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The Least Fixed Point Property for ω-chain Continuous Functions

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THE LEAST FIXED POINT PROPERTY FOR ω-CHAIN CONTINUOUS FUNCTIONS

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I. Introduction. A partially ordered set P is ω-chain complete if every countable chain (including the empty set) in P has a supremum. A function f from P to a partially ordered set Q is ω-chain continuous if for every nonempty countable chain C in P which has a supremum, $f(\sup_P C) = \sup_Q f(C)$. Notice that an ω-chain continuous function must preserve order. P has the (least) fixed point property for ω-chain continuous functions if every ω-chain continuous function from P to itself has a (least) fixed point.

It has been shown [6] that a partially ordered set does not have to be ω-chain complete to have the least fixed point property for ω-chain continuous functions. This answers a question posed by G. Plotkin in 1978. I. I. Kolodner has shown [4] that an ω-chain complete partially ordered set has the least fixed point property for ω-chain continuous functions. Plotkin and Smythe [11] and others have used ω-chain complete partially ordered sets in their study of models for theoretical computer science in order to have fixed or least fixed point properties. The result should also be compared with G. Markowsky's result [5] that to have the least fixed point property (every order preserving function has a least fixed point) a partially ordered set must be chain complete. It is the purpose of this paper to look at some cases in which ω-chain completeness and the least fixed point property for ω-chain continuous functions are equivalent.

An ω-chain continuous function from a partially ordered set, P, to a nonempty subset, X, of P is a retraction of P if it is the identity map on X. A nonempty subset of P is a retract of P if it is the image of a retraction on P. X is closed in P if it contains the suprema of all of its nonempty countable chains which have suprema in P. This first lemma from [6] will be used later.

LEMMA 1. *An unbounded countable chain* $C = \{c_n | n \in N\}$ *of a partially ordered set* P *is a retract of* P *if and only if there is a collection of disjoint closed subsets of* P, $\{D_n | n \in N\}$ *such that*

(1) $P = \cup_{n \in N} D_n$,

(2) *For all* $n \in N$, $c_n \in D_n$,

(3) *If* $p \in D_m$; $q \in D_n$; *and* $p \leqslant q$, *then* $m \leqslant n$.

II. Finite width. The partially ordered sets in the first three examples of [6] all contained infinite antichains. It is also clear that any chain has the least fixed point property for ω-chain continuous functions if and only if it is ω-chain complete. So it seems reasonable that any partially ordered set in which the cardinalities of all its antichains are finite and bounded would have the least fixed point property for ω-chain continuous functions if and only if it is ω-chain complete. Before this is shown to be true, some notation and a definition are in order.

DEFINITION. A partially ordered set P has finite width if there is a positive integer N such that, for every antichain A of P, the cardinality of A is less than or equal to N.

For every element x of P, the lower end generated by x, denoted $\downarrow x$, is the set of all elements of P less than or equal to x. For a subset X of P, the lower end generated by X, denoted $\downarrow X$, is $\cup_{x \in X} \downarrow x$. The sets $\uparrow x$ and $\uparrow X$ are defined dually.

LEMMA 2. *Let* C *be a countable chain in a partially ordered set* P *and let* U *be the set of upper bounds of* C. *If* U *contains a chain* D *which is not bounded below by any element of* U *and if* C *is a retract of* $\cap_{d \in D} \downarrow d$, *then* P *does not have the fixed point property for* ω*-chain continuous functions.*

PROOF. Let D′ be a subchain of D such that every subset of D′ has a largest element and every element of D has an element of D′ below it. Denote D′ by $\{d_\alpha : \alpha < \sigma\}$ for some ordinal σ. Set E_0 equal to $P - \downarrow d_0$ and for every $\alpha > 0$ let E_α be the set $(\downarrow d_\alpha) - (\downarrow d_{\alpha+1})$. Let f be a retraction of $\cap_{d \in D} \downarrow d$ onto C. Define a function g: $P \to D' \cup C$ by

$$g(x) = \begin{cases} d_\alpha & \text{if } x \in E_\alpha \\ f(x) & \text{if } x \in \cap_{d \in D} \downarrow d. \end{cases}$$

Let x and y be lements of P such that $x \leqslant y$. If x and y are elements of $\cap_{d \in D} \downarrow d$, then $g(x) = f(x) \leqslant f(y) = g(y)$. If x is an element of $\cap_{d \in D} \downarrow d$ and y is an element o $P - \cap_{d \in D} \downarrow d$, then g(x) is in C; g(y) is in D′; and $g(x) \leqslant g(y)$. If x and y are elements o $P - \cap_{d \in D} \downarrow d$, then let $\alpha < \sigma$ such that y is an element of E_α. Then if x is in E_β, β mus be at least as large as α. Thus $g(x) = d_\beta \leqslant d_\alpha = g(y)$. So g preserves order.

Let X be a nonempty countable chain in P which has a supremum in P. If sup(X is an element of $\cap_{d \in D} \downarrow d$, then X is a subset of $\cap_{d \in D} \downarrow d$ and g(sup(X)) = f(sup(X)) sup(f(X)) = sup(g(X)). If sup(X) is an element of $P - \cap_{d \in D} \downarrow d$ then, because g preserve order and $\cap_{d \in D} \downarrow d$ is closed, it may be assumed that X is contained in $P - \cup_{d \in D} \downarrow d$ Let $\alpha < \sigma$ such that sup(X) is in E_α. Since $\downarrow E_{\alpha+1}$ equals $\downarrow d_{\alpha+1}$ and is closed, X canno contained in $\downarrow E_{\alpha+1}$. Thus, from some point on, every element of X is in E_α. Hence g(sup(X)) equals sup(g(X)). Therefore, C ∪ D′ is a retract of P.

Because C ∪ D′ does not have the fixed point proprety for ω-chain continuou functions, P does not either.

LEMMA 3. *Let* C *be a countable chain in a partially ordered set* P *and let* U *b the set of upper bounds of* C. *If* U *contains two minimal elements* x *and* y *with common upper bound* z *and if* C *is a retract of* $(\downarrow x) \cap (\downarrow y)$, *then* P *does not have th least fixed point property for* ω*-chain continuous functions.*

PROOF. Let f be a retraction from $(\downarrow x) \cap (\downarrow y)$ onto C and define a functio $g: P \to C \cup \{x,y,z\}$ by

$$g(p) = \begin{cases} f(p) & \text{if } p \in (\downarrow x) \cap (\downarrow y) \\ x & \text{if } p \in (\downarrow x) - (\downarrow y) \\ y & \text{if } p \in (\downarrow y) - (\downarrow x) \\ z & \text{if } p \in P - [(\downarrow x) \cup (\downarrow y)] \end{cases}$$

Let P and q be elements of P such that $p \leqslant q$. The function g obviously preserve order in each of the subsets of P given in the definition of g. If p is an element c $(\downarrow x) \cap (\downarrow y)$ and q is not, then g(p) is an element of C and g(q) is in {x,y,z}. Therefor g(p) is less than g(q). Assume that p is an element of $(\downarrow x) - (\downarrow y)$. Then q is an elemen either of $(\downarrow x) - (\downarrow y)$ or of $P - [(\downarrow x) \cup (\downarrow y)]$. In either case, g(p) is less than or equal t g(q). The other case being analogous, g preserves order.

Since each of the subsets given in the definition of g is closed and g is ω-chai continuous on all of them, g is ω-chain continuous on P. Because C ∪ {x,y,z} doe not have the least fixed point property for ω-chain continuous functions, P does not either.

The two previous theorems should be compared with Rival's result in [8] that if P has the fixed point property, C is a chain in P, and U is the set of upper bounds of C, then U has the fixed point property.

A subset X of a partially ordered set P is an upper end if and only if X equals $\uparrow X$. It is a lower end if and only if X equals $\downarrow X$.

LEMMA 4. *Let* P *be a partially ordered set. Let* A *and* B *be nonempty disjoint subsets of* P *where* A *has finite width and* B *is an upper end of* P. *Then there are only a finite number of elements of* B *which are suprema of countable chains in* A.

PROOF. Let A have width n and let $C_1, C_2, \ldots, C_n$ be disjoint chains whose union is A. Let $b \in B$ be the supremum of a countable chain C of A. For some i, $C \cap C_i$ is cofinal in C, so $b = \sup_P(C \cap C_i)$. In fact, $b = \sup_P C_i$. If b is not an upper bound of C_i then there is a $y \in C_i$ such that $b \not\geqslant y$. If $y \geqslant x$ for all $x \in C$, then $y > b$. But B is an upper end and $A \cap B = \phi$, so this is a contradiction. Thus there is some $x \in C$ such that $y \not\geqslant x$. Now $x \notin C_i$ since $x \in C_i$ implies $b > x > y$. Because $C \cap C_i$ is cofinal in C there is a $z \in C \cap C_i$ with $z > x$. Then $y \geqslant z$ contradicts $y \not\geqslant x$, so $z > y$. Thus $b > z > y$, b is an upper bound of C_i and $b = \sup_P C_i$.

THEOREM 1. *If* P *is a partially ordered set of finite width, then any unbounded chain in* P *of order type* ω *is a retract of* P.

PROOF. Let $C = \{c_n : n \in N\}$ be an unbounded chain of P. Set E_1 equal to $P - (\uparrow c_2)$ and for every $n \geqslant 2$ let E_n be the set $(\uparrow c_n) - (\uparrow c_{n+1})$. For every n in N, let L_n' be the set of all the elements of $P - E_n$ which are suprema of nonempty countable chains in E_n. Let L_n equal $L_n' - \cup_{k<n} L_k'$. By Lemma 4 each L_n is finite. For every $m,n \in N$, let K_{nm} be the set $L_n \cap E_m$. Notice that K_{nm} is empty if $n \geqslant m$.

Let p be an element of K_{nm} and let V_p'' be an open upper end containing p. Because it is an upper end, V_p'' is also closed. Let V_p' be the intersection of V_p'' with $\uparrow c_n$. There may be elements of V_p' which are suprema of nonempty countable chains in $P - (\uparrow c_n)$. Let A be the collection of all such elements of V_p'. By Lemma 4, A is finite, so $V_p = V_p' - \cup_{a \in A}(\downarrow a)$ is an open upper end. Now let U_p be the set $(V_p - \cup_{n \leqslant k \leqslant m}(\downarrow c_k)) \cap (\cup_{n \leqslant k \leqslant m} E_k)$. This is an open set contained in $\cup_{n \leqslant k \leqslant m} E_k$ and is an upper end in that set. Hence it is open and closed with respect to $\cup_{n\leq k\leq m}E_k$. Define D_n by

$$D_n = [E_n \cup (\cup_{k=1}^{n} \cup_{p\in K_{kn}} U_p)] - \cup_{k=1}^{n} \cup_{n<j} \cup_{q\in K_{kj}} U_q.$$

The following properties will be shown to be true.

(1) $P = \cup_{n\in N} D_n$,

(2) if $n \neq m$, then $D_n \cap D_m = \phi$,

(3) for all n, $c_n \in D_n$,

(4) if $x,y \in P$; $x \leq y$; $x \in D_m$; and $y \in D_n$, then $m \leq n$,

(5) each D_n is closed.

(1) Let x be an element of P. Since P equals $\cup_{n\in N}E_n$, x must be in E_n for some n in N. If x is not in D_n, then there is some $1 \leq a \leq n$ and some $b > n$ and some $p \in K_{ab}$ such that x is an element of U_p. Let M be the set $\{j > n$: $\exists\, 1 \leq k \leq n$, $\exists\, q \in K_{kj} \ni x \in U_q\}$ and set m equal to max(M). Then x is an element of $E_m \cup (\cup_{k=1}^{n} \cup_{p\in K_{km}} U_p)$. Let $1 \leq a \leq m$; let $b > m$; and let p be an element of K_{kj}. Then U_q is a subset of $\cup_{a\leq i<b}E_i$. So if x is an element of U_q, then $a \leq n$. But if $k \leq n$, then b is an element of M, contradicting the choice of m. Thus x is not an element of $\cup_{k=1}^{m} \cup_{j>m} \cup_{q\in K_{km}} U_q$ and x is in D_m.

(2) Let m and n be positive integers such that $m < n$. Then the intersection of E_m with E_n is empty. Let $1 \leq a \leq m$ and let p be an element of K_{am}. Then U_p is a subset of $\cup_{a\leq i<m}E_i$ which has an empty intersection with E_n. Therefore the intersection of $\cup_{k=1}^{m} \cup_{p\in K_{km}} U_p$ with E_n is empty. Assume that $E_m \cap (\cup_{k=1}^{m} \cup_{p\in K_{kn}} U_p)$ is not empty and let x be an element of K_{an} where $1 \leq a \leq m$ such that $E_m \cap U_x$ is not empty. But then U_x is a subset of $\cup_{k=1}^{m} \cup_{j>m} \cup_{q\in K_{kj}} U_q$; so $(E_m - (\cup_{k=1}^{m} \cup_{j>m} \cup_{q\in K_{kj}} U_q)) \cap (\cup_{k=1}^{n} \cup_{p\in K_{kn}} U_p)$ is empty. Now assume that $(\cup_{k=1}^{m} \cup_{p\in K_{km}} U_p) \cap (\cup_{j=1}^{n} \cup_{q\in K_{jn}} U_q)$ is not empty. Then there exist $1 \leq a \leq m$, $1 \leq b \leq n$, $x \in K_{am}$, and $y \in K_{bn}$ such that $U_x \cap U_y$ is not empty. If $b \leq m$, then U_y is a subset of $\cup_{k=1}^{m} \cup_{j>m} \cup_{q\in K_{kj}} U_q$ and $D_m \cap U_p$ is empty. Because U_y is a subset of $\cup_{b\leq i\leq n}E_i$ and U_x is a subset of $\cup_{a\leq i\leq m}E_i$, b is less than or equal to m. Therefore, $D_m \cap U_p$ is empty. Thus, $D_m \cap D_n$ is empty.

(3) For every n in N, c_n is an element of E_n. By the way they were defined, no U_p contains a c_n. So c_n is an element of D_n.

(4) Let x and y be elements of P such that $x \leqslant y$, x is an element of D_m and y is an element of D_n. Let x be an element of E_a and y an element of E_b. Since x is less than y, a must be less than or equal to b. Obviously a is less than or equal to m and b is less than or equal n. If m is equal to a, then $m \leqslant b \leqslant n$. If m is greater than a, then there exists $1 \leqslant k \leqslant a$ and $j > a$ and an element p of K_{kj} such that x is an element of U_p. If there are no such j's greater than b, then $m \leqslant b \leqslant n$. If $j > b$, then U_p is an upper end of $\cup_{k \leqslant i < j} E_i$, so y is an element of U_p. Therefore, m is less than or equal to n.

(5) Let n be an element of N and let X be a nonempty countable chain in D_n. If X is a subset of E_n, then either sup(X) is an element of E_n or sup(X) is an element of K_{kj} for some $1 \leqslant k \leqslant n$ and $j > n$. If sup(X) is an element of K_{kj}, then the intersection of X with $U_{\sup(X)}$ is not empty and X is not a subset of D_n. For the same reason, sup(X) cannot belong to U_p for any element p of K_{kj}, where $1 \leqslant k \leqslant n$ and $j > n$. Thus, sup(X) is an element of D_n. If X is a subset of $\cup_{k=1}^{n} \cup_{p \in K_{kn}} U_p$, then it may be assumed that X is a subset of U_p for some element p of K_{kn} where $1 \leqslant k \leqslant n$. Assume that sup(X) is not in U_p and that sup(X) is not in E_n. Let sup(X) be an element of E_m. Since U_p is a subset of $\cup_{k \leqslant i < n} E_i$ and U_p is closed in this set, m must be greater than n. Again, it may be assumed that X is a subset of E_j for some $k \leqslant j \leqslant n$. But then sup(X) is an element of K_{im} for some $1 \leqslant i \leqslant n$ and the intersection of X with $\cup_{k=1}^{n} \cup_{j > n} \cup_{q \in K_{kj}} U_q$ is not empty, contradicting the assumption that X is a subset of D_n. Therefore, sup(X) is an element of D_n.

It follows from Lemma 1 that C is a retract of P.

Since there is a retract of P which does not have the fixed point property for ω-chain continuous functions, it doesn't either. This result is similar to that of Duffus, et. al., [2], which states that for any partially ordered set P every maximal chain C is the image of an order preserving function on P which is the identity on C.

THEOREM 2. *If a partially ordered set with finite width has the least fixed point property for ω-chain continuous functions, then it is ω-chain complete.*

PROOF. By Theorem 1, it may be assumed that every countable chain in a partially ordered set P is bounded. Let C be a nonempty countable chain in P which does not have a supremum. Let U be the set of upper bounds of C.

By Lemmas 2 and 3 it may be assumed that U contains a set of minimal elements M, that every element of U is above some element of M, and that no two elements of M have a common upper bound. Denote the elements of M by $m_1, m_2, \ldots, m_r$. Let D be the set of all elements x of P such that $\uparrow x$ is not contained in $\downarrow(\uparrow m_i) - \cup_{j \neq i} \downarrow(\uparrow m_j)$ for any $1 \leqslant i \leqslant r$.

Let $X = \{x_n : n \in N\}$ be a chain in D which has a supremum in P. There are two cases to consider. First, there could be a cofinal subchain of X each element of which lies below an element of $\uparrow m_i$ and an element of $\uparrow m_j$ where $i \neq j$. Second, from some point on each element of X may lie below an element of P which is not itself below an upper bound of C.

Assume that for every n in N, there are $i \geqslant n$, $1 \leqslant j_i \neq k_i \leqslant r$, and $y_{j_i} \geqslant m_{j_i}$, and $y_{k_i} \geqslant m_{k_i}$ such that x_i is less than both y_{j_i} and y_{k_i}. Then there are an infinite number of pairs, (j_i, k_i), and one of the pairs, (j,k), must be repeated an infinite number of times. Let Y_j be the set $\{y_{j_i} : j_i = j\}$ and Y_k the set $\{y_{k_i} : k_i = k\}$. If Y_j if finite, then it contains an element which is an upper bound of X. If it is infinite, it must contain an infinite chain which, by assumption, is bounded. Any upper bound of this chain is, of course, an upper bound of X. In either case there is a $y_j \geqslant m_j$ which is an upper bound of X. Similarly, there is a $y_k \geqslant m_k$ which is an upper bound of X. Thus, sup(X) is less than both y_j and y_k and sup(X) is an element of D.

Assume that for every n in N, there is a y_n in $P - \cup_{i=1}^{r} \downarrow(\uparrow m_i)$ such that $x_n \leqslant y_n$. Let Y be the set of all these y_n's. If Y is finite, one of its elements must be an upper bound of X. If Y is infinite, it must contain an infinite chain which, by assumption, is bounded. Any upper bound of this chain will be an upper bound of X. In either case there is an element y of $P - \cup_{i=1}^{r} \downarrow(\uparrow m_i)$ which is an upper bound of X. Then sup(X) is less than or equal to y and sup(X) is an element of D. Therefore D is closed.

Since C is unbounded in D, there is, by Theorem 1, a retraction f of D onto a cofinal subchain C′ of C. Define a function, $g: P \to C' \cup M$ by

$$g(x) = \begin{cases} m_i & \text{if } x \in \downarrow(\uparrow m_i) - D,\ 1 \leqslant i \leqslant r \\ f(x) & \text{if } x \in D \end{cases}$$

Let x and y be elements of P with $x \leqslant y$. If x is in D, then $g(x) \leqslant g(y)$. If x is not in D, then there is some $1 \leqslant i \leqslant r$ such that $\uparrow x$ is a subset of $\downarrow(\uparrow m_i) - \cup_{j \neq i} \downarrow(\uparrow m_j)$. But then y is an element of $\downarrow(\uparrow m_i) - D$, so $g(x) = m_i = g(y)$. Therefore, g preserves order.

Clearly, g is ω-chain continuous on D and on $\downarrow(\uparrow m_i) - D$ for each $1 \leqslant i \leqslant r$. Thus g is ω-chain continuous on P; $C' \cup M$ is a retract of P; and P cannot have the least fixed point property for ω-chain continuous function.

COROLLARY 1. *A partially ordered set* P *having finite width is ω-chain complete if and only if it has the least fixed point property for ω-chain continuous functions.*

III. Layered partially ordered sets. The proof of Theorem 2 relied heavily on Lemma 4, which is no longer true if partially ordered sets not of finite width are considered. Another way must be found if Theorem 3 is to be generalized.

In the first three examples of [6], the partially ordered sets all contained a countable chain which sneaked around an antichain, that is, they all contained an antichain which generates a nonclosed lower end. Notice also that the lower end generated by a finite antichain will always be closed. Thus no partially ordered set of finite width could have an antichain which generates a lower end that is not closed. Let P be a partially ordered set containing an unbounded chain $C = \{C_n : n \in \mathbf{N}\}$. Let $\{A_n : n \in \mathbf{N}\}$ be a collection of anitchains of P such that $C_n \in \downarrow A_n$. If $P = \cup_{n \in \mathbf{N}} \downarrow A_n$ and for each n, $\downarrow A_n$ is closed, then C is a retract of P. This is the approach which will be taken next.

The first question which arises concerning this method is: can a partially ordered set acutally be partitioned in such a manner? The answer is that not all can be. For example, if a partially ordered set contains a maximal chain which has no countable cofinal subchain, then it cannot be the union of a countable number of lower ends generated by antichains. Such chains must therefore be avoided. Even so, it is not true that such partitions occur in general.

DEFINITION. A partially ordered set P is layered if and only if there is a collection $\{A_n : n \in \mathbf{N}\}$ of antichains of P such that P is equal to $\cup_{n \in \mathbf{N}} \downarrow A_n$.

Assume that any partially ordered set in which all chains are countable is layered. Well order the real numbers, **R**, and define a new order on **R** by setting $x \leqslant y$ if and only if x is less than or equal to y in the usual order and in the well order. F. B. Jones has shown [3] that every chain and antichain in **R** with this new order is countable.

Also he shows that every chain is well-ordered. By assumption, this set is layered. Thus there is antichain which generates an uncountable lower end. Since the antichain is countable, one of its elements, x_1, also generates an uncountable lower end. Again, this lower end is assumed to be layered. In this way an infinite chain $x_1 > x_2 > x_3 > \cdots$ can be generated. But this contradicts the well-ordering of all chains. Thus, only assuming the well-ordering of **R**, not all partially ordered sets are layered.

THEOREM 3. *If every antichain of a layered partially ordered set* P *generates a closed lower end, then any unbounded chain in* P *of order type* ω *is a retract of* P.

PROOF. Let $\{A_n: n \in \mathbf{N}\}$ be a collection of antichains of P such that P equals $\cup_{n\in\mathbf{N}} \downarrow A_n$. First, assume that for every n in **N**, $\downarrow A_n$ is a subset of $\downarrow A_{n+1}$. Let $C = \{c_n: n \in \mathbf{N}\}$ be an unbounded chain in P and assume that C is not contained in $\downarrow A_n$ for any n. Set d_n equal to the largest element of C contained in $\downarrow A_n$, if there are any elements of C in $\downarrow A_n$, and c_1 if there are not. Define a function, $f: P \to C$, by

$$f(x) = \begin{cases} c_1 & \text{if } x \in \downarrow c_1 \\ d_1 & \text{if } x \in (P - \downarrow C) \cap (\downarrow A_1) \\ \min(c_n, d_k) & \text{if } x \in [(\downarrow c_n) - (\downarrow c_{n-1})] \cap [(\downarrow A_k) - (\downarrow A_{k-1})] \text{ and } n,k > 1 \\ d_n & \text{if } x \in (P - \downarrow C) \cap ((\downarrow A_n) - (\downarrow A_{n-1})) \text{ and } n > 1 \end{cases} .$$

Then f preserves order and is the identity on C. Let X be a nonempty countable chain in P which has a supremum in P. If $\sup(X) \in \downarrow c_1$, then $X \subset \downarrow c_1$ and $f(\sup(X)) = \sup(f(X))$.

Assume $\sup(X) \in [(\downarrow c_n) - (\downarrow c_{n-1})] \cap [(\downarrow A_k) - (\downarrow A_{k-1})]$ for some $n,k > 1$. Assume further that $f(\sup(X)) = c_n$. Either f maps a cofinal subchain of X to $\{c_j: j \leqslant n\}$ or it maps a cofinal subchain of X to $\{d_j: d_j \leqslant c_n\}$.

Let Y be a cofinal subchain of X which f maps to $\{c_j: j \leqslant n\}$. If no element of Y is mapped to c_n, then $Y \subset \downarrow c_{n-1}$. But $\sup(Y) = \sup(X) \notin \downarrow c_{n-1}$, a contradiction. Thus $\sup(f(X)) = \sup(f(Y)) = f(\sup(X))$.

Let Z be a cofinal subchain of X which f maps to $\{d_j: d_j \leqslant c_n\}$. Since $c_n < d_k$, $Z \subset \downarrow A_{k-1}$. But $\sup(Z) = \sup(X) \notin \downarrow A_k$, a contradiction. So no such chain can exist.

Assume that $f(\sup(X)) = d_k$. Let Y be a cofinal subchain of X which f maps to $\{c_j: j \leqslant n\}$. Since $d_k < c_n$, $Y \subset \downarrow c_{n-1}$. But $\sup(Y) = \sup(X) \notin \downarrow c_{n-1}$, a contradiction.

So no such chain can exist.

Let Z be a cofinal subchain of X which f maps to $\{d_j\colon d_j \leqslant c_n\}$. If no element of Z is mapped to d_k then $Z \subset \downarrow A_i$ for some $i < k$. But $\sup(Z) = \sup(X) \notin \downarrow A_i$ for any $i < k$. Thus $\sup(f(X)) = \sup(f(Z)) = f(\sup(X))$.

Similar arguments can be used for the cases $\sup(X) \in (P - \downarrow C) \cap (\downarrow A_1)$ and $\sup(X) \in (P - \downarrow C) \cap ((\downarrow A_n) - (\downarrow A_{n-1}))$ where $n > 1$.

If there is an n in **N** such that C is a subset of $\downarrow A_n$, then it may be assumed that C is a subset of $\downarrow A_n$ for every n in **N**. Set D_1 equal to $\cup_{n\in N}\downarrow(A_n - (\uparrow c_2))$. For all $m > 1$, set D_m equal to $(\cup_{n\in N}\downarrow((A_n - (\uparrow c_{m+1})) \cap (\uparrow c_m))) - \cup_{j<m}D_j$. Let x be an element of P, n an element of **N**, and a an element of A_n such that x is less than a. Now a is not an upper bound of C, so either there is an m in **N** such that a is an element of $(A_n - (\uparrow c_{m+1})) \cap (\uparrow c_m)$ or it is an element of $A_n - (\uparrow c_2)$. Thus, either x is in D_m or x is in D_1. Hence P equals $\cup_{n\in N}D_n$. Let X be a nonempty countable chain in D_m which has a supremum in P. Since P equals $\cup_{n\in N}\downarrow A_n$ there is a k in **N** such that X is contained in $\downarrow A_k$. Then X is contained in $(\downarrow A_k) \cap D_m$ which is equal to $\downarrow((A_k - (\uparrow c_{n+1})) \cap (\uparrow c_n))$ for some $n > 1$ or $\downarrow(A_k - (\uparrow c_2))$. In either case, X is a subset of a lower end generated by an antichain. But such a lower end is closed, so sup(X) is an element of D_m. Thus, D_m is closed. Let x and y be elements of P such that x is less than y. Assume that x is an element of D_m and that y is an element of D_n. Clearly, if y is an element of $\downarrow((A_k - (\uparrow c_{n+1})) \cap (\uparrow c_n))$, then x must be also. Therefore, m is less than or equal to n. By Lemma 1, C is a retract of P.

The requirement that every antichain of P generate a closed lower end is not necessary for the first part of the previous proof. There it is only needed that each of the A_n's generate a closed lower end. The requirement is necessary in the second part, however, as is shown by the following example.

EXAMPLE 1. Let Y be the collection of infinite countable subsets of ω_1 which don't have a largest element. Let P be the set

$$\omega_1 \cup (\cup_{\alpha\in\omega_1-\omega}\cup_{A\in Y}(\alpha \times A \times \{A\})).$$

Let $\{X_\alpha\colon \alpha \in \omega\}$ be a partition of $\omega_1 - \omega$ such that, for every α, X_α is infinite. If x and y are elements of P, set $x \leqslant y$ if and only if one of the following conditions hold.

(1) $x \in \omega$; $y \in X_\alpha$ for some $\alpha \in \omega$; and $x \leqslant \alpha$ in ω,

(2) $x = (\alpha,\beta,A) \in \alpha \times A \times \{A\}$; $y = (\alpha,\delta,A) \in \alpha \times A \times \{A\}$; and $\beta \leqslant \delta$,

(3) $x = (\alpha,\beta,A) \in \alpha \times A \times \{A\}$ and $y = \alpha$ or $y = \beta$,

(4) $x,y \in \omega$ and $x \leqslant y$ in ω,

(5) $x = 0$ (the least element of ω).

Clearly, ω is an unbounded chain in P; $\omega_1 - \omega$ is an antichain; and $\downarrow(\omega_1 - \omega)$ is all of P and is closed. Let f be an ω-chain continuous function from P to itself and assume that f(P) is a subset of ω. Then there is an element m of ω such that $f^{-1}(m)$ is infinite. Let A be an element of Y which is contained in $f^{-1}(m)$. Then for every α in $\omega_1 - \omega$, $f(\alpha \times A \times \{A\})$ is a subset of $\{1,2,...,m\}$. Since α is the supremum of $\alpha \times A \times \{A\}$, it must be less than or equal to m. Hence for every $n \geqslant m$, f(n) is less than or equal to m. Thus, no cofinal subchain of ω is a retract of P.

In fact, P has the least fixed point property for ω-chain continuous functions. Assume that f(0) is not 0. Since $P - \omega$ is ω-chain complete, it may be assumed that $\{f^n(0): n \in N\}$ is contained in ω. By the argument above, it may also be assumed that there is an α in $\omega_1 - \omega$ such that $f(\alpha)$ is an element of $\omega_1 - \omega$. Let $f(\alpha)$ be an element of X_m. For every $n \geqslant m$, let A_n be the element of Y contained in X_n. Since α is the supremum of $\alpha \times A_n \times \{A_n\}$, there is an element β of A_n such that $f((\alpha,\beta,A))$ equals $f(\alpha)$. Therefore, $f(\beta)$ equals $f(\alpha)$ and f(n) is less than or equal to m. It follows that f has a least fixed point.

THEOREM 4. *Let* P *be a partially ordered set in which every antichain generates a closed lower end and every lower end which contains a countable chain but none of its upper bounds is layered. If* P *has the least fixed point property for ω-chain continuous functions, then* P *is ω-chain complete.*

PROOF. Assume that P contains a nonempty countable chain C which does not have a supremum. By Theorem 3 and Lemmas 2 and 3, it may be assumed that every countable chain in P is bounded; that the set U of upper bounds of C has a set M of minimal elements; that every element of U is above an element of M; and that no two elements of M have a common upper bound.

Let D be the set of all the elements x of P such that $\uparrow x$ is not contained in $(\downarrow(\uparrow m)) - \cup_{n \in M - \{m\}} \downarrow(\uparrow n)$ for any m in M. Let $X = \{x_n: n \in N\}$ be a chain in D which has a supremum in P. Assume that for every k in N, there is a $j \geqslant k$, elements

$m_{r_j} \neq m_{s_j}$ of M, and y_{r_j} of $\uparrow m_{r_j}$ and y_{s_j} of $\uparrow m_{s_j}$ such that x_j is less than both y_{r_j} and y_{s_j}. Let Y_r be the set of all the y_{r_j}'s and Y_s the set of all the y_{s_j}'s.

Assume that both Y_r and Y_s are infinite and that Y_r contains an infinite antichain, A_r. Since the lower end generated by an antichain is closed, sup(X) is below all but a finite number of the elements of A_r. Let Z_s be those elements of Y_s which are paired with the elements of A_r. If Z_s has an infinite antichain A_s then sup(X) is below all but a finite number of the elements of A_s. Thus, there is a j in N such that sup(X) is less than y_{r_j} and y_{s_j}. Then sup(X) is an element of D.

If every antichain of Z_s is finite, then it contains an infinite chain, C_s, which, by assumption, is bounded. Let y_s be an upper bound for this chain. Then sup(X) is less than or equal to y_s. Let B_r be the set of elements of A_r which are paired with the elements of C_s. Since B_r is an antichain and X is contained in $\downarrow B_r$, sup(X) is in $\downarrow B_r$. Let y_r be an element of B_r which is above sup(X). Then y_r and y_s are elements of U which are above distinct elements of M and are both greater than sup(X). Thus sup(X) is in D.

If Y_r is finite, then there is a j in N such that sup(X) $\leqslant y_{r_j}$. If Y_s is finite, then there is a k in N such that sup(X) $\leqslant y_{s_k}$. If both are finite then we take j = k. If only one is finite, then the arguments above show that sup(X) is in D.

Assume that all antichains in both Y_r and Y_s are finite. Let C_r be an infinite chain in Y_r; let y_r be an upper bound of C_r; and let m_r be the element of M less than or equal to y_r. Let Z_s be the set of elements of Y_s which are paired with the elements of C_r. Then no element of Z_s is greater than m_r. Since every antichain of Z_s is finite, it must contain an infinite chain C_s. Let y_s be an upper bound of C_s and let m_s be the element of M less than or equal to y_s. Then m_r does not equal m_s and sup(X) is less than both y_r and y_s. Therefore, sup(X) is in D.

Assume that for every k in N, there is a $y_k \geqslant x_k$ such that y_k is not contained in $\downarrow(\uparrow m)$ for any m in M. Let Y be the set of all the y_k's. Assume that Y is infinite. If Y contains an infinite antichain, then sup(X) is in the lower end it generates and is therefore in D. If Y does not contain an infinite antichain, then it contains an infinite chain which is bounded. Let y be an upper bound for this chain. Then y is above sup(X) and is not in $\downarrow(\uparrow m)$ for any m in M. Therefore, D is closed.

Since C is unbounded in D, there is, by Theorem 3, a retraction f of D onto C′, cofinal subchain of C of order type ω. Let m_1 and m_2 be distinct elements of M Define a function g: $P \rightarrow C' \cup \{m_1, m_2\}$, by

$$g(x) = \begin{cases} m_1 & \text{if } \uparrow x \subset (\downarrow(\uparrow m_1)) - D \\ m_2 & \text{if } \uparrow x \subset (\downarrow(\uparrow m)) - D \text{ and } m \neq m_1 \\ f(x) & \text{if } x \in D \end{cases}$$

Then f preserves order. Since the sets $(\downarrow(\uparrow m_1)) - D$ an $\cup_{m \in M - \{m_1\}}((\downarrow(\uparrow m)) - D)$ are closed and g is ω-chain continuous on them and on D it is ω-chain continuous on P. Thus, $C' \cup \{m_1, m_2\}$ is a retract of P and P cannot hav the least fixed point property for ω-chain continuous functions.

COROLLARY 2. *If every antichain of a partially ordered set* P *generates closed lower end and every lower end of* P *which contains a countable chain but non of its upper bounds is layered, then* P *is ω-chain complete if and only if it has th least fixed point property for ω-chain continuous functions.*

The following theorem by Edwin Miller appears as Theorem B in [4].

THEOREM 5. *If every antichain of an uncountable partially ordered set* P *i finite, then* P *contains an uncountable chain.*

COROLLARY 3. *If every antichain of a partially ordered set* P *is finite an every chain contains a countable cofinal subchain, then* P *is layered.*

PROOF. Assume that there is no countable collection Y of antichains of P sucl that P equals $\cup_{A \in Y} \downarrow A$. Let A_1 be a maximal antichain of P. If β is less than ω_1 and for every α less than β, A_α has been defined, then let A_β be a maximal antichain o $P - \cup_{\alpha < \beta} \downarrow A_\alpha$. The set $A = \cup_{\alpha < \omega_1} A_\alpha$ is an uncountable partially ordered set and, b Theorem 5, contains an uncountable chain, C. But C must intersect an uncountabl number of A_α's and so cannot contain a countable cofinal subchain.

COROLLARY 4. *If every antichain of a partially ordered set* P *is finite an every chain contains a countable cofinal subchain, then* P *is ω-chain complete if an only if it has the least point property for ω-chain continuous functions.*

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