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COUNTABLE POSITIVE SOLUTIONS OF A CONJUGATE BOUNDARY VALUE PROBLEM

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Abstract

In this paper we consider the conjugate type nonlinear boundary value problem

$$(-1)^{n-k}u^{(n)}(t) = f(u(t)),\ 0 < t < 1,$$

$$\begin{cases} u^{(i)}(0) = 0, & i = 0, \dots, k-1 \\ u^{(j)}(1) = 0, & j = 0, \dots, n-k-1 \end{cases},$$

where $n \geq 2$ and $k \in \{1, \dots, n-1\}$, which under certain growth assumptions on $f(u)$ has countably many positive solutions. The results are based on applications of the fixed point theorems due to Krasnosel'skiĭ and Leggett-Williams.

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1 Introduction

Let $n \geq 2$ and $k \in \{1, ..., n-1\}$. We consider the class of $(k, n-k)$ conjugate BVPs,

$$(-1)^{n-k}u^{(n)}(t) = f(u(t)), \tag{1}$$

$$\begin{cases} u^{(i)}(0) = 0, & i = 0, ..., k-1, \\ u^{(j)}(1) = 0, & j = 0, ..., n-k-1. \end{cases} \tag{2}$$

Let $G(t,s)$ be the Green's function of $u^{(n)} = 0$ subject to (2). Define an integral operator, T, as follows

$$Tu(t) = (-1)^{n-k}\int_0^1 G(t,s)f(u(s))ds, \ 0 \leq t \leq 1. \tag{3}$$

Now, if we establish that (3) has a fixed point in a subset of a certain Banach space, then this will imply that there exists a solution of (1), (2).

The Green's function, $G(t,s)$, of (1) subject to the $(k, n-k)$ conjugate boundary conditions (2) satisfies $(-1)^{n-k}G(t,s) > 0, \ (t,s) \in (0,1) \times (0,1)$. For all $s \in [0,1]$, define

$$\|G(\cdot, s)\| = max_{t\in[0,1]}|G(t,s)|.$$

It was shown in [6], that for each fixed $s \in (0,1)$ and all $t \in [\frac{1}{4}, \frac{3}{4}]$,

$$(-1)^{n-k}G(t,s) \geq \sigma\|G(\cdot, s)\|, \tag{4}$$

where $\sigma = min\{\frac{1}{4^k}, \frac{1}{4^{n-k}}\}$. Moreover, for each fixed $s \in (0,1)$ and all $t \in [\frac{1}{4}, \frac{3}{4}]$, and $\tau \in [0,1]$,

$$(-1)^{n-k}G(t,s) \geq \sigma G(\tau, s). \tag{5}$$

More on the above stated and other properties of Green's functions can be found in Eloe and Henderson [6].

Let $\mathcal{B}$ be a real Banach space.

Definition 1 *A nonempty, closed set $\mathcal{P} \subset \mathcal{B}$ is said to be a cone provided:*

(i) $\alpha u + \beta v \in \mathcal{P}$ for all $u, v \in \mathcal{P}$ and $\alpha, \beta \geq 0$, and

(ii) $u, -u \in \mathcal{P}$ implies $u = 0$.

Definition 2 *A Banach space $\mathcal{B}$ is called a partially ordered Banach space provided there exists a partial ordering $\preceq$ on $\mathcal{B}$ satisfying:*

(i) $u \preceq v$ for $u, v \in \mathcal{B}$ implies $tu \preceq tv$ for all $t \geq 0$, and

(ii) $u_1 \preceq v_1$ and $u_2 \preceq v_2$ for $u_1, u_2, v_1, v_2 \in \mathcal{B}$ imply $u_1 + u_2 \preceq v_1 + v_2$.

One can point out that there is a convenient characterization of cones in terms of partial orderings. Let $\mathcal{P} \subset \mathcal{B}$ be a cone and define $u \preceq v$ if and only if $u - v \in \mathcal{P}$. Then $\preceq$ is a partial ordering on $\mathcal{B}$ and we will say that $\preceq$ is the partial ordering induced by $\mathcal{P}$. Moreover, $\mathcal{B}$ is a partially ordered Banach space with respect to $\preceq$.

Let the following hypotheses on the righthand side of (1), f, be satisfied:

(H1) $f(u(t))$ is not identically equal to zero on any subinterval of $[0,1]$ for all $u: [0,1] \to [0,\infty)$.

(H2) $f(u)$ is a continuous nonnegative function on $[0,\infty)$.

Let $\mathcal{B} = C[0,1]$ with norm $\|u\| = \max_{t\in[0,1]} |u(t)|$.
Define a cone, $\mathcal{P}_\sigma \subset \mathcal{B}$, by

$$\mathcal{P}_\sigma = \{u(t) \in \mathcal{B} | u(t) \geq 0 \text{ on } [0,1], \text{ and } min_{t\in[\frac{1}{4},\frac{3}{4}]} \geq \sigma\|u\|\}.$$

Definition 3 *An operator T is completely continuous if T is continuous and compact.*

Based on property (4) of the Green's function for (1), (2) and the hypotheses (H1) and (H2), one can easily establish that, as defined by (3), $T: \mathcal{P}_\sigma \to \mathcal{P}_\sigma$. An application of the Arzelá-Ascoli Theorem readily shows that T is a completely continuous operator. Our first two results will follow from the following central theorem due to Guo [9] and Krasnosel'skiĭ [11].

Theorem 1 *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume Ω_1, Ω_2 are open subsets of $\mathcal{B}$ with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, and let*

$$T: \mathcal{P} \cap (\bar{\Omega}_2 \setminus \Omega_1) \to \mathcal{P}$$

be a completely continuous operator such that, either

(i) $\|Tu\| \leq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \geq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$ *or*

(ii) $\|Tu\| \geq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \leq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$.

Then T has a fixed point in $\mathcal{P} \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

Recently Erbe and Tang [8] established several multiplicity results for positive radial solutions of a nonlinear Laplace's equation on an annular domain. In proving their results they used the above theorem. Other recent multiplicity results belong to Davis, Eloe and Henderson [4], Eloe and Henderson [7], Avery [2], Henderson and Thompson [10], Davis and Henderson [5], and Chyan and Henderson [3]. They are based on cone-theoretic methods provided by the Leggett- Williams Fixed Point Theorem [12] and cover a broad range of nonlinear boundary value problems. Just recently Agarwal, O'Reagan and Wong published a book [1] containing many results on applications of fixed point methods to a variety of BVP's.

In addition to the first two countability results, we also develop an alternative approach based on an application of the Leggett-Williams Fixed Point Theorem. To this end, we need to introduce more definitions.

Definition 4 *The map α is a nonnegative continuous concave functional on $\mathcal{P}$ provided $\alpha: \mathcal{P} \to [0, \infty)$ is continuous and*

$$\alpha(tu + (1-t)v) \geq t\alpha(u) + (1-t)\alpha(v)$$

for all $x, y \in \mathcal{P}$ and $0 \leq t \leq 1$.

Definition 5 *Let $0 < a < b$ be given and α be a nonnegative continuous concave functional on the cone $\mathcal{P}$. Define the convex sets B_r and $\mathcal{P}(\alpha, a, b)$ by*

$$B_r = \{u \in \mathcal{P}: \|u\| < r\}$$

and

$$\mathcal{P}(\alpha, a, b) = \{u \in \mathcal{P}: a \leq \alpha(u), \|u\| \leq b\}.$$

Theorem 2 *(Leggett-Williams Fixed Point Theorem) Let $T: \overline{B}_c \to \overline{B}_c$ be a completely continuous operator and let α be a nonnegative continuous concave functional on $\mathcal{P}$ such that $\alpha(u) \leq \|u\|$ for all $u \in \overline{B}_c$. Suppose there exist $0 < a < b < d \leq c$ such that*

(C1) $\{\mathcal{P}(\alpha, a, b): \alpha(u) > b\} \neq \emptyset$ and $\alpha(Tu) > b$ for $u \in \mathcal{P}(\alpha, a, b)$,

(C2) $\|Tu\| < a$ for $\|u\| \leq a$, and

(C3) $\alpha(Tu) > b$ for $u \in \mathcal{P}(\alpha, a, b)$, with $\|Tu\| > d$.

Then T has at least three fixed points u_1, u_2, and u_3 such that $\|u_1\| < a$, $b < \alpha(u_2)$, and $\|u_3\| > a$ with $\alpha(u_3) < b$.

2 Main Results

Theorem 3 *Assume that $(H1)$ and $(H2)$ are satisfied. Assume also that there exist sequences $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ such that*

$$a_{i+1} < \sigma b_i < b_i < \frac{B}{A} b_i < a_i \text{ for each } i \in N, \tag{6}$$

where

$$A = \left(\int_0^1 \|G(\cdot, s)\| ds\right)^{-1} \tag{7}$$

and

$$B = \left(\sigma \int_{\frac{1}{4}}^{\frac{3}{4}} (-1)^{n-k} G\left(\frac{1}{2}, s\right) ds\right)^{-1}. \tag{8}$$

Let, in addition, f satisfy the following conditions:

(H3) $f(u(s)) \leq Aa_i$ on $[0, 1]$ for all $u \in [0, a_i]$ all $i \in N$,

(H4) $f(u(s)) \geq Bb_i$ *on* $[\frac{1}{4}, \frac{3}{4}]$ *for all* $u \in [\sigma b_i, b_i]$ *all* $i \in N$.

Then (3) *has countably infinitely many fixed points in the cone* $\mathcal{P}_\sigma$.

Proof. Note that since $A < B$, the last inequality in (6) is to simply guarantee the existence of a function $f(u)$ satisfying the hypotheses (H3) and (H4).

Consider the sequence $\{\Omega_i\}_{i=1}^{\infty}$ of open sets in $\mathcal{B}$ defined by

$$\Omega_{2i-1} = \{u \in \mathcal{B}: \|u\| < a_i\},$$

$$\Omega_{2i} = \{u \in \mathcal{B}: \|u\| < b_i\}$$

for $i \in N$.

Then $\|u\| = a_i$ on $\partial\Omega_{2i-1}$ for each $i \in N$. So,

$$0 \leq u(s) \leq a_i$$

on $\partial\Omega_{2i-1} \cap \mathcal{P}_\sigma$ for $s \in [0, 1]$. Hence, since $(H3)$ is satisfied, by (7)

$$\begin{aligned} \|Tu\| &= \|\int_0^1 (-1)^{n-k} G(\cdot, s) f(u(s)) ds\| \\ &\leq \int_0^1 max_{t\in[0,1]} |G(t, s)| f(u(s)) ds \\ &\leq A(\int_0^1 \|G(\cdot, s)\| ds) a_i \\ &= a_i \\ &= \|u\| \end{aligned}$$

on $\partial\Omega_{2i-1} \cap \mathcal{P}_\sigma$ for each $i \in N$.

If $u \in \mathcal{P}_\sigma \cap \partial\Omega_{2i}$, then

$$b_i = \|u\| \geq u(s) \geq min_{t\in[\frac{1}{4},\frac{3}{4}]} u(s) \geq \sigma\|u\| = \sigma b_i$$

for all $s \in [\frac{1}{4}, \frac{3}{4}]$. Then, since $(H4)$ is satisfied, with the use of (8) we get

$$\begin{aligned} (Tu)(\frac{1}{2}) &= \int_0^1 (-1)^{n-k} G(\frac{1}{2}, s) f(u(s)) ds \\ &\geq \int_{\frac{1}{4}}^{\frac{3}{4}} (-1)^{n-k} G(\frac{1}{2}, s) f(u(s)) ds \\ &\geq B(\int_{\frac{1}{4}}^{\frac{3}{4}} (-1)^{n-k} G(\frac{1}{2}, s) ds) b_i \\ &= b_i \\ &= \|u\| \end{aligned}$$

on $\partial\Omega_{2i} \cap \mathcal{P}_\sigma$ for each $i \in N$.

We have $\|Tu\| \leq \|u\|$ on $\partial\Omega_{2i-1} \cap \mathcal{P}_\sigma$ and $\|Tu\| \geq \|u\|$ on $\partial\Omega_{2i} \cap \mathcal{P}_\sigma$ with Ω_{2i-1} and Ω_{2i} such that $\Omega_{2i-1} \subset \bar{\Omega}_{2i-1} \subset \Omega_{2i}$ for all $i \in N$. Therefore, by Theorem 1, T has a fixed point in $(\bar{\Omega}_{2i} \setminus \Omega_{2i-1}) \cap \mathcal{P}_\sigma$, $i \in N$. that is, T has countably many fixed points in the cone $\mathcal{P}_\sigma$. Furthermore, $\|u_{i+1}\| < \|u_i\|$ for all $i \in N$. The proof is complete.

From the above theorem we immediately obtain the following corollary.

Corollary 1 *Under the assumptions of Theorem* 3

$$\lim_{i\to\infty} \|u_i\| = 0.$$

Proof. It follows from (6) that $a_{i+1} < \sigma^i a_1$. So,

$$\lim_{i\to\infty} a_i = 0.$$

Hence

$$\lim_{i\to\infty} \|u_i\| = 0.$$

Theorem 3 and Corollary 1 establish the existence of countably many positive solutions of (1), (2) such that the sequence of their norms is monotone decreasing and converges to zero. In order to obtain a monotone increasing (in norm) sequence of positive solutions we just need to modify slightly our condition (6). At this point we present our next theorem.

Theorem 4 *Assume that that* $(H1)$ *and* $(H2)$ *are satisfied. Assume also that there exist sequences* $\{a_i\}_{i=1}^{\infty}$ *and* $\{b_i\}_{i=1}^{\infty}$ *such that*

$$a_i < \sigma b_i < b_i < \frac{B}{A} b_i < a_{i+1}, \tag{9}$$

where A *and* B *are given by* (7) *and* (8), *respectively. Let, in addition,* f *satisfy* $(H3)$ *and* $(H4)$.

Then (3) *has countably many infinitely fixed points* u_i *in the cone* $\mathcal{P}_\sigma$. *Moreover,* $\|u_{i+1}\| > \|u_i\|$ *for all* $i \in N$ *and* $\lim_{i\to\infty} \|u_i\| = \infty$.

Proof. The proof is almost identical to that of Theorem 3 except the part employing condition (9). The last assertion is fulfilled simply because $b_i > \sigma^{-i} a_1$ for all $i \in N$ (which is a trivial consequence of (9)).

Remark 1: Note that a function f satisfying the hypotheses of Theorem 4 is unbounded by construction, while a bounded function f can be selected to fulfill the assumptions of Theorem 3.

Remark 2: In the situation of Theorem 3, the limit point of a sequence of solutions $\{u_i\}_{i=1}^{\infty}$, $u(t) = 0$, is also a solution of (1), (2). Now define a new cone $\mathcal{P}$ by

$$\mathcal{P} = \{u \in \mathcal{B}: u(t) \geq 0,\ 0 \leq t \leq 1\}.$$

As before, the operator T preserves the cone $\mathcal{P}$. We can now prove prove the following theorem.

Theorem 5 *Assume that* $(H1)$ *and* $(H2)$ *are satisfied. Let there exist sequences* $\{a_i\}_{i=1}^{\infty}$, $\{b_i\}_{i=1}^{\infty}$, *and* $\{c_i\}_{i=1}^{\infty}$ *such that*

$$\lim_{i\to\infty} c_i = 0$$

and

$$c_{i+1} < a_i < b_i < \frac{B}{A} b_i < \sigma c_i \text{ for each } i \in N \tag{10}$$

Let, in addition, the function f *satisfy the following hypothesis:*

(H5) $f(u) < Aa_i$ *on* $[0, a_i]$,

(H6) $f(u) > Cb_i$ *on* $[b_i, \frac{1}{\sigma}b_i]$, *and*

(H7) $f(u) < Ac_i$ *on* $[0, c_i]$, *where* A *is given by* (7) *and*

$$C = (\sigma \min_{t\in[0,1]} \int_{\frac{1}{4}}^{\frac{3}{4}} (-1)^{n-k} G(t,s) ds)^{-1}. \tag{11}$$

Then the boundary value problem (1), (2) *has three infinite families of positive solutions* $\{u_{1_i}\}_{i=1}^{\infty}$, $\{u_{2_i}\}_{i=1}^{\infty}$, *and* $\{u_{3_i}\}_{i=1}^{\infty}$ *satisfying* $\|u_{1_i}\| < a_i$, $b_i < \alpha(u_{2_i})$, *and* $\|u_{3_i}\| > a_i$, $b_i > \alpha(u_{3_i})$.

Proof. Observe that (9) makes the use of (H5)-(H7) consistent. As in Definition 5, consider for each $i \in N$

$$B_{a_i} = \{u \in \mathcal{P}: \|u\| < a_i\}$$

and

$$B_{c_i} = \{u \in \mathcal{P}: \|u\| < c_i\}.$$

Repeating the argument in the proof of Theorem 3, we can show that $T: \overline{B}_{a_i} \to \underline{B}_{a_i}$ and $T: \overline{B}_{c_i} \to \overline{B}_{c_i}$, $i \in N$.

Define the nonnegative functional $\alpha: \mathcal{P} \to [0, \infty)$ by

$$\alpha(u) = \min_{t\in[\frac{1}{4},\frac{3}{4}]} u(t).$$

A direct computation verifies that α is a nonnegative continuous concave functional on $\mathcal{P}$, and $\alpha(u) \le \|u\|$ for all $u \in \overline{B}_{c_i}$ for all $i \in N$.

We would like to apply Theorem 2. First, observe that the assumptions on T and condition (C2) of Theorem 2 are satisfied since $T: \overline{B}_{a_i} \to B_{a_i}$ and $T: \overline{B}_{c_i} \to \overline{B}_{c_i}$, $i \in N$.

Now, as in Definition 5, set

$$\mathcal{P}(\alpha, b_i, \frac{1}{\sigma}b_i) = \{u \in \mathcal{P}: b_i \le \alpha(u), \|u\| \le \frac{1}{\sigma}b_i\},$$

and

$$\mathcal{P}(\alpha, b_i, c_i) = \{u \in \mathcal{P}: b_i \le \alpha(u), \|u\| \le c_i\},\ i \in N.$$

We show that $\{\mathcal{P}(\alpha, b_i, \frac{1}{\sigma}b_i): \alpha(u) > b_i\} \ne \emptyset$ and $\alpha(Tu) > b_i$ for $u \in \mathcal{P}(\alpha, b_i, \frac{1}{\sigma}b_i)$. To this end, note that setting $u = \frac{1}{\sigma}b_i$, we trivially have $b_i < \frac{1}{\sigma}b_i = \alpha(u)$ and $\|u\| = \frac{1}{\sigma}$, that is $\{\mathcal{P}(\alpha, b_i, \frac{1}{\sigma}b_i): \alpha(u) > b_i\} \ne \emptyset$. In addition, by (H6), for any $u \in \mathcal{P}(\alpha, b_i, \frac{1}{\sigma}b_i)$ we have

$$\begin{aligned}
\alpha(Tu) &= \min_{t\in[\frac{1}{4},\frac{3}{4}]} Tu(t) \\
&= \min_{t\in[\frac{1}{4},\frac{3}{4}]} \int_0^1 (-1)^{n-k} G(t,s) f(u(s)) ds \\
&\ge \min_{t\in[\frac{1}{4},\frac{3}{4}]} \int_{\frac{1}{4}}^{\frac{3}{4}} (-1)^{n-k} G(t,s) f(u(s)) ds \\
&> (\min_{t\in[\frac{1}{4},\frac{3}{4}]} \int_{\frac{1}{4}}^{\frac{3}{4}} (-1)^{n-k} G(t,s) ds) C b_i \\
&= b_i.
\end{aligned}$$

Thus, $\alpha(u) > b_i$ for all $u \in \mathcal{P}(\alpha, b_i, \frac{1}{\sigma}b_i)$, which verifies assumption (C1) of Theorem 2.

Assume now that $u \in \mathcal{P}(\alpha, b_i, c_i)$ and $\|Tu\| > \frac{1}{\sigma}b_i$. Then, by (5), for all $\tau \in [0,1]$ we have

$$\begin{aligned}
\alpha(Tu) &= \min_{t\in[\frac{1}{4},\frac{3}{4}]} Tu(t) \\
&= \min_{t\in[\frac{1}{4},\frac{3}{4}]} \int_0^1 (-1)^{n-k} G(t,s) f(u(s)) ds \\
&\geq \int_0^1 \min_{t\in[\frac{1}{4},\frac{3}{4}]} (-1)^{n-k} G(t,s) f(u(s)) ds \\
&\geq \int_0^1 (-1)^{n-k} \sigma G(\tau,s) f(u(s)) ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\alpha(Tu) &\geq \max_{\tau\in[0,1]} \int_0^1 (-1)^{n-k} \sigma G(t,s) f(s,u(s)) ds \\
&= \sigma \max_{\tau\in[0,1]} |Tu(\tau)| \\
&= \sigma \|Tu\| \\
&> b_i.
\end{aligned}$$

That is, $\alpha(Tu) > b_i$ for all $u \in \mathcal{P}(\alpha, b_i, c_i)$ with $\|Tu\| > \frac{1}{\sigma}b_i$. This verifies assumption (C3) of Theorem 2.

Since the hypotheses of Theorem 2 are satisfied, the assertion follows for each $i \in N$. Together with (9), this proves the existence of three families of positive solutions $\{u_{1_i}\}_{i=1}^{\infty}$, $\{u_{2_i}\}_{i=1}^{\infty}$, and $\{u_{3_i}\}_{i=1}^{\infty}$ such that $\|u_{1_i}\| < a_i$, $b_i < \alpha(u_{2_i})$, and $\|u_{3_i}\| > a_i$, $b_i > \alpha(u_{3_i})$ for all $i \in N$. The proof is complete.

3 Concluding Remarks

We have been able to find conditions under which the BVP (1), (2) admits countably many positive solutions. Under our assumptions sequences of their norms turn out to be either convergent to zero or divergent at infinity. This poses the following question: Is it possible to determine a function $f(u)$ such that (1), (2) would allow a monotone in $C[0,1]$-norm sequence of solutions with $\lim_{k\to\infty}\|u_k\| = d > 0$?

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