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On the decomposition of order-separable posets of countable width into chains

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Abstract. A partially ordered set X has countable width if and only if every collection of pairwise incomparable elements of X is countable. It is order-separable if and only if there is a countable subset D of X such that whenever $p, q \in X$ and $p < q$, there is $r \in D$ such that $p \leq r \leq q$. Can every order-separable poset of countable width be written as the union of a countable number of chains? We show that the answer to this question is "no" if there is a 2-entangled subset of $\mathbb{R}$, and "yes" under the Open Coloring Axiom.

Keywords: countable width, order-separable, chain, k-entangled subset, Open Coloring Axiom

Mathematics Subject Classification (1991): 06A06, 03E05

1. Introduction

The decomposition of partially ordered sets into chains has been a significant part of the study of the structure of partially ordered sets. The success in this area has come primarily using posets with the property that there is $n \in \omega$ such that for every antichain (by which we mean a set of incomparable elements) has cardinality $\leq n$. See, for example, these references: [1], and [3]–[14]. In [13], Peles constructed an example of a poset P such that every antichain of P is finite, but P is not the union of a countable number of chains. In [15] a problem was studied which required posets to be order-separable and have countable width. These posets seemed to be good candidates for decomposition into a countable number of chains. This led to the question which we will

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answer in this paper. That is, can every order-separable poset having countable width be written as the union of a countable number of chains? In Section 2, we use a type of subset of $\mathbb{R}$ called a 2-entangled set to show that under certain axioms, such as the continuum hypothesis (CH), there are order-separable posets of countable width that cannot be written as the union of a countable number of chains. On the other hand, we show in Section 3 that under the Open Coloring Axiom, every order-separable poset can indeed be written as the union of a countable number of chains.

Let us define some of the terminology that we have used, then consider a couple of related questions. The definition of entangled sets and the statement of the Open Coloring Axiom will be left to the appropriate sections.

Definition. A poset X is said to have *countable width* if and only if every antichain of X is countable.

Definition. A poset X is *order-separable* if and only if there is a countable $C \subseteq X$ such that for every $p, q \in X$ with $p < q$ there is $r \in C$ such that $p \leq r \leq q$.

Two questions related to the concepts investigated in this paper which were asked in [15] also have both positive and negative answers, depending on your set theory. A structure introduced in [15] is the collection of nonoverlapping subsets of X.

Definition. A collection $\mathcal{A}$ of subsets of a poset X is called a *collection of nonoverlapping subsets* of X if and only if $\mathcal{A}$ satisfies the following conditions.

1. $\mathcal{A}$ is a collection of pairwise disjoint sets, each having at least two elements.
2. The transitive closure of the relation

$$\left\{\langle A, B\rangle \in \mathcal{A}^2 : A \neq B \wedge \exists p \in A \exists q \in B (p < q)\right\}$$

   is a partial order.

We use $\nu(X)$ to represent the supremum of the cardinalities of collections of nonoverlapping subsets of X. Obviously, if $\nu(X) \leq \omega$ then X has countable width. To see why these kinds of collections are of any interest, we must make one more definition.

Definition. A poset $\langle X, < \rangle$ is *pliable* if and only if for every linear extension $\prec$ of $<$, there is a strictly $\prec$-increasing function $f : X \to I\!R$.

In [15] it was shown that X is pliable if and only if $\nu(X) \leq \omega$ and X is order-separable, and the following two questions appeared.

Question. If $\nu(X) \leq \omega$, can X necessarily be written as the union of a countable number of chains?

Question. If X is pliable, can X necessarily be written as the union of a countable number of chains?

It will be shown in Section 2 that the answer to both questions is "no" if there is a 4-entangled set. An immediate corollary to Theorem 2 in Section 3 is that the answer to the second question is "yes" under OCA. Since it was shown in [15] that Souslin's Hypothesis (SH) is equivalent to the statement that every poset X with $\nu(X) \leq \omega$ is pliable, it also follows that the answer to the first question is "yes" under OCA+SH (and so, in particular, under the Proper Forcing Axiom (PFA)).

2. Entangled Sets

How could one construct an order-separable poset of countable width that is not the union of a countable number of chains? One approach would be to find an uncountable poset in which all chains are countable. If one could then introduce a countable order-dense set, the resulting poset would have the desired properties. This is precisely what 2-entangled sets do. For $k \in \omega$, k-entangled sets were introduced by Shelah and were shown in [2] to follow from CH and to be consistent with $\mathtt{MA}_{\omega_1}$. He defined them as follows.

Definition. Let $k \in \omega$. An uncountable subset A of $I\!R$ is a *k-entangled* subset of $I\!R$ if and only if for every uncountable set $\mathcal{A}$ of k-tuples from A such that $\alpha(i) \neq \beta(j)$ if $\alpha, \beta \in \mathcal{A}$ and $\alpha \neq \beta$ or $i \neq j$, and every $\sigma : k \to 2$, there are $\alpha, \beta \in \mathcal{A}$ such that $\alpha(i) < \beta(i) \iff \sigma(i) = 1$

This means that for every uncountable collection of k-tuples from A having distinct coordinates and disjoint as unordered sets, there are pairs illustrating every possible ordering between the coordinates. An equivalent, and simpler, definition of 2-entangled sets is the following.

Definition. An uncountable subset A of $\mathbb{R}$ is a *2-entangled* subset of $\mathbb{R}$ if and only if there is no uncountable monotone function from a subset of A to A with no fixed points.

This means that if f is an uncountable function from a subset of A to A, and the graph of f is given the usual product ordering (i.e., $\langle a, b\rangle \leq \langle a', b'\rangle$ iff $a \leq a'$ and $b \leq b'$), then there are $p, q, r, s \in A$ such that $\langle p, f(p)\rangle$ and $\langle q, f(q)\rangle$ are comparable, while $\langle r, f(r)\rangle$ and $\langle s, f(s)\rangle$ are not. This gives us the desired properties that chains and antichains are countable. If we also require that f is one-to-one, then the addition of $\mathbb{Q} \times \mathbb{Q}$ will give us an order-separable poset.

THEOREM 1. *If there is a 2-entangled subset of $\mathbb{R}$, then there is an order-separable poset of countable width which is not the union of a countable number of chains.*

Proof. Let A be a 2-entangled subset of $\mathbb{R}$ and let $f : A \to A$ be a one-to-one function with no fixed points. Set $X = f \cup (\mathbb{Q}^2)$ with the usual product order. As was noted above, every chain and antichain of X is countable, so X has countable width and is not the union of a countable number of chains.

Let $\langle p, q\rangle$ and $\langle r, s\rangle$ be elements of X with $\langle p, q\rangle < \langle r, s\rangle$. Assume that $\langle p, q\rangle \notin \mathbb{Q}^2$ and $\langle r, s\rangle \notin \mathbb{Q}^2$. Then $\langle p, q\rangle, \langle r, s\rangle \in f$ and, since f is a one-to-one function, $p = r$ if and only if $q = s$. Therefore $p < r$ and $q < s$. There are $a, b \in \mathbb{Q}$ such that $p < a < r$ and $q < b < s$. So $\langle p, q\rangle < \langle a, b\rangle < \langle r, s\rangle$, and $\mathbb{Q}^2$ is order-dense in X.

To obtain negative answers to the questions at the end of the previous section, we need 4-entangled sets.

THEOREM 2. *If there is a 4-entangled subset of $\mathbb{R}$, then there is an order-separable poset X with $\nu(X) \leq \omega$ which is not the union of a countable number of chains.*

Proof. Let X be as in the previous example, but with A 4-entangled instead of 2-entangled. As before, X is order-separable and not a countable union of chains.

It remains to prove that $\nu(X) \leq \omega$. Suppose on the contrary that $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ is an uncountable collection of nonoverlapping subsets of X. Then, by definition, each A_α has at least two points, say p_α and q_α. Applying the definition of 4-entangled to the 4-tuples p_α followed by q_α, we see that there are $\alpha, \beta < \omega_1$ such that $p_\alpha < p_\beta$ and $q_\alpha > q_\beta$. But then the transitive closure of the relation given in part 2 of the definition of a nonoverlapping collection is not antisymmetric, hence not a partial order. That completes the proof.

3. Open Coloring Axiom

If there is to be a model of set theory in which order-separable posets of countable width will be the union of a countable number of chains, we will have to use something that kills 2-entangled sets. One of the things that does this is the Open Coloring Axiom (OCA). The statement of the axiom, in fact, indicates a strong connection with the problem at hand. But the axiom applies to subsets of $\mathbb{R}$. Can we extend it to general posets? Here is a statement of OCA.

[OCA] *If* $[X]^2 = K_0 \cup K_1$ *is a given partition where* $X \subseteq \mathbb{R}$ *and where* K_0 *is open in* $[X]^2$*, then either there is an uncountable 0-homogeneous set, or else* X *is the union of countably many 1-homogeneous sets.*

Here $[X]^2$ is the set of all subsets of X having exactly two elements, and it is identified with $\{\langle x, y\rangle : x < y \text{ in } X\}$ where the topology is inherited from $\mathbb{R}^2$. A subset Y of X is 0-homogeneous if $[Y]^2 \subseteq K_0$ and is 1-homogeneous is $[Y]^2 \subseteq K_1$. OCA is a known consequence of the Proper Forcing Axiom (PFA). See [16] for more information.

How does OCA help? Suppose that K_0 consists of pairs of incomparable elements. Then a set that is 0-homogeneous must be an antichain. If there can't be an uncountable antichain, then X is a countable union of the other kinds of sets, chains. To use this, we must somehow represent a poset as a subset of $\mathbb{R}$. It is well known that every separable zero-dimensional metric space is, in fact, a subset of $\mathbb{R}$. So the thing to do is to show that order-separable posets of countable width have a suitable separable zero-dimensional metric topology so that, when embedded in $\mathbb{R}$, the set K_0 is really open in $\mathbb{R}^2$. This is not obvious, since the orders of the poset X and the order inherited from $\mathbb{R}$ need not have a lot in common.

In the following theorem, we will use $\downarrow p$ to represent the set of all $q \in X$ such that $q \leq p$ and $\uparrow p$ to represent the set of all $q \in X$ such that $p \leq q$. We will also use $p \| q$ to represent the fact that p and q are incomparable.

THEOREM 3. (OCA). *Every order-separable poset of countable width is the union of a countable number of chains.*

Proof. Let X be an order-separable poset of countable width, and let D be a countable order-dense subset of X. We are going to use D to define a suitable 0-dimensional separable metric topology on X. But in order for this to work, we first need to enlarge D and thin out X.

Define an equivalence relation $\sim$ on X by setting $p \sim q$ if and only if for every $r \in D$, $r \in \downarrow p \Leftrightarrow r \in \downarrow q$ and $r \in \uparrow p \Leftrightarrow r \in \uparrow q$. Then every equivalence class $[p]$ is an antichain and is therefore countable. We may thus assume that $[p] = \{p\}$ for every $p \in X$.

For every $p \in X$ let

$$A_p^+(D) = \{q \in X : q \,\|\, p \text{ and } \forall r \in D (r \geq q \Rightarrow r \geq p)\}$$

and let

$$A_p^-(D) = \{q \in X : q \,\|\, p \text{ and } \forall r \in D (r \leq q \Rightarrow r \leq p)\}.$$

Note that $A_p^+(D) \cap D = \emptyset$ and $A_p^-(D) \cap D = \emptyset$ for every $p \in X$.

Claim 1. For every $p \in X$, $|A_p^+(D)| \leq \omega$ and $|A_p^-(D)| \leq \omega$.

We will show that $A_p^+(D)$ is, in fact, an antichain, and thus it must be countable. If there are $q, r \in A_p^+(D)$ such that $q < r$ then there is $s \in D$ such that $q \leq s \leq r$. But $q \leq s$ implies that $p \leq s$, so $p \leq r$, a contradiction. A similar argument shows that $|A_p^-(D)| \leq \omega$.

Let $D_0 = D$, and for $n < \omega$, let $D_{n+1} = D_n \cup [\bigcup_{p \in D_n} (A_p^+(D) \cup A_p^-(D))$. Let $D_\omega = \bigcup_{n \in \omega} D_n$. Then $A_p^+(D_\omega) \subset A_p^+(D) \subset D_\omega$ for every $p \in D_\omega$, and since $A_p^+(D_\omega) \cap D_\omega = \emptyset$, it follows that $A_p^+(D_\omega) = \emptyset$ (and similarly $A_p^-(D_\omega) = \emptyset$) for every $p \in D_\omega$.

We will henceforth assume, then, without loss of generality, that D itself has the property that $A_p^+(D) = A_p^-(D) = \emptyset$ for every $p \in D$.

Now for every $p \in X$ let

$$B_p^+ = \{q \in X : q \,\|\, p \text{ and } \forall r \in D (r \geq p \Rightarrow r \geq q)\}$$

and let

$$B_p^- = \{q \in X : q \,\|\, p \text{ and } \forall r \in D (r \leq p \Rightarrow r \leq q)\}$$

Claim 2. For every $p \in X$, $|B_p^+| \leq \omega$ and $|B_p^-| \leq \omega$.

We will again show that these sets are antichains. If there are q, $s \in B_p^+$ such that $q < s$, then there is $t \in D$ such that $q \leq t \leq s$. If $r \in D$ and $r \geq p$, then $r \geq s \geq t$. Also, $p \,\|\, t$, so $p \in A_t^+(D)$; but $A_t^+(D) = \emptyset$ since $t \in D$, a contradiction. The proof that $|B_p^-| \leq \omega$ is similar.

Now we can write X as a union of countable sets M_α, $\alpha < \kappa$, such that $p \in M_\alpha$ implies that $B_p^- \cup B_p^+ \subset M_\alpha$. Let $M'_\alpha = M_\alpha \setminus \bigcup_{\beta < \alpha} M_\beta$. Without loss of generality, each $M'_\alpha \neq \emptyset$. Note that X is the union of countable many sets Y such that $|Y \cap M'_\alpha| = 1$ for each $\alpha < \kappa$. Thus it will suffice to prove that if $Y = \{p_\alpha : \alpha < \kappa\}$, where $p_\alpha \in M'_\alpha$, then Y is a countable union of chains, and that is what we now do.

It is not hard to show that the topology on X generated by the subbase consisting of all $\downarrow p$ and $\uparrow p$ for $p \in D$ and their complements is a zero-dimensional topology with a countable base. It is also T_1 under the assumption that $[p] = \{p\}$ for all $p \in X$. Therefore X is metrizable and is homeomorphic to a subset of $\mathbb{R}$. Let Y be as above, and give Y the subspace topology; this is the space to which we apply OCA.

Let $K_0 = \big\{\{p_\alpha, p_\beta\} \in [Y]^2 : p_\alpha \| p_\beta\big\}$. We'll be done if we show that K_0 is open, for then OCA implies that Y is a countable union of chains. To this end, let $\{p_\alpha, p_\beta\} \in K_0$. We may assume that $\alpha \in \beta$. Then $p_\beta \notin B^-_{p_\alpha} \cup B^+_{p_\alpha}$, so there are q, $r \in D$ such that $q \le p_\alpha$, $q \not\le p_\beta$, $p_\alpha \le r$, and $p_\beta \not\le r$. Since $q \| p_\beta$ and $p_\beta \notin A^+_q(D)$, there is $s \in D$ such that $s \ge p_\beta$ and $s \not\ge q$. Similarly, since $p_\beta \notin A^-_r(D)$, there is $t \in D$ such that $t \le p_\beta$ and $t \not\le r$. Let $\langle x, y\rangle \in (\uparrow q \cap \downarrow r) \times (\uparrow t \cap \downarrow s)$. If $x \le y$ then $q \le x \le y \le s$, a contradiction. If $y \le x$ then $t \le y \le x \le r$, another contradiction. Therefore, $[Y]^2 \cap [(\uparrow q \cap \downarrow r) \times (\uparrow t \cap \downarrow s)]$ is a neighborhood of $\{p_\alpha, p_\beta\}$ that is contained in K_0, so K_0 is open. That completes the proof.

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