2015

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International Journal of Difference Equations
ISSN 0973-6069, Volume 10, Number 1, pp. 91–106 (2015)
http://campus.mst.edu/ijde

Multi-Term Linear Fractional Nabla Difference Equations with Constant Coefficients

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Abstract

We shall consider a linear fractional nabla (backward) fractional difference equation of Riemann–Liouville type with constant coefficients. We apply a transform method to construct solutions. Sufficient conditions in terms of the coefficients are given so that the solutions are absolutely convergent. The method is known for two-term fractional difference equations; the method is new for fractional equations with three or more terms. As a corollary, we exhibit new summation representations of a discrete exponential function, a^t, $t = 0, 1, \ldots$.

AMS Subject Classifications: 39A12, 34A25, 26A33.
Keywords: Discrete fractional calculus, transform methods, Riemann–Liouville fractional difference.

1 Introduction

In this article we shall apply the transform method to obtain solutions of a linear fractional backward difference equation of the form

$$\nabla_0^\nu y(t) + A_1 \nabla y(t) + A_2 y(t) = f(t), \quad t = 1, 2 \ldots, \tag{1.1}$$

Received November 24, 2014; Accepted March 11, 2015
Communicated by Agnieszka Malinowska

where $1 < \nu \leq 2$. The fractional difference operator, ∇_0^ν, is of Riemann–Liouville type and is defined below. The solution algorithm is standard in the case $A_1 = 0$; see [2]. For the sake of exposition, we shall illustrate the solution algorithm with $A_1 = 0$ in the paper as well.

The motivation for this work is to continue to develop the theory of linear fractional difference equations as an analogue of the theory of linear difference equations. Progress on what we shall call three term equations, (1.1), is limited. An equation of the form

$$\nabla_0^{2\mu} y(t) + A_1 \nabla_0^\mu y(t) + A_2 y(t) = f(t), \quad t = 1, 2 \ldots,$$

is called a sequential fractional difference equation; important progress has been made in the study of sequential fractional equations (see [15], [14], for fractional differential equations, and see [13], [1], for fractional difference equations). For a more general equation,

$$\nabla_0^{\nu_2} y(t) + A_1 \nabla_0^{\nu_1} y(t) + A_2 y(t) = f(t), \quad t = 1, 2 \ldots,$$

we prefer to assume $0 < \nu_1 \leq 1 < \nu_2 \leq 2$ as the only relation between ν_1 and ν_2. For simplicity in the development that follows, we have taken $\nu_1 = 1$.

Let us recall the notation that will be employed throughout. The nabla operator commonly represents the backward difference operator and in this article

$$\nabla y(t) = y(t) - y(t-1), \quad \nabla^k y(t) = \nabla \nabla^{k-1} y(t), \quad k = 1, 2, \ldots.$$

The operator ∇_a^ν, a Riemann–Liouville fractional difference operator, is defined as follows. If $\mu > 0$, define the μth fractional sum by

$$\nabla_a^{-\mu} y(t) = \sum_{s=a}^{t} \frac{\big(t - \rho(s)\big)^{\overline{\mu-1}}}{\Gamma(\mu)} y(s) \tag{1.2}$$

where $\rho(s) = s - 1$, and the raising factorial power function is defined by

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}.$$

Then if $0 \leq n - 1 < \nu \leq n$, define the νth fractional difference (a Riemann–Liouville fractional difference) by

$$\nabla_a^\nu y(t) = \nabla^n \nabla_a^{\nu - n} y(t)$$

where ∇^n denotes the standard $n-$th order backward difference.

In Section 2, so that the article is self-contained, we shall give the definition of the discrete Laplace transform ($\mathcal{N}$-transform) and provide the basic properties employed in this work. We shall provide some basic algebra properties related to the function $t^{\overline{\alpha}}$. We shall also apply the transform method to find solutions of (1.1) in the case $A_1 = 0$. We obtain explicit sufficient conditions as a function of A_2 for the absolute convergence of these solutions. In Section 3, we shall apply the transform method to (1.1) and obtain solutions; again, we shall obtain explicit sufficient conditions as a function of A_1 and A_2 for convergence of the solutions. We believe these series representations are new and so, in Section 4, in an effort to recognize the series solutions, we apply the algorithm in the case when a solution is 2^t and verify independently that the series represents the known function.

For further reading in this area, we refer the reader to the books on fractional differential equations [11, 16, 17] and the articles on fractional difference equations [4–6, 8].

2 Algebraic Properties and the Discrete Transform

For the sake of exposition, we first introduce algebra and calculus properties related to the raising factorial power function.

Lemma 2.1. *The following identities are valid:*
(i) $\nabla t^{\overline{\alpha}} = \alpha t^{\overline{\alpha-1}}$;
(ii) $t^{\overline{\alpha}}(t+\alpha)^{\overline{\beta}} = t^{\overline{\alpha+\beta}}$;
(iii) $\nabla_1^{-\nu} t^{\overline{\mu}} = \dfrac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} t^{\overline{\mu+\nu}}$;
(iv) $\nabla_0^{-\nu}(t+1)^{\overline{\mu}} = \dfrac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}(t+1)^{\overline{\mu+\nu}}$;
(v) $\displaystyle\sum_{n=0}^{k} \frac{t^{\overline{n}}}{\Gamma(n+1)} = \frac{(t+1)^{\overline{k}}}{\Gamma(k+1)}$.
(vi) $\nabla_a^{-(\mu)} f(t)|_{t=a} = f(a)$, *if* $0 < \mu$.
(vii) $\nabla_a^{-(\mu)} f(t)|_{t=a+1} = \mu f(a) + f(a+1)$, *if* $0 < \mu$.

Proof. (i) and (ii) are easily observed by applying standard gamma function identities. (iii) is proved in [2] and is referred to as the power rule. (iv) is another form of the power rule and is stated for clarity since it is used in the specific applications in this article. (v) is obtained by induction on k and is in fact one of the identities found in Pascal's triangle since $\dfrac{(t+1)^{\overline{k}}}{\Gamma(k+1)} = \dbinom{t+k}{t}$. Thus, by induction on k,

$$\sum_{n=0}^{k+1} \frac{t^{\overline{n}}}{\Gamma(n+1)} = \binom{t+k}{t} + \frac{t^{\overline{k+1}}}{\Gamma(k+2)} = \binom{t+k}{t} + \binom{t+k}{t-1} = \binom{t+k+1}{t}.$$

(vi) and (vii) are obtained directly from the definition given by (1.2). □

Transform methods are commonly used to solve fractional differential and fractional difference equations. We have need to develop further transform properties to produce the applications given here.

The following definition and the derivation of most of the following properties are found in [2]. Define the discrete Laplace transform ($\mathcal{N}$-transform) by

$$\mathcal{N}_{t_0}\big(f(t)\big)(s) = \sum_{t=t_0}^{\infty}(1-s)^{t-1}f(t). \tag{2.1}$$

Lemma 2.2. *For any* $\nu \in \mathbb{R} \setminus \{..., -2, -1, 0\}$,

(i) $\mathcal{N}_1(t^{\overline{\nu-1}})(s) = \dfrac{\Gamma(\nu)}{s^\nu}$, $|1-s| < 1$.

(ii) $\mathcal{N}_1(t^{\overline{\nu-1}}\alpha^{-t})(s) = \dfrac{\alpha^{\nu-1}\Gamma(\nu)}{(s+\alpha-1)^\nu}$, $|1-s| < \alpha$.

(iii) $\mathcal{N}_1(t^{\overline{\nu}})(s) = \dfrac{\nu}{s}\mathcal{N}_1(t^{\overline{\nu-1}})$.

(iv) $\mathcal{N}_a f(t+1) = (1-s)^{-1}\mathcal{N}_{a+1}f(t)$.

(v) $A\mathcal{N}_1 f(t) = -\dfrac{Af(0)}{1-s} + A\mathcal{N}_0 f(t)$.

(vi) $\mathcal{N}_a\big(\nabla_a^{-\nu}f(t)\big) = s^{-\nu}\mathcal{N}_a\big(f(t)\big)(s)$.

(vii) *If* $0 < \nu \leq 1$,

$$\mathcal{N}_{a+1}\big(\nabla_a^\nu f(t)\big)(s) = s^\nu\mathcal{N}_a\big(f(t)\big)(s) - (1-s)^{a-1}f(a).$$

$(viii)$

$$\mathcal{N}_{a+2}\big(\nabla^2 f(t)\big)(s) = s^2\mathcal{N}_a\big(f(t)\big)(s) - s(1-s)^{a-1}f(a) - (1-s)^a\nabla f(a+1).$$

(ix) *If* $1 < \nu \leq 2$,

$$\begin{aligned}\mathcal{N}_{a+2}\big(\nabla_a^\nu f(t)\big)(s) &= s^\nu\mathcal{N}_a\big(f(t)\big)(s) - s(1-s)^{a-1}f(a)\\ &\quad - (1-s)^a\nabla_a^{\nu-1}f(a+1)\\ &= s^\nu\mathcal{N}_a\big(f(t)\big)(s) - s(1-s)^{a-1}f(a)\\ &\quad - (1-s)^a\big(f(a+1) - (\nu-1)f(a)\big).\end{aligned}$$

(x) $\mathcal{N}_0\Big(\big(\dfrac{1}{1-a^2}\big)^{(t+1)}\Big) = \dfrac{1}{(1-s)(s-a^2)}$.

We point out that one can obtain $(viii)$ using repeated applications of (vii) with $\nu = 1$. (ix) is then obtained by applying $(viii)$ with f replaced by $\nabla_a^{-(2-\nu)}f$ and employing (vi) and (vii) of Lemma 2.1 to compute the initial values at $t = a$ and $t = a + 1$.

We shall also tabulate some inverse transforms which are obtained from Lemma 2.2 and geometric series expansions.

Lemma 2.3. *For any* $\nu \in \mathbb{R} \setminus \{..., -2, -1, 0\}$,

$$(i) \quad \frac{1}{s^\nu + A} = \sum_{n=0}^{\infty}(-1)^n A^n s^{-\nu(n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma\big(\nu(n+1)\big)} \mathcal{N}_1(t^{\overline{\nu(n+1)-1}}).$$

$$(ii) \quad \frac{1}{s(s^\nu + A)} = \sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma\big(\nu(n+1)+1\big)} \mathcal{N}_1(t^{\overline{\nu(n+1)}}).$$

$$(iii) \quad \frac{1}{(1-s)(s^\nu + A)} = \sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma\big(\nu(n+1)\big)} \mathcal{N}_0\Big((t+1)^{\overline{\nu(n+1)-1}}\Big).$$

Proof. Statements (ii) and (iii) follow from Lemma 2.2 (iii) and (iv) respectively; in particular,

$$\sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma\big(\nu(n+1)\big)} \frac{1}{s} \mathcal{N}_1(t^{\overline{\nu(n+1)-1}}) = \sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma\big(\nu(n+1)+1\big)} \mathcal{N}_1(t^{\overline{\nu(n+1)}})$$

and

$$\sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma\big(\nu(n+1)\big)} (1-s)^{-1} \mathcal{N}_1(t^{\overline{\nu(n+1)-1}}) = \sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma\big(\nu(n+1)\big)} \mathcal{N}_0\Big((t+1)^{\overline{\nu(n+1)-1}}\Big).$$

□

An initial value problem for a two term linear equation of the form

$$\nabla_0^\nu y(t) + Ay(t) = 0, \quad y(0) = a, \quad 0 < \nu \leq 1, \text{ for } \quad t = 1, 2, ...,$$

was solved in [3]. In that article, the authors expressed solutions as functions of Mittag–Leffler type functions and obtained series solutions in the case, $|A| < 1$. The method is readily extended to an initial value problem for a two-term, linear, nonhomogeneous

$$\nabla_0^\nu y(t) + Ay(t) = k, \quad y(0) = a, \quad 0 < \nu \leq 1, \quad t = 1, 2, ... \tag{2.2}$$

and is illustrated here.

Apply the $\mathcal{N}_1$-transform to each side of the equation (2.2) to obtain

$$\mathcal{N}_1\big(\nabla_0^\nu y(t)\big) + A\mathcal{N}_1 y(t) = \mathcal{N}_1 k.$$

Then

$$s^\nu \mathcal{N}_0 y(t) - (1-s)^{-1} a + A\mathcal{N}_1 y(t) = \frac{k}{s}$$

or

$$(s^\nu + A)\mathcal{N}_0 y(t) - \frac{1+A}{(1-s)} a = \frac{k}{s}.$$

Thus,

$$\mathcal{N}_0 y(t) = \frac{k}{s(s^\nu + A)} + \frac{(1+A)a}{(1-s)(s^\nu + A)}.$$

Use Lemma 2.3 to calculate $y(t)$.

$$\begin{aligned}
\mathcal{N}_0 y(t) &= \frac{k}{s(s^\nu + A)} + \frac{(1+A)a}{(1-s)(s^\nu + A)} \\
&= k \sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma\big(\nu(n+1)+1\big)} \mathcal{N}_1(t^{\overline{\nu(n+1)}}) \\
&+ (1+A)a \sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma\big(\nu(n+1)\big)} \mathcal{N}_0\big((t+1)^{\overline{\nu(n+1)-1}}\big) \\
&= k \sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma\big(\nu(n+1)+1\big)} \mathcal{N}_0(t^{\overline{\nu(n+1)}}) \\
&+ (1+A)a \sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma\big(\nu(n+1)\big)} \mathcal{N}_0\big((t+1)^{\overline{\nu(n+1)-1}}\big).
\end{aligned}$$

Note, we claim $\mathcal{N}_1(t^{\overline{\nu(n+1)}}) = \mathcal{N}_0(t^{\overline{\nu(n+1)}})$. We have employed Lemma 2.2 (v) with $f(t) = t^{\overline{\nu(n+1)}}$ and with the convention, $\dfrac{1}{\Gamma(0)} = 0$. Thus,

$$y(t) = k \sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma\big(\nu(n+1)+1\big)} t^{\overline{\nu(n+1)}} + (1+A)a \sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma\big(\nu(n+1)\big)} (t+1)^{\overline{\nu(n+1)-1}} \tag{2.3}$$

provides a solution of (2.2).

As noted in a number of references, [7] or [12], the asymptotic property

$$\lim_{k\to\infty} \frac{\Gamma(k+\alpha)}{k^\alpha \Gamma(k)} = 1, \quad \alpha \in \mathbb{R}, \tag{2.4}$$

and the ratio test imply that y, given by (2.3), converges absolutely if $|A| < 1$. In order that the paper is self-contained, we outline how (2.4) is employed with the ratio test. Consider the term

$$\sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma\big(\nu(n+1)+1\big)} t^{\overline{\nu(n+1)}}$$

and consider the limit of the ratio

$$\lim_{n\to\infty} \frac{t^{\overline{\nu(n+2)}}}{\Gamma\big(\nu(n+2)+1\big)} \Big/ \frac{t^{\overline{\nu(n+1)}}}{\Gamma\big(\nu(n+1)+1\big)}.$$

Write

$$\begin{aligned}
&\frac{t^{\overline{\nu(n+2)}}}{\Gamma\big(\nu(n+2)+1\big)}\frac{\Gamma\big(\nu(n+1)+1\big)}{t^{\overline{\nu(n+1)}}}\\
=&\frac{\Gamma\big(t+\nu(n+2)\big)}{\Gamma\big(\nu(n+2)+1\big)}\left(\frac{(\nu(n+1)+1)^{\nu}(t+\nu(n+1))^{\nu}}{(\nu(n+1)+1)^{\nu}(t+\nu(n+1))^{\nu}}\right)\frac{\Gamma\big(\nu(n+1)+1\big)}{\Gamma\big(t+\nu(n+1)\big)}\\
=&\frac{\Gamma\big(t+\nu(n+2)\big)}{(\nu(n+1)+1)^{\nu}\Gamma\big(t+\nu(n+1)\big)}\frac{(t+\nu(n+1))^{\nu}}{(\nu(n+1)+1)^{\nu}}\frac{(\nu(n+1)+1)^{\nu}\Gamma\big(\nu(n+1)+1\big)}{\Gamma\big(\nu(n+2)+1\big)}.
\end{aligned}$$

Apply (2.4) to

$$\frac{\Gamma\big(t+\nu(n+2)\big)}{(\nu(n+1)+1)^{\nu}\Gamma\big(t+\nu(n+1)\big)}$$

and

$$\frac{(\nu(n+1)+1)^{\nu}\Gamma\big(\nu(n+1)+1\big)}{\Gamma\big(\nu(n+2)+1\big)}$$

with

$$k=t+\nu(n+1),\quad \alpha=\nu,\quad \text{and}\quad k=\nu(n+1)+1,\quad \alpha=\nu,$$

respectively. Then

$$\lim_{n\to\infty}\frac{t^{\overline{\nu(n+2)}}}{\Gamma\big(\nu(n+2)+1\big)}\frac{\Gamma\big(\nu(n+1)+1\big)}{t^{\overline{\nu(n+1)}}}=1.$$

So, one can apply the ratio test and the convergence reduces to the condition $|A|<1$.

The method is valid regardless of the order of ν. For example, consider an initial value problem

$$\nabla_0^{\nu}y(t)+Ay(t)=k,\quad y(0)=a_0,\quad y(1)=a_1\quad 1<\nu\leq 2,\quad t=2,3,.... \tag{2.5}$$

Thus,

$$\begin{aligned}
&\mathcal{N}_2\big(\nabla_0^{\nu}y(t)\big)+A\mathcal{N}_2y(t)\\
&\qquad=\mathcal{N}_2\big(\nabla_0^{\nu}y(t)\big)+A\mathcal{N}_2y(t)+A\big(y(1)+\frac{y(0)}{1-s}\big)-A\big(a_1-\frac{a_0}{1-s}\big)\\
&\qquad=\mathcal{N}_2\big(\nabla_0^{\nu}y(t)\big)+A\mathcal{N}_0y(t)-\big(Aa_1-A\frac{a_0}{1-s}\big)=\mathcal{N}_2k=k\Big(\frac{1-s}{s}\Big).
\end{aligned}$$

Apply Condition (ix) of Lemma 2.2 to obtain

$$(s^{\nu}+A)\mathcal{N}_0y(t)-\frac{a_0s}{1-s}-\big(a_1-(\nu-1)a_0\big)-Aa_1-\frac{A}{1-s}a_0=k\Big(\frac{1-s}{s}\Big).$$

Apply Lemma 2.3, and

$$\begin{aligned}\mathcal{N}_0 y(t) &= k\sum_{n=0}^{\infty}\frac{(-1)^n A^n}{\Gamma\big(\nu(n+1)+1\big)}\mathcal{N}_1\big(t^{\overline{\nu(n+1)}}\big)\\
&\quad - k\sum_{n=0}^{\infty}\frac{(-1)^n A^n}{\Gamma\big(\nu(n+1)\big)}\mathcal{N}_1\big(t^{\overline{\nu(n+1)-1}}\big)\\
&\quad + \frac{a_0}{1-s}\sum_{n=0}^{\infty}\frac{(-1)^n A^n}{\Gamma\big(\nu(n+1)-1\big)}\mathcal{N}_1\big(t^{\overline{\nu(n+1)-2}}\big)\\
&\quad + \frac{Aa_0}{1-s}\sum_{n=0}^{\infty}\frac{(-1)^n A^n}{\Gamma\big(\nu(n+1)\big)}\mathcal{N}_1\big(t^{\overline{\nu(n+1)-1}}\big)\\
&\quad + \Big((1+A)a_1-(\nu-1)a_0\Big)\sum_{n=0}^{\infty}\frac{(-1)^n A^n}{\Gamma\big(\nu(n+1)\big)}\mathcal{N}_1\big(t^{\overline{\nu(n+1)-1}}\big).\end{aligned}$$

Apply Lemma 2.2 (v) and the convention, $\frac{1}{\Gamma(0)}=0$ in the first, second and last terms, combine the second and last terms, and in the third and fourth terms, apply Lemma 2.2 (iv) to obtain

$$\begin{aligned}\mathcal{N}_0 y(t) &= k\sum_{n=0}^{\infty}\frac{(-1)^n A^n}{\Gamma\big(\nu(n+1)+1\big)}\mathcal{N}_0\big(t^{\overline{\nu(n+1)}}\big)\\
&\quad + a_0\sum_{n=0}^{\infty}\frac{(-1)^n A^n}{\Gamma\big(\nu(n+1)-1\big)}\mathcal{N}_0\big((t+1)^{\overline{\nu(n+1)-2}}\big)\\
&\quad + Aa_0\sum_{n=0}^{\infty}\frac{(-1)^n A^n}{\Gamma\big(\nu(n+1)\big)}\mathcal{N}_0\big((t+1)^{\overline{\nu(n+1)-1}}\big)\\
&\quad + \Big((1+A)a_1-(\nu-1)a_0-k\Big)\sum_{n=0}^{\infty}\frac{(-1)^n A^n}{\Gamma\big(\nu(n+1)\big)}\mathcal{N}_0(t^{\overline{\nu(n+1)-1}}).\end{aligned}$$

Thus,

$$\begin{aligned}y(t) &= k\sum_{n=0}^{\infty}\frac{(-1)^n A^n}{\Gamma\big(\nu(n+1)+1\big)}t^{\overline{\nu(n+1)}}\\
&\quad + a_0\sum_{n=0}^{\infty}\frac{(-1)^n A^n}{\Gamma\big(\nu(n+1)-1\big)}(t+1)^{\overline{\nu(n+1)-2}} + Aa_0\sum_{n=0}^{\infty}\frac{(-1)^n A^n}{\Gamma\big(\nu(n+1)\big)}(t+1)^{\overline{\nu(n+1)-1}}\\
&\quad + \Big((1+A)a_1-(\nu-1)a_0-k\Big)\sum_{n=0}^{\infty}\frac{(-1)^n A^n}{\Gamma\big(\nu(n+1)\big)}t^{\overline{\nu(n+1)-1}}\end{aligned}$$

provides a solution of (2.5). Again, (2.4) and the ratio test imply that y converges absolutely if $|A|<1$, for $t=0,1,\ldots$.

3 A Three Term Equation

In this section, we begin by describing an algorithm to construct a solution of an initial value problem for a three term linear fractional difference equation of the form

$$\nabla_0^\nu y(t) + A_1\nabla y(t) + A_2 y(t) = 0, \quad y(0) = a_0, y(1) = a_1, \text{ for } \quad t = 1, 2, ..., \tag{3.1}$$

where $1 < \nu \leq 2$. Apply the operator $\mathcal{N}_2$ to (3.1) to obtain

$$\mathcal{N}_2\big(\nabla_0^\nu y(t)\big) + A_1\mathcal{N}_2\big(\nabla y(t)\big) + A_2\mathcal{N}_2 y(t) = 0. \tag{3.2}$$

Apply Lemma 2.2 (ix) and note

$$\mathcal{N}_2\big(\nabla_0^\nu y(t)\big) = s^\nu\mathcal{N}_0\big(f(t)\big)(s) - s(1-s)^{-1}a_0 - (a_1 - (\nu-1)a_0); \tag{3.3}$$

apply Lemma 2.2 (vii) and note

$$\begin{aligned} \mathcal{N}_2\big(\nabla y(t)\big) &= \mathcal{N}_2\big(\nabla y(t)\big) + \nabla y(1) - \nabla y(1) = \mathcal{N}_1\big(\nabla y(t)\big) - \nabla y(1) \\ &= s\mathcal{N}_1\big(y(t)\big) - (1-s)^{-1}a_0 - (a_1 - a_0) \\ &= s\mathcal{N}_0\big(y(t)\big) - \frac{s}{1-s}a_0 - a_1. \end{aligned} \tag{3.4}$$

Similarly,

$$\mathcal{N}_2 y(t) = \mathcal{N}_2 y(t) + a_1 + (1-s)^{-1}a_0 - \big(a_1 + (1-s)^{-1}a_0\big);$$

in particular,

$$\mathcal{N}_2 y(t) = \mathcal{N}_0 y(t) - a_1 - (1-s)^{-1}a_0. \tag{3.5}$$

Substitute (3.3), (3.4), and (3.5) into (3.2) and obtain

$$\mathcal{N}_0 y(t) = \frac{(s + A_1 s + A_2)a_0}{(1-s)(s^\nu + A_1 s + A_2)} + \frac{(1 + A_1 + A_2)a_1 + (1-\nu)a_0}{(s^\nu + A_1 s + A_2)}. \tag{3.6}$$

Write

$$\begin{aligned} \frac{1}{s^\nu + A_1 s + A_2} &= \frac{1}{s^\nu(1 + A_1 s^{1-\nu})}\left(\frac{1}{1 + \frac{A_2}{s^\nu(1+A_1 s^{1-\nu})}}\right) \\ &= \sum_{n=0}^{\infty}(-1)^n s^{-\nu(n+1)}A_2^n\left(\frac{1}{1 + A_1 s^{1-\nu}}\right)^{n+1}. \end{aligned}$$

Note that

$$\left(\frac{1}{1 + A_1 s^{1-\nu}}\right)^{n+1} = \sum_{m=n}^{\infty}(-1)^{m+n}\binom{m}{n}(A_1 s^{1-\nu})^{m-n}$$

and so,

$$\frac{1}{s^\nu + A_1 s + A_2} = \sum_{n=0}^{\infty}\sum_{m=n}^{\infty}(-1)^m\binom{m}{n}A_1^{m-n}A_2^n s^{(1-\nu)m-n-\nu} \qquad (3.7)$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m\binom{m}{n}A_1^{m-n}A_2^n s^{(1-\nu)m-n-\nu}.$$

Since

$$s^{(1-\nu)m-n-\nu} = \mathcal{N}_1\Big(\frac{t^{\overline{(\nu-1)m+n+(\nu-1)}}}{\Gamma\big((\nu-1)m+n+\nu\big)}\Big) = \mathcal{N}_0\Big(\frac{t^{\overline{(\nu-1)m+n+(\nu-1)}}}{\Gamma\big((\nu-1)m+n+\nu\big)}\Big),$$

employ Lemma 2.2 (v) to reexpress $\mathcal{N}_1$ as $\mathcal{N}_0$, and we have

$$\frac{1}{s^\nu + A_1 s + A_2} = \mathcal{N}_1\sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m A_1^{m-n}A_2^n\binom{m}{n}\frac{t^{\overline{(\nu-1)m+n+(\nu-1)}}}{\Gamma\big((\nu-1)m+n+\nu\big)} \qquad (3.8)$$
$$= \mathcal{N}_0\sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m A_1^{m-n}A_2^n\binom{m}{n}\frac{t^{\overline{(\nu-1)m+n+(\nu-1)}}}{\Gamma\big((\nu-1)m+n+\nu\big)};$$

it follows from Lemma 2.2 (iv) that

$$\frac{(1-s)^{-1}}{s^\nu + A_1 s + A_2} = \mathcal{N}_0\sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m A_1^{m-n}A_2^n\binom{m}{n}\frac{(t+1)^{\overline{(\nu-1)m+n+(\nu-1)}}}{\Gamma\big((\nu-1)m+n+\nu\big)}. \qquad (3.9)$$

Moreover, from (3.7),

$$\frac{s}{s^\nu + A_1 s + A_2} = \sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m\binom{m}{n}A_1^{m-n}A_2^n s^{(1-\nu)m-n-\nu+1}$$

and so,

$$\frac{s(1-s)^{-1}}{s^\nu + A_1 s + A_2} = \mathcal{N}_0\sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m A_1^{m-n}A_2^n\binom{m}{n}\frac{(t+1)^{\overline{(\nu-1)m+n+(\nu-2)}}}{\Gamma\big((\nu-1)m+n+(\nu-1)\big)}. \qquad (3.10)$$

Substitute (3.8), (3.9), (3.10) directly into (3.6), to obtain

$$y(t) = A_2 a_0\sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m A_1^{m-n}A_2^n\binom{m}{n}\frac{(t+1)^{\overline{(\nu-1)m+n+(\nu-1)}}}{\Gamma\big((\nu-1)m+n+\nu\big)} \qquad (3.11)$$
$$+ (1+A_1)a_0\sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m A_1^{m-n}A_2^n\binom{m}{n}\frac{(t+1)^{\overline{(\nu-1)m+n+(\nu-2)}}}{\Gamma\big((\nu-1)m+n+(\nu-1)\big)}$$
$$+ K\sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m A_1^{m-n}A_2^n\binom{m}{n}\frac{t^{\overline{(\nu-1)m+n+(\nu-1)}}}{\Gamma\big((\nu-1)m+n+\nu\big)},$$

where $K = \big((1 + A_1 + A_2)a_1 + (1 - \nu)a_0\big)$.

To simplify this representation, note that

$$\frac{(t+1)^{\overline{(\nu-1)m+n+(\nu-1)}}}{\Gamma\big((\nu-1)m+n+\nu\big)} = \frac{t^{\overline{(\nu-1)m+n+(\nu-1)}}}{\Gamma\big((\nu-1)m+n+\nu\big)} + \frac{(t+1)^{\overline{(\nu-1)m+n+(\nu-2)}}}{\Gamma\big((\nu-1)m+n+\nu-1\big)}$$

since

$$\begin{aligned}\frac{(t+1)^{\overline{(\nu-1)m+n+(\nu-1)}}}{\Gamma\big((\nu-1)m+n+\nu\big)} &= \frac{\Gamma\big(t+1+(\nu-1)m+n+(\nu-1)\big)}{\Gamma(t+1)\Gamma\big((\nu-1)m+n+\nu\big)} \\ &= t\frac{\Gamma\big(t+(\nu-1)m+n+(\nu-1)\big)}{\Gamma(t+1)\Gamma\big((\nu-1)m+n+\nu\big)} \\ &\quad + \big((\nu-1)m+n+(\nu-1)\big)\frac{\Gamma\big(t+(\nu-1)m+n+(\nu-1)\big)}{\Gamma(t+1)\Gamma\big((\nu-1)m+n+\nu\big)}.\end{aligned}$$

Thus, (3.11) can be expressed as

$$\begin{aligned} y(t) = K_1 \sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m A_1^{m-n}A_2^n\binom{m}{n}\frac{(t+1)^{\overline{(\nu-1)m+n+(\nu-2)}}}{\Gamma\big((\nu-1)m+n+(\nu-1)\big)} \\ + K_2 \sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m A_1^{m-n}A_2^n\binom{m}{n}\frac{(t)^{\overline{(\nu-1)m+n+(\nu-1)}}}{\Gamma\big((\nu-1)m+n+\nu\big)},\end{aligned} \tag{3.12}$$

where

$$K_1 = a_0(1+A_1+A_2), \quad K_2 = K + a_0A_2 = (1+A_1+A_2)a_1 + (1-\nu+A_2)a_0. \tag{3.13}$$

Note that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m A_1^{m-n}A_2^n\binom{m}{n}\frac{(t+1)^{\overline{(\nu-1)m+n+(\nu-2)}}}{\Gamma\big((\nu-1)m+n+(\nu-1)\big)} \tag{3.14}$$

and

$$\sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m A_1^{m-n}A_2^n\binom{m}{n}\frac{t^{\overline{(\nu-1)m+n+(\nu-1)}}}{\Gamma\big((\nu-1)m+n+\nu\big)} \tag{3.15}$$

are two linear independent solutions of (3.1). We provide the details to obtain conditions for absolute convergence in (3.15) for fixed t; the details for (3.14) are analogous. First note that for each $t \geq 1$,

$$\frac{t^{\overline{(\nu-1)m+n+(\nu-1)}}}{\Gamma\big((\nu-1)m+n+\nu\big)}$$

is an increasing function in n. To see this, consider the ratio

$$\frac{t^{\overline{(\nu-1)m+n+1+(\nu-1)}}}{\Gamma\big((\nu-1)m+n+1+\nu\big)}\Big/\frac{(t)^{\overline{(\nu-1)m+n+(\nu-1)}}}{\Gamma\big((\nu-1)m+n+\nu\big)}.$$

Then

$$\frac{\Gamma\big(t+(\nu-1)m+n+1+(\nu-1)\big)}{\Gamma(t)\Gamma\big((\nu-1)m+n+1+\nu\big)}\frac{\Gamma(t)\Gamma\big((\nu-1)m+n+\nu\big)}{\Gamma(t+(\nu-1)m+n+(\nu-1)\big)}$$

$$=\frac{t+(\nu-1)m+n+(\nu-1)}{(\nu-1)m+n+\nu}\geq 1,$$

and the inequality is strict if $t>1$.

Thus, compare (3.15) to

$$\sum_{m=0}^{\infty}\frac{t^{\overline{(\nu m+(\nu-1))}}}{\Gamma(\nu m+\nu)}\Big|\sum_{n=0}^{m}A_1^{m-n}A_2^{n}\binom{m}{n}\Big|=\sum_{m=0}^{\infty}\frac{(t)^{\overline{(\nu m+(\nu-1))}}}{\Gamma(\nu m+\nu)}|A_1+A_2|^m$$

and apply (2.4) and the ratio test. Thus, each of (3.14) and (3.15) converge absolutely if $|A_1+A_2|<1$.

4 Representations of Known Functions

We illustrate this method with an initial value problem for a classical second order finite difference equation. The unique solution of the initial value problem is $y(t)=2^t$; thus we obtain a series representation of 2^t as a linear combination of the forms (3.14) and (3.15). We believe this representation is new.

Consider the initial value problem

$$10\nabla^2 y(t)-\nabla y(t)-2y(t)=0,\quad t=2,3,\ldots,\quad y(0)=1,\quad y(1)=2. \tag{4.1}$$

Then, apply (3.6) with

$$A_1=\frac{-1}{10},\quad A_2=\frac{-1}{5},\quad a_0=1,\quad a_1=2,\quad \nu=2,$$

to obtain

$$\mathcal{N}_0 y(t)=\frac{\frac{9}{10}s-\frac{1}{5}}{(1-s)(s^2-\frac{1}{10}s-\frac{5}{10})}+\frac{\frac{4}{10}}{(s^2-\frac{1}{10}s-\frac{1}{5})};$$

from (3.13),

$$K_1=\frac{9}{10}-\frac{1}{5},\quad K_2=\frac{4}{10}-\frac{1}{5}$$

and

$$y(t) = \frac{7}{10}\sum_{m=0}^{\infty}\sum_{n=0}^{m}(\frac{1}{5})^n(\frac{1}{10})^{m-n}\binom{m}{n}\frac{(t+1)^{\overline{m+n}}}{\Gamma(m+n+1)} \tag{4.2}$$
$$+\frac{1}{5}\sum_{m=0}^{\infty}\sum_{n=0}^{m}(\frac{1}{5})^n(\frac{1}{10})^{m-n}\binom{m}{n}\frac{(t)^{\overline{m+n+1}}}{\Gamma(m+n+2)}.$$

The unique solution of (4.1) is 2^t and the series given in (4.2) converges absolutely for each $t = 0, 1, \ldots$. Thus, (4.2) provides a series representation of 2^t, $t = 0, 1, \ldots$. However, we claim the representation is new and so we further validate the representation. Write (4.2) as

$$y(t) = \frac{7}{10}A(t) + \frac{1}{5}B(t)$$

where

$$A(t) = \sum_{m=0}^{\infty}\sum_{n=0}^{m}(\frac{1}{5})^n(\frac{1}{10})^{m-n}\binom{m}{n}\frac{(t+1)^{\overline{m+n}}}{\Gamma(m+n+1)}$$
$$B(t) = \sum_{m=0}^{\infty}\sum_{n=0}^{m}(\frac{1}{5})^n(\frac{1}{10})^{m-n}\binom{m}{n}\frac{t^{\overline{m+n+1}}}{\Gamma(m+n+2)}.$$

To show $y(t+1) = 2y(t)$, or

$$\frac{7}{10}A(t+1) + \frac{1}{5}B(t+1) = 2\Big(\frac{7}{10}A(t) + \frac{1}{5}B(t)\Big)$$

it is sufficient to show

$$B(t+1) = A(t) + B(t) \text{ and } \frac{7}{10}A(t+1) = \big(\frac{6}{5}A(t) + \frac{1}{5}B(t)\big).$$

Treating $B(t+1)$ is straightforward since,

$$\begin{aligned}
B(t+1) &= \sum_{m=0}^{\infty}\sum_{n=0}^{m}(\frac{1}{5})^n(\frac{1}{10})^{m-n}\binom{m}{n}\frac{(t+1)^{\overline{m+n+1}}}{\Gamma(m+n+2)}\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{m}(\frac{1}{5})^n(\frac{1}{10})^{m-n}\binom{m}{n}\frac{\Gamma(t+1+m+n+1)}{\Gamma(t+1)\Gamma(m+n+2)}\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{m}(\frac{1}{5})^n(\frac{1}{10})^{m-n}\binom{m}{n}(t+m+n+1)\frac{\Gamma(t+m+n+1)}{\Gamma(t+1)\Gamma(m+n+2)}\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{m}(\frac{1}{5})^n(\frac{1}{10})^{m-n}\binom{m}{n}t\frac{\Gamma(t+m+n+1)}{\Gamma(t+1)\Gamma(m+n+2)}\\
&+ \sum_{m=0}^{\infty}\sum_{n=0}^{m}(\frac{1}{5})^n(\frac{1}{10})^{m-n}\binom{m}{n}(m+n+1)\frac{\Gamma(t+m+n+1)}{\Gamma(t+1)\Gamma(m+n+2)}\\
&= B(t) + A(t).
\end{aligned}$$

We have yet to obtain a straightforward approach to treat $A(t+1)$. We begin by showing directly that $B(t)$ satisfies

$$10\nabla^2 y(t) - \nabla y(t) - 2y(t) = 0, \quad t = 2, 3, \ldots.$$

Apply the power rule and

$$\begin{aligned}
B(t) &= \sum_{m=0}^{\infty}\sum_{n=0}^{m}(\frac{1}{5})^n(\frac{1}{10})^{m-n}\binom{m}{n}\frac{t^{\overline{m+n+1}}}{\Gamma(m+n+2)},\\
\nabla B(t) &= \sum_{m=0}^{\infty}\sum_{n=0}^{m}(\frac{1}{5})^n(\frac{1}{10})^{m-n}\binom{m}{n}\frac{t^{\overline{m+n}}}{\Gamma(m+n+1)},\\
\nabla^2 B(t) &= \sum_{m=1}^{\infty}\sum_{n=0}^{m}(\frac{1}{5})^n(\frac{1}{10})^{m-n}\binom{m}{n}\frac{t^{\overline{m+n-1}}}{\Gamma(m+n)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\nabla^2 B(t) =& \sum_{m=0}^{\infty}\sum_{n=0}^{m+1}(\frac{1}{5})^n(\frac{1}{10})^{m+1-n}\binom{m+1}{n}\frac{t^{\overline{m+n}}}{\Gamma(m+n+1)}\\
=& \sum_{m=0}^{\infty}\Big(\frac{1}{10}\sum_{n=0}^{m}(\frac{1}{5})^n(\frac{1}{10})^{m-n}\Big(\binom{m}{n}+\binom{m}{n-1}\Big)\frac{t^{\overline{m+n}}}{\Gamma(m+n+1)}\\
&+ (\frac{1}{5})^{m+1}\frac{t^{\overline{2m+1}}}{\Gamma(2m+2)}\Big)\\
=& \sum_{m=0}^{\infty}\frac{1}{10}\sum_{n=0}^{m}(\frac{1}{5})^n(\frac{1}{10})^{m-n}\binom{m}{n}\frac{t^{\overline{m+n}}}{\Gamma(m+n+1)}\\
&+ \sum_{m=0}^{\infty}\sum_{n=0}^{m+1}(\frac{1}{5})^n(\frac{1}{10})^{m+1-n}\binom{m}{n-1}\frac{t^{\overline{m+n}}}{\Gamma(m+n+1)}\\
=& \frac{1}{10}\nabla B(t) + \frac{1}{5}\sum_{m=0}^{\infty}\sum_{n=0}^{m}(\frac{1}{5})^n(\frac{1}{10})^{m-n}\binom{m}{n}\frac{t^{\overline{m+n+1}}}{\Gamma(m+n+2)}\\
=& \frac{1}{10}\nabla B(t) + \frac{1}{5}B(t).
\end{aligned}$$

A similar calculation shows that $A(t)$ satisfies

$$10\nabla^2 y(t) - \nabla y(t) - 2y(t) = 0, \quad t = 2, 3, \ldots.$$

We close by arguing that

$$\frac{7}{10}A(t+1) = \frac{6}{5}A(t) + \frac{1}{5}B(t).$$

Simplify

$$\begin{aligned}0 &= 10\nabla^2 A(t+1) - \nabla A(t+1) - 2A(t+1)\\ &= 10\big(A(t+1) - 2A(t) + A(t-1)\big) - \big(A(t+1) - A(t)\big) - 2A(t+1)\end{aligned}$$

to obtain

$$\frac{7}{10}A(t+1) = \frac{19}{10}A(t) - A(t-1) = \frac{6}{5}A(t) + \big(\frac{7}{10}A(t) - A(t-1)\big).$$

Now it is sufficient to show

$$\frac{7}{10}A(t) - A(t-1) = \frac{1}{5}B(t).$$

Employ $A(t) = B(t+1) - B(t)$ and $A(t-1) = B(t) - B(t-1)$ to obtain

$$\begin{aligned}\frac{7}{10}A(t) - A(t-1) &= \frac{7}{10}\big(B(t+1) - B(t)\big) - \big(B(t) - B(t-1)\big)\\ &= \frac{7}{10}B(t+1) - \frac{17}{10}B(t) + B(t-1)\\ &= \Big(\frac{7}{10}B(t+1) - \frac{19}{10}B(t) + B(t-1)\Big) + \frac{2}{10}B(t) = \frac{1}{5}B(t).\end{aligned}$$

Remark 4.1. A duality between delta and nabla finite differences has been established for both classical finite differences and finite differences on time scales (see [10]). To our knowledge, the calculations performed here for the multi-term nabla fractional equations have not been carried out for analogous multi-term delta fractional equations. Even in the two-term delta fractional case, one considers two independent time domains (see [7], for example), one for the function space $y(t)$ and one for the function space $\Delta_0^\nu y(t)$; for the multi-term delta fractional equation it is feasible that one introduces multiple time domains. The nabla fractional equation avoids the issue of multiple time domains; we have not yet considered the multi-term delta fractional equation.

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