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LINEARLY ORDERED TOPOLOGICAL SPACES AND WEAK DOMAIN REPRESENTABILITY

JOE MASHBURN

ABSTRACT. It is well known that domain representable spaces, that is topological spaces that are homeomorphic to the space of maximal elements of some domain, must be Baire. In this paper it is shown that every linearly ordered topological space (LOTS) is homeomorphic to an open dense subset of a weak domain representable space. This means that weak domain representable spaces need not be Baire.

1. INTRODUCTION

For some time domains have been useful in the study and modeling of information systems. See [1], [6], and [17]. They have more recently become interesting to topologists because of a connection between domains and certain topological spaces. A topological space T is said to be domain representable if there is a domain X such that the set $\max X$ of maximal elements of X under the relative Scott topology is homeomorphic to T. It is well known that all domain representable spaces must be Baire so that, in particular, the set of rational numbers is not domain representable, even though the set of real numbers is. See [9] for a survey of basic results on domain representability.

In 2002 Coecke and Martin [6] created an ordered set to model finite dimensional quantum states. Their goal was to create a domain, which would enable them to treat quantum space as an information system. But their model failed to satisfy a basic property of domains: the ordered set was not continuous under the way below relation. The set did behave enough like a domain to justify creating a new relation, here called the weakly way below relation, under which the model had properties very similar to those of a domain. This new relation generates a topology which can differ from the Scott topology usually assigned to domains.

We can use the weakly way below relation to define a new structure, called a weak domain. If X is a weak domain, $a \in X$, and $b \in \max X$, then a is weakly way below b if and only if a is way below b. For this reason, one might expect that the class of topological spaces which are weak domain representable is the same as the class of topological spaces which are domain representable. That turns out not to be the case. We will prove the following theorem.

Theorem. *If T is a linearly ordered topological space then T is homeomorphic to an open dense subset of a weak domain representable space.*

This generalizes the example in [14] in which the set of rational numbers was shown to be an open dense subset of a weak domain representable space. Therefore weak domain representable spaces need not be Baire.

In Section 2 we will give the relevant definitions and compare domains and weak domains. In Section 3 we will prove the main theorem.

2. Definitions

The definitions of the concepts needed for weak domains will be given simultaneously with those needed for domains so that the two can be easily compared. For a more detailed comparison of the two structures and the topologies that they generate see [14]. Throughout this section X is a partially ordered set with order $<$.

Definition 1. *A subset D of X is* directed *if and only if for every $a, b \in D$ there is $c \in D$ such that $a \leq c$ and $b \leq c$.*

Definition 2. *X is* directed complete *(or a* dcpo*) if and only if every nonempty directed subset of X has a supremum.*

We will require weak domains to be directed complete. It is this property which will lead to the addition of extra points to a LOTS to obtain a weak domain representable space rather than the LOTS itself being weak domain representable.

As is customary we will use $\uparrow_X a$ to represent the set $\{b \in X : a \leq b\}$ and $\downarrow_X a$ to represent the set $\{b \in X : b \leq a\}$. In this section, where only one general poset X is used, we will drop the subscript from the arrows. This notation can also be applied to subsets of X, so that if $A \subseteq X$ then $\downarrow_X A = \{p \in x : \exists q \in A(p \leq q)\}$ and $\uparrow_X A = \{q \in X : \exists p \in A(p \leq q)\}$. In Section 3 we will consider closed bounded subsets of a LOTS L both as subsets of L and as elements of new poset $\mathcal{C}$. Then the subscripts on the arrows are needed to show whether the down- and up-sets are being generated in L or in $\mathcal{C}$.

Definition 3. *For all $a, b \in X$, a is* way below *b, denoted $a \ll b$, if and only if for every directed subset D of X if $\sup D \geq b$ then $D \cap \uparrow a \neq \emptyset$.*

Definition 4. *For all $a, b \in X$, a is* weakly way below *b, denoted $a \ll_w b$, if and only if for every directed subset D of X if $\sup D = b$ then $D \cap \uparrow a \neq \emptyset$.*

We will use the standard notation $\Uparrow a = \{b \in X : a \ll b\}$ and $\Downarrow a = \{b \in X : b \ll a\}$. Similarly, $\Uparrow_w a = \{b \in X : a \ll_w b\}$ and $\Downarrow_w a = \{b \in X : b \ll_w a\}$.

Definition 5. *X is* continuous *if and only if for every $a \in X$, $\Downarrow a$ is directed and $\sup \Downarrow a = a$.*

Definition 6. *X is* exact *if and only if for every $a \in X$, $\Downarrow_w a$ is directed and $\sup \Downarrow_w a = a$.*

The way below relation is increasing. That is, if $a \ll b \leq c$ then $a \ll c$. But the weakly way below relation need not be. In fact, if $\ll_w$ is increasing then it is the same as $\ll$. That $\ll$ is increasing contributes towards showing that it is also interpolative. That is, if $a \ll c$ then there is $b \in X$ such that $a \ll b \ll c$. Because $\ll_w$ is not increasing, it need not be interpolative. Now interpolation is an important property. Among other things, it helps to guarantee the

existence of a topology based on the relation. To make sure that $\ll_w$ is interpolative we want it to be close to increasing, without actually being increasing. So we will require weak domains to satisfy a property which we will call weakly increasing.

Definition 7. *A relation $\sim$ is* weakly increasing *on X if and only if for every $a, b, c \in X$, if $a \sim b \leq c$ and there is $d \in X$ such that $c \sim d$ then $a \sim c$.*

So $\ll_w$ is weakly increasing if and only if $a \ll_w c$ whenever $a \ll_w b \leq c$ and c is not maximal with respect to $\ll_w$. In [6] Coecke and Martin speak of this property as insuring that a and c have the same *context* in order to compare them with the weakly way below relation. If $\ll_w$ is weakly increasing on X then $\ll_w$ is interpolative on X. See Example 3.7 of [14] for a partially ordered set in which the weakly way below relation is not weakly increasing, but which is interpolative. Example 2.2 of the same paper is a poset in which $\ll_w$ is not interpolative. Obviously $\ll_w$ is not weakly increasing in this poset.

Definition 8. *A* weak domain *is an exact dcpo on which $\ll_w$ is weakly increasing.*

Compare these conditions to those of a domain, which is a continuous directed complete ordered set.

Definition 9. *A topological space T is* domain representable *if and only if there is a domain X such that T is homeomorphic to $\max X$ in the relative Scott topology.*

In a domain the set $\{\Uparrow a : a \in X\}$ is a basis for the Scott topology. This is not necessarily the case in a weak domain. That is, $\{\Uparrow_w a : a \in X\}$ generates a topology, but it might not be the same as the Scott topology. In fact, if $\Uparrow_w a$ is Scott open for every $a \in X$ then $\ll_w$ is increasing and $\ll_w = \ll$. The topology generated by $\{\Uparrow_w a : a \in X\}$ is called the *weakly way below topology* or the *wwb-topology*. We will use this new topology when discussing weak domain representability.

Definition 10. *A topological space T is* weak domain representable *if and only if there is a weak domain X such that T is homeomorphic to $\max X$ in the relative wwb-topology.*

Even when the topology generated by $\ll_w$ is not the Scott topology, it seems as though they should give the same topology to $\max X$. For if $a \in X$, $b \in \max X$, then $a \ll_w b$ if and only if $a \ll b$. However, we will see that the topology given to $\max X$ by the Scott topology can be different from that given to $\max X$ by the wwb-topology. This is also shown by Theorem 4.2 of [14]. The author will take this opportunity to correct an error in this theorem. The theorem *should* state that if T is a T_1 first countable topological space then there is a weak domain X such that $\max X$, in its relative Scott topology, is homeomorphic to T. The separation property was omitted in the statement of the theorem in [14]. Note that the topology used is the Scott topology, not the wwb-topology. There is a weak domain X such that $\max X$ is the real line in its usual topology when the Scott topology is used. But $\max X$ will be discrete under the wwb-topology. See Example 26 below.

3. Proof of the Main Theorem

Let L be a linearly ordered topological space (LOTS). A standard approach when showing that certain LOTS's, such as the real line, are domain representable is to use the set of closed bounded intervals as the elements of the domain. These are ordered by the superset relation ($A \leq B$ if and only if $A \supseteq B$) and the singletons form the set of maximal elements. In this case, the supremum of a directed set of such intervals is its intersection. But in the general case, when L has gaps, the intersection of a directed set can be empty, and the directed set won't have a supremum. To solve this problem, we will introduce new elements which will fill these gaps. We will use cuts (essentially Dedekind cuts) for this.

Definition 11. *A* cut *of L is a pair $\langle P, Q \rangle$ of nonempty convex subsets of L such that $P \cup Q = L$, P has no greatest element, Q has no least element, and every element of P is strictly less than every element of Q.*

The last condition ensures that P and Q are disjoint. If $L = \mathbb{R}^* = \mathbb{R} - \{0\}$ then $\langle \mathbb{R}^-, \mathbb{R}^+ \rangle$ is a cut of $\mathbb{R}^*$, where $\mathbb{R}^-$ is the set of negative real numbers and $\mathbb{R}^+$ is the set of positive real numbers. This example will be used to illustrate some of the concepts and definitions as we proceed.

Let K be the set of all cuts of L. We will next identify those directed sets of convex subsets of L which will "converge" to a cut.

Let $\mathcal{C}$ be the set of all nonempty bounded convex closed subsets of L, ordered by the superset relation. When we say that C is bounded we mean that there are $a, b \in L$ such that $a \leq x \leq b$ for all $x \in C$. Let $\mathbf{Dir}$ be the collection of all nonempty subsets of $\mathcal{C}$ which are directed. That is, if $\mathcal{E} \in \mathbf{Dir}$ and if $A, B \in \mathcal{E}$ then there is $C \in \mathcal{E}$ such that $C \subseteq A$ and $C \subseteq B$.

Definition 12. *For every $\mathcal{E} \in \mathbf{Dir}$ let $i(\mathcal{E}) = \cap\{\downarrow_L A : A \in \mathcal{E}\}$ and $f(\mathcal{E}) = \cap\{\uparrow_L A : A \in \mathcal{E}\}$.*

Here $\downarrow_L A$ and $\uparrow_L A$ are taken with respect to L, so $\downarrow_L A = \{x \in L : \exists y \in A(x \leq y)\}$ and $\uparrow_L A = \{y \in L : \exists x \in A(x \leq y)\}$. Thus $i(\mathcal{E})$ is the initial or left-most part of L determined from $\mathcal{E}$ by taking those elements of L which lie to the left of at least one element of A for every $A \in \mathcal{E}$, and $f(\mathcal{E})$ is the final or right-most part of L determined by taking those elements of L which lie to the right of at least one element of A for every $A \in \mathcal{E}$. If $\mathcal{E} = \{(0, 2^{-n}] : n \in \omega\}$ in $\mathbb{R}^*$ then $i(\mathcal{E}) = \mathbb{R}^-$ and $f(\mathcal{E}) = \mathbb{R}^+$.

Lemma 13. *If $\mathcal{E} \in \mathbf{Dir}$ and $\cap\mathcal{E} = \emptyset$ then $\langle i(\mathcal{E}), f(\mathcal{E}) \rangle \in K$.*

Proof. That $i(\mathcal{E})$ and $f(\mathcal{E})$ are both convex follows directly from the definitions. Suppose that there are $b \in i(\mathcal{E})$ and $a \in f(\mathcal{E})$ such that $a \leq b$. Let $A \in \mathcal{E}$. There are $p, q \in A$ such that $p \leq a \leq b \leq q$. Since A is convex, we have $a, b \in A$. Therefore $a, b \in \cap\mathcal{E}$, a contradiction. Thus every element of $i(\mathcal{E})$ is strictly less than every element of $f(\mathcal{E})$.

Let $A \in \mathcal{E}$. Since A is bounded there are $a, b \in L$ such that $A \subseteq [a, b]$. If $B \in \mathcal{E}$ then there is $C \in \mathcal{E}$ such that $C \subseteq A$ and $C \subseteq B$. Since $C \subseteq A$ we have $C \subseteq [a, b]$. Since $C \subseteq B$ we know that $a \in\downarrow_L B$ and $b \in\uparrow_L B$. Therefore $i(\mathcal{E})$ and $f(\mathcal{E})$ are not empty.

Let $x \in L$. There is $A \in \mathcal{E}$ such that $x \notin A$. Either $x \in\downarrow_L A$ or $x \in\uparrow_L A$. Assume that $x \in\downarrow_L A$. Then $x < y$ for all $y \in A$ because A is convex. If $B \in \mathcal{E}$ then $B \cap A \neq \emptyset$ because $\mathcal{E}$ is directed, so $x \in\downarrow_L B$. Thus $x \in i(\mathcal{E})$. Similarly, if $x \in\uparrow_L A$ then $x \in f(\mathcal{E})$. Therefore $i(\mathcal{E}) \cup f(\mathcal{E}) = L$.

Let $a \in i(\mathcal{E})$. Since $a \notin f(\mathcal{E})$ there is $B \in \mathcal{E}$ such that $a \notin\uparrow_L B$. But B is closed, so there is $b \in L$ such that $a < b$ and $B \subseteq\uparrow_L b$. If $A \in \mathcal{E}$ then $A \cap B \neq \emptyset$ so $b \in\downarrow_L A$. Thus $b \in i(\mathcal{E})$ and a is not

the greatest element of $i(\mathcal{E})$. Similarly, $f(\mathcal{E})$ does not have a least element. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

For every $k \in K$ let $\mathbf{Dir}(k)$ be the set of all $\mathcal{E} \in \mathbf{Dir}$ such that $\langle i(\mathcal{E},) f(\mathcal{E}) \rangle = k$. If we know that $k = \langle P, Q \rangle$ then we will use the notation $\mathbf{Dir}(P, Q)$. The elements of $\mathbf{Dir}(k)$ are those elements of $\mathbf{Dir}$ which "congregate" around the cut k. If $L = \mathbb{R}^*$ and $k = \langle \mathbb{R}^-, \mathbb{R}^+ \rangle$ then $\{[-2^{-n}, 0) : n \in \omega\}$, $\{[-2^{-n}, 2^{-n}] : n \in \omega\}$, and $\{(0, 2^{-n}] : n \in \omega\}$ are all elements of $\mathbf{Dir}(k)$. Every element of $\mathbf{Dir}(k)$ is a directed subset of $\mathcal{C}$ which does not have an upper bound in $\mathcal{C}$. So $\mathcal{C}$ by itself is not directed complete. By adding these cuts to $\mathcal{C}$ we will obtain an ordered set which is directed complete. For now we will think of this ordered set X as consisting of the cuts along with the closed bounded convex subsets of L. The order on this expanded set must ensure that the cuts appear in the maximal elements of the ordered set X. However, we wish to make the set of cuts closed in $\max X$. In order for this to happen, we need to make sure that the singletons of L which will appear in $\max X$ are the suprema of directed sets whose elements are not weakly way below any cut. To obtain this, we will add new directed sets which will converge to the cuts, but none of whose elements are above any of the bounded closed convex subsets of L already in X. These new directed sets will also ensure that X is exact. The easiest way to accomplish this is to add a copy of $\mathcal{C}$. Before we do this, we need to further develop the properties of cuts in relation to directed sets from $\mathcal{C}$, the set of closed bounded convex subsets of L.

For every $k \in K$ let $\mathcal{A}(k) = \cup \mathbf{Dir}(k)$. Again, if we know that $k = \langle P, Q \rangle$ then we will use the notation $\mathcal{A}(P, Q)$. This set will determine which elements of $\mathcal{C}$ or its copy will be less than the cut k. $\mathcal{A}(k)$ is the set of all closed bounded subsets of L which are included in some element of $\mathbf{Dir}(k)$. These are subsets of L which are adjacent to the cut k. If $L = \mathbb{R}^*$ and $k = \langle \mathbb{R}^-, \mathbb{R}^+ \rangle$ then $[-1, 0) \in \mathcal{A}(k)$ and $[-1, 1] \in \mathcal{A}(k)$, but $[1, 2] \notin \mathcal{A}(k)$.

We will next select a special representative from each $\mathbf{Dir}(k)$. For every $\langle P, Q \rangle \in K$ let $\mathcal{S}(P, Q) = \{A \in \mathcal{C} : A \cap P \neq \emptyset \text{ and } A \cap Q \neq \emptyset\}$. In other words, $\mathcal{S}(k)$ is the set of those closed bounded convex subsets of L which straddle the cut k. Then $\mathcal{S}(k)$ is a nonempty element of $\mathbf{Dir}(k)$ for every $k \in K$. These representatives will

determine the elements of X that should be weakly way below the cut k.

Lemma 14. *If $A, B \in \mathcal{C}$ with $A \subseteq B$ and $A \in \mathcal{A}(k)$ for some $k \in K$ then $B \in \mathcal{A}(k)$.*

This follows from the fact that if $A \in \mathcal{E} \in \mathbf{Dir}(k)$ and $A \subseteq B$ then $\mathcal{E} \cup \{B\} \in \mathbf{Dir}(k)$.

Lemma 15. *Let $\mathcal{B} \subseteq \mathcal{C}$ such that $\cap \mathcal{B} \neq \emptyset$. If there is $k \in K$ such that $\mathcal{B} \subseteq \mathcal{A}(k)$ then $\cap \mathcal{B} \in \mathcal{A}(k)$.*

Proof. Let $k = \langle P, Q \rangle$ and let $B = \cap \mathcal{B}$. Then $B \in \mathcal{C}$. If, for every $A \in \mathcal{S}(k)$, $A \cap B \neq \emptyset$ then $B \in \mathcal{A}(k)$. We will show that this must be the case. Assume that there is $A \in \mathcal{S}(k)$ such that $A \cap B = \emptyset$. In particular, $B \notin \mathcal{S}(k)$, so either $B \subseteq P$ or $B \subseteq Q$. Assume that $B \subseteq P$. Then $x < y$ for all $x \in B$ and $y \in A$. Fix $y \in A \cap P$. There is $C \in \mathcal{B}$ such that $y \notin C$ and therefore $x < y$ for all $x \in C$. Let $\mathcal{E} \in \mathbf{Dir}(k)$ such that $C \in \mathcal{E}$. Then $y \notin i(\mathcal{E})$. This contradicts the fact that $i(\mathcal{E}) = P$. We get a similar contradiction when $B \subseteq Q$. $\square$

We are now ready to define the weak domain and its order. Let $\mathcal{C}_0 = \mathcal{C} \times \{0\}$ and $\mathcal{C}_1 = \mathcal{C} \times \{1\}$. Set $X = K \cup \mathcal{C}_0 \cup \mathcal{C}_1$. The set $\mathcal{C}_1$ corresponds to the original set $\mathcal{C}$ of closed bounded subsets of L and $\mathcal{C}_0$ is its copy. To shorten the notation we will follow set-theoretic convention and use 2 to represent the set $\{0, 1\}$. For every $a = \langle A, j \rangle \in \mathcal{C} \times 2$ let $\pi_1(a) = A$ and $\pi_2(a) = j$. Define the order $\sqsubseteq$ on X by setting $a \sqsubseteq b$ if and only if one of the following conditions is satisfied.

(1) $a = b$
(2) $a \in \mathcal{C} \times 2$, $b \in K$, and $\pi_1(a) \in \mathcal{A}(b)$.
(3) $a, b \in \mathcal{C} \times 2$, $\pi_1(b) \subseteq \pi_1(a)$, and $\pi_2(a) \leq \pi_2(b)$.

Property 3 is basically the superset relation. Note that no element of $\mathcal{C}_1$ lies below an element of $\mathcal{C}_0$. In Property 2 we are comparing convex subsets of L to cuts. A cut will be larger than a convex set if the convex set is adjacent to the cut.

It is straightforward to check that $\sqsubseteq$ is an order. The set of maximal elements of X is $K \cup \{\langle \{x\}, 1 \rangle : x \in L\}$ and every element of X is below one of these maximal elements.

We next show that X is directed complete. Note that if $\mathcal{E}$ is a directed subset of X and $\mathcal{E}$ contains a maximal element m of X then $m = \sup\mathcal{E}$. So in Lemma 16 through Lemma 18 we will assume that $\mathcal{E}$ is a nonempty directed subset of X which contains no maximal elements of X. In particular, $\mathcal{E} \subseteq \mathcal{C} \times 2$. Also note that $\pi_1(\mathcal{E})$ is a directed subset of $\mathcal{C}$.

Lemma 16. *If* $\cap\pi_1[\mathcal{E}] = \emptyset$ *then* $\sup\mathcal{E} = \langle i(\pi_1[\mathcal{E}]), f(\pi_1[\mathcal{E}])\rangle$.

Proof. Since $\cap\pi_1[\mathcal{E}] = \emptyset$ we know that no element of $\mathcal{C} \times 2$ is an upper bound of $\mathcal{E}$. Furthermore, $k = \langle i(\pi_1[\mathcal{E}]), f(\pi_1[\mathcal{E}])\rangle$ is a cut which is an upper bound of $\mathcal{E}$. Let $\langle P, Q\rangle \in K$ be an upper bound of $\mathcal{E}$. Let $a \in \mathcal{E}$. There is $\mathcal{F} \in \mathbf{Dir}(P, Q)$ such that $\pi_1(a) \in \mathcal{F}$. Then $P = i(\mathcal{F}) \subseteq \downarrow_L \pi_1(a)$ and $Q = f(\mathcal{F}) \subseteq \uparrow_L \pi_1(a)$. So $P \subseteq i(\pi_1[\mathcal{E}])$ and $Q \subseteq f(\pi_1[\mathcal{E}])$. It follows that $k = \langle P, Q\rangle$ and $k = \sup\mathcal{E}$. □

Lemma 17. *If* $\mathcal{E} \cap \mathcal{C}_1 \neq \emptyset$ *and* $\cap\pi_1[\mathcal{E}] \neq \emptyset$ *then* $\sup\mathcal{E} = \langle\cap\pi_1[\mathcal{E}], 1\rangle$.

Proof. Note that $\cap\pi_1[\mathcal{E}] \in \mathcal{C}$ and $\langle\cap\pi_1[\mathcal{E}], 1\rangle$ is an upper bound of $\mathcal{E}$. Let b be an upper bound of $\mathcal{E}$. Either $b \in \mathcal{C}_1$ or $b \in K$. If $b \in \mathcal{C}_1$ then $\pi_1(b) \subseteq \cap\pi_1[\mathcal{E}]$ and $\langle\cap\pi_1[\mathcal{E}], 1\rangle \sqsubseteq b$. If $b \in K$ then $\pi_1[\mathcal{E}] \subseteq \mathcal{A}(b)$. By Lemma 15, $\cap\pi_1[\mathcal{E}] \in \mathcal{A}(b)$ so $\langle\cap\pi_1[\mathcal{E}], 1\rangle \sqsubseteq b$. □

Note that the fact that $\mathcal{E}$ is directed was not used in the preceding proof and the lemma is true for subsets of X which are not directed. The proof of the following lemma is the same.

Lemma 18. *If* $\mathcal{E} \subseteq \mathcal{C}_0$ *and* $\cap\pi_1[\mathcal{E}] \neq \emptyset$ *then* $\sup\mathcal{E} = \langle\cap\pi_1[\mathcal{E}], 0\rangle$.

The previous three lemmas show that every nonempty directed subset of X has a supremum, so X is directed complete. In fact, X has the stronger property of being bounded complete. An ordered set is *bounded complete* if and only if it is directed complete and every nonempty subset which is bounded above has a supremum.

Proposition 19. X *is bounded complete.*

Proof. Let $\mathcal{U}$ be a nonempty subset of X which has an upper bound. If $\mathcal{U}$ contains a maximal element of X then that maximal element is the only possible upper bound of $\mathcal{U}$, so we may assume that $\mathcal{U} \subseteq \mathcal{C} \times 2$. If an element of $\mathcal{C} \times 2$ is an upper bound of $\mathcal{U}$ then $\cap\pi_1[\mathcal{U}] \neq \emptyset$. The comments after Lemma 17 show that Lemmas 17 and 18 apply, so $\mathcal{U}$ has a supremum.

Assume that no element of $\mathcal{C} \times 2$ is an upper bound of $\mathcal{U}$. Then the only upper bounds of $\mathcal{U}$ are cuts and $\cap \pi_1[\mathcal{U}] = \emptyset$. We will show that there is exactly one cut which is an upper bound of $\mathcal{U}$. Assume not, and let $j, k \in K$ such that $j \neq k$ and both j and k are upper bounds of $\mathcal{U}$. Let $j = \langle P, Q \rangle$ and $k = \langle R, S \rangle$. Since $j \neq k$ either $Q \cap R \neq \emptyset$ or $P \cap S \neq \emptyset$. That is, one cut must occur sooner within L than the other cut. Assume that $Q \cap R \neq \emptyset$. Let $a \in \mathcal{U}$ and let $\mathcal{D} \in \mathbf{Dir}(j)$ and $\mathcal{E} \in \mathbf{Dir}(k)$ such that $a \in \mathcal{D} \cap \mathcal{E}$. Then $Q = f(\mathcal{D}) \subseteq \uparrow_L \pi_1(a)$ and $R = i(\mathcal{E}) \subseteq \downarrow_L \pi_1(a)$ so $Q \cap R \subseteq \pi_1(a)$. This contradicts $\cap \pi_1[\mathcal{U}] = \emptyset$. Therefore $\mathcal{U}$ is bounded by a unique cut and has a supremum. The assumption that $P \cap S \neq \emptyset$ yields the same result. $\square$

In order to finish the proof that X is a weak domain we need to know what $\ll_w$ looks like in X. The next lemmas will establish the criteria for $a \ll_w b$.

Lemma 20. *If $a \in X$ and $b \in K$ then $a \ll_w b$ if and only if $a \in \mathcal{S}(b) \times \{0\}$.*

Proof. If $a \ll_w b$ then $a \sqsubseteq b$. But $\mathcal{S}(b) \times \{0\}$ is a directed subset of X whose supremum is b and none of the elements of $\mathcal{S}(b) \times \{0\}$ is above any element of $K \cup \mathcal{C}_1$, so $a \in \mathcal{C}_0$. Also, there is $c \in \mathcal{S}(b) \times \{0\}$ such that $a \sqsubseteq c$. Therefore $\pi_1(c) \subseteq \pi_1(a)$ and $\pi_1(a) \in \mathcal{S}(b)$.

Now assume that $a \in \mathcal{S}(b) \times \{0\}$ and let $\mathcal{E}$ be a directed subset of X such that $\sup \mathcal{E} = b$. If $\mathcal{E} \cap K \neq \emptyset$ then $b \in \mathcal{E}$ and $a \sqsubset b$. So we may assume that $\mathcal{E} \subseteq \mathcal{C} \times 2$ and that $\cap \pi_1[\mathcal{E}] = \emptyset$. Therefore $\langle i(\mathcal{E}), f(\mathcal{E}) \rangle = b$. Let $b = \langle P, Q \rangle$. Since $\pi_1(a) \in \mathcal{A}(b)$, $\pi_1(a) \cap P \neq \emptyset$ and $\pi_1(a) \cap Q \neq \emptyset$. Let $p \in \pi_1(a) \cap P$ and $q \in \pi_1(a) \cap Q$. Since $i(\mathcal{E}) = P$ and $f(\mathcal{E}) = Q$ we know that $p \notin f(\mathcal{E})$ and that there is $e \in \mathcal{E}$ such that $\pi_1(e) \subseteq \uparrow_L p$. Also, since $q \notin i(\mathcal{E})$ there is $g \in \mathcal{E}$ such that $\pi_1(g) \subseteq \downarrow_L q$. Let $c \in \mathcal{E}$ such that $\pi_1(c) \subseteq \pi_1(e) \cap \pi_1(g)$. Then $\pi_1(c) \subseteq [p, q] \subseteq \pi_1(a)$ and $a \sqsubset c$. $\square$

Lemma 21. *If $a \in X$ and $b \in \mathcal{C}_1$ then $a \ll_w b$ if and only if $a \in \mathcal{C} \times 2$ and there are $p, q \in \pi_1(a)$ such that $\pi_1(b) \subseteq \mathrm{Int}[p, q]$.*

Proof. If $a \notin \mathcal{C} \times 2$ then $a \not\sqsubseteq b$. Assume that $a \in \mathcal{C} \times 2$ and there are no $p, q \in \pi_1(a)$ such that $\pi_1(b) \subseteq \mathrm{Int}[p, q]$. We may also assume that $a \sqsubseteq b$. This means that $\pi_1(b) \subseteq \pi_1(a)$. Let $\mathcal{E}$ be the set of all $c \in \mathcal{C}_1$ such that there are $p, q \in \pi_1(c)$ with $\pi_1(b) \subseteq \mathrm{Int}[p, q]$. Then $\mathcal{E}$ is a nonempty directed subset of $\mathcal{C}_1$ and b is an upper bound

of $\mathcal{E}$. We will show that $\pi_1(b) = \cap \pi_1[\mathcal{E}]$ so that $b = \sup \mathcal{E}$. Since no element of $\mathcal{E}$ is above a it follows that $a \not\ll_w b$.

Let $p, q \in L$ such that $\pi_1(b) \subseteq [p, q]$. If $\pi_1(b) = [p, q] = L$ then $\pi_1(a) = [p, q]$ and $\pi_1(b) = \text{Int}[p, q]$, contradicting our assumption that $\pi_1(a)$ contains no such p and q. We may therefore assume that $L - \pi_1(b) \neq \emptyset$. The elements of $L - \pi_1(b)$ could all lie below $\pi_1(b)$, or they could all lie above $\pi_1(b)$, or some could be below and others above. First assume that if $x \in L - \pi_1(b)$ then $x < y$ for all $y \in \pi_1(b)$, and let $x \in L - \pi_1(b)$. Since $\pi_1(b)$ is closed there is $z \in L$ such that $x < z$ and $z \leq y$ for all $y \in \pi_1(b)$. If $(x, z) = \emptyset$ then set $r = z$. If not, pick $r \in (x, z)$. In either case, $\pi_1(b) \subseteq \text{Int}[r, q]$ so $\langle [r, q], 1 \rangle \in \mathcal{E}$ and $x \notin \cap \pi_1[\mathcal{E}]$. Similar arguments show that $x \notin \cap \pi_1[\mathcal{E}]$ for all $x \in L - \pi_1(b)$ no matter how the elements of $L - \pi_1(b)$ lie in relation to $\pi_1(b)$. Therefore $\pi_1(b) = \cap \pi_1[\mathcal{E}]$.

To prove the other direction assume that $a \in \mathcal{C} \times 2$ and that there are $p, q \in \pi_1(a)$ such that $\pi_1(b) \subseteq \text{Int}[p, q]$. Let $\mathcal{E}$ be a directed subset of X such that $\sup \mathcal{E} = b$. Then $\mathcal{E} \cap \mathcal{C}_1 \neq \emptyset$ and $\pi_1(b) = \cap \pi_1[\mathcal{E}]$. In fact, since $\mathcal{E}$ is directed, we may assume that $\mathcal{E} \subseteq \mathcal{C}_1$. If there are $x, y \in \pi_1(a)$ such that $x < p$ and $q < y$ then there is $c \in \mathcal{E}$ such that $\pi_1(c) \subseteq (x, y)$. Thus $a \sqsubset c$. If not then either $\pi_1(a) \subseteq \uparrow_L p$ or $\pi_1(a) \subseteq \downarrow_L q$.

If $\pi_1(a) \not\subseteq \downarrow_L q$ then $\pi_1(a) \subseteq \uparrow_L p$ and there is $y \in \pi_1(a)$ such that $q < y$. If $p \notin \pi_1(b)$ then there is $c \in \mathcal{E}$ such that $\pi_1(c) \subseteq (p, y)$. Then $\pi_1(c) \subsetneq \pi_1(a)$ and $a \sqsubset c$. If $p \in \pi_1(b)$ then $p \in \text{Int}[p, q]$ so either p is the least element of L or there is $x < p$ such that $(x, p) = \emptyset$. If p is the least element of L choose any element c of $\mathcal{E}$ such that $y \notin \pi_1(c)$. Then $\pi_1(c) \subseteq [p, y) \subsetneq \pi_1(a)$. If there is $x < p$ such that $(x, p) = \emptyset$ then there is $c \in \mathcal{E}$ such that $\pi_1(c) \subseteq (x, y) \subsetneq \pi_1(a)$. In either case $a \sqsubset c$. A similar argument shows that if $\pi_1(a) \not\subseteq \uparrow_L p$ then there is $c \in \mathcal{E}$ such that $a \sqsubset c$.

Finally, assume that $\pi_1(a) \subseteq (\uparrow_L p) \cap (\downarrow_L q) = [p, q]$. Then $\pi_1(a) = [p, q]$. If $p \in \pi_1(b)$ then $p \in \text{Int}[p, q]$ so either p is the least element of L or there is $x < p$ such that $(x, p) = \emptyset$. If p is the least element of L then $\pi_1(c) \subseteq \uparrow_L p$ for every $c \in \mathcal{E}$. Assume that there is $x < p$ such that $(x, p) = \emptyset$. Since $x \notin \pi_1(b)$ there is $e \in \mathcal{E}$ such that $x \notin \pi_1(e)$ and $\pi_1(e) \subseteq \uparrow_L p$. But if $p \notin \pi_1(b)$ then there is $e \in \mathcal{E}$ such that $p \notin \pi_1(e)$ and $\pi_1(e) \subseteq \uparrow_L p$. Similarly, there is $f \in \mathcal{E}$ such that $\pi_1(f) \subseteq \downarrow_L q$. Since $\mathcal{E}$ is directed, there is $c \in \mathcal{E}$

such that $\pi_1(c) \subseteq \pi_1(e) \cap \pi_1(f) \subseteq (\uparrow_L p) \cap (\downarrow_L q) = \pi_1(a)$. Thus $a \sqsubseteq c$ and $a \ll_w b$. $\qquad\square$

The following lemma has essentially the same proof.

Lemma 22. *If $a \in X$ and $b \in \mathcal{C}_0$ then $a \ll_w b$ if and only if $a \in \mathcal{C}_0$ and there are $p, q \in \pi_1(a)$ such that $\pi_1(b) \subseteq \mathrm{Int}[p, q]$.*

We can now complete the proof that X is a weak domain.

Lemma 23. *X is exact.*

Proof. For every $k \in K$, $\downarrow_w k = \mathcal{S}(k) \times \{0\}$ which is directed and has supremum k. For every $b \in \mathcal{C}_1$, $\downarrow_w b$ is the set of all $a \in \mathcal{C} \times 2$ such that there are $p, q \in \pi_1(a)$ with $\pi_1(b) \subseteq \mathrm{Int}[p, q]$. Then $\downarrow_w b$ is directed and $\sup \downarrow_w b = b$. For every $b \in \mathcal{C}_0$, $\downarrow_w b$ is the set of all $a \in \mathcal{C}_0$ such that there are $p, q \in \pi_1(a)$ with $\pi_1(b) \subseteq \mathrm{Int}[p, q]$. Then $\downarrow_w b$ is directed and $\sup \downarrow_w b = b$. $\qquad\square$

Lemma 24. *$\ll_w$ is weakly increasing in X.*

Proof. Let $a, b, c \in X$ with $a \ll_w b \sqsubseteq c$. Assume that there is $d \in X$ such that $c \ll_w d$. Then $a, b, c \in \mathcal{C} \times 2$. There are $p, q \in \pi_1(a)$ such that $\pi_1(b) \subseteq \mathrm{Int}[p, q]$. But $\pi_1(c) \subseteq \pi_1(b)$, so $a \ll_w c$. $\qquad\square$

So X is a weak domain. The final step is to show that L is homeomorphic to $\{\{x\} : x \in L\} \times \{1\} = M$ with the topology it inherits from the *wwb*-topology on X.

Lemma 25. *L is homeomorphic to M, which is an open dense subset of X.*

Proof. The obvious candidate for a homeomorphism is the function $\phi : L \to M$ given by $\phi(x) = \langle \{x\}, 1 \rangle$. It is clearly one-to-one and onto. We just need to show that it and its inverse are continuous.

Let U be an open subset of L and let $\langle \{x\}, 1 \rangle \in \phi[U]$. Then $x \in U$ and there are $p, q \in L$ such that $x \in \mathrm{Int}[p, q] \subseteq U$. So $[p, q] \in \mathcal{C}$ and $\langle \{x\}, 1 \rangle \in M \cap \hat{\uparrow}_w \langle [p, q], 1 \rangle \subseteq \phi[U]$, and $\phi[U]$ is open in M.

Let U be an open subset of M and let $x \in \phi^{-1}[U]$. Then $\langle \{x\}, 1 \rangle \in U$. There is $a \in X$ such that $\langle \{x\}, 1 \rangle \in M \cap \hat{\uparrow}_w a \subseteq U$. There are $p, q \in A$ such that $x \in \mathrm{Int}[p, q]$. Then $\mathrm{Int}[p, q] \subseteq \phi^{-1}[U]$ so $\phi^{-1}[U]$ is open in L and ϕ is a homeomorphism.

The set M is an open dense subset of $\max X$ in this topology. For if $a \in C_0$ and $\Uparrow_w a \cap \max X \neq \emptyset$ then there are $p, q \in \pi_1(a)$ such that $\text{Int}[p, q] \neq \emptyset$ and $\Uparrow_w a$ must contain at least one element of M. But $\Uparrow_w a \subseteq M$ for all $a \in \mathcal{C}_1$. $\hfill\square$

We can use this construction to show that the Scott topology and the wwb-topology can generate different topologies on the set of maximal elements.

Example 26. *If $L = \mathbb{Q}$ then the Scott topology and the wwb-topology generate different topologies on $\max X$.*

Let L be a LOTS in which the cuts are dense. That is, if $p, q \in L$ and $p < q$ then there is a cut $\langle P, Q \rangle$ of L such that $p \in P$ and $q \in Q$. The set of rational numbers is, of course, such a LOTS. We have seen that under the wwb-topology there are open subsets of $\max X$ which do not contain any cuts. When $L = \mathbb{Q}$ these open subsets do not contain any irrationals. Let U be a Scott open subset of X. Then U is increasing, so it must contain a maximal element b of X. If $b \in K$ then we already have $U \cap K \neq \emptyset$. Assume that $b \notin K$. Then $b \in \mathcal{C}_1$, so there is $x \in L$ such that $b = \langle \{x\}, 1 \rangle$. The set $\mathcal{E} = \{a \in \mathcal{C}_1 : x \in \pi_1(a)\}$ is a directed subset of X whose supremum is a. But U must capture an element of every directed subset of X whose supremum is in U, so there are $p, q \in L$ such that $p < q$ and $\langle [p, q], 1 \rangle \in U$. But there is a cut $\langle P, Q \rangle$ such that $p \in P$ and $q \in Q$. In other words, $\langle [p, q], 1 \rangle \sqsubset \langle P, Q \rangle$. Therefore $\langle P, Q \rangle \in U$. So if V is an open subset of M generated by the Scott topology then V must contain a cut. If we have $L = \mathbb{Q}$ then M is the set of real numbers, the topology on M generated by the Scott topology is the usual topology, and in the topology on M generated by the wwb-topology the rational numbers form an open subset. So these two topologies are very different.

4. QUESTIONS

What is the real difference between weak domains and domains? The thing which seems to allow the difference is the fact that $\Uparrow_w a$ need not be increasing. Because of this, the weak domain X constructed above is not necessarily a domain. Here is why. If the LOTS in our example contains a closed convex interval A which itself contains a cut, that is $A \in \mathcal{S}(k)$ for some cut k, then $\Downarrow\langle A, 1 \rangle$,

the set of all elements of X which are way below $\langle A, 1 \rangle$, is a subset of C_0. This is because in trying to determine which elements of X are way below $\langle A, 1 \rangle$, one must consider directed sets whose suprema are cuts that lie within A. Only elements of $\mathcal{C}_0$ lie below the elements of such directed sets that are nontrivial. Therefore $\sup \downarrow\langle A, 1\rangle \sqsubset \langle A, 1\rangle$ and X is not continuous.

For elements a of $\mathcal{C}_1$, $\Uparrow_w a$ is not increasing. Again, the cuts mess things up. But if $a \in \mathcal{C}_0$ then $\Uparrow_w a$ *is* increasing, which is easy to show based on the results of the previous section. This means that the ordered set $Y = K \cup \mathcal{C}_0$ *is* a domain and that $\Uparrow_w a = \Uparrow a$ for all $a \in Y$. So $\{\Uparrow a : a \in Y\}$ generates the Scott topology on $\max Y$. The set $\max Y$ is the set $K \cup \{\langle\{x\}, 0\rangle : x \in L\}$ which is seen to be in one-to-one correspondence with $\max X$. The topology we get for $\max Y$ can be achieved on $\max X$ by taking the set $\{\Uparrow_w a : a \in \mathcal{C}_0\}$ as the basis for the topology on X. So by eliminating the elements of the weak domain which keep it from being a domain, we get a domain. The space of maximal elements that results is the same as the space of maximal elements of the original weak domain, but with a courser topology. For example, if L is the set of rationals then $\max Y$ becomes the set of reals with its usual topology while in $\max X$ the rationals have neighborhoods containing only rationals and the irrationals have their usual neighborhoods.

Question 1. *Does every weak domain representable topological space have a courser topology under which it is domain representable?*

We know that weak domain representable spaces need not be Baire, but is there some other property which they must have? That is some property, other than T_1, which all weak domain representable spaces must have.

Question 2. *Is there a LOTS or an incomplete metric space or even just a T_1 space which is not weak domain representable?*

Added in Proof: The author has shown that the answer to the question above is yes: the rational numbers are not weak domain representable. Details will appear in a future paper.

The main theorem of this paper provides a process whereby a LOTS is, in some sense, completed. Can such a process be extended to other types of spaces?

Question 3. *Is every T_1 topological space an open dense subset of a weak domain representable space?*

If the cuts of the LOTS are sufficiently "dense" then the space constructed from the LOTS will not be locally compact. It is well known that locally compact T_2 spaces are domain representable.

Question 4. *Will every locally compact weak domain representable space be domain representable?*

Obviously any locally compact space that is not domain representable cannot be T_2.

Question 5. *Will every LOTS that is a weak domain representable space be domain representable?*

Question 6. *Must every Baire weak domain representable space be domain representable?*

Added in Proof: In a private communication the author learned that Bennet and Lutzer answered Question 6 in the negative. See [4] for details.

Finally, the question naturally arises whether a GO-space will have the same properties as a LOTS. A GO-space, or generalized order space, is a subspace of a LOTS with the topology generated by the order topology of the LOTS. Duke and Lutzer [7] have recently shown that every GO-space constructed on a locally compact LOTS is Scott domain representable. A Scott domain is a domain in which every pair of elements which is bounded above has a least upper bound. Since every GO-space is a dense subset of a LOTS, we know that it is the dense subset of a weak domain representable space. But it is not necessarily an open subset. Is the structure of a GO-space and LOTS different in this respect?

Question 7. *Is every GO-space a dense open subset of a weak domain representable space?*

References

[1] S. Abramsky and A. Jung, *Domain Theory.* In S. Abramsky, D. M. Gabbay, T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume III, Oxford University Press, 1994.

[2] H. Bennett and D. Lutzer, *Domain-Representable Spaces*, Fundamenta Math. **189** (2006), 255-268

[3] H. Bennett and D. Lutzer, *Domain-representability of certain complete spaces*, Houston Journal of Mathematics, **34** (2008), no. 3, 753–772

[4] H. Bennett and D. Lutzer, *Domain representable spaces and completeness*, Topology Proceedings, To Appear

[5] K. Ciesielski, R. Flagg, and R. Kopperman, *Polish spaces, computable approximations, and bitopological spaces*, Topology Appl. **119** (2002), 241–256.

[6] B. Coecke and K. Martin, *A partial order on classical and quantum states*, Oxford University Computing Laboratory, Research Report PRG-RR-02-07, August 2002,
     `http://web.comlab.ox.ac.uk/oucl/publications/tr/rr-02-07.html`.

[7] K. Duke and D. Lutzer, *Scott Domain Representability of a Class of Generalized Ordered Spaces*, Topology Proceedings **32** (2008), 1–14.

[8] G. Gierz, K. Hoffman, K. Keimel, J. Lawson, M. Mislove, and D. Scott, *A Compendium of Continuous Lattices*, Springer-Verlag, Berlin, 1980

[9] K. Martin, *Non-classical techniques for models of computation*, Topology Proceedings **24** (1999), 375–405.

[10] K. Martin, *The space of maximal elements of a compact domain*, Electonic Notes in Theoretical Computer Science **40** (2001), 10 pp.

[11] K. Martin, *Ideal models of spaces*, Theoretical Computer Science **305** (2003), 277–297.

[12] K. Martin, *Topological games in domain theory*, Topology Appl. **129** (2003), 177–186.

[13] K. Martin, M. Mislove, and G. Reed, *Topology and domain theory*, in *Recent Progress in General Topology II* edited by J. Hušek and J. van Mill, Elsevier, Amsterdam, 2002.

[14] J. Mashburn, *A comparison of three topologies on ordered sets*, Toplogy Proceedings **31** (2007), 197–218.

[15] M. Mislove, *Topology, domain theory and theoretical computer science*, Topology Appl. **89** (1998), 3–59.

[16] C. Mummert and F. Stephan, *Toplogical aspects of poset spaces*, preprint

[17] D. Scott, *Outline of a mathematical theory of computation*, Technical Monograph PRG-2, November 1970.

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