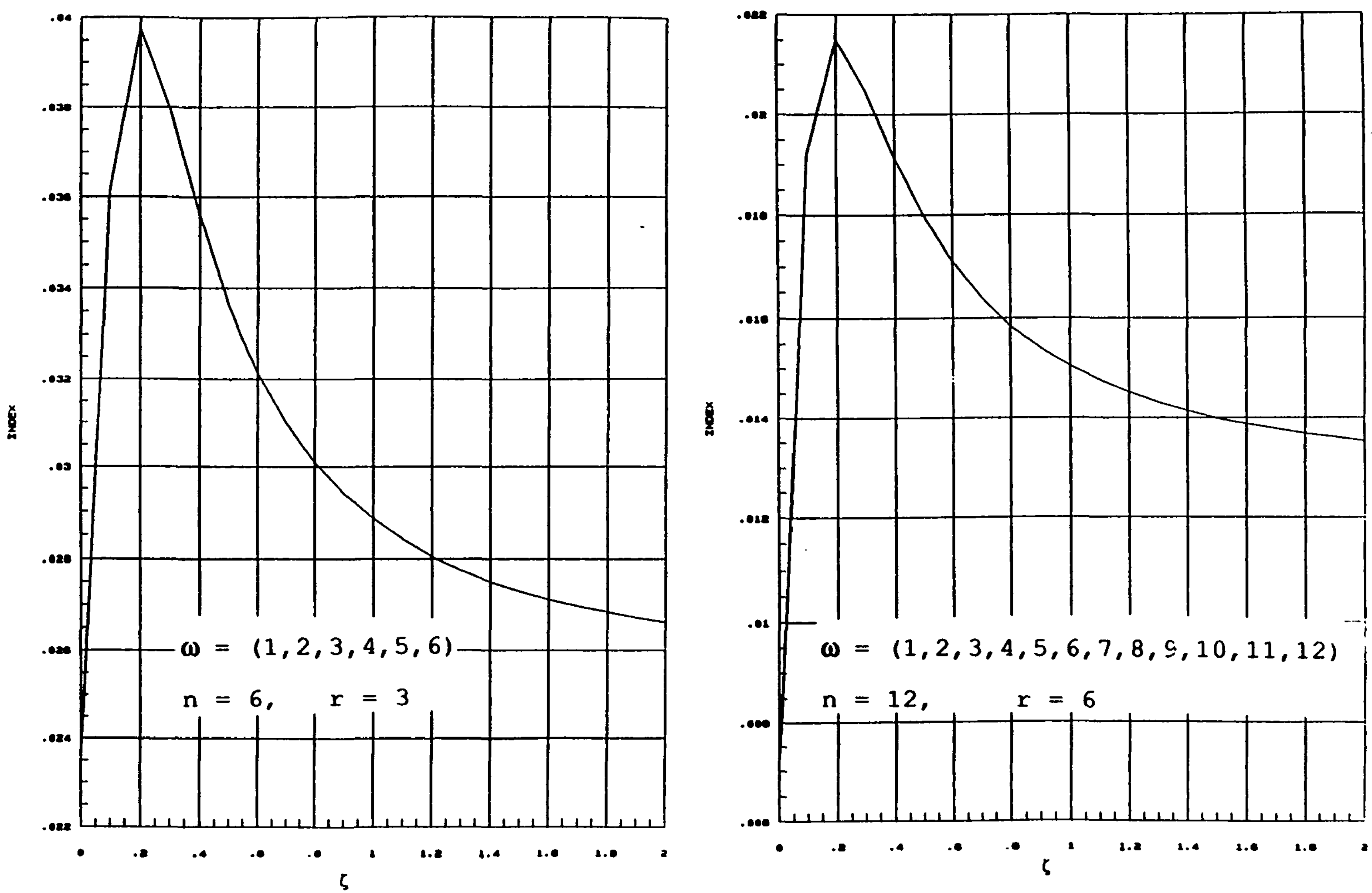


Figure 5.4(a) - Index Maximum at 0.2

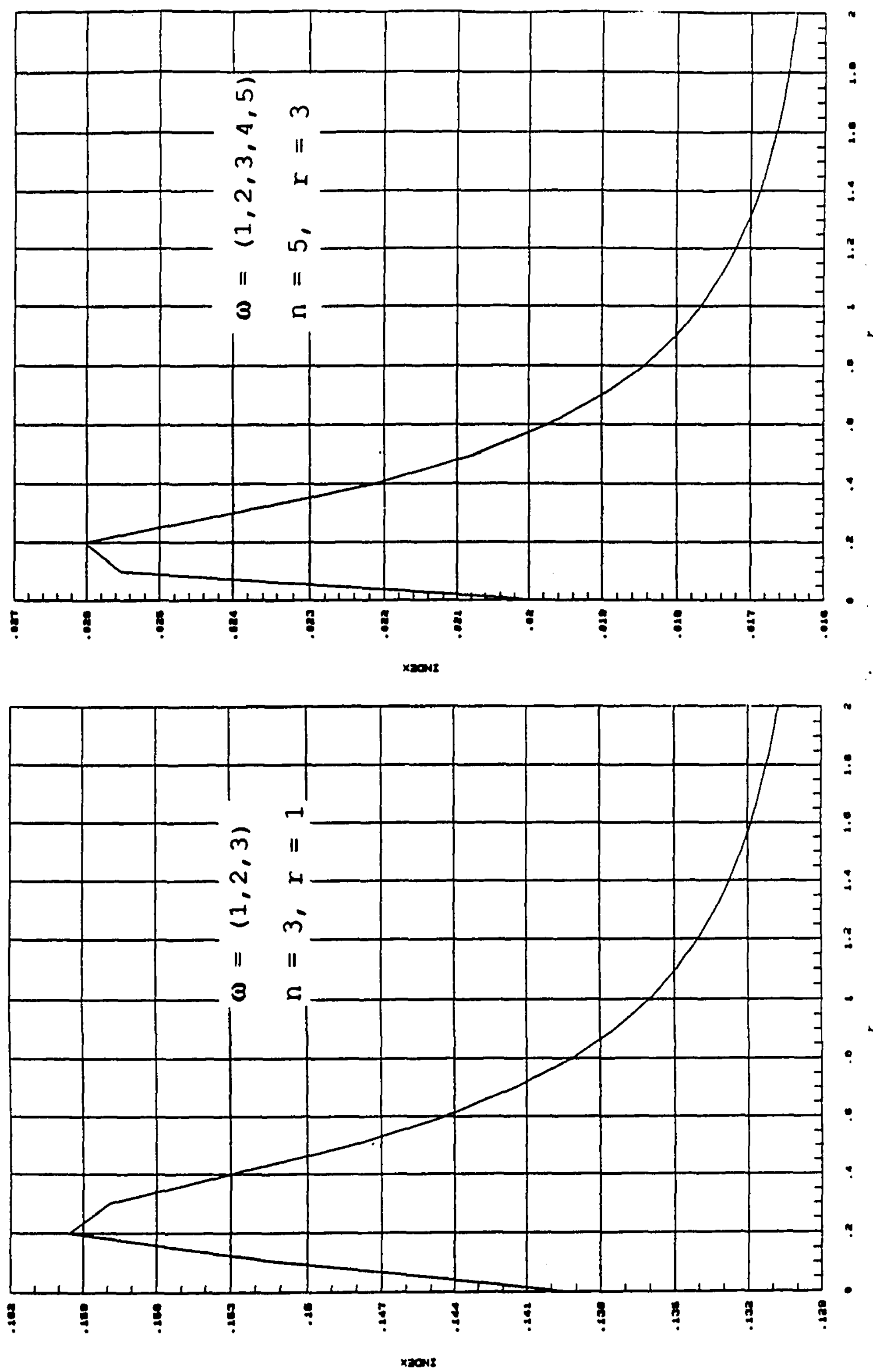


Figure 5.4(b) - Index Maximum at 0.2

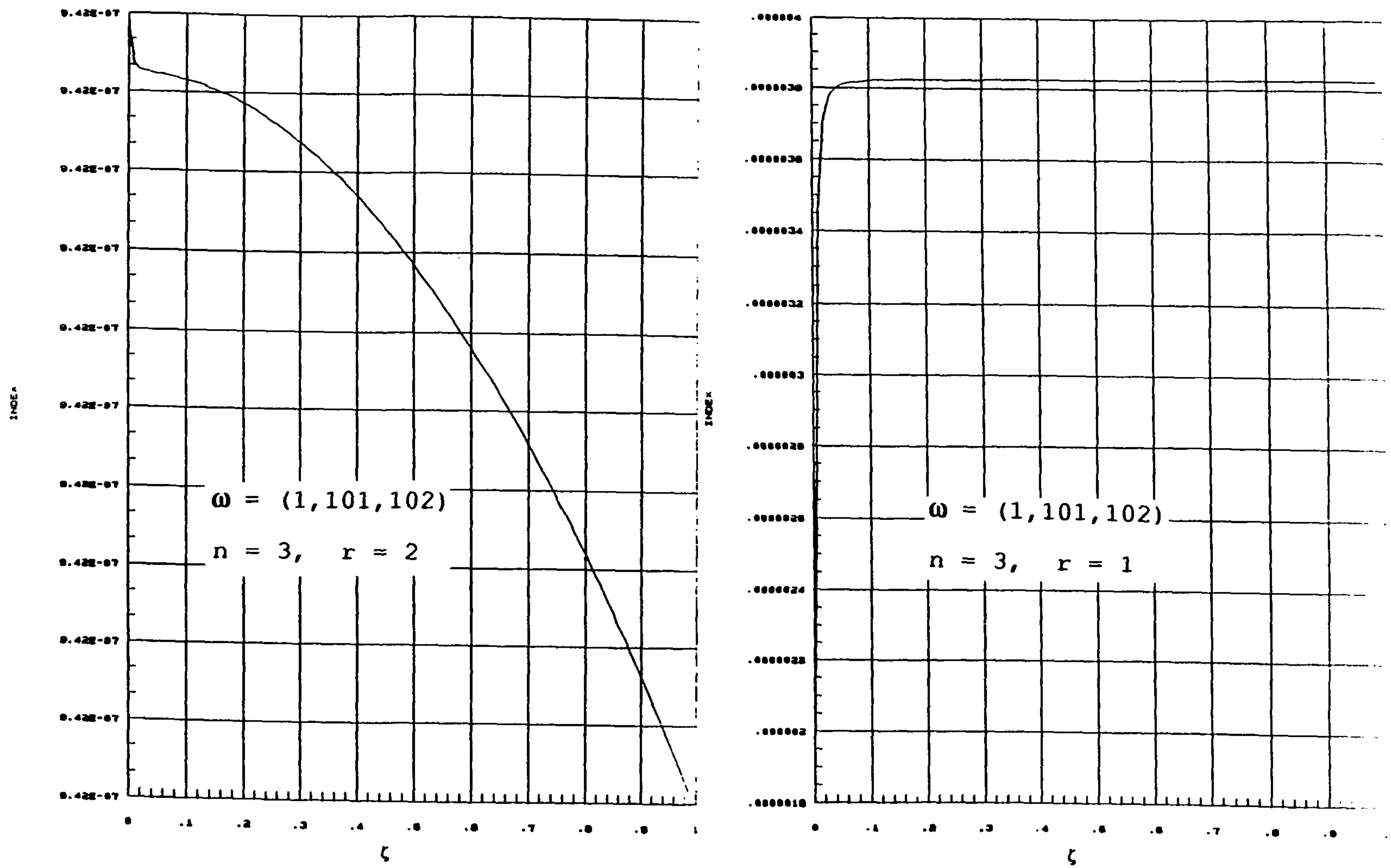


Figure 5.5(a) - Index Invariant

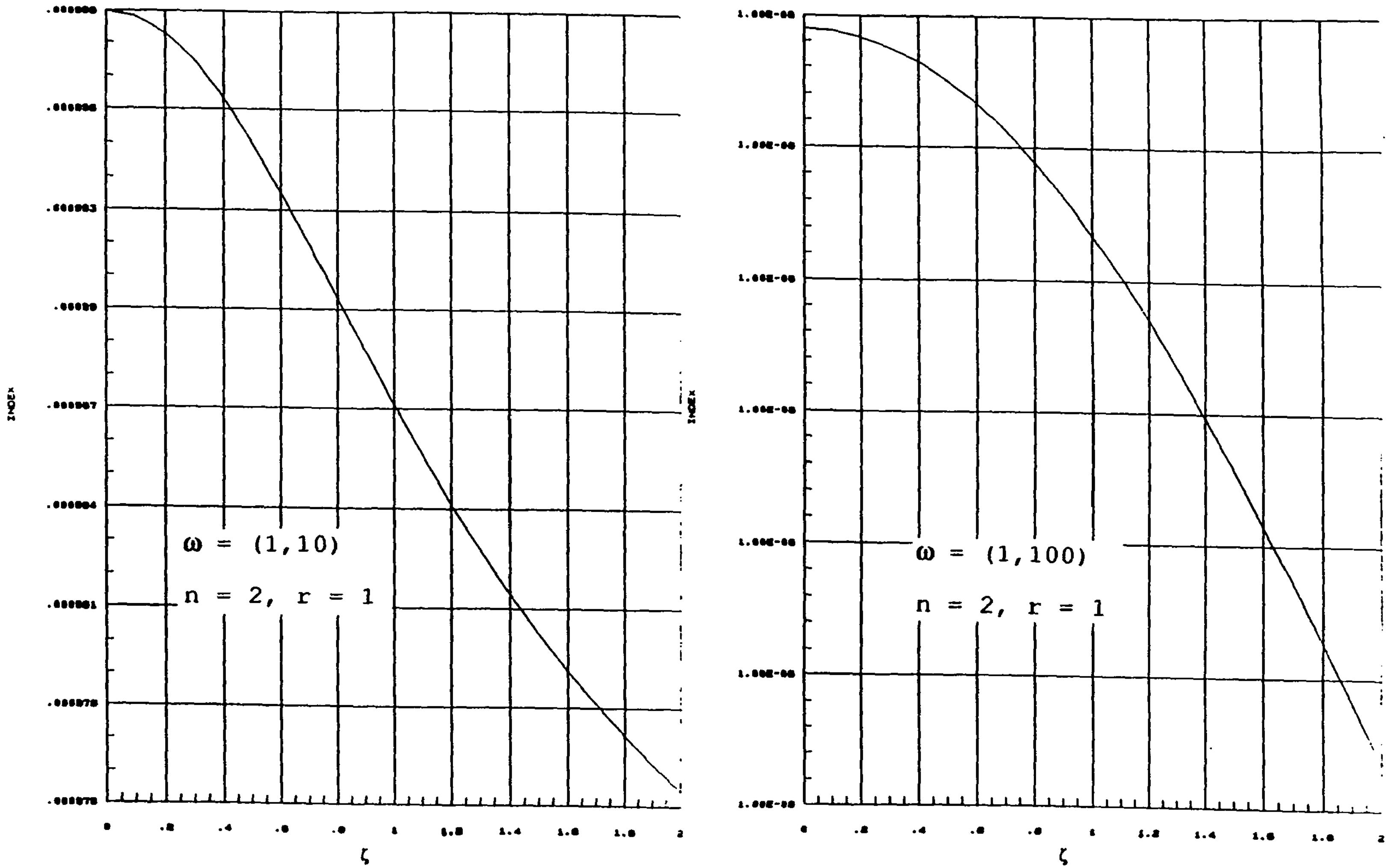


Figure 5.5(b) - Index Invariant

Conclusions

Generally speaking, it appears that passive damping does in fact offer only a glimmer of hope in the quest for reducing the size of plant models when using modal coordinates. We have shown:

1. When damping is light and constant, the model reduction index (MRI) is invariant to changes in damping.
2. Damping can be beneficial in the case in which the damping ratio increases progressively through the higher modes, but at present this is not possible technologically.
3. When damping is general in magnitude, the MRI can be made to decrease, but graphs indicate
   a) this decrease is monotonic only in the cases in which the modal frequencies are clustered, or the reduction involves a truncation of only one mode.
   b) reduction occurs after an increase to approximately $\zeta = 0.2$, which is already technologically a very high value for ζ.
   c) reduction is insignificant when the modal frequencies are relatively widely separated and the MRI can actually increase.

CHAPTER SIX

SUMMARY AND CONCLUSIONS

In this dissertation we have followed up on what is now the generally recognized fact that some form of passive mechanical damping will be needed to augment currently conceived active controllers in the control of multi-dimensional systems. The concept of normal modes played an integral role throughout this work. Two new expressions (eqns. 2.14 and 2.16) were derived relating modal damping values to modal frequencies. From these expressions, it became apparent that the much used proportional (or Rayleigh) damping is nothing but a two degree-of-freedom special case of the actual n degree-of-freedom general relationship. If this idea is coupled with Meirovitch's [26] research demonstrating that modal control is very effective even in cases in which there is control spillover due to small amounts of non-proportional damping, modal coordinates can be utilized without great loss in generality.

After deriving equation (2.14), it was discovered that this same equation could be used to measure the relative ability of a general linear matrix system to uncouple into normal modes. Using least squares estimation, a formula was derived for the correlation between a general system and the closest possible modal system.

After the general analysis on normal modes, the investigation turned to the role played by passive damping in the optimal control of a damped linear matrix system. The system was assumed to separate (or approximately separate) into normal modes. Independent modal space control was selected to be the control method used. This was done for two reasons. First, optimal feedback parameters could be obtained by taking the appropriate partial derivatives of the newly derived closed-form system performance function. This process alleviated having to solve a Riccati matrix differential equation. Secondly, data for damping values associated with certain passive damping inserts was available in modal coordinates. By treating the performance function as our objective function , and using the passive damping data as a constraint, a particular damping design was derived that minimized the performance function. The control system was now optimized with respect to the feedback parameters as well as the system damping parameters. Even though it was shown that the damping design derived from this process would change somewhat if a different type of control is used, it was not altered materially under modal truncation in IMSC. An example was done showing the beneficial effects of this synergistic optimal control methodology. The optimal damping design, depending on control effort weighting a parameter, was shown to produce savings in the performance function relative to sub-optimal damping designs as well as to the undamped system case.

Finally, the research centered on the possible effects of passive damping on plant model reduction. The well-known results of Skelton showed that for simple continuous structures,damping played no role in model reduction if it was constant in magnitude throughout the modes and relatively light. Progressive damping, damping whose ratios increase with mode number, was shown to create significant model reduction in these same structures, even though the assumption of light damping was retained. The light damping assumption was then abandoned, and a new expression for modal costs was derived on the assumption that damping was constant throughout the modes but general in magnitude. This formula, unlike the corresponding expression for light damping, contained coupled frequencies and a non-linear damping term. This cost function offered at least some hope that passive damping may, in fact, provide some plant reduction. As it turned out, graphs of the model reduction index did indicate a reduction as the damping increased, but only for cases of clustered modes, reductions by a single mode, and reductions at high, impractical values of damping. Passive damping does not at present seem to be the agent for model reduction that it was hoped to be.

It appears that passive mechanical damping will play a critical role in the control of multi-d: .sional systems as a defense against both modelling unc. ainty and limited actuator bandwidths. Variations in the different designs of damping inserts permits one to optimize the system performance with respect to the damping parameters as well as the usual feedback coefficients. The savings in both performance and dollar costs due to the optimal damping design could determine the difference between a damping design that is economically feasible and one that is not. While passive damping did not provide dramatic general results in the area of plant model reduction, certain specific results were encouraging as well as the general trend toward model reduction as the damping ratios increased. Future research in these areas should be aimed at determining the degree by which an optimal damping design based on IMSC changes with different types of active control, and some analysis to explain why so many of the model reduction cu ves are maximum at a particular value of damping. The answer to this last question may give us the clue we need to resurrect passive damping as a model reducing agent.

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APPENDIX

COMPUTER PROGRAMS

PROGRAM ONE - Cost Function Vs. Optir Frequencies

```
      update 7-13-90 for Joe Harrell Vari..les:
      F(MM) --- frequency
      N ------- No. of natural frequency
      NG ------ viscoelastic material loss factor
      GT ------ viscoelastic material shear modulus

DIM THETA(40), NG(40), NGT(40), F(100), ZETA(11), W(11),
G(40) DIM GT(40), REB(11), B(11), N1(11), NL(11)
CLS : SCREEN 0, 0, 0: WIDTH 80
REM ******************************************************
REM     PARAMETER VALUES {H=1 AND B=12^(1/3) 50 THAT I=1}
REM ******************************************************
H = 1!: B = 12^(1/3): T1 = .002: T2 = .01: EO =1!: E2 =
101: N = 6 CS = 1: TN = 3: FLAG = 0: BU - 1000:
     'ALPHA = .2
A(1) = 1.309: A(2) = 1.809: A(3) = .191
A(4) = .691: A(5) = 2: A(6) = .691
CLS
PRINT " THIS PROGRAM CALCULATES THE DAMPING ASSOCIATED"
PRINT " WITH EACH MODE OF A BERNOULLI TYPE BEAM WHEN A"
PRINT " CONSTRAINED, SECTIONED VISCOELASTIC DAMPING"
PRINT " TREATMENT IS APPLIED. (RECTANGULAR OR HOLLOW
        CIRCULAR BEAM)"
PRINT : PRINT
PRINT " THIS IS A REVISION TO COMPUTE A COST FUNCTION FOR
        VARIOUS"
PRINT "       OPTIMIZATION FREqUENCIES. (1-1OOOHz)"
REM ******************************************************
REM           TABLES CONTAIN DATA FOR FREQUECIES F(MM)
REM ******************************************************
FOR MM = 1 TO 30
READ F(MM)
DATA 1,2,3,4,5,6,7,8,9,10,15,20,30,40,50,60,70,80,90,100,
DATA 150,200,300,400,500,600,700,800,900,1000,1500,
DATA 2000,3000,4000,5000,6000,7000,8000,9000,10000
NEXT MM
P1 = 3.141592

REM ******************************************************
REM                  Set Parameter of ALPHA
REM ******************************************************
FOR ALPH = 1 TO 5
CLOSE #2
IF ALPH = 1 THEN ALPHA = .1:OPEN "ZETA_O1.DAT" FOR OUTPUT
     AS #2
IF ALPH = 2 THEN ALPHA = .5:OPEN "ZETA_05.DAT" FOR OUTPUT
     AS #2
IF ALPH = 3 THEN ALPHA = 11: OPEN "ZETA_1.DAT" FOR OUTPUT
```

```
     AS #2
IF ALPH = 4 THEN ALPHA = 21: OPEN "ZETA_:    :T" FOR OUTPUT
     AS #2
IF ALPH = 5 THEN ALPHA = 51: OPEN "ZETA      AT" FOR OUTPUT
     AS #2

20 GOSUB 100
     'call for Inputting Section for Rectangular Cross
          Section Beam
PL$ = "N"
FLAG = 1
REM ****************************************************
30 REM    RETRIEVE SCOTCHDAMP DATA FROM EXTERNAL FILES
REM ****************************************************
IF TN = 1 THEN OPEN "I", #1, "DATA2"
IF TN = 2 THEN OPEN "I", #1, "A:DATA1"
IF TN = 3 THEN OPEN "I", #1, "DATA3"
IF TN = 4 THEN OPEN "I", #1, "DATA5"
IF TN = 5 THEN OPEN "I", #1, "DATA4"
IF TN = 6 THEN OPEN "I", #1, "DATA6"
IF NOT ((TN = 1) OR (TN = 2) OR (TN = 3) OR (TN = 4) OR
     (TN = 5) OR (TN = 6)) THEN GOTO 30
FOR J = 1 TO 30
INPUT #1, GT(J), NGT(J):
NEXT J:   CLOSE #1
FLAG = O
REM ****************************************************
REM                SECTION LENGTH INPUT DATA
REM: LET U(N+1) BE THE OPTIMIZATION FREQUENCY
REM PRINT:PRINT "UTIMIZE SECTION LENGTH FOR WHAT
     FREQUENCY?":INPUT W(N+1)
GOSUB 200     'Call freq.interpolation
GOSUB 300     ' Call opti. freq. interpolation
PRINT " SUB 400"
GOSUB 400           ' Call output
NEXT ALPH
50 PRINT : INPUT "RUN AGAIN (Y or N)"; R$
IF NOT ((R$ = "Y") OR (R$ = "N")) THEN GOTO 50
KEY ON

999 END'~~~~~~~~~~~End of Main Program ~~~~~~~~~~~~~~~~~


REM ****************************************************
REM   INPUT SECTION FOR RECTANGULAR CROSS SECTION BEAM
REM ****************************************************
100 REM PRINT "WIDTH OF COATED SURFACE B (IN.) [DEFAULT
     "B"]":INPUT B
I = (B * H^3)/12
GAMMA = (E2 * T2 * B * H^2)/(4 * PI * EO * I)
199 RETURN
```

```
REM*************************************    ************
REM              FREOUENCIES FOR INTERPOLA    N
REM ************************************.  ************
200 PRINT : PRINT " Frequencies for Inte   ation": PRINT
IF PL$ = "Y" THEN 210
FOR M = 1 TO N
W(M) = PI * PI * M * M / (2 * PI)
NEXT M
REM *********************************************
REM                    INTERPOLATION ROUTINE
REM *********************************************
210 IF N < 10 THEN 220 ELSE WIDTH 40: KEY OFF
LOCATE 12, 1
COLOR 31: PRINT "   INTERPOLATING": COLOR 7: LOCATE , , 0
220 FOR NN = 1 TO 6
FOR MM = 1 TO 30
IF W(NN) <= F(MM) THEN MAX = F(MM): MIN = F(MM - 1): GOTO
     230
NEXT MM
230 G(NN)=GT(MM-1)+((W(NN)-MIN)/(MAX-MIN))*(GT(MM)-
     GT(MM - 1))
NG(NN) = NGT(MM-1)+((W(NN)-MIN)/(MAX-MIN))*(NGT(MM)-
     NGT(MM- 1))
THETA(NN) = ATN(NG(NN))
' PRINT W(NN), MAX, GT(MM), MIN, GT(MM-1)
NEXT NN
WIDTH 80: LOCATE ,,1
RETURN ''''''''''''   End of Subprogram 200    ''''''''''

300 REM *****************************************
REM          INTERPOLATION FOR OPTIMIZATION FREOUENCIES
REM *********************************************
PRINT "SUB 300"
CLOSE #3
IF ALPHA = .01 THEN UEN "COST_001.DAT" FOR OUTPUT AS #3
IF ALPHA = .1 THEN OPEN "COST_01.DAT" FOR OUTPUT AS #3
IF ALPHA = .5 THEN OPEN "COST_05.DAT" FOR OUTPUT AS #3
IF ALPHA = 1! THEN OPEN "COST_1.DAT" FOR OUTPUT AS #3
IF ALPHA = 2! THEN OPEN "COST_2.DAT" FOR OUTPUT AS #3
IF ALPHA = 5! THEN OPEN "COST 5.DAT" FOR OUTPUT AS #3
PRINT : PRINT " Calculating Optimal Section Length and
     Damping": PRINT
FOR NNN = 1 TO 100 STEP 2          'Optimal frequency
FOR MM = 1 TO 30                 'material freq.
IF NNN <= F(MM) THEN MAX = F(MM): MIN = F(MM -1): GOTO
     310
NEXT MM
310 G(N + 1) = GT(MM - 1) + ((NNN - MIN) / (MAX - MIN)) *
     (GT(MM) - GT(MM - 1))
NG(N + 1) = NGT(MM - 1) + ((NNN - MIN) / (MAX - MIN)) *
     (NGT(MM) - NGT(MM - 1))
THETA(N + 1) = ATN(NG(N + 1))
```

```
PRINT NNN, N, G(N + 1), NG(N + 1)
GOSUB 3100 ' Call calcu. opt. section len    and damping

COST = 0
FOR PP = 1 TO N
COST = COST + A(PP) * (-2 * PI * SIG(PP)    ((2 * PI *
          SIG(PP))^2 + ALPHA^2)^.5)
NEXT PP
COST = COST / (2 * ALPHA^2)
PRINT USING "#### #.#### #.#### #.#### #.#### #.####
     #.####  ###.###"; NNN, ZETA(1), ZETA(2), ZETA(3),
     ZETA(4), ZETA(5), ZETA(6)
PRINT #3, NNN, COST
NEXT NNN
399 RETURN '''''''' End of Subprogram 300 ''''''''''''

REM ****************************************************
3100 REM    CALULATION OF OPTIMAL SECTION LENGTH AND
            DAMPING
REM ****************************************************
REB(N + 1) = (((T1 * T2 * E2)/G(N + 1))^.5)/(1+NG(N+1)^2)^.25
BETA = 3.28
L2 = BETA * REB(N + 1)
FOR P = 1 TO N
REM REB(P)=((T1*T2*E2*(1+(1+NG(P)^2)^.5))/(G(P)*(1+NG(P)^2)*
     2))^.5
REB(P)= (((T1 * T2 * E2) / G(P))^.5)/(1+NG(P)^2)^.25
B(P) = L2 / REB(P)
S = SIN(THETA(P) / 2): C = COS(THETA(P)/2)
REM ****************************************************
REM IF STATEMENT IN NEXT LINE IS FOR OVERFLOW PREVENTION.
REM IT DOES NOT AFFECT THE CALCULATION BECAUSE LIM SNH
REM GOES TO CSH FOR LARGE X.
REM ****************************************************
X = B(P) * C:IF X > 85 THEN N1(P) = 4*PI * S/B(P): GOTO
     3110
CSH = .5 * (EXP(X) + EXP(-X)): SNH = .5 * (EXP(X) -
     EXP(-X))
N1(P) = (4 * P1 / B(P)) * (SNH * S - SIN(B(P) * S) *
     C)/(CSH + COS(B(P)*S))
3110 NL(P) = N1(P) * GAMMA
ZETA(P) = NL(P) / 2
PRINT #2, W(P), NNN, ZETA(P)
SIG(P) = ZETA(P) * W(P) * 2 * PI
NEXT P
RETURN '''End of subprogram of 3000 for subprogram 300'''
REM ****************************************************
REM                    OUTPUT SECTION
REM ****************************************************
400 'WIDTH 80: LOCATE , , 1
IF PL$ = "Y" THEN GOSUB 3000
KEY OFF: CLS : SCREEN 0, 0, 0: WIDTH 80
```

```
PRINT "INPUT DATA GIVEN:"
PRINT " T1", "T2", "EO", "E2", "N"
PRINT USING "#.###    #.###    ###.#^^^^ #    ‡^^^^ ###;
      T1,T2,EO,E2,N
PRINT
IF CS=1 THEY PRINT "RECTANGULAR BEAM WITH B =";B;"AND
     H =";H
IF CS=2 THEN PRINT "CIRCULAR BEAM WITH DO="; DO; ",
     D1 = "; D1; " AND PSI=";PSI
PRINT
IF (OO$ = "Y" AND PL$ = "Y") THEN PRINT "TO OPTIMIZE
DAMPING AT "; FO; " HZ USE "; L2; " INCH SECTIONS.":
     GOTO 410
IF OO$ = "Y" THEN PRINT "TO OPTIMIZE DAMPING WITH RESPECT
TO MODE"; O; "USE "; L2; "INCH SECTIONS~': GOTO 410
PRINT "SECTION LENGTH PRESCRIBED BY OPERATOR
     WAS";L2;"INCH."
410 PRINT
'1246 IF ((DPS = "N") AND (PLS = "Y")) THEN 1290 'there
     no 1290 in old program
PRINT "MODE", "FREQ", "G", "Ng", "ZETA"
FOR Q = 1 TO N
PRINT USING " ### #####.##  #####.##   #.##    #.####";
          Q, U(Q), G(Q), NG(Q), ZETA(Q)
NEXT Q
499 RETURN '''' End of subprogram 400 for output ''''''''

1700 REM ***********************************************
1710 REM    INPUT SECTION FOR CIRCULAR CROSS SECTION BEAM
1720 REM ***********************************************
1730 REM PRINT
1810 RETURN '''''''''''''''''''''''''''''''''''''''''''''

3000 REM ***********************************************
REM                  PLOTTING SUBOUTINE
REM ***************************************************
PRINT : PRINT " Plotting ": PRINT
OPEN "O", #1, "POINTS"
XMIN = 0: XMAX = BW: YMIN = 0: YMAX = O
FOR I = 1 TO N
IF ZETA(I) < YMIN THEN YMIN = ZETA(I)
IF ZETA(I) > YMAX THEN YMAX = ZETA(I)
WRITE #1, W(I), ZETA(I)
NEXT I
CLOSE #1
ON ERROR GOTO O
SCREEN 1
OPEN "I", #1, "POINTS"
KEY OFF: CLS
REM ***** DRAW THE AXES ******
LINE (40, 165)-(300, 165): LINE (40, 165)-(40, 10)
3770 FOR LN = 1 TO 10
```

```
3780   XONE = 40 + 26 * LN: YONE = 165 - 13.5 * LN
3790   LINE (XONE, 162)-(XONE, 165)
3795   LINE (40, YONE)-(43, YONE)
3800 NEXT LN
3870 SCX = 260 / (XMAX - XMIN): SCY = 155 / (YMAX - YHIN)
4060 REM
4070 V = EOF(1) + 1
4080 WHILE V
4090 INPUT #1, X, y
4100 XONE = 40 + SCX ~ X: YONE = 165 - SCY * y
4110 PSET (XONE, YONE)
4120   V = EOF(1) + 1
4130 WEND
4140 CLOSE
4150 KILL "POINTS"
4155 XX = 40 + SCX * W(O): YY = 165 - SCY * ZETA(O)
4156 IF OO$ = "N" THEN 4165
4157 JP = INT((165 - YY) * 15 / 155)
4158 FOR JJ = O TO JP
4159 LINE (XX, YY + JJ * 10)-(XX, YY + JJ * 10 + 5)
4160 NEXT JJ
4165 PRINT "    PRESS ANY KEY TO CONTINUE EXECUTION":
     PRINT USING ".###"; YMAX: LOCATE 7, 1:
     PRINT "DAMP": PRINT "RATIO": LOCATE 11,
4170 PRINT USING "    #            ####           #####";
          0, U(50), U(100): LOCATE 23, 1:
     PRINT "             FREQUENCY [HZ]"
4190 KK$ = INKEYS: IF KK$ = "" THEN 4190
4200 RETURN
```

PROGRAM TWO - Optimal System Performance, Energy, Effort and Impulse Response

```
NIL=O*EYE(6)
RIN=0.7, ROUT=0.5
FOR J=1:6, W(J)=(P1**2)*(J**2),...
PSIRIN(J)=SQRT(2)*SIN(PI*J*RIN),...
PS1ROUT(J)=SQRT(2)*S1N(PI*J*ROUT),END
OMEG=DIAG(W)
D=O
B=[O,O,O,O,O,O,PSIRIN']'
C=[PSIROUT',O,O,O,O,O,O]
FOR 1=1:4,...
FOR J=1:6,...
H1(1,J)=-2*ZETA1(1,J)*OMEG(J)+2*SQRT((ZETA1(1,J)*
      OMEG(J))**2+ALPH(1)**2),...
H2(1,J)=-2*ZETA2(1,J)*OMEG(J)+2*SQRT((ZETA2(1,J)*
      OMEG(J))**2+ALPH(2)**2),...
H3(1,J)=-2*ZETA3(1,J)*OMEG(J)+2*SQRT((ZETA3(1,J)*
      OMEG(J))**2+ALPH(3)**2),...
END,END
A1=[NIL,EYE(6);-1*0MEG**2,-2*DIAG(ZETA1(1,:))*
      0MEG-DIAG(H1(1,:))];
A2=[N1L,ErE(6);-1*0MEG**2,-2*D1AG(ZETA1(2,:))*
      0MEG-D1AG(H1(2,:))]; A3=[N1L,ErE(6);-1*
      0MEG**2,-2*D1AG(ZETA1(3,:))*0MEG-D1AG(H1(3,:))];
A4=[N1L,ErE(6);-1*0MEG**2,-2*D1AG(ZETA1(4,:))*
      0MEG-D1AG(H1(4,:))];
A5=[N1L,ErE(6);-1*0MEG**2,-2*D1AG(ZETA2(1,:))*
      0MEG-D1AG(H2(1,:))];
A6=[N1L,ErE(6);-1*0MEG**2,-2*D1AG(ZETA2(2,:))*
      0MEG-D1AG(H2(2,:))];
A7=[N1L,ErE(6);-1*0MEG**2,-2*D1AG(ZETA2(3,:))*
      0MEG-D1AG(H2(3,:))];
A8=[111L,ErE(6);-1*0MEG**2,-2*D1AG(ZETA2(4,:))*
      0MEG-D1AG(H2(4,:))];
A9=[NIL,EYE(6);-1*0MEG**2,-2*DIAG(ZETA3(1,:))*
      0MEG-DIAG(H3(1,:))];
A10=[NIL,EYE(6);-1*0MEG**2,-2*DIAG(ZETA3(2,:))*
      0MEG-DIAG(H3(2,:))]; A11=[NIL,EYE(6);-1*0MEG**2,-
2*DIAG(ZETA3(3,:))*0MEG-DIAG(H3(3,:))];
A12=[NIL,EYE(6);-1*0MEG**2,-2*DIAG(ZETA3(4,:))*
0MEG-DIAG(H3(4,:))];
S1=[A1,B;C,D];
S2=[A2,B;C,D];
S3=[A3,B;C,D];
S4=[A4,B;C,D];
S5=[AS,B;C,D];
S6=[A6,B;C,D];
S7=[A7,B;C,D];
```

```
S8=[A8,B;C,D];
S9=[A9,B;C,D];
S10=[A10,B;C,D];
S11=[A11,B;C,D];
S12=[A12,B;C,D];

[T,Y1]=1MPULSE(S1,12,10);
[T,Y2]=1MPULSE(S2,12,10);
[T,Y3]=1MPULSE(S3,12,10);
[T,Y4]=IMPULSE(S4,12,10);
[T,YS]=IMPULSE(S5,12,10);
[T,Y6]=1MPULSE(S6,12,10);
[T,Y7]=1MPULSE(S7,12,10);
[T,Y8]=1MPULSE(S8,12,10);
[T,Y9]=1MPULSE(S9,12,2);
[T,Y10]=1MPULSE(S10,12,2);
[T,Y11]=1MPULSE(S11,12,2);
[T,Y12]=1MPULSE(S12,12,2);
FOR 1=1:4,...
FOR K=1:6,...
EN1(I,K)=0.25*ACOEFF(K)/SQRT((ZETA1(I,K)*
      U(K))**2+ALPH(1)**2),...
EN2(I,K)=0.25*ACOEFF(K)/SORT((ZETA2(I,K)*
      U(K))**2+ALPH(2)**2),...
EN3(I,K)=0.25*ACOEFF(K)/SQRT((ZETA3(I,K)*
      U(K))**2+ALPH(3)**2),...
EFF1(I,K)=0.5*ACOEFF(K)*(2*ZETA1(1,K)*U(K)+(2*
      (ZETA1(1,K)*U(K))**2...
+ALPH(1)**2)/SQRT((ZETA1(1,K)*U(K))**2+ALPH(1)**2)),...
EFF2(I,K)=0.5*ACOEFF(K)*(2*ZETA2(1,K)*U(K)+(2*
      (ZETA2(1,K)*U(K))**2...
+ALPH(2)**2)/SQRT((ZETA2(1,K)*U(K))**2+ALPH(2)**2)),...
EFF3(1,K)=0.5*ACOEFF(K)*(2*ZETA3(1,K)*U(K)+(2*
      (ZETA3(1,K)*U(K))**2...
+ALPH(3)**2)/SQRT((ZETA3(1,K)*U(K))**2+ALPH(3)**2)),..
ENERGY1(I)=SUM(EN1(1,:)),...
ENERGY2(I)=SUM(EN2(1,:)),...
ENERGY3(I)=SUM(EN3(1,:)),...
EFFORT1t1)=SUMtEFF1(1,:)),...
EFFORT2t1)=SUM(EFF2(1,:)),...
EFFORT3(1)=SUM(EFF3(1,:)),...
PERF1(1)=ENERGr1(1)+0.5*ALPH(1)**(-2)*EFFORT1(1), . . .
PERF2(1)=ENERGY2(1)+0.5*ALPH(2)**(-2)*EFFORT2(1),...
PERF3(1)=ENERGr3(1)+0.5*ALPH(3)**(-2)*EFFORT3(1),END,END
```

PROGRAM THREE - Total Modal Code

```
10 LET W1 = 1
20 LET W2 = 4
30 INPUT R
40 INPUT Z
50 J1 = (SIN (R)**2)/(4*Z*W1**3)
60 J2 = (4*SIN(R)**2)/(4*Z*W2**3)
70 JC = (8*Z*SIN(R)*SIN(2*R)/((W1 + W2))*((W1 - W2)**2 +
        4*Z**2*W1*W2))
80 PRINT J1, J2, JC
RUN
```

PROGRAM FOUR - Model Reduction Index

```
JNUM = O
JDEN = O
S = R + T
FOR J = S to N, ...
FOR K = S to N, ...
T(J,K) = (W(J) + W(K)) * (W(J) * W(J) + W(K) * W(K) ... +
     2 *(2*Z*Z - 1) * W(J) * W(K)), ...
JNUM = JNUM + 1/T(J,K), END, END
FOR J = 1 to N,
FOR K = 1 to N,
T(J,K) = (W(J) + W(K)) * (W(J) * W(J) + W(K) * W(K) ... +
     2 *(2*Z*Z - 1) * W(J) * W(K)), ...
JDEN = JDEN + 1/T(J,K), END, END
I = JNUM/JDEN
```