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ESTIMATES OF LIFE SPAN OF SOLUTIONS OF A CAUCHY PROBLEM

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Abstract: In this paper we get estimates of life span of a Cauchy problem

$$u_t(x,t) = \Delta u(x,t) + u(x,t)^p, \quad x \in R^n, t > 0,$$
$$u(x,0) = \lambda\phi(x), \quad x \in R^n$$

in terms of the positive constant parameter λ when $\phi(x) \in L^q$ is a nonnegative bounded continuous function in R^n but not identically zero, where q is large enough. The technique we used in this paper is the Comparison Principle.

1. Introduction

In this paper we consider the Cauchy problem

$$\begin{aligned} u_t(x,t) &= \Delta u(x,t) + u(x,t)^p, \quad x \in R^n, t > 0, \\ u(x,0) &= \lambda\phi(x), \quad x \in R^n, \end{aligned} \tag{1}$$

where $\Delta = \sum_{i=1}^{n}(\frac{\partial^2}{\partial x_i^2})$ is the Laplace operator, $p > 1, \phi \in L^q$ is a nonnegative bounded continuous function in R^n but not identically zero, where q is large enough, and λ is a positive constant parameter. It is well known that there exists an $T_\lambda > 0$ such that (1) possesses a unique classical solution $u(x,t,\lambda)$ in $[0,T_\lambda)$, i.e., $u(x,t,\lambda) \in C^{2,1}(R^n \times (0,T_\lambda)) \bigcap C(R^n \times [0,T_\lambda))$ is bounded in $[0,T']$ for any $T' < T_\lambda$ and $\| u(\cdot,t,\lambda) \|_{L^\infty} \to \infty$ when $t \to T_\lambda$ if T_λ is finite. We call T_λ the life span of the solution $u(x,t,\lambda)$ and say that $u(x,t,\lambda)$ blows up in

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finite time if $T_\lambda < \infty$.

Since Fujita's classic work [1], (1) has been studied extensively in a lot of directions. For stability and instability results, the interested readers are refered to [2] for a survey and some new developments; You can also refer to [4] for some other related results. Motivated by a paper of Lee and Ni [5], we are concerned with asymptotic behavior of the life span T_λ as $\lambda \to \infty$ or $\lambda \to 0$. The following was proved in [5]:

Theorem 1.1. *(1) $T_\lambda \sim \lambda^{-(p-1)}$ as $\lambda \to \infty$; i.e., there exist positive constants C_1 and C_2 such that $C_1\lambda^{-(p-1)} \leq T_\lambda \leq C_2\lambda^{-(p-1)}$ for large λ.*

(2) If $\liminf_{|x|\to\infty} \phi(x) > 0$, then $T_\lambda < \infty$ for any $\lambda > 0$ and $T_\lambda \sim \lambda^{-(p-1)}$ as $\lambda \to 0$.

In [3] they improved the 1.1. The following was the result.

Theorem 1.2. *(1) $\lim_{\lambda\to\infty} T_\lambda \lambda^{p-1} = \frac{1}{p-1} \parallel \phi \parallel_{L^\infty}^{-(p-1)}$.*

(2) If $\lim_{|x|\to\infty} \phi(x) = \phi_\infty > 0$, then

$$\lim_{\lambda\to 0} T_\lambda \lambda^{p-1} = \frac{1}{p-1}\phi_\infty^{-(p-1)}.$$

In this research, we prove other estimates of the life span of (1) when we assumed $\phi \in L^q$ for large enough q using the Comparison Principle.

2. Main Results

Theorem 2.1. *(1) $T_\lambda \geq \frac{1}{2^{p-1}(p-1)}\lambda^{1-p} \parallel \phi \parallel_{L^q}^{-(p-1)}$ for some large enough q. So, $\lim_{\lambda\to 0} T_\lambda = \infty$.*

(2) $T_\lambda \leq \frac{1}{2(p-1)}\lambda^{1-p} \parallel \phi \parallel_{L^r}^{-(p-1)}$ for some large enough r. So, $\lim_{\lambda\to\infty} T_\lambda =$ 0.

Proof. (1) Let $v_\lambda(x,t) = \frac{\lambda^{-1}}{2}u(\lambda^{\frac{1-p}{2}}x, \lambda^{1-p}t)$. Then v_λ satisfies

$$\begin{aligned}
\frac{\partial v_\lambda(x,t)}{\partial t} &= \frac{\lambda^{-1}}{2}\lambda^{1-p}\frac{\partial u(z,\tau)}{\partial \tau} \\
&= \frac{\lambda^{-p}}{2}\frac{\partial u(z,\tau)}{\partial \tau} \\
&= \frac{\lambda^{-p}}{2}[\Delta_z u(z,\tau) + u^p(z,\tau)] \\
&= \frac{\lambda^{-p}}{2}\Delta_z u(z,\tau) + \frac{\lambda^{-p}}{2}u^p(z,\tau) \\
&= \frac{\lambda^{-1}}{2}\lambda^{1-p}\Delta_z u(z,\tau) + \frac{\lambda^{-p}}{2}u^p(z,\tau) \\
&= \Delta v_\lambda(x,t) + 2^{p-1}v_\lambda^p(x,t), x \in R^n \times (0, \tilde{T_\lambda}), \\
v_\lambda(x,0) &= \frac{1}{2}\phi(\lambda^{\frac{1-p}{2}}x), x \in R^n,
\end{aligned}$$

where $\tilde{T_\lambda} = \lambda^{p-1}T_\lambda$ is the life span of v_λ, $z = \lambda^{\frac{1-p}{2}}x$ and $\tau = \lambda_{1-p}t$. Since $\lim_{s\to\infty} \| \phi \|_{L^s} = \| \phi \|_{L^\infty}$, there is a large enough q such that $\frac{1}{2} \| \phi \|_{L^\infty} < \| \phi \|_{L^q}$. Then we may consider the following ordinary differential equation.

$$\frac{dv(t)}{dt} = 2^{p-1}v(t)^p, t > 0,$$
$$v(0) = \| \phi \|_{L^q} .$$

The ordinary differential equation implies that

$$\frac{1}{v^p}dv = 2^{p-1}dt,$$

so

$$\int v^{-p}dv = \int 2^{p-1}dt + C,$$

so

$$\frac{1}{1-p}v^{1-p} = 2^{p-1}t + C.$$

But, by applying the initial condition $v(0) = \| \phi \|_{L^q}$, we have

$$C = \frac{1}{1-p} \| \phi \|_{L^q}^{1-p},$$

and so

$$\frac{1}{1-p}v^{1-p} = 2^{p-1}t + \frac{1}{1-p} \| \phi \|_{L^q}^{1-p} .$$

Therefore,

$$v^{p-1} = \frac{1}{2^{p-1}(1-p)t + \| \phi \|_{L^q}^{1-p}},$$

and so the life span of v is $T = \frac{1}{2^{p-1}(p-1)} \| \phi \|_{L^q}^{-(p-1)}$.

By the Comparison Principle(see [6]), we have

$$\tilde{T_\lambda} = \lambda^{p-1}T_\lambda \geq \frac{1}{2^{p-1}(p-1)} \| \phi \|_{L^q}^{-(p-1)},$$

and so we conclude (1).

(2) Let $v_\lambda(x,t) = 2\lambda^{-1}u(\lambda^{\frac{1-p}{2}}x, \lambda^{1-p}t)$. Then v_λ satisfies

$$\begin{aligned}
\tfrac{\partial v_\lambda(x,t)}{\partial t} &= 2\lambda^{-1}\lambda^{1-p}\tfrac{\partial u(z,\tau)}{\partial \tau} \\
&= 2\lambda^{-p}\tfrac{\partial u(z,\tau)}{\partial \tau} \\
&= 2\lambda^{-p}[\Delta_z u(z,\tau) + u^p(z,\tau)] \\
&= 2\lambda^{-p}\Delta_z u(z,\tau) + 2\lambda^{-p}u^p(z,\tau) \\
&= 2\lambda^{-1}\lambda^{1-p}\Delta_z u(z,\tau) + \tfrac{2}{2^p}u^p(z,\tau) \\
&= 2\lambda^{-1}\Delta u(\lambda^{\frac{1-p}{2}}x, \lambda^{1-p}t) + \tfrac{1}{2^{p-1}}v_\lambda^p(x,t) \\
&= \Delta v_\lambda(x,t) + \tfrac{1}{2^{p-1}}v_\lambda(x,t), x \in R^n \times (0, \tilde{T_\lambda}), \\
v_\lambda(x,0) &= 2\phi(\lambda^{\frac{1-p}{2}}x), x \in R^n,
\end{aligned}$$

where $\tilde{T_\lambda} = \lambda^{p-1}T_\lambda$ is the life span of v_λ, $z = \lambda^{\frac{1-p}{2}}x$ and $\tau = \lambda_{1-p}t$. Since $\lim_{s\to\infty} \parallel \phi \parallel_{L^s} = \parallel \phi \parallel_{L^\infty}$, there is a large enough r such that

$$2 \parallel \phi \parallel_{L^\infty} > \parallel \phi \parallel_{L^r} .$$

Then we may consider the following ordinary differential equation.

$$\begin{aligned}
&\tfrac{dv(t)}{dt} = \tfrac{1}{2^{p-1}}v(t)^p, t > 0, \\
&v(0) = \parallel \phi \parallel_{L^r} .
\end{aligned}$$

The ordinary differential equation implies that

$$\frac{1}{v^p}dv = \frac{1}{2^{p-1}}dt,$$

so

$$\int \frac{1}{v^{-p}}dv = \int \frac{1}{2^{p-1}}dt + C,$$

so

$$\frac{1}{1-p}v^{1-p} = \frac{1}{2^{p-1}}t + C.$$

But, by applying the initial condition $v(0) = \parallel \phi \parallel_{L^r}$, we have

$$C = \frac{1}{1-p} \parallel \phi \parallel_{L^r}^{1-p},$$

and so

$$\frac{1}{1-p}v^{1-p} = \frac{1}{2^{p-1}}t + \frac{1}{1-p} \parallel \phi \parallel_{L^r}^{1-p} .$$

Therefore,

$$v^{1-p} = \frac{1-p}{2^{p-1}}t + \parallel \phi \parallel_{L^r}^{1-p},$$

so

$$v^{p-1} = \frac{1}{\frac{1-p}{2^{p-1}}t + \parallel \phi \parallel_{L^r}^{1-p}},$$

and so the life span of v is $T = \frac{2^{p-1}}{p-1} \parallel \phi \parallel_{L^r}^{-(p-1)}$.

By the Comparison Principle(see [6]), we have

$$\tilde{T_\lambda} = \lambda^{p-1} T_\lambda \leq \frac{2^{p-1}}{p-1} \parallel \phi \parallel_{L^r}^{-(p-1)},$$

and so we conclude (2).

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