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Topology of Fractals

Honors Thesis

Amelia Pompilio

Department: Mathematics

Advisor: Lynne Yengulalp, Ph.D.

May 2019

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Acknowledgements

By November 10, 2017, I had to answer a very important question. The University Honors Program knew of my intent to pursue an undergraduate honors thesis, but they needed to know what I planned to study and who would advise me in doing so. I'd like to first thank Dr. Lynne Yengulalp for responding to a certain e-mail I sent on November 9th, 2017.

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Thank you to all who have molded me into the type of person who loves doing math; I hope to pay it forward through projects like this thesis and beyond.

Contents

Chapter 1

Introduction

The purpose of this thesis is to explore fractals both topologically and computationally. What makes fractals so peculiar, as Benoit Mandelbrot noted, is that they can be either naturally occurring or purely abstract. Such is the basis for the two-pronged approach in this thesis. We consider the numerous characteristics a fractal may or may not have, and how we use different tools for studying different types of fractals.

The first tool we pick up is topology. Informally, studying the topological properties of an object gives us insight into how that object behaves spatially. For example, we can think of geometry as a helpful tool for understanding the properties of objects like squares and spheres as well as giving us some language to describe what makes them different. Likewise, topology gives us an environment for studying fractals that live in the plane but are not very straightforward geometrically. We will define properties such as connectedness and compactness and apply them to our study of fractals. One of the most important topological concepts throughout the rest of the thesis is that of open covers and finite subcovers. This section equips us with the foundation we will need to understand different notions of dimension.

The main criterium that qualifies an object as a fractal is an inequality involving two different notions of dimension; if an object's Hausdorff dimension exceeds its topological dimension, then it is a fractal. In Chapter 3 we discuss how both of these dimensions are constructed. The difficulty of calculating the Hausdorff dimension of a fractal in part comes from the completely unrestricted collection of open covers that we can consider in the construction its Hausdorff measure. However, a property of some fractals called self-similarity allows us to side-step this process with a simple formula. We will see examples of this special case as well as a third type of dimension, the Box-Counting dimension, that allows us to perform a similar computation much more easily. The Box-Counting dimension is similar to the Hausdorff dimension as they both rely on the construction of open covers, but is different in that the elements of the cover are restricted to all being the same size. These two constructions of dimension coincide for self-similar objects, but can differ on irregular sets.

Though not the only fractal studied in this thesis, the Mandelbrot set is certainly a personal favorite. One of the first questions that popped up during our work was "what is the Hausdorff dimension of the boundary of the Mandelbrot set?" The answer is known to be 2, but since this particular fractal is not self-similar, there is no convenient formula to verify this. So we turned to a numerical approach. We outline our process for generating fractal images in R, employ the Box-Counting method to generate a cover of a fractal boundary, and analyze the output to determine the approximate dimension value.

Chapter 2

Topological Background

The objects of study in this thesis primarily live in the plane. So, we present some topological preliminaries to describe subsets of the plane.

2.1 Metric Spaces and Topological Spaces

Definition 2.1.1. A function $d : X \times X \to [0, \infty)$ is a **metric** if the following properties are satisfied:

i) $\forall\, x, y \in X,\ d(x, y) \geq 0$,

ii) $d(x, y) = 0 \leftrightarrow x = y$,

iii) $d(x, y) = d(y, x)$, and

iv) $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 2.1.2. We say (X, d) is a **metric space** if d is a metric on X.

Definition 2.1.3. An **open ball** [2] of radius ϵ about $x \in X$ is defined as

$$B_\delta(x, \epsilon) = \{x \in \mathbb{R} : d(x, \epsilon) < \delta\} \tag{2.1}$$

Definition 2.1.4. A **closed ball** of radius ϵ about $x \in X$ is defined as

$$C_\delta(x, \epsilon) = \{x \in \mathbb{R} : d(x, \epsilon) \geq \delta\} \tag{2.2}$$

Example 2.1.5. If X is a set,

$$d(x, y) = \begin{cases} 1 & if\ x \neq y \\ 0 & if\ x = y \end{cases}$$

is a metric on X called the *discrete metric*.

Example 2.1.6. $\mathbb{R}^2$ with the Euclidean distance $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ is a metric space.

Definition 2.1.7. Let (X, d) be a metric space. Let A be a subset of X. We say A is **open** if $\forall\, x \in A, \exists\, \epsilon > 0$ such that $B(x, \epsilon) \subseteq A$, where $B(x, \epsilon)$ is an open ball of radius ϵ about x.

We say that a set $C \subseteq X$ is closed if its complement, denoted $X \backslash C$, is open. The following are important properties related to openness.

1) $\emptyset$ is open,

2) X is open since $\forall\, x \in X,\ B(x, \epsilon) \subseteq X$ for any $\epsilon > 0$,

3) If $U_1, U_2, \cdots, U_n$ are open, then $U_1 \cap U_2 \cap U_3 \cap \cdots \cap U_n$ is open.

   Proof. Suppose $U_1, U_2, \ldots, U_n$ are open. Suppose $x \in U_1 \cap U_2 \cap U_3 \cap \cdots \cap U_n$. Then $\exists\, \epsilon_1, \epsilon_2, \ldots, \epsilon_n$ such that $B(x, \epsilon_i) \subseteq U_i$, for each $i = 1, \ldots n$. Let $\epsilon = min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$. Then $B(x, \epsilon) \subseteq U_i$, $\forall\, i$, and thus $B(x, \epsilon) \subseteq U_1 \cap U_2 \cap U_3 \cap \cdots \cap U_n$. Hence the finite intersection of open sets is open. □

4) If $\mathcal{U}$ is a collection of open sets, then $\bigcup \mathcal{U}$ is open.

   Proof. Let $\mathcal{U}$ be a collection of open sets, and let $x \in \bigcup \mathcal{U}$. Then $x \in U$ for some $U \in \mathcal{U}$. Since U is open, $\exists\, \epsilon > 0$ such that $B(x, \epsilon) \subseteq U$. Since $U \subseteq \bigcup \mathcal{U}$, then $B(x, \epsilon) \subseteq \bigcup \mathcal{U}$ and thus $\bigcup \mathcal{U}$ is open. □

When working in metric spaces, the following is the working definition of continuity.

Definition 2.1.8. A function $f : (X, \mu) \to (Y, \rho)$ is **continuous** at a if, for all $\epsilon > 0$, there exists $\delta > 0$ such that for all x, $\mu(x, a) < \delta$ implies that $\rho(f(x), f(a)) < \epsilon$.

We can reconceptualize the definition of continuity with open balls. This leads us to the abstract definition that one uses in topology.

Definition 2.1.9. A function $f : (X, \mu) \to (Y, \rho)$, is **continuous** if and only if for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $f[B_\mu(x, \delta)] \subseteq B_\rho(f(x), \epsilon)$. In other words, $f : X \to Y$ is continuous if and only if $f^{-1}[U]$ is open for each open set U in Y. The definition also holds for closed sets and their inverse images [3].

The definition of a topology, below, is reminiscent of the properties of openness.

Definition 2.1.10. Suppose X is a set. A **topology** is a collection, τ, of subsets of X that satisfies:

i) $\emptyset \in \tau$,

ii) $X \in \tau$,

iii) if $U_1, \cdots, U_n \in \tau$, then $U_1 \cap \cdots \cap U_n \in \tau$, and

iv) if $\mathcal{U} \subseteq \tau$, then $\bigcup \mathcal{U} \in \tau$

We say (X, τ) is a **topological space**. Just like how the finite intersection of open sets is open, the finite intersection of subsets of τ is also in τ. And similarly again, the infinite union of open sets is open, and the infinite union of subsets of τ is still in τ. This justifies us referring to the sets $U \in \tau$ as "open" sets. Likewise, a set $C \subseteq X$ is closed if $X \backslash C\ \in \tau$.

Definition 2.1.11. Let (X, τ) be a topological space. We say (X, τ) is **Hausdorff** if $\forall x, y \in X$, if $x \neq y$, there exist $U, V \in \tau$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$

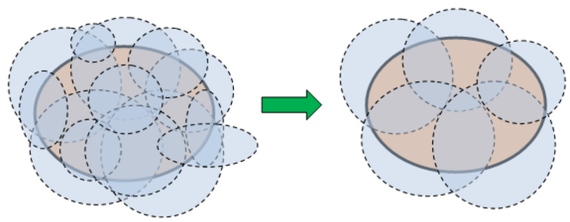
Remark. *Metric spaces are Hausdorff*

2.2 Compactness

The following definitions will be especially helpful in our discussion of the Mandelbrot set's topological properties.

Definition 2.2.1. An **open cover** $\mathcal{U}$ of a topological space X is a collection of open subsets of X such that $X \subseteq \bigcup \mathcal{U}$. An open cover $\mathcal{U}$ of a subset K of a topological space X is a collection $\mathcal{U}$ of open subsets of X such that $K \subseteq \bigcup \mathcal{U}$.

Definition 2.2.2. A subset K of a topological space X is **compact** if, for every open cover $\mathcal{U}$ of a subset K, there is a finite subcover $K \subseteq U_1 \cup U_2 \cup \cdots \cup U_n$ where $\{U_1, U_2, \ldots, U_n\} \subseteq \mathcal{U}$.



Proposition 2.2.3. *If K is compact, and $C \subseteq K$ and C is closed, then C is also compact. In other words, a closed subset of a compact space is compact.*

Proof. Suppose K is compact, and $C \subseteq K$ is closed. Let $\mathcal{U}$ be an open cover of C. That is, $C \subseteq \bigcup \mathcal{U}$. Set $\mathcal{W} = \mathcal{U} \cup \{K \backslash C\}$. Note that $\mathcal{W}$ consists of open sets because C is closed. Also, $\mathcal{W}$ covers K. Since K is compact, there is a finite subcover W_1, $W_2, \cdots$, $W_n \in \mathcal{W}$. Thus $\{W_1$, $W_2, \cdots$, $W_n\} \backslash \{X \backslash C\}$ is a finite subcover of C generated from $\mathcal{U}$, so C is compact. $\square$

Using this definition of compactness can be clunky – we have to construct a finite subcover with whatever information we have about the set that we want to show is compact. Luckily, since the Mandelbrot set is an object in the complex plane, we can use a well-known theorem that gives us simpler conditions for compactness. We need one more definition and a lemma before we get to it, though.

Definition 2.2.4. A subset K of a metric space X is **bounded** if there is some $x \in X$ and some $R \in \mathbb{R}$ such that $K \subseteq B(x, R)$.

We prove the following lemma by using the "completeness" of $\mathbb{R}^n$.

Lemma 1. *In $\mathbb{R}^n$, given any x and any $r > 0$, $C(x, r)$ is compact.*

Proof. (of Lemma). Suppose by way of contradiction that $C(x, r)$ is not compact. Then there exists an open cover $\mathcal{U}$ of $C(x, r)$ with no finite subcover. We choose $x_0 \in C(x, r)$. Since $\mathcal{U}$ covers $C(x, r)$, there is some $U_0 \in \mathcal{U}$ with $x_0 \in U_0$. In particular, $\{U_0\}$ is not a cover of $C(x, r)$. Let $x_1 \in C(x, r) \backslash U_0$. Since $\mathcal{U}$ covers $C(x, r)$, there exists some $U_1 \in \mathcal{U}$ with $x_1 \in U_1$, and $\{U_0, U_1\}$ does not cover $C(x, r)$. We then let $x_2 \in C(x, r) \backslash (U_1 \cup U_2)$, and so on as we construct a sequence $\{x_n\}$ in $C(x, r)$, which is bounded. But we also know that any bounded sequence in $\mathbb{R}^n$ has a convergent subsequence, and thus $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ which converges to some $L \in \mathbb{R}^n$. It must be the case that $L \in C(x, r)$ since $C(x, r)$ is closed, and so there is some $U \in \mathcal{U}$ with $L \in \mathcal{U}$. Take $\epsilon > 0$ with $L \in B(x, \epsilon) \subseteq \mathcal{U}$. Since $\{x_{n_k}\}$ converges to L, there must be $M \in \mathbb{N}$ such that $\{x_{n_k}\} \in B(L, \epsilon)\ \forall\ k \geq M$. But then this gives us a finite subcover $U_{n_1}, U_{n_2}, \ldots, U_{n_m}$. Thus $C(x, r)$ is compact. $\square$

Theorem 1. The Heine-Borel Theorem. A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof. ($\Longrightarrow$) Suppose K, a subset of $\mathbb{R}^n$, is compact. First we will show that K is closed. Recall that K is closed when, for all $x \notin K$, there exists an $\epsilon > 0$ such that $B(x, \epsilon) \cap K = \emptyset$.
Let $x \notin K$ be arbitrary. Define $\mathcal{U} = \{\mathbb{R}^n \backslash C(x, r) : r \in \mathbb{R}\}$.

<u>Claim:</u> $\mathcal{U}$ covers K. That is, $K \subseteq \bigcup \mathcal{U}$.

Proof. (of claim). Let $y \in K$. Since we know that $x \notin K$, then $y \neq x$. Let $r = \frac{1}{2}d(x, y) > 0$. Notice that $y \notin C(x, r)$. Thus we know that $y \in \mathbb{R}^n \backslash C(x, r)$. So, we have shown that there exists an $r \in \mathbb{R}$ such that $y \in X \backslash C(x, r)$. Thus $\mathcal{U} = \{\mathbb{R}^n \backslash C(x, r) : r \in \mathbb{R}\}$ covers K. □

Now, since K is compact, U has a finite subcover: $\mathbb{R}^n \backslash C(x, r_1), \mathbb{R}^n \backslash C(x, r_2), \ldots, \mathbb{R}^n \backslash C(x, r_m)$. Let $\epsilon = \min\{r_1, \ldots, r_m\}$.

<u>Claim:</u> $B(x, \epsilon) \cap K = \emptyset$. That is, the ball of radius ϵ does not hit K.

Proof. (of claim). We have that $K \subseteq \mathbb{R}^n \backslash C(x, r_1) \cup \mathbb{R}^n \backslash C(x, r_2) \cup \cdots \cup \mathbb{R}^n \backslash C(x, r_m) = \mathbb{R}^n \backslash C(x, \epsilon)$. Thus since $K \subseteq \mathbb{R}^n \backslash C(x, \epsilon)$, then $K \cap C(x, \epsilon) = \emptyset$. And since $B(x, \epsilon) \subseteq C(x, \epsilon)$ by definition, then $K \cap B(x, \epsilon) = \emptyset$. □

Thus we have shown that K is closed. We now show that K is bounded.
Let $x = (0, 0, \ldots, 0)$. Define $\mathcal{U} = \{B(x, r) : r > 0, r \in \mathbb{R}\}$.

<u>Claim:</u> $\mathcal{U}$ covers K. ($K \subseteq \bigcup \mathcal{U}$).

Proof. (of claim). Let $y \in K, y \neq x$. Then $r_0 = d(x, y) \geq 0$. It is clear that $y \in B(x, r_0 + 1)$. Thus $\mathcal{U}$ covers K. □

Since K is compact, $\mathcal{U}$ has a finite subcover $B(x, r_1), B(x, r_2), \ldots, B(x, r_n)$. Let $\epsilon = max\{r_1, \ldots, r_n\}$. Then $K \subseteq B(x, r_1) \cup B(x, r_2) \cup \cdots \cup B(x, r_n) \subseteq B(x, \epsilon)$, and therefore K is bounded.

($\Longleftarrow$) Suppose K is closed and bounded. We will show that K is also compact.
Since K is bounded, there exists an $x \in \mathbb{R}^n$ and an $r > 0$ such that $K \subseteq B(x, r) \subseteq C(x, r)$. By the Lemma, $C(x, r)$ is compact.

So now, since we know that $C(x, r)$ is compact, K is closed, and $K \subseteq C(x, r)$, by Proposition 2.2.3 K is compact. □

Phew! For the following results, we need to introduce the finite intersection property.

Definition 2.2.5. A collection $\mathcal{F}$ of subsets of a set X has the **finite intersection property** if the intersection of any finite subcollection of $\mathcal{F}$ is nonempty.

Proposition 2.2.6. *If $\mathcal{F}$ is a collection of nonempty closed subsets of a compact space X with the finite intersection property, then $\bigcap \mathcal{F} \neq \emptyset$.*

Proof. Suppose by way of contradiction that $\bigcap \mathcal{F} = \emptyset$. The following is an open cover of X: $\mathcal{U} := \{X \backslash F : F \in \mathcal{F}\}$. We know that each $X \backslash F \in \mathcal{U}$ is open because it is the complement of a closed set. We know that $\mathcal{U}$ is a cover of X because $\bigcap \mathcal{F} = \emptyset$, so by DeMorgan's laws the union of the complements of the sets in $\mathcal{F}$ must be equal to the whole space. That is, $\bigcup\limits_{F \in \mathcal{F}} X \backslash F = X$. Now, $\mathcal{U}$ must have a finite subcover $\{X \backslash F_1, X \backslash F_2, \cdots, X \backslash F_n\}$ since X is compact. So we have $X \backslash F_1 \cup X \backslash F_2 \cup \cdots \cup X \backslash F_n = X$. By DeMorgan's law, this is equivalent to $F_1 \cap F_2 \cap \cdots \cap F_n = \emptyset$. But this contradicts the fact that $\mathcal{F}$ has the finite intersection property. So, $\bigcap \mathcal{F} \neq \emptyset$. □

Proposition 2.2.7. *If $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$ is a decreasing sequence of nonempty closed sets in a compact space X, then $F_1 \cap F_2 \cdots \neq \emptyset$.*

Proof. Since $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$ is a decreasing sequence of nonempty closed sets, this implies that $\mathcal{F} = \{F_1, F_2, \dots\}$ has the finite intersection property. Thus the proof follows as above. $\square$

For our final compactness related result, we define two more fundamentals of topology - connectedness and closure.

Definition 2.2.8. A subset A of a metric space X is **connected** if there do not exist open sets U and V such that

1) $U \cap A \neq \emptyset$,$V \cap A \neq \emptyset$,

2) $A \subseteq U \cup V$, and

3) $U \cap V = \emptyset$.

Definition 2.2.9. For a subset A of a topological space X, we say $x \in A$ is a cluster point of A if, for every open set U containing x, $U \cap A \neq \emptyset$. The **closure** of A is the set $\overline{A} = \operatorname{cl} A = \{x \in X : x$ is a cluster point of $A\}$

Proposition 2.2.10. *Suppose $\{K_n\colon n \in \mathbb{N}\}$ is a decreasing sequence of compact connected subsets of X. Then $\bigcap\limits_{n\in\mathbb{N}} K_n$ is connected.*

Proof. Suppose K is not connected. So there exists U such that $K \subset U \cup (X \setminus \operatorname{cl} U), U \cap K \neq \emptyset$, and $(X \setminus \operatorname{cl} U) \cap K \neq \emptyset$. We know that each individual K_i is connected. So, U does not separate any K_i. So for each $i, K_i \nsubseteq U \cup (X \setminus \operatorname{cl} U), U \cap K_i \neq \emptyset$, or $(X \setminus \operatorname{cl} U) \cap K_i \neq \emptyset$. Set $B_i = X \setminus (U \cup X \setminus \operatorname{cl} U) \cap K_i$. Note that the B_i's are nested, so their intersection is nonempty since each $\{K_n : n \in \mathbb{N}\}$ is compact. Since $\bigcap B_i \neq \emptyset$, each $K_i \nsubseteq U \cup (X \setminus \operatorname{cl} U)$. But $K \subset U \cup (X \setminus \operatorname{cl} U)$. So, K is connected. $\square$

Chapter 3

Dimension

One of the most important topological aspects of the Mandelbrot set, as with the study of any particular fractal, is its dimension. We can intuitively understand 1, 2, and 3-dimensional objects with simple examples such as lines, squares, and cubes. But when we consider these objects, what is it exactly that we are "counting" in order to determine that a cube is *three* dimensional?

Usually, we think of dimension as the number of coordinates or variables we need in order to locate a point or describe an object. For our cube, we need length, width, and height to describe it, and we need a value for each one of those dimensions in order to find a particular point in that 3-dimensional space. This definition of dimension will suffice for most of the geometry one might see out in the real world.

This is where things start to sound a little bit like science fiction. There are more types of dimension than just the intuitive one we've known since second grade. We will see three different types of dimension in this chapter.

3.1 Topological Dimension

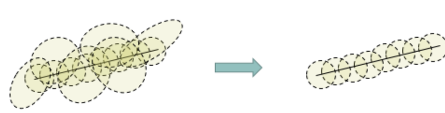
To define topological dimension, we'll need two properties of open covers:

1. The *order* of a cover is the smallest number n such that each point of the covered topological space belongs to at most n members of the cover.

2. A *refinement* $\mathcal{V}$ of a cover $\mathcal{U}$ is another cover, such that for all $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ with $V \subseteq U$. The order of a refinement may be lower than the orignal cover.

Definition 3.1.1. The **topological dimension**, or the Lebesgue covering dimension, of a topological space X is the minimum value n such that every open cover of X has an open refinement of order $n + 1$ or less.

Let's look at an example to help us sort this out.



If the line segment above is covered with some arbitrary collection of open sets like the picture on the left, we can arrange a refinement like the one on the right. We can see that a point on the line is covered by at most two open sets of the refinement. The order of this refinement is therefore 2, so we have that the topological dimension of this line segment is 1.

3.2 Hausdorff Dimension

Hausdorff dimension is a little more complicated than the topological dimension. We present the construction of Hausdorff dimension in a three step process.

Step 1. Hausdorff Outer Measure

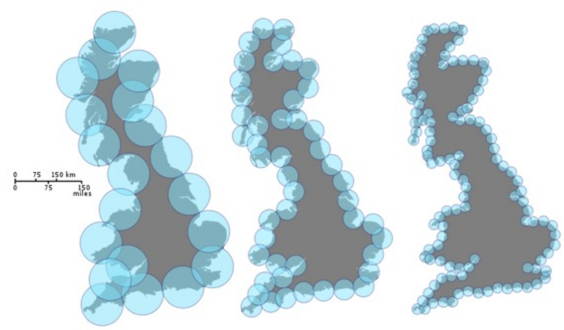
Definition 3.2.1. The **Hausdorff Outer Measure** [4] of dimension d bounded by δ, written H_δ^d, is defined by

$$H_\delta^d = \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} U_i)^d \colon \bigcup_{i=1}^{\infty} U_i \supseteq S, \operatorname{diam} U_i < \delta \right\} \tag{3.1}$$

where the infimum is taken over all countable covers of S by open sets $U_i \subseteq X$ satisfying $\operatorname{diam}(U_i) < \delta$.

Remark. $\operatorname{diam} U_i = \sup\{d(x, y) : x, y \in U_i\}$, *where distance is defined by the metric,* d *of the given metric space.*

It is easier to conceptualize Hausdorff Outer Measure with a picture. Below is a visual of three different covers of the coastline of Great Britain, with δ decreasing from left to right.



Imagine we want to compute H_δ^d for $d = 1$ and $d = 2$ and some small δ. We would cover the coastline with open sets of diameter less than δ and compute the sums $\sum_{i=1}^{\infty} \operatorname{diam} U_i$ and $\sum_{i=1}^{\infty} (\operatorname{diam} U_i)^2$. With the first sum, we're effectively computing a rough tracing of the boundary we're trying to measure. With the second sum, we are getting approximately the area of the ragged coastline's cover. Decreasing δ is analogous to increasing the resolution of the picture. Note that as δ becomes very small, the number of sums we consider in the infimum decreases. Thus the infimum itself may *increase* as delta shrinks.

Step 2. Limit Hausdorff Outer Measure

One can imagine that the rougher the boundary gets at higher and higher resolutions, the larger the length ($d = 1$) computation gets. On the other hand, the area ($d = 2$) computation is giving zero, since we expect a boundary to have no area. Both of these situations are describing what we would expect to see as we let δ approach zero.

Definition 3.2.2. The **limit Hausdorff Outer Measure** is defined as $\lim_{\delta \to 0} H_\delta^d(S) = H^d(S)$ and may be infinite.

Step 3. Hausdorff Dimension

If Great Britain's coastline was wiggly at every resolution, using $d = 1$ might lead to a limit Hausdorff outer measure of infinity, while $d = 2$ returns "zero area". So, in some sense, the coastline is two wiggly to be 1 dimensional, but to thin to be 2 dimensional. But we do not want such a region simply to have *no* dimension. There is no reason we can't compute H^d_δ for non-integer d. This leads us to the definition of Hausdorff dimension.

We'll need the following theorem to understand the definition of Hausdorff dimension.

Theorem 2 (Proposition 10.22, [1])**.** If $H^p(S) < \infty$, then $H^q(S) = 0$ for all $q > p$. If $H^p(S) > 0$, then $H^q(S) = \infty$ for all $q < p$.

Definition 3.2.3. The **Hausdorff dimension** of a set S, denoted $\dim_H(S)$, is the supremum of numbers q such that $H^q(S) = \infty$. Equivalently, $\dim_H(S) = \inf\{q : H^q(S) = 0\}$

Remark. *If $H^p(S)$ is a finite nonzero value, then for all $k \in [0, +\infty)$ such that $k \neq p$, $H^k(A)$ is either infinite or zero. If the set S has Hausdorff dimension $\dim_H(S) = p$, then*

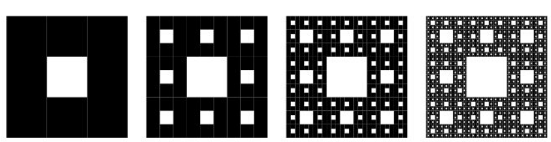
$$H^p(S) = \begin{cases} 0 & if\ k < p \\ +\infty & if\ k > p \end{cases}$$

Informally, the Hausdorff dimension of a set is the point of inflection, p, where all values of k less than $\dim_H(S) = p$ return a Hausdorff outer measure of $+\infty$, and all values of k greater than $\dim_H(S) = p$ return a Hausdorff outer measure of zero. Let's imagine that we want to find the Hausdorff dimension of a square in the plane (spoiler alert: it's 2). If we were to let $d = 1$, our Hausdorff outer measure would approach infinity since we are basically trying to measure an object's area with an object that has no area, i.e., a 1-dimensional line. We would need an infinitely long line to even begin to "fill up" the square. Since we have a tangible understanding of what integer dimensions represent in our real world, finding the Hausdorff outer measure of a square using $d = 1$ is like asking how *long* the square is – the problem here is that a square has two dimension, both its length and its height.

Now, if we let $d = 3$ in our Hausdorff outer measure computation for this square, we are asking how much *volume* the square has. Of course a 2-dimensional object has 0 volume since its third dimension, width, is 0. Note that an object of study can still have an infinite Hausdorff outer measure while still having finite Hausdorff dimension. For example, the real line is infinitely long but still has Hausdorff dimension 1.

So we are a bit more comfortable now with the idea that there are different ways to define dimension. But Hausdorff dimension, or *fractal dimension*, will be important to our discussion of fractals for a fascinating reason: the Hausdorff dimension of a particular object need not be an integer. Who needs Star Trek when we can study interdimensional beings right here?

Example 3.2.4. Let's try and find the Hausdorff dimension of the Sierpinski carpet, denoted S. The Sierpinski carpet is a fractal that is constructed by taking a square, dividing it into 9 congruent squares, and then removing the innermost square. This process is then repeated on the remaining eight squares and continues on recursively. The following figure shows the first four iterations.



It's important here to note that the Sierpinski carpet is *self-similar*. This means that it decomposes into pieces that look exactly like the original set. For example, the upper left square of the second iteration looks exactly like the entire first iteration.

Let's explore how dimension behaves in self-similar objects. Consider a (two dimensional) square with area 4. When we scale the square by a factor of 3, the area is scaled by 9. Now, consider a (one dimensional) line segment of length 2. When we scale the square by a factor of 3, the length is scaled by 3. We then observe the following relationship:

$$\text{measure of scaled object} = \text{measure of original} * (\text{scale factor})^{\text{dimension}}. \tag{3.2}$$

Suppose we have a Sierpinski carpet of area 1. Since each Sierpinski carpet has 8 copies of itself, when we scale by a factor of 3, the resulting carpet will have exactly 8 copies of the original carpet, i.e., the area has been scaled by 8. Following the equation above, we get

$$8 = 1 * (3)^{\text{dimension}}. \tag{3.3}$$

In other words, we can find the dimension of the Sierpinski carpet by solving the equation

$$\dim(S) = \frac{\log 8}{\log 3} \approx 1.89 \tag{3.4}$$

Briggs and Tyree's derivation of this result for general self-similar sets [4], which we apply to S here, ends with the following equation:

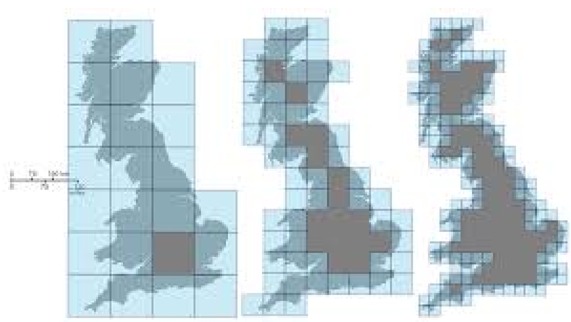
$$\dim_H(C) = p = \frac{\log m}{\log 1/r} \tag{3.5}$$

where m is the number of copies belonging to the similitude family that are created at each iteration of the recursion, and r is the scaling factor. In this way we can calculate a self-similar object's Hausdorff dimension without using any open covers.

We will see in the next section that not all Hausdorff dimension computations are as feasible as that of the Sierpinski carpet. The simple formula that allowed us to find the carpet's Hausdorff dimension was a direct result of its own self-similarity. Luckily, there are still other approaches that allow us to study the dimensions of different fractals.

3.3 Box-Counting Dimension

The box-counting dimension has a lot in common with Hausdorff dimension, and is commonly used as a simpler alternative [5]. The name gives us some intuition – we use boxes of decreasing size to cover the object whose dimension we want to study.



Seem familiar? On the surface it seems as though we've just substituted our various-sized open covers with boxes. But the mechanics of what we do with these boxes is a bit different. We see that the box covers consist of boxes all of the *same size*, whereas the open covers in the definition of Hausdorff dimension have no "same size" requirement.

Definition 3.3.1. We define $N_\epsilon(S)$ to be the minimum number of boxes of side length ϵ we need in order to construct a cover of a set S. The **upper and lower box-counting dimensions** are defined as

$$\overline{\dim}_{box}(S) := \overline{\lim}_{\epsilon\to 0}\frac{\log N_\epsilon(S)}{\log(1/\epsilon)} \tag{3.6}$$

$$\underline{\dim}_{box}(S) := \underline{\lim}_{\epsilon\to 0}\frac{\log N_\epsilon(S)}{\log(1/\epsilon)} \tag{3.7}$$

When the upper and lower box-counting dimensions coincide, the **box-counting dimension**, or the Minkowski-Bouligand dimension, is defined as

$$\dim_{box}(S) := \lim_{\epsilon\to 0}\frac{\log N_\epsilon(S)}{\log(1/\epsilon)} \tag{3.8}$$

Remark. *$\overline{\dim}_{box}(S)$ and $\underline{\dim}_{box}(S)$ always exist and can be useful when $\dim_{box}(S)$ does not exist [6]. $\dim_{box}(S)$ is well-defined when the upper and lower box dimensions are equal [6].*

The construction here looks similar to equation 3.5, but now the fractal being measured need not be self-similar. In place of m and r, we have one value that gives information about the size of each box in the open cover.

Remark. *Hausdorff dimension is related to the Box-Counting dimension through the following inequality [5]:*

$$\dim_H(S) \le \underline{\dim}_{Box}(S) \le \overline{\dim}_{Box}(S) \tag{3.9}$$

The following examples help us see the distinction between these different notions of dimension.

Example 3.3.2. The Hausdorff dimension of the set of rational numbers in the unit interval, $\mathbb{Q}\cap[0,1]$, is 0.

This result lies in the fact that the rationals are countable. Suppose we are computing $H^1(\mathbb{Q}\cap[0,1])$. We can construct an open cover that is a sequence of open balls, each centered on a rational, whose radii are any sequence of positive real numbers that converges to zero rapidly. Hence their sum, from Calculus II $\left(\text{think } \sum_{n=1}^{\infty}\frac{\epsilon}{2^n}\right)$, can be as small as we like. In other words, $H^1(\mathbb{Q}) = 0$. Pushing this idea further, we can see that $H^d(\mathbb{Q}) = 0$ for any $d > 0$, $\left(\text{think } \sum_{n=1}^{\infty}\left(\frac{\epsilon}{2^{n/d}}\right)^d\right)$. Using $d = 0$ in the computation returns infinity since we would be adding infinitely many 1's. Thus the Hausdorff dimension of the rationals is 0.

Remark. *If there were a "same-size" restriction of the sets in the open covers in the computation of H^d_δ, the result would be much different. Each sum would necessarily evaluate to infinity. Applying routine infinite series facts, we see that under this type of restriction, the dimension would be 1.*

Example 3.3.3. The box-counting dimension of, $\mathbb{Q}\cap[0,1]$, is 1.

To see this, observe that it requires n many boxes of size $1/n$ to cover the rationals in $[0,1]$.

Upper and lower box counting dimensions may differ for fractals and other objects that behave differently at different scaling. Imagine a line in space that coils like a spring with a large radius. If we attempt to find the Box-Counting dimension of this object with a very small initial value of ϵ, our dimension computation will pick up on the fact that we are looking at a single line. The expected Box-Counting dimension would then be 1. However, at an intermediate value of ϵ, our computation can lose just enough detail to result in an interpretation of the object as a solid cylinder and reports that the object's dimension is 2. "Zooming out" even further, we can choose an ϵ large enough such that the entire coil's circumference is contained in one box. At this resolution, the coil looks like a single line itself and thus the Box-Counting dimension again becomes 1.

Chapter 4

Fractals

4.1 Definition

> "Defining fractals for mathematicians is a bit like defining life for biologists: We all know what it is that we're trying to encapsulate, but every time you try to nail it down with a concrete definition it seems to neglect some of what you want or admit some of what you don't." - Grant Sanderson [13]

You may be wondering why Great Britain keeps popping up in the middle of this math thesis. And you'd be in good company, since a famous mathematician by the name Benoit Mandelbrot wondered why math kept popping up in the middle of nature [11]. He was one of the first to further examine a phenomenon known as the coastline paradox: the smaller the ruler one uses to measure the coastline of a landmass, the longer the coastline seems to get [12]. We've already encountered this idea in our discussion of Hausdorff measure when we saw that a smaller δ returned a more detailed image. Objects with this property have been in our world long before any mathematicians came along, but mathematicians were the first to call them "fractals".

At this point, our calculus alarm bells might be ringing. Usually things get smoother and more linear as we zoom in, so what gives [13]? Recall our discussion of self-similarity in Chapter 3. We can characterize fractals by their ability to retain self-similarity while being altered by a scaling factor. Yet not all fractals are self-similar – an exact but smaller copy of the coastline of Great Britain will likely never emerge as we continue to zoom in on the original. So what makes a non-self-similar object a fractal?

Mandelbrot dubbed such objects "fractals" because they appear to be fractured in some way. To him, a fractal had the property of *roughness*, which becomes obvious when we look at the rough edges of nature. Even from a wide range of vantage points, mountain tops remain jagged, clouds remain lumpy, and coastlines remain fractured. These objects aren't *exactly* self-similar, but their *roughness* is preserved when scaled.

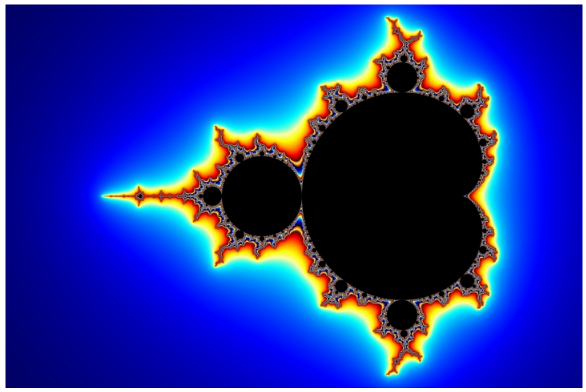
If one-dimensionality is length, two-dimensionality is area, and three-dimensionality is volume, the analogous interpretation of fractal dimensionality is roughness. Finally we can understand and compare fractals like the Sierpinski carpet and a coastline: the one with the larger Hausdorff dimension is *more rough* than the other. This brings us to the widely accepted definition of fractals that puts both the natural and mathematically abstract ones in the same family.

Definition 4.1.1. A **fractal** is a subset of a Euclidean space whose Hausdorff dimension strictly exceeds its topological dimension.

4.2 The Mandelbrot Set

Now we are ready to define the main fractal of this thesis, discovered by and named after Benoit Mandelbrot himself.

Definition 4.2.1. The **Mandelbrot set** is the set of complex numbers c for which the function $f_c(z) = z^2 + c$ does not diverge when iterated from $z = 0$.

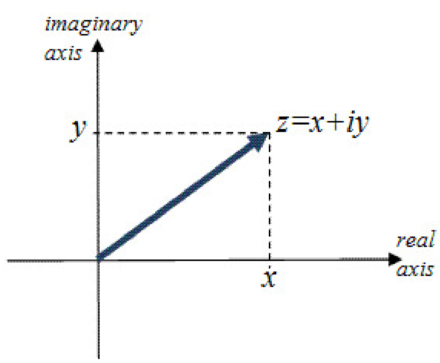


The Mandelbrot set lives in the complex plane, so it is useful to review the basics of complex numbers here. Complex numbers are ordered pairs of real numbers, so topologically the set of complex numbers is the same as $\mathbb{R}^2$. However, the complex numbers have an additional algebraic structure.

Complex Preliminaries

Definition 4.2.2. A **complex number**, written $a + bi$, is a number comprised of a real part $a \in \mathbb{R}$ and an imaginary part $b \in \mathbb{R}$, where $i^2 = -1$.

To create the complex plane, we plot the real part along the x-axis and the imaginary part along the y-axis. This way of expressing a complex number is rectangular form. Unlike the real numbers, there is no "order" among the complex numbers – 3+4i is no greater or less than 2-7i. But one way we can create some order is to define the norm of a complex number.



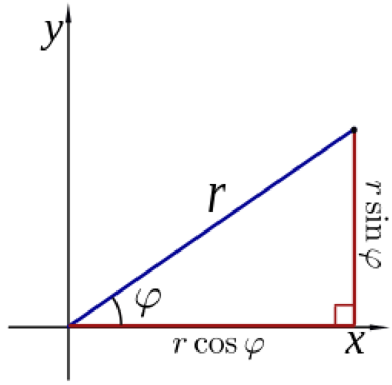
[7]

Definition 4.2.3. The **norm** of a complex number $a + bi$ is$\sqrt{a^2 + b^2}$.

The norm, which is the distance of $a + bi$ to the origin, is also known as the magnitude and is denoted $|a + bi|$. Instead of orienting a complex number graphically using its real and imaginary parts as coordinates, we can transform it using the norm and some trigonometry. This is the polar form of a complex number.

Definition 4.2.4. The **polar form** of a complex number $a + bi$ is $r(\cos\varphi + \sin i\varphi)$, or $re^{i\varphi}$ where $r = \sqrt{a^2 + b^2}$.

To graph a complex number in polar form, we use the polar coordinate system. We think of r as the length of a ray originating from the origin, and φ is the angle that this ray makes with the x-axis.



[8]

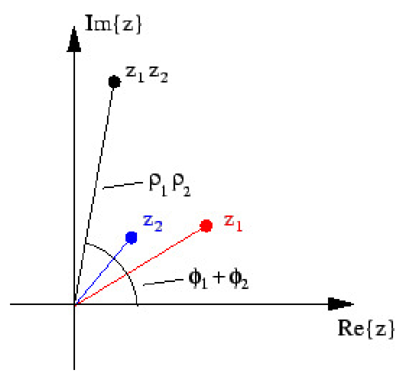
These two ways of presenting complex numbers consequently give us two ways to think about multiplication. In rectangular form, we foil two complex numbers together to get their product.

$$(a + bi)(c + di) = ac + bci + adi + bdi^2 = ac + bci + adi - bd \tag{4.1}$$

But this doesn't tell us much about how the new product relates to what we had before. Multiplying complex numbers in polar form gives us more information.

$$r_1 e^{i\varphi} \times r_2 e^{i\varphi} = r_1 r_2 e^{i\varphi + i\varphi} \tag{4.2}$$

This product gives us more insight into what complex multiplication is really doing. Out front we have $r_1 r_2$, our norms, where the first number's norm has been scaled up by a factor of the second number's norm. Thus the distance from the origin has changed. In the exponent, we have angles being added together. The ray we started with may be making a different angle with the x-axis now. To locate where the product of these two complex numbers landed in space, we can understand its movement as the aftermath of scaling and rotating the original ray [9]. That's much more information than we get from finding a new rectangular coordinate.



[9]

Properties of the Mandelbrot Set

The Mandelbrot set is certainly a rough looking thing. It is interesting that something so "natural" looking can come from a simply defined formula. And while that roughness has been shown to be preserved at higher and higher resolutions, the set is not perfectly self-similar [11]. Modern graphics have shown that almost identical looking objects do arise (and they're affectionately known as mini-brots), but they are not an exact copy of the original. Further, the set's boundary is so rough and detailed that it has an unexpected *integer* Hausdorff dimension. The Japanese mathematician Mitsuhiro Shishikura gave a proof of the following result, which we will not reproduce here [10].

Theorem 3. The boundary of the Mandelbrot set has Hausdorff dimension 2.

Proposition 4.2.5. *The Mandelbrot set is compact.*

Proof. We will show that the Mandelbrot set is compact by using the Heine-Borel theorem since it lives in Euclidean space. We can define the Mandelbrot set M in the following way:

$$M = \left\{ c \in \mathbb{C} : \langle f_c^{(n)}(0) \rangle_{n=1}^{\infty} \text{ is bounded} \right\} \tag{4.3}$$

First we will show that M is bounded. Say we fix $|c| > 2$ and let $c_n = f_c^{(n)}(0)$. Then we have the following inequality for all n:

$$|c_{n+1}| = |c_n^2 + c| \geq |c_n^2| - |c_n| \geq 2|c_n| - |c_n| = |c_n|$$

From this we can conclude that $|c_n| > 2$ for all n.

Now, let $\epsilon = |c| - 2 = |c_0| - 2 > 0$. Observe that

$$|c_1| - |c_0| = |c_0^2 + c_0| - |c_0| \geq |c_0^2| - |c_0| - |c_0| = |c_0|(|c_0| - 2) \geq |c_0| - 2 = \epsilon \tag{4.4}$$

Now, suppose that for some n, $|c_{n+1}| \geq |c_n| + \epsilon$. A similar sequence of inequalities to (4.4) gives us $|c_{n+2}| - |c_{n+1}| \geq |c_{n+1}| - 2$. By assumption, $|c_{n+1}| - 2 \geq |c_n| + \epsilon - 2$ and since $|c_n| > 2$, we have $|c_{n+2}| - |c_{n+1}| \geq \epsilon$. This means that the sequence $\{|c_n|\}$ increases by at least ϵ at each term. Therefore, $\{c_n\}$ is unbounded. Therefore, we have shown that any c with $|c| > 2$ is not in the Mandelbrot set.

Thus we have that $c \leq 2$ for all $c \in M$, or that M is contained within the closed ball of radius 2.

Now we want to show that M is closed. Well, by the previous observation, we know the following:

$$M = \bigcap_{n \in \mathbb{N}} f_c^{(n)^{-1}}[B] \tag{4.5}$$

where B is the closed ball of radius 2. Since $f_c(z) = z^2 + c$ is a polynomial, it is continuous and thus by definition 2.1.9, the preimage of the closed set B is also closed. Since the intersection of closed sets is also closed, M is closed.

$\square$

Chapter 5

Approximating Fractal Dimension

We present a script for approximating the fractal dimension of the boundary of the Mandelbrot set. We first generated the Mandelbrot set using a portion of the following R script written by Myles Harrison [14]. Parts of this script were used in our counting function which will be defined later.

```
1  # parameters
2  cols=colorRampPalette(c("blue","yellow","red","black"))(2)
3  xmin = -2
4  xmax = 2
5  nx = 150
6  ymin = -1.5
7  ymax = 1.5
8  ny = 150
9  n=200
10
11 # variables
12 x <- seq(xmin, xmax, length.out=nx)
13 y <- seq(ymin, ymax, length.out=ny)
14 c <- outer(x,y*1i,FUN="+")
15 z <- matrix(0.0, nrow=length(x), ncol=length(y))
16 k <- matrix(0.0, nrow=length(x), ncol=length(y))
17
18 for (rep in 1:n) {
19   print(rep)
20   for (i in 1:nx) {
21     for (j in 1:ny) {
22       if(Mod(z[i,j]) < 2 && k[i,j] < n) {
23         z[i,j] <- z[i,j]^2 + c[i,j]
24         k[i,j] <- k[i,j] + 1
25       }
26     }
27   }
28 }
29
30 image(x,y,k, col=cols)
```

MylesHarrisonGenerator.R

The parameter in line 2 specifies that the resulting image will have only two colors, blue and black. This was done so that we could more easily distinguish the boundary of the Mandelbrot set without its signature multi-colored aura. The more we iterate the function that generates the Mandelbrot set, the more detailed the boundary becomes. We use ‘n’ to control the number of iterations.

To determine which points comprise the boundary of the Mandelbrot set, we create a matrix that tracks how many times a point of the set has been iterated to determine its “escape speed”. We create a loop that can be adjusted based on what range of values will determine the cutoff escape speed of a boundary point. In this case, the range of iterations 10 to $n-1$ defines the boundary.
We created a function “CountMeADelta” that accepts a value of δ and returns a total count of how many boxes of side length δ intersect the boundary.

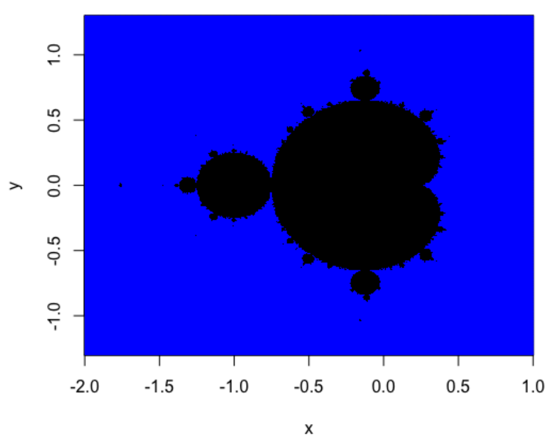
```r
#function that accepts a delta
CountMeADelta <- function(delta){
  #divide window up into delta-sized blocks
  nx <- floor((xmax-xmin)/delta)
  ny <- floor((xmax-xmin)/delta)
  x <- seq(xmin, xmax, length.out=nx)
  y <- seq(ymin, ymax, length.out=ny)

  c <- outer(x,y*1i,FUN="+")
  z <- matrix(0.0, nrow=length(x), ncol=length(y))
  k <- matrix(0.0, nrow=length(x), ncol=length(y))
  boundary <- matrix(0.0, nrow=length(x), ncol=length(y))

  #iterating the Mandelbrot function
  for (rep in 1:n) {
    print(rep)
    for (i in 1:nx) {
      for (j in 1:ny) {
        if(Mod(z[i,j]) < 2 && k[i,j] < n) {
          z[i,j] <- z[i,j]^2 + c[i,j]
          k[i,j] <- k[i,j] + 1
        }
      }
    }
  }

  count <- 0
  for (s in 1:nx){
    for (t in 1:ny){
      if (k[s,t]>10 && k[s,t]< (n-1)){
        count = count + 1
        boundary[s,t] = 1
      }
    }
  }
  image(x,y,k, col=cols)
  image(x,y,boundary, col=cols)
  View(k)
  return(count)
}
```

Mandelbrot.Hausdorff.R



It is fair to say that we have taken a blended approach to this approximation. Our construction is clearly the Box-Counting method since we are dividing the window into boxes of equal size and counting how many intersect the boundary. In this way we obtain our "Hausdorff measure" analog. However, the way we analyze the resulting dimension data is more in line with the definition of Hausdorff dimension. We refer to the resulting dimension as "fractal dimension" since we have not

strictly stuck to one definition in our computation and analysis.
Recall that the Hausdorff dimension of an object can be thought of, very informally, as the pivotal value of dimension d for which the coinciding Hausdorff measure changes from blowing up to infinity to approaching zero. To verify that the fractal dimension we chose is in fact the approximated Hausdorff dimension, we need to see empirically where this change occurs. Since we know that the Hausdorff dimension of the boundary is 2, we tested values of d from 1 to 3, incrementing by 0.05 each time.
To see the change in Hausdorff measure at each of these dimension values, we need to change the resolution. We chose 10 different values of δ, in increasing order given the construction of our loop, and computed the Hausdorff measure at each of these resolutions. The following code snippet shows how we constructed this computation.

```
colnum <- 0
rownum <- 0
haus <- 0
HausMatrix <- matrix(0.0, nrow = 10, ncol = 41)
d <- 1
for (increment in seq(0.001,0.01, by = 0.001)){
  rownum = rownum + 1
  colnum <- 0
  C <- CountMeADelta(increment)
  for (d in seq(1, 3, by = 0.05)){
    colnum = colnum + 1
    haus <- C*(increment)^d
    HausMatrix[rownum,colnum] = haus
    print(d)
    print(haus)
  }
}
write.csv(HausMatrix, "Haus.csv")
```

Mandelbrot.Hausdorff.R

The resulting matrix gives us a table for the approximated measure at every chosen δ and d. The table below shows a portion of this data.

Table 5.1: Measure Results for Dimensions 1.80 through 2.20

	d=1.80	d=1.85	d=1.90	d=1.95	d=2.00	d=2.05	d=2.10	d=2.15	d=2.20
$\delta = .001$	1.05552	0.74725	0.52901	0.37451	0.26513	0.18770	0.13288	0.09407	0.06660
$\delta = .002$	0.92052	0.67466	0.49447	0.36240	0.26561	0.19467	0.14267	0.10457	0.07664
$\delta = .003$	0.84365	0.63098	0.47192	0.35296	0.26399	0.19744	0.14767	0.11045	0.08261
$\delta = .004$	0.79854	0.60590	0.45973	0.34882	0.26467	0.20082	0.15238	0.11562	0.08772
$\delta = .005$	0.76247	0.58502	0.44887	0.34440	0.26425	0.20275	0.15557	0.11936	0.09158
$\delta = .006$	0.73213	0.56689	0.43894	0.33987	0.26316	0.20376	0.15777	0.12216	0.09459
$\delta = .007$	0.70136	0.54727	0.42703	0.33320	0.25999	0.20287	0.15830	0.12352	0.09638
$\delta = .008$	0.69626	0.54692	0.42962	0.33747	0.26509	0.20823	0.16357	0.12849	0.10093
$\delta = .009$	0.67492	0.53329	0.42138	0.33296	0.26309	0.20788	0.16426	0.12979	0.10255
$\delta = .010$	0.65560	0.52076	0.41366	0.32858	0.26100	0.20732	0.16468	0.13081	0.10391

We now explain why this table shows us that the approximated fractal dimension of the boundary of the Mandelbrot set is 2. Recall that using a value of d less than the actual dimension of an object will return a Hausdorff measure of infinity, and a value of d greater than the actual dimension returns zero. In our program, we will not see infinity or zero directly, but we can see the shift between these two behaviors reflected in the data.
For example, we see that $d = 1.85$ is not the approximated dimension of the boundary because the values in this column monotonically increase as the resolution increases. If we were to imagine the limit as δ approaches zero, we would see that the values in this column would continue to grow towards infinity. Similarly, $d = 2.10$ is not the approximated dimension either, since the values in this column are monotonically decreasing as δ approaches zero.

Notice, however, what happens when we set $d = 2.00$. We see values that neither monotonically increase nor decrease. We have thus experimentally determined the pivotal point where the behavior of our measure stabilizes and found the fractal dimension of the boundary.

Bibliography

[1] Gerald B. Folland. "Real analysis : modern techniques and their applications." New York, Wiley, c1984

[2] Eric W. Weisstein. "Open Ball." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/OpenBall.html

[3] David Grossier."Continuity, Images, and Inverse Images". University of Florida. https://people.clas.ufl.edu/groisser/files/cont_img_preimg.pdf

[4] Jimmy Briggs and Tim Tyree. "Hausdorff Measure". 3 December 2016.

[5] Johan Lindström. "A Method for Estimating the Fractal Dimension of Digital Color Images". Umeå University. 21 June 2008.

[6] Erin Pearse. "An Introduction to Dimension Theory and Fractal Geometry: Fractal Dimensions and Measures". http://pi.math.cornell.edu/ erin/docs/dimension.pdf. Cornell University.

[7] Bert G. Wachsmuth. "Interactive Complex Analysis". http://pirate.shu.edu/ wachsmut/complex/numbers/plane.html. 29 May 2007.

[8] "Polar Coordinate System". Wikipedia. Last updated 23 November 2018.

[9] Lewis A. Riley. "Review of Complex Numbers". http://webpages.ursinus.edu/lriley/ref/complex/node1.html. 19 January 2004.

[10] Mitsuhiro Shishikura. "The Hausdorff Dimension of the Boundary of the Mandelbrot Set and Julia Sets." Annals of Mathematics, vol. 147, no. 2, 1998, pp. 225–267. JSTOR, www.jstor.org/stable/121009.

[11] IBM. "Icons of Progress", Fractal Geometry. https://www.ibm.com/ibm/history/ibm100/us/en/icons/fractal/.

[12] Eric W. Weisstein. "Coastline Paradox". From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/CoastlineParadox.html.

[13] Grant Sanderson. "Fractals Are Typically Not Self-Similar". https://www.3blue1brown.com/videos/2017/5/26/fractals-are-typically-not-self-similar. 3blue1brown. 26 January 2017.

[14] Myles Harrison. "The Mandelbrot Set in R". https://www.r-bloggers.com/the-mandelbrot-set-in-r/. R Bloggers. 16 December 2014.