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RESEARCH PAPER

CONJUGATE POINTS FOR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract

Let $b > 0$. Let $1 < \alpha \leq 2$. The theory of u_0-positive operators with respect to a cone in a Banach space is applied to study the conjugate boundary value problem for Riemann-Liouville fractional linear differential equations $D^{\alpha}_{0+}u + \lambda p(t)u = 0$, $0 < t < b$, satisfying the conjugate boundary conditions $u(0) = u(b) = 0$. The first extremal point, or conjugate point, of the conjugate boundary value problem is defined and criteria are established to characterize the conjugate point. As an application, a fixed point theorem is applied to give sufficient conditions for existence of a solution of a related boundary value problem for a nonlinear fractional differential equation.

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Key Words and Phrases: fractional boundary value problem, u_0-positive operator, conjugate point, fractional differential inequalities

1. Introduction

Let $1 < \alpha \leq 2$. For each $0 < b$, we shall consider the family of boundary value problems (BVPs) of the form

$$D^{\alpha}_{0+}u + p(t)u = 0, \quad 0 < t < b, \tag{1.1}$$

$$u(0) = u(b) = 0, \tag{1.2}$$

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where D^{α}_{0+} is the standard Riemann-Liouville derivative, defined in Section 2, and $p(t)$ is a continuous nonnegative function on $[0,\infty)$ which does not vanish identically on any nondegenerate compact subinterval of $[0,\infty)$.

The purpose of this article is to define and characterize a unique $b_0 \in (0,\infty)$ such that the BVP, (1.1), (1.2), has a nontrivial solution that exists in a cone, and if $0 < b < b_0$, the BVP, (1.1), (1.2), is uniquely solvable (with unique solution $u \equiv 0$.) In particular, we shall define the conjugate point of (1.1) corresponding to the Dirichlet or conjugate boundary conditions, (1.2), to be this value b_0. Since b is a variable in this article, we shall refer to the BVP(b), (1.1), (1.2).

Disconjugacy of ordinary differential equations is well-studied and has enjoyed a rich history; we refer the reader to the monograph of Coppel [7] or landmark papers of Hartman [20], Levin [25] or Nehari [28]. Very little progress has been made to develop analogous theories for linear fractional differential equations, although a Fite type result [1] has recently been obtained for an iterated Riemann-Liouville fractional differential operator. The contribution we make here is modest since we consider the analogue of the second order conjugate two point conjugate problem for the ordinary differential equation, and one of the roots is pre-specified at $t = 0$, due to the nature of the definitions in fractional calculus. However, the characterization of the conjugate point, even in this case, is new.

Recently, the authors [14] showed the existence and compared smallest eigenvalues of the boundary value problem, (1.1), (1.2), using the theory of u_0-positive operators. As pointed out in that article, significant progress has been made with the study of nonlinear eigenvalue problems for fractional BVPs through the application of fixed point theory (see, for example, [3, 4, 16, 17, 21]). We are unaware of other advancements for the linear eigenvalue problem for fractional BVPs. In particular, in [14], a Banach space was defined and a cone with nonempty interior was defined so that the theory of u_0-positive operators could be applied. Once these definitions were constructed, a usual technique that has been employed for a wide variety of boundary value problems for various functional equations was used; we refer the reader to the following citations, [6, 8, 11, 12, 15, 19, 26, 27, 31, 32].

For ordinary differential equations, a classical result is that if $p(t) > 0$ and continuous on $(0,\infty)$ and b_0 is the conjugate point of the BVP, $y''(t) + p(t)y(t) = 0$, $y(0) = y(b) = 0$, then there exists a nontrivial solution y of the BVP, $y''(t) + p(t)y(t) = 0$, $y(0) = y(b_0) = 0$, and y does not vanish on $(0, b_0)$. The classical result is obtained by elementary methods, although we refer the reader to [7] or [28]. Schmitt and Smith [30] applied the theory of cones to extend this theory to second order, $m-$dimensional systems of two-point conjugate BVPs. Their method [30] was further extended to higher order scalar equations [18], [13]. We extend these arguments here to the BVP(b_0), (1.1), (1.2).

In Section **2**, we provide definitions and theorems so that the article is self-contained. In Section **3**, we introduce a family of Green's functions and observe appropriate sign properties. Some of the properties are obvious and merely tabulated, some have been essentially obtained in [5], and some are developed here and employed in Section **4**. In Section **4**, appropriate Banach spaces, cones and fixed point operators are defined. With the introduction of the Banach spaces, the contraction mapping principle is employed to obtain a $\delta > 0$, such that (1.1), (1.2) is uniquely solvable if $0 < b < \delta$. We close Section **4** with a statement and proof of the main result of the paper, a characterization of the conjugate point of the BVP(b_0), (1.1), (1.2). Finally, in Section **5**, we apply a fixed point theorem and obtain sufficient conditions for existence of a solution of a boundary value problem for a nonlinear fractional differential equation.

2. Preliminary Definitions and Theorems

Definition **2.1**. Let n denote a positive integer and assume $n-1 < \alpha \le n$. The α-th Riemann-Liouville fractional derivative of the function $u : [0,\infty) \to \mathbb{R}$, denoted $D_{0+}^{\alpha}u$, is defined as

$$D_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1}u(s)ds,$$

provided the right-hand side exists.

For this article, $n = 2$ and $1 < \alpha \le 2$. Recall [10], the $\alpha-$th fractional integral, I_{0+}^{α}, is defined as

$$I_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds,$$

and

$$D_{0+}^{\alpha}I_{0+}^{\alpha}u(t) = u(t), 0 < t, \text{ if } u \in L_1[0,b],$$
$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t) = u(t) + c_1t^{\alpha-2} + c_2t^{\alpha-1}, 0 < t, \text{ if } D_{0+}^{\alpha}u \in L_1[0,b].$$

Note that if $D_{0+}^{\alpha}u \in C[0,b]$ and $u(0) = 0$, then $u \in C[0,1]$.

Definition **2.2**. We shall say that $0 < b_0$ is the first extremal point or conjugate point of the BVP(b), (1.1), (1.2), if

$$b_0 = \inf\{b > 0 : (1.1), (1.2) \text{ has a nontrivial solution}\}.$$

DEFINITION **2.3**. Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$. A closed nonempty subset $\mathcal{P}$ of $\mathcal{B}$ is said to be a cone provided:
(i) $\alpha u + \beta v \in \mathcal{P}$, for all $u, v \in \mathcal{P}$ and all $\alpha, \beta \geq 0$, and
(ii) $u \in \mathcal{P}$ and $-u \in \mathcal{P}$ implies $u = 0$.

DEFINITION **2.4**. A cone $\mathcal{P}$ is solid if the interior, $\mathcal{P}^\circ$, of $\mathcal{P}$, is nonempty. A cone $\mathcal{P}$ is reproducing if $\mathcal{B} = \mathcal{P} - \mathcal{P}$; i.e., given $w \in \mathcal{B}$, there exist $u, v \in \mathcal{P}$ such that $w = u - v$.

REMARK **2.1**. Krasnosel'skii [24] showed that every solid cone is reproducing.

DEFINITION **2.5**. Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{B}$. If $u, v \in \mathcal{B}$, $u \leq v$ with respect to $\mathcal{P}$ if $v - u \in \mathcal{P}$. If both $M, N : \mathcal{B} \to \mathcal{B}$ are bounded linear operators, $M \leq N$ with respect to $\mathcal{P}$ if $Mu \leq Nu$ for all $u \in \mathcal{P}$.

DEFINITION **2.6**. A bounded linear operator $M : \mathcal{B} \to \mathcal{B}$ is u_0-positive with respect to $\mathcal{P}$ if there exists $u_0 \in \mathcal{P}\backslash\{0\}$ such that for each $u \in \mathcal{P}\backslash\{0\}$, there exist $k_1(u) > 0$ and $k_2(u) > 0$ such that $k_1 u_0 \leq Mu \leq k_2 u_0$ with respect to $\mathcal{P}$.

Throughout the paper we shall denote the spectral radius of a bounded linear operator N by $r(N)$.

The following four theorems are fundamental to our results. The first result is found in [29]. The last three results and proofs are found in [2], [24], [23], or [22]. In each of the following theorems, assume that $\mathcal{P}$ is a reproducing cone, and that $N, N_1, N_2 : \mathcal{B} \to \mathcal{B}$ are compact, linear, and positive with respect to $\mathcal{P}$.

THEOREM **2.1**. *Let N_b, $\eta \leq b \leq \beta$ be a family of compact, linear operators on a Banach space such that the mapping $b \mapsto N_b$ is continuous in the uniform operator topology. Then the mapping $b \mapsto r(N_b)$ is continuous.*

THEOREM **2.2**. *Assume $r(N) > 0$. Then $r(N)$ is an eigenvalue of N, and there is a corresponding eigenvector in $\mathcal{P}$.*

THEOREM **2.3**. *If $N_1 \leq N_2$ with respect to $\mathcal{P}$, then $r(N_1) \leq r(N_2)$.*

THEOREM **2.4**. *Suppose there exists $\gamma > 0$, $u \in \mathcal{B}$, $-u \notin \mathcal{P}$, such that $\gamma u \leq Nu$ with respect to $\mathcal{P}$. Then N has an eigenvector in $\mathcal{P}$ which corresponds to an eigenvalue λ with $\lambda \geq \gamma$.*

3. Green's Functions and Associated Properties

The Green's function for $-D^{\alpha}_{0+}u(t) = 0$, (1.2) is given by [5] as

$$G(b;t,s) = \begin{cases} \dfrac{[t(b-s)]^{\alpha-1}}{b^{\alpha-1}\Gamma(\alpha)} - \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq b, \\ \\ \dfrac{[t(b-s)]^{\alpha-1}}{b^{\alpha-1}\Gamma(\alpha)}, & 0 \leq t \leq s \leq b. \end{cases} \tag{3.1}$$

In particular, if $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous, if $u \in C[0,b]$ and

$$u(t) = \int_0^b G(b;t,s) f(s,u(s))ds,$$

then u is a solution of

$$D^{\alpha}_{0+}u + f(t,u(t)) = 0, \;\; 0 < t < b, \quad u(0) = 0, \; u(b) = 0.$$

We remark that Bai and Lu [5] assume $b = 1$. The construction of G for arbitrary b is straightforward.

We point out some sign properties of G, which we summarize in a theorem.

THEOREM **3.1**. *The following properties hold:*

(1) $G(b;t,s) > 0$ *on* $(0,b) \times (0,b)$.

(2) $\dfrac{\partial}{\partial t}G(b;t,s) > 0, \dfrac{\partial^2}{(\partial t)^2}G(b;t,s) < 0, 0 < t < s.$

(3) $\dfrac{\partial}{\partial t}G(b;t,s) < 0, \dfrac{\partial^2}{\partial t^2}G(b;t,s) > 0, 0 < s < t.$

(4) $\displaystyle\lim_{t\to s^+} \frac{\partial}{\partial t}G(b;t,s) = -\infty.$

(5) $G(b;t,s) = t^{\alpha-1}v(b;t,s)$ *where* $v(b;0,s) > 0$, $\dfrac{\partial}{\partial b}v(b;0,s) > 0$,

for $0 < s < b$, *and* $\dfrac{\partial}{\partial t}v(b;b,s) < 0$, *for* $0 < s < b$.

(6) $\dfrac{\partial}{\partial t}G(b;b,s) = -\dfrac{\alpha-1}{\Gamma(\alpha)}(b-s)^{\alpha-2}(\dfrac{s}{b}) < 0, 0 < s < b.$

(7) $y^{\alpha-1}G(b;w,s) \leq w^{\alpha-1}G(b;y,s), \quad 0 \leq y \leq w \leq b.$

(8) $0 \leq G(b;t,s) \leq G(b;s,s) = \displaystyle\max_{0\leq t\epsilon b} G(b;t,s).$

(9) *If* $\frac{b}{4} \leq t \leq \frac{3b}{4}, 0 < s < b$, *then there exists* $\gamma_b(s) > 0$ *such that*

$$\gamma_b(s) G(b; s, s) \leq G(b; t, s).$$

P r o o f. The proofs are straightforward calculations. To verify (1), first note the inequality is obvious on the triangle, $t < s$. On the triangle, $s \leq t$, write

$$\frac{[t(b-s)]^{\alpha-1}}{b^{\alpha-1}} = \left(t - \left(\frac{t}{b}\right) s\right)^{\alpha-1}.$$

Then

$$\left(t - \left(\frac{t}{b}\right) s\right)^{\alpha-1} > (t-s)^{\alpha-1}, \quad \text{on} \quad 0 < s < t < b$$

and the inequality is valid on the triangle, $s \leq t$.

To verify (5), note

$$v(b; t, s) = \begin{cases} \dfrac{[(b-s)]^{\alpha-1}}{b^{\alpha-1}\Gamma(\alpha)} - \dfrac{(1-\frac{s}{t})^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq b, \\[2ex] \dfrac{[(b-s)]^{\alpha-1}}{b^{\alpha-1}\Gamma(\alpha)}, & 0 \leq t \leq s \leq b. \end{cases}$$

So,

$$v(b; 0, s) = \frac{[(b-s)]^{\alpha-1}}{b^{\alpha-1}\Gamma(\alpha)} > 0,$$

$$-\frac{\partial}{\partial b} v(b; 0, s) = \frac{\partial}{\partial t} v(b; b, s) = -\frac{\alpha-1}{\Gamma(\alpha)} \left(1 - \frac{s}{b}\right)^{\alpha-2} \frac{s}{b^2} < 0.$$

To verify (7), consider each of the three cases, $y < w < s$, $y < s < w$, $s < y < w$, independently. If $y < w < s$,

$$\frac{G(b; w, s)}{w^{\alpha-1}} = \frac{G(b; y, s)}{y^{\alpha-1}}.$$

If $y < s < w$,

$$\frac{G(b; w, s)}{w^{\alpha-1}} = \frac{(b-s)^{\alpha-1}}{b^{\alpha-1}\Gamma(\alpha)} - \frac{(w-s)^{\alpha-1}}{w^{\alpha-1}\Gamma(\alpha)} \leq \frac{(b-s)^{\alpha-1}}{b^{\alpha-1}\Gamma(\alpha)} = \frac{G(b; y, s)}{y^{\alpha-1}}.$$

Finally, if $s < y < w$,

$$\frac{(y-s)^{\alpha-1}}{y^{\alpha-1}} \leq \frac{(w-s)^{\alpha-1}}{w^{\alpha-1}} \Longrightarrow \frac{G(b; w, s)}{w^{\alpha-1}} \leq \frac{G(b; y, s)}{y^{\alpha-1}}.$$

Property (9) has been obtained in [5] for the case $b = 1$. For arbitrary b,

$$\gamma_b(s) = \begin{cases} \dfrac{[\frac{3b}{4}(b-s)]^{\alpha-1} - (\frac{3b}{4} - s)^{\alpha-1}}{(s(b-s))^{\alpha-1}}, & 0 < s \le r(b), \\ \left(\frac{b}{4s}\right)^{\alpha-1}, & r(b) \le s \le b, \end{cases}$$

where $r(b) \in \left(\frac{b}{4}, \frac{3b}{4}\right)$ is the unique solution of

$$\left(\frac{3b}{4}(b-s)\right)^{\alpha-1} - \left(\frac{3b}{4} - s\right)^{\alpha-1} = \left(\frac{b(b-s)}{4}\right)^{\alpha-1}.$$

In [5], Bai and Lu point out the important observation that if $\alpha \neq 2$, $\lim_{s\to 0} \gamma_b(s) = 0$. □

Now, define $H(b;t,s) = \frac{\partial}{\partial b} G(b;t,s)$ for $b \in (0,\infty)$. We make three observations.

THEOREM **3.2**. *The following properties hold:*

(1) $H(b;t,s) = \dfrac{\alpha-1}{\Gamma(\alpha)}(1-\frac{s}{b})^{\alpha-2}(\dfrac{s}{b^2})t^{\alpha-1}.$

(2) $H(b;t,s) > 0$ *on* $(0,b) \times (0,b)$.

(3) For each $s \in (0,b)$, H *is the solution of the boundary value problem*

$$-D^{\alpha}_{0+}u(t) = 0, \quad 0 < t < b,$$

$$u(0) = 0, \quad u(b) = -(\frac{\partial}{\partial t})G(b;b,s) > 0.$$

P r o o f. (1) and (3) are obtained through straightforward calculations and (1) implies (2). However, we offer a more qualitative proof in the spirit of developing disconjugacy of fractional linear differential equations.

Let $\epsilon > 0$, and let $u(\epsilon, b, t)$ denote the solution of the BVP,

$$D^{\alpha}_{0+}u(t) + \epsilon = 0, \quad 0 < t < b,$$

$$u(0) = 0, \quad u(b) = -\left(\frac{\partial}{\partial t}\right)G(b;b,s) > 0.$$

Then

$$u(\epsilon, b, t) = \left(-\left(\frac{\partial}{\partial t}\right)G(b;b,s)\right)\frac{t^{\alpha-1}}{b^{\alpha-1}} + \epsilon\int_0^b G(b;t,s)ds.$$

Let $0 < y \le w \le b$, and employ property (7), Theorem **3.1** to obtain

$$\begin{aligned} w^{\alpha-1}u(\epsilon,b,y) &= w^{\alpha-1}\left(-\left(\frac{\partial}{\partial t}\right)G(b;b,s)\right)\frac{y^{\alpha-1}}{b^{\alpha-1}} + w^{\alpha-1}\epsilon\int_0^b G(b;y,s)ds \\ &\ge y^{\alpha-1}\left(-\left(\frac{\partial}{\partial t}\right)G(b;b,s)\right)\frac{w^{\alpha-1}}{b^{\alpha-1}} + y^{\alpha-1}\epsilon\int_0^b G(b;w,s)ds \\ &= y^{\alpha-1}u(\epsilon,b,w). \end{aligned}$$

Then

$$\begin{aligned} wu(\epsilon,b,y) = w^{2-\alpha}w^{\alpha-1}u(\epsilon,b,y) &\ge w^{2-\alpha}y^{\alpha-1}u(\epsilon,b,w) \\ &\ge y^{2-\alpha}y^{\alpha-1}u(\epsilon,b,w) = yu(\epsilon,b,w). \end{aligned}$$

In particular, $bu(\epsilon,b,t) \ge tu(\epsilon,b,b) = t(-(\frac{\partial}{\partial t})G(b;b,s))$, or

$$u(\epsilon,b,t) \ge \frac{(-(\frac{\partial}{\partial t})G(b;b,s))}{b}t. \tag{3.2}$$

(3.2) is true for all $\epsilon > 0$. Continuous dependence on ϵ will be valid once we define appropriate Banach spaces and fixed point operators in Section 4. Hence, (2) is verified. □

4. Criteria for Conjugate Points

To apply Theorems (**2.1**)-(**2.4**) we define a family of Banach spaces and cones. First define

$$\mathcal{B} = \{u : [0,\infty) \to \mathbb{R} : u = t^{\alpha-1}v, v \in BC[0,\infty)\}, \tag{4.1}$$

where $BC[0,\infty)$ is the Banach space of bounded continuous functions on $[0,\infty)$ and

$$\|u\| = \sup_{t\in[0,\infty)} |v(t)|.$$

Define the cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \{u \in \mathcal{B} : u(t) \ge 0 \text{ for } 0 \le t < \infty\}. \tag{4.2}$$

The cone $\mathcal{P}$ is a reproducing cone since if $u \in \mathcal{B}$,

$$u_1(t) = \max\{0, u(t)\}, \quad u_2(t) = \max\{0, -u(t)\},$$

are in $\mathcal{P}$ and $u = u_1 - u_2$.

For each $0 < b$, define the Banach Space

$$\mathcal{B}_b = \{u : [0,b] \to \mathbb{R} : u = t^{\alpha-1}v, v \in C^1[0,b], v(b) = 0\},$$

with the norm

$$\|u\|_b = \max_{t\in[0,b]} |v'(t)| = |v'|_0.$$

We point out some easy but useful relations between common supremum norms and the norm $\|u\|_b$. Note that for $v \in C^1[0,b]$, $v(b)=0$,

$$|v(t)| = |v(t)-v(b)| = \left|\int_b^t v'(s)ds\right| \le (b-t)|v'|_0 \le b\|u\|_b, \quad 0 \le t \le b.$$

Therefore, using sup norm notation and assuming $0 \le t \le b$,

$$|v|_0 \le b\|u\|_b = b|v'|_0 \text{ and } \max_{t\in[0,b]} |u(t)| = |u|_0 = |t^{\alpha-1}v|_0 \le b^{\alpha}\|u\|_b.$$

For each $0<b$ define the cone $\mathcal{P}_b \subset \mathcal{B}_b$ to be

$$\mathcal{P}_b = \{u \in B_b : u(t) \ge 0 \text{ for } t \in [0,b]\}.$$

In [14], the authors showed

$$\Omega_1 = \{u = t^{\alpha-1}v \in \mathcal{P}_1 : u(t) > 0, \text{ for } 0<t<1, \ v(0)>0, v'(1)<0\} \subset \mathcal{P}_1^{\circ},$$

where $\mathcal{P}_1^{\circ}$ denotes the interior of $\mathcal{P}_1$. So, $\mathcal{P}_1$ has nonempty interior implying $\mathcal{P}_1$ is solid and hence, reproducing.

Remark **4.1**. The proof in [14] is easily modified to show that for each $b>0$,

$$\Omega_b = \{u = t^{\alpha-1}v \in \mathcal{P}_b : u(t) > 0, \text{ for } 0<t<b, \ v(0)>0, v'(b)<0\} \subset \mathcal{P}_b^{\circ}.$$

So, the results obtained in [14] on $[0,1]$ are valid on $[0,b]$ for each $0<b$ and are employed in this work without proof. Hence, we shall state without proof that $\mathcal{P}_b$ has nonempty interior and is reproducing.

Define $N_0u(t)=0$, $0 \le t$, and for each $0<b$, and define $N_b : \mathcal{B} \to \mathcal{B}$ by

$$N_bu(t) = \begin{cases} \int_0^b G(b;t,s)p(s)u(s)ds, & 0 \le t \le b, \\ 0, & b \le t < \infty. \end{cases} \tag{4.3}$$

We shall also refer to $N_b : \mathcal{B}_b \to \mathcal{B}_b$ where N_b is defined by

$$N_bu(t) = \int_0^b G(b;t,s)p(s)u(s)ds, \ 0 \le t \le b.$$

Before proceeding, we argue that b_0, the first conjugate point, is positive.

Theorem **4.1**. *Assume $p \in C[0,\beta]$ for some $\beta>0$. Then there exists $\delta>0$ such that if $0<b<\delta$, there exists a unique solution of the BVP(b), (1.1), (1.2); in particular, if $0<b<\delta$, then $u \equiv 0$ is the only solution of (1.1), (1.2).*

P r o o f. Let $P = \max_{0\le t\le\beta} |p(t)|$. We show there exists $\delta > 0$ such that if $0 < b < \delta$, $N_b : \mathcal{B}_b \to \mathcal{B}_b$ is a contraction map. Let $u_1, u_2 \in \mathcal{B}_b$ and consider

$$(N_b u_1 - N_b u_2)(t) = t^{\alpha-1}\left(\int_0^b \frac{(b-s)^{\alpha-1}}{b^{\alpha-1}\Gamma(\alpha)} p(s)(u_1-u_2)(s)ds \right.$$
$$\left. - t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)(u_1-u_2)(s)ds\right).$$

Set

$$v(t) = \left(\int_0^b \frac{(b-s)^{\alpha-1}}{b^{\alpha-1}\Gamma(\alpha)} p(s)(u_1-u_2)(s)ds \right.$$
$$\left. - t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)(u_1-u_2)(s)ds\right).$$

So $\|N_b u_1 - N_b u_2\|_b = |v'|_0$. In [14], the authors showed $v'(0) = 0$ for the case $b = 1$; the argument is valid for $b > 0$. For $t > 0$,

$$\begin{aligned}
|v'(t)| &= \left|(1-\alpha)t^{-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(u_1-u_2)(s)ds\right. \\
&\quad \left. + (\alpha-1)t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)} p(u_1-u_2)(s)ds\right| \\
&\le \left|(1-\alpha)t^{-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(u_1-u_2)(s)ds\right| \\
&\quad + \left|(\alpha-1)t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)} p(u_1-u_2)(s)ds\right| \\
&\le \frac{\alpha-1}{\Gamma(\alpha)} P|u_1-u_2|_0 t^{-\alpha}\int_0^t (t-s)^{\alpha-1}ds \\
&\quad + \frac{\alpha-1}{\Gamma(\alpha)} P|u_1-u_2|_0 t^{1-\alpha}\int_0^t (t-s)^{\alpha-2}ds \\
&= \left(\frac{\alpha-1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)}\right) P|u_1-u_2|_0 \\
&\le \left(\frac{\alpha-1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)}\right) Pb^\alpha \|u_1-u_2\|_b.
\end{aligned}$$

Thus, if $\left(\frac{\alpha-1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)}\right) Pb^\alpha < 1$, then N_b is a contraction map. Choose $\delta > 0$ such that $\left(\frac{\alpha-1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)}\right) P\delta^\alpha = 1$ and the proof is complete. □

LEMMA **4.1**. *For each $0 < b$, the linear operator N_b is positive with respect to $\mathcal{P}$ and with respect to $\mathcal{P}_b$.*

This is an easy consequence of Theorem 3.1 (1) and the sign condition on p.

LEMMA **4.2**. *Let $\beta > 0$. Consider N_b, $0 \le b \le \beta$, defined on $\mathcal{B}$. The map $b \mapsto N_b$ is continuous in the uniform topology.*

This is, of course, an application of Theorem **2.1**. We point out that the operator, N_1, defined on $\mathcal{B}_1$, was proved to be compact in [14]. This argument is readily modified to show each N_b, defined on $\mathcal{B}_b$, is a compact map. Finally, since each $u \in \mathcal{B}_b$ is extended by 0 to construct $u \in \mathcal{B}$, it follows that each N_b, defined now on $\mathcal{B}_\beta$, is compact. To see that the mapping $b \mapsto N_b$ is continuous in the uniform operator topology, assume $u = t^{\alpha-1}v \in \mathcal{B}_\beta$ with $\|u\|_\beta = 1$. Assume for simplicity that $0 < b_1 < b_2 \le \beta$. Let $P = \max\limits_{0\le t\le\beta} |p(t)|$. Then

$$\begin{aligned}
&|(N_{b_2} - N_{b_1})u(t)| \\
&\le \frac{Pt^{\alpha-1}\beta^\alpha}{\Gamma(\alpha)}\left(\int_0^{b_1}\left|\frac{(b_2-s)^{\alpha-1}}{b_2^{\alpha-1}} - \frac{(b_1-s)^{\alpha-1}}{b_1^{\alpha-1}}\right| ds + \int_{b_1}^{b_2}\frac{(b_2-s)^{\alpha-1}}{b_2^{\alpha-1}}ds\right).
\end{aligned}$$

Note that

$$\begin{aligned}
&\frac{Pt^{\alpha-1}\beta^\alpha}{\Gamma(\alpha)}\int_0^{b_1}\left|\frac{(b_2-s)^{\alpha-1}}{b_2^{\alpha-1}} - \frac{(b_1-s)^{\alpha-1}}{b_1^{\alpha-1}}\right| ds \\
&\qquad\le \frac{P\beta^{2\alpha-1}}{\Gamma(\alpha)}\int_0^{b_1}\left(\frac{(b_2-s)^{\alpha-1}}{b_2^{\alpha-1}} - \frac{(b_1-s)^{\alpha-1}}{b_1^{\alpha-1}}\right) ds \\
&\qquad\le \frac{P\beta^{2\alpha-1}}{\Gamma(\alpha)}\left(-\frac{(b_2-s)^\alpha}{\alpha b_2^{\alpha-1}} + \frac{(b_2-s)^\alpha}{\alpha b_2^{\alpha-1}}\right)\Bigg|_0^{b_1} \le \frac{P\beta^{2\alpha-1}}{\Gamma(\alpha+1)}|b_2-b_1|
\end{aligned}$$

and

$$\begin{aligned}
\frac{Pt^{\alpha-1}\beta^\alpha}{\Gamma(\alpha)}\int_{b_1}^{b_2}\frac{(b_2-s)^{\alpha-1}}{b_2^{\alpha-1}}ds &\le \frac{P\beta^{2\alpha-1}}{\Gamma(\alpha)}\frac{(b_2-b_1)^\alpha}{b_2^{\alpha-1}} \\
&= \frac{P\beta^{2\alpha-1}}{\Gamma(\alpha)}\frac{(b_2-b_1)^{\alpha-1}}{b_2^{\alpha-1}}(b_2-b_1) \le \frac{P\beta^{2\alpha-1}}{\Gamma(\alpha)}|b_2-b_1|.
\end{aligned}$$

Thus,

$$|(N_{b_2} - N_{b_1})u(t)| \le P\beta^{2\alpha-1}\left(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)}\right)|b_2-b_1|.$$

If $b_1 = 0$,

$$|(N_{b_2} - N_0)u(t)| \leq \frac{P\beta^{2\alpha-1}}{\Gamma(\alpha)}|b_2 - 0|.$$

In particular, Theorem **2.1** now applies on compact domains $[0, \beta]$.

THEOREM **4.2**. *Consider N_b, $0 \leq b$, defined on $\mathcal{B}$. $r(N_b)$ is strictly increasing as a function of b.*

P r o o f. In [14], the authors showed that there exists $\lambda > 0$ and $u \in \mathcal{P}_1 \setminus \{0\}$ such that $N_1 u(t) = \lambda u(t)$, $0 \leq t \leq 1$. Referring to Remark 4.1, the proof is easily extended to show that for each $0 < b$, there exists $\lambda_b > 0$ and $u \in \mathcal{P}_b \setminus \{0\}$ such that $N_b u(t) = \lambda_b u(t)$, $0 \leq t \leq b$. Extend u by $u(t) = 0$, for $b < t$. Then for $0 \leq t$, $N_b u(t) = \lambda_b u(t)$, and $r(N_b) \geq \lambda_b > 0$.

Assume $0 < b_1 < b_2$. $r(N_{b_1}) > 0$ implies there exists $u_1 \in \mathcal{P}_{b_1} \setminus \{0\}$ such that $N_{b_1} u_1 = r(N_{b_1}) u_1$. Set $y_1 = N_{b_1} u_1$ and $y_2 = N_{b_2} u_1$. Let $t \in [0, b_1]$. Then

$$\begin{aligned}(y_2 - y_1)(t) &= \int_0^{b_2} G(b_2; t, s)p(s)u_1(s)ds - \int_0^{b_1} G(b_1; t, s)p(s)u_1(s)ds \\ &= \int_0^{b_1} (G(b_2; t, s) - G(b_1; t, s))p(s)u_1(s)ds.\end{aligned}$$

Since $u_1(s) \neq 0$ for some $s \in [0, b_1]$, it follows from Theorem **3.2** (2) that $(y_2 - y_1)(t) > 0$ for $0 < t < b_1$. Moreover, it follows by Theorem **3.1** (2) that if $u_i(t) = t^{\alpha-1} v_i(t)$, $i = 1, 2$, then $(v_2 - v_1)(0) > 0$. In particular, the restriction of $y_2 - y_1$ to $[0, b_1]$ is an element of the interior of P_{b_1}. Thus, there exists $\delta > 0$ such that $y_2 - y_1 \geq \delta u_1$, where this inequality is with respect to the cone P_{b_1}. Since $y_1(t) = 0$ for $t > b_1$ and $y_2 \in \mathcal{P}_{b_2}$, it follows that $y_2 - y_1 \geq \delta u_1$, where the inequality is now with respect to the cone, $\mathcal{P}_{b_2}$. Thus, $y_2 \geq y_1 + \delta u_1 = (r(N_{b_1}) + \delta)u_1$ and by Theorem **2.4**, $r(N_{b_2}) \geq r(N_{b_1}) + \delta > r(N_{b_1})$.

□

We now state and prove the main result of the paper.

THEOREM **4.3**. *The following are equivalent:*

(1) b_0 is the first extremal point of the BVP corresponding to (1.1), (1.2);

(2) there exists a nontrivial solution u of the BVP(b_0) (1.1), (1.2) such that $u \in \mathcal{P}_{b_0}$;

(3) $r(N_{b_0}) = 1$.

P r o o f. (3) $\implies$ (2) is an immediate consequence of Theorem **2.2**.

We prove (2) $\implies$ (1). Let $u \in \mathcal{P}_{b_0} \setminus \{0\}$ satisfy (1.1), (1.2). Extend u by $u(t) = 0$ for $b_0 < t$. Then $r(N_{b_0}) \geq 1$. If $r(N_{b_0}) = 1$, the proof is complete by Theorem **4.2**. To see this, let $0 < b < b_0$. Then $r(N_b) < 1$ and the BVP(b), (1.1), (1.2), has only the trivial solution.

So assume for the sake of contradiction that $r(N_{b_0}) > 1$. Let $\hat{u} \in \mathcal{P} \setminus \{0\}$ such that $N_{b_0}\hat{u} = r(N_{b_0})\hat{u}$. Again, referring to Remark 4.1, the methods and results in [14] are readily modified to show that the restriction of $\hat{u}$ to $[0, b_0]$ (denoted by $\hat{u}$) satisfies $\hat{u} \in \mathcal{P}^{\circ}_{b_0}$. Thus, there exists $\delta > 0$ such that $u \geq \delta\hat{u}$ where $\leq$ is with respect to the cone $\mathcal{P}_{b_0}$. Thus, $u \geq \delta\hat{u}$ where u and $\hat{u}$ have been extended to $[0, \infty)$ and $\leq$ is now with respect to the cone $\mathcal{P}$. Let $\hat{\delta} = \sup\{\delta > 0 | u \geq \delta\hat{u}\}$. Then

$$u = N_{b_0}u \geq N_{b_0}\hat{\delta}\hat{u} = \hat{\delta}N_{b_0}\hat{u} = \hat{\delta}r(N_{b_0})\hat{u}.$$

This contradicts the definition of $\hat{\delta}$ if $r(N_{b_0}) > 1$, so, $r(N_{b_0}) = 1$.

To prove (1) $\implies$ (3) observe that $\lim_{b \to 0^+} r(N_b) = 0$. Thus, (1) $\implies$ (3) follows from Theorem **2.1**. Notice since (1) implies $r(N_{b_0}) \geq 1$ and if $r(N_{b_0}) > 1$, then by continuity of r, there exists $0 < b < b_0$ such that $r(N_b) = 1$, contradicting (1). □

5. Application to a Nonlinear Problem

In this section, we shall apply Theorems **4.2** and **4.3** to a nonlinear problem in a standard way. This application employs a fixed point result, proved in [9] or [30] and this particular application has been employed in [30] or [13], for example. Webb [33, 34, 35] has recently produced several applications of u_0 positive operators to nonlinear problems.

Consider a boundary value problem for a nonlinear fractional differential equation of the form

$$D^{\alpha}_{0+}u + f(t, u) = 0, \quad 0 < t < b, \tag{5.1}$$

with boundary conditions, (1.2), where $f(t, u) : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous, and $f(t, 0) \equiv 0$. In addition, assume f is differentiable in u at $u = 0$. Assume $p(t) \equiv (\frac{\partial f}{\partial u})(t, 0)$ is continuous on $[0, \infty)$ and does not vanish identically on each compact subinterval of $[0, \infty)$. Then the variational equation along the zero solution of (5.1) is

$$D^{\alpha}_{0+}u + p(t)u = 0, \quad 0 < t < b. \tag{5.2}$$

Thus, assume in addition that $p(t) \geq 0$, $t \in (0, \infty)$ so that if $u \in \mathcal{P}$, then $p(t)u(t) \geq 0$, for $0 \leq t$.

To obtain sufficient conditions for the existence of solutions of the BVP, (5.1), (1.2), we shall apply the following fixed point theorem; see [9] or [30].

THEOREM **5.1**. *Let $\mathcal{B}$ be a Banach space and let $\mathcal{P} \subset \mathcal{B}$ be a reproducing cone. Let $M : \mathcal{B} \to \mathcal{B}$ be a completely continuous, nonlinear operator such that $M : \mathcal{P} \to \mathcal{P}$ and $M(0) = 0$. Assume M is Fréchet differentiable at $u = 0$ whose Fréchet derivative $N = M'(0)$ has the property:*

(A) There exist $w \in \mathcal{P}$ and $\mu > 1$ such that $Nw = \mu w$, and $Nu = u$ implies $u \notin \mathcal{P}$. Further, there exists $\rho > 0$ such that, if $u = (\frac{1}{\lambda})Mu, u \in \mathcal{P}$ and $||u|| = \rho$, then $\lambda \leq 1$.

Then, the equation, $u = Mu$ has a solution, $u \in \mathcal{P} \setminus \{0\}$.

We shall apply Theorem 5.1 and the results of Section 4 to prove the following theorem. So, now assume $\mathcal{B}$ and $\mathcal{P}$ are as in (4.1) and (4.2), respectively.

THEOREM **5.2**. *Assume b_0 is the first extremal point of (5.2), (1.2). For each $b > b_0$ assume the property:*

($\hat{A}$) There exists $\rho(b) > 0$ such that if $u(t)$ is a nontrivial solution of the BVP,

$$D^{\alpha}_{0+}u + \left(\frac{1}{\lambda}\right) f(t,u) = 0, \quad 0 < t < b,$$

with boundary conditions, (1.2) and if $u \in \mathcal{P}_b$, with $\|u\|_b = \rho(b)$, then $\lambda \leq 1$.

Then for all $b > b_0$, the BVP(b), (5.1), (1.2), has a nontrivial solution, $u \in \mathcal{P}_b$.

P r o o f. For each $b > b_0$, let $N_b : \mathcal{B} \to \mathcal{B}$ be defined by (4.3), where $p(t) \equiv (\frac{\partial f}{\partial u})(t, 0)$, and define the nonlinear operator, $M_b : \mathcal{B} \to \mathcal{B}$ by

$$M_b u(t) = \begin{cases} \displaystyle\int_0^b G(b;t,s) f(s,u(s))ds, & 0 \leq t \leq b, \\ 0, & b \leq t < \infty. \end{cases}$$

The differentiability of f with respect to u at $u = 0$ is sufficient to argue that M_b is Fréchet differentiable at $u = 0$ since

$$\begin{aligned} &\left| \int_0^b G(b;t,s)[f(s,u(s)) - p(s)u(s)]ds \right| \\ &\qquad = \left| \int_0^b G(b;t,s)[f_u(s,\tilde{u}(s)) - p(s)]u(s)ds \right| \\ &\qquad \leq K b^{\alpha} \|u\|_b \int_0^b |f_u(s,\tilde{u}(s)) - p(s)|ds, \end{aligned}$$

where $K = \frac{b^{\alpha-1}}{\Gamma(\alpha)} \geq |G(b;t,s)|$ for $(t,s) \in (0,b) \times (0,b)$. Moreover, $M_b'(0) = N_b$.

By Theorems **4.2** and **4.3**, it follows that $r(N_{b_0}) = 1$ and $r(N_b) > 1$ if $b_0 < b$. Moreover, since b_0 is the first extremal point of (5.2), (1.2), it also follows from Theorem **4.3** that if $b_0 < b$, if $N_b u = u$ and u is nontrivial, then $u \notin \mathcal{P}$. Thus, with Condition $(\hat{A})$, we can apply Theorem 5.1 and obtain the existence of a $u \in \mathcal{P} \setminus \{0\}$ such that $u = M_b u$ and the proof is complete. □

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