Spring 2-2016

Positive Solutions to a General Elliptic System with Smooth Functions

Timothy Robertson
St Andrews Presbyterian College, robertsont@andrews.edu

Joon Hyuk Kang
Andrews University, kang@andrews.edu

Andrews University
Department of Mathematics

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Office of Research & Creative Scholarship
Andrews University

Timothy Robertson, Joon Hyuk Kang

Department of Mathematics, Andrews University
Berrien Springs, Michigan

INTRODUCTION

The purpose of this research is to give a sufficient condition for the existence and nonexistence of positive solutions to a rather general type of elliptic system of the Dirichlet problem on a bounded domain. This research was supported by the Office of Research and Creative Scholarship.

The techniques used include upper-lower solutions, eigenvalues of operators, maximum principles, and spectrum estimates.

The arguments also rely on some detailed properties of the solutions of logistic equations.

These results yields algebraically computable criteria for the positive coexistence of competing species of animals in many biological models.

$$\begin{cases} 0=\Delta u+g(u,v) \\ 0=\Delta v+h(u,v) \\ (u,v)\big|_{\partial\Omega}=(0,0) \end{cases} \quad \text{in } \Omega$$

In this model, $g, h \in C^2$ and $\Omega \subset R^n$.

For our purposes, we assume the following growth conditions:

$(G_1)\, g_v<0, g_{uu}<0, g_{uv}<0, h_u<0, h_{uv}<0, h_{vv}<0$

$(G_2)\, g(0,v)\geq 0, h(u,0)\geq 0$

(G_3) There is $c_0>0$ such that $u, v > c_0 \Longrightarrow$

$g(u,0)<0, g_u(u,0)<0, h(0,v)<0, h_v(0,v)<0$

APPLICATION

One of the prominent subjects of study and analysis in mathematical biology concerns the competition of two or more species of animals in the same environment.

Especially pertinent areas of investigation include the conditions under which the species can coexist, as well as the conditions under which any one of the species becomes extinct, that is, one of the species is excluded by the others.

We focus on the general population model to better understand the interactions between two species. Specifically, we investigate the conditions needed for the coexistence of two species when the factors affecting them are perturbed.

Within the academia of mathematical biology, extensive work has been devoted to investigation of the simple population model, commonly known as the Lotka-Volterra model. This system describes the interaction of two species residing in the same environment in the following manner:

$$u_t(x,t)=\Delta u(x,t)+u(x,t)(a-bu(x,t)-cv(x,t))$$
$$v_t(x,t)=\Delta v(x,t)+v(x,t)(d-eu(x,t)-fv(x,t)$$

In this model, u(x,t) and v(x,t) designate the population densities of the two species. The constant coefficients in this system represent growth rates (a and d), self-limitation rates (b and f) and competition rates (c and e).

The mathematical community has already established several results for the existence, uniqueness and stability of the positive steady state solution to the time-independent system. Our research represents a significant generalization of the existing results relating to this model.

RESULTS

(A) $g_u(0,c_0)>\lambda_1, h_v(c_0,0)>\lambda_1 \Rightarrow$ It has a positive solution. Any positive solution (u,v) satisfies

$$\theta_{g_u(\bullet,c_0)} < u \leq c_0$$
$$\theta_{h_v(c_0,\bullet)} < v \leq c_0$$

(B) $g_u(0,0)\leq\lambda_1, h_v(0,0)\leq\lambda_1 \Longrightarrow$ No positive solution

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