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POSITIVE SOLUTIONS OF NONLOCAL BOUNDARY VALUE PROBLEM FOR HIGHER ORDER FRACTIONAL DIFFERENTIAL SYSTEM

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ABSTRACT. In this paper, we study existence and multiplicity results for a coupled system of nonlinear nonlocal boundary value problems for higher order fractional differential equations of the type

$$\begin{cases} {}^c\mathcal{D}_{0+}^{\alpha}u(t) = \lambda a(t)f(u(t),v(t)), \quad {}^c\mathcal{D}_{0+}^{\beta}v(t) = \mu b(t)g(u(t),v(t)), \\ u'(0) = u''(0) = u'''(0) = \cdots = u^{(n-1)}(0) = 0, \;\; u(1) = \xi_1 u(\eta_1), \\ v'(0) = v''(0) = v'''(0) = \cdots = v^{(n-1)}(0) = 0, \quad v(1) = \xi_2 v(\eta_2), \end{cases}$$

where $\lambda,\ \mu > 0$, $n-1 < \alpha, \beta \leq n$ for $n \in \mathbb{N}$; $\xi_i,\ \eta_i \in (0,1)$ for $i = 1,2$ and ${}^c\mathcal{D}_{0+}^{\alpha}$ is Caputo fractional derivative. We employ the Guo-Krasnosel'skii fixed point theorem to establish existence and multiplicity results for positive solutions. We derive explicit intervals for the parameters λ and μ for which the system possess the positive solutions or multiple positive solutions. Examples are included to show the applicability of the main results.

1. INTRODUCTION

Fractional differential equations is rapidly growing both in theory and in applications to real world problems. Many problems in engineering, physics and other applied sciences can be modeled as differential equations of fractional order. It has been established that, in many situations, these models provide more suitable results than analogous models with integer derivatives.

The attention drawn to boundary value problems for nonlinear fractional differential equations is evident from an increased number of recent publications. Existence and uniqueness of solutions of fractional differential equations is well studied, see for

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example [1, 2, 3, 6, 12, 13, 15] and references therein. Recently, existence and multiplicity results for positive solutions of boundary value problems involving fractional order derivatives have received much attention, for example, [4, 5, 7, 8, 9, 14, 16, 18]. In contrast, boundary value problems for coupled systems of fractional differential equations have received less attention; only a few results can be found in the literature concerning existence and multiplicity of positive solutions of boundary value problems for coupled systems of fractional differential equations. See for example [4, 9, 18].

In this paper, we study existence and multiplicity results for positive solutions to nonlinear three-point boundary value problems corresponding to a higher order coupled system of fractional differential equations of the type

$$\begin{cases} {}^c\mathcal{D}^{\alpha}_{0+}u(t) = \lambda a(t) f(u(t), v(t)), \quad {}^c\mathcal{D}^{\beta}_{0+}v(t) = \mu b(t) g(u(t), v(t)), \\ u'(0) = u''(0) = u'''(0) = \cdots = u^{(n-1)}(0) = 0, \;\; u(1) = \xi_1 u(\eta_1), \\ v'(0) = v''(0) = v'''(0) = \cdots = v^{(n-1)}(0) = 0, \;\; v(1) = \xi_2 v(\eta_2), \end{cases} \tag{1.1}$$

where $\lambda,\ \mu > 0$, $n-1 < \alpha, \beta \leq n$ for $n \in \mathbb{N}$; $\xi_i,\ \eta_i \in (0,1)$ for $i = 1,2$ and ${}^c\mathcal{D}^{\alpha}_{0+}$ is Caputo fractional derivative. We employ the Guo-Krasnosel'skii fixed point theorem to study existence and multiplicity results. We point out that the results in this article extend those obtained in [3] as we study a system of higher order equations and more importantly, our results extend those obtained in [3, 9] as we consider $f(u(t), v(t))$, $g(u(t), v(t))$ instead of $f(v(t))$, $g(u(t))$.

The paper is organized as follows: In Section 2, we recall the definitions of fractional integral and fractional derivative and some basic lemmas. In Section 3, properties of the Green's function for the associated linear problem are studied. By employing the Guo-Krasnosel'skii fixed point theorem, some sufficient conditions for the existence or multiplicity of positive solutions to the system (1.1) are established. We obtain explicit intervals for λ and μ such that for any value of λ and μ in the intervals, existence or multiplicity of positive solution is guaranteed. Examples are included to demonstrate the application the main results.

2. PRELIMINARIES

We recall some basic definitions and lemmas from fractional calculus.

Definition 2.1. [17] The fractional integral of order $\alpha > 0$ of a function $g : (a, \infty) \to \mathbb{R}$ is defined by

$$\mathcal{I}^{\alpha}_{a+} g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g(s)}{(t-s)^{1-\alpha}} ds,$$

provided the integral converges.

Definition 2.2. [17] The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $g : (0, \infty) \to \mathbb{R}$ is defined by

$$^{c}\mathcal{D}^{\alpha}_{a+}g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{g^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$, (the notation $[a]$ stands for the largest integer not greater than α).

Remark 2.3. Under the natural conditions on $g(t)$ the Caputo fractional derivative becomes conventional integer order derivative of function $g(t)$ as $\alpha \to n$.

Lemma 2.4. *For $\alpha > 0$, $g(t) \in C(0, 1) \cap L(0, 1)$, the homogenous fractional differential equation $^{c}\mathcal{D}^{\alpha}_{0+}g(t) = 0$, has a solution $g(t) = c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1}$, where, $c_i \in \mathbb{R}$, $i = 0, ..., n$ and $n = [\alpha] + 1$.*

Lemma 2.5. *Assume that $g(t) \in C(0, 1) \cap L(0, 1)$, with derivative of order n that belongs to $C(0, 1) \cap L(0, 1)$, then $\mathcal{I}^{\alpha}_{0+}{}^{c}\mathcal{D}^{\alpha}_{0+}g(t) = g(t) + c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1}$, where, $c_i \in \mathbb{R}$, $i = 0, ..., n$ and $n = [\alpha] + 1$.*

Lemma 2.6. [10] *If $\alpha > \beta > 0$, then $^{c}\mathcal{D}^{\beta}_{0+}\mathcal{I}^{\alpha}_{0+}\phi(t) = \mathcal{I}^{\alpha-\beta}_{0+}\phi(t)$. In particular, if m is positive integer and $\delta > m$, then $\frac{d^m}{dt^m}(\mathcal{I}^{\delta}_{0+}\phi(t)) = \mathcal{I}^{\delta-m}_{0+}\phi(t)$.*

The proofs of our main results are based on the following fixed point theorem.

Theorem 2.7 ([11] Guo-Krasnosel'skii Fixed Point Theorem)**.** *Let $\mathcal{B}$ be Banach space and $\mathcal{E} \subset \mathcal{B}$ be a cone. Assume that Ω_1, Ω_2 are open disks in $\mathcal{B}$ such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Let $\mathcal{A} : \mathcal{E} \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \mathcal{E}$ be completely continuous such that either*

(i) $\|\mathcal{A}u\| \leq \|u\|$, *for $u \in \mathcal{E} \cap \partial\Omega_1$ and $\|\mathcal{A}u\| \geq \|u\|$, for $u \in \mathcal{E} \cap \partial\Omega_2$ or*
(ii) $\|\mathcal{A}u\| \geq \|u\|$, *for $u \in \mathcal{E} \cap \partial\Omega_1$ and $\|\mathcal{A}u\| \leq \|u\|$, for $u \in \mathcal{E} \cap \partial\Omega_2$.*

Then, $\mathcal{A}$ has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. MAIN RESULTS

Lemma 3.1. *Let $h \in C[0, 1]$, then the three-point linear boundary value problem*

$$\begin{aligned} &^{c}\mathcal{D}^{\alpha}_{0+}u(t) + h(t) = 0,\ t \in (0, 1), n-1 < \alpha \leq n, \\ &u'(0) = u''(0) = u'''(0) = \cdots = u^{(n-1)}(0) = 0,\ \ u(1) = \xi_1 u(\eta_1), \end{aligned} \tag{3.1}$$

has a unique unique solution given by

$$u(t) = \int_0^1 \mathcal{H}_{\alpha}(t, s)h(s)ds,$$

where

$$\mathcal{H}_\alpha(t,s)=\begin{cases}\frac{(1-s)^{\alpha-1}-(1-\xi_1)(t-s)^{\alpha-1}-\xi_1(\eta_1-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)}, & s\le t,\ \eta_1\ge s,\\ \frac{(1-s)^{\alpha-1}-(1-\xi_1)(t-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)}, & \eta_1\le s\le t\le 1,\\ \frac{(1-s)^{\alpha-1}-\xi_1(\eta_1-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)}, & 0\le t\le s\le \eta_1,\\ \frac{(1-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)}, & t\le s,\ s\ge \eta_1.\end{cases} \tag{3.2}$$

Proof. In view of Lemma 2.5, the general solution of differential equation in (3.1) is given by

$$u(t)=-\mathcal{I}_{0+}^{\alpha}h(t)+\sum_{k=1}^{n}c_kt^{k-1},\, c_k\in\mathbb{R},\ k=1,2,\dots,n. \tag{3.3}$$

By Lemma 2.6 and equation (3.3), we obtain

$$u^{(m)}(t)=-\mathcal{I}_{0+}^{\alpha-m}h(t)+\sum_{k=m+1}^{n}\frac{(k-1)!c_k}{(k-m-1)!}t^{k-m-1}, \tag{3.4}$$

where $m=1,2,\dots,n-1$. The boundary conditions imply that $c_k=0, k=2,3,\dots,n$, and

$$c_1=\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\xi_1)}h(s)ds-\int_0^{\eta_1}\frac{\xi_1(\eta_1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\xi_1)}h(s)ds.$$

Substitute c_1 into (3.3) to obtain

$$\begin{aligned}u(t)=&\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\xi_1)}h(s)ds\\&-\int_0^{\eta_1}\frac{\xi_1(\eta_1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\xi_1)}h(s)ds-\int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds.\end{aligned}$$

Thus,

$$u(t)=\int_0^1\mathcal{H}_\alpha(t,s)h(s)ds$$

where $\mathcal{H}_\alpha(t,s)$ is given by (3.2). □

Similarly, the general solution of ${}^c\mathcal{D}_{0+}^{\beta}v(t)+h(t)=0,\, t\in(0,1), 1<\alpha<2,\ v'(0)=v''(0)=v'''(0)=\cdots=v^{(n-1)}(0)=0,\ v(1)=\xi_2u(\eta_2)$, is given by $v(t)=\int_0^1\mathcal{H}_\beta(t,s)h(s)ds$, where

$$\mathcal{H}_\beta(t,s)=\begin{cases}\frac{(1-s)^{\beta-1}-(1-\xi_2)(t-s)^{\beta-1}-\xi_2(\eta_2-s)^{\beta-1}}{(1-\xi_2)\Gamma(\beta)}, & s\le t,\ \eta_2\ge s,\\ \frac{(1-s)^{\beta-1}-(1-\xi_2)(t-s)^{\beta-1}}{(1-\xi_2)\Gamma(\beta)}, & \eta_2\le s\le t\le 1,\\ \frac{(1-s)^{\beta-1}-\xi_2(\eta_2-s)^{\beta-1}}{(1-\xi_2)\Gamma(\beta)}, & 0\le t\le s\le \eta_2,\\ \frac{(1-s)^{\beta-1}}{(1-\xi_2)\Gamma(\beta)}, & t\le s,\ s\ge \eta_2.\end{cases}$$

Lemma 3.2. *The Green's function $\mathcal{H}_\alpha(t,s)$ defined by (3.2) satisfies the following properties:*

$(P1)$ $\mathcal{H}_\alpha(t,s)>0$ *for all* $t,s\in(0,1)$*;*

$(P2)$ *For each* $s \in [0,1]$, $\mathcal{H}_\alpha(t,s)$ *is nonincreasing in* t*;*

$(P3)$ *For* $\ell \in (0,1)$, $\Phi_\alpha(s) \geq \mathcal{H}_\alpha(t,s) \geq \gamma_1 \min\limits_{\ell \leq t \leq 1} \mathcal{H}_\alpha(t,s) \geq \gamma_1 \Phi_\alpha(s)$ *where*

$$\gamma_1 = \xi_1(1-\eta_1^{\alpha-1}), \quad \Phi_\alpha(s) = \frac{(1-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)}.$$

Proof. $(P1)$: For $0 \leq t \leq 1$, $\eta_1 \geq s$,

$$\mathcal{H}_\alpha(t,s) = \frac{(1-s)^{\alpha-1} - (1-\xi_1)(t-s)^{\alpha-1} - \xi_1(\eta_1 - s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)}.$$

For $t < \eta_1$,

$$\mathcal{H}_\alpha(t,s) > \frac{(1-s)^{\alpha-1} - (\eta_1 - s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)} > 0$$

and for $t \geq \eta_1$,

$$\mathcal{H}_\alpha(t,s) \geq \frac{(1-s)^{\alpha-1} - (t-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)} > 0.$$

For $\eta_1 \leq s \leq t \leq 1$,

$$\mathcal{H}_\alpha(t,s) = \frac{(1-s)^{\alpha-1} - (1-\xi_1)(t-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)} \geq \frac{(1-s)^{\alpha-1} - (t-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)} > 0.$$

Thus, for each case, $\eta_1 \leq s \leq t \leq 1$ and $0 \leq t \leq s$, $s \geq \eta_1$, $\mathcal{H}_\alpha(t,s) > 0$.

$(P2)$: From (3.2), if $t > s$,

$$\frac{\partial}{\partial t}\mathcal{H}_\alpha(t,s) = -\frac{(\alpha-1)}{\Gamma(\alpha)}(t-s)^{\alpha-2} = -\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \leq 0.$$

If $t \leq s$, then

$$\frac{\partial}{\partial t}\mathcal{H}_\alpha(t,s) \equiv 0.$$

Thus, $\mathcal{H}_\alpha(t,s)$ is nonincreasing in t.

$(P3)$: Clearly, $\mathcal{H}_\alpha(t,s) \leq \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\xi_1)} = \Phi_\alpha(s)$.

Case (i): $s \leq \eta_1$: For $t \geq s$, apply $(P2)$ so that

$$\mathcal{H}_\alpha(t,s) \geq \mathcal{H}_\alpha(1,s) = \frac{\xi_1(1-\eta_1^{\alpha-1})(1-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)} = \gamma_1 \Phi_\alpha(s).$$

For $t \leq s$, we have

$$\mathcal{H}_\alpha(t,s) \geq \frac{(1-s)^{\alpha-1} - \xi_1 \eta_1^{\alpha-1}(1-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)} \geq \xi_1(1-\eta_1^{\alpha-1})\Phi_\alpha(s) = \gamma_1 \Phi_\alpha(s).$$

Case (ii): $s > \eta_1$: For $t \geq s$,

$$\mathcal{H}_\alpha(t,s) \geq \mathcal{H}_\alpha(1,s) = \frac{\xi_1(1-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)} \geq \gamma_1 \Phi_\alpha(s).$$

For $t < s$, we have $\mathcal{H}_\alpha(t,s) = \frac{(1-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)} = \Phi_\alpha(s) \geq \gamma_1 \Phi_\alpha(s)$.

Hence, $\min\limits_{\ell \leq t \leq 1} \mathcal{H}_\alpha(t,s) \geq \gamma_1 \Phi_\alpha(s)$ where $\ell \in (0,1)$, $\gamma_1 = \xi_1(1-\eta_1^{\alpha-1})$ and $\Phi_\alpha(s) = \frac{(1-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)}$. □

Similarly, we can prove that $\mathcal{H}_\beta(t,s) > 0$ for all $t,s \in (0,1)$ and for $\ell \in (0,1)$, $\min\limits_{\ell \le t \le 1} \mathcal{H}_\beta(t,s) \ge \gamma_2 \Phi_\beta(s)$ where $\gamma_2 = \xi_2(1-\eta_2^{\beta-1})$ and $\Phi_\beta(s) = \frac{(1-s)^{\beta-1}}{(1-\xi_2)\Gamma(\beta)}$. Furthermore $\mathcal{H}_\beta(t,s)$ is decreasing in t.

Now we write the system of boundary value problem (1.1) as an equivalent system of integral equations

$$\begin{cases} u(t) = \lambda \int_0^1 \mathcal{H}_\alpha(t,s)a(s)f(u(s),v(s))ds \\ v(t) = \mu \int_0^1 \mathcal{H}_\beta(t,s)b(s)g(u(s),v(s))ds, \end{cases} \tag{3.5}$$

and consider the Banach space $C[0,1]$ equipped with the norm

$$\|(u,v)\| = \|u\| + \|v\| = \max_{0\le t\le 1}|u(t)| + \max_{0 \le t \le 1}|v(t)|.$$

Define operators $\mathcal{A}_\lambda, \mathcal{A}_\mu : C[0,1]\times C[0,1] \to C[0,1]\times C[0,1]$ by

$$\mathcal{A}_\lambda(u,v)(t) = \lambda \int_0^1 \mathcal{H}_\alpha(t,s)a(s)f(u(s),v(s))ds,$$

$$\mathcal{A}_\mu(u,v)(t) = \mu \int_0^1 \mathcal{H}_\beta(t,s)b(s)g(u(s),v(s))ds,$$

respectively, and define an operator $\mathcal{A} : C[0,1]\times C[0,1] \to C[0,1]\times C[0,1]$ by

$$\mathcal{A}(u,v) = (\mathcal{A}_\lambda(u,v),\ \mathcal{A}_\mu(u,v)).$$

By a solution of (1.1), we mean a fixed point of the operator $\mathcal{A}$.

Let $\gamma = \min\{\gamma_1,\gamma_2\}$, fix $\ell \in (0,1)$, and define a cone in $C[0,1]\times C[0,1] = (C[0,1])^2$ by

$$P = \Big\{(u,v) \in (C[0,1])^2 : u(t) \ge 0,\ v(t) \ge 0,\ \min_{l \le t \le 1}(u(t)+v(t)) \ge \gamma\|(u,v)\|\Big\}.$$

For the remainder of the paper, we shall employ the following assumptions when appropriate:

(H1) $f,g \in C([0,1]\times\mathbb{R}^+,\mathbb{R})$, and the limits

$$f_0 = \lim_{u+v\to 0}\frac{f(u,v)}{u+v}, \qquad f_\infty = \lim_{u+v\to\infty}\frac{f(u,v)}{u+v},$$
$$g_0 = \lim_{u+v\to 0}\frac{g(u,v)}{u+v}, \qquad g_\infty = \lim_{u+v\to\infty}\frac{g(u,v)}{u+v}$$

exist and $f_0,\ f_\infty,\ g_0,\ g_\infty \in [0,\infty)$;

(H2) $a,b \in C([0,1],(0,\infty))$ such that $\mathcal{I}_{0+}^\alpha a(1)$ and $\mathcal{I}_{0+}^\beta b(1)$ exist and are finite;

(H3)

$$(1-\xi_1)\mathcal{I}_{0+}^\alpha a(1) f_0 < \gamma_1\gamma\mathcal{I}_{l+}^\alpha a(1) f_\infty,$$

$$(1-\xi_2)\mathcal{I}_{0+}^\beta b(1) g_0 < (1-\gamma)\gamma_2\mathcal{I}_{l+}^\beta b(1) g_\infty;$$

(H4)
$$(1-\xi_1)\mathcal{I}^{\alpha}_{0+}a(1)f_\infty < \gamma\gamma_1\mathcal{I}^{\alpha}_{l+}a(1)f_0,$$
$$(1-\xi_2)\mathcal{I}^{\beta}_{0+}b(1)g_\infty < (1-\gamma)\gamma_2\mathcal{I}^{\beta}_{0+}b(1)g_0;$$

(H5) there exist constants r, Λ, $\mathcal{N}$ satisfying
$$(1-\xi_1)\mathcal{N}\mathcal{I}^{\alpha}_{0+}a(1) < \gamma^2\Lambda\mathcal{I}^{\alpha}_{l+}a(1), \quad (1-\xi_2)\mathcal{N}\mathcal{I}^{\beta}_{0+}b(1) < \gamma(1-\gamma)\Lambda\mathcal{I}^{\beta}_{l+}b(1),$$
and such that
  (*i*) $f_0 = 0$, $g_0 = 0$, $f_\infty = 0$, $g_\infty = 0$;
  (*ii*) $f(u,v) \geq \Lambda r$, $g(u,v) \geq \Lambda r$, for $\|(u,v)\| \in [\gamma r, r]$;

(H6) there exist constants r, Λ, $\mathcal{N}$ satisfying
$$(1-\xi_1)\Lambda\mathcal{I}^{\alpha}_{0+}a(1) < \gamma^2\mathcal{N}\mathcal{I}^{\alpha}_{l+}a(1) \quad (1-\xi_2)\Lambda\mathcal{I}^{\beta}_{0+}b(1) < \gamma(1-\gamma)\mathcal{N}\mathcal{I}^{\beta}_{l+}b(1),$$
and such that
  (*i*) $f_0 = \infty$, $g_0 = \infty$, $f_\infty = \infty$, $g_\infty = \infty$;
  (*ii*) $f(u,v) \leq \Lambda r$, $g(u,v) \leq \Lambda r$, for $\|(u,v)\| \in [0, r]$.

Lemma 3.3. *Assume that (H1), (H2) hold. Then $\mathcal{A} : P \to P$ is completely continuous.*

Proof. First, we prove that $\mathcal{A}P \subset P$. For any $(t,s) \in [l,1]\times[0,1]$, by Lemma 3.2, we have
$$\begin{aligned}
&\min_{l\leq t\leq 1}\left(\mathcal{A}_\lambda(u,v)(t) + \mathcal{A}_\mu(u,v)(t)\right)\\
&= \min_{l\leq t\leq 1}\left(\lambda\int_0^1 \mathcal{H}_\alpha(t,s)a(s)f(u(s),v(s))ds, + \mu\int_0^1 \mathcal{H}_\beta(t,s)b(s)g(u(s),v(s))ds\right)\\
&\geq \lambda\gamma_1\int_0^1 \Phi_\alpha(s)a(s)f(u(s),v(s))ds + \mu\gamma_2\int_0^1 \Phi_\beta(s)b(s)g(u(s),v(s))ds\\
&= \max_{0\leq t\leq 1}\left(\gamma_1\mathcal{A}_\lambda(u,v)(t) + \gamma_2\mathcal{A}_\mu(u,v)(t)\right) = \gamma_1\|\mathcal{A}_\lambda(u,v)\| + \gamma_2\|\mathcal{A}_\mu(u,v)\|.
\end{aligned}$$
Hence $\min_{l\leq t\leq 1}\left(\mathcal{A}_\lambda(u,v)(t) + \mathcal{A}_\mu(u,v)(t)\right) \geq \gamma\|\mathcal{A}(u,v)\|$, and $\mathcal{A}P \subset P$.

Next we prove that $\mathcal{A}$ maps bounded sets into uniformly bounded sets. For fixed $M > 0$, consider a bounded subset $\mathcal{M}$ of P defined by $\mathcal{M} = \Big\{(u,v) \in P : \|(u,v)\| \leq M\Big\}$. Define
$$L_1 = \max\Big\{f(u(t),v(t)) : \|(u,v)\| \leq M\Big\}, \quad L_2 = \max\Big\{g(u(t),v(t)) : \|(u,v)\| \leq M\Big\}.$$
Then for $(u,v) \in \mathcal{M}$, employ $(H2)$ and Lemma 3.2 to obtain obtain
$$\begin{aligned}
|\mathcal{A}_\lambda(u,v)| &= \left|\lambda\int_0^1 \mathcal{H}_\alpha(t,s)a(s)f(u(s),v(s))ds\right|\\
&\leq \lambda L_1\int_0^1 \frac{(1-s)^{\alpha-1}a(s)}{\Gamma(\alpha)(1-\xi_1)}ds = \frac{L_1\lambda}{1-\xi_1}\mathcal{I}^{\alpha}_{0+}a(1) < +\infty.
\end{aligned}$$

Similarly, $|\mathcal{A}_\mu(u,v)| < +\infty$. Therefore $|\mathcal{A}_\lambda(u,v)| + |\mathcal{A}_\mu(u,v)| < +\infty$, which implies that $\mathcal{A}(\mathcal{M})$ is uniformly bounded.

Finally, we show that $\mathcal{A}(\mathcal{M})$ is equicontinuous. By $(P2)$ of Lemma 3.2, we have following estimate

$$\begin{aligned}\Big|\frac{d}{dt}\left(\mathcal{A}_\lambda(u,v)(t)\right)\Big| &\le \lambda \int_0^1 \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} a(s)|f(u(s),v(s))|ds\\ &\le L_1\lambda \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} a(s)ds = L_1\lambda \mathcal{I}_{0+}^{\alpha-1}a(1) < +\infty.\end{aligned}$$

Define $\delta = \left(L_1\lambda\mathcal{I}_{0+}^{\alpha-1}a(1) + L_2\mu\mathcal{I}_{0+}^{\beta-1}b(1)\right)^{-1}$, and choose $t_1, t_2 \in [0,1]$ such that $t_1 < t_2$ and $t_2 - t_1 < \delta$. Then for all $\varepsilon > 0$ and $(u,v) \in \mathcal{M}$, we obtain

$$\begin{aligned}\left|\mathcal{A}_\lambda(u,v)(t_2) - \mathcal{A}_\lambda(u,v)(t_1)\right| &= \Big|\int_{t_1}^{t_2} \frac{d}{ds}\left(\mathcal{A}_\lambda(u,v)(s)\right)ds\Big|\\ &\le L_1\lambda\mathcal{I}_{0+}^{\alpha-1}a(1)(t_2-t_1).\end{aligned}$$

Similarly,

$$\left|\mathcal{A}_\mu(u,v)(t_2) - \mathcal{A}_\mu(u,v)(t_1)\right| \le L_2\mu\mathcal{I}_{0+}^{\beta-1}b(1)(t_2-t_1).$$

Hence, it follows that

$$\left\|\mathcal{A}(u,v)(t_2) - \mathcal{A}(u,v)(t_1)\right\| \le \left(L_1\lambda\mathcal{I}_{0+}^{\alpha-1}a(1) + L_2\mu\mathcal{I}_{0+}^{\beta-1}b(1)\right)(t_2-t_1) < \varepsilon.$$

Therefore, by means of Arzela-Ascoli Theorem, $\mathcal{A} : P \to P$ is completely continuous. □

3.1. **Existence of at least one positive solution.**

Theorem 3.4. (i) *Assume that $(H1)$-$(H3)$ hold. Then for every*

$$\lambda \in \left(\frac{1-\xi_1}{\gamma_1\mathcal{I}_{l+}^{\alpha}a(1)f_\infty}, \frac{\gamma}{\mathcal{I}_{0+}^{\alpha}a(1)f_0}\right), \quad \mu \in \left(0, \frac{1-\gamma}{\mathcal{I}_{0+}^{\beta}b(1)g_0}\right),$$

or

$$\lambda \in \left(0, \frac{\gamma}{\mathcal{I}_{0+}^{\alpha}a(1)f_0}\right), \quad \mu \in \left(\frac{1-\xi_2}{\gamma_2\mathcal{I}_{l+}^{\beta}b(1)g_\infty}, \frac{1-\gamma}{\mathcal{I}_{0+}^{\beta}b(1)g_0}\right),$$

the boundary value problem (1.1) *has at least one positive solution.*

(ii) *Assume that $(H2),(H3)$ hold and assume $f_0 = 0$, $g_0 = 0$, $f_\infty = \infty$, $g_\infty = \infty$. Then for each $\lambda, \mu \in (0,\infty)$ the boundary value problem* (1.1) *has at least one positive solution.*

Proof. Assume that $(H2),(H3)$ hold and assume either $f_\infty < \infty$ and $g_\infty < \infty$ or $f_\infty = \infty$ and $g_\infty = \infty$. Choose $\varepsilon > 0$ such that

$$\frac{1-\xi_1}{\gamma_1\mathcal{I}_{l+}^{\alpha}a(1)(f_\infty-\varepsilon)} \le \lambda \le \frac{\gamma}{\mathcal{I}_{0+}^{\alpha}a(1)(f_0+\varepsilon)}, \quad 0 < \mu \le \frac{1-\gamma}{\mathcal{I}_{0+}^{\beta}b(1)(g_0+\varepsilon)}.$$

By the definition of f_0, g_0 , there exists $r > 0$ such that

$$f(u,v) \le (f_0+\varepsilon)(u+v), \;\; g(u,v) \le (g_0+\varepsilon)(u+v), \;\; \text{for } u+v \in [0,r].$$

Define $\Omega_r = \{(u,v) \in P : \|(u,v)\| \leq r\}$. For any $(u,v) \in P \cap \partial\Omega_r$, by Lemma 3.2, we have

$$\begin{aligned}\mathcal{A}_\lambda(u,v)(t) &= \lambda \int_0^1 H_\alpha(t,s)a(s)f(u(s),v(s))ds \\ &\leq \lambda \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} a(s)(f_0+\varepsilon)(u+v)ds \\ &\leq \lambda(f_0+\varepsilon)\mathcal{I}_{0+}^{\alpha}a(1)\|(u,v)\| \leq \gamma\|(u,v)\|,\end{aligned}$$

$$\begin{aligned}\mathcal{A}_\mu(u,v)(t) &= \mu \int_0^1 H_\beta(t,s)b(s)f(u(s),v(s))ds \\ &\leq \mu \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} b(s)(g_0+\varepsilon)(u+v)ds \\ &\leq \mu(g_0+\varepsilon)\mathcal{I}_{0+}^{\beta}b(1)\|(u,v)\| \leq (1-\gamma)\|(u,v)\|.\end{aligned}$$

Hence,

$$\begin{aligned}\|\mathcal{A}(u,v)\| &\leq \gamma\|(u,v)\| + (1-\gamma)\|(u,v)\| \\ &= \|(u,v)\|, \text{ for all } (u,v) \in P \cup \partial\Omega_r.\end{aligned} \tag{3.6}$$

Now we consider two different cases:

Case 1. Assume $f_\infty < \infty$ and $g_\infty < \infty$. Choose $\varepsilon_1 > 0$ such that $0 < \frac{1-\xi_1}{\gamma_1 \mathcal{I}_{l+}^{\alpha}a(1)(f_\infty-\varepsilon_1)} \leq \lambda$. By the definition of f_∞, g_∞ there exists a constant $r^* > r$, such that

$$f(u,v) \geq (f_\infty - \varepsilon_1)(u+v), \; g(u,v) \geq (g_\infty - \varepsilon_1)(u+v), \text{ for } u+v \geq \gamma r^*.$$

Define $\Omega_{r^*} = \{(u,v) \in P : \|(u,v)\| \leq r^*\}$. Then for $(u,v) \in P \cap \partial\Omega_{r^*}$, we have

$$u(t)+v(t) \geq \min_{l \leq t \leq 0}(u(t)+v(t)) \geq \gamma\|(u,v)\| = \gamma r^*.$$

Therefore, by Lemma 3.2, we have

$$\begin{aligned}\mathcal{A}_\lambda(u,v)(t) &= \lambda \int_0^1 H_\alpha(t,s)a(s)f(u(s),v(s))ds \\ &\geq \lambda\gamma_1 \int_l^1 \frac{(1-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)} a(s)(f_\infty-\varepsilon)(u+v)ds \\ &\geq \frac{\lambda\gamma_1}{1-\xi_1}\mathcal{I}_{l+}^{\alpha}a(1)(f_\infty-\varepsilon)\|(u,v)\| \geq \|(u,v)\|.\end{aligned}$$

Therefore, we have

$$\|\mathcal{A}(u,v)\| \geq \|\mathcal{A}_\lambda(u,v)\| \geq \|(u,v)\|, \text{ for all } (u,v) \in P \cap \partial\Omega_{r^*}. \tag{3.7}$$

Hence, by Theorem 2.7 and equations (3.6), (3.7), the operator $\mathcal{A}$ has a fixed point $(u,v) \in P \cap \bar{\Omega}_{r^*} \backslash \Omega_r$ such that $r \leq \|(u,v)\| \leq r^*$.

Case 2. Assume $f_\infty = \infty$ and $g_\infty = \infty$. Let $\lambda > 0$, Choose a constant $\mathcal{N} > 0$ such that $\mathcal{N} \geq \min\left\{\frac{1-\xi_1}{\lambda\gamma_1\mathcal{I}^{\alpha}_{l+}a(1)}, \frac{1-\xi_2}{\mu\gamma_2\mathcal{I}^{\beta}_{l+}b(1)}\right\}$. Since $f_\infty = \infty$, $g_\infty = \infty$, there exists $r^* > r$, such that

$$f(u,v) \geq \mathcal{N}(u+v), \text{ for } u+v \geq \gamma r^*.$$

For $(u,v) \in P \cap \partial\Omega_{r^*}$, we have $u(t)+v(t) \geq \min_{l\leq t\leq 0}(u(t)+v(t)) \geq \gamma\|(u,v)\| = \gamma r^*$. Hence,

$$f(u,v) \geq \mathcal{N}(u+v), \text{ for any } (u,v) \in P \cap \partial\Omega_{r^*}.$$

By Lemma 3.2, we have

$$\begin{aligned}\mathcal{A}_\lambda(u,v)(t) &= \lambda\int_0^1 H_\alpha(t,s)a(s)f(u(s),v(s))ds \\ &\geq \lambda\gamma_1\mathcal{N}\int_l^1 \frac{(1-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)}a(s)(u+v)ds \\ &\geq \frac{\lambda\gamma_1\mathcal{N}}{1-\xi_1}\mathcal{I}^{\alpha}_{l+}a(1)\|(u,v)\| \geq \|(u,v)\|.\end{aligned}$$

Thus, we conclude that

$$\|\mathcal{A}(u,v)\| \geq \|\mathcal{A}_\lambda(u,v)\| \geq \|(u,v)\|, \text{ for all } (u,v) \in P \cap \partial\Omega_{r^*}. \tag{3.8}$$

Hence, Theorem 2.7 and equations (3.6), (3.8), the operator $\mathcal{A}$ has a fixed point $(u,v) \in P \cap \bar{\Omega}_{r^*}\backslash\Omega_r$ such that $r \leq \|(u,v)\| \leq r^*$. □

Theorem 3.5. (i) *Assume that* $(H1)$, $(H2)$ *and* $(H4)$ *hold. Then for every*

$$\lambda \in \Big(\frac{1-\xi_1}{\gamma_1\mathcal{I}^{\alpha}_{l+}a(1)f_0}, \frac{\gamma}{\mathcal{I}^{\alpha}_{0+}a(1)f_\infty}\Big), \quad \mu \in \Big(0, \frac{1-\gamma}{\mathcal{I}^{\beta}_{0+}b(1)g_\infty}\Big),$$

or

$$\lambda \in \Big(\frac{\gamma}{\mathcal{I}^{\alpha}_{0+}a(1)f_\infty}\Big), \quad \mu \in \Big(\frac{1-\xi_2}{\gamma_2\mathcal{I}^{\beta}_{0+}b(1)g_0}, \frac{1-\gamma}{\mathcal{I}^{\beta}_{0+}b(1)g_\infty}\Big),$$

the boundary value problem (1.1) *has at least one positive solution.*

(ii) *Assume* $(H2), (H4)$ *hold and assume* $f_0 = \infty$, $g_0 = \infty$, $f_\infty = 0$, $g_\infty = 0$. *Then for each* $\lambda, \mu \in (0,\infty)$ *the boundary value problem* (1.1) *has at least one positive solution.*

Proof. The proof is similar to the Theorem 3.4, so we omit it. □

3.2. **Multiplicity Results.**

Theorem 3.6. *Assume that* $(H1)$, $(H2)$ *and* $(H5)$ *hold. Then for any*

$$\lambda \in \Big[\frac{1-\xi_1}{\gamma\Lambda\mathcal{I}^{\alpha}_{l+}a(1)}, \frac{\gamma}{\mathcal{N}\mathcal{I}^{\alpha}_{0+}a(1)}\Big], \quad \mu \in \Big(0, \frac{1-\gamma}{\mathcal{N}\mathcal{I}^{\beta}_{0+}b(1)}\Big],$$

or

$$\lambda \in \Big(0, \frac{\gamma}{\mathcal{N}\mathcal{I}^{\alpha}_{0+}a(1)}\Big], \quad \mu \in \Big[\frac{1-\xi_2}{\gamma\Lambda\mathcal{I}^{\beta}_{l+}b(1)}, \frac{1-\gamma}{\mathcal{N}\mathcal{I}^{\beta}_{0+}b(1)}\Big],$$

the boundary value problem (1.1) *has at least two positive solutions* (u_1, v_1), (u_2, v_2) *such that* $0 < \|(u_1, v_1)\| < r < \|(u_2, v_2)\|$.

Proof. Since $f_0 = 0$, $g_0 = 0$, there exists $r_1 \in (0, r)$ such that

$$f(u, v) \le \mathcal{N}(u + v), \ g(u, v) \le \mathcal{N}(u + v), \ \text{for } u + v \in (0, r_1).$$

Define $\Omega_{r_1} = \{(u, v) \in P : \|(u, v)\| < r_1\}$. For any $(u, v) \in P \cap \partial\Omega_{r_1}$, by Lemma 3.2 we have

$$\begin{aligned}\mathcal{A}_\lambda(u, v)(t) &= \lambda \int_0^1 H_\alpha(t, s)a(s)f(u(s), v(s))ds \\ &\le \lambda\mathcal{N} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} a(s)(u + v)ds \\ &\le \lambda\mathcal{N}\mathcal{I}^\alpha_{0+}a(1)\|(u, v)\| \le \theta\|(u, v)\|.\end{aligned}$$

$$\begin{aligned}\mathcal{A}_\mu(u, v)(t) &= \lambda \int_0^1 H_\beta(t, s)b(s)g(u(s), v(s))ds \\ &\le \mu\mathcal{N} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} b(s)(u + v)ds \\ &\le \mu\mathcal{N}\mathcal{I}^\beta_{0+}b(1)\|(u, v)\| \le (1-\theta)\|(u, v)\|,\end{aligned}$$

which implies that

$$\|\mathcal{A}(u, v)\| \le \|(u, v)\|, \ \text{for all } (u, v) \in P \cap \partial\Omega_{r_1}. \tag{3.9}$$

Since $f_\infty = 0$, $g_\infty = 0$, there exists $r_2 > r$ such that for some positive constant $\mathcal{N}$, we have

$$f(u, v) \le \mathcal{N}(u + v), \ g(u, v) \le \mathcal{N}(u + v), \ \text{for } u + v \ge r_2. \tag{3.10}$$

Set $\Omega_{r_2} = \{(u, v) \in P : \|(u, v)\| < r_2\}$. For any $(u, v) \in P \cap \partial\Omega_{r_2}$, by Lemma 3.2 we have

$$\|\mathcal{A}(u, v)\| \le \|(u, v)\|, \ \text{for all } (u, v) \in P \cap \partial\Omega_{r_2}. \tag{3.11}$$

Next define $\Omega_r = \{(u, v) \in P : \|(u, v)\| < r\}$. For any $(u, v) \in P \cap \partial\Omega_r$, by Lemma 3.2 we have

$$u(t) + v(t) \ge \|(u, v)\| \ge \gamma r, \ \text{for all } t \in [l, 1].$$

Therefore we have following estimate

$$\begin{aligned}\mathcal{A}_\lambda(u, v)(t) &= \lambda \int_0^1 H_\alpha(t, s)a(s)f(u(s), v(s))ds \\ &\ge \lambda\gamma \int_l^1 \frac{(1-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)} a(s)\Lambda r ds \\ &= \lambda\gamma\mathcal{I}^\alpha_l a(1)\Lambda r = r = \|(u, v)\|.\end{aligned}$$

Finally, we have

$$\|\mathcal{A}(u,v)\| \geq \|(u,v)\|, \text{ for all } (u,v) \in P \cap \partial\Omega_r. \tag{3.12}$$

Hence, by (3.9), (3.11), (3.12) and Theorem 2.7, it follows that the operator $\mathcal{A}$ has two fixed points $u_1 \in \bar{\Omega}_r \backslash \Omega_{r_1}$ and $u_2 \in \bar{\Omega}_{r_2} \backslash \Omega_r$. □

Theorem 3.7. *Assume that* $(H1)$, $(H2)$ *and* $(H6)$ *hold. Then for any*

$$\lambda \in \left[\frac{1-\xi_1}{\gamma \mathcal{N} \mathcal{I}_{l+}^{\alpha} a(1)}, \frac{\gamma}{\Lambda \mathcal{I}_{0+}^{\alpha} a(1)}\right], \quad \mu \in \left(\frac{1-\gamma}{\Lambda \mathcal{I}_{0+}^{\beta} b(1)}\right],$$

or

$$\lambda \in \left(0, \frac{\gamma}{\Lambda \mathcal{I}_{0+}^{\alpha} a(1)}\right], \quad \mu \in \left[\frac{1-\xi_2}{\gamma \mathcal{N} \mathcal{I}_{l+}^{\beta} b(1)}, \frac{1-\gamma}{\Lambda \mathcal{I}_{0+}^{\beta} b(1)}\right],$$

the boundary value problem (1.1) *has at least two positive solutions* (u_1, v_1), (u_2, v_2) *such that* $0 < \|(u_1, v_1)\| < r < \|(u_2, v_2)\|$.

Proof. The proof is similar to the Theorem 3.6, so we omit it. □

Example 3.8. Consider the system of fractional differential equations

$$\begin{cases} {}^c\mathcal{D}_{0+}^{\frac{5}{2}} u(t) = \lambda(\frac{243}{578} + e^{\pi t})\big(\pi - 6(u + v + \frac{\pi}{2})^{-\frac{3}{2}}\big), \\ {}^c\mathcal{D}_{0+}^{\frac{27}{10}} v(t) = \mu(\sin t + 256 e^{\pi t})\Big(\big(1 + \frac{23}{\sqrt{u+v}}\big)^{-\frac{5}{2}} + \frac{\pi}{257}\Big), \\ u'(0) = u''(0) = 0,\ u(1) = \frac{1}{2} u(\frac{2}{3}), \\ v'(0) = v''(0) = 0,\ v(1) = \frac{3}{4} u(\frac{3}{5}), \end{cases} \tag{3.13}$$

where $\alpha = \frac{5}{2}$, $\beta = \frac{27}{10}$, $a(t) = \frac{243}{578} + e^{\pi t}$, $b(t) = \sin t + 256 e^{\pi t}$, $f(u,v) = \pi - 6(u + v + \frac{\pi}{2})^{-\frac{3}{2}}$, $g(u,v) = \Big(1 + \frac{23}{\sqrt{u+v}}\Big)^{-\frac{5}{2}} + \frac{\pi}{257}$, $\xi_1 = \frac{1}{2}$, $\xi_2 = \frac{3}{4}$, $\eta_1 = \frac{2}{3}$, $\eta_2 = \frac{3}{5}$. Obviously $f_0 = \pi - 6(\frac{2}{\pi})^{\frac{3}{2}}$, $f_\infty = \pi$, $g_0 = \frac{\pi}{257}$ and $g_\infty = 1 + \frac{\pi}{257}$. Choose $l = \frac{1}{4}$. By direct calculations $\gamma_1 \approx 0.227834$, $\gamma_2 \approx 0.435284$, $\gamma \approx 0.227834$; $\mathcal{I}_{0+}^{\alpha} a(1) \approx 1.079424$, $\mathcal{I}_{l+}^{\alpha} a(1) \approx 0.786560$, $\mathcal{I}_{0+}^{\beta} b(1) \approx 182.201363$ and $\mathcal{I}_{l+}^{\beta} b(1) \approx 133.391627$.

Since, $(1 - \xi_1)\mathcal{I}_{0+}^{\alpha} a(1) f_0 \approx 0.050678 < \gamma_1 \gamma \mathcal{I}_{l+}^{\alpha} a(1) f_\infty \approx 0.128268$ and $(1 - \xi_2)\mathcal{I}_{0+}^{\beta} b(1) g_0 \approx 0.556812 < (1 - \gamma)\gamma_2 \mathcal{I}_{l+}^{\beta} b(1) g_\infty \approx 45.382788$. Therefore for each $\lambda \in (0.888114, 2.247865)$ and $\mu \in (0, 0.346692)$ or $\lambda \in (0, 2.247865)$ and $\mu \in (0.004254, 0.346692)$, by Theorem 3.4, the system of fractional differential equations (3.13) has at least one positive solution.

Example 3.9. Consider the system of fractional differential equations

$$\begin{cases} {}^c\mathcal{D}_{0+}^{\frac{14}{5}} u(t) = \frac{19^4 \lambda t^2}{1+t}\Big(\frac{12\pi}{u+v+12\pi} - e^{-\pi(u+v)}\Big), \\ {}^c\mathcal{D}_{0+}^{\frac{12}{5}} v(t) = \frac{78125 \mu t^2}{1+t^2}\Big(\frac{1}{\sqrt{u+v}} - e^{-\pi(u+v)}\Big), \\ u'(0) = u''(0) = 0,\ u(1) = \frac{37}{50} u(\frac{11}{25}), \\ v'(0) = v''(0) = 0,\ v(1) = \frac{11}{20} u(\frac{19}{50}), \end{cases} \tag{3.14}$$

where $\alpha = \frac{14}{5}$, $\beta = \frac{12}{5}$, $a(t) = \frac{t^2}{1+t}$, $b(t) = \frac{t^2}{1+t^2}$, $f(u,v) = 19^4\Big(\frac{12\pi}{u+v+12\pi} - e^{-\pi(u+v)}\Big)$, $g(u,v) = 78125\Big(\frac{12\pi}{u+v+12\pi} - e^{-\pi(u+v)}\Big)$, $\xi_1 = \frac{37}{50}$, $\xi_2 = \frac{11}{20}$, $\eta_1 = \frac{11}{25}$, $\eta_2 = \frac{19}{50}$. Obviously $f_0 = 0$, $f_\infty = 0$, $g_0 = 0$ and $g_\infty = 0$. Choose constants $l = \frac{1}{2}$, $r = 5$, $\mathcal{N} = 12$, $\Lambda = 2449$, then by computations

$$\mathcal{I}^{\alpha}_{0+}a(1) \approx 0.015652, \quad \mathcal{I}^{\alpha}_{l+}a(1) \approx 0.007547,$$

$$\mathcal{I}^{\beta}_{0+}b(1) \approx 0.034126, \quad \mathcal{I}^{\beta}_{l+}b(1) \approx 0.018676,$$

$$(1-\xi_1)\mathcal{N}\mathcal{I}^{\alpha}_{0+}a(1) \approx 0.326817 < \gamma^2\Lambda\mathcal{I}^{\alpha}_{l+}a(1) \approx 0.959355$$

and

$$(1-\xi_2)\mathcal{N}\mathcal{I}^{\beta}_{0+}b(1) \approx 0.184282 < \gamma(1-\gamma)\Lambda\mathcal{I}^{\beta}_{l+}b(1) \approx 8.046569.$$

Also $f(u,v) > 12245$, for $\|(u,v)\| \in (1.1397, 5)$ and $g(u,v) > 12245$, for $\|(u,v)\| \in (1.1397, 5)$. Therefore, for any $\lambda \in [0.061746, 1.213008]$ and
$\mu \in (0, 1.885560]$ or $\lambda \in (0, 1.213008]$ and $\mu \in [0.043183, 1.885560]$, by Theorem 3.6 the boundary value problem (3.14) has at least two positive solutions.

REFERENCES

[1] R. P. Agarwal, V. Lakshmikantham, J. J. Nieto, On the concept of solution for fractional differential equations with uncertainty, *Nonlinear Anal.*, 72:59–2862, 2010.

[2] B. Ahmad and J. J. Nieto, Existence of solutions for nonlocal boundary value problems of higher-order nonlinear fractional differential equations, *Abs. Appl. Anal.*, Volume 2009, Article ID 494720, 9 pages doi:10.1155/2009/494720.

[3] B. Ahmad and J. J. Nieto, Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions, *Comp. Math. Appl.*, 58:1838–1843, 2009.

[4] C. Bai and J. Fang, The existence of positive solution for singular coupled system of nonlinear fractional equations, *Appl. Math. Comput.* 150:611–621, 2004.

[5] Z. Bai and L. Haishen, Positive solutions for boundary-value problem of nonlinear fractional differential equation, *J. Math. Anal. Appl.* 311: 495–505, 2005.

[6] M. Belmekki, J. J. Nieto, and Rosana Rodriguez-Lopez, Existence of periodic solution for a nonlinear fractional differential equation, *Bound. Value Probl.* Volume 2009, Article ID 324561, 18 pages doi:10.1155/2009/324561.

[7] C. S. Goodrich, Existence of a positive solution to a class of fractional differential equations, *Appl. Math. Lett.* 2010. doi:10.1016/j.aml.2010.04.035.

[8] D. Jiang, C. Yuan, The positive properties of the Green function for Dirichlet-type boundary value problems of nonlinear fractional differential equations and its application, *Nonlinear Anal.*, 72:710–719, 2010.

[9] R. A. Khan and Mujeeb ur Rehman, Existence of multiple positive solutions for a general system of fractional differential equations, *Commun. Appl. Anal.*, 18: 25-35, 2011.

[10] A. A. Kilbas, H. M. Srivastava and J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier Science, Amsterdam, 2006.

[11] M. A. Krasnosel'skii, Positive Solutions of Operator Equations, *Noordhoff Gronigen, Netherland*, 1964.

[12] V. Lakshmikantham and A. S. Vatsala, Basic theory of fractional differential equations, *Nonlinear Anal.*, 69:2677–2682, 2010.

[13] V. Lakshmikantham, Theory of fractional functional differential equations, *Nonlinear Analysis*, 69:3337–3347, 2008.

[14] M. E. Shahed, Positive Solutions for Boundary Value Problem of Nonlinear Fractional Differential Equation, *Abst. Appl. Analy.*, 2007, Article ID 10368.

[15] X. Su, Boundary value problem for a coupled system of nonlinear fractional differential equations, *Appl. Math. Lett.*, 22:64–69, 2009.

[16] S. Liang, J. Zhang, Positive solutions for boundary value problems of nonlinear fractional differential equation, *Nonlinear Anal.*, 71:5545–5550, 2009.

[17] I. Podlubny, Fractional Differential Equations, Mathematics in Science and Engineering, Academic Press, New York, London, Toronto 1999.

[18] A. Yang and W. Ge, Positive solutions for boundary value problems of N-dimension nonlinear fractional differential system, *Bound. Value Probl.*, 2008, Article ID 437453.