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SMOOTH POSITIVE SOLUTIONS TO AN ELLIPTIC MODEL WITH C^2 FUNCTIONS

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Abstract: Two species of animals are competing or cooperating in the same environment. Under what conditions do they coexist peacefully? Or under what conditions does either one of the two species become extinct, that is, is either one of the two species excluded by the other? We investigate this phenomena in mathematical point of view.

In this paper we concentrate on coexistence solutions of the competition or cooperation model

$$\begin{cases} \Delta u + ug(u,v) = 0, \\ \Delta v + vh(u,v) = 0, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \end{cases} \quad \text{in } \Omega,$$

This system is the general model for the steady state of a competitive or cooperative interacting system depending on growth conditions for g and h. The techniques used in this paper are elliptic theory, super-sub solutions, maximum principles, and spectrum estimates. The arguments also rely on some detailed properties of the solution of logistic equations.

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1. Introduction

A lot of research has been focused on reaction-diffusion equations modeling various systems in mathematical biology, especially the elliptic steady states of competitive and cooperative interacting processes with various boundary conditions. In earlier literature, investigations into mathematical biology models were concerned with studying those with homogeneous Neumann boundary conditions. Later on, the more important Dirichlet problems, which allow flux across the boundary, became the subject of study.

Suppose two species of animals, rabbits and squirrels for instance, are competing or cooperating in a bounded domain Ω. Let $u(x,t)$ and $v(x,t)$ be densities of the two habitats in the place x of Ω at time t. Then we have the following biological interpretation of terms.

(A) The partial derivatives $u_t(x,t)$ and $v_t(x,t)$ mean the rate of change of densities with respect to time t.

(B) The laplacians $\Delta u(x,t)$ and $\Delta v(x,t)$ stand for the diffusion or migration rates.

(C) The rates of self-reproduction of each species of animals are expressed as multiples of some positive constants a, d and current densities $u(x,t), v(x,t)$, i.e. $au(x,t)$ and $dv(x,t)$ which will increase the rate of change of densities in (A), where $a > 0, d > 0$ are called the self-reproduction constants.

(D) The rates of self-limitation of each species of animals are multiples of some positive constants b, f and the frequency of encounters among themselves $u^2(x,t), v^2(x,t)$, i.e. $bu^2(x,t)$ and $fv^2(x,t)$ which will decrease the rate of change of densities in (A), where $b > 0, f > 0$ are called the self-limitation constants.

(E) The rates of competition or cooperation of each species of animals are multiples of some positive constants c, e and the frequency of encounters of each species with the other $u(x,t)v(x,t)$, i.e. $cu(x,t)v(x,t)$ and $eu(x,t)v(x,t)$ which will decrease in competition and increase in cooperation the rate of change of densities in (A), where $c > 0, e > 0$ are called the competition constants.

(F) We assume that none of the species of animals is staying on the boundary of Ω.

Combining all those together, we have the dynamic competition model

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + au(x,t) - bu^2(x,t) - cu(x,t)v(x,t) \\ v_t(x,t) = \Delta v(x,t) + dv(x,t) - fv^2(x,t) - eu(x,t)v(x,t) \\ u(x,t) = v(x,t) = 0 \ \text{ for } \ x \in \partial\Omega, \end{cases} \quad \text{in } \ \Omega \times [0,\infty),$$

or equivalently,

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + u(x,t)(a - bu(x,t) - cv(x,t)) \\ v_t(x,t) = \Delta v(x,t) + v(x,t)(d - fv(x,t) - eu(x,t)) \\ u(x,t) = v(x,t) = 0 \ \text{ for } \ x \in \partial\Omega, \end{cases} \quad \text{in } \ \Omega \times [0,\infty),$$

or cooperation model

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + au(x,t) - bu^2(x,t) + cu(x,t)v(x,t) \\ v_t(x,t) = \Delta v(x,t) + dv(x,t) - fv^2(x,t) + eu(x,t)v(x,t) \\ u(x,t) = v(x,t) = 0 \ \text{ for } \ x \in \partial\Omega, \end{cases} \quad \text{in } \ \Omega \times [0,\infty),$$

or equivalently,

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + u(x,t)(a - bu(x,t) + cv(x,t)) \\ v_t(x,t) = \Delta v(x,t) + v(x,t)(d - fv(x,t) + eu(x,t)) \\ u(x,t) = v(x,t) = 0 \ \text{ for } \ x \in \partial\Omega. \end{cases} \quad \text{in } \ \Omega \times [0,\infty),$$

Here we are interested in the time independent, positive solutions, i.e. the positive solutions $u(x), v(x)$ of

$$\begin{cases} \Delta u(x) + u(x)(a - bu(x) - cv(x)) = 0 \\ \Delta v(x) + v(x)(d - fv(x) - eu(x)) = 0 \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases} \quad \text{in } \ \Omega, \tag{1}$$

or

$$\begin{cases} \Delta u(x) + u(x)(a - bu(x) + cv(x)) = 0 \\ \Delta v(x) + v(x)(d - fv(x) + eu(x)) = 0 \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases} \quad \text{in } \ \Omega, \tag{2}$$

which are called the coexistence state or the steady state. The coexistence state is the positive density solution depending only on the spatial variable x, not on the time variable t, and so its existence means the two species of animals can live peacefully and forever.

A lot of work about the existence and uniqueness of the coexistence state of the above steady state models has already been done during the last decades.(See [2], [3], [4], [7], [6], [12], [13].)

In [4], Cosner and Lazer established a sufficient and necessary conditions for the existence of positive solution to the competing system.

The following is their result:

Theorem 1.1. *In order that there exist positive smooth functions u and v in Ω satisfying (1) with $a = d$, it is necessary and sufficient that one of the*

following three sets of conditions hold, where λ_1 is as described in the Lemma 2.2.

$$(1)a > \lambda_1, b > e, c < f$$
$$(2)a > \lambda_1, b = e, c = f$$
$$(3)a > \lambda_1, b < e, c > f$$

Biologically, the Theorem 1.1 implies that they can coexist peacefully if their reproduction rates are large enough and their self-limitation and competition rates are balanced each other.

In [13], Korman and Leung established a sufficient and necessary conditions for the existence of positive solution to the cooperation system. The following is their result:

Theorem 1.2. *For existence of a positive solution to (2) it is necessary and sufficient that $ce < 1$.*

Biologically, the Theorem 1.2 indicates that their strong cooperation may decrease reproduction capacities and increase self-limitation abilities that cause their extinction.

However, in reality, the rates of change of population densities may vary in a more complicated and irregular manner than can be described by the simple competition model. In this paper we study rather general types of the system. We are concerned with the existence of positive coexistence when the relative growth rates are nonlinear, more precisely, the existence and uniqueness of a positive steady state of

$$\begin{cases} \Delta u + ug(u,v) = 0 \\ \Delta v + vh(u,v) = 0 \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases} \quad \text{in } \Omega,$$

where g and h are C^1 functions, Ω is a bounded domain in R^n and u, v are densities of the two competitive or cooperative species.

The functions g and h describe how species 1 (u) and 2 (v) interact among themselves and with each other.

The followings are questions raised in the general model with nonlinear growth rates.

Problem 1: Under what conditions do the species coexist?

Problem 2: When does either one of the species become extinct?

In Section 3, some sufficient and necessary conditions for the existence of positive solution in the competition system are obtained that generalizes the Theorem 1.1, and we can also see some nonexistence result. In Section 4. we

establish sufficient and necessary conditions for the existence of positive solution in the cooperating system that generalizes the Theorem 1.2.

2. Preliminaries

Before entering into our primary arguments and results, we must first present a few preliminary items that we later employ throughout the proofs detailed in this paper. The following definition and lemmas are established and accepted throughout the literature on our topic.

Definition 2.1. (Super and sub solutions) Consider

$$\begin{cases} \Delta u + f(x,u) = 0 \ \text{ in } \ \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{3}$$

where $f \in C^{\alpha}(\bar{\Omega} \times R)$ and Ω is a bounded domain in R^n.

(A) A function $\bar{u} \in C^{2,\alpha}(\bar{\Omega})$ satisfying

$$\begin{cases} \Delta \bar{u} + f(x,\bar{u}) \leq 0 \ \text{ in } \ \Omega, \\ \bar{u}|_{\partial\Omega} \geq 0 \end{cases}$$

is called a super solution to (3).

(B) A function $\underline{u} \in C^{2,\alpha}(\bar{\Omega})$ satisfying

$$\begin{cases} \Delta \underline{u} + f(x,\underline{u}) \geq 0 \ \text{ in } \ \Omega, \\ \underline{u}|_{\partial\Omega} \leq 0 \end{cases}$$

is called a sub solution to (3).

Lemma 2.1. *Let $f(x,\xi) \in C^{\alpha}(\bar{\Omega} \times R)$ and let $\bar{u}, \underline{u} \in C^{2,\alpha}(\bar{\Omega})$ be, respectively, super and sub solutions to (3) which satisfy $\underline{u}(x) \leq \bar{u}(x), x \in \bar{\Omega}$. Then (3) has a solution $u \in C^{2,\alpha}(\bar{\Omega})$ with $\underline{u}(x) \leq u(x) \leq \bar{u}(x), x \in \bar{\Omega}$.*

In our proof, we also employ accepted conclusions concerning the solutions of the following logistic equations.

Lemma 2.2. *(Established in [15]) Consider*

$$\begin{cases} \Delta u + uf(u) = 0 \ \text{ in } \ \Omega, \\ u|_{\partial\Omega} = 0, u > 0, \end{cases}$$

where f is a decreasing C^1 function such that there exists $c_0 > 0$ such that $f(u) \leq 0$ for $u \geq c_0$ and Ω is a bounded domain in R^n.

If $f(0) > \lambda_1$, then the above equation has a unique positive solution, where λ_1 is the first eigenvalue of $-\Delta$ with homogeneous boundary conditions whose corresponding eigenfunction is denoted by ϕ_1. We denote this unique positive solution as θ_f.

The most important property of this positive solution is that θ_f is increasing as f is increasing.

We specifically note that for $a > \lambda_1$, the unique positive solution of

$$\begin{cases} \Delta u + u(a-u) = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, u > 0, \end{cases}$$

is denoted by $\omega_a \equiv \theta_{a-x}$. Hence, θ_a is increasing as $a > 0$ is increasing.

Consider the system

$$\begin{aligned} \Delta u + f(x,u) &= 0 \text{ in, } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{4}$$

where $u = (u_1, ..., u_m)$ and $f = (f_1, ..., f_m)$ is quasimonotone increasing, i.e. $f_i(x,u)$ is increasing in u_j for all $j \neq i$.

Lemma 2.3. *([13]) Let w_λ be a family of subsolutions($\alpha \leq \lambda \leq \beta$) to (4), increasing in λ such that*

$$\Delta w_\lambda + f(x, w_\lambda) \geq 0 \text{ in } \Omega, w_\lambda = 0 \text{ on } \partial\Omega.$$

Assume also $u \geq w_\alpha$, w_λ does not satisfy (4) for any λ, and $\frac{\partial w_\lambda}{\partial n}$ changes continuously in λ on $\partial\Omega$. Then $u \geq \sup w_\lambda$.

3. Competing Species

Consider the system for two competing species of anmials

$$\begin{cases} \Delta u(x) + u(x)g(u,v) = 0 \\ \Delta v(x) + v(x)h(u,v) = 0 \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases} \text{ in } \Omega, \tag{5}$$

where $g, h \in C^1$ are such that $g_u < 0, g_v < 0, h_u < 0, h_v < 0$, there exist constants $c_0 > 0, c_1 > 0$ such that $g(u,0) \leq 0$ for $u \geq c_0$ and $h(0,v) \leq 0$ for $v \geq c_1$.

If there were no competition between the species, that is, if we consider

$$\begin{cases} \Delta u + ug(u,0) = 0 \\ \Delta v + vh(0,v) = 0 \\ u = v = 0 \text{ on } \partial\Omega, \end{cases} \text{ in } \Omega,$$

then the condition $g(0,0) > \lambda_1, h(0,0) > \lambda_1$(i.e. reproductions are relatively large) were sufficient to guarantee the existence of a positive density solution $\theta_{g(\cdot,0)}, \theta_{h(0,\cdot)}$. But, if there is some competition between them, then as we see in the following Theorem 3.1, we should have the balance conditions for self-limitation and competition rates.

The following theorem provides a sufficient condition for the existence of a positive smooth solution to (5).

Theorem 3.1. *Suppose $g(0,0) = h(0,0)$ and one of the following three sets of conditions holds.*

$$\begin{array}{ll} (1) g(0,0) > \lambda_1, & \inf(g_u) < \inf(h_u), \quad \inf(g_v) > \inf(h_v) \\ (2) g(0,0) > \lambda_1, & \inf(g_u) = \inf(h_u), \quad \inf(g_v) = \inf(h_v) \\ (3) g(0,0) > \lambda_1, & \inf(g_u) > \inf(h_u), \quad \inf(g_v) < \inf(h_v) \end{array}$$

Then (5) has a positive smooth solution.

Proof. By the Theorem 1.1, if one of the above three sets of conditions holds, then there is a positive smooth solution (u,v) to

$$\begin{array}{l} \Delta u + u[g(0,0) - (-\inf(g_u))u - (-\inf(g_v))v] = 0 \\ \Delta v + v[h(0,0) - (-\inf(h_u))u - (-\inf(h_v))v] = 0 \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \end{array} \text{ in } \Omega$$

But, by the Mean Value Theorem,

$$\begin{aligned} & \Delta u + ug(u,v) \\ = \ & \Delta u + u[g(0,0) + g(u,v) - g(0,0)] \\ = \ & \Delta u + u[g(0,0) + g(u,v) - g(0,v) + g(0,v) - g(0,0)] \\ \geq \ & \Delta u + u[g(0,0) + \inf(g_u)u + \inf(g_v)v] \\ = \ & \Delta u + u[g(0,0) - (-\inf(g_u))u - (-\inf(g_v))v] \\ = \ & 0, \end{aligned}$$

and

$$\begin{aligned} & \Delta v + vh(u,v) \\ = \ & \Delta v + v[h(0,0) + h(u,v) - h(0,0)] \\ = \ & \Delta v + v[h(0,0) + h(u,v) - h(0,v) + h(0,v) - h(0,0)] \\ \geq \ & \Delta v + v[h(0,0) + \inf(h_u)u + \inf(h_v)v] \\ = \ & \Delta v + v[h(0,0) - (-\inf(h_u))u - (-\inf(h_v))v] \\ = \ & 0. \end{aligned}$$

Hence, (u, v) is a subsolution to (5).

But by the conditions of g, h, any large positive constant M satisfying $u < M, v < M$ in Ω is a supersolution to (5).

Therefore, by the Lemma 2.1, (5) has a positive smooth solution.

The next theorem establishes a necessary condition for the existence of a positive smooth solution to (5).

Theorem 3.2. *Suppose $g(0,0) = h(0,0)$. If (5) has a positive smooth solution, then $g(0,0) > \lambda_1$ and one of the following six sets of conditions holds.*

$$
\begin{array}{l}
(1) g_u \equiv h_u \text{ are constants, } \inf(h_v) \le \sup(g_v), \sup(h_v) \ge \inf(g_v) \\
(2) \inf(h_u) = \sup(g_u), \sup(h_u) > \inf(g_u), \inf(h_v) \le \sup(g_v) \\
(3) \inf(h_u) > \sup(g_u), \inf(h_v) < \sup(g_v) \\
(4) \inf(h_u) < \sup(g_u), \sup(h_u) = \inf(g_u), \sup(h_v) \ge \inf(g_v) \\
(5) \inf(h_u) < \sup(g_u), \sup(h_u) < \inf(g_u), \sup(h_v) > \inf(g_v) \\
(6) \inf(h_u) < \sup(g_u), \sup(h_u) > \inf(g_u)
\end{array}
$$

Proof. Suppose (u, v) is a positive smooth solution to (5).

By the Mean Value Theorem, there are $\tilde{u}, \tilde{v}$ with $0 \le \tilde{u} \le u, 0 \le \tilde{v} \le v$ such that

$$
\begin{aligned}
g(u,0) - g(0,0) &= g_u(\tilde{u}, 0)u, \\
g(u,v) - g(u,0) &= g_v(u, \tilde{v})v.
\end{aligned}
$$

Hence, by the Green's Identity,

$$
\begin{aligned}
& \int_\Omega u\phi_1[\lambda_1 - g(0,0) - g_u(\tilde{u},0)u - g_v(u,\tilde{v})v]dx \\
= & \int_\Omega u\phi_1[\lambda_1 - g(0,0) + g(0,0) - g(u,0) + g(u,0) - g(u,v)]dx \\
= & \int_\Omega u\phi_1[\lambda_1 - g(u,v)]dx \\
= & \int_\Omega \phi_1[-ug(u,v)] + u\lambda_1\phi_1 dx \\
= & \int_\Omega \phi_1\Delta u - u\Delta\phi_1 dx \\
= & \ 0.
\end{aligned}
$$

But, since $-g_u(\tilde{u}, 0)u - g_v(u, \tilde{v})v > 0$ in Ω, $g(0,0) > \lambda_1$.

By the Mean Value Theorem again, there are u_1, u_2, v_1, v_2 with $0 \le u_1, u_2 \le u, 0 \le v_1, v_2 \le v$ such that

$$
\begin{aligned}
g(u,v) - g(0,v) &= g_u(u_1, v)u, \\
h(u,v) - h(0,v) &= h_u(u_2, v)u, \\
g(0,v) - g(0,0) &= g_v(0, v_1)v, \\
h(0,v) - h(0,0) &= h_v(0, v_2)v.
\end{aligned}
$$

Therefore, by the Green's Identity again,

$$\begin{aligned}
& \int_\Omega uv([h_u(u_2,v)-g_u(u_1,v)]u+[h_v(0,v_2)-g_v(0,v_1)]v)dx \\
= \; & \int_\Omega uv[h_u(u_2,v)u+h_v(0,v_2)v-g_u(u_1,v)u-g_v(0,v_1)v]dx \\
= \; & \int_\Omega uv[h(u,v)-h(0,v)+h(0,v)-h(0,0)+g(0,v)-g(u,v)-g(0,v) \\
& +g(0,0)]dx \\
= \; & \int_\Omega uv[h(u,v)-g(u,v)]dx \\
= \; & \int_\Omega v\Delta u-u\Delta v dx \\
= \; & 0,
\end{aligned}$$

and so,

$$\begin{aligned}
&\int_\Omega uv([\inf(h_u)-\sup(g_u)]u+[\inf(h_v)-\sup(g_v)]v)dx\le 0,\\
&\int_\Omega uv([\sup(h_u)-\inf(g_u)]u+[\sup(h_v)-\inf(g_v)]v)dx\ge 0,
\end{aligned}$$

which derives

$$\begin{aligned}
&(A)\inf(h_u)=\sup(g_u),\inf(h_v)\le\sup(g_v),\\
&(B)\inf(h_u)>\sup(g_u),\inf(h_v)<\sup(g_v),\\
&(C)\inf(h_u)<\sup(g_u),
\end{aligned}$$

and

$$\begin{aligned}
&(A')\sup(h_u)=\inf(g_u),\sup(h_v)\ge\inf(g_v),\\
&(B')\sup(h_u)<\inf(g_u),\sup(h_v)>\inf(g_v),\\
&(C')\sup(h_u)>\inf(g_u).
\end{aligned}$$

Combining $(A),(B),(C)$ and $(A'),(B'),(C')$ together, we may have

$$\begin{aligned}
&(A")g_u\equiv h_u \text{ are constants, } \inf(h_v)\le\sup(g_v),\sup(h_v)\ge\inf(g_v),\\
&(B")\inf(h_u)=\sup(g_u),\sup(h_u)<\inf(g_u),\inf(h_v)\le\sup(g_v),\sup(h_v)>\inf(g_v),\\
&(C")\inf(h_u)=\sup(g_u),\sup(h_u)>\inf(g_u),\inf(h_v)\le\sup(g_v),\\
&(D")\inf(h_u)>\sup(g_u),\sup(h_u)=\inf(g_u),\inf(h_v)<\sup(g_v),\sup(h_v)\ge\inf(g_v),\\
&(E")\inf(h_u)>\sup(g_u),\sup(h_u)<\inf(g_u),\inf(h_v)<\sup(g_v),\sup(h_v)>\inf(g_v),\\
&(F")\inf(h_u)>\sup(g_u),\sup(h_u)>\inf(g_u),\inf(h_v)<\sup(g_v),\\
&(G")\inf(h_u)<\sup(g_u),\sup(h_u)=\inf(g_u),\sup(h_v)\ge\inf(g_v),\\
&(H")\inf(h_u)<\sup(g_u),\sup(h_u)<\inf(g_u),\sup(h_v)>\inf(g_v),\\
&(I")\inf(h_u)<\sup(g_u),\sup(h_u)>\inf(g_u).
\end{aligned}$$

However, it is clear that $(B"),(D"),(E")$ are not possible, so we establish the result of the Theorem.

We easily recognize that combining the Theorems 3.1 and 3.2 generalizes the result of Theorem 1.1 with linear growth rates.

We also prove a nonexistence result. In biological terms, this result confirms that sufficiently large reproduction capacity of the species 1 with fixed reproduction rate of the species 2 results in their extinction.

Theorem 3.3. *If* $g(0,0) > \frac{\nu}{\mu}h(0,0), -1 \le g_u < 0$, *and* $h_v \le -1$, *where* $\mu = \min[-\sup(h_u), 1]$ *and* $\nu = \max[-\inf(g_v), 1]$, *then there is no positive solution to (5).*

Proof. Suppose there is a positive solution (u, v) to (5).

Then by the Mean Value Theorem, the Green's Identity and the inequality conditions,

$$\begin{aligned} & \int_\Omega (g(0,0) - h(0,0) + [-\sup(h_u) - 1]u + [1 + \inf(g_v)]v)uvdx \\ \le\ & \int_\Omega (g(0,0) - h(0,0) + [\inf(g_u) - \sup(h_u)]u + [\inf(g_v) \\ & - \sup(h_v)]v)uvdx \\ \le\ & \int_\Omega [g(0,0) - h(0,0) + g(u,0) - g(0,0) - h(u,v) + h(0,v) \\ & + g(u,v) - g(u,0) \\ & - h(0,v) + h(0,0)]uvdx \\ =\ & \int_\Omega [g(u,v) - h(u,v)]uvdx \\ =\ & \int_\Omega (v\Delta u - u\Delta v)dx \\ =\ & 0. \end{aligned} \tag{6}$$

But, if $g(0,0) > \frac{\nu}{\mu}h(0,0)$, then since $g(0,0) \ge u$ and $h(0,0) \ge v$,

$$g(0,0) - h(0,0) + [-\sup(h_u) - 1]u + [1 + \inf(g_v)]v \ge \mu g(0,0) - \nu h(0,0) > 0,$$

which contradicts to (6).

4. Cooperating Species

Consider the system for two cooperating species of animals

$$\begin{cases} \Delta u(x) + u(x)g(u(x), v(x)) = 0 \\ \Delta v(x) + v(x)h(u(x), v(x)) = 0 \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases} \quad \text{in } \Omega, \tag{7}$$

where $g, h \in C^1$ are such that $g_u < 0, g_v > 0, h_u > 0, h_v < 0$.

The following Theorem proves a necessary condition for the existence of a positive solution to (7).

Theorem 4.1. *If* $g(0,0) > \lambda_1, h(0,0) > \lambda_1, \inf(h_v) \geq -1$, *and* $\inf(h_u) > 0$, *then for the existence of a positive solution to (7) implies*

$$\inf(g_v)\inf(h_u) + \inf(g_u) < 0.$$

Proof. Suppose $\inf(g_v)\inf(h_u) + \inf(g_u) \geq 0$. Consider a family $(u_\lambda, v_\lambda) = (\lambda\phi_1, \lambda\inf(h_u)\phi_1)$ with any $\lambda > 0$.

Then by the assumption and Mean Value Theorem,

$$\begin{aligned}
& \Delta u_\lambda + u_\lambda g(u_\lambda, v_\lambda) \\
= \;& -\lambda\lambda_1\phi_1 + \lambda\phi_1 g(\lambda\phi_1, \lambda\inf(h_u)\phi_1) \\
= \;& \lambda\phi_1(-\lambda_1 + g(\lambda\phi_1, \lambda\inf(h_u)\phi_1) \\
= \;& \lambda\phi_1(-\lambda_1 + g(\lambda\phi_1, \lambda\inf(h_u)\phi_1) - g(\lambda\phi_1, 0) + g(\lambda\phi_1, 0) - g(0,0) + g(0,0)) \\
= \;& \lambda\phi_1(g(0,0) - \lambda_1 + g(\lambda\phi_1, \lambda\inf(h_u)\phi_1 - g(\lambda\phi_1, 0) + g(\lambda\phi_1, 0) - g(0,0)) \\
\geq \;& \lambda\phi_1(g(0,0) - \lambda_1 + \inf(g_v)\inf(h_u)\lambda\phi_1 + \inf(g_u)\lambda\phi_1) \\
> \;& 0,
\end{aligned}$$

and

$$\begin{aligned}
& \Delta v_\lambda + v_\lambda h(u_\lambda, v_\lambda) \\
= \;& -\lambda\inf(h_u)\lambda_1\phi_1 + \lambda\inf(h_u)\phi_1 h(\lambda\phi_1, \lambda\inf(h_u)\phi_1) \\
= \;& \lambda\inf(h_u)\phi_1(-\lambda_1 + h(\lambda\phi_1, \lambda\inf(h_u)\phi_1)) \\
= \;& \lambda\inf(h_u)\phi_1(-\lambda_1 + h(\lambda\phi_1, \lambda\inf(h_u)\phi_1) - h(\lambda\phi_1, 0) + h(\lambda\phi_1, 0) - h(0,0) \\
& +h(0,0)) \\
= \;& \lambda\inf(h_u)\phi_1(h(0,0) - \lambda_1 + h(\lambda\phi_1, \lambda\inf(h_u)\phi_1) - h(\lambda\phi_1, 0) + h(\lambda\phi_1, 0) \\
& -h(0,0)) \\
\geq \;& \lambda\inf(h_u)\phi_1(h(0,0) - \lambda_1 + \inf(h_v)\lambda\inf(h_u)\phi_1 + \inf(h_u)\lambda\phi_1) \\
> \;& 0.
\end{aligned}$$

Therefore, $(u_\lambda, v_\lambda) = (\lambda\phi_1, \lambda\inf(h_u)\phi_1)$ with any $\lambda > 0$ is a family of subsolutions to (7).

Furthermore, if (u, v) is a positive solution to (7), then $u > \lambda_0\phi_1$ and $v > \lambda_0\inf(h_u)\phi_1$ for sufficiently small $\lambda_0 > 0$, and so by the lemma 2.3, we conclude that $u \geq \lambda\phi_1$ and $v \geq \lambda\inf(h_u)\phi_1$ for any $\lambda \geq \lambda_0$.

Hence, there is no positive solution to (7).

For a sufficient condition for the existence of a positive solution to (7), we need the following Lemma.

Lemma 4.2. *If* $bf > ce$, *then we can choose arbitrary large* $M, N > 0$ *such that*

$$\begin{aligned}
a - bM + cN < 0, \\
d + eu - fN < 0.
\end{aligned}$$

We now establish a sufficient condition for the existence of a positive solution to (7).

Theorem 4.3. *If* $g(0,0) > \lambda_1, h(0,0) > \lambda_1$ *and*

$$\sup(g_v)\sup(h_u) < \sup(g_u)\sup(h_v),$$

then (7) has a positive solution.

Proof. Let $\underline{u} = \alpha\phi_1, \underline{v} = \beta\phi_1$, where $\alpha, \beta > 0$.

Then since $g(0,0) > \lambda_1$ and $h(0,0) > \lambda_1$, by the Mean Value Theorem, for small enough $\alpha, \beta > 0$,

$$\begin{aligned}
& \Delta\underline{u} + \underline{u}g(\underline{u},\underline{v}) \\
= \;& -\alpha\lambda_1\phi_1 + \alpha\phi_1 g(\alpha\phi_1, \beta\phi_1) \\
= \;& \alpha\phi_1[-\lambda_1 + g(\alpha\phi_1, \beta\phi_1)] \\
= \;& \alpha\phi_1[-\lambda_1 + g(0,0) + g(\alpha\phi_1, \beta\phi_1) - g(0, \beta\phi_1) + g(0, \beta\phi_1) - g(0,0)] \\
\geq \;& \alpha\phi_1[-\lambda_1 + g(0,0) + \inf(g_u)\alpha\phi_1 + \inf(g_v)\beta\phi_1] \\
\geq \;& 0,
\end{aligned}$$

and

$$\begin{aligned}
& \Delta\underline{v} + \underline{v}h(\underline{u},\underline{v}) \\
= \;& -\beta\lambda_1\phi_1 + \beta\phi_1 h(\alpha\phi_1, \beta\phi_1) \\
= \;& \beta\phi_1[-\lambda_1 + h(\alpha\phi_1, \beta\phi_1)] \\
= \;& \beta\phi_1[-\lambda_1 + h(0,0) + h(\alpha\phi_1, \beta\phi_1) - h(0, \beta\phi_1) + h(0, \beta\phi_1) - h(0,0)] \\
\geq \;& \beta\phi_1[-\lambda_1 + h(0,0) + \inf(h_u)\alpha\phi_1 + \inf(h_v)\beta\phi_1] \\
\geq \;& 0,
\end{aligned}$$

and so, $(\underline{u}, \underline{v}) = (\alpha\phi_1, \beta\phi_1)$ is a subsolution to (7) for sufficiently small $\alpha, \beta > 0$.

But, for all (u, v), by the Mean Value Theorem again,

$$\begin{aligned}
g(u,v) &= g(0,0) + g(u,v) - g(u,0) + g(u,0) - g(0,0) \\
&\leq g(0,0) + \sup(g_v)v + \sup(g_u)u,
\end{aligned}$$

and

$$\begin{aligned}
h(u,v) &= h(0,0) + h(u,v) - h(u,0) + h(u,0) - h(0,0) \\
&\leq h(0,0) + \sup(h_v)v + \sup(h_u)u,
\end{aligned}$$

so by the condition and the Lemma 4.2, there are constants $M, N > 0$ with $\alpha\phi_1 < M, \beta\phi_1 < N$ such that

$$\begin{aligned}
&\Delta M + Mg(M,N) \leq M[g(0,0) + \sup(g_v)N + \sup(g_u)M] < 0, \\
&\Delta N + Nh(M,N) \leq N[h(0,0) + \sup(h_v)N + \sup(h_u)M] < 0,
\end{aligned}$$

in other words, (M, N) is a supersolution to (7).

We conclude by the Lemma 2.1 that there is a positive solution (u, v) to (7) with $\alpha\phi_1 \leq u \leq M, \beta\phi_1 \leq v \leq N$.

In biological terms, the Theorems 4.1 and 4.3 illustrate that the reproduction rates must be large enough, and self-limitation rates should be relatively larger that cooperation ones for their peaceful coexistence.

We easily recognize that combining the Theorems 4.1 and 4.3 generalizes the result of Theorem 1.2 with linear growth rates.

5. Conclusions

In this paper, our investigation of the effects of nonlinear growth rates on the competition and cooperation models resulted in the development and proof of Theorems 3.1, 3.2, 3.3, 4.1 and 4.3, as detailed above. The results together assert that right choice of functions $g(u, v)$ and $h(u, v)$ will maintain the existence of the positive steady state. Indeed, our results specifically outline conditions sufficient and necessary to maintain the positive, steady state solution when rates of reproduction, self-limitation, competition and cooperation are nonlinear.

Applying this mathematical results to real world situations, our results establish that two species residing in the same environment can vary their interactions, within certain limitations, and continue to survive together indefinitely at densities.The conditions necessary for coexistence, as described in the Theorems, simply require that competing members' rates of reproduction, self-limitation and competition are well-balanced, and cooperating members of each species interact strongly with themselves and weakly with members of the other species.

While our research in this paper therefore represents a progression in the field, the results obtained have an important limitation. Our model describes the interactions of only two species who reside in the same environment, so the parallel conditions required for the coexistence of more than two species need to be investigated in future research(For example, see [8] that has results of perturbation of the model with arbitrary N species of animals, and is the development of [14] that has the results with two species of animals.). Mathematically, analysis of competition and cooperation models for N populations would expand the community's understanding of the behavior of functions and extend established theory in the field. Biologically, the investigation of models for N species would increase knowledge on the nature of interactions between

any number of species within the same environment. Thus, the results achieved through our research will enable both fields to continue the development of theory on interaction of populations.

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