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Optimal Intervals for Uniqueness of Solutions for Nonlocal Boundary Value Problems

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Abstract

For the nth order differential equation, $y^{(n)} = f(t, y, y', \ldots, y^{(n-1)})$, where $f(t, r_1, r_2, \ldots, r_n)$ is Lipschitz continuous in terms of $r_i, 1 \leq i \leq n$, we obtain optimal bounds on the length of intervals on which solutions are unique for certain nonlocal three point boundary value problems. These bounds are obtained through an application of the Pontryagin Maximum Principle from the theory of optimal control.

Key words: Nonlocal boundary value problem, optimal length intervals, Pontryagin Maximum Principle.

AMS Subject Classification: 34B15

1 Introduction

In this paper, we shall be concerned with the nth order differential equation,

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)}), \quad a < t < b, \tag{1.1}$$

for which the assumptions in the following hypothesis hold throughout.

Hypothesis. $f(t, r_1, \dots, r_n) : (a, b) \times \mathbb{R}^n \to \mathbb{R}$ is continuous, and for nonnegative constants $k_i, 1 \le i \le n$, satisfies the Lipschitz condition,

$$|f(t, r_1, \dots, r_n) - f(t, s_1, \dots, s_n)| \le \sum_{i=1}^{n} k_i |r_i - s_i|, \tag{1.2}$$

for each $(t, r_1, \dots, r_n), (t, s_1, \dots, s_n) \in (a, b) \times \mathbb{R}^n$.

We characterize optimal length for subintervals of (a, b), in terms of the Lipschitz coefficients $k_i, 1 \le i \le n$, on which solutions are unique for problems involving (1.1) and satisfying the *nonlocal* three point boundary conditions,

$$y^{(i-1)}(t_1) = y_i, \quad 1 \le i \le n-1, \quad y^{(n-2)}(t_2) - y^{(n-2)}(t_3) = y_n, \tag{1.3}$$

where $a < t_1 < t_2 < t_3 < b$, and $y_1, \dots, y_n \in \mathbb{R}$.

More precisely, we characterize optimal length for subintervals of (a, b) on which solutions of (1.1), (1.3) are unique. Such uniqueness results are of interest, because in many cases, uniqueness of solutions implies existence of solutions for boundary value problems; see, for example, the papers [5, 7, 9, 18, 19, 22, 24, 32] and the references therein.

There is a close connection between the boundary value problem (1.1), (1.3) and certain right focal boundary value problems for (1.1). Because of this relationship, we will eventually establish that it suffices for us to characterize optimal length subintervals of (a, b) on which solutions are unique for (1.1) satisfying the right focal boundary conditions,

$$y^{(i-1)}(t_1) = y_i, \quad 1 \le i \le n-1, \quad y^{(n-1)}(t_2) = y_n, \tag{1.4}$$

where $a < t_1 < t_2 < b$, and $y_1, \dots, y_n \in \mathbb{R}$. The connection between this characterization and the characterization for our three point nonlocal problems is through a simple application of the Mean Value Theorem.

Theorem 1.1 *If solutions for* (1.1), (1.4) *are unique, when they exist on* (a, b)*, then solutions for* (1.1), (1.3) *are unique, when they exist on* (a, b).

Thus, in view of Theorem 1.1, conditions sufficient to provide uniqueness of solutions, when they exist on (a, b), for two point right focal boundary value problems (1.1), (1.4), are sufficient to provide uniqueness of solutions, when they exist on (a, b) for three point nonlocal boundary value problems (1.1), (1.3).

Our process will involve development of a scenario in which the Pontryagin Maximum Principle can be applied. We follow a pattern that has an extensive history, with first motivation found in the papers by Melentsova [36] and Melentsova and Mil'shtein [37, 38].

Those papers were subsequently adapted to the context of several types of boundary value problems by Jackson [28, 29], Eloe and Henderson [8], Hankerson and Henderson [17] and Henderson *et al.* [6, 20, 21, 23, 25].

Interest in nonlocal boundary value problems also has a long history, both in application and theory, as can be seen in this list of papers and the references therein: [1] -[4], [10, 11], [13] - [16], [23], [26, 27], [30, 31], [34, 35], [39] - [47].

2 Optimal Intervals for Uniqueness of Solutions

In this section, we apply the Pontryagin Maximum Principle to obtain a characterization, in terms of the Lipschitz constants $k_i, 1 \le i \le n$, for the optimal length of subintervals of (a, b) on which solutions are unique, when they exist for the right focal boundary value problem (1.1), (1.4). This length, it will be argued later, is optimal for uniqueness of solutions for the three point nonlocal boundary value problem (1.1), (1.3).

We first introduce a set of vector-valued *control functions*

$$\begin{aligned} \mathcal{U} \quad := \quad & \{\mathbf{v}(t) = (v_1(t), \ldots, v_n(t))^T \in \mathbb{R}^n \mid v_i(t) \text{ are Lebesgue} \\ & \text{measurable and } |v_i(t)| \le k_i \text{ on } (a, b), i = 1, \ldots, n\}. \end{aligned}$$

We will be concerned with boundary value problems associated with linear differential equations of the form

$$x^{(n)} = \sum_{i=1}^{n} u_i(t) x^{(i-1)}, \tag{2.1}$$

where $\mathbf{u}(t) = (u_1(t), \ldots, u_n(t))^T \in \mathcal{U}$. We immediately make a connection of these linear differential equations with solutions of (1.1), (1.4). Much of our analysis will be based upon our choosing, if they exist, distinct solutions $y(t)$ and $z(t)$ of (1.1), (1.4).

If $y(t)$ and $z(t)$ are distinct solutions of (1.1), (1.4), then their difference $x(t) := y(t) - z(t)$ satisfies

$$x^{(i-1)}(t_1) = x^{(n-1)}(t_2) = 0, \quad 1 \le i \le n-1, \tag{2.2}$$

for some $a < t_1 < t_2 < b$, and if for $1 \le i \le n$,

$$u_i(t) := \begin{cases} \dfrac{f(t, z(t), \ldots, z^{(i-2)}(t), y^{(i-1)}(t), \ldots, y^{(n-1)}(t)) - f(t, z(t), \ldots, z^{(i-1)}(t), y^{(i)}(t), \ldots, y^{(n-1)}(t))}{y^{(i-1)}(t) - z^{(i-1)}(t)}, \\ \hfill y^{(i-1)}(t) \ne z^{(i-1)}(t), \\ 0, \hfill y^{(i-1)}(t) = z^{(i-1)}(t), \end{cases}$$

then $u_i(t)$ is Lebesgue measurable, and $|u_i(t)| \le k_i$, $i = 1, \ldots, n$, so that $\mathbf{u}(t) = (u_1(t), \ldots, u_n(t))^T \in \mathcal{U}$, and $x(t)$ is a nontrivial solution of the boundary value problem (2.1), (2.2). It follows from optimal control theory (cf. Gamkrelidze [12, p. 147] and Lee and Markus [33, p. 259]), there is a boundary value problem in the class (2.1), (2.2), which has a nontrivial time optimal solution; that is, there exists at least one nontrivial $\mathbf{u}^* \in \mathcal{U}$ and points $t_1 \le c < d \le t_2$ such that

$$x^{(n)} = \sum_{i=1}^{n} u_i^*(t) x^{(i-1)}, \tag{2.3}$$

$$x^{(i-1)}(c) = x^{(n-1)}(d) = 0, \quad 1 \leq i \leq n, \tag{2.4}$$

has a nontrivial solution, $x_0(t)$, and $d-c$ is a minimum over all such solutions. For this time optimal solution, $x_0(t)$, set $\mathbf{x_0}(t) = (x_0(t), \ldots, x_0^{(n-1)}(t))^T$. Then $\mathbf{x_0}(t)$ is a solution of a first order system,

$$\mathbf{r}' = A[\mathbf{u}^*(t)]\mathbf{r}.$$

By the Pontryagin Maximum Principle, the adjoint system, whose form is given by

$$\mathbf{x}' = -A^T[\mathbf{u}^*(t)]\mathbf{x}, \quad a < t < b, \tag{2.5}$$

has a nontrivial solution, $\mathbf{x}^*(t) = (x_1^*(t), \ldots, x_n^*(t))^T$ such that, for *a. e.* $t \in [c, d]$,

(i) $\sum_{i=1}^n x_0^{(i)}(t)x_i^*(t) = \langle \mathbf{x_0}'(t), \mathbf{x}^*(t)\rangle = \max_{\mathbf{u}\in\mathcal{U}}\{\langle A[\mathbf{u}(t)]\mathbf{x_0}(t), \mathbf{x}^*(t)\rangle\}$,

(ii) $\langle \mathbf{x_0}'(t), \mathbf{x}^*(t)\rangle$ is a nonnegative constant,

(iii) $x_n^*(c) = x_1^*(d) = \cdots = x_{n-1}^*(d) = 0$.

The maximum condition in (i) can be rewritten as

$$x_n^*(t)\sum_{i=1}^n u_i^*(t)x_0^{(i-1)}(t) = \max_{\mathbf{u}\in\mathcal{U}}\big\{x_n^*(t)\sum_{i=1}^n u_i(t)x_0^{(i-1)}(t)\big\}, \tag{2.6}$$

for *a. e.* $t \in [c, d]$.

By its time optimality and repeated applications of Rolle's Theorem, $x_0(t) \neq 0$, $t \in (c, d]$. In fact, for each $1 \leq i \leq n$, $x_0^{(i-1)}(t) \neq 0$ on (c, d). We may assume without loss of generality that $x_0(t) > 0$ on $(c, d]$. If $x_n^*(t)$ has no zeros on (c, d), then we can use (2.6) to determine an optimal control $\mathbf{u}^*(t)$, for *a. e.* $t \in [c, d]$. We now examine the sign of $x_n^*(t)$ on (c, d).

In that direction, if $\overline{\mathbf{u}} \in \mathcal{U}$ is such that the boundary value problem given by (2.1) and (2.2), for some $a < t_1 < t_2 < b$, has a nontrivial solution, then the adjoint system

$$\alpha' = -A^T[\overline{\mathbf{u}}(t)]\alpha, \quad t \in (a, b), \tag{2.7}$$

$$\alpha_n(t_1) = \alpha_1(t_2) = \cdots = \alpha_{n-1}(t_2) = 0, \tag{2.8}$$

also has a nontrivial solution, and conversely. That is, the Pontryagin Maximum Principle associates with a time optimal solution of boundary value problem (2.1), (2.2) a time optimal solution of boundary value problem (2.7), (2.8), and conversely. Hence, it follows by its own time optimality that $x_n^*(t)$ does not vanish on (c, d).

Now, $x_0(t) > 0$ on $(c, d]$, and so we have from (2.6) that, if $x_n^*(t) < 0$ on (c, d), then the time optimal solution $x_0(t)$ is a solution of

$$x^{(n)} = -k_1x - \sum_{i=2}^n k_i|x^{(i-1)}| \tag{2.9}$$

on $[c, d]$, while if $x_n^*(t) > 0$ on (c, d), then the time optimal solution $x_0(t)$ is a solution of

$$x^{(n)} = k_1x + \sum_{i=2}^n k_i|x^{(i-1)}| \tag{2.10}$$

on $[c,d]$. Since $n-(n-1)=1$ is odd, the result by Jackson [29, Theorem 2] yields that $x_n^*(t)<0$ on (c,d), so that that $x_0(t)$ is a solution of (2.9) on $[c,d]$. Moreover, from the assumed positivity of $x_0(t)$ and the nature of the boundary conditions (2.4), along with the fact that $x_0^{(i-1)}(t)\neq 0$ on (c,d), $1\le i\le n$, it follows that $x_0^{(i-1)}(t)>0$ on (c,d), $1\le i\le n$. As a consequence, not only is $x_0(t)$ is a solution of (2.9), but also where (2.9) takes the form

$$x^{(n)}=-\sum_{i=1}^{n}k_i x^{(i-1)}. \tag{2.11}$$

Recall that our discussion is based on (1.1) having distinct solutions whose difference satisfies (2.2). In addition, if sufficient sign conditions are satisfied by the optimal solution $x_0(t)$ of the boundary value problem (2.1), (2.2) and by the component $x_n^*(t)$ of the solution of the associated adjoint system (2.5), then optimal intervals can be determined on which only trivial solutions exist for boundary value problems (2.9), (2.2) or (2.10), (2.2). Ultimately, a more detailed sign analysis led to determination of optimal intervals on which only trivial solutions exist for only the boundary value problem (2.11), (2.2). As a consequence, solutions of the boundary value problem (1.1), (1.4) will be unique on such subintervals.

Theorem 2.1 *If there is a vector-valued* $\mathbf{u}(t)\in\mathcal{U}$ *for all* $a<t<b$*, for which the boundary value problem* (2.1), (2.2) *has a nontrivial solution for some* $a<t_1<t_2<b$*, and if* $x_0(t)$ *is a time optimal solution satisfing* (2.4)*, where* $d-c$ *is a minimum, then* $x_0(t)$ *is a solution of* (2.11) *on* $[c,d]$.

Theorem 2.2 *Let* $\ell=\ell(k_1,\dots,k_n)>0$ *be the smallest positive number such that there exists a solution* $x(t)$ *of the boundary value problem for* (2.11) *satisfying*

$$x^{(i-1)}(0)=0,\ 1\le i\le n-1,\ x^{(n-1)}(\ell)=0, \tag{2.12}$$

with $x(t)>0$ *on* $(0,\ell]$*, or* $\ell=\infty$ *if no such solution exists. If* $y(t)$ *and* $z(t)$ *are solutions of the boundary value problem* (1.1), (1.4)*, for some* $a<t_1<t_2<b$*, and if* $t_2-t_1<\ell$*, it follows that* $y(t)\equiv z(t)$ *on* $[t_1,t_2]$*, and this is best possible for the class of all differential equations satisfying the Lipschitz condition* (1.2).

Proof: Since equation (2.11) is autonomous, translations of solutions are again solutions of (2.11). Hence, it suffices to apply Theorem 2.1 with respect to the boundary conditions at 0 and ℓ.

Now, if $y(t)$ and $z(t)$ are distinct solutions of (1.1) whose difference $w(t):=y(t)-z(t)$ satisfies (2.2), where $t_2-t_1<\ell$, then $w(t)$ is a nontrivial solution of the boundary value problem (2.1), (2.2), for appropriately defined $\mathbf{u}\in\mathcal{U}$. Then, from the discussion and Theorem 2.1, equation (2.11) has a nontrivial solution on a subinterval of length less than ℓ. But, by the minimality of ℓ, such a boundary value problem can have only the trivial solution; this is a contradiction. Therefore, solutions of the boundary value problem (1.1), (1.4) are unique, whenever $t_2-t_1<\ell$.

That this is best possible from the fact that (2.11) satisfies the Lipschitz condition (1.2), and if $\ell\neq\infty$, then $x(t)$ is a nontrivial solution of (2.11) and (2.2) on $[0,\ell]$. The boundary value problem also has the trivial solution. □

Remark 2.1 *Since* (2.11) *is a linear equation, we observe that, if $x(t)$ is the solution, of the initial value problem for* (2.11)*, satisfying,*

$$x^{(i-1)}(0) = 0,\ 1 \le i \le n-1, \quad x^{(n-1)}(0) = 1,$$

and if $\eta > 0$ is the first positive number such that $x^{(n-1)}(\eta) = 0$, then $\eta = \ell(k_1, \ldots, k_n)$ of Theorem 2.2.

Because of the uniqueness relationships stated in Theorem 1.1, we can apply Theorem 2.2 to obtain optimal intervals for uniqueness of solutions of the boundary value problem (1.1), (1.3).

Theorem 2.3 *Let ℓ be as in Theorem 2.2. If $y(t)$ and $z(t)$ are solutions of the boundary value problem* (1.1), (1.3)*, for some $a < t_1 < t_2 < t_3 < b$, and if $t_3 - t_1 \le \ell$, it follows that $y(t) \equiv z(t)$ on $[t_1, t_3]$, and this is best possible for the class of all differential equations satisfying the Lipschitz condition* (1.2)*.*

Proof: In view of Theorem 1.1 and Theorem 2.2, solutions of the boundary value problem (1.1), (1.3) are unique, when $t_3 - t_1 \le \ell$. To see again that this is best possible, consider the nontrivial solution $x(t)$ of (2.11) and (2.12) in Theorem 2.2.

Let $\epsilon > 0$ be sufficiently small that $x(t)$ is a solution of (2.11) on $[0, \ell+\epsilon]$. Now, $x^{(n)}(t) < 0$ on $[0, \ell+\epsilon]$. From (2.12), $x^{(n-1)}(\ell) = 0$, and since $x^{(n)}(\ell) < 0$, we have that $x^{(n-2)}(t)$ has a positive maximum at ℓ. So, there exist $0 < \tau_1 < \ell < \tau_2 < \ell + \epsilon$ such that $x(t)$ is a nontrivial solution of (2.11) satisfying $x^{(i-1)}(0) = 0, 1 \le i \le n-1$, and $x^{(n-2)}(\tau_1) - x^{(n-2)}(\tau_2) = 0$. This boundary value problem also has the trivial solution. Since $\epsilon > 0$ was arbitrary, the "best possible" statement follows for uniqueness of solutions of the boundary value problem (1.1), (1.3). □

3 Optimal Intervals of Existence for Linear Equations

In the case of boundary value problem (1.1), (1.3), we do not have a "uniqueness implies existence" theorem to appeal to, since this is an open question for this type of boundary value problem. However, uniqueness does imply existence for linear differential equations, and so the following corollary can be stated.

Corollary 3.1 *Let ℓ be as in Theorem 2.2. Assume $p_i(t), 1 \le i \le n$, and $q(t)$ are continuous on (a, b) and that $|p_i(t)| \le k_i$ on $(a, b), 1 \le i \le n$. If $a < t_1 < t_2 < t_3 < b$ and $t_3 - t_1 < \ell$, then the boundary value problem,*

$$y^{(n)} = \sum_{i=1}^{n} p_i(t) y^{(i-1)} + q(t),$$

$$y^{(i-1)}(t_1) = y_i,\ 1 \le i \le n-1, \quad y^{(n-2)}(t_2) - y^{(n-2)}(t_3) = y_n,$$

has a solution for any assignment of values of $y_i \in \mathbb{R}, 1 \le i \le n$.

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