Sufficient Conditions for the Existence of Positive Solutions to an Elliptic Model

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INTRODUCTION

One of the prominent subjects of study and analysis in mathematical biology concerns the competition, predator-prey, cooperation of two or more species of animals in the same environment.

Especially pertinent areas of investigation include the conditions under which the species can coexist, as well as the conditions under which any one of the species becomes extinct, that is, one of the species is excluded by the others.

We focus on the general population model to better understand the interactions between two species. Specifically, we investigate the conditions needed for the coexistence of two species when the factors affecting them are perturbed.

Within the academia of mathematical biology, extensive academic work has been devoted to investigation of the simple population model, commonly known as the Lotka-Volterra model. This system describes the interaction of two species residing in the same environment in the following manner:

$$u_t(x,t) = \Delta u(x,t) + u(x,t)(a - bu(x,t) - cv(x,t))$$

$$v_t(x,t) = \Delta v(x,t) + v(x,t)(d - eu(x,t) - fv(x,t))$$

Here, u(x,t) and v(x,t) designate the population densities for the two species. The constant coefficients in this system represent growth rates(a and d), self-limitation rates(b and f) and competition rates(c and e).

The mathematical community has already established several results for the existence, uniqueness and stability of the positive steady state solution to the time-independent system.

$$\Delta u(x,t) + u(x,t)(a - bu(x,t) - cv(x,t)) = 0$$

$$\Delta v(x,t) + v(x,t)(d - eu(x,t) - fv(x,t)) = 0$$

GENERALIZATION

Their results are somewhat limited by a few key assumptions. In the Lotka-Volterra model that they studied, the rate of change of densities largely depends on constant rates of reproduction, self-limitation, and competition. The model also assumes a linear relationship of the terms affecting the rate of change for both population densities.

However, in reality, the rates of change of population densities may vary in a more complicated and irregular manner than can be described by the simple model. Therefore, significant research has been focused on the existence and uniqueness of the positive steady state solution of the general population model for two species,

$$u_t(x,t) = \Delta u(x,t) + g(u(x,t), v(x,t))$$

$$v_t(x,t) = \Delta v(x,t) + h(u(x,t), v(x,t))$$

or, equivalently, the positive solution to

$$0 = \Delta u(x,t) + g(u(x,t), v(x,t))$$

$$0 = \Delta v(x,t) + h(u(x,t), v(x,t))$$

where g, h designate reproduction, self-limitation and competition rates that satisfy the growth conditions .

(1)Competition Case

$$g_{uu} < 0, g_{uv} < 0, h_{vv} < 0, h_{vu} < 0$$

(2) Predator-Prey Case

$$g_{uu} < 0, g_{uv} < 0, h_{vv} < 0, h_{vu} > 0$$

(3) Cooperation Case

$$g_{uu} < 0, g_{uv} > 0, h_{vv} < 0, h_{vu} > 0$$

RESULTS

$$\left.\begin{array}{l} g_u(0,0) > \lambda_1 \\ g(0,v) \geq 0 \\ g_{uu} < 0 \\ g_{uv} > 0 \\ u,v > c_0 \Rightarrow g_u(u,0), g(u,v) < 0 \end{array}\right\} \Rightarrow \exists M(g)$$

$$\left.\begin{array}{l} h_v(0,0) > M(g) \\ h(u,0) \geq 0 \\ h_{vv} < 0 \\ h_{vu} > 0 \\ u,v > c_0 \Rightarrow h(u,v), h_v(0,v) < 0 \end{array}\right\} \Rightarrow \text{a positive solution}$$

$$\begin{array}{l} g_u(0,0) > \lambda_1 \\ h_v(0,0) > \lambda_1 \\ u,v > c_0 \Rightarrow g_u(u,0), h_v(0,v) < 0 \\ \inf(h_{uv})\inf(g_{uv}) + \inf(h_{uv}) + \inf(h_{vv})\sup(h_{uv}) + \inf(h_{vv}) \geq 0 \end{array}$$

$$\Downarrow$$

no positive solution