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John M. Davis
Baylor University

Paul W. Eloe
University of Dayton, peloe1@udayton.edu

John R. Graef
University of Tennessee at Chattanooga

Johnny Henderson
Baylor University

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POSITIVE SOLUTIONS FOR A SINGULAR FOURTH ORDER NONLOCAL BOUNDARY VALUE PROBLEM

John M. Davis[1], Paul W. Eloe[2], John R. Graef[3], Johnny Henderson[4]

[1,4]Department of Mathematics
Baylor University
Waco, Texas, 76798-7328, USA

[2]Department of Mathematics
University of Dayton
300 College Park, Dayton, Ohio, 45469-2317, USA

[3]Department of Mathematics
The University of Tennessee at Chattanooga
Chattanooga, Tennessee, 37403, USA
e-mail: John-Graef@utc.edu

Abstract: Positive solutions are obtained for the fourth order nonlocal boundary value problem, $u^{(4)} = f(x,u)$, $0 < x < 1$, $u(0) = u''(0) = u'(1) = u''(1) - u''(2/3) = 0$, where $f(x,y)$ is singular at $x = 0, x = 1, y = 0$, and may be singular at $y = \infty$. The solutions are shown to exist at fixed points for an operator that is decreasing with respect to a cone.

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In Memory of Professor Drumi Bainov
July 2, 1933 – July 1, 2011.

1. Introduction

We obtain positive solutions to the singular fourth order nonlocal boundary value problem,

$$u^{(4)} = f(x, u), \quad 0 < x < 1, \tag{1}$$

$$u(0) = u''(0) = u'(1) = u''(1) - u''(2/3) = 0, \tag{2}$$

where $f(x, y)$ is singular at $x = 0$, $x = 1$, $y = 0$, and may be singular at $y = \infty$.

Throughout, we assume the following conditions on f:

(A1) $f(x, y) : (0, 1) \times (0, \infty) \to (0, \infty)$ is continuous, and $f(x, y)$ is decreasing in y, for every x.

(A2) $\lim_{y \to 0^+} f(x, y) = +\infty$ and $\lim_{y \to +\infty} f(x, y) = 0$ uniformly on compact subsets of $(0, 1)$.

Equation (1), which is often referred to as the *beam equation*, has been studied under a variety of boundary conditions. Physical interpretations of some of the boundary conditions for the linear beam equation can be found in Zill and Wright [40]. Contributions to the literature for the beam equation involving boundary conditions different from the boundary conditions (2) include the papers [14, 16, 17, 18, 26, 27, 35, 38]. The beam equation, with the nonlocal boundary conditions like (2) has been studied by Graef *et al.* [13, 15]. Some of the results from these latter two papers will play major roles in this work.

Singular boundary value problems for ordinary differential equations have used to model glacial advance and transport of coal slurries down conveyor belts as examples of non-Newtonian fluid theory in studies of pseudoplastic fluids [9], for problems involving draining flows [1, 5], and semipositone and positone problems [2], and as models in boundary layer applications, Emden-Fowler boundary value problems, and reaction-diffusion applications [6, 7, 8, 25].

There has been substantial theoretical interest in singular boundary value problems; we suggest the studies in [4, 21, 22, 31, 32, 33, 36, 37, 39]. In this work, we will convert the problem (1)–(2) into an integral equation problem, from which we define a sequence of decreasing integral operators associated with a sequence of perturbed integral equations. Applications of a Gatica, Oliker, and Waltman [12] fixed point theorem, for operators that are decreasing with respect to a cone, yield a sequence of fixed points of the integral operators. A solution of (1)–(2) is then obtained from a subsequence of the fixed points.

This method has been used to obtain positive solutions for other singular boundary value problems by DaCunha, Davis and Singh [10], Eloe and Henderson [11], Henderson *et al.* [23, 24], Maroun [29, 30] and Singh [34]. Important to our obtaining positive solutions of (1)–(2) are positivity results by Graef *et al.* in [13, 15].

2. Definitions, Cone Properties and the Gatica, Oliker and Waltman Fixed Point Theorem

In this section, we state some definitions and properties of Banach space cones, and we state the fixed point theorem on which the paper's main result depends.

Let $(B, ||\cdot||)$ be a real Banach space. A nonempty closed $K \subset B$ is called a *cone* if the following hold:

(i) $\alpha u + \beta v \in K$, for all $u, v \in K$, and for all $\alpha, \beta \in [0, \infty)$.

(ii) $K \cap (-K) = \{0\}$.

Given a cone K, a *partial order*, $\leq$, is induced on B by $x \leq y$, for $x, y \in B$ if, and only if, $y - x \in K$. (We sometimes will write $x \leq y$ (w.r.t.K).) If $x, y \in B$ with $x \leq y$, let $\langle x, y\rangle$ denote the *closed order interval between x and y* and be defined by, $\langle x, y\rangle := \{z \in B \mid x \leq z \leq y\}$. A cone K is *normal* in B provided there exists a $\delta > 0$ such that $||e_1 + e_2|| \geq \delta$, for all $e_1, e_2 \in K$ with $||e_1|| = ||e_2|| = 1$.

Remark 2.1. If K is a normal cone in B, then closed order intervals are norm bounded.

We now state the Gatica, Oliker, and Waltman [12] fixed point theorem on which the main result of this paper depends.

Theorem 2.2. *Let B be a Banach space, K a normal cone, J a subset of K such that, if $x, y \in J$, $x \leq y$, then $\langle x, y\rangle \subseteq J$, and let $T : J \to K$ be a continuous decreasing mapping which is compact on any closed order interval contained in J. Suppose there exists $x_0 \in J$ such that T^2x_0 is defined, and furthermore, Tx_0 and T^2x_0 are order comparable to x_0.*

Then T has a fixed point in J provided that, either

(I) $Tx_0 \le x_0$ *and* $T^2x_0 \le x_0$*, or* $Tx_0 \ge x_0$ *and* $T^2x_0 \ge x_0$*, or*

(II) The complete sequence of iterates $\{T^n x_0\}_{n=0}^{\infty}$ *is defined, and there exists* $y_0 \in J$ *such that* $y_0 \le T^n x_0$*, for every* n*.*

We shall also make extensive use of the following theorem due to Graef, Qian and Yang [15].

Theorem 2.3. *Let* $\beta(x) \in C^{(4)}[0,1]$*. If* $\beta(x)$ *satisfies the boundary conditions* (2) *and* $\beta^{(4)}(x) \ge 0$ *on* $[0,1]$*, then*

$$\max_{0\le s\le 1} \beta(s) = \beta(1) > \beta(x),\ 0 \le x < 1,$$

$$\beta'(x) > 0,\ 0 \le x < 1,$$

$$\beta''(x) < 0,\ 0 < x \le 1,$$

and

$$\left(\tfrac{3x-x^3}{2}\right)\beta(1) \le \beta(x),\ 0 \le x \le 1.$$

3. Properties of Positive Solutions

In preparing to apply Theorem 2.2, we define the Banach space $(B, ||\cdot||)$ by

$$B := \{u : [0,1] \to \mathbb{R} \mid u \text{ is continuous}\}, \quad ||u|| := \sup_{0\le x\le 1} |u(x)|.$$

And, we define a cone $K \subset B$ by

$$K := \{u \in B \mid u(x) \ge 0 \text{ on } [0,1]\}.$$

We observe that, if $y(x)$ is a positive solution of (1)–(2), then $y^{(4)}(x) \ge 0$, and moreover from Theorem 2.3,

$$y(x) \ge 0,\ y'(x) \ge 0,\ \text{and}\ y''(x) \le 0.$$

Next, we define $g(x) : [0,1] \to [0,1]$ by

$$g(x) := \frac{3x - x^3}{2},$$

and for $\theta > 0$, we define

$$g_\theta(x) := \theta g(x).$$

Notice that $g(x) > 0$, $g'(x) > 0$, $g''(x) < 0$ on $(0,1)$, and

$$\max_{0\le x\le 1} g(x) = 1 \text{ and } \max_{0\le x\le 1} g_\theta(x) = \theta.$$

We will assume hereafter:

(A3) $\int_0^1 f(x, g_\theta(x))dx < \infty$, for all $\theta > 0$.

It follows from Theorem 2.3 that, for each positive solution $u(x)$ of (1)–(2), there exists a $\theta > 0$ such that

$$g_\theta(x) \le u(x), \quad 0 \le x \le 1.$$

In particular, with $\theta = \sup_{0\le x\le 1} |u(x)| = u(1)$, then

$$u(x) \ge \left(\frac{3x - x^3}{2}\right)\theta = g_\theta(x), \quad 0 \le x \le 1.$$

Next, we define a subset $D \subset K$ by

$$D := \{v \in K \mid \text{there exists } \theta(v) > 0 \text{ such that } g_\theta(x) \le v(x),\ 0 \le x \le 1\}.$$

We observe that, for each $v \in D$ and $\frac{2}{3} \le x \le 1$,

$$v(x) \ge g_\theta(x) = \left(\frac{3x - x^3}{2}\right)\theta \ge \frac{23}{27}\theta, \tag{3}$$

and for each positive solution $u(x)$ of (1)–(2),

$$u(x) \ge g(x) \sup_{0\le x\le 1} |u(x)| \ge \frac{23}{27} \sup_{0\le x\le 1} |u(x)| = \frac{23}{27}u(1), \ \frac{2}{3} \le x \le 1. \tag{4}$$

There is a Green's function, $G(x, s)$, for $y^{(4)} = 0$ satisfying (2) which will play the role of a kernel for certain compact operators meeting the requirements of Theorem 2.2.

First, the Green's function $G_1(x, s)$ for

$$-y'' = 0, \ y(0) = y'(1) = 0,$$

is given by

$$G_1(x,s) = \begin{cases} x, & x \le s, \\ s, & s \le x, \end{cases}$$

and second, the Green's function $G_2(x,s)$ for

$$-y'' = 0,\ y(0) = y(1) - y(2/3) = 0,$$

is given by

$$G_2(x,s) = \begin{cases} x, & x \le s \le 2/3, \\ s, & s \le x \text{ and } s \le 2/3, \\ 3(1-s)x, & s \ge 2/3 \text{ and } x \le s, \\ s + (2-3s)x, & x \ge s \ge 2/3. \end{cases}$$

Both G_1 and G_2 are positive valued on $(0,1] \times (0,1)$.

It follows that $G(x,s) : [0,1] \times [0,1] \to [0,\infty)$ defined by

$$G(x,s) := \int_0^1 G_1(x,r)G_2(r,s)dr$$

is the Green's function for $y^{(4)} = 0$ and satisfying (2).

Remark 3.1. Graef, Kong, and Yang [13] by direct computation have also given the closed form expression

$$\begin{aligned} G(x,s) = \frac{x(1-s)}{2}\left(2 - x^2 + s\right) + \frac{(x-s)^3}{6} H(x-s) \\ + \frac{(2-3s)}{6}\left(x^3 - 3x\right) H\left(\frac{2}{3} - s\right), \end{aligned}$$

where $H(\cdot)$ denotes the Heaviside function.

Now we define an integral operator $T : D \to K$ by

$$(Tu)(x) := \int_0^1 G(x,s)f(s,u(s))ds, \quad u \in D.$$

We shall show that T is well-defined on D, is decreasing, and $T : D \to D$. To that end, let $v,\ u \in D$ be given, with $v(x) \le u(x)$. Then, there exists $\theta > 0$ such that $g_\theta(x) \le v(x)$. By Assumptions (A1) and (A3), and the positivity of G,

$$0 \le \int_0^1 G(x,s)f(x,u(x))dx \le \int_0^1 G(x,s)f(x,v(x))dx$$
$$\le \int_0^1 G(x,s)f(x,g_\theta(x))dx < \infty.$$

Therefore, T is well-defined on D and T is a decreasing operator.

Next, for $v \in D$, let $w(x) := (Tv)(x) = \int_0^1 G(x,s)f(s,v(s))ds \ge 0,\ 0 \le x \le 1$. From properties of the Green's functions, $w^{(4)}(x) = f(x,v(x)) > 0,\ 0 < x < 1$, and $w(0) = w''(0) = w'(1) = w''(1) - w''(2/3) = 0$, and so by Theorem 2.3, $w = Tv \in D$. So, we also have $T : D \to D$.

Remark 3.2. It is well-known that $Tu = u$ if, and only if, u is a solution of (1)–(2). Therefore, we seek solutions of (1)–(2) that belong to D. It follows from (4) and (5), in the context of our Banach space B, that for each positive solution $u(x)$ of (1)–(2),

$$u(x) \ge g(x)||u|| \ge \frac{23}{27}||u|| = \frac{23}{27}u(1),\ \frac{2}{3} \le x \le 1. \tag{5}$$

4. *A priori* Bounds on Norms of Solutions

In this section, we exhibit that solutions of (1)–(2) have positive *a priori* upper and lower bounds on their norms.

Lemma 4.1. *If f satisfies* (A1) – (A3), *then there exists an $S > 0$ such that $||u|| \le S$, for any solution u of* (1)–(2) *in D.*

Proof. Assume the conclusion is false. Then there exists a sequence $\{u_m\}_{m=1}^{\infty}$ of solutions of (1)–(2) in D such that $u_m(x) > 0$, for all $0 < x \le 1$, and

$$||u_m|| \le ||u_{m+1}|| \text{ and } \lim_{m\to\infty} ||u_m|| = \infty.$$

From (4) or (5),

$$u_m(x) \ge \frac{23}{27}||u_m|| = \frac{23}{27}u_m(1),\quad \frac{2}{3} \le x \le 1.$$

Therefore,

$$\lim_{m\to\infty} u_m(x) = \infty \text{ uniformly on } \left[\frac{2}{3}, 1\right].$$

Next, let $M > 0$ be defined by

$$M := \max\{G(x,s) \mid (x,s) \in [0,1]\times[0,1]\}.$$

(A2) implies there exists $m_0 \in \mathbb{N}$ such that, for each $m \geq m_0$ and $\frac{2}{3} \leq x \leq 1$,

$$f(x, u_m(x)) \leq \frac{3}{M}.$$

Let

$$\theta := u_{m_0}(1).$$

Then, for $m \geq m_0$,

$$u_m(x) \geq g_{||u_m||}(x) \geq g_{||u_{m_0}||}(x) = g_\theta(x),\ 0 \leq x \leq 1.$$

So, for $m \geq m_0$ and $0 \leq x \leq 1$, we have

$$\begin{aligned} u_m(x) &= Tu_m(x) \\ &= \int_0^1 G(x,s)f(s,u_m(s))ds \\ &= \int_0^{\frac{2}{3}} G(x,s)f(s,u_m(s))ds + \int_{\frac{2}{3}}^1 G(x,s)f(s,u_m(s))ds \\ &\leq \int_0^{\frac{2}{3}} G(x,s)f(s,u_m(s))ds + \int_{\frac{2}{3}}^1 M\cdot\frac{3}{M}ds \\ &\leq \int_0^{\frac{2}{3}} G(x,s)f(s,g_\theta(s))ds + 1 \\ &\leq M\int_0^1 f(s,g_\theta(s))ds + 1, \end{aligned}$$

which, in view of (A3), contradicts $\lim_{m\to\infty} ||u_m|| = \infty$. Therefore, there exists an $S > 0$ such that $||u|| \leq S$, for any solution $u \in D$ of (1)–(2). □

Following that, we now exhibit positive *a priori* lower bounds on the solution norms.

Lemma 4.2. *If f satisfies* (A1) – (A3), *then there exists an $R > 0$ such that $||u|| \geq R$, for any solution u of* (1)–(2) *in D.*

Proof. Again, we assume the conclusion to the lemma is false. Then, there exists a sequence $\{u_m\}_{m=1}^{\infty}$ of solutions of (1)–(2) in D such that $u_m(x) > 0$, for $0 < x \leq 1$, and

$$||u_m|| \geq ||u_{m+1}|| \text{ and } \lim_{m\to\infty} ||u_m|| = 0.$$

That is,

$$\lim_{m\to\infty} u_m(x) = 0 \text{ uniformly on } [0,1].$$

Now, define

$$\overline{m} := \min\left\{G(x,s) \mid (x,s) \in \left[\frac{2}{3},\frac{5}{6}\right] \times \left[\frac{2}{3},\frac{5}{6}\right]\right\} > 0.$$

(A2) implies $\lim_{y\to 0^+} f(x,y) = \infty$ uniformly on compact subsets of $(0,1)$, and so, there exists a $\delta > 0$ such that, for $\frac{2}{3} \leq x \leq \frac{5}{6}$ and $0 < y < \delta$,

$$f(x,y) > \frac{6}{\overline{m}}.$$

Also, there exists $m_0 \in \mathbb{N}$ such that, for $m \geq m_0$ and $0 < x < 1$,

$$0 < u_m(x) < \frac{\delta}{2}.$$

For $m \geq m_0$ and $\frac{2}{3} \leq x \leq \frac{5}{6}$, we have

$$\begin{aligned}
u_m(x) &= Tu_m(x) \\
&= \int_0^1 G(x,s)f(s,u_m(s))ds \\
&\geq \int_{\frac{2}{3}}^{\frac{7}{9}} G(x,s)f(s,u_m(s))ds \\
&\geq \overline{m}\int_{\frac{2}{3}}^{\frac{5}{6}} f(s,u_m(s))ds \\
&\geq \overline{m}\int_{\frac{2}{3}}^{\frac{5}{6}} f(s,\frac{\delta}{2})ds \\
&\geq \overline{m}\int_{\frac{2}{3}}^{\frac{5}{6}} \frac{6}{\overline{m}}ds \\
&= 1.
\end{aligned}$$

This contradicts $\lim_{m\to\infty} u_m(x) = 0$ uniformly on $[0,1]$. Therefore, there exists an $R > 0$ such that $R \le ||u||$ for any solution $u \in D$ of (1)–(2). □

In summary, there exist $0 < R < S$ such that, for each solution $u \in D$ of (1)–(2), we have

$$R \le ||u|| \le S.$$

5. Existence of Positive Solutions

In this section, we will construct a sequence of operators, $\{T_m\}_{m=1}^{\infty}$, each of which is defined on all of K. Applications of Theorem 2.2 yield that, for each $m \in \mathbb{N}$, T_m has a fixed point $\phi_m \in K$. Then, we will extract a subsequence from the fixed points $\{\phi_m\}_{m=1}^{\infty}$ that converges to a fixed point of the operator T.

Theorem 5.1. *If f satisfies* (A1) – (A3), *then* (1)–(2) *has at least one positive solution $u \in D$.*

Proof. For each $m \in \mathbb{N}$, we define a function $u_m(x)$ by

$$u_m(x) := T(m) := \int_0^1 G(x,s)f(s,m)ds, \quad 0 \le x \le 1.$$

Since f is decreasing with respect to its second component, we have

$$0 < u_{m+1}(x) < u_m(x), \text{ for } 0 < x < 1,$$

and by (A2), $\lim_{m\to\infty} u_m(x) = 0$ uniformly on $[0,1]$.

Next, we define $f_m(x,y) : (0,1) \times [0,\infty) \to (0,\infty)$ by

$$f_m(x,y) := f(x, \max\{y, u_m(x)\}).$$

Then, f_m is continuous and f_m does not have the singularity at $y = 0$ possessed by f. Moreover, for $(x,y) \in (0,1) \times (0,\infty)$,

$$f_m(x,y) \le f(x,y) \text{ and } f_m(x,y) \le f(x, u_m(x)).$$

Next, we define a sequence of operators, $T_m : K \to K$, for $\phi \in K$ and $0 \le x \le 1$, by

$$T_m\phi(x) := \int_0^1 G(x,s)f_m(s,\phi(s))ds.$$

Then standard arguments yield that each T_m is a compact operator on K. Furthermore,

$$\begin{aligned} T_m(0) &= \int_0^1 G(x,s) f_m(s,0)ds \\ &= \int_0^1 G(x,s) f(s, \max\{0, u_m(s)\})ds \\ &= \int_0^1 G(x,s) f(s, u_m(s))ds \\ &\geq 0, \end{aligned}$$

and

$$T_m^2(0) = T_m\left(\int_0^1 G(x,s) f_m(s,0)ds\right) \geq 0.$$

Theorem 2.2 implies, with $J = K$ and $x_0 = 0$, that T_m has a fixed point in K, for each m. That is, for each m, there exists $\phi_m \in K$ such that

$$T_m\phi_m(x) = \phi_m(x), \quad 0 \leq x \leq 1.$$

So, for each $m \geq 1$, ϕ_m satisfies the boundary conditions (2), and also,

$$\begin{aligned} T_m\phi_m(x) &= \int_0^1 G(x,s) f_m(s, \phi_m(s))ds \\ &\leq \int_0^1 G(x,s) f(s, u_m(s))ds \\ &= Tu_m(x). \end{aligned}$$

That is, for each $0 \leq x \leq 1$ and for each m,

$$\phi_m(x) = T_m\phi_m(x) \leq Tu_m(x).$$

Using arguments similar to those in the proofs of Lemmas 4.1 and 4.2, there exist $R > 0$ and $S > 0$ such that

$$R \leq ||\phi_m|| \leq S, \quad \text{for every } m.$$

Now, let $\theta := R$. Since ϕ_m belongs to K and is a fixed point of T_m, the conditions of Theorem 2.3 hold. So, for every m and $0 \leq x \leq 1$,

$$\phi_m(x) \geq g(x)||\phi_m|| \geq g(x) \cdot R = g_\theta(x).$$

So, the sequence $\{\phi_m\}_{m=1}^{\infty}$ is contained in the closed order interval $\langle g_\theta, S\rangle$, and therefore, the sequence is contained in D. Since T is a compact mapping, we may assume without loss of generality that $\lim_{m\to\infty} T\phi_m$ exists; let us call the limit ϕ^*.

To complete the proof, it suffices to show that

$$\lim_{m\to\infty} (T\phi_m(x) - \phi_m(x)) = 0$$

uniformly on $[0,1]$, from which it will follow that $\phi^* \in \langle g_\theta, S\rangle$.

In that direction, let $\epsilon > 0$ be given, and choose $0 < \delta < \frac{1}{2}$ such that

$$\int_0^\delta f(s, g_\theta(s))ds + \int_{1-\delta}^1 f(s, g_\theta(s))ds < \frac{\epsilon}{2M},$$

where as before $M := \max\{G(x,s) \mid (x,s) \in [0,1]\times[0,1]\}$. Then, there exists m_0 such that, for $m \geq m_0$ and for $\delta \leq x \leq 1-\delta$,

$$u_m(x) \leq g_\theta(x) \leq \phi_m(x).$$

So, for $m \geq m_0$ and for $\delta \leq x \leq 1-\delta$,

$$f_m(x, \phi_m(x)) = f(x, \max\{\phi_m(x), u_m(x)\}) = f(x, \phi_m(x)).$$

Then, for $m \geq m_0$ and $0 \leq x \leq 1$,

$$\begin{aligned}
|T\phi_m(x) - \phi_m(x)| &= |T\phi_m(x) - T_m\phi_m(x)| \\
&= \left|\int_0^1 G(x,s)[f(s,\phi_m(s)) - f_m(s,\phi_m(s))]ds\right| \\
&= \left|\int_0^\delta G(x,s)[f(s,\phi_m(s)) - f_m(s,\phi_m(s))]ds\right. \\
&\quad \left. + \int_{1-\delta}^1 G(x,s)[f(s,\phi_m(s)) - f_m(s,\phi_m(s))]ds\right| \\
&\leq M\int_0^\delta [f(s,\phi_m(s)) + f_m(s,\phi_m(s))]ds \\
&\quad + M\int_{1-\delta}^1 [f(s,\phi_m(s)) + f_m(s,\phi_m(s))]ds \\
&\leq M\int_0^\delta [f(s,\phi_m(s)) + f(s,\phi_m(s))]ds \\
&\quad + M\int_{1-\delta}^1 [f(s,\phi_m(s)) + f(s,\phi_m(s))]ds
\end{aligned}$$

$$\begin{aligned}
&= 2M\left[\int_0^{\delta} f(s,\phi_m(s))ds + \int_{1-\delta}^{1} f(s,\phi_m(s))ds\right]\\
&\leq 2M\left[\int_0^{\delta} f(s,g_\theta(s))ds + \int_{1-\delta}^{1} f(s,g_\theta(s))ds\right]\\
&< 2M\cdot\frac{\epsilon}{2M}\\
&= \epsilon.
\end{aligned}$$

So, for $m \geq m_0$,

$$||T\phi_m - \phi_m|| < \epsilon.$$

That is, $\lim_{m\to\infty}(T\phi_m(x)-\phi_m(x)) = 0$ uniformly on $[0,1]$. Hence, for $0 \leq x \leq 1$,

$$\begin{aligned}
T\phi^*(x) &= T(\lim_{m\to\infty} T\phi_m(x))\\
&= T(\lim_{m\to\infty} \phi_m(x))\\
&= \lim_{m\to\infty} T\phi_m(x)\\
&= \phi^*(x),
\end{aligned}$$

and ϕ^* is a desired positive solution of (1)–(2) belonging to D. $\square$

Example. Define $f(x,y) : (0,1)\times(0,\infty)\to(0,\infty)$ by

$$f(x,y) := \frac{1}{\sqrt[4]{x(1-x)y}}.$$

Clearly, Assumptions (A1) and (A2) are satisfied with respect to f. Next, let $\theta > 0$ be given. Then,

$$\begin{aligned}
\int_0^1 f(x,g_\theta(x))dx &= \int_0^1 \frac{1}{\sqrt[4]{x(1-x)g_\theta(x)}}dx\\
&\leq \sqrt[4]{\frac{2}{\theta}}\left[\int_0^{\frac{2}{3}} \frac{1}{\sqrt[4]{x^2}\sqrt[4]{(1-x)(3-x^2)}}dx\right.\\
&\qquad \left.+\int_{\frac{2}{3}}^{1} \frac{1}{\sqrt[4]{x^2(1-x)}}\sqrt[4]{\frac{1}{2}}dx\right]\\
&\leq \sqrt[4]{\frac{2}{\theta}}\left[\int_0^{\frac{2}{3}} x^{-\frac{1}{2}}\cdot\sqrt[4]{3}\cdot\sqrt[4]{\frac{9}{23}}dx + \int_{\frac{2}{3}}^{1}\sqrt{\frac{3}{2}}\sqrt[4]{\frac{1}{2}}\cdot(1-x)^{-\frac{1}{4}}dx\right]\\
&< \infty,
\end{aligned}$$

and so Assumption (A3) is also satisfied. By Theorem 5.1, the boundary value problem

$$u^{(4)} = \frac{1}{\sqrt[4]{x(1-x)u}}, \quad 0 < x < 1, \tag{6}$$

$$u(0) = u''(0) = u'(1) = u''(1) - u''(2/3) = 0, \tag{7}$$

has at least one positive solution.

6. Dependence on Higher Order Derivatives

The techniques of proof of Theorem 5.1 can be extended to a boundary value problem of the form

$$u^{(4)} = f(x, u, u', u''), \quad 0 < x < 1, \tag{8}$$

with boundary conditions (2) using methods developed by Henderson and Yin [24] if one extends Theorem 2.3 in the following way.

Theorem 6.1. *Let $\beta(x) \in C^{(4)}[0,1]$. If $\beta(x)$ satisfies the boundary conditions* (2) *and $\beta^{(4)}(x) \geq 0$ on $[0,1]$, and $\beta^{(4)}(x) > 0$ for some $x \in [0,1]$, then*

$$\max_{0 \leq s \leq 1} \beta(s) = \beta(1) > \beta(x),\ 0 \leq x < 1,$$

$$\beta'(x) > (-\beta''(1)) \left(\tfrac{1-x^2}{2}\right) > 0,\ 0 \leq x < 1,$$

$$\beta''(x) \leq \beta''(1)x < 0,\ 0 < x \leq 1,$$

and

$$\left(\tfrac{3x-x^3}{6}\right)(-\beta''(1)) \leq \beta(x),\ 0 \leq x \leq 1.$$

Proof. In the proof of Theorem 2.3, Graef, Qian and Yang [15] have shown

$$\beta''(x) < 0,\ 0 < x \leq 1.$$

Set $v(x) = \beta''(x) - \beta''(1)x$. Then v satisfies,

$$v''(x) \geq 0,\ 0 < x < 1,\ v''(x) > 0 \text{ for some } x \in (0,1),$$

and

$$v(0) = 0,\ v(1) = 0.$$

So,

$$v(x) < 0,\ 0 < x < 1,$$

or

$$\beta''(x) \leq \beta''(1)x < 0,\ 0 < x \leq 1. \tag{9}$$

The inequality in β' is obtained by integrating (9) from 1 to x and then the inequality in β is obtained by integrating again, now from 0 to x.

□

To state and prove a theorem analogous to Theorem 5.1 for the boundary value problem, (8), (2), conditions (A1) and (A2) are replaced by conditions:

(B1) $f(x,y) : (0,1) \times (0,\infty)^2 \times (-\infty,0) \to (0,\infty)$ is continuous,
$f(x,y_1,y_2,y_3)$ is decreasing in y_1, for every x, y_2, y_3,
$f(x,y_1,y_2,y_3)$ is decreasing in y_2, for every x, y_1, y_3,
$f(x,y_1,y_2,y_3)$ is increasing in y_3, for every x, y_1, y_2.

(B2) $\lim_{y_1\to 0^+} f(x,y_1,y_2,y_3) = +\infty$ and $\lim_{y_1\to+\infty} f(x,y_1,y_2,y_3) = 0$ uniformly on compact subsets of $(0,1)\times(0,\infty)\times(-\infty,0)$,
$\lim_{y_2\to 0^+} f(x,y_1,y_2,y_3) = +\infty$ and $\lim_{y_2\to+\infty} f(x,y_1 y_2,y_3) = 0$ uniformly on compact subsets of $(0,1)\times(0,\infty)\times(-\infty,0)$,
$\lim_{y_3\to 0^+} f(x,y_1,y_2,y_3) = -\infty$ and $\lim_{y_3\to-\infty} f(x,y_1 y_2,y_3) = 0$ uniformly on compact subsets of $(0,1)\times(0,\infty)^2$.

In this section, the definition of $g(x)$ is motivated by Theorem 6.1 and so for this section, define $g(x) : [0,1] \to [0,1]$ by

$$g(x) := \frac{3x - x^3}{6}.$$

Again for $\theta > 0$, define

$$g_\theta := \theta g(x).$$

Condition (A3) is replaced by the condition:

(B3) $\int_0^1 f(x, g_\theta(x),\ g_\theta'(x),\ g_\theta''(x))dx < \infty,$ for all $\theta > 0$.

The Banach space for this section is $C^2[0,1]$ equipped with the standard norm

$$||u|| := \max\{\sup_{0\leq x\leq 1}|u(x)|,\ \sup_{0\leq x\leq 1}|u'(x)|,\ \sup_{0\leq x\leq 1}|u''(x)|\}$$

and, we define a cone $K \subset B$ by

$$K := \{u \in B \mid u(x) \geq 0 \text{ on } [0,1],\ u'(x) \geq 0 \text{ on } [0,1],\ u''(x) \leq 0 \text{ on } [0,1]\}.$$

It follows from Theorem 6.1 that, for each solution $u(x) \in K$ of (8), (2), there exists a $\theta > 0$ such that

$$g_\theta(x) \leq u(x), \quad g'_\theta(x) \leq u'(x), \quad g''_\theta(x) \geq u''(x), \ \ 0 \leq x \leq 1. \tag{10}$$

In particular, with $\theta = |u''(1)|$, (10) is valid. So, define $D \subset K$ by

$$D := \{v \in K \mid \text{there exists } \theta(v) > 0 \text{ such that} \\ g_\theta(x) \leq v(x),\ g'_\theta(x) \leq v'(x),\ g''_\theta(x) \geq v''(x),\ 0 \leq x \leq 1\}$$

and define the fixed point operator $T : D \to K$

$$(Tu)(x) := \int_0^1 g(x,s) f(s, u(s), u'(s), u''(s))ds, \quad u \in D.$$

Then the proof of Theorem 5.1 can be adapted along the lines of the proof of Theorem 3.3 in [24] so that Theorem 2.3 applies to the boundary value problem (8), (2). The construction of the *a priori* bounds R and S begins by obtaining the bounds on $\sup_{0 \leq x \leq 1} |u''(x)|$ followed by successive integration, using the boundary conditions, to obtain the bounds on $||u||$.

Theorem 6.2. *If f satisfies* (B1) – (B3), *then* (8), (2) *has at least one solution $u \in D$.*

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