2011

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SIAM J. APPL. MATH.
Vol. 71, No. 4, pp. 1354–1373

UPPER AND LOWER SOLUTIONS FOR REGIME-SWITCHING DIFFUSIONS WITH APPLICATIONS IN FINANCIAL MATHEMATICS*

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Abstract. This paper develops a method of upper and lower solutions for a general system of second-order ordinary differential equations with two-point boundary conditions. Our motivation of study stems from a class of financial mathematics problems under regime-switching diffusion models. Two examples are double barrier option valuation and optimal selling rules in asset trading. We establish the existence of a unique C^2 solution of the two-point boundary value problem. We construct monotone sequences of upper and lower solutions that are shown to converge to the unique solution of the boundary value problem. This construction provides a feasible numerical method to compute approximate solutions. An important feature of the proposed numerical method is that the unique solution is bracketed by the upper and lower approximate solutions, which provide an interval estimate of the unique solution function. We apply the general results to a regime-switching mean-reverting model and improve related results already reported in the literature. For the mean-reverting model, explicit upper and lower solutions are obtained and numerical integration methods are employed. In another case (Example 3 in section 5) a different regime-switching model is considered, where the general results apply, but only the upper solution is explicitly obtained. In that example, only the sequence of upper solutions is numerically constructed using finite difference methods. Numerical results are reported.

Key words. regime-switching diffusion, upper and lower solutions, boundary value problem, mean-reverting process, double barrier option, selling rule

AMS subject classifications. 91G20, 91G60, 91G80, 34B60, 65L10

DOI. 10.1137/100799691

1. Introduction.

Regime-switching models have drawn considerable attention in recent decades in a variety of application fields, due to their capability of modeling complex systems with uncertainty. A common setup of a regime-switching model is to use a number of continuous stochastic differential equations, each for a specific regime, together with a Markov chain for the random switching among regimes. Consequently, both continuous dynamics and discrete events are present in the switching model formulation, providing realistic models in applications. We refer the reader to Yin and Zhu [29] on the recent studies of regime-switching systems.

One area of applications of regime-switching models with growing interest is financial mathematics. Aiming to include the influence of macroeconomic factors on the individual asset price behavior, regime-switching models have been introduced for describing asset price changes. In this setting, asset prices are dictated by a number of stochastic differential equations coupled by a finite-state Markov chain, which represents various randomly changing economical factors. Model parameters (drift and volatility coefficients) are assumed to depend on the Markov chain. Regime-switching models have been used in various studies in financial mathematics, including equity options [3, 13, 14, 26, 21, 17, 9, 20], interest rate derivatives and bond prices [1, 19], energy and commodity derivatives [4, 22], portfolio selection [32], trading rules [30, 27, 31, 28, 10, 15], and others.

*Received by the editors June 22, 2010; accepted for publication (in revised form) May 26, 2011; published electronically August 4, 2011.
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In this paper we study a system of second-order ordinary differential equations with two-point boundary conditions. The primary motivation stems from a class of financial mathematics problems using regime-switching diffusion models. Two examples are double barrier option valuation and selling rules in asset trading. See the next section for descriptions of these two problems and their connections with the general problem formulation. We establish the existence of a unique C^2 solution of the two-point boundary value problem. For existence, the method of upper and lower solutions is employed. Monotone sequences of upper and lower solutions are shown to converge to C^2 solutions of the boundary value problem. The uniqueness of the solution can be obtained by an application of the Dynkin's formula. Moreover, this proof by construction provides a feasible numerical method to compute approximate solutions. An important feature of the proposed numerical method is that the unique solution is bracketed by the upper and lower approximate solutions, which provide an interval estimate of the unique solution. We next apply the general results to a regime-switching mean-reverting model. Mean-reverting diffusion processes have been used in modeling financial variables such as stochastic volatilities, stochastic interest rates, energy and commodity prices. A closed-form initial lower solution in terms of Weber's functions is obtained, and an explicit expression of the Green's function is constructed. This Green's function is then used in constructing the sequences of upper and lower approximations of the value functions. A closed-form initial upper solution is also obtained. Two numerical examples are provided for this model in which the sequences of upper and lower solutions are obtained with numerical integration methods. Moreover, a different regime-switching model is considered in another example (Example 3 in section 5) for which only the upper solution is explicitly obtained. The general results still apply, but only the sequence of upper solutions is constructed and finite difference methods are employed.

The main contributions of this paper are twofold. First, we develop the method of upper and lower solutions for a fairly general stochastic diffusion model that includes as special cases a number of commonly used models in financial mathematics (for examples, geometric Brownian motion for equities, and mean-reverting diffusion for interest rates and commodities). The results presented in this paper generalize our results in [10] and [9], and can be applied to a broad class of models. Second, for the regime-switching mean-reverting model, the application of the general results enables us to remove an assumption on model parameters that was made in our previous works ([10, Assumption 3.4] and [9, Assumption 1]). Hence, the upper and lower solutions developed in this paper for the mean-reverting diffusion processes permit much more freedom in choosing model parameters, providing more chances for model calibration in practical applications.

The paper is organized as follows. Section 2 presents the general problem formulation with two application examples in financial mathematics. Section 3 presents the general results for the method of upper and lower solutions. We establish the existence of a C^2 solution of the two-point boundary value problem by using the method of upper and lower solutions. We construct sequences of upper and lower approximate solutions that converge monotonically to C^2 solutions in an appropriate Banach space. The uniqueness of the C^2 solution follows by applying Dynkin's formula. Section 4 is concerned with a regime-switching mean-reverting model. Closed-form initial upper and lower solutions are exhibited and an explicit Green's function is constructed. The Green's function is used in constructing the sequences of upper and lower approximation solutions. Numerical results are reported in section 5. Section 6 provides further remarks and concludes the paper.

2. Problem formulation and motivation examples.

2.1. Problem formulation. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be the underlying probability space, upon which all stochastic processes are defined. Let α_t be a continuous-time Markov chain taking values in $\mathcal{M} := \{1, \dots, m\}$, a finite state space. The states of α_t represent general market trends and other economic factors (also known as state of the world or regime) and are labeled by integers 1 to m where m is the total number of states considered for the economy. For example, with $m = 2$, $\alpha_t = 1$ may stand for an up market (a bullish market) and $\alpha_t = 2$ a down market (a bearish market). Let B_t be a real-valued standard Brownian motion. Assume α_t is independent of B_t.

We consider the following one-dimensional stochastic differential equation modulated by the Markov chain α_t,

$$dZ_t = \mu(Z_t, \alpha_t)dt + \sigma(Z_t, \alpha_t)dB_t, \quad t \ge 0, \quad Z_0 = z_0, \tag{2.1}$$

where $\mu(\cdot,\cdot) : \mathbb{R}\times\mathcal{M} \to \mathbb{R}$ and $\sigma(\cdot,\cdot) : \mathbb{R}\times\mathcal{M} \to \mathbb{R}$ are appropriate functions satisfying the Lipschitz condition

$$|\mu(z_1, i) - \mu(z_2, i)| \le K|z_1 - z_2|, \qquad |\sigma(z_1, i) - \sigma(z_2, i)| \le K|z_1 - z_2|, \tag{2.2}$$

and the linear growth condition,

$$|\mu(z, i)| \le K(1 + |z|), \qquad |\sigma(z, i)| \le K(1 + |z|), \tag{2.3}$$

for all $z, z_1, z_2 \in \mathbb{R}$ and for each $i \in \mathcal{M}$, where K is a positive constant. The conditions (2.2) and (2.3) ensure the existence of a unique solution to (2.1) (see Yin and Zhu [29]). In addition, we assume $\sigma(z, i) \neq 0$ for all $z \in \mathbb{R}$ and for each $i \in \mathcal{M}$.

Given two numbers z_1 and z_2 satisfying $-\infty < z_1 \le z_2 < \infty$, define a stopping time τ by

$$\tau = \inf\{t \ge 0 : \; Z_t \notin (z_1, z_2)\}. \tag{2.4}$$

τ is the first time that the process Z_t hits either the upper bound z_2 or the lower bound z_1. We consider the value functions $v(z, i)$, $i = 1, \dots, m$, defined by

$$v(z, i) = E\left\{\exp(-\rho\tau)\Phi(Z_\tau, \alpha_\tau)\Big| Z_0 = z, \alpha_0 = i\right\}, \tag{2.5}$$

where $\Phi(\cdot,\cdot) : \mathbb{R} \times \mathcal{M} \to \mathbb{R}$ is a prespecified utility function and $\rho > 0$ is a discount factor.

Let matrix $Q = (q_{ij})_{m\times m}$ denote the generator of α_t. Then its entries q_{ij} satisfy: (I) $q_{ij} \ge 0$ if $i \neq j$; (II) $q_{ii} \le 0$ and $q_{ii} = -\sum_{j\neq i} q_{ij}$ for each $i = 1, \dots, m$. The value functions $v(z, i)$, $i = 1, \dots, m$ satisfy a system of second order differential equations:

$$\frac{\sigma^2(z, i)}{2}\frac{d^2 v(z, i)}{dz^2} + \mu(z, i)\frac{dv(z, i)}{dz} - \rho v(z, i) + \sum_{j\neq i} q_{ij}[v(z, j) - v(z, i)] = 0, \tag{2.6}$$

for $z \in (z_1, z_2)$. The associated boundary conditions are given by

$$v(z_1, i) = \Phi(z_1, i), \qquad v(z_2, i) = \Phi(z_2, i). \tag{2.7}$$

If the boundary value problem (2.6), (2.7) has a smooth solution $v(z, i)$, $i = 1, \dots, m$, then using Dynkin's formula, we can show that the solution must be given by (2.5), which implies the uniqueness of the solution. In the next section, we shall establish the existence of a C^2 solution to (2.6) and (2.7) via the method of upper and lower solutions.

The motivation for studying this general problem stems from a number of application problems in financial mathematics. We next present two examples: one in double barrier option valuation and another in selling rules for trading assets.

Double barrier option. Barrier options have attracted increasing attention in derivative markets due to the fact that they are usually cheaper than standard options and can serve the same hedging purposes in risk management. Barrier options have been treated in a number of articles (see [23, 12, 18, 11, 25, 24, 9], among others). We consider double barrier options under the general regime-switching model (2.1). Let S_t be the risk-neutral price of a traded asset at time $t \geq 0$. Assume that $S_t = \varphi(Z_t)$, where Z_t follows the regime-switching diffusion (2.1), and $\varphi(\cdot) : \mathbb{R} \to \mathbb{R}^+$ is a continuous and strictly increasing function. Consider a perpetual double barrier knockout option written on the asset S_t that pays a rebate at the time when either one of the two barriers is hit. Let S_U and S_L denote the up and down barriers for S_t that are specified in the option contract and satisfy $0 < S_L \leq S_0 \leq S_U < \infty$. In terms of the process Z_t, we introduce

$$z_1 = \varphi^{-1}(S_L)\ , \qquad z_2 = \varphi^{-1}(S_U). \tag{2.8}$$

Then $-\infty < z_1 \leq z \leq z_2 < \infty$. Define the hitting time τ as in (2.4), or equivalently,

$$\tau = \inf\{t > 0 :\ S_t \notin (S_L, S_U)\}. \tag{2.9}$$

τ is the knockout time for the option. Let $\Phi(z, i)$ be the rebate payment function; that is, upon knocking out of the option, the option holder will receive a rebate determined by $\Phi(Z_\tau, \alpha_\tau)$. In particular, a cash rebate option is given by $\Phi(z, i) = K_1$ if $z = z_1$ and $\Phi(z, i) = K_2$ if $z = z_2$, where K_1 and K_2 are the cash rebate amounts corresponding to the upper and lower barriers, respectively.

Let $v(z, i)$ denote the barrier option value function when $Z_0 = z$ and $\alpha_0 = i$. Let $\rho > 0$ be the risk-free interest rate. Then by the risk-neutral valuation principle we have

$$v(z, i) = E\left\{\exp(-\rho\tau)\Phi(Z_\tau, \alpha_\tau)\Big|Z_0 = z, \alpha_0 = i\right\}. \tag{2.10}$$

By solving the two-point boundary value problem (2.6), (2.7), we calculate the option price $v(z, i)$.

Remark 1. In connection with the boundary value ODE problem (2.6), (2.7) studied in this paper, we focus on a perpetual double barrier knockout option (infinite maturity) in this example. We note that double barrier options with finite maturity (either with or without rebate at hitting time) have been introduced and studied extensively in the literature (e.g., Geman and Yor [11] and Sepp [25]), among others). Options with finite maturity in regime-switching models would lead to a system of second-order partial differential equations (PDE) with boundary value conditions. Developing upper and lower approximation solutions for such boundary value PDE problems is an interesting topic for future work.

Optimal selling rules in asset trading. Selling decisions in asset trading are very important for successful investment. Optimal selling rules using regime-switching diffusion models are studied in a number of papers (see [30, 27, 31, 28, 10, 15]). In these works, a selling rule is specified by two threshold levels: an upper level (greater than the purchase price) for profit target and a lower level (less than the purchase price) for stop-loss limit. The asset is sold once its price hits either level. In this case, z_1 and z_2 denote the two thresholds, and τ is the selling time. An investor sells the asset at time τ for the price S_τ, either to take a profit (if z_2 is reached) or to prevent from further loss (if z_1 is reached). The optimal selling rule problem is to find a pair of numbers (z_1^*, z_2^*) that maximize an objective function $V(z_1, z_2)$ defined by

$$V(z_1, z_2) = E\left\{\exp(-\rho\tau)\Phi(Z_\tau, \alpha_\tau)\right\}, \tag{2.11}$$

where $\Phi(z, i)$, $i = 1, \ldots, m$ are prespecified utility functions and $\rho > 0$ is a discount factor.

For given real numbers z_1, z_2, and $z \in [z_1, z_2]$, consider the solution Z_t of (2.1) with the initial value $Z_0 = z$. For each $z \in [z_1, z_2]$, define a stopping time:

$$\tau(z) = \inf\{t > 0 :\ Z_t \notin (z_1, z_2)\}.$$

Note that we use $\tau(z)$ to indicate the z dependence of the stopping time. Let

$$v(z, i) = E\left\{\exp(-\rho\tau(z))\Phi(Z_{\tau(z)}, \alpha_{\tau(z)})\Big| Z_0 = z, \alpha_0 = i\right\}. \tag{2.12}$$

Then $v(z, i)$, $i = 1, \ldots, m$, satisfy the boundary value problem (2.6), (2.7). The objective function (2.11) can be written in terms of $v(z, i)$ as

$$V(z_1, z_2) = \sum_{i=1}^{m} p_i v(z_0, i), \tag{2.13}$$

where $p_i = P\{\alpha_0 = i\}, i = 1, \ldots, m$, assumed given, is the initial probability distribution of the Markov chain α_t. An optimization technique can be employed to find the optimal thresholds z_1^* and z_2^*.

Remark 2. We note that the selling rules specified by (2.11) are of threshold type. In other words, we are seeking the optimal selling rules within the class of threshold type rules. A closely related and more general problem is the optimal stopping problem defined by

$$v(z, i) = \max_{\tau} E\left\{\exp(-\rho\tau)\Phi(Z_\tau, \alpha_\tau)\Big| Z_0 = z, \alpha_0 = i\right\}, \tag{2.14}$$

over all stopping times τ. These two problems may not be equivalent, depending on the form of the utility functions $\Phi(z, i)$, the discount factor ρ, as well as the asset model (2.1) for Z_t. In some cases, the optimal selling rules are, in fact, of threshold type (see Guo and Zhang [15] for results obtained for a regime-switching geometric Brownian motion model with two regimes). However, for the general regime-switching model (2.1), to our best knowledge, there are no clear results in the literature on the structure of the optimal selling rules, which remains as an interesting open problem. Our contribution in this paper is the development of the upper and lower approximation solutions that can be used to calculate the suboptimal selling rules for a fairly general asset model with regime switching.

3. General results. For the sake of self-containment, we present in this section an abridged development of the method of upper and lower solutions coupled with monotone methods for boundary value problems for ordinary differential equations. This theory can be developed in the context of systems and partial orders on $\mathbb{R}^m$

and is done so, very nicely in [6, Chapter 3]. For our purposes, since the differential operator in (2.6) is decoupled, the corresponding coefficient matrix in the differential system is diagonal. Hence, we shall present a scalar version with details of the method of upper and lower solutions coupled with monotone methods; we shall then state the corresponding results for vector-valued functions.

Consider the following two-point boundary value problem:

$$Ly = g(z)y''(z) + f(z)y'(z) - ry(z) = 0, \qquad \text{for } z \in (a,b), \tag{3.1}$$

$$y(a) = y_1, \qquad y(b) = y_2, \tag{3.2}$$

where $f, g \in C[a,b]$, $r > 0$ is a constant, and $-\infty < a < b < \infty$.

LEMMA 1. *Suppose that $g(z) > 0$ for $z \in [a,b]$, and then* (3.1) *is disconjugate on $[a,b]$; that is, if y is a solution of* (3.1) *and y has two roots counting multiplicity in $[a,b]$, then $y(z) \equiv 0$ for $z \in [a,b]$.*

Proof. Uniqueness of solutions of initial value problems implies that if y has repeated roots, then $y(z) \equiv 0$. If y is a solution of (3.1) and $y(z_1) = y(z_2) = 0$ for some $a \le z_1 < z_2 \le b$, then y has an extreme point at some $z_0 \in (z_1, z_2)$. Since, $y'(z_0) = 0$, $g(z_0) > 0$ and $r > 0$, it follows from (3.1) that $y''(z_0)y(z_0) > 0$, which is a contradiction. ◻

LEMMA 2. *If* (3.1) *is disconjugate on $[a,b]$, and y_1, $y_2 > 0$, then there is a positive solution $B(z)$ of the boundary value problem,* (3.1), (3.2).

Proof. Let u, v denote linearly independent solutions of (3.1). Lemma 1 implies that if

$$A = \begin{pmatrix} u(a) & v(a) \\ u(b) & v(b) \end{pmatrix},$$

then $\det A \neq 0$. Hence, the solution y of the boundary value problem, (3.1), (3.2), exists. Since y_1 and y_2 are positive, Lemma 1 also implies that $y > 0$ on $[a,b]$. ◻

LEMMA 3. *If* (3.1) *is disconjugate on $[a,b]$, then there exists a corresponding Green's function, $G(z,s)$, defined on $[a,b] \times [a,b]$ such that if $h \in C[a,b]$, then the unique solution of the boundary value problem,*

$$Ly = h, \qquad a < z < b,$$

satisfying the boundary conditions, (3.2), *is*

$$y(z) = B(z) + \int_a^b G(z,s)h(s)ds, \qquad a \le z \le b.$$

Moreover, $G(z,s) < 0$ on $(a,b) \times (a,b)$.

Remark 3. Constructions for $G(z,s)$ are given in Coppel [7], in Coddington and Levinson [5], and in (4.17) and (4.18). An important feature in this article is that the existence of G is important and the sign property of G is important; however, we shall be successful to proceed theoretically and numerically without an explicit representation of $G(z,s)$. In section 5 we shall numerically study two models, one for which the Green's function is explicitly constructed and one for which the Green's function is not explicitly constructed.

The introduction of the Green's function introduces applications of fixed point theory as methods to analyze existence of solutions of nonlinear problems. Let $h : [a,b] \times \mathbb{R} \to \mathbb{R}$ be a continuous map. Consider a nonlinear boundary value problem of the form,

$$Ly(z) = h(z, y(z)), \qquad a < z < b, \tag{3.3}$$

with boundary conditions, (3.2).

Corollary 1. *$y \in C^2[a,b]$ is a solution of the boundary value problem,* (3.3), (3.2)*, if, and only if, $y \in C[a,b]$ and*

$$y(z) = B(z) + \int_a^b G(z,s)h(s,y(s))ds, \qquad a \le z \le b.$$

A function $v_0 \in C^2[a,b]$ is called a *lower solution* of the boundary value problem, (3.3), (3.2), if

$$Lv_0(z) \ge h(z, v_0(z)), \quad a < z < b, \quad v_0(a) \le y_1, \quad v_0(b) \le y_2.$$

A function $u_0 \in C^2[a,b]$ is called an *upper solution* of the boundary value problem, (3.3), (3.2), if

$$Lu_0(z) \le h(z, u_0(z)), \quad a < z < b, \quad u_0(a) \ge y_1, \quad u_0(b) \ge y_2.$$

In Lemma 3 we point out that the kernel, $G(z,s) < 0$ on $(a,b) \times (a,b)$. Assume, in addition, the function $h(s,y)$ is monotone decreasing in y. Define an operator $T : C[a,b] \to C[a,b]$ by

$$Ty(z) = B(z) + \int_a^b G(z,s)h(s,y(s))ds, \qquad a \le z \le b. \tag{3.4}$$

Then T is monotone on $C[a,b]$; that is, if $y, w \in C[a,b]$, $y(z) \le w(z), a \le z \le b$, then $Ty(z) \le Tw(z), a \le z \le b$.

Finally, assume the lower and upper solutions satisfy

$$v_0(z) \le u_0(z), \quad a \le z \le b.$$

Note that Lemma 3 implies

$$v_0(z) \le Tv_0(z), \quad Tu_0(z) \le u_0(z), \qquad a \le z \le b.$$

To see this, let B_{v_0} denote the solution of

$$\begin{gathered} Ly = 0, \qquad a \le z \le b, \\ y(a) = v_0(a), \qquad y(b) = v_0(b). \end{gathered}$$

By Lemma 3,

$$v_0(z) = B_{v_0}(z) + \int_a^b G(z,s)Lv_0(s)ds, \qquad a < z < b.$$

By Lemma 1, $B_{v_0}(z) \le B(z), a \le z \le b$, and it follows by the definition of lower solution and the sign of the Green's function that

$$v_0(z) = B_{v_0}(z) + \int_a^b G(z,s)Lv_0(s)ds \le B(z) + \int_a^b G(z,s)h(s,v_0(s))ds \le Tv_0(s),$$

for $a < z < b$. Employ that T is monotone, and we obtain the inequality

$$v_0(z) \le Tv_0(z) \le Tu_0(z) \le u_0(z), \qquad a \le z \le b. \tag{3.5}$$

At this point, a straightforward application of the Schauder fixed point theorem (see Jackson [16]) implies the existence of a solution y of the boundary value problem, (3.3), (3.2), satisfying

$$v_0(z) \leq y(z) \leq u_0(z), \qquad a \leq z \leq b.$$

Moreover, define recursively, sequences $\{v_n(z)\}$ and $\{u_n(z)\}$ by

$$v_{n+1}(z) = Tv_n(z), \qquad u_{n+1}(z) = Tu_n(z). \tag{3.6}$$

Then, it follows inductively from (3.5) that for each n,

$$v_n(z) \leq v_{n+1}(z) \leq u_{n+1}(z) \leq u_n(z);$$

v_n converges monotonically and uniformly to a solution v of the boundary value problem, (3.3), (3.2), and u_n converges monotonically and uniformly to a solution u of the boundary value problem, (3.3), (3.2), where

$$v(z) \leq u(z), \qquad a \leq z \leq b.$$

Also note that by Corollary 1, v_{n+1} can be constructed as the solution of the boundary value problem,

$$Ly = h(z, v_n(z)), \qquad a < z < b, \tag{3.7}$$

satisfying the boundary conditions (3.2). The sequence $\{u_{n+1}\}$ can be constructed analogously. In particular, if the Green's function is explicitly constructed, we prefer to employ quadrature methods and (3.6) to compute numerical solutions. If the Green's function is not explicitly constructed, we can employ numerical methods for ordinary differential equations on boundary value problems of the form, (3.7), (3.2).

The application of differential inequalities developed above readily generalizes to systems of m equations. Define a Banach space $C_m[z_1, z_2]$ by

$$C_m[z_1, z_2] = \Big\{U = (u_1, \ldots, u_m)' : [z_1, z_2] \to \mathbb{R}^m, u_i \in C[z_1, z_2], i = 1, \ldots, m\Big\}$$

with norm $\| U \| = \max_{1\leq i\leq m}\{\| u_i \|_0\}$, where $\| \cdot \|_0$ denotes the usual supremum norm, and $C[z_1, z_2]$ denotes the space of continuous functions from $[z_1, z_2]$ to $\mathbb{R}$. Consider the partial order on $\mathbb{R}^m$:

$$W \leq Y \Longleftrightarrow w_i \leq y_i, \qquad i = 1, \ldots, m,$$

where $W = (w_1, \ldots, w_m)' \in \mathbb{R}^m$, $Y = (y_1, \ldots, y_m)' \in \mathbb{R}^m$. Define a partial order on $C_m[z_1, z_2]$:

$$V \leq U \Longleftrightarrow V(z) \leq U(z), \quad z \in [z_1, z_2], \quad \text{where } U, V \in C_m.$$

Assume $T : C_m[z_1, z_2] \to C_m[z_1, z_2]$ is a continuous map that maps closed convex sets into relatively compact sets. Assume there exist $V_0, U_0 \in C_m[z_1, z_2]$ satisfying

$$V_0 \leq TV_0 \leq TU_0 \leq U_0,$$

with respect to the partial order on $C_m[z_1, z_2]$. Then the application of the Schauder fixed point theorem and the monotone methods described above readily adapt to the operator T; a sequence $V_{n+1} = TV_n$ converges to a fixed point $V \in C_m[z_1, z_2]$ of T from below and a sequence $U_{n+1} = TU_n$ converges to a fixed point $U \in C_m[z_1, z_2]$ of T from above.

We now show how this theory applies to the boundary value problem for the system of equations (2.6), (2.7). For each $i = 1, \ldots, m$, write

$$L_i v(z,i) = \frac{\sigma^2(z,i)}{2}\frac{d^2 v(z,i)}{dz^2} + \mu(z,i)\frac{dv(z,i)}{dz} - (\rho - q_{ii})v(z,i).$$

In view of $q_{ii} = -\sum_{j\neq i} q_{ij}$, we can rewrite (2.6) as

$$L_i v(z,i) = -\sum_{j\neq i} q_{ij} v(z,j), \quad z \in (z_1, z_2), \quad i = 1, \ldots, m, \tag{3.8}$$

$$v(z_1, i) = \Phi(z_1, i), \quad v(z_2, i) = \Phi(z_2, i), \quad i = 1, \ldots, m. \tag{3.9}$$

Write the boundary value problem (3.8), (3.9) in matrix form,

$$LV(z) = Q_0 V(z), \qquad z_1 < z < z_2, \tag{3.10}$$

$$V(z_1) = \Phi(z_1), \qquad V(z_2) = \Phi(z_2), \tag{3.11}$$

where

$$\begin{aligned} V(z) &= (v(z,1), \ldots, v(z,m))', \\ LV(z) &= \operatorname{diag}\left(L_1 v(z,1), \ldots, L_m v(z,m)\right), \\ \Phi(z) &= (\Phi(z,1), \ldots, \Phi(z,m))', \end{aligned}$$

and

$$Q_0 = \begin{pmatrix} 0 & -q_{12} & \cdots & -q_{1m} \\ -q_{21} & 0 & \cdots & -q_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ -q_{m1} & -q_{m2} & \cdots & 0 \end{pmatrix}. \tag{3.12}$$

The boundary value problem, (3.10), (3.11), is now equivalent to a fixed point problem

$$V(z) = TV(z) = B(z) + \int_{z_1}^{z_2} G(z,s) Q_0 V(s) ds, \tag{3.13}$$

where $B(z) = (b(z,1), b(z,2), \ldots, b(z,m))'$, $b(z,i)$ is the solution of the scalar boundary value problem

$$L_i v(z) = 0, \quad z_1 < z < z_2, \quad v(z_1) = \Phi(z_1, i), \quad v(z_2) = \Phi(z_2, i),$$

and $G(z,s) = \operatorname{diag}\{G_{ii}(z,s)\}$, where $G_{ii}(z,s) = G(z,s,i)$ denotes the scalar Green's function for the boundary value problem,

$$L_i y(z) = 0, \quad z_1 < z < z_2, \quad y(z_1) = 0, \quad y(z_2) = 0.$$

Note that $\frac{\sigma^2(z,i)}{2} > 0$, $r = \rho - q_{ii} > 0$, for each $i = 1, \ldots, m$; it follows from Lemma 2 that $B(z) > 0$ on $[z_1, z_2]$. For the purposes of introducing upper and lower solutions, we shall relabel

$$B(z) = \mathbf{V}_0(z),$$

where $b(z,i) = v_0(z,i), i = 1, \ldots, m$.

Let K be a constant satisfying

$$K \geq \max_{i=1,\ldots,m} \{||v_0(z,i)||\}. \tag{3.14}$$

Let $u_0(z,i) = K$, $z_1 \leq z \leq z_2$, $i = 1, \ldots, m$, and denote

$$\mathbf{U}_0(z) = (u_0(z,1), u_0(z,2), \ldots, u_0(z,m))'. \tag{3.15}$$

Thus,

$$\mathbf{V}_0(z) \leq \mathbf{U}_0(z), \qquad z_1 \leq z \leq z_2.$$

Note that $0 = L\mathbf{V}_0 \geq Q_0\mathbf{V}_0$. It follows that $\mathbf{V}_0$ is a lower solution and if $\mathbf{V}_1$ is defined as $\mathbf{V}_1 = T\mathbf{V}_0$, where T is defined in (3.13), then

$$\mathbf{V}_0(z) \leq \mathbf{V}_1(z), \qquad z_1 \leq z \leq z_2.$$

To see that $\mathbf{U}_0$ serves as an appropriate upper solution, note that $\mathbf{U}_0$ is a constant vector and recall that $q_{ii} = -\sum_{j\neq i} q_{ij}$. So, for each $i = 1, \ldots, m$,

$$L_i u_0(z,i) = -(\rho - q_{ii})u_0(z,i) \leq -\sum_{j\neq i} q_{ij}u_0(z,j).$$

In particular, $L\mathbf{U}_0 \leq Q_0\mathbf{U}_0$. $\mathbf{U}_0$ satisfies appropriate boundary conditions so that $\mathbf{U}_0$ is an upper solution. Define $\mathbf{U}_1$ by $\mathbf{U}_1 = T\mathbf{U}_0$, and then

$$\mathbf{U}_1(z) \leq \mathbf{U}_0(z), \qquad z_1 \leq z \leq z_2.$$

In summary, we have shown

$$\mathbf{V}_0(z) \leq \mathbf{V}_1(z) \leq \mathbf{U}_1(z) \leq \mathbf{U}_0(z), \qquad z_1 \leq z \leq z_2.$$

Define inductively

$$\mathbf{V}_{n+1}(z) = T\mathbf{V}_n(z), \qquad \mathbf{U}_{n+1}(z) \leq T\mathbf{U}_n(z),$$

and it follows from the monotonicity of T that

$$\mathbf{V}_n(z) \leq \mathbf{V}_{n+1}(z) \leq \mathbf{U}_{n+1}(z) \leq \mathbf{U}_n(z), \qquad z_1 \leq z \leq z_2.$$

In addition to the construction of a numerical algorithm, we have proved the following existence and uniqueness of solution result.

THEOREM 1. *Assume for each $i = 1, \ldots, m$, $\sigma(z,i)$ and $\mu(z,i)$ are continuous on $[z_1, z_2]$ and assume $\sigma(z,i) \neq 0, z \in [z_1, z_2]$. Assume $\rho > 0$. Assume the matrix $Q = (q_{ij})_{m\times m}$ denotes the generator of α_t; in particular, its entries q_{ij} satisfy:* (I) *$q_{ij} \geq 0$ if $i \neq j$;* (II) *$q_{ii} \leq 0$ and $q_{ii} = -\sum_{j\neq i} q_{ij}$ for each $i = 1, \ldots, m$. Then there exists a unique solution, $V(z) = (v(z,1), \ldots, v(z,m))'$ of the boundary value problem for the system of ordinary differential equations* (2.6), (2.7). *Moreover, each value function $v(z,i) \in C^2[z_1, z_2]$.*

4. Upper and lower solutions for a regime-switching mean-reverting model. In this section we deal with a regime-switching mean-reverting diffusion model for the underlying stochastic process. This model was considered in Eloe, Liu, and Sun [9] for pricing double barrier options and in Eloe et al. [10] for optimal selling rules. However, an assumption on model parameters was made ([9, Assumption 1], [10, Assumption 3.4]). In this section we obtain closed-form lower and upper solutions without making the assumption. Thus, the results presented in this section generalize our previous work. The upper and lower solutions are then used to approximate the unique solution of the boundary value problem.

Let $\mu(z,i) = \kappa(i)[b(i) - z]$, $\sigma(z,i) = \sigma(i)$ in (2.1). Then we have the following regime-switching mean-reverting process for Z_t:

$$dZ_t = \kappa(\alpha_t)[b(\alpha_t) - Z_t]dt + \sigma(\alpha_t)dB_t, \qquad Z_0 = z, \tag{4.1}$$

where $b(\alpha_t)$ denotes the mean reverting level, $\kappa(\alpha_t)$ denotes the rate at which Z_t is pulled back to the level $b(\alpha_t)$, and $\sigma(\alpha_t)$ is the volatility. Note that the parameters $b(\cdot)$, $\kappa(\cdot)$, and $\sigma(\cdot)$ in (4.1) depend on α_t, indicating that they can take different values for different regimes. We assume that $\kappa(i) > 0$ and $\sigma(i) > 0$ for $i = 1, \ldots, m$. In this case, the system (2.6), (2.7) becomes

$$\frac{\sigma^2(i)}{2}\frac{d^2v(z,i)}{dz^2} + \kappa(i)[b(i) - z]\frac{dv(z,i)}{dz} - (\rho - q_{ii})v(z,i) = -\sum_{j\neq i} q_{ij}v(z,j), \quad i = 1, \ldots, m, \tag{4.2}$$

for $z \in (z_1, z_2)$, and,

$$v(z_1,i) = \Phi(z_1,i), \quad v(z_2,i) = \Phi(z_2,i), \quad i = 1, \ldots, m. \tag{4.3}$$

To find a lower solution of (4.2), (4.3), consider the following one-dimensional differential equation:

$$\frac{\sigma^2}{2}V_{zz}(z) + \kappa(b - z)V_z(z) - rV(z) = 0, \qquad \text{for } z \in (z_1, z_2), \tag{4.4}$$

where κ, b, r, σ are constants with $\sigma > 0$, $\kappa > 0$, and $r > 0$. Set $x = \frac{\sqrt{2\kappa}}{\sigma}(z - b)$ and let $\widetilde{V}(x) = V(z)$. Then (4.4) is transformed to

$$\widetilde{V}_{xx}(x) - x\widetilde{V}_x(x) - \lambda\widetilde{V}(x) = 0, \qquad \text{for } x \in (\bar{x}_1, \bar{x}_2), \tag{4.5}$$

where $\lambda := \frac{r}{\kappa}$, $\bar{x}_1 = \frac{\sqrt{2\kappa}}{\sigma}(z_1 - b)$, and $\bar{x}_2 = \frac{\sqrt{2\kappa}}{\sigma}(z_2 - b)$. To solve (4.5), we employ the following transform:

$$\widetilde{V}(x) = \exp\left(\frac{x^2}{4}\right)D(x).$$

Then $D(x)$ satisfies

$$D_{xx}(x) + \left[\frac{1}{2} - \frac{x^2}{4} - \lambda\right]D(x) = 0. \tag{4.6}$$

Equation (4.6) is known as Weber's equation, and its two independent solutions are given by (see [2, 8]) $D(x)$ and $D(-x)$, where

$$D(x) = \frac{1}{\Gamma(\lambda)}\exp\left(-\frac{x^2}{4}\right)\int_0^\infty t^{\lambda-1}\exp\left(-\frac{t^2}{2} - xt\right)\,dt,$$

where $\Gamma(\cdot)$ is the Gamma function. Consequently, the general solution to (4.5) is

$$\widetilde{V}(x) = C_1 \int_0^\infty t^{\lambda-1} \exp\left(-\frac{t^2}{2} - xt\right) dt + C_2 \int_0^\infty t^{\lambda-1} \exp\left(-\frac{t^2}{2} + xt\right) dt, \tag{4.7}$$

where C_1 and C_2 are constants. It then follows that the general solution of (4.4) is given by

$$\begin{aligned} V(z) := V(z;\kappa,b,\sigma,r) &= \widetilde{V}\left(\frac{\sqrt{2\kappa}}{\sigma}(z-b)\right) \\ &= C_1 \int_0^\infty t^{\frac{r}{\kappa}-1} \exp\left(-\frac{t^2}{2} - \frac{\sqrt{2\kappa}}{\sigma}(z-b)t\right) dt \\ &\quad + C_2 \int_0^\infty t^{\frac{r}{\kappa}-1} \exp\left(-\frac{t^2}{2} + \frac{\sqrt{2\kappa}}{\sigma}(z-b)t\right) dt. \end{aligned} \tag{4.8}$$

For (4.2), $i = 1, \dots, m$, consider the homogeneous equation

$$\frac{\sigma^2(i)}{2}\frac{d^2 v(z,i)}{dz^2} + \kappa(i)[b(i)-z]\frac{dv(z,i)}{dz} - (\rho - q_{ii})v(z,i) = 0, \tag{4.9}$$

for $z \in (z_1, z_2)$ and the boundary value conditions

$$v(z_1,i) = \Phi(z_1,i), \qquad v(z_2,i) = \Phi(z_2,i). \tag{4.10}$$

Its solution is given by

$$v_0(z,i) = V(z;\kappa(i),b(i),\sigma(i),(\rho - q_{ii})), \tag{4.11}$$

in which the two constants C_1 and C_2 are determined uniquely by the boundary conditions (4.10). A closed-form lower solution for (4.2), (4.3) is then given by

$$\mathbf{V}_0(z) = (v_0(z,1), v_0(z,2), \dots, v_0(z,m))'. \tag{4.12}$$

On the other hand, let K be a constant given by

$$K = \max\left\{\Phi(z_1,i), \Phi(z_2,i), i = 1,\dots,m\right\}. \tag{4.13}$$

Let $u_0(z,i) = K$, $i = 1, \dots, m$, and denote

$$\mathbf{U}_0(z) = (u_0(z,1), u_0(z,2), \dots, u_0(z,m))'. \tag{4.14}$$

Then it can be seen that $\mathbf{U}_0$ is an upper solution of (4.2), (4.3).

Next we construct an explicit Green's function $G(z,s)$ for (4.2), (4.3). Let

$$D_1(z) := D_1(z;\kappa,b,\sigma,r) = \int_0^\infty t^{\frac{r}{\kappa}-1} \exp\left(-\frac{t^2}{2} - \frac{\sqrt{2\kappa}}{\sigma}(z-b)t\right) dt \tag{4.15}$$

and

$$D_2(z) := D_2(z;\kappa,b,\sigma,r) = \int_0^\infty t^{\frac{r}{\kappa}-1} \exp\left(-\frac{t^2}{2} + \frac{\sqrt{2\kappa}}{\sigma}(z-b)t\right) dt \tag{4.16}$$

denote the two independent solutions of (4.4). The Green's function G is given as follows:

$$G(z,s;\kappa,b,\sigma,r)=\begin{cases}\pi_1(s)D_1(z)+\pi_2(s)D_2(z), & z_1\le s<z\le z_2,\\ \pi_3(s)D_1(z)+\pi_4(s)D_2(z), & z_1\le z<s\le z_2,\end{cases} \tag{4.17}$$

where $\pi_1(s)$, $\pi_2(s)$, $\pi_3(s)$, and $\pi_4(s)$ satisfy [5]

$$\begin{pmatrix} 0 & 0 & D_1(z_1) & D_2(z_1)\\ D_1(z_2) & D_2(z_2) & 0 & 0\\ D_1(s) & D_2(s) & -D_1(s) & -D_2(s)\\ D_{1,z}(s) & D_{2,z}(s) & -D_{1,z}(s) & -D_{2,z}(s)\end{pmatrix}\begin{pmatrix}\pi_1(s)\\ \pi_2(s)\\ \pi_3(s)\\ \pi_4(s)\end{pmatrix}=\begin{pmatrix}0\\0\\0\\ \frac{2}{\sigma^2}\end{pmatrix}, \tag{4.18}$$

where $D_{1,z}$ and $D_{2,z}$ denote the derivatives of D_1 and D_2 with respect to z. It follows that the Green's function for (4.2), (4.3) is given as $G(z,s)=\operatorname{diag}\{G_{ii}(z,s)\}$, where

$$G_{ii}(z,s):=G(z,s;\kappa(i),b(i),\sigma(i),(\rho-q_{ii}))$$

is the scalar Green's function for the ith equation in (4.2). This Green's function G is used in the next section in numerically constructing sequences of upper and lower solutions.

5. Numerical examples. In this section we provide three numerical examples to validate the upper and lower solutions developed in this paper. The computations were performed using MATLAB on a notebook PC with the following system specifications: Intel(R) Core(TM)2 CPU T7400 2.16 GHz 1GB RAM. The iterative process stops when the difference between two successive upper (respectively, lower) approximation solutions is within a prespecified error tolerance $\varepsilon>0$.

Example 1. We price perpetual double barrier options in a regime-switching mean-reverting diffusion model. Let S_t be the price of the underlying asset at time $t\ge 0$ with initial price $S_0>0$. Let $S_t=\exp(Z_t)$, where Z_t follows (4.1) and $Z_0=\ln S_0$.

Case 1. We consider the case of two regimes, i.e., $m=2$. The following parameter values are used. For the regime-switching model (4.1),

$$\kappa(1)=3,\quad \kappa(2)=2,\quad \sigma(1)=0.6,\quad \sigma(2)=0.8,\quad b(1)=0.05,\quad b(2)=0.08.$$

The risk-free interest rate is $\rho=0.07$. The jump rates between the two regimes are specified by the following generator matrix:

$$Q=(q_{ij})=\begin{pmatrix}-2 & 2\\ 3 & -3\end{pmatrix}.$$

The two barriers are chosen as $z_1=\ln(0.5)$, $z_2=\ln(2)$; that is, if the initial asset price $S_0=1$, then the option would be knocked out whenever the asset price is doubled or halved. We assume that upon the knocking out time, the option holder receives a cash rebate equal to 2 units of currency (dollar, for example), i.e., $\Phi(z_1,i)=\Phi(z_2,i)=2$, $i=1,2$. Figure 1 plots the initial upper and lower solutions corresponding to the two regimes, constructed using (4.14) and (4.12), respectively. Figure 2 plots a number of selected upper and lower approximation solutions of the option price functions $v(z,1)$ and $v(z,2)$. These approximation solutions are obtained by using the Green's function (4.17) and numerical integration methods. Figure 2 clearly suggests that the

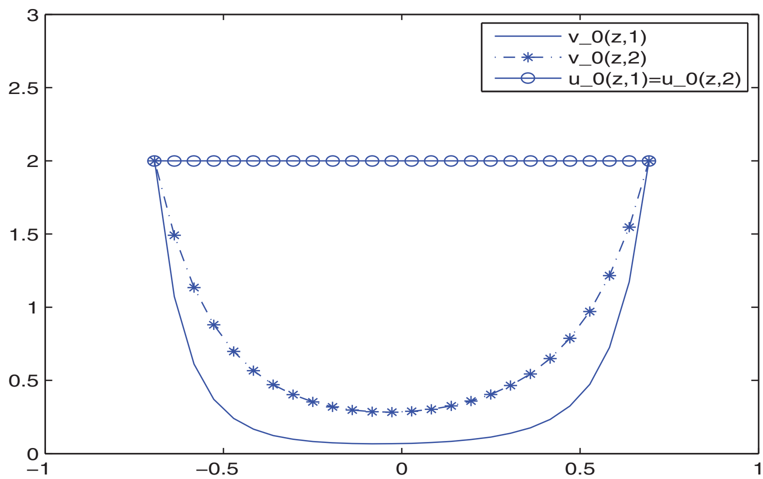

FIG. 1. *Initial upper and lower solutions—two regimes.*

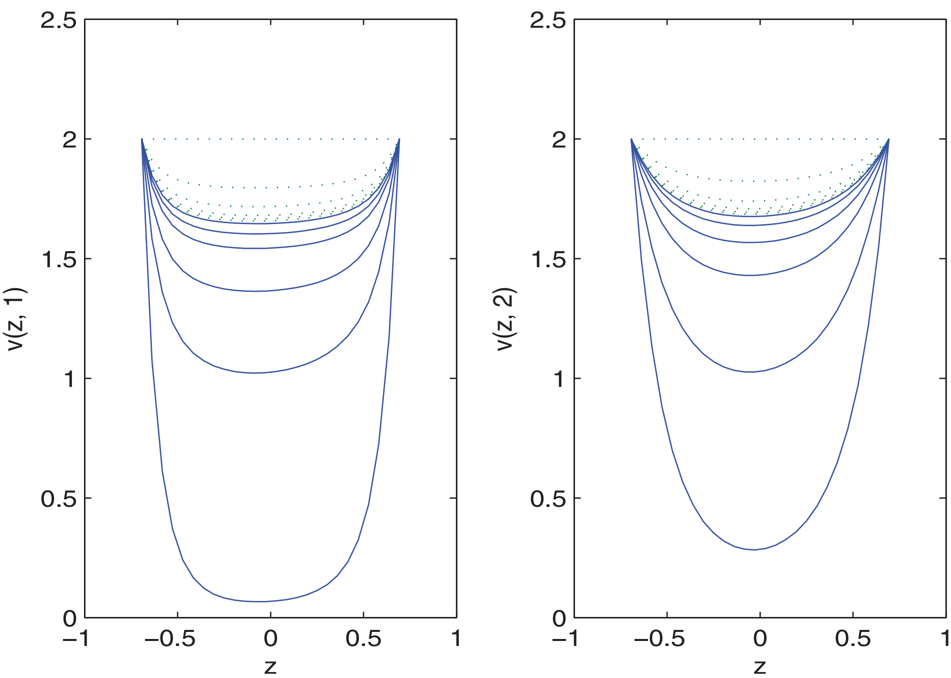

FIG. 2. *Approximation sequences and convergence for $V(z)$. The dotted lines represent the upper approximation sequences, and the solid lines represent the lower approximation sequences. The left graph is for $v(z,1)$ (regime 1) and the right graph is for $v(z,2)$ (regime 2).*

upper and lower approximation sequences converge to a common solution, which is the unique solution to the boundary value system.

For comparison, we directly approximate the expectation in (2.10) by implementing a Monte Carlo (MC) simulation algorithm for the double barrier options considered in this example. To this end, a time step h is used to discretize the continuous-time

TABLE 1
Approximate prices of double barrier options computed by upper and lower solutions and by MC simulations (two regimes).

z	-0.5545	-0.4159	-0.2773	-0.1386	0	0.1386	0.2773	0.4159	0.5545
$S = e^z$	0.5743	0.6598	0.7579	0.8706	1.0000	1.1487	1.3195	1.5157	1.7411
LP ($\alpha_0 = 1$)	1.7429	1.6800	1.6592	1.6525	1.6530	1.6595	1.6739	1.7030	1.7727
UP ($\alpha_0 = 1$)	1.7431	1.6802	1.6595	1.6528	1.6533	1.6598	1.6741	1.7033	1.7729
MC ($\alpha_0 = 1$)	1.7417	1.6790	1.6583	1.6516	1.6517	1.6584	1.6734	1.7039	1.7721
st. dev	0.0025	0.0024	0.0025	0.0026	0.0024	0.0023	0.0026	0.0025	0.0025
LP ($\alpha_0 = 2$)	1.8240	1.7396	1.6998	1.6834	1.6817	1.6923	1.7172	1.7636	1.8474
UP ($\alpha_0 = 2$)	1.8241	1.7398	1.7000	1.6837	1.6820	1.6926	1.7174	1.7637	1.8475
MC ($\alpha_0 = 2$)	1.8234	1.7382	1.6994	1.6839	1.6810	1.6906	1.7162	1.7629	1.8461
st. dev	0.0024	0.0027	0.0026	0.0025	0.0026	0.0025	0.0026	0.0025	0.0024

process (4.1). The resultant discrete process is

$$Z_{n+1} = Z_n + \kappa(\alpha_n)[b(\alpha_n) - Z_n]h + \sigma(\alpha_n)\sqrt{h}\,\xi_n, \tag{5.1}$$

where for $n = 0, 1, 2, \ldots$, $Z_n = Z_{nh}$, $\alpha_n = \alpha_{nh}$, $\{\xi_n\}$ is a sequence of independent normal random variables with mean 0 and variance 1. The MC algorithm proceeds as follows: Generate a sample path of the random sequence (Z_n, α_n), $n = 1, 2, \ldots$. Find the stopping time $\tau = \inf\{n \geq 0 : \ Z_n \notin (z_1, z_2)\}$. Calculate the payoff $\exp(-\rho\tau)\Phi(Z_\tau, \alpha_\tau)$ for the sample path. Repeat the process for N times and compute the average of the N payoff values.

Table 1 reports the upper approximate prices (labeled as UP), the lower approximate prices (labeled as LP), and the Monte Carlo approximations (labeled as MC) for a range of asset prices. The first and second rows list the z values and the corresponding asset prices S, respectively; the third to fifth rows list the approximate option prices for regime $\alpha_0 = 1$; the sixth to eighth rows list the results for $\alpha_0 = 2$. For the specified precision $\varepsilon = 0.0005$, it took 45 iterations for the lower solution and 54 iterations for the upper solution. For the MC simulations, $h = 0.00001$ was used. The standard deviation was computed based on 100 approximations, each being done using 10000 sample paths. The MC simulations are very time consuming (hours were spent to complete the process). In contrast, the upper and lower solutions were done within 10 seconds.

Case 2. We report the results of using the upper and lower solutions approximation method for the case of four regimes, i.e., $m = 4$. The following parameter values are used. For the regime-switching model (4.1),

$$\begin{array}{llll} \kappa(1) = 3, & \kappa(2) = 2.5, & \kappa(3) = 2, & \kappa(4) = 1.5, \\ \sigma(1) = 0.4, & \sigma(2) = 0.5, & \sigma(3) = 0.6, & \sigma(4) = 0.7, \\ b(1) = 0.05, & b(2) = 0.07, & b(3) = 0.08, & b(4) = 0.09. \end{array}$$

The generator of the Markov chain α_t is

$$Q = \begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{pmatrix}.$$

The other parameters for the barrier options are the same as in Case 1.

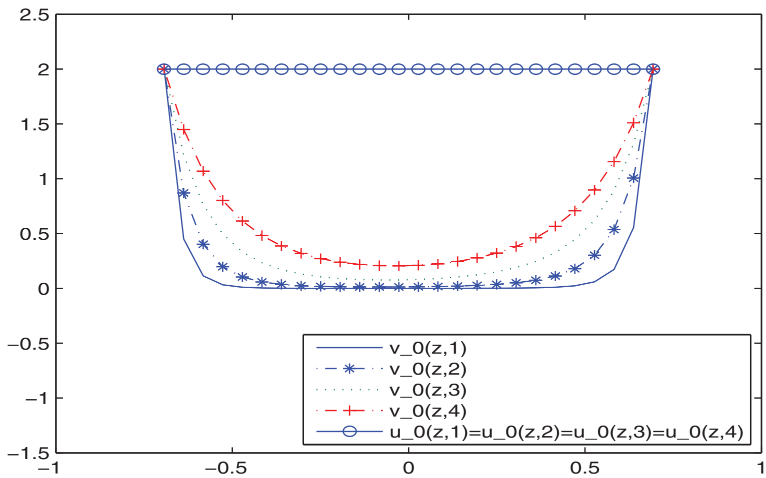

FIG. 3. *Initial upper and lower solutions—four regimes.*

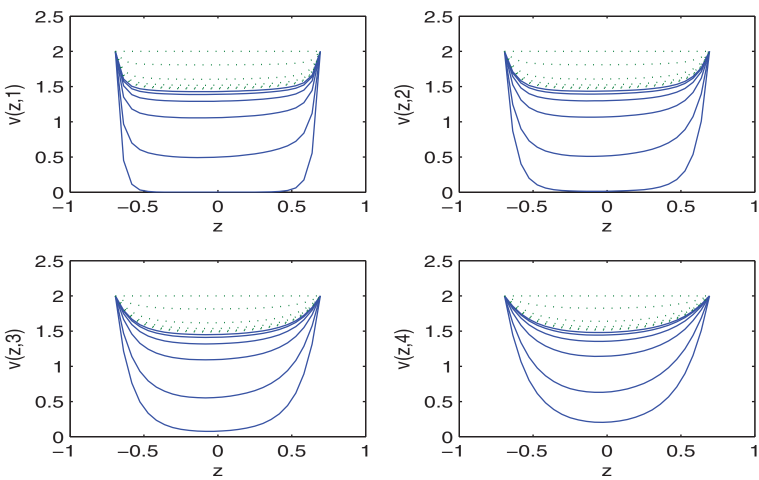

FIG. 4. *Approximation sequences and convergence for $V(z)$. The dotted lines represent the upper approximation sequences, and the solid lines represent the lower approximation sequences.*

Figure 3 plots the initial upper and lower solutions corresponding to the four regimes. Figure 4 plots a number of selected upper and lower approximation solutions of the option price functions $v(z,1)$, $v(z,2)$, $v(z,3)$, and $v(z,4)$. We see again that both the upper and lower approximate solutions converge to the solution of the boundary value system satisfied by the option value functions.

Table 2 reports the upper approximation prices and the lower approximation prices for a range of asset prices, for the four regimes. The same precision $\varepsilon = 0.0005$ is used. A total of 97 iterations are required for the lower approximations, and 86 iterations are required for the upper approximations. The convergence rate is slower for the four-regime case than the two-regime case.

Example 2. In this example we study the optimal selling rule problem given by (2.11)–(2.13) using the same model parameters as in Example 1, Case 1. The prob-

Table 2
Upper and lower prices of the double barrier options in a four-regime mean-reverting model.

z	-0.5545	-0.4159	-0.2773	-0.1386	0	0.1386	0.2773	0.4159	0.5545
$S = e^z$	0.5743	0.6598	0.7579	0.8706	1.0000	1.1487	1.3195	1.5157	1.7411
LP ($\alpha_0 = 1$)	1.5007	1.4635	1.4524	1.4493	1.4513	1.4579	1.4700	1.4918	1.5452
UP ($\alpha_0 = 1$)	1.5012	1.4640	1.4530	1.4499	1.4519	1.4584	1.4706	1.4923	1.5456
LP ($\alpha_0 = 2$)	1.5596	1.4820	1.4609	1.4554	1.4577	1.4666	1.4848	1.5220	1.6187
UP ($\alpha_0 = 2$)	1.5601	1.4825	1.4615	1.4560	1.4582	1.4672	1.4854	1.5225	1.6191
LP ($\alpha_0 = 3$)	1.6411	1.5260	1.4852	1.4721	1.4734	1.4863	1.5146	1.5715	1.6951
UP ($\alpha_0 = 3$)	1.6414	1.5265	1.4857	1.4727	1.4739	1.4869	1.5151	1.5720	1.6954
LP ($\alpha_0 = 4$)	1.7157	1.5851	1.5261	1.5030	1.5014	1.5176	1.5549	1.6254	1.7560
UP ($\alpha_0 = 4$)	1.7160	1.5855	1.5266	1.5035	1.5019	1.5181	1.5554	1.6258	1.7562

Table 3
Optimal selling rules using a two-regime mean-reverting model.

$[l_1, u_1] \times [l_2, u_2]$	$V(z_1^*, z_2^*)$	(z_1^*, z_2^*)	% up	% down
$[-0.2, -0.01] \times [0.01, 1.0]$	0.04	$(-0.2, 0.31)$	36.3%	18.1%
$[-0.3, -0.01] \times [0.01, 1.0]$	0.08	$(-0.3, 0.36)$	43.3%	25.9%
$[-0.4, -0.01] \times [0.01, 1.0]$	0.14	$(-0.4, 0.42)$	52.2%	33.0%
$[-0.5, -0.01] \times [0.01, 1.0]$	0.22	$(-0.5, 0.48)$	61.6%	39.4%
$[-0.6, -0.01] \times [0.01, 1.0]$	0.31	$(-0.6, 0.54)$	71.6%	45.1%
$[-0.8, -0.01] \times [0.01, 1.0]$	0.52	$(-0.8, 0.66)$	93.5%	55.1%
$[-1.0, -0.01] \times [0.01, 1.0]$	0.68	$(-1.0, 0.73)$	107.5%	63.2%

ability distribution of the initial Markov chain α_0 is assumed to be $p_1 = p_2 = \frac{1}{2}$. We use $\Phi(z, 1) = \Phi(z, 2) = e^z - 1$. By using this utility function, one seeks for the maximum percentage return of investment (see [10, Remark 2.4]). We numerically search the maximum objective function value $V(z_1, z_2)$ over a closed rectangular region $[l_1, u_1] \times [l_2, u_2]$ of (z_1, z_2). We use a grid size 0.01 to discretize the region $[l_1, u_1] \times [l_2, u_2]$, resulting in a collection of discrete values for (z_1, z_2). For example, for $[l_1, u_1] \times [l_2, u_2] = [-0.40, -0.01] \times [0.01, 1.0]$, we have a total of 4000 points for (z_1, z_2), representing a discrete approximation of the region $[-0.40, -0.01] \times [0.01, 1.0]$. For each point, which specifies the boundaries z_1 and z_2, the boundary value problem (4.2), (4.3) is solved and the approximate solutions $v(z, 1)$, $v(z, 2)$ are obtained. In view of (2.13), the objective function $V(z_1, z_2)$ is then calculated by $V(z_1, z_2) = [v(0, 1) + v(0, 2)]/2$. The optimal thresholds (z_1^*, z_2^*) and the maximum objective function value $V(z_1^*, z_2^*)$ are then identified.

Table 3 reports the results of the optimal selling rules over a number of search regions for (z_1, z_2). The first column is the rectangular region over which the optimal thresholds are sought; the second and third columns are the optimal objective function value $V(z_1^*, z_2^*)$ and the optimal thresholds (z_1^*, z_2^*), respectively; the fourth and fifth columns are the percentage increase (% up) and decrease (% down) for the underlying asset prices, computed using (z_1^*, z_2^*). From Table 3 we see that the optimal thresholds for selling the asset are strongly influenced by the prespecified stop-loss limit l_1. For instance, if a 33% drop in asset price (given by $l_1 = -0.4$) is used by an investor for the stop-loss limit, then the optimal percentage increase in asset price is 52.2%. Following the selling rule, the investor would sell the asset he/she has held whenever the price goes up by 52.2% or down by 33%.

Example 3. In this example we consider a different regime-switching diffusion process:

$$dZ_t = \sin(a(\alpha_t)Z_t)dt + \sigma(\alpha_t)dB_t. \tag{5.2}$$

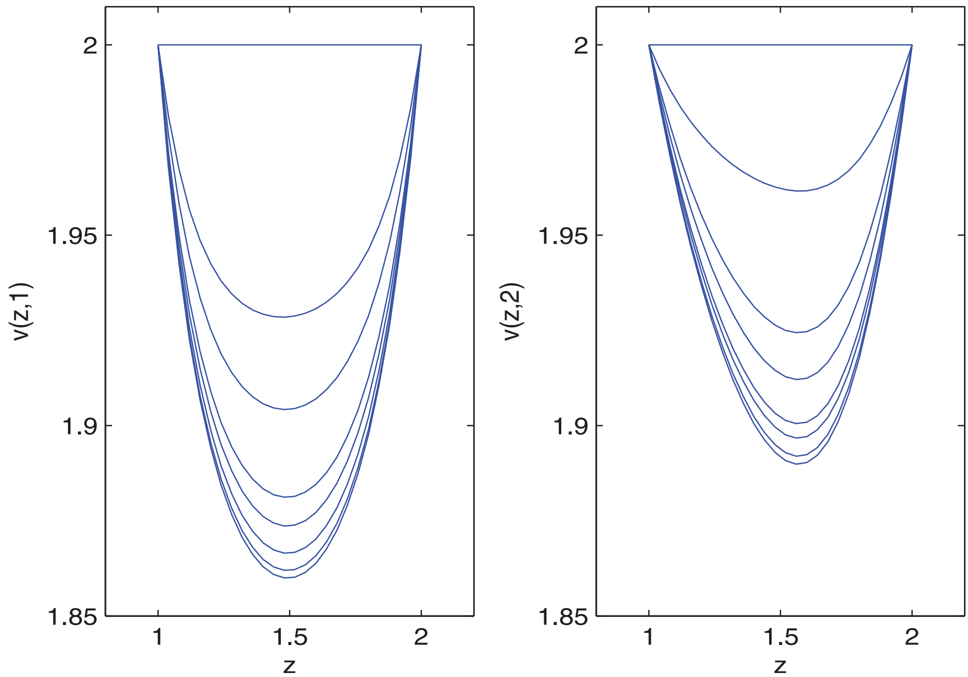

FIG. 5. *Upper solution sequences in Example* 3.

The corresponding boundary value problem is given as

$$\tag{5.3} \begin{cases} \dfrac{\sigma^2(i)}{2}\dfrac{d^2v(z,i)}{dz^2} + \sin(a(i)z)\dfrac{dv(z,i)}{dz} - (\rho - q_{ii})v(z,i) \\ \qquad\qquad\qquad = -\displaystyle\sum_{j\neq i} q_{ij}v(z,j), \quad \text{for } z \in (z_1, z_2), \\ v(z_1,i) = \Phi(z_1,i), \quad v(z_2,i) = \Phi(z_2,i), \quad i = 1,\dots,m. \end{cases}$$

For (5.3), we do not have a closed-form lower solution $\mathbf{V}_0(z)$ and an explicit Green's function $G(z,s)$. However, an upper solution is still given by $\mathbf{U}_0(z) = (K, K, \dots, K)'$ where $K = \max\{\Phi(z_1,i), \Phi(z_2,i), i = 1,\dots,m\}$. Hence we proceed with the construction of an upper solution sequence to approximate the value function. A finite difference method is used. We choose the following parameters for the numerical implementation: $m = 2$, $a(1) = 2$, $a(2) = 3$, $\sigma(1) = 0.6$, $\sigma(2) = 0.8$, $\rho = 0.1$, $q_{12} = 2$, $q_{21} = 3$, $z_1 = 1$, $z_2 = 2$, and $\Phi(z_1,i) = \Phi(z_2,i) = 2$, $i = 1, 2$. Figure 5 plots several upper approximation solutions of the value functions $v(z,1)$ and $v(z,2)$, beginning at the initial constant upper solutions. We see again that the approximation sequences converge to the solution of the boundary value system (5.3).

6. Concluding remarks. We develop a method of upper and lower solutions for a general regime-switching model in this paper. The method is used to prove the existence of a unique C^2 solution of the associated system of ordinary differential equations with two-point boundary conditions. It is also used to numerically compute approximation solutions. An outstanding feature of the proposed numerical method is that the true solution is bracketed by the upper and lower approximation solutions. The convergence of the approximation sequences is validated both theoretically and numerically. The numerical experiments show that the upper and lower approximations method is much faster than the MC simulations, which are commonly used to determine a confidence interval for the true value. Our method can be applied to a broad class of asset models typically used in financial mathematics and financial engineering. In particular, we apply the method to a regime-switching mean-reverting model and generalize early results.

The method of upper and lower solutions is an important and appealing approach in the study of boundary value problems. While in this paper we focus on systems of ordinary differential equations with boundary conditions, an interesting topic for future research is to develop the upper and lower solutions for systems of partial differential equations with boundary conditions, which will have many important applications, particularly in financial mathematics and financial engineering.

Acknowledgment. We are grateful to the referees for their valuable comments, which helped to improve the exposition of this paper.

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